A note on the logarithmic (p, p') fusion

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Abstract

The procedure in [Fuchs et al.] to obtain fusion algebra from the modular transformation of characters in logarithmic conformal field models is extended to the (p, p') logarithmic models.

1 Introduction

This paper is a remark on fusion in a class of logarithmic models of conformal field theory [1–3]. In rational conformal field models, fusion is related to modular transformations of characters by the celebrated Verlinde formula [4,5]. Because the Verlinde formula relies on the fact that the fusion algebra is semisimple, it does not immediately extend to logarithmic conformal field theories, where fusion algebras (starting with the pioneering results in [6]) are typically nonsemisimple. The known extensions of the Verlinde formula to the nonsemisimple realm rely on some extra input, in one form or another [7] (also see [8]). In the prescription proposed in [9], this extra input can be related to a quantum-group formulation.

The role of quantum groups in logarithmic conformal field theory gradually emerged in [10–13] (see [14] for a summary and [15] for some further development), leading to a version of the Kazhdan–Lusztig "duality" between the extended algebra W in a logarithmic conformal field model and the corresponding quantum group $\mathfrak{g}$.[1] The most remarkable result related to the Kazhdan–Lusztig duality is the coincidence of modular group representations (the one generated from the W characters and the one carried by the center of $\mathfrak{g}$); also, the Grothendieck ring of $\mathfrak{g}$ is a natural candidate for the fusion algebra of W-representations (we speak of the K_0-type fusion, see [9,18]).

For the $(p, 1)$ logarithmic models, in particular, this "quantum-group candidate fusion" coincides with the fusion derived in [9] from the characters, thus giving additional support to the procedure proposed in [9]. The aim of this paper is to extend the existing state of consistency to (p, p') logarithmic models: we propose a prescription whereby the modular transformations of the characters of the extended algebra in the (p, p') logarithmic model [12] are converted into a nonsemisimple fusion algebra coinciding with the Grothendieck ring of the corresponding quantum group $\mathfrak{g}$ [13]. For this, we follow the approach in [9] (also see [7]) very closely. In Sec. , we describe our starting point, the modular group representation generated from the characters of the extended algebra of the (p, p') models. In Sec. , we formulate the procedure to convert these modular

[1] These are factorizable ribbon quantum groups at even roots of unity; see [16] for their other use and [17] for an interesting precursor.

transformations to the following fusion algebra on $2pp'$ elements $\mathscr{K}^{\pm}_{r,r'}$ [13]:

$$\mathscr{K}^{\alpha}_{r,r'}\mathscr{K}^{\beta}_{s,s'} = \sum_{\substack{u=|r-s|+1\\ \text{step}=2}}^{r+s-1} \sum_{\substack{u'=|r'-s'|+1\\ \text{step}=2}}^{r'+s'-1} \widetilde{\mathscr{K}}^{\alpha\beta}_{u,u'}, \tag{1}$$

where $\alpha, \beta = \pm 1$ and

$$\widetilde{\mathscr{K}}^{\alpha}_{r,r'} = \begin{cases} \mathscr{K}^{\alpha}_{r,r'}, & 1 \leqslant r \leqslant p, \quad 1 \leqslant r' \leqslant p', \\ \mathscr{K}^{\alpha}_{2p-r,r'} + 2\mathscr{K}^{-\alpha}_{r-p,r'}, & p+1 \leqslant r \leqslant 2p-1, \quad 1 \leqslant r' \leqslant p', \\ \mathscr{K}^{\alpha}_{r,2p'-r'} + 2\mathscr{K}^{-\alpha}_{r,r'-p'}, & 1 \leqslant r \leqslant p, \quad p'+1 \leqslant r' \leqslant 2p'-1, \\ \mathscr{K}^{\alpha}_{2p-r,2p'-r'} + 2\mathscr{K}^{-\alpha}_{2p-r,r'-p'} & \\ \quad + 2\mathscr{K}^{-\alpha}_{r-p,2p'-r'} + 4\mathscr{K}^{\alpha}_{r-p,r'-p'}, & p+1 \leqslant r \leqslant 2p-1, \quad p'+1 \leqslant r' \leqslant 2p'-1. \end{cases}$$

The identity of this associative commutative algebra is given by $\mathscr{K}^{+}_{1,1}$. We also recall from [13] that this algebra is generated by two elements $\mathscr{K}^{+}_{1,2}$ and $\mathscr{K}^{+}_{2,1}$ and can also be described as the quotient of $\mathbb{C}[x, y]$ by the ideal generated by the polynomials

$$\begin{aligned} &U_{2p+1}(x) - U_{2p-1}(x) - 2, \\ &U_{2p'+1}(y) - U_{2p'-1}(y) - 2, \\ &U_{p+1}(x) - U_{p-1}(x) - U_{p'+1}(y) + U_{p'-1}(y), \end{aligned}$$

where

$$U_s(2\cos t) = \frac{\sin st}{\sin t}, \quad s \geqslant 1,$$

are Chebyshev polynomials of the second kind.

2 Modular transformations of the (p, p') characters [12]

For each pair of coprime positive integers p, p', the extended algebra of the logarithmic (p, p') model is the W-algebra $\mathscr{W}_{p,p'}$ identified and studied in [12]. It has $\frac{1}{2}(p-1)(p'-1) + 2pp'$ irreducible representations, the $\frac{1}{2}(p-1)(p'-1)$ of which are just the Virasoro representations in the corresponding (p, p') minimal model and the other are "genuine" $\mathscr{W}_{p,p'}$-representations (such that the radical of $\mathscr{W}_{p,p'}$ acts nontrivially). In what follows, the characters of irreducible $\mathscr{W}_{p,p'}$-representations are denoted as

$$\underset{(r,r')\in\mathscr{I}_0}{\chi_{r,r'}(\tau)}, \quad \underset{1\leqslant r\leqslant p,\ 1\leqslant r'\leqslant p'}{\chi^{+}_{r,r'}(\tau), \quad \chi^{-}_{r,r'}(\tau)} \tag{2}$$

where we introduce the index set

$$\mathscr{I}_0 = \{(r, r') \mid 1 \leqslant r \leqslant p-1,\ 1 \leqslant r' \leqslant p'-1,\ p'r + pr' \leqslant pp'\}, \tag{3}$$

with $|\mathscr{I}_0| = \frac{1}{2}(p-1)(p'-1)$ (we recall the well-known symmetry $\chi_{r,r'}(\tau) = \chi_{p-r,p'-r'}(\tau)$ of the minimal-model Virasoro characters).

The modular (specifically, S-) transformation properties of the characters are as follows. First, the minimal-model characters $\chi_{r,r'}$ are well-known to S-transform as

$$\chi_{r,r'}(-\tfrac{1}{\tau}) = -\frac{2\sqrt{2}}{\sqrt{pp'}} \sum_{(s,s')\in\mathscr{I}_0} (-1)^{rs'+sr'} \sin\frac{\pi p' rs}{p} \sin\frac{\pi p r' s'}{p'} \chi_{s,s'}(\tau), \quad (r, r') \in \mathscr{I}_0. \tag{4}$$

Next, it follows from [12] that (for $1 \leqslant r \leqslant p$ and $1 \leqslant r' \leqslant p'$)

$$\chi^{+}_{r,r'}(-\tfrac{1}{\tau}) = \sum_{s=1}^{p}\sum_{s'=1}^{p'} \mathscr{S}_{r,r';s,s'}(\tau)\big(\chi^{+}_{s,s'}(\tau) + (-1)^{p'r+pr'}\chi^{-}_{s,s'}(\tau)\big) + \sum_{(s,s')\in\mathscr{I}_0} \widetilde{\mathscr{S}}^{+}_{r,r';s,s'}(\tau)\chi_{s,s'}(\tau), \quad (5)$$

$$\chi^{-}_{r,r'}(-\tfrac{1}{\tau}) = \sum_{s=1}^{p}\sum_{s'=1}^{p'} (-1)^{ps'+p's}\mathscr{S}_{r,r';s,s'}(\tau)\big(\chi^{+}_{s,s'}(\tau) + (-1)^{p'r+pr'}\chi^{-}_{s,s'}(\tau)\big) + \sum_{(s,s')\in\mathscr{I}_0} \widetilde{\mathscr{S}}^{-}_{r,r';s,s'}(\tau)\chi_{s,s'}(\tau), \quad (6)$$

where the matrix elements $\mathscr{S}_{r,r',s,s'}(\tau)$ that interest us in what follows are given by

$$\begin{aligned}
\mathscr{S}_{r,r';s,s'}(\tau) &= \tfrac{2\sqrt{2}}{\sqrt{pp'}}(-1)^{rs'+sr'}\big(\tfrac{r}{p}\cos\tfrac{\pi p' r s}{p} - i\tau\tfrac{p-s}{p}\sin\tfrac{\pi p' r s}{p}\big) \\
&\quad\times\big(\tfrac{r'}{p'}\cos\tfrac{\pi p r' s'}{p'} - i\tau\tfrac{p'-s'}{p'}\sin\tfrac{\pi p r' s'}{p'}\big), \qquad 1\leqslant s\leqslant p-1,\ 1\leqslant s'\leqslant p'-1, \\
\mathscr{S}_{r,r';s,p'}(\tau) &= \tfrac{\sqrt{2}}{\sqrt{pp'}}\tfrac{r'}{p'}(-1)^{sr'+pr'+p'r}\big(\tfrac{r}{p}\cos\tfrac{\pi p' r s}{p} - i\tau\tfrac{p-s}{p}\sin\tfrac{\pi p' r s}{p}\big), \qquad 1\leqslant s\leqslant p-1 \\
\mathscr{S}_{r,r';p,s'}(\tau) &= \tfrac{\sqrt{2}}{\sqrt{pp'}}\tfrac{r}{p}(-1)^{s'r+p'r+pr'}\big(\tfrac{r'}{p'}\cos\tfrac{\pi p r' s'}{p'} - i\tau\tfrac{p'-s'}{p'}\sin\tfrac{\pi p r' s'}{p'}\big), \quad 1\leqslant s'\leqslant p'-1 \\
\mathscr{S}_{r,r';p,p'}(\tau) &= \tfrac{1}{\sqrt{2pp'}}\tfrac{rr'}{pp'},
\end{aligned} \quad (7)$$

and the other matrix elements are

$$\begin{aligned}
\widetilde{\mathscr{S}}^{+}_{r,r';s,s'}(\tau) = (-1)^{rs'+sr'}\tfrac{\sqrt{2}}{p^2p'^2\sqrt{pp'}}\Big(&pp'rr'\cos\tfrac{\pi p' r s}{p}\cos\tfrac{\pi p r' s'}{p'} \\
&+ ip'r\tau(ps'-p's)\cos\tfrac{\pi p' r s}{p}\sin\tfrac{\pi p r' s'}{p'} + ipr'\tau(p's-ps')\sin\tfrac{\pi p' r s}{p}\cos\tfrac{\pi p r' s'}{p'} \\
&+ \big(\tfrac{(ps'-p's)^2}{2}\tau^2 - 2i\pi pp'\tau + \tfrac{p^2r'^2+p'^2r^2}{2}\big)\sin\tfrac{\pi p' r s}{p}\sin\tfrac{\pi p r' s'}{p'}\Big),
\end{aligned}$$

$$\widetilde{\mathscr{S}}^{-}_{r,r';s,s'}(\tau) = (-1)^{sp'+s'p}\widetilde{\mathscr{S}}^{+}_{r,r';s,s'}(\tau) - (-1)^{rs'+sr'+sp'+s'p}\tfrac{1}{\sqrt{2pp'}}\sin\tfrac{\pi p' r s}{p}\sin\tfrac{\pi p r' s'}{p'}.$$

3 "Logarithmic" (p,p')-fusion

3.1 The procedure

The steps leading from (5) and (6) to (1), in much the same way as in [9], are as follows.

1. We view the characters in (2) as a column vector and write the S-transformation formulas as

$$\chi(-\tfrac{1}{\tau}) = \mathscr{S}(\tau)\chi(\tau),$$

with the corresponding $N\times N$ τ-dependent matrix $\mathscr{S}(\tau)$, where

$$N = \tfrac{1}{2}(p-1)(p'-1) + 2pp'$$

is the total number of characters.

We then take $\mathbb{S}(\tau)$ to be the $(2pp')\times(2pp')$ block of $\mathscr{S}(\tau)$ corresponding to the $2pp'$ characters $\chi^{\pm}_{r,r'}(\tau)$, $1\leqslant r\leqslant p$, $1\leqslant r'\leqslant p'$. That is, we deal only with the $\mathscr{S}_{r,r';s,s'}(\tau)$ in (7). From now on, $\chi=(\chi_J)$ denotes the $2pp'$ characters ordered as

$$\chi=(\chi^{+}_{p,p'},\chi^{-}_{p,p'},\underbrace{\chi^{+}_{r,p'},\chi^{-}_{p-r,p'}}_{1\leqslant r\leqslant p-1},\underbrace{\chi^{+}_{p,r'},\chi^{-}_{p,p'-r'}}_{1\leqslant r'\leqslant p'-1},\underbrace{\chi^{+}_{r,r'},\chi^{-}_{p-r,r'},\chi^{-}_{r,p'-r'},\chi^{+}_{p-r,p'-r'}}_{(r,r')\in\mathscr{I}_0}). \tag{8}$$

In accordance with this ordering of characters, we fix the block structure of matrices as follows: 2 blocks of size 1×1, $(p-1)+(p'-1)$ blocks of size 2×2, and $\frac{1}{2}(p-1)(p'-1)$ blocks of size 4×4. The matrices used in what follows are square matrices of size $(2pp')\times(2pp')$ with this block structure.

2. Totally similarly to [9], there exists a ($(2pp')\times(2pp')$-matrix) automorphy factor $J(\gamma,\tau)$, for $\gamma\in SL(2,\mathbb{Z})$, satisfying the cocycle condition and a commutativity property formulated in [9], such that $\mathsf{S}=J(S,\tau)\mathbb{S}(\tau)$ is a numerical (τ-independent) matrix, and in fact

$$\mathsf{S}=\mathbb{S}(i). \tag{9}$$

It then follows, in particular, that $\mathsf{S}^2=1$.

Let $\mathsf{S}_\Omega=(\mathsf{S}_\Omega{}^J)$ be the row of S corresponding to the vacuum-representation character $\chi_\Omega=\chi^{+}_{1,1}$, i.e.,

$$\chi^{+}_{1,1}(-\tfrac{1}{\tau})=\mathsf{S}_\Omega{}^J\chi_J(\tau)$$

(the sum is taken over the $2pp'$ values of J in accordance with (8)). With the chosen ordering, χ_Ω occupies position $2p+2p'-1$ in (8) and, accordingly, S_Ω is the $(2p+2p'-1)$th row. Explicitly (see (7)), the segment of S_Ω corresponding to $(\chi^{+}_{s,s'},\chi^{-}_{p-s,s'},\chi^{-}_{s,p'-s'},\chi^{+}_{p-s,p'-s'})$ is given by $\frac{2\sqrt{2}}{pp'\sqrt{pp'}}(-1)^{s'+s}$ times

$$\begin{aligned}\Big(&\big(\cos\tfrac{\pi p's}{p}+(p-s)\sin\tfrac{\pi p's}{p}\big)\big(\cos\tfrac{\pi ps'}{p'}+(p'-s')\sin\tfrac{\pi ps'}{p'}\big),\\ &\big(\cos\tfrac{\pi p's}{p}-s\sin\tfrac{\pi p's}{p}\big)\big(\cos\tfrac{\pi ps'}{p'}+(p'-s')\sin\tfrac{\pi ps'}{p'}\big),\\ &\big(\cos\tfrac{\pi p's}{p}+(p-s)\sin\tfrac{\pi p's}{p}\big)\big(\cos\tfrac{\pi ps'}{p'}-s'\sin\tfrac{\pi ps'}{p'}\big),\\ &\big(\cos\tfrac{\pi p's}{p}-s\sin\tfrac{\pi p's}{p}\big)\big(\cos\tfrac{\pi ps'}{p'}-s'\sin\tfrac{\pi ps'}{p'}\big)\Big).\end{aligned} \tag{10}$$

We also define a special row

$$\mathsf{P}_\Omega=(1,1,\underbrace{1,0,\ldots,1,0}_{2(p-1)\text{ elements}},\underbrace{1,0,\ldots,1,0}_{2(p'-1)\text{ elements}},\underbrace{1,0,0,0,\ldots,1,0,0,0}_{4\cdot\frac{1}{2}(p-1)(p'-1)\text{ elements}}).$$

3. Let K be a block-diagonal matrix of the specified form (with zeros outside the

blocks),

$$\mathsf{K} = \begin{pmatrix} \kappa_1 & & & & & & \\ & \kappa_2 & & & & & \\ & & \boxed{K_{2\times 2}} & & & & \\ & & & \ddots & & & \\ & & & & \boxed{K_{2\times 2}} & & \\ & & & & & \boxed{K_{4\times 4}} & \\ & & & & & & \ddots \\ & & & & & & & \boxed{K_{4\times 4}} \end{pmatrix}, \tag{11}$$

where the two 1×1 blocks are arbitrary, the 2×2 blocks are as in [9], i.e., have the structure

$$K_{2\times 2} = \begin{pmatrix} a & \lambda \\ -a & b\lambda \end{pmatrix}$$

(it is understood that $K^{(i)}_{2\times 2} = \begin{pmatrix} a^{(i)} & \lambda^{(i)} \\ -a^{(i)} & b^{(i)}\lambda^{(i)} \end{pmatrix}$ for each block, but the block dependence is not indicated for brevity), and the 4×4 blocks have the structure

$$K_{4\times 4} = \begin{pmatrix} a & \mu & \nu & \frac{1}{a}\mu\nu \\ -a & -\mu & c\nu & \frac{c}{a}\mu\nu \\ -a & b\mu & -\nu & \frac{b}{a}\mu\nu \\ a & -b\mu & -c\nu & \frac{bc}{a}\mu\nu \end{pmatrix}$$

(again, with the block dependence omitted).

The nonzero factors λ, μ, and ν, rescaling columns 2 through 4 in each block, are irrelevant in what follows (because nilpotent elements have no canonical normalization). The unknowns a and b in each 2×2 block and a, b, and c in each 4×4 block are determined from the equation generalizing the one in [9]:

$$\mathsf{P}_\Omega = \mathsf{S}_\Omega \mathsf{K}. \tag{12}$$

That is, if (s_1, s_2, s_3, s_4) is a segment of S_Ω corresponding to a 4×4 block, then

$$a = \frac{1}{s_1 - s_2 - s_3 + s_4}, \quad b = \frac{s_2 - s_1}{s_3 - s_4}, \quad c = \frac{s_3 - s_1}{s_2 - s_4}$$

in this block; the equations are compatible because $s_1 s_4 = s_2 s_3$, as is readily seen from (10). (By (12), the two elements of K that constitute the 1×1 blocks are just the inverse of the corresponding S-matrix coefficients, just as the denominators in the semisimple Verlinde formula.)

4. We set

$$\mathsf{P} = \mathsf{S}\mathsf{K}.$$

The fusion algebra is reconstructed from the P matrix in much the same way as in [9], as follows. Clearly, the $(2p + 2p' - 1)$th row of P is just P_Ω. We define M_I, $I = 1, \ldots, 2pp'$, to be block-diagonal matrices that solve the equation

$$\mathsf{P}_I = \mathsf{P}_\Omega \, \mathsf{M}_I \tag{13}$$

(where P_I is the Ith row of P) and whose 2×2 blocks are of the form (just as in [9])

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$$

and the 4×4 blocks are

$$\begin{pmatrix} \alpha & \beta & \gamma & \zeta \\ & \alpha & 0 & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix}$$

(with zeros below the diagonal).

The M_I are then determined *uniquely*; in particular, the 4×4 blocks are given by

$$\begin{pmatrix} p_I & q_I & r_I & s_I \\ & p_I & 0 & r_I \\ & & p_I & q_I \\ & & & p_I \end{pmatrix},$$

where (p_I, q_I, r_I, s_I) is a segment of P_I corresponding to the chosen block.

The result is then that the M_I satisfy the algebra

$$\mathsf{M}_I\mathsf{M}_J = \sum_K n_{IJ}^K \mathsf{M}_K, \tag{14}$$

where the nonnegative integer coefficients n_{IJ}^K turn out to be those read off from (1). (Simultaneously, the matrix $\mathsf{N}_I = \mathsf{P}\mathsf{M}_I\mathsf{P}^{-1}$ for each I gives the fusion structure constants as $(\mathsf{N}_I)_J{}^K = n_{IJ}^K$.)

3.2 Examples

The illustrative power of examples is hampered by the rapidly growing matrix size and the general clumsiness of explicit expressions. We consider only the "percolation" and "Lee–Yang" cases, where explicit values of the various matrix entries may be useful for comparison with the studies of these cases by more direct methods (see [19]).

3.2.1 (3, 2) For $(p,p') = (3,2)$, the 12×12 matrix $\mathsf{S} = \mathbb{S}(i)$ explicitly evaluates as

$$\mathsf{S} = \begin{pmatrix}
\frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{2}{\sqrt3} & \frac{2}{\sqrt3} & \frac{2}{\sqrt3} & \frac{2}{\sqrt3} \\
\frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} & \frac{-1}{\sqrt3} & \frac{-2}{\sqrt3} & \frac{-2}{\sqrt3} & \frac{-2}{\sqrt3} & \frac{-2}{\sqrt3} \\
\frac{1}{6\sqrt3} & \frac{1}{6\sqrt3} & \frac{6-\sqrt3}{18} & \frac{-3-\sqrt3}{18} & \frac{-3-\sqrt3}{18} & \frac{6-\sqrt3}{18} & \frac{1}{3\sqrt3} & \frac{1}{3\sqrt3} & \frac{6-\sqrt3}{9} & \frac{-3-\sqrt3}{9} & \frac{6-\sqrt3}{9} & \frac{-3-\sqrt3}{9} \\
\frac{1}{3\sqrt3} & \frac{1}{3\sqrt3} & \frac{-3-\sqrt3}{9} & \frac{3-2\sqrt3}{18} & \frac{3-2\sqrt3}{18} & \frac{-3-\sqrt3}{9} & \frac{2}{3\sqrt3} & \frac{2}{3\sqrt3} & \frac{-2(3+\sqrt3)}{9} & \frac{3-2\sqrt3}{9} & \frac{-2(3+\sqrt3)}{9} & \frac{3-2\sqrt3}{9} \\
\frac{1}{3\sqrt3} & \frac{1}{3\sqrt3} & \frac{-3-\sqrt3}{9} & \frac{3-2\sqrt3}{18} & \frac{3-2\sqrt3}{18} & \frac{-3-\sqrt3}{9} & \frac{-2}{3\sqrt3} & \frac{-2}{3\sqrt3} & \frac{2(3+\sqrt3)}{9} & \frac{2\sqrt3-3}{9} & \frac{2(3+\sqrt3)}{9} & \frac{2\sqrt3-3}{9} \\
\frac{1}{6\sqrt3} & \frac{1}{6\sqrt3} & \frac{6-\sqrt3}{18} & \frac{-3-\sqrt3}{18} & \frac{-3-\sqrt3}{18} & \frac{6-\sqrt3}{18} & \frac{-1}{3\sqrt3} & \frac{-1}{3\sqrt3} & \frac{\sqrt3-6}{9} & \frac{3+\sqrt3}{9} & \frac{\sqrt3-6}{9} & \frac{3+\sqrt3}{9} \\
\frac{1}{4\sqrt3} & \frac{-1}{4\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{-1}{2\sqrt3} & \frac{-1}{2\sqrt3} & \frac{-1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{-1}{\sqrt3} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\
\frac{1}{4\sqrt3} & \frac{-1}{4\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{-1}{2\sqrt3} & \frac{-1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{-1}{2\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} & -\frac{1}{\sqrt3} \\
\frac{1}{12\sqrt3} & \frac{-1}{12\sqrt3} & \frac{6-\sqrt3}{36} & \frac{-3-\sqrt3}{36} & \frac{3+\sqrt3}{36} & \frac{\sqrt3-6}{36} & \frac{-1}{6\sqrt3} & \frac{1}{6\sqrt3} & \frac{\sqrt3-6}{18} & \frac{3+\sqrt3}{18} & \frac{6-\sqrt3}{18} & \frac{-3-\sqrt3}{18} \\
\frac{1}{6\sqrt3} & \frac{-1}{6\sqrt3} & \frac{-3-\sqrt3}{18} & \frac{3-2\sqrt3}{36} & \frac{2\sqrt3-3}{36} & \frac{3+\sqrt3}{18} & \frac{-1}{3\sqrt3} & \frac{1}{3\sqrt3} & \frac{3+\sqrt3}{9} & \frac{2\sqrt3-3}{18} & \frac{-3-\sqrt3}{9} & \frac{3-2\sqrt3}{18} \\
\frac{1}{12\sqrt3} & \frac{-1}{12\sqrt3} & \frac{6-\sqrt3}{36} & \frac{-3-\sqrt3}{36} & \frac{3+\sqrt3}{36} & \frac{\sqrt3-6}{36} & \frac{1}{6\sqrt3} & \frac{-1}{6\sqrt3} & \frac{6-\sqrt3}{18} & \frac{-3-\sqrt3}{18} & \frac{\sqrt3-6}{18} & \frac{3+\sqrt3}{18} \\
\frac{1}{6\sqrt3} & \frac{-1}{6\sqrt3} & \frac{-3-\sqrt3}{18} & \frac{3-2\sqrt3}{36} & \frac{2\sqrt3-3}{36} & \frac{3+\sqrt3}{18} & \frac{1}{3\sqrt3} & \frac{-1}{3\sqrt3} & \frac{-3-\sqrt3}{9} & \frac{3-2\sqrt3}{18} & \frac{3+\sqrt3}{9} & \frac{2\sqrt3-3}{18}
\end{pmatrix}.$$

Here, S_Ω is the 9th row of S. The matrix K in (11) is then given by

$$\mathsf{K}=\begin{pmatrix}
12\sqrt{3} &&&&&&&&&&& \\
& -12\sqrt{3} &&&&&&&&&& \\
&& 4 & 1 &&&&&&&& \\
&& -4 & \frac{7-3\sqrt{3}}{2} &&&&&&&& \\
&&&& 4 & 1 &&&&&& \\
&&&& -4 & \frac{7+3\sqrt{3}}{11} &&&&&& \\
&&&&&& -3\sqrt{3} & 1 &&&& \\
&&&&&& 3\sqrt{3} & 1 &&&& \\
&&&&&&&& -1 & 1 & 1 & -1 \\
&&&&&&&& 1 & -1 & \frac{7-3\sqrt{3}}{2} & \frac{3\sqrt{3}-7}{2} \\
&&&&&&&& 1 & 1 & -1 & -1 \\
&&&&&&&& -1 & -1 & \frac{3\sqrt{3}-7}{2} & \frac{3\sqrt{3}-7}{2}
\end{pmatrix},$$

which gives rise to the fusion-algebra eigenmatrix

$$\mathsf{P}=\mathsf{SK}=\begin{pmatrix}
6 & -6 & 0 & \frac{3(\sqrt{3}-1)}{2} & 0 & \frac{3(1+2\sqrt{3})}{11} & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 6(1-\sqrt{3}) \\
6 & -6 & 0 & \frac{3(\sqrt{3}-1)}{2} & 0 & \frac{3(1+2\sqrt{3})}{11} & 0 & \frac{-2}{\sqrt{3}} & 0 & 0 & 0 & 6(\sqrt{3}-1) \\
2 & -2 & 2 & 0 & -2 & 0 & 0 & \frac{2}{3\sqrt{3}} & 0 & 2 & 0 & 0 \\
4 & -4 & -2 & \frac{3(1-\sqrt{3})}{4} & 2 & \frac{-3(1+2\sqrt{3})}{22} & 0 & \frac{4}{3\sqrt{3}} & 0 & -2 & 0 & 3(\sqrt{3}-1) \\
4 & -4 & -2 & \frac{3(1-\sqrt{3})}{4} & 2 & \frac{-3(1+2\sqrt{3})}{22} & 0 & \frac{-4}{3\sqrt{3}} & 0 & 2 & 0 & 3(1-\sqrt{3}) \\
2 & -2 & 2 & 0 & -2 & 0 & 0 & \frac{-2}{3\sqrt{3}} & 0 & -2 & 0 & 0 \\
3 & 3 & 0 & \frac{3(\sqrt{3}-1)}{4} & 0 & \frac{-3(1+2\sqrt{3})}{22} & 3 & 0 & 0 & 0 & 3(1-\sqrt{3}) & 0 \\
3 & 3 & 0 & \frac{3(\sqrt{3}-1)}{4} & 0 & \frac{-3(1+2\sqrt{3})}{22} & -3 & 0 & 0 & 0 & 3(\sqrt{3}-1) & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
2 & 2 & -1 & \frac{3(1-\sqrt{3})}{8} & -1 & \frac{3(1+2\sqrt{3})}{44} & 2 & 0 & -1 & 0 & \frac{3(\sqrt{3}-1)}{2} & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
2 & 2 & -1 & \frac{3(1-\sqrt{3})}{8} & -1 & \frac{3(1+2\sqrt{3})}{44} & -2 & 0 & 1 & 0 & \frac{3(1-\sqrt{3})}{2} & 0
\end{pmatrix}$$

The fusion relations result in accordance with (13)–(14); this $(3,2)$ specialization of (1) is explicitly written in [12].

3.2.2 $(5,2)$ For $(p,p')=(5,2)$, all of the entries of the 20×20 matrix S can be easily evaluated from the $\mathscr{S}_{r,r';s,s'}(i)$ in (7). In particular, the vacuum-representation row is

$$\mathsf{S}_\Omega=\mathsf{S}_{13}=\Big(\frac{1}{20\sqrt{5}},-\frac{1}{20\sqrt{5}},\frac{5-\sqrt{5}+4\sqrt{10(5+\sqrt{5})}}{200},\frac{5-\sqrt{5}-\sqrt{10(5+\sqrt{5})}}{200},\frac{5+\sqrt{5}-3\sqrt{10(5-\sqrt{5})}}{200},$$
$$\frac{5+\sqrt{5}+2\sqrt{10(5-\sqrt{5})}}{200},\frac{-5-\sqrt{5}-2\sqrt{10(5-\sqrt{5})}}{200},\frac{-5-\sqrt{5}+3\sqrt{10(5-\sqrt{5})}}{200},\frac{\sqrt{5}-5+\sqrt{10(5+\sqrt{5})}}{200},$$
$$\frac{\sqrt{5}-5-4\sqrt{10(5+\sqrt{5})}}{200},\frac{1}{10\sqrt{5}},-\frac{1}{10\sqrt{5}},\frac{5-\sqrt{5}+4\sqrt{10(5+\sqrt{5})}}{100},\frac{5-\sqrt{5}-\sqrt{10(5+\sqrt{5})}}{100},$$
$$\frac{\sqrt{5}-5-4\sqrt{10(5+\sqrt{5})}}{100},\frac{\sqrt{5}-5+\sqrt{10(5+\sqrt{5})}}{100},\frac{5+\sqrt{5}-3\sqrt{10(5-\sqrt{5})}}{100},$$
$$\frac{5+\sqrt{5}+2\sqrt{10(5-\sqrt{5})}}{100},\frac{-5-\sqrt{5}+3\sqrt{10(5-\sqrt{5})}}{100},\frac{-5-\sqrt{5}-2\sqrt{10(5-\sqrt{5})}}{100}\Big).$$

The matrix K in (11) then consists of the blocks

$$
\begin{aligned}
\mathsf{K} = \mathrm{diag}\Bigg(&20\sqrt{5}, -20\sqrt{5}, \begin{bmatrix} 2\sqrt{2(5-\sqrt{5})} & 1 \\ -2\sqrt{2(5-\sqrt{5})} & \frac{5\sqrt{5}-2+5\sqrt{5-2\sqrt{5}}}{2} \end{bmatrix},\\
&\begin{bmatrix} -2\sqrt{2(5+\sqrt{5})} & 1 \\ 2\sqrt{2(5+\sqrt{5})} & \frac{109+20\sqrt{5}-5\sqrt{365+158\sqrt{5}}}{41} \end{bmatrix}, \begin{bmatrix} -2\sqrt{2(5+\sqrt{5})} & 1 \\ 2\sqrt{2(5+\sqrt{5})} & \frac{142+15\sqrt{5}+5\sqrt{485+202\sqrt{5}}}{158} \end{bmatrix},\\
&\begin{bmatrix} 2\sqrt{2(5-\sqrt{5})} & 1 \\ -2\sqrt{2(5-\sqrt{5})} & \frac{379-40\sqrt{5}-5\sqrt{6245-2558\sqrt{5}}}{1121} \end{bmatrix}, \begin{bmatrix} 5\sqrt{5} & 1 \\ -5\sqrt{5} & 1 \end{bmatrix},\\
&\begin{bmatrix} \sqrt{\frac{5-\sqrt{5}}{2}} & 1 & 1 & \sqrt{\frac{5+\sqrt{5}}{10}} \\ -\sqrt{\frac{5-\sqrt{5}}{2}} & -1 & \frac{5\sqrt{5}-2+5\sqrt{5-2\sqrt{5}}}{2} & \frac{25(\sqrt{5}-1)+\sqrt{5450+290\sqrt{5}}}{20} \\ -\sqrt{\frac{5-\sqrt{5}}{2}} & 1 & -1 & \sqrt{\frac{5+\sqrt{5}}{10}} \\ \sqrt{\frac{5-\sqrt{5}}{2}} & -1 & \frac{2-5\sqrt{5}-5\sqrt{5-2\sqrt{5}}}{2} & \frac{25(\sqrt{5}-1)+\sqrt{5450+290\sqrt{5}}}{20} \end{bmatrix},\\
&\begin{bmatrix} -\sqrt{\frac{5+\sqrt{5}}{2}} & 1 & 1 & -\sqrt{\frac{5-\sqrt{5}}{10}} \\ \sqrt{\frac{5+\sqrt{5}}{2}} & -1 & \frac{109+20\sqrt{5}-5\sqrt{365+158\sqrt{5}}}{41} & \frac{425+125\sqrt{5}-\sqrt{476050+79190\sqrt{5}}}{410} \\ \sqrt{\frac{5+\sqrt{5}}{2}} & 1 & -1 & -\sqrt{\frac{5-\sqrt{5}}{10}} \\ -\sqrt{\frac{5+\sqrt{5}}{2}} & -1 & \frac{-109-20\sqrt{5}+5\sqrt{365+158\sqrt{5}}}{41} & \frac{425+125\sqrt{5}-\sqrt{476050+79190\sqrt{5}}}{410} \end{bmatrix}\Bigg).
\end{aligned}
$$

This gives rise to the fusion-algebra eigenmatrix $\mathsf{P} = \mathsf{SK}$, shown (at about the limit of reasonable typesetting capabilities) in Fig. 1. The $(5,2)$-case of algebra (1) follows from this P in accordance with (13)–(14).

4 Conclusions

The procedure proposed here is of course not a replacement for the "honest" derivation of fusion (cf. [19]). We also reiterate that the success of this procedure is apparently rooted in the quantum group structure of the corresponding logarithmic conformal field models [12, 13] (and actually amounts to no more than establishing the coincidence with the quantum group Grothendieck ring). For the logarithmic (p, p') models, anyway, the existence of a relation between modular transformations of characters and the fusion additionally supports the "quantum-group candidate" for the fusion of representations of the extended algebra in [12] (in fact, Kazhdan–Lusztig-dual quantum groups "know" not only about the numerology and modular group transformations of extended-algebra characters in logarithmic conformal field models but also about the asymptotic form of the characters [20]). But the much more complicated "logarithmic" modular transformations in [21] are not likely to yield a fusion algebra similarly.

Acknowledgments

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$$
\begin{pmatrix}
10 & -10 & 0 & \frac{5+\sqrt{5(5-2\sqrt{5})}}{2} & 0 & \frac{\sqrt{5(5-2\sqrt{5})}}{1+2\sqrt{5-2\sqrt{5}}} & 0 & \frac{\sqrt{5(5-2\sqrt{5})}}{-1+3\sqrt{5-2\sqrt{5}}} & 0 & \frac{300\sqrt{5}-40-\sqrt{5(6245-2558\sqrt{5})}}{1121} & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 5-\sqrt{5}+\sqrt{10(5+\sqrt{5})} & 0 & 0 & 0 & \frac{50+24\sqrt{5}-4\sqrt{1850+110\sqrt{5}}}{41} \\
10 & -10 & 0 & \frac{5+\sqrt{5(5-2\sqrt{5})}}{2} & 0 & \frac{\sqrt{5(5-2\sqrt{5})}}{1+2\sqrt{5-2\sqrt{5}}} & 0 & \frac{\sqrt{5(5-2\sqrt{5})}}{-1+3\sqrt{5-2\sqrt{5}}} & 0 & \frac{-40+300\sqrt{5}-\sqrt{5(6245-2558\sqrt{5})}}{1121} & 0 & -\frac{2}{\sqrt{5}} & 0 & 0 & 0 & \sqrt{5}-5-\sqrt{10(5+\sqrt{5})} & 0 & 0 & 0 & \frac{2(-25-17\sqrt{5}+2\sqrt{1850+110\sqrt{5}})}{41} \\
2 & -2 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & \frac{2}{5\sqrt{5}} & 0 & \sqrt{2+\frac{2}{\sqrt{5}}} & 0 & 0 & 0 & \sqrt{2-\frac{2}{\sqrt{5}}} & 0 & 0 \\
8 & -8 & -2 & \frac{5\sqrt{5}-5+\sqrt{250-110\sqrt{5}}}{8} & 2 & \frac{-170-30\sqrt{5}+\sqrt{18850+8390\sqrt{5}}}{164} & 2 & \frac{-315-75\sqrt{5}-\sqrt{24650+10910\sqrt{5}}}{632} & 2 & \frac{1540-240\sqrt{5}-\sqrt{316250-130130\sqrt{5}}}{4484} & 0 & \frac{8}{5\sqrt{5}} & 0 & -\sqrt{2+\frac{2}{\sqrt{5}}} & 0 & \frac{3\sqrt{5}-5+\sqrt{50-10\sqrt{5}}}{2} & 0 & -\sqrt{2-\frac{2}{\sqrt{5}}} & 0 & \frac{-55-21\sqrt{5}+2\sqrt{3050+1090\sqrt{5}}}{41} \\
4 & -4 & \sqrt{5}-1 & \frac{-5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{8} & 1+\sqrt{5} & \frac{10-26\sqrt{5}+\sqrt{610+218\sqrt{5}}}{164} & 1-\sqrt{5} & \frac{45-57\sqrt{5}-\sqrt{890+242\sqrt{5}}}{632} & 1-\sqrt{5} & \frac{\sqrt{11890-2858\sqrt{5}}-4(65+73\sqrt{5})}{4484} & 0 & -\frac{4}{5\sqrt{5}} & 0 & -\sqrt{2-\frac{2}{\sqrt{5}}} & 0 & 1+\sqrt{5+2\sqrt{5}} & 0 & \sqrt{2+\frac{2}{\sqrt{5}}} & 0 & \frac{2(3+5\sqrt{5}-2\sqrt{125-38\sqrt{5}})}{41} \\
6 & -6 & 1-\sqrt{5} & \frac{-\sqrt{5}-\sqrt{5-2\sqrt{5}}}{2} & -1-\sqrt{5} & \frac{30+4\sqrt{5}-\sqrt{365+158\sqrt{5}}}{41} & 1+\sqrt{5} & \frac{60+3\sqrt{5}+\sqrt{485+202\sqrt{5}}}{158} & \sqrt{5}-1 & \frac{-300+8\sqrt{5}+\sqrt{6245-2558\sqrt{5}}}{1121} & 0 & -\frac{6}{5\sqrt{5}} & 0 & \sqrt{2-\frac{2}{\sqrt{5}}} & 0 & \sqrt{5}-1+\sqrt{2(5+\sqrt{5})} & 0 & -\sqrt{2+\frac{2}{\sqrt{5}}} & 0 & -\frac{2(17+5\sqrt{5}-2\sqrt{370+22\sqrt{5}})}{41} \\
6 & -6 & 1-\sqrt{5} & \frac{-\sqrt{5}-\sqrt{5-2\sqrt{5}}}{2} & -1-\sqrt{5} & \frac{30+4\sqrt{5}-\sqrt{365+158\sqrt{5}}}{41} & 1+\sqrt{5} & \frac{60+3\sqrt{5}+\sqrt{485+202\sqrt{5}}}{158} & \sqrt{5}-1 & \frac{-300+8\sqrt{5}+\sqrt{6245-2558\sqrt{5}}}{1121} & 0 & \frac{6}{5\sqrt{5}} & 0 & -\sqrt{2-\frac{2}{\sqrt{5}}} & 0 & 1-\sqrt{5}-\sqrt{2(5+\sqrt{5})} & 0 & \sqrt{2+\frac{2}{\sqrt{5}}} & 0 & \frac{2(17+5\sqrt{5}-2\sqrt{370+22\sqrt{5}})}{41} \\
4 & -4 & \sqrt{5}-1 & \frac{-5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{8} & 1+\sqrt{5} & \frac{10-26\sqrt{5}+\sqrt{610+218\sqrt{5}}}{164} & 1-\sqrt{5} & \frac{45-57\sqrt{5}-\sqrt{890+242\sqrt{5}}}{632} & 1-\sqrt{5} & \frac{\sqrt{11890-2858\sqrt{5}}-4(65+73\sqrt{5})}{4484} & 0 & \frac{4}{5\sqrt{5}} & 0 & \sqrt{2-\frac{2}{\sqrt{5}}} & 0 & -1-\sqrt{5+2\sqrt{5}} & 0 & -\sqrt{2+\frac{2}{\sqrt{5}}} & 0 & -\frac{2(2+3\sqrt{5}-2\sqrt{125-38\sqrt{5}})}{41} \\
8 & -8 & -2 & \frac{5\sqrt{5}-5+\sqrt{250-110\sqrt{5}}}{8} & 2 & \frac{-170-30\sqrt{5}+\sqrt{18850+8390\sqrt{5}}}{164} & 2 & \frac{-315-75\sqrt{5}-\sqrt{24650+10910\sqrt{5}}}{632} & 2 & \frac{1540-240\sqrt{5}-\sqrt{313250-133190\sqrt{5}}}{4484} & 0 & -\frac{8}{5\sqrt{5}} & 0 & \sqrt{2+\frac{2}{\sqrt{5}}} & 0 & \frac{5-3\sqrt{5}-\sqrt{50-10\sqrt{5}}}{2} & 0 & \sqrt{2-\frac{2}{\sqrt{5}}} & 0 & \frac{55+21\sqrt{5}-2\sqrt{3050+1090\sqrt{5}}}{41} \\
2 & -2 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & -\frac{2}{5\sqrt{5}} & 0 & -\sqrt{2+\frac{2}{\sqrt{5}}} & 0 & 0 & 0 & -\sqrt{2-\frac{2}{\sqrt{5}}} & 0 & 0 \\
5 & 5 & 0 & \frac{5+\sqrt{5(5-2\sqrt{5})}}{4} & 0 & -\frac{\sqrt{5(5-2\sqrt{5})}}{2+4\sqrt{5-2\sqrt{5}}} & 0 & \frac{\sqrt{5(5-2\sqrt{5})}}{-2+6\sqrt{5-2\sqrt{5}}} & 0 & \frac{40-300\sqrt{5}+\sqrt{5(6245-2558\sqrt{5})}}{2242} & 5 & 0 & 0 & 0 & 5+\sqrt{5(5-2\sqrt{5})} & 0 & 0 & 0 & -\frac{2\sqrt{5(5-2\sqrt{5})}}{1+2\sqrt{5-2\sqrt{5}}} & 0 \\
5 & 5 & 0 & \frac{5+\sqrt{5(5-2\sqrt{5})}}{4} & 0 & -\frac{\sqrt{5(5-2\sqrt{5})}}{2+4\sqrt{5-2\sqrt{5}}} & 0 & \frac{\sqrt{5(5-2\sqrt{5})}}{-2+6\sqrt{5-2\sqrt{5}}} & 0 & \frac{40-300\sqrt{5}+\sqrt{5(6245-2558\sqrt{5})}}{2242} & -5 & 0 & 0 & 0 & -5-\sqrt{5(5-2\sqrt{5})} & 0 & 0 & 0 & \frac{2\sqrt{5(5-2\sqrt{5})}}{1+2\sqrt{5-2\sqrt{5}}} & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 4 & -1 & \frac{5\sqrt{5}-5+\sqrt{250-110\sqrt{5}}}{16} & -1 & \frac{\sqrt{\frac{5}{2}(5-\sqrt{5})}}{4+8\sqrt{5-2\sqrt{5}}} & -1 & \frac{\sqrt{\frac{5}{2}(5-\sqrt{5})}}{4-12\sqrt{5-2\sqrt{5}}} & -1 & \frac{340\sqrt{5}-1540+\sqrt{315250-139190\sqrt{5}}}{8968} & 4 & 0 & -1 & 0 & \frac{5\sqrt{5}-5+\sqrt{250-110\sqrt{5}}}{4} & 0 & -1 & 0 & \frac{170+30\sqrt{5}-\sqrt{18850+8390\sqrt{5}}}{82} & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
4 & 4 & -1 & \frac{5\sqrt{5}-5+\sqrt{250-110\sqrt{5}}}{16} & -1 & \frac{\sqrt{\frac{5}{2}(5-\sqrt{5})}}{4+8\sqrt{5-2\sqrt{5}}} & -1 & \frac{\sqrt{\frac{5}{2}(5-\sqrt{5})}}{4-12\sqrt{5-2\sqrt{5}}} & -1 & \frac{340\sqrt{5}-1540+\sqrt{315250-139190\sqrt{5}}}{8968} & -4 & 0 & 1 & 0 & \frac{5-5\sqrt{5}-\sqrt{250-110\sqrt{5}}}{4} & 0 & 1 & 0 & \frac{-170-30\sqrt{5}+\sqrt{18850+8390\sqrt{5}}}{82} & 0 \\
2 & 2 & \frac{\sqrt{5}-1}{2} & \frac{-5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{16} & \frac{-1-\sqrt{5}}{2} & \frac{-10+26\sqrt{5}-\sqrt{610+218\sqrt{5}}}{328} & \frac{1-\sqrt{5}}{2} & \frac{45-57\sqrt{5}-\sqrt{890+242\sqrt{5}}}{1264} & \frac{\sqrt{5}-1}{2} & \frac{260+292\sqrt{5}-\sqrt{11890-2858\sqrt{5}}}{8968} & -2 & 0 & \frac{1-\sqrt{5}}{2} & 0 & \frac{5+\sqrt{5}+\sqrt{10-2\sqrt{5}}}{4} & 0 & \frac{1+\sqrt{5}}{2} & 0 & \frac{10-26\sqrt{5}+\sqrt{610+218\sqrt{5}}}{82} & 0 \\
3 & 3 & \frac{1-\sqrt{5}}{2} & \frac{-\sqrt{5}-\sqrt{5-2\sqrt{5}}}{4} & \frac{1+\sqrt{5}}{2} & \frac{-30-4\sqrt{5}+\sqrt{365+158\sqrt{5}}}{82} & \frac{1+\sqrt{5}}{2} & \frac{60+3\sqrt{5}+\sqrt{485+202\sqrt{5}}}{316} & \frac{1-\sqrt{5}}{2} & \frac{300-8\sqrt{5}-\sqrt{6245-2558\sqrt{5}}}{2242} & -3 & 0 & \frac{\sqrt{5}-1}{2} & 0 & \sqrt{5}+\sqrt{5-2\sqrt{5}} & 0 & \frac{-1-\sqrt{5}}{2} & 0 & \frac{2(30+4\sqrt{5}-\sqrt{365+158\sqrt{5}})}{41} & 0 \\
2 & 2 & \frac{\sqrt{5}-1}{2} & \frac{-5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{16} & \frac{-1-\sqrt{5}}{2} & \frac{-10+26\sqrt{5}-\sqrt{610+218\sqrt{5}}}{328} & \frac{1-\sqrt{5}}{2} & \frac{45-57\sqrt{5}-\sqrt{890+242\sqrt{5}}}{1264} & \frac{\sqrt{5}-1}{2} & \frac{260+292\sqrt{5}-\sqrt{11890-2858\sqrt{5}}}{8968} & 2 & 0 & \frac{\sqrt{5}-1}{2} & 0 & \frac{-5-\sqrt{5}-\sqrt{10-2\sqrt{5}}}{4} & 0 & \frac{-1-\sqrt{5}}{2} & 0 & \frac{-10+26\sqrt{5}-\sqrt{610+218\sqrt{5}}}{82} & 0 \\
3 & 3 & \frac{1-\sqrt{5}}{2} & \frac{-\sqrt{5}-\sqrt{5-2\sqrt{5}}}{4} & \frac{1+\sqrt{5}}{2} & \frac{-30-4\sqrt{5}+\sqrt{365+158\sqrt{5}}}{82} & \frac{1+\sqrt{5}}{2} & \frac{60+3\sqrt{5}+\sqrt{485+202\sqrt{5}}}{316} & \frac{1-\sqrt{5}}{2} & \frac{300-8\sqrt{5}-\sqrt{6245-2558\sqrt{5}}}{2242} & 3 & 0 & \frac{1-\sqrt{5}}{2} & 0 & -\sqrt{5}-\sqrt{5-2\sqrt{5}} & 0 & \frac{1+\sqrt{5}}{2} & 0 & -\frac{2(30+4\sqrt{5}-\sqrt{365+158\sqrt{5}})}{41} & 0
\end{pmatrix}
$$

FIGURE 1. The 20×20 eigenmatrix $\mathsf{P} = \mathsf{SK}$ for $(p, p') = (5, 2)$. The vacuum-representation row is the 13th.

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