



# Nariai-like black holes in light of scale-dependent gravity

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Received: 10 January 2023 / Accepted: 5 September 2023  
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**Abstract** In this work, we investigate for the first time Nariai-like black hole solutions in four dimensional space time with spherical symmetry in the context of scale-dependent gravity. In particular, we construct self-consistent solutions in such a way that the classical one that coming from Einstein theory is also contained into our new results. We start by considering a truncated effective Einstein–Hilbert action with cosmological constant in which both the gravitational and the cosmological couplings are elevated to scale-dependent functions (instead of being coupling constants), taking advantage of the scale-dependent scenario, which is a concrete and simple approach strongly inspired by asymptotically safe gravity. We briefly discuss black hole thermodynamics and compare it with its classical counterpart. The impact of non-Einsteinian features is encoded by the scale-dependent parameter,  $\epsilon$ . A few comments regarding the topology of the solution and their impact are also mentioned.

## 1 Introduction

General Relativity (GR) is considered the most successful theory to describe the gravitational interaction [1,2]. In this respect, exact solutions to Einstein field equations (EFE hereafter) are of paramount importance to reveal fundamental problems which already persist, such that, for instance, the existence of space time singularities. Thus, albeit GR is robust both from a theoretical and an experimental perspective, there are several good reasons to investigate alternative theories of gravity. First, from a theoretical point of view, we can reinforce two problems still present in GR: (i) the presence of singularities [3,4], and (ii) the impossibility of a renormalization (following standard processes of quantization) [5]. Second, from an observational angle, the discov-

ery of the dark sectors of the Universe makes evident the necessity of fundamental physics. The latter reveals that the inclusion of new physics to connect both the ultraviolet (UV) and the infrared (IR) sectors is, therefore, mandatory. To get insights about a possible modification of GR by the inclusion of quantum features, the best approach is always take advantage of exact solutions of EFE.

Some of the most-relevant solutions to EFE, albeit being well understood and explored in detail, are not free of problems. To face these, we can assume alternative theories of gravity as, for instance, (i) asymptotically safe gravity applied to black holes, [6–9], (ii) improved black hole formalisms [10–12], (iii) scale-dependent gravity [13–33] among others. In what follows we will focus on the last case, i.e., scale-dependent gravity. Such formalism is a well-known approach used to introduce semi-classical corrections in black hole solutions in  $2+1$  and  $3+1$  dimensional space-times.

The main idea behind the scale-dependent scenario can be summed up as follows: (i) taking as inspiration Weinberg's Asymptotic Safety program, we consider that the effective action (a quantum corrected version of the classical action) is the fundamental object; (ii) we derive the corresponding equations of motion using this improved action; (iii) accepting that the coupling constants can evolve, we take it into account and supplement EFE with a some supplementary condition. We have used a NEC-like constraint, given that such a condition is the less restrictive of the energy conditions. At this point, it should be mentioned that, albeit scale-dependent gravity in  $3+1$  dimensions has been significantly investigated, the case of Nariai black holes (or, more precisely, Nariai-like black hole,) is still pending to be studied. Thus, taking these last ingredients into account, we shall analyse how Nariai-like black holes suffer deviations with respect to their classical counterparts.

This manuscript is organized as follows: after a short introduction, Sect. 2 summarizes the main points of the scale-

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dependent gravitational theory. In Sect. 3, we give the very basics of the classical Nariai black holes in  $3+1$  dimensions and, subsequently, in Sect. 4, we apply the idea of scale-dependent gravity to obtain Nariai-like solutions, where the solution is investigated in detail, including the computation of some invariants, the corresponding asymptotic behavior, and the Hawking temperature as well as the Bekenstein–Hawking entropy. Finally, in Sect. 5 we point out the main features of the obtained solution.

## 2 Scale-dependent gravity

In the context of quantum-inspired theories of gravity, there are a variety of possibilities in which the basic idea of incorporate quantum effects on classical gravity has been applied on black hole physics. As the literature in this topic is vast, we will only mention a few papers, the seminal ones and other recent applications, where the inclusion of quantum features are present. Of particular interest is the work of Bonanno and Reuter [10], where a detailed study of how the renormalization group effects could deviate the classical Schwarzschild black hole solution is performed. Subsequently, following same spirit, other works have been done, see [8, 34, 35] to name a few. However, irrespectively of the physics we will consider, the construction of a self-consistent and well-defined solution should start by considering the object which describes the “fundamental theory”, i.e., the action. As a difference with respect to the classical formalism, where EFE are obtained from a classical action  $I_0[g_{\mu\nu}]$ , scale-dependent gravity is obtained from effective field equations coming from an average effective action,  $\Gamma[k, g_{\mu\nu}, \dots]$ , being  $k$  a renormalization scale. Within this formalism based on an effective action, some suitable properties of quantum field theories should be present. Thus, it is known that the effective action for the gravitational field, at low-energy, acquires a scale dependence. Such feature is included at the level of the action: the coupling parameters are now functions which run according to certain energy scale. Asymptotically safe gravity is based on the existence of a non-trivial ultra-violet fixed point for the leading dimensionless gravitational couplings and that is, by far, where these ideas have been best implemented up to now. In particular, it was Weinberg, in his seminal work [36], who introduced this program and, after that, substantial improvement has been made up to now [37–59]. The underlying idea has been considered in closely related approaches, one of them the so-called scale-dependent gravity, usually used to construct black hole solutions both by improving classical solutions with the scale dependent couplings from Asymptotic Safety [60–72] and by solving the gap equations of a generic scale-dependent action [13–29]. Particularly, scale-dependent regular black holes [28] and traversable (vacuum) scale-dependent wormholes [26] have

been found, showing that, in some sense and in particular situations, scale-dependent gravity might offer us light on how to cure, or at least improve, certain classical problems related to singularities and the appearance of exotic matter inside wormholes. In the cosmological scenario, the idea of scale-dependent couplings been used in Refs. [73–85]. In what follows, we will show the details regarding how the simplest formulation of scale-dependent gravity can be made. Let us start by considering the average effective Einstein–Hilbert action

$$\Gamma[g_{\mu\nu}, k] = \int d^n x \sqrt{-g} \left[ \frac{1}{2\kappa_k} \left( R - 2\Lambda_k \right) + \mathcal{L}_M \right], \quad (1)$$

being  $\mathcal{L}_M$  the Lagrangian matter density and  $\kappa_k \equiv 8\pi G_k$  is the Einstein coupling.  $G_k$  and  $\Lambda_k$  refer to the scale-dependent gravitational and cosmological couplings, respectively. For simplicity, we will consider only vacuum solutions, i.e.,  $\mathcal{L}_M \equiv 0$ . We proceed by obtaining the corresponding differential equations, i.e., the modified EFE, by taking variations with respect to the metric field,  $g_{\mu\nu}$ :

$$G_{\mu\nu} + g_{\mu\nu} \Lambda_k = -\Delta t_{\mu\nu} \equiv \kappa_k T_{\mu\nu}^{\text{SD}}. \quad (2)$$

Notice that the so-called scale-dependent energy-momentum tensor,  $\Delta t_{\mu\nu}$ , is defined according to [78, 86]

$$\Delta t_{\mu\nu} = G_k \left( g_{\mu\nu} \square - \nabla_\mu \nabla_\nu \right) G_k^{-1}. \quad (3)$$

Subsequently, the second step to obtain the complete set of equations consists in taking the variation of the average effective action with respect to the additional field,  $k(x)$ , i.e.,

$$\frac{\delta \Gamma[g_{\mu\nu}, k]}{\delta k} = 0, \quad (4)$$

or, more precisely,

$$-\left( R - 2\Lambda \right) \frac{d}{dk} \ln(G) = 2 \frac{d\Lambda}{dk}. \quad (5)$$

Such a computation provide us a supplementary equation to close the set of equations and, in particular, using (4), we are able to connect  $G_k$  with  $\Lambda_k$ . The auxiliary scalar field  $k(x)$  is a real physical scale, identified with the momentum as a function of any space-time point,  $x^\mu$ . The scale-dependence of the theory is then modulated by  $k(x)$ , and its modifies the classical theory via the evolution of the classical couplings to scale-dependent couplings. At this point we should be careful: the dependence of  $k$  on the physical coordinates is not unique and, therefore, we will avoid the problem in light of the following observations: First we recognize that  $\mathcal{O}(k(x)) \rightarrow \mathcal{O}(x)$ , and then we can “ignore” a concrete relation between  $k$  and the spatial (temporal) coordinates. In what follows, we will consider that the couplings  $G_k$  and  $\Lambda_k$  inherit the dependence on the space-time coordinates from

the space-time dependence of the scale field,  $k(x)$ . Therefore, these couplings are written as  $G(x)$  and  $\Lambda(x)$  [18,87]. Taking into account the last idea, together with an appropriate choice for the line element, we can solve the equations in certain situations with a high degree of symmetry.

### 3 Classical four dimensional Nariai solution

In this section we will briefly summarize the main ingredients of the four-dimensional Nariai black holes obtained within Einstein's theory. Our starting point is the classical Einstein–Hilbert action with cosmological constant, i.e.,

$$I_0[g_{\mu\nu}] = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa_0} \left( R - 2\Lambda_0 \right) \right], \quad (6)$$

where the parameters involved have the usual meaning, i.e.,  $\kappa_0 \equiv 8\pi G_0$  is the gravitational coupling,  $G_0$  is Newton's constant and  $\Lambda_0$  is the cosmological constant. EFE are obtained as usual, i.e., by varying the classical action (6) with respect to the metric field, obtaining,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda_0 g_{\mu\nu}. \quad (7)$$

The Nariai spacetime, as first presented in [88], can be understood as a special limit of the Schwarzschild–de Sitter solution when the event and cosmological horizons coincide. Under an appropriate coordinate transformation, the solution can be expressed as the product  $\text{AdS}^2 \times S^2$ . For instance, in certain coordinates one can write the solution as

$$ds^2 = \frac{1}{\Lambda_0^2} \left[ -d\tau^2 + \sin^2 \tau d\rho^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right], \quad (8)$$

or, introducing new coordinates for the  $\text{AdS}^2$  sector, as

$$ds^2 = - \left( 1 - \frac{r^2}{l^2} \right) dt^2 + \left( 1 - \frac{r^2}{l^2} \right)^{-1} dr^2 + l^2 d\Omega_2, \quad (9)$$

where  $l$  is certain (constant) length scale.

In the following section, we will obtain scale-dependent Nariai-like black holes by considering a spherically symmetric line element given by

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + L^2 d\Omega_2^2, \quad (10)$$

where  $L$  is a constant value with units of length. In order to do that, both the Newton and cosmological couplings will be taken to be functions of the radial coordinate,  $G(r)$  and  $\Lambda(r)$ , respectively. Let us note that, although scale-dependent generalizations of the Schwarzschild–de Sitter solutions exist

[13], we will not try to construct a scale-dependent generalization of Nariai's solution from the aforementioned scale-dependent Schwarzschild–de Sitter one. The route we are proposing is to impose an ansatz given by Eq. (10) in order to look for vacuum scale-dependent solutions which generalize the classical Nariai metric expressed by Eq. (9).

### 4 Scale-dependent four dimensional black hole solutions

The scale-dependent formalism implies that the classical couplings evolve to scale-dependent functions which, in general, depend on certain energy scale,  $k$ . Due to the symmetries we want to explore, we can make progress by assuming that such a dependency is inherited to the radial coordinate. Therefore, we will assume that

$$G_0 \rightarrow G_k \rightarrow G(r), \quad (11)$$

$$\Lambda_0 \rightarrow \Lambda_k \rightarrow \Lambda(r), \quad (12)$$

identifying such quantities as the quasi-dynamical Newton and cosmological couplings, respectively. Now, having accepted that we have two additional functions, we have to solve the effective EFE. To complete our set of differential equations, we will use the following fact: for a Schwarzschild-like ansatz,  $G_{\mu\nu}\ell^\mu\ell^\nu = 0$ , being  $\ell^\mu$  a radial null vector. Therefore, consistency of the vacuum scale-dependent equations imply that  $\Delta t_{\mu\nu}\ell^\mu\ell^\nu = 0$ , which is the additional equation to be included (usually referred to as the NEC-like condition). Thus, as the unknown function are  $G(r)$ ,  $\Lambda(r)$  and  $V(r)$ , using two EFE plus the NEC-like condition we can completely solve the problem.

#### 4.1 NEC-like conditions in spherical symmetry

In four dimensional spacetime and in presence of spherical symmetry, we can consider a convenient trick as advantage to found the explicit form of the Newton's gravitational coupling. From the saturated version of the NEC we known that  $T_{\mu\nu}\ell^\mu\ell^\nu = 0$ , being  $\ell^\mu$  a radial null vector. Applying such constraint on the right-hand side of Eq. (2) we obtain the following second order differential equations

$$2 \left[ \frac{dG(r)}{dr} \right]^2 = G(r) \frac{d^2G(r)}{dr^2}, \quad (13)$$

which produces the following scale-dependent gravitational coupling

$$G(r) = \frac{G_0}{1 + \epsilon r}. \quad (14)$$

The running coupling found in [16–19] corresponds to  $g_{tt} \cdot g_{rr} = -1$ , which is consistent with Eq. 14. Notice that the ordinary differential equation for Newton’s coupling is of second order, which means two free parameters to be considered. The first one,  $\epsilon$ , encodes the scale-dependent effect, i.e., this parameter controls the strength of the scale-dependent effect. Thus, when  $\epsilon$  goes to zero we are in the classical regimen, and when  $\epsilon$  increases, the scale-dependent effect increases too. After setting this parameter, the second free parameter,  $G_0$ , is taken in such a way that when  $\epsilon$  goes to zero, we recover a constant Newton’s coupling constant. Having clarified and obtained the concrete value of  $G(r)$ , we now will obtain the Nariai-like solution in spherical symmetry in the context of scale-dependent gravity.

#### 4.2 Compatibility of scale-dependent scenario and the asymptotic safety program

Scale-dependent gravity and asymptotically safe gravity are two distinct approaches that address the issue of how quantum effects perturb classical and widely recognized solutions. Both approaches take advantage of an effective action in which the coupling parameters are presumed to be spatially and temporally varying functions, deviating from the conventional assumption of fixed values. In both scenarios, the impact of the renormalization scale  $k$  becomes evident. Specifically, within spherically symmetric black holes, such a function encapsulates how the quantum corrections manifest in terms of the radial coordinate only. Therefore, when examining asymptotically safe gravity within black holes, by employing the effective action, utilizing the Wetterich equation, and ultimately deriving the solution, we essentially disregard the specific form of  $k$ . When incorporating quantum corrections into a classical solution, we substitute the classical coupling constant,  $G_0$ , with a position-dependent running coupling,  $G(r)$ . This final step necessitates an additional scale identification between  $k$  and the coordinate variable,  $r$ , which is commonly approximated as  $k r = \text{constant}$  for simplicity (albeit different parametrizations are still possible, for instance,  $kr^{3/2} = \text{constant}$  [10] or  $kr^{\alpha/2} = \text{constant}$  [92]). In this regard, such a choice guarantees that: (i) Quantum effects manifest in the “appropriate” sector, namely the UV sector. (ii) The standard sector remains unaltered, corresponding to the IR sector. When considering scale-dependent gravity, on the other hand, notable corrections emerge within the lower range of  $k$  values, particularly in the IR region. Consequently, this leads to modifications occurring in the opposite sector compared to what is anticipated in approaches presented within the framework of the Asymptotic Safety program. While one might initially assume that scale-dependent gravity and the asymptotic safety programs are incompatible, certain studies within the asymptotic safety program have addressed significant corrections for small values of  $k$  (see

for instance [89,90]). This suggests that in certain regions, the two approaches can coexist and, therefore, the solution (14) can be a well-defined solution for Newton’s coupling (see also the discussion in [91]).

#### 4.3 Black hole solution

Using Eq. (2) plus the concrete form of the Newton’s coupling, (14), we can obtain the lapse function,  $V(r)$ , and the cosmological function,  $\Lambda(r)$ . After some manipulations, the relevant equation for the lapse function takes the simple form

$$\frac{d^2 V(r)}{dr^2} + \left( \frac{\epsilon}{1 + \epsilon r} \right) \frac{dV(r)}{dr} + \frac{2}{L^2} = 0, \quad (15)$$

which implies the exact solution

$$V(r) = B - \frac{(r\epsilon + 2)r}{2L^2\epsilon} + \frac{(1 + AL^2\epsilon)}{L^2\epsilon^2} \ln(1 + \epsilon r), \quad (16)$$

where we have two free parameters to be specified. Thus,  $A, B$  are selected in such a way that the classical solution is recovered when  $\epsilon$  goes to zero. So, to maintain the classical solution as limit case, we take  $A = 0$  and  $B = 1$  and, therefore, the solution acquires the simple form

$$V(r) = 1 - \frac{r^2}{2L^2} - \frac{1}{L^2\epsilon^2} \left( r\epsilon - \ln(1 + \epsilon r) \right). \quad (17)$$

Finally, the cosmological function can be found by simple substitution, i.e.,

$$\Lambda(r) = \frac{1}{L^2} + \frac{d}{dr} \ln \left( \sqrt{G(r)} \right) \frac{d}{dr} V(r), \quad (18)$$

obtaining

$$\Lambda(r) = \frac{1}{L^2} \left( \frac{3}{2} - \frac{1}{2(1 + \epsilon r)^2} \right). \quad (19)$$

As a final check, we observe that we effectively recover the classical case when  $\epsilon$  goes to zero, i.e.,

$$\lim_{\epsilon \rightarrow 0} G(r) = G_0, \quad (20)$$

$$\lim_{\epsilon \rightarrow 0} V(r) = 1 - \frac{r^2}{L^2} \equiv V_0(r), \quad (21)$$

$$\lim_{\epsilon \rightarrow 0} \Lambda(r) = \frac{1}{L^2} \equiv \Lambda_0. \quad (22)$$

At this point, let us mention that the cosmological function,  $\Lambda(r)$ , tends to be a constant different to its classical value, when  $\epsilon$  is taken to be large (see Fig. 1 left). Thus, the scale-dependent scenario not only modifies  $\Lambda_0$  by  $\Lambda(r)$  but also affects the cosmological coupling when the scale-dependent

effect is strong ( $\epsilon$  large). Similarly, we show, in Fig. 1 right, how the metric potential  $V(r)$  presents a single horizon. What is more, such horizon could be larger than its classical counterpart. Thus, the modified potential is consistent with the classical case but, in this case, the black hole horizon is larger than its classical counterpart.

#### 4.4 Some invariants

It is usually convenient to check how a new solution looks in terms of its invariants. Specifically, we will compute the Ricci and Kretschmann scalars,  $R$  and  $K$ , respectively. They are given as follows:

$$R = \frac{3}{L^2} \left( 1 + \frac{1}{3(r\epsilon + 1)^2} \right), \quad (23)$$

$$K = \frac{8}{L^4} \left( 1 - \frac{r\epsilon(r\epsilon + 2)(3r\epsilon(r\epsilon + 2) + 4)}{8(1 + \epsilon r)^4} \right). \quad (24)$$

In both cases, we observe that the classical solution is preserved and also recovered when we demand  $\epsilon \rightarrow 0$  as it should be. We can observe that some critical points could appear, however, given that we are interested in solutions with  $\epsilon > 0$  and as  $r > 0$ , the combination  $\epsilon r = -1$  is not possible.

The asymptotic behavior is usually required to interpretate our results and check, if possible, differences between the classical and the scale-dependent solutions. Thus, we compute the previous invariants and lapse function in two different regimes: (i) for small values of the running parameter  $\epsilon$  and (ii) for large values of the radial coordinate. In concrete, we have:

$$R = \frac{4}{L^2} \left( 1 - \frac{1}{2}\epsilon r \right) + \mathcal{O}(\epsilon^2), \quad (25)$$

$$K = \frac{8}{L^4} \left( 1 - \epsilon r \right) + \mathcal{O}(\epsilon^2), \quad (26)$$

$$V = V_0(r) + \frac{\epsilon r^3}{3L^2} + \mathcal{O}(\epsilon^2), \quad (27)$$

and

$$R = \frac{3}{L^2} \left( 1 + \frac{1}{3} \left( \frac{1}{\epsilon r} \right)^2 \right) + \mathcal{O} \left( \frac{1}{r} \right), \quad (28)$$

$$K = \frac{5}{L^4} \left( 1 + \frac{2}{5} \left( \frac{1}{\epsilon r} \right)^2 \right) + \mathcal{O} \left( \frac{1}{r} \right), \quad (29)$$

$$V = -\frac{r^2}{2L^2} - \frac{r}{\epsilon L^2} + \mathcal{O} \left( \frac{1}{r} \right). \quad (30)$$

At this point, some comments are in order. First, we notice that the invariants for  $\epsilon \sim 0$  decrease with respect to the classical solution but for  $r \sim \infty$ , they increase with respect

to their classical counterpart. Thus, the asymptotic structure is slightly disturbed when Newton's coupling can vary.

#### 4.5 Thermodynamics

In this subsection we will briefly review the basic thermodynamic properties and its differences with respect to the classical Nariai black hole solution. Firstly, we should compute the black hole horizon given that all the thermodynamic quantities are evaluated there. To obtain the horizon we need to compute  $V(r_H) = 0$ , however, albeit simple, the logarithm term prevents a simple computation to be performed. Despite of that, the lapse function can be rewritten conveniently and subsequently identified as a generalized Lambert  $W(\cdot)$  function. Thus, using the change of variable  $z = 1 + \epsilon r$ , replacing it into the lapse function at the horizon, we obtain the following more friendly equation

$$w(z_H) e^{w(z_H)} = \xi(L, \epsilon), \quad (31)$$

where we used the following definitions:  $w(z) \equiv z^2$  and  $\xi(L, \epsilon) \equiv -e^{-1-2L^2\epsilon^2}$ . Thus, solving for  $w(z_H)$  we obtain the now trivial solution:

$$w(z_H) = W \left( \xi(L, \epsilon) \right), \quad (32)$$

and restoring the original variables we have the two solutions:

$$r_{\pm} = -\frac{1}{\epsilon} \left( 1 \pm \sqrt{-W \left( \xi(L, \epsilon) \right)} \right), \quad (33)$$

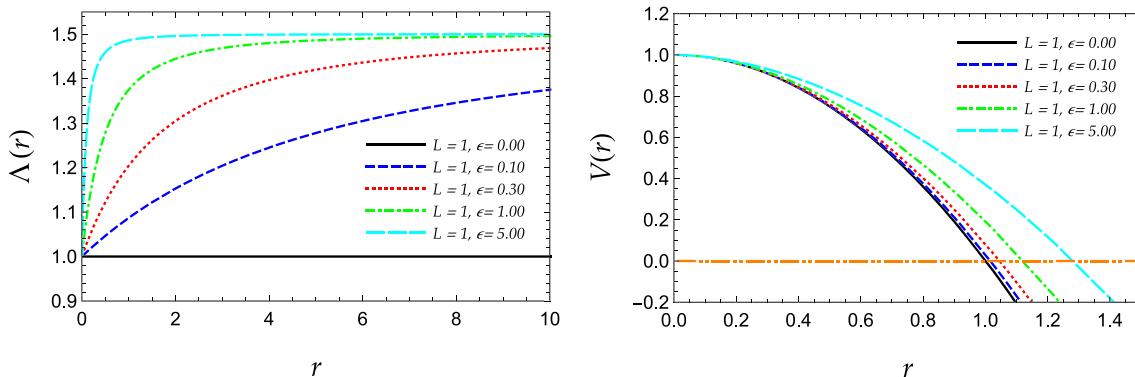
where the black hole hole horizon corresponds to the external solution, i.e.,  $r_H \equiv \max\{r_{\pm}\}$ . Finally, note that Lambert  $W(\cdot)$  function is a multi-valued function and we should specify the branch of interest for us. In particular, for arbitrary branches, we use the symbol  $W_k(\cdot)$ , being  $k$  an integer number. Taking into account only real numbers, there are just two branches:  $W_0$  and  $W_{-1}$ . We then just choose the solution with  $k = -1$ , given the range of parameters involved.

To make progress, we could take a first order approximation of the lapse function to obtain closed expressions for the horizon, temperature and entropy. We prefer to maintain the discussion in general terms, given that we can express all the relevant quantities implicitly. The temperature can be obtained using the well-known expression

$$T = \frac{1}{4\pi} \left| \lim_{r \rightarrow r_H} \frac{\partial_r g_{tt}}{\sqrt{-g_{tt}g_{rr}}} \right|, \quad (34)$$

or more precisely,

$$T_H = \left( \frac{1}{2} + \frac{1}{2(1 + \epsilon r_H)} \right) T_0, \quad (35)$$



**Fig. 1** Left panel: cosmological running coupling  $\Lambda(r)$  against the radial coordinate assuming  $L = 1$ ,  $G_0 = 1$ , for different values of the scale-dependent parameter  $\epsilon$ . Right panel: lapse potential  $V(r)$  against

being  $T_0$  the black hole temperature in the classical case. The last expression converges to its classical counterpart after demanding  $\epsilon \rightarrow 0$ . The Bekenstein–Hawking entropy,  $S$ , can be computed directly using the expression from a Brans–Dicke theory, i.e.,

$$S_H = \frac{\mathcal{A}_H}{4G(r_H)}, \quad (36)$$

being  $\mathcal{A}_H$  the black hole area, defined as follows:

$$\mathcal{A}_H = \oint d^3x \sqrt{h} = 4\pi r_H^2. \quad (37)$$

## 5 Concluding remarks

In this work we have obtained, for the first time, a four dimensional Nariai-like black hole solution assuming (i) varying Newton and (ii) cosmological couplings, in the context of scale-dependent gravity. The constants of integration have been chosen in such a way that the classical Nariai solution is obtained when the scale-dependent parameter,  $\epsilon$ , is taking to be zero. After a brief study pointing out the finiteness of some geometric invariants, we have explored some essential features of black hole thermodynamics in presence of running couplings, showing their main differences with respect to their classical counterparts. We conclude by noting that it would constitute a nice problem to see if the black hole solution we have presented in the present work coincides with that obtained from the scale-dependent Schwarzschild–de Sitter black hole of Ref. [13] when both cosmological and event horizons coincide.

**Acknowledgements** We acknowledge financial support from the Generalitat Valenciana through PROMETEO PROJECT CIPROM/2022/13. P. B. is funded by the Beatriz Galindo contract BEAGAL 18/00207 (Spain). A.R. is funded by the Maria Zambrano contract ZAMBRANO 21-25 (Spain). P. B. dedicates this work to Ana Bargueño-Dorta.

the radial coordinate obtained in our model assuming  $L = 1$ ,  $G_0 = 1$ , for different values of the scale-dependent parameter  $\epsilon$

**Funding** Funding was provided by Ministerio de Universidades (Programa Beatriz Galindo, Programa María Zambrano) and Generalitat Valenciana (PROMETEO).

**Data availability statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: This work is a theoretical study and no data was used for its development].

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Funded by SCOAP<sup>3</sup>. SCOAP<sup>3</sup> supports the goals of the International Year of Basic Sciences for Sustainable Development.

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