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Lattice Fermions with Solvable Wide Range Interactions

Ryu Sasaki *

Department of Physics and Astronomy, Tokyo University of Science, Noda 278-8510, Japan
*Email: ryu@yukawa.kyoto-u.ac.jp

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Exactly solvable (spinless) lattice fermions with wide range interactions are constructed explicitly based on *exactly solvable stationary and reversible Markov chains* $\mathcal{K}^R$ reported a few years earlier by Odake and myself. The reversibility of $\mathcal{K}^R$ with the stationary distribution π leads to a positive classical Hamiltonian $\mathcal{H}^R$. The exact solvability of $\mathcal{H}^R$ warrants that of a spinless lattice fermion c_x, $c_x^\dagger$, $\mathcal{H}_f^R = \sum_{x,y\in\mathcal{X}} c_x^\dagger \mathcal{H}^R(x, y)c_y$ based on the principle advocated recently by myself. The reversible Markov chains $\mathcal{K}^R$ are constructed by convolutions of the orthogonality measures of the discrete orthogonal polynomials of the Askey scheme. Several explicit examples of the fermion systems with wide range interactions are presented.

Subject Index A10, A43, A60, B81

1. Introduction

Here I report a simple construction of exactly solvable (spinless) fermions with wide range interactions on a 1D integer lattice. Compared to fermion systems with the nearest neighbor interactions, exactly solvable and wide range interactions are relatively hard to fathom. The goal is achieved, following the general principle advocated in Ref. [1], by rewriting the stationary and reversible (detail balanced) Markov chain matrix $\mathcal{K}^R(x, y)$ into a positive classical Hamiltonian $\mathcal{H}^R(x, y)$ by a similarity transformation in terms of the square root of the stationary distribution $\pi(x)$. The construction of many exactly solvable, stationary, and reversible Markov chain matrices $\mathcal{K}^R(x, y)$ on 1D integer lattices has been reported a few years earlier by Odake and myself [2]. Various convolutions of the orthogonality measures of the discrete orthogonal polynomials of the Askey scheme [3–6] provide the desired forms of the reversible Markov chain matrix $\mathcal{K}^R(x, y)$. The corresponding eigen polynomials are the Krawtchouk (K), Hahn (H), q-Hahn (qH), Meixner (M), and Charlier (C).

This paper is organized as follows. In Section 2, the general setting of the stationary and reversible Markov chains is recapitulated with the simple derivation of the classical Hamiltonian $\mathcal{H}^R$. In Section 3.1 three types of convolutions for constructing the reversible Markov chains $\mathcal{K}^R$ are displayed. In Section 3.2 the exactly solvable fermion Hamiltonian $\mathcal{H}_f^R$ with wide range interactions is trivially constructed from the classical Hamiltonian $\mathcal{H}^R$. The main results, several explicit forms of the classical Hamiltonians $\mathcal{H}^R$ with the eigensystems, are displayed in Section 4. They are grouped for each orthogonal polynomial. Those for the finite polynomials are shown first, with three forms of $\mathcal{K}^R$ corresponding to the convolution types. The corresponding Hamiltonians belonging to the infinite polynomials are shown after them. They are obtained by certain limit procedures from those of the finite polynomials.

2. General setting; stationary and reversible Markov chains $\mathcal{K}^R$

Let us start with a brief recapitulation of the general setting of classical stationary Markov chains $\mathcal{K}$ on a 1D integer lattice $\mathcal{X}$, finite or semi-infinite,

$$x, y, z \in \mathcal{X} = \{0, 1, \ldots, N\}, \quad N \in \mathbb{N}, \quad x, y, z \in \mathcal{X} = \mathbb{Z}_{\geq 0}, \tag{1}$$

and points on $\mathcal{X}$ are denoted by x, y, z for analytical treatment. I use the convention that the transition probability matrix per unit time interval $\mathcal{K}(x, y)$ on $\mathcal{X}$ means the transition from an initial point y to a final point x with $\mathcal{K}(x, y) > 0$ and it satisfies the conservation of the probability

$$\sum_{x\in X} \mathcal{K}(x, y) = 1. \tag{2}$$

The positive $\mathcal{K}$ means that all the points on $\mathcal{X}$ are connected with each other by $\mathcal{K}$ and this translates into wide range interactions.

Definition 1. Markov chain $\mathcal{K}^R$ is reversible (detail balanced) if it has a reversible distribution π satisfying

$$\mathcal{K}^R(x, y)\pi(y) = \mathcal{K}^R(y, x)\pi(x), \quad x, y \in \mathcal{X}; \quad \pi(x) > 0, \quad \sum_{x\in\mathcal{X}} \pi(x) = 1. \tag{3}$$

Taking y summation of the reversibility definition (3),

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y) = \pi(x) \sum_{y\in\mathcal{X}} \mathcal{K}^R(y, x) = \pi(x), \tag{4}$$

leads to the following.

Proposition 1. *The reversible $\mathcal{K}^R(x, y)$ has a Perron–Frobenius eigenvector $\pi(x)$ with the maximal and simple eigenvalue 1 and the range of spectrum*

$$-1 < \text{The moduli of the eigenvalues of } \mathcal{K}^R(x, y) \leq 1.$$

This is a consequence of the positivity, i.e. Perron–Frobenius theorem, and the probability conservation (2).

Proposition 2. *A positive Hamiltonian $\mathcal{H}^R$, a real symmetric $|\mathcal{X}| \times |\mathcal{X}|$ matrix, is obtained by dividing the definition of the reversible Markov chain matrix $\mathcal{K}^R$ (3) by $\sqrt{\pi(x)}\sqrt{\pi(y)}$,*

$$\mathcal{H}^R(x, y) \stackrel{def}{=} \frac{1}{\sqrt{\pi(x)}} \mathcal{K}^R(x, y)\sqrt{\pi(y)} = \frac{1}{\sqrt{\pi(y)}} \mathcal{K}^R(y, x)\sqrt{\pi(x)} = \mathcal{H}^R(y, x), \quad x, y \in \mathcal{X}. \tag{5}$$

It has the Perron–Frobenius eigenvector $\sqrt{\pi(x)}$,

$$\sum_{y\in\mathcal{X}} \mathcal{H}^R(x, y)\sqrt{\pi(y)} = \frac{1}{\sqrt{\pi(x)}} \sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y) = \sqrt{\pi(x)} \sum_{y\in\mathcal{X}} \mathcal{K}^R(y, x) = \sqrt{\pi(x)}, \tag{6}$$

and its eigenvalues are all real,

$$-1 < \text{The eigenvalues of } \mathcal{H}^R(x, y) \leq 1. \tag{7}$$

3. Construction of fermion Hamiltonian $\mathcal{H}_f^R$ with wide range interactions

Here I explain the construction method of reversible and *finite* Markov chains $\mathcal{K}^R$ adopted in Ref. [2]. This gives a classical Hamiltonian $\mathcal{H}^R$ (5) and the general principle of Ref. [1] provides the fermion Hamiltonian $\mathcal{H}_f^R$ with wide range interactions. Those for the *infinite* Markov chains are derived by certain limiting procedures including $N \to \infty$, as will be shown shortly. The method is based on certain convolutions of the orthogonality measures of discrete orthogonal polynomials of the Askey scheme. The normalized orthogonality measure, being positive, can always be a probability distribution. Let us denote the normalized orthogonality measure with the explicit N dependence by

$$\pi(x, N, \boldsymbol{\lambda}) > 0, \quad \sum_{x\in\mathcal{X}} \pi(x, N, \boldsymbol{\lambda}) = 1, \tag{8}$$

in which $\boldsymbol{\lambda}$ stands for the set of parameters.

3.1. *Three types of convolutions*

Here I present three types of convolutions among the five reported in Ref. [2] for simplicity and clarity:

$$\text{(i)}: \quad \mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \stackrel{\text{def}}{=} \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, \boldsymbol{\lambda}_2)\pi(z, y, \boldsymbol{\lambda}_1), \tag{9}$$

$$\text{(ii)}: \quad \mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \stackrel{\text{def}}{=} \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z, N-y, \boldsymbol{\lambda}_2)\pi(z, y, \boldsymbol{\lambda}_1), \tag{10}$$

$$\text{(iii)}: \quad \mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \stackrel{\text{def}}{=} \sum_{z=\max(x,y)}^{N} \pi(x, z, \boldsymbol{\lambda}_2)\pi(z-y, N-y, \boldsymbol{\lambda}_1). \tag{11}$$

It is easy to verify the positivity and the conservation of the probability (2) for each convolution. The strategy is to find a good set of parameters $\boldsymbol{\lambda}_i$, $i = 1, 2, 3$, such that the reversibility condition

$$\mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\pi(y, N, \boldsymbol{\lambda}_3) = \mathcal{K}^R(y, x; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\pi(x, N, \boldsymbol{\lambda}_3), \tag{12}$$

is satisfied. Obviously $\boldsymbol{\lambda}_3$ is a function of the parameters $\boldsymbol{\lambda}_i$, $i = 1, 2$ in $\mathcal{K}^R$.

The main result of Ref. [2] is the following.

Theorem 1. *(Odake–Sasaki)*

The finite discrete orthogonal polynomials $\{\check{P}_n(x, \boldsymbol{\lambda}_3)\}$*, whose orthogonality measure* $\pi(x, N, \boldsymbol{\lambda}_3)$ *provides the reversible distribution of* $\mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$*, constitute the left eigenvectors of* $\mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$*,*

$$\sum_{x\in\mathcal{X}} \mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\check{P}_n(x, \boldsymbol{\lambda}_3) = \kappa(n)\check{P}_n(y, \boldsymbol{\lambda}_3), \ -1 < \kappa(n) \le 1, \ x, n \in \mathcal{X}, \ \kappa(0) = 1. \tag{13}$$

The right eigenvectors are $\{\pi(x, N, \boldsymbol{\lambda}_3)\check{P}_n(x, \boldsymbol{\lambda}_3)\}$*,*

$$\begin{aligned}\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\pi(y, N, \boldsymbol{\lambda}_3)\check{P}_n(y, \boldsymbol{\lambda}_3) &= \pi(x, N, \boldsymbol{\lambda}_3)\sum_{y\in\mathcal{X}} \mathcal{K}^R(y, x; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\check{P}_n(y, \boldsymbol{\lambda}_3)\\ &= \kappa(n)\pi(x, N, \boldsymbol{\lambda}_3)\check{P}_n(x, \boldsymbol{\lambda}_3), \quad x, n \in \mathcal{X}.\end{aligned} \tag{14}$$

The Hamiltonian $\mathcal{H}^R$ *has the eigenvectors* $\{\sqrt{\pi(x, N, \boldsymbol{\lambda}_3)}\,\check{P}_n(x, \boldsymbol{\lambda}_3)\}$*,*

$$\begin{aligned}&\sum_{y\in\mathcal{X}} \mathcal{H}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\sqrt{\pi(y, N, \boldsymbol{\lambda}_3)}\,\check{P}_n(y, \boldsymbol{\lambda}_3)\\ &\quad= \frac{1}{\sqrt{\pi(x, N, \boldsymbol{\lambda}_3)}}\sum_{y\in\S} \mathcal{K}^R(x, y; \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\pi(y, N, \boldsymbol{\lambda}_3)\,\check{P}_n(y, \boldsymbol{\lambda}_3)\\ &\quad= \kappa(n)\sqrt{\pi(x, N, \boldsymbol{\lambda}_3)}\,\check{P}_n(y, \boldsymbol{\lambda}_3).\end{aligned} \tag{15}$$

The normalization constant of the polynomial $\check{P}_n(x)$ is determined by the universal normalization condition

$$\check{P}_n(0,\boldsymbol{\lambda}_3)=1, \qquad \forall n\in\mathcal{X}, \tag{16}$$

$$\sum_{x\in\mathcal{X}}\pi(x,N,\boldsymbol{\lambda}_3)\check{P}_m(x,\boldsymbol{\lambda}_3)\check{P}_n(x,\boldsymbol{\lambda}_3)=\frac{\delta_{m,n}}{d_n^2}, \quad d_n>0, \quad m,n\in\mathcal{X}. \tag{17}$$

Of course, the constant d_n also depends on $\boldsymbol{\lambda}_3$ but its dependence is suppressed for simplicity of presentation. It should be noted that, because of the context, the present definition of d_n^2 is slightly different from the previous one [1,6]. The orthonormal eigenvectors of the classical Hamiltonian $\mathcal{H}^R$ are $\hat{\phi}_n(x)$,

$$\mathcal{H}^R\hat{\phi}_n=\kappa(n)\hat{\phi}_n \;\Leftrightarrow \sum_{y\in\mathcal{X}}\mathcal{H}^R(x,y;\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\hat{\phi}_n(y)=\kappa(n)\hat{\phi}_n(x),$$

$$\hat{\phi}_n(x)\stackrel{\text{def}}{=}d_n\sqrt{\pi(x,N,\boldsymbol{\lambda}_3)}\,\check{P}_n(x,\boldsymbol{\lambda}_3)\in\mathbb{R}, \tag{18}$$

$$\sum_{x\in\mathcal{X}}\hat{\phi}_m(x)\hat{\phi}_n(x)=\delta_{m,n}, \quad \sum_{n\in\mathcal{X}}\hat{\phi}_n(x)\hat{\phi}_n(y)=\delta_{x,y}. \tag{19}$$

3.2. *Fermion Hamiltonian* $\mathcal{H}_f^R$

The fermion Hamiltonian $\mathcal{H}_f^R$ with wide range interactions is defined from the classical Hamiltonian $\mathcal{H}^R$ as a bilinear form of the lattice fermions $\{c_x\}$, $\{c_x^\dagger\}$ on $\mathcal{X}$, obeying the canonical anticommutation relations,

$$\{c_x^\dagger,c_y\}=\delta_{x,y}, \quad \{c_x^\dagger,c_y^\dagger\}=0=\{c_x,c_y\}, \quad x,y\in\mathcal{X}, \tag{20}$$

$$\mathcal{H}_f^R\stackrel{\text{def}}{=}\sum_{x,y\in\mathcal{X}}c_x^\dagger\mathcal{H}^R(x,y)c_y, \tag{21}$$

in which the parameter dependence is suppressed for simplicity of presentation.

Theorem 2. *The Hamiltonian $\mathcal{H}_f^R$ is diagonalized by the introduction of the momentum space fermion operators $\{\hat{c}_n\}$, $\{\hat{c}_n^\dagger\}$, $n\in\mathcal{X}$,*

$$\hat{c}_n\stackrel{def}{=}\sum_{x\in\mathcal{X}}\hat{\phi}_n(x)c_x, \;\; \hat{c}_n^\dagger=\sum_{x\in\mathcal{X}}\hat{\phi}_n(x)c_x^\dagger \;\Leftrightarrow\; c_x=\sum_{n\in\mathcal{X}}\hat{\phi}_n(x)\hat{c}_n, \;\; c_x^\dagger=\sum_{m\in\mathcal{X}}\hat{\phi}_m(x)\hat{c}_m^\dagger, \tag{22}$$

$$\Longrightarrow\; \{\hat{c}_m^\dagger,\hat{c}_n\}=\delta_{m\,n}, \;\; \{\hat{c}_m^\dagger,\hat{c}_n^\dagger\}=0=\{\hat{c}_m,\hat{c}_n\}, \tag{23}$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{H}_f^R&=\sum_{m,n,x,y\in\mathcal{X}}\hat{\phi}_m(x)\mathcal{H}^R(x,y)\hat{\phi}_n(y)\hat{c}_m^\dagger\hat{c}_n=\sum_{m,n,x\in\mathcal{X}}\kappa(n)\hat{\phi}_m(x)\hat{\phi}_n(x)\hat{c}_m^\dagger\hat{c}_n\\&=\sum_{n\in\mathcal{X}}\kappa(n)\hat{c}_n^\dagger\hat{c}_n,\end{aligned} \tag{24}$$

$$\Longrightarrow\; [\mathcal{H}_f^R,\hat{c}_n^\dagger]=\kappa(n)\hat{c}_n^\dagger, \quad [\mathcal{H}_f^R,\hat{c}_n]=-\kappa(n)\hat{c}_n. \tag{25}$$

4. Explicit forms of the classical Hamiltonians $\mathcal{H}^R$

Here I present the explicit forms of the reversible Markov chain matrices $\mathcal{K}^R(x,y)$ belonging to a certain subset of the discrete orthogonal polynomials of the Askey scheme [3–6]. They are all reported in Ref. [2] and reproduced here for self-containedness. The polynomials are the Krawtchouk (K), Charlier (C), Hahn (H), Meixner (M), and q-Hahn (qH). For each polynomial, after listing the basic data, at most three types of reversible Markov chain matrices $\mathcal{K}^R(x, y)$ are displayed. The classical Hamiltonian $\mathcal{H}^R$ (5) and the fermion Hamiltonian $\mathcal{H}_f^R$ (21) are obtained straightforwardly. They could be used to calculate many interesting quantities of the fermions with wide range interactions, e.g. entanglement entropy etc. [7–12].

4.1. *Krawtchouk*

The polynomial depends on one positive parameter $\boldsymbol{\lambda} = p\ (0 < p < 1)$,

$$\pi(x, N, p) = \binom{N}{x} p^x (1-p)^{N-x}, \quad \binom{N}{x} = \frac{N!}{x!\,(N-x)!}, \quad d_n^2 = \binom{N}{n}\Big(\frac{p}{1-p}\Big)^n, \tag{26}$$

$$\check{P}_n(x, p) = P_n(x, p) = {}_2F_1\Big(\begin{matrix}-n,\ -x\\ -N\end{matrix}\ \Big|\ p^{-1}\Big), \quad P_n(x, p) = P_x(n, p), \quad \text{(self-dual)}. \tag{27}$$

4.1.1. *Type (i) convolution.* This convolution has been applied to (K) and (H) in many papers [13–15] in connection with “cumulative Bernoulli trials.” By taking $\boldsymbol{\lambda}_1 = a$, $\boldsymbol{\lambda}_2 = b$, and $\boldsymbol{\lambda}_3 = p \stackrel{\text{def}}{=} \frac{b}{1-a+ab}$, the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b)\pi(z, y, a), \quad 0 < a, b, p < 1, \tag{28}$$

satisfying

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y, N, p)\check{P}_n(y, p) = \kappa(n)\pi(x, N, p)\check{P}_n(x, p),$$

$$\kappa(n) = a^n(1-b)^n = {}_1F_0\Big(\begin{matrix}-n\\ -\end{matrix}\ \Big|\ bp^{-1}\Big), \quad n \in \mathcal{X}. \tag{29}$$

$$\Rightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi(x, N, p)}} \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b)\,\pi(z, y, a)\sqrt{\pi(y, N, p)}, \tag{30}$$

$$\hat{\phi}_n(x) = d_n\sqrt{\pi(x, N, p)}\,{}_2F_1\Big(\begin{matrix}-n,\ -x\\ -N\end{matrix}\ \Big|\ p^{-1}\Big), \quad d_n^2 = \tbinom{N}{n}\big(\tfrac{p}{1-p}\big)^n, \quad p = \tfrac{b}{1-a+ab}. \tag{31}$$

4.1.2. *Type (ii) convolution.* By taking $\boldsymbol{\lambda}_1 = a$, $\boldsymbol{\lambda}_2 = b$, and $\boldsymbol{\lambda}_3 = p \stackrel{\text{def}}{=} \frac{b}{1-a+b}$, the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z, N-y, b)\pi(z, y, a), \quad 0 < a, b, p < 1, \tag{32}$$

satisfying

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y, N, p)\check{P}_n(y, p) = \kappa(n)\pi(x, N, p)\check{P}_n(x, p),$$

$$\kappa(n) = (a-b)^n = {}_1F_0\Big(\begin{matrix}-n\\ -\end{matrix}\ \Big|\ bp^{-1}\Big). \quad n \in \mathcal{X}, \tag{33}$$

$$\Rightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi(x, N, p)}} \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z, N-y, b)\pi(z, y, a)\sqrt{\pi(y, N, p)}, \tag{34}$$

$$\hat{\phi}_n(x) = d_n\sqrt{\pi(x, N, p)}\,{}_2F_1\Big(\begin{matrix}-n,\ -x\\ -N\end{matrix}\ \Big|\ p^{-1}\Big), \quad d_n^2 = \tbinom{N}{n}\big(\tfrac{p}{1-p}\big)^n, \quad p = \tfrac{b}{1-a+b}. \tag{35}$$

It is interesting to note that odd eigenvalues are all negative if $0 < a < b < 1$.

4.1.3. *Type (iii) convolution.* By taking $\boldsymbol{\lambda}_1 = a$, $\boldsymbol{\lambda}_2 = b$, and $\boldsymbol{\lambda}_3 = p \stackrel{\text{def}}{=} \frac{ab}{1-b+ab}$, the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=\max(x,y)}^{N} \pi(x, z, b)\pi(z-y, N-y, a) \quad 0 < a, b, p < 1, \tag{36}$$

satisfying

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y, N, p)\check{P}_n(y, p) = \kappa(n)\pi(x, N, p)\check{P}_n(x, p),$$

$$\kappa(n) = (1-a)^n b^n = {}_1F_0\Big(\begin{matrix}-n\\ -\end{matrix}\ \Big|\ abp^{-1}\Big), \quad n \in \mathcal{X}, \tag{37}$$

$$\Rightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi(x, N, p)}} \sum_{z=\max(x,y)}^{N} \pi(x, z, b)\pi(z-y, N-y, a)\sqrt{\pi(y, N, p)}, \tag{38}$$

$$\hat{\phi}_n(x) = d_n\sqrt{\pi(x, N, p)}\,{}_2F_1\Big(\begin{matrix}-n,\ -x\\ -N\end{matrix}\ \Big|\ p^{-1}\Big), \quad d_n^2 = \binom{N}{n}\big(\tfrac{p}{1-p}\big)^n, \quad p = \tfrac{ab}{1-a+ab}. \tag{39}$$

4.2. *Charlier*

This polynomial is defined on a semi-infinite integer lattice $\mathcal{X} = \mathbb{Z}_{\geq 0}$ depending on one positive parameter $\boldsymbol{\lambda} = a$ $(a > 0)$,

$$\pi(x, a) = \frac{a^x e^{-a}}{x!}, \quad d_n^2 = \frac{a^n}{n!}, \tag{40}$$

$$\check{P}_n(x, a) = P_n(x, a) = {}_2F_0\Big(\begin{matrix}-n,\ -x\\ -\end{matrix}\ \Big|\ -a^{-1}\Big), \quad P_n(x, a) = P_x(n, a), \quad \text{(self-dual)}. \tag{41}$$

By the replacement $p \to pN^{-1}$ and in the limit $N \to \infty$, the Krawtchouk (K) goes to Charlier (C) [4],

$$\check{P}_{\mathrm{K}\,n}(x, p) \to \check{P}_{\mathrm{C}\,n}(x, p), \quad \pi_{\mathrm{K}}(x, N, p) \to \pi_{\mathrm{C}}(x, p), \quad d_{\mathrm{K}\,n}^2 \to d_{\mathrm{C}\,n}^2.$$

4.2.1. *Type (i) convolution.* This is achieved by $b \to bN^{-1}$, $N \to \infty$ in $\mathcal{K}^R$ of Krawtchouk type (i) convolution (28),

$$\check{P}_n(x, p) \to \check{P}_{\mathrm{C}\,n}(x, p'), \quad p' \stackrel{\text{def}}{=} \tfrac{b}{1-a}, \qquad 0 < a < 1,$$

$$\pi(x, N, p) \to \pi_{\mathrm{C}}(x, p'), \quad \kappa(n) \to \kappa_{\mathrm{C}}(n) = a^n,$$

$$\mathcal{K}^R(x, y) \to \mathcal{K}^R_{\mathrm{C}}(x, y) = \sum_{z=0}^{\min(x,y)} \pi_{\mathrm{C}}(x-z, b)\pi_{\mathrm{K}}(z, y, a), \tag{42}$$

$$\Longrightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi_{\mathrm{C}}(x, p')}} \sum_{z=0}^{\min(x,y)} \pi_{\mathrm{C}}(x-z, b)\pi_{\mathrm{K}}(z, y, a)\sqrt{\pi_{\mathrm{C}}(y, p')}, \tag{43}$$

$$\hat{\phi}_n(x) = d_n\pi_{\mathrm{C}}(x, p')\,{}_2F_0\Big(\begin{matrix}-n,\ -x\\ -\end{matrix}\ \Big|\ -p'^{-1}\Big), \quad d_n^2 = p'^n/n!. \tag{44}$$

The infinite limit of Krawtchouk $\mathcal{K}^R(x, y)$ of type (ii) convolution gives the same result as this one.

4.2.2. *Type (iii) convolution.* This is achieved by $a \to aN^{-1}$, $N \to \infty$ in $\mathcal{K}^R$ of Krawtchouk type (iii) convolution (36),

$$\check{P}_n(x, p) \to P_{\mathrm{C}\,n}(x, p'), \quad p' \stackrel{\text{def}}{=} \tfrac{ab}{1-b}, \qquad 0 < b < 1,$$

$$\pi(x, N, p) \to \pi_{\mathrm{C}}(x, p'), \quad \kappa(n) \to \kappa_{\mathrm{C}}(n) = b^n = {}_1F_0\Big(\begin{matrix}-n\\ -\end{matrix}\Big|\, abp'^{-1}\Big),$$

$$\mathcal{K}^R(x, y) \to \mathcal{K}^R_{\mathrm{C}}(x, y) = \sum_{z=\max(x,y)}^{\infty} \pi_{\mathrm{K}}(x, z, b)\pi_{\mathrm{C}}(z-y, a), \tag{45}$$

$$\Longrightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi_{\mathrm{C}}(x, p')}} \sum_{z=\max(x,y)}^{\infty} \pi_{\mathrm{K}}(x, z, b)\pi_{\mathrm{C}}(z-y, a)\sqrt{\pi_{\mathrm{C}}(y, p')}, \tag{46}$$

$$\hat{\phi}_n(x) = d_n\pi_{\mathrm{C}}(x, p')\,{}_2F_0\Big(\begin{matrix}-n, -x\\ -\end{matrix}\Big|\, -p'^{-1}\Big), \quad d_n^2 = p'^n/n!. \tag{47}$$

4.3. *Hahn*

The polynomial depends on two positive parameters $\boldsymbol{\lambda} = (a, b)$ $(a, b > 0)$,

$$\pi(x, N, a, b) = \binom{N}{x}\frac{(a)_x\,(b)_{N-x}}{(a+b)_N}, \quad d_n^2 = \binom{N}{n}\frac{(a)_n\,(2n+a+b-1)(a+b)_N}{(b)_n\,(n+a+b-1)_{N+1}}, \tag{48}$$

$$\check{P}_n(x, a, b) = P_n(x, a, b) = {}_3F_2\Big(\begin{matrix}-n,\ n+a+b-1,\ -x\\ a,\ -N\end{matrix}\Big|\, 1\Big). \tag{49}$$

4.3.1. *Type (i) convolution.* For $\boldsymbol{\lambda}_1 = (a, b)$, $\boldsymbol{\lambda}_2 = (b, c)$, and $\boldsymbol{\lambda}_3 = (a+b, c)$, the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b, c)\pi(z, y, a, b), \quad 0 < a, b, c, \tag{50}$$

satisfying

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y, N, a+b, c)\check{P}_n(y, a+b, c) = \kappa(n)\pi(x, N, a+b, c)\check{P}_n(x, a+b, c),$$

$$\kappa(n) = \frac{(a)_n(c)_n}{(a+b)_n(b+c)_n} = {}_3F_2\Big(\begin{matrix}-n,\ n+a+b+c-1,\ b\\ a+b,\ b+c\end{matrix}\Big|\, 1\Big), \quad n \in \mathcal{X}, \tag{51}$$

$$\Longrightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi(x, N, a+b, c)}} \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b, c)\pi(z, y, a, b)\sqrt{\pi(y, N, a+b, c)}, \tag{52}$$

$$\hat{\phi}_n(x) = d_n\pi(x, N, a+b, c)\,{}_3F_2\Big(\begin{matrix}-n,\ n+a+b+c-1,\ -x\\ a+b,\ -N\end{matrix}\Big|\, 1\Big), \tag{53}$$

$$d_n^2 = \binom{N}{n}\frac{(a+b)_n\,(2n+a+b+c-1)(a+b+c)_N}{(c)_n\,(n+a+b+c-1)_{N+1}}. \tag{54}$$

4.3.2. *Type (ii) convolution.* For $\boldsymbol{\lambda}_1 = (a, b)$, $\boldsymbol{\lambda}_2 = (b, c)$, and $\boldsymbol{\lambda}_3 = (a+b, b+c)$ the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z, N-y, b, c)\pi(z, y, a, b), \tag{55}$$

satisfying

$$\sum_{y\in\mathcal{X}}\mathcal{K}^R(x,y)\pi(y,N,a+b,b+c)\check{P}_n(y,a+b,b+c)$$

$$=\kappa(n)\pi(x,N,a+b,b+c)\check{P}_n(x,a+b,b+c),$$

$$\kappa(n)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(b)_k(n+a+2b+c-1)_k}{(a+b)_k(b+c)_k} \tag{56}$$

$$={}_3F_2\Bigl(\begin{matrix}-n,\ n+a+2b+c-1,\ b\\ a+b,\ b+c\end{matrix}\Bigm|1\Bigr),\quad n\in\mathcal{X}, \tag{57}$$

$$\Longrightarrow \mathcal{H}^R(x,y)=\frac{1}{\sqrt{\pi(x,N,a+b,b+c)}}\sum_{z=\max(0,x+y-N)}^{\min(x,y)}\pi(x-z,N-y,b,c)\pi(z,y,a,b)\times\sqrt{\pi(y,N,a+b,b+c)}, \tag{58}$$

$$\hat{\phi}_n(x)=d_n\pi(x,N,a+b,b+c)\,{}_3F_2\Bigl(\begin{matrix}-n,\ n+a+2b+c-1,\ -x\\ a+b,\ -N\end{matrix}\Bigm|1\Bigr), \tag{59}$$

$$d_n^2=\binom{N}{n}\frac{(a+b)_n\,(2n+a+2b+c-1)(a+2b+c)_N}{(b+c)_n\,(n+a+2b+c-1)_{N+1}}. \tag{60}$$

4.3.3. *Type (iii) convolution.* For $\boldsymbol{\lambda}_1=(a,b)$, $\boldsymbol{\lambda}_2=(c,a)$, and $\boldsymbol{\lambda}_3=(c,a+b)$, the matrix $\mathcal{K}^R(x,y)$ is

$$\mathcal{K}^R(x,y)=\sum_{z=\max(x,y)}^{N}\pi(x,z,c,a)\pi(z-y,N-y,a,b)\quad 0<a,b,c, \tag{61}$$

satisfying

$$\sum_{y\in\mathcal{X}}\mathcal{K}^R(x,y)\pi(y,N,c,a+b)\check{P}_n(y,c,a+b)$$

$$=\kappa(n)\pi(x,N,c,a+b)\check{P}_n(x,c,a+b),$$

$$\kappa(n)=\frac{(b)_n(c)_n}{(a+b)_n(a+c)_n}={}_3F_2\Bigl(\begin{matrix}-n,\ n+a+b+c-1,\ a\\ a+b,\ a+c\end{matrix}\Bigm|1\Bigr),\quad n\in\mathcal{X}, \tag{62}$$

$$\Longrightarrow \mathcal{H}^R(x,y)=\frac{1}{\sqrt{\pi(x,N,c,a+b)}}\sum_{z=\max(x,y)}^{N}\pi(x,z,c,a)\pi(z-y,N-y,a,b)\sqrt{\pi(y,N,c,a+b)}, \tag{63}$$

$$\hat{\phi}_n(x)=d_n\pi(x,N,c,a+b)\,{}_3F_2\Bigl(\begin{matrix}-n,\ n+a+b+c-1,\ -x\\ c,\ -N\end{matrix}\Bigm|1\Bigr), \tag{64}$$

$$d_n^2=\binom{N}{n}\frac{(c)_n\,(2n+a+b+c-1)(a+b+c)_N}{(a+b)_n\,(n+a+b+c-1)_{N+1}}. \tag{65}$$

4.4. *Meixner*

This self-dual polynomial is defined on a semi-infinite integer lattice $\mathcal{X}=\mathbb{Z}_{\geq 0}$ with two positive parameters $\boldsymbol{\lambda}=(a,b)$ $(0<a,\ 0<b<1)$,

$$\pi(x,a,b)=\frac{(a)_x\,b^x(1-b)^a}{x!},\quad d_n^2=\frac{(a)_n\,b^n}{n!}, \tag{66}$$

$$\check{P}_n(x,a,b)=P_n(x,a,b)={}_2F_1\Bigl(\begin{matrix}-n,\ -x\\ a\end{matrix}\Bigm|1-b^{-1}\Bigr),\quad P_n(x,a,b)=P_x(n,a,b). \tag{67}$$

By the replacement $b\to N(1-b)b^{-1}$ and in the limit $N\to\infty$, the Hahn (H) goes to Meixner (M),

$$\check{P}_{\mathrm{H}\,n}(x,a,b)\to\check{P}_{\mathrm{M}\,n}(x,a,b),\quad \pi_{\mathrm{H}}(x,N,a,b)\to\pi_{\mathrm{M}}(x,a,b),\quad d_{\mathrm{H}\,n}^2\to d_{\mathrm{M}\,n}^2.$$

By the replacement $b\to b/(a+b)$ and in the limit $a\to\infty$, the Meixner (M) goes to Charlier (C),

$$\check{P}_{\mathrm{M}\,n}(x,a,b)\to\check{P}_{\mathrm{C}\,n}(x,b),\quad \pi_{\mathrm{M}}(x,a,b)\to\pi_{\mathrm{C}}(x,b),\quad d_{\mathrm{M}\,n}^2\to d_{\mathrm{C}\,n}^2.$$

4.4.1. *Type (i) convolution.* This is achieved by fixing a and b with $c\to N(1-c)c^{-1}$ ($\Rightarrow 0<c<1$), $N\to\infty$ in $\mathcal{K}^R$ of Hahn type (i) convolution (50),

$$\check{P}_n(x,a+b,c)\to\check{P}_{\mathrm{M}\,n}(x,a+b,c),$$

$$\pi(x,N,a+b,c)\to\pi_{\mathrm{M}}(x,a+b,c),\quad \kappa(n)\to\kappa_{\mathrm{M}}(n)=\frac{(a)_n}{(a+b)_n}={}_2F_1\Bigl(\begin{matrix}-n,\ b\\ a+b\end{matrix}\Bigm|1\Bigr), \tag{68}$$

$$\mathcal{K}^R(x,y)\to\mathcal{K}^R_{\mathrm{M}}(x,y,a,b,c)=\sum_{z=0}^{\min(x,y)}\pi_{\mathrm{M}}(x-z,b,c)\pi_{\mathrm{H}}(z,y,a,b), \tag{69}$$

$$\Longrightarrow \mathcal{H}^R(x,y)=\frac{1}{\sqrt{\pi_{\mathrm{M}}(x,a+b,c)}}\sum_{z=0}^{\min(x,y)}\pi_{\mathrm{M}}(x-z,b,c)\pi_{\mathrm{H}}(z,y,a,b)\sqrt{\pi_{\mathrm{M}}(y,a+b,c)}, \tag{70}$$

$$\hat{\phi}_n(x)=d_n\pi_{\mathrm{M}}(x,a+b,c)\,{}_2F_1\Bigl(\begin{matrix}-n,\ -x\\ a+b\end{matrix}\Bigm|1-c^{-1}\Bigr),\quad d_n^2=\frac{(a+b)_n\,c^n}{n!}. \tag{71}$$

4.4.2. *Type (ii) convolution.* By fixing a, b with $c\to N(1-c)c^{-1}$ ($\Rightarrow 0<c<1$) and taking the limit $N\to\infty$ in $\mathcal{K}^R$ of Hahn type (ii) convolution (55), one obtains the same Meixner limit $\mathcal{K}^R_{\mathrm{M}}(x,y)$ as in Eq. (69),

$$\mathcal{K}^R(x,y)\to\mathcal{K}^R_{\mathrm{M}}(x,y,a,b,c)=\sum_{z=0}^{\min(x,y)}\pi_{\mathrm{M}}(x-z,b,c)\pi_{\mathrm{H}}(z,y,a,b).$$

4.4.3. *Type (iii) convolution.* This is achieved by fixing a and c with $b\to N(1-b)b^{-1}$ ($\Rightarrow 0<b<1$), $N\to\infty$ in $\mathcal{K}^R$ of Hahn type (iii) convolution (61),

$$\check{P}_n(x,c,a+b)\to\check{P}_{\mathrm{M}\,n}(x,c,b),$$

$$\pi(x,N,c,a+b)\to\pi_{\mathrm{M}}(x,c,b),\quad \kappa(n)\to\kappa_{\mathrm{M}}(n)=\frac{(c)_n}{(a+c)_n}={}_2F_1\Bigl(\begin{matrix}-n,\ a\\ a+c\end{matrix}\Bigm|1\Bigr), \tag{72}$$

$$\mathcal{K}^R(x,y)\to\mathcal{K}^R_{\mathrm{M}}(x,y,a,b,c)=\sum_{z=\max(x,y)}^{\infty}\pi_{\mathrm{H}}(x,z,c,a)\pi_{\mathrm{M}}(z-y,a,b), \tag{73}$$

$$\Longrightarrow \mathcal{H}^R(x,y)=\frac{1}{\sqrt{\pi_{\mathrm{M}}(x,c,b)}}\sum_{z=\max(x,y)}^{\infty}\pi_{\mathrm{H}}(x,z,c,a)\pi_{\mathrm{M}}(z-y,a,b)\sqrt{\pi_{\mathrm{M}}(y,c,b)}, \tag{74}$$

$$\hat{\phi}_n(x)=d_n\pi_{\mathrm{M}}(x,c,b)\,{}_2F_1\Bigl(\begin{matrix}-n,\ -x\\ c\end{matrix}\Bigm|1-b^{-1}\Bigr),\quad d_n^2=\frac{(c)_n\,b^n}{n!}. \tag{75}$$

4.5. *q-Hahn*

The q-Hahn is defined on a finite integer lattice with two positive parameters $\boldsymbol{\lambda} = (a, b)$ $(0 < a < 1, b < 1)$ on top of q, $0 < q < 1$, and the q dependence of π and $\check{P}_n$ is suppressed. It should be stressed that the polynomial $\check{P}_n(x)$ is a degree n polynomial in $\eta(x) \stackrel{\text{def}}{=} q^{-x} - 1$, not in x,

$$\pi(x, N, a, b) = \begin{bmatrix} N \\ x \end{bmatrix} \frac{(a\,;q)_x\,(b\,;q)_{N-x}a^{N-x}}{(ab\,;q)_N}, \quad \begin{bmatrix} N \\ x \end{bmatrix} \stackrel{\text{def}}{=} \frac{(q\,;q)_N}{(q\,;q)_x\,(q\,;q)_{N-x}}, \tag{76}$$

$$d_n^2 = \begin{bmatrix} N \\ n \end{bmatrix} \frac{(a, abq^{-1}\,;q)_n}{(abq^N, b\,;q)_n\,a^n} \frac{1 - abq^{2n-1}}{1 - abq^{-1}}, \tag{77}$$

$$\check{P}_n(x, a, b) = P_n\big(\eta(x), a, b\big) = {}_3\phi_2\Big(\begin{matrix} q^{-n},\ abq^{n-1},\ q^{-x} \\ a,\ q^{-N} \end{matrix}\ \Big|\ q\,;q\Big). \tag{78}$$

4.5.1. *Type (i) convolution.* By taking $\boldsymbol{\lambda}_1 = (a, b)$, $\boldsymbol{\lambda}_2 = (b, c)$, and $\boldsymbol{\lambda}_3 = (ab, c)$ the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=0}^{\min(x,y)} \pi(x - z, N - z, b, c)\pi(z, y, a, b), \tag{79}$$

satisfying

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y, N, ab, c)\check{P}_n(y, ab, c) = \kappa(n)\pi(x, N, ab, c)\check{P}_n(x, ab, c),$$

$$\kappa(n) = \frac{b^n(a\,;q)_n(c\,;q)_n}{(ab\,;q)_n(bc\,;q)_n} = {}_3\phi_2\Big(\begin{matrix} q^{-n},\ abcq^{n-1},\ b \\ ab,\ bc \end{matrix}\ \Big|\ q\,;q\Big), \tag{80}$$

$$\Longrightarrow \mathcal{H}^R(x, y) = \frac{1}{\sqrt{\pi(x, N, ab, c)}} \sum_{z=0}^{\min(x,y)} \pi(x - z, N - z, b, c)\pi(z, y, a, b)\sqrt{\pi(y, N, ab, c)}, \tag{81}$$

$$\hat{\phi}_n(x) = d_n\pi(x, N, ab, c)\,{}_3\phi_2\Big(\begin{matrix} q^{-n},\ abcq^{n-1},\ q^{-x} \\ ab,\ q^{-N} \end{matrix}\ \Big|\ q\,;q\Big), \tag{82}$$

$$d_n^2 = \begin{bmatrix} N \\ n \end{bmatrix} \frac{(ab, abcq^{-1}\,;q)_n}{(abcq^N, c\,;q)_n\,(ab)^n} \frac{1 - abcq^{2n-1}}{1 - abcq^{-1}}. \tag{83}$$

The type (ii) convolution does not exist for q-Hahn.

4.5.2. *Type (iii) convolution.* By taking $\boldsymbol{\lambda}_1 = (a, b)$, $\boldsymbol{\lambda}_2 = (c, a)$, and $\boldsymbol{\lambda}_3 = (c, ab)$ the matrix $\mathcal{K}^R(x, y)$ is

$$\mathcal{K}^R(x, y) = \sum_{z=\max(x,y)}^{N} \pi(x, z, c, a)\pi(z - y, N - y, a, b), \tag{84}$$

satisfying

$$\sum_{y\in\mathcal{X}} \mathcal{K}^R(x, y)\pi(y, N, c, ab)\check{P}_n(y, c, ab) = \kappa(n)\pi(x, N, c, ab)\check{P}_n(x, c, ab),$$

$$\kappa(n) = \frac{a^n(b\,;q)_n(c\,;q)_n}{(ab\,;q)_n(ac\,;q)_n} = {}_3\phi_2\Big(\begin{matrix} q^{-n},\ abcq^{n-1},\ a \\ ac,\ ab \end{matrix}\ \Big|\ q\,;q\Big), \tag{85}$$

$$\Longrightarrow \mathcal{H}^R(x,y) = \frac{1}{\sqrt{\pi(x,N,c,ab)}} \sum_{z=\max(x,y)}^{N} \pi(x,z,c,a)\pi(z-y,N-y,a,b)\sqrt{\pi(y,N,c,ab)}, \tag{86}$$

$$\hat{\phi}_n(x) = d_n \pi(x,N,c,ab)\,{}_3\phi_2\Big(\begin{matrix} q^{-n},\ abcq^{n-1},\ q^{-x} \\ c,\ q^{-N} \end{matrix}\ \Big|\ q\,;q\Big), \tag{87}$$

$$d_n^2 = \begin{bmatrix} N \\ n \end{bmatrix} \frac{(c, abcq^{-1}\,;q)_n}{(abcq^N, ab\,;q)_n\, c^n} \frac{1-abcq^{2n-1}}{1-abcq^{-1}}. \tag{88}$$

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Data availability statement

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

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