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Simple approximate model for pion masses

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Abstract Closed expressions for the masses of the charged and neutral pions are derived based on a simple approximate model for electrostatic screening when a proton is located adjacent to a neutron. The estimated mass values agree to a few percent of experimentally measured values. The electric dipole moment of the deuteron is also estimated based on the phenomenological toy model used here.

1 Introduction

While the Standard Model agrees with most high energy physics experimental data, it still requires at least 19 input parameters, including 9 mass parameters for the six quarks and three leptons. The bare masses of the valence quarks in nucleons and mesons, such as protons, neutrons and pions (pi-mesons), account for only a small fraction of the experimental baryon and meson masses. It is thought that the measured masses of the neutron, proton and pions come primarily from the gluons and sea quarks. Theoretical estimates of the masses of the pions have been derived using a number of approaches, including lattice QCD [1,2] and methods via the Higgs field [3]. This article proposes a simple phenomenological toy model to estimate the pion masses based on electrostatic screening in nuclei when protons and neutrons are adjacent. Experiments have demonstrated the impact within nucleons due to adjacent nucleons in atomic nuclei, which includes the EMC effect [4].

2 The model

Atomic nuclei consists of protons and neutrons and here we will analyze the potential effect on the neutron substructure due to the electrostatic field from an adjacent proton. The geometry of the two adjacent nucleons are shown in Fig. 1, where the origin is at the center of the proton and the x-axis passing through the both nucleon centers. Assume that the quarks are confined in the neutron and relatively free to move. Outside the proton, from Gauss' law, the proton charge $+e$ is concentrated at $r = 0$. The geometric equations for the two nucleons are

$$x^2 + y^2 + z^2 = r^2 \tag{1}$$
$$(x - 2a)^2 + y^2 + z^2 = a^2 \tag{2}$$

combining equations yield

$$y^2 + z^2 = 4ax - x^2 - 3a^2 \tag{3}$$
$$r^2 = 4ax - 3a^2 \tag{4}$$

where a is the radius of the neutron (and proton), $a = r_n \approx r_p$.

Since the quarks are confined within the neutron, we know from the uncertainty principle an estimate of the momentum, which in turn can be used to estimate the matter wavelength of the quarks:

$$\Delta p = \frac{\hbar}{\Delta x} \tag{5}$$
$$\lambda \approx \frac{\hbar}{\Delta p} \approx \Delta x \approx a. \tag{6}$$

The quarks then span the volume of the neutron and given their high speeds, they can be approximated with a charged gas or cloud of uniform density ρ^- for the two negatively charged down quarks $2\,q_d = -\frac{2e}{3}$ and ρ^+ for the positively charged up quark $q_u = \frac{2e}{3}$.

$$\rho^- = -2 * \frac{e/3}{\frac{4}{3}\pi a^3} = -\frac{e}{2\pi a^3} = -\rho^+$$

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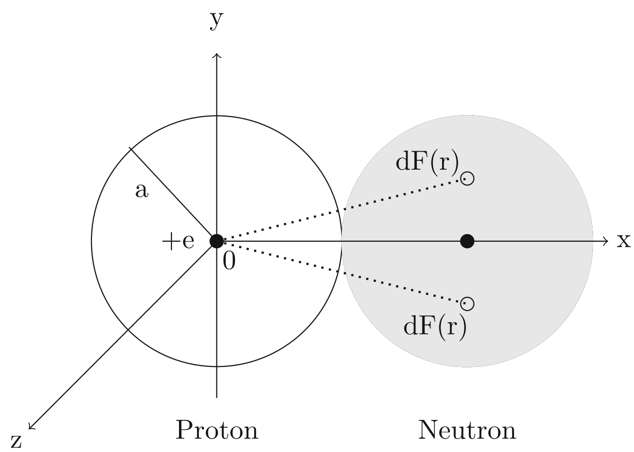

Fig. 1 Proton and Neutron adjacent where x-axis runs through each center. The radius a of each nucleon is assumed to be approximately the same

In the unpolarized state, these densities are assumed to be uniform inside the neutron and cancel out electrically. Once the proton is brought in next to the neutron, we assume there is an induced density fluctuation, where the motion and cloud of the q_d quarks are assumed to move slightly towards the proton and the q_u quark cloud to shift slightly away, causing a polarization within the neutron.

3 Analysis of Charged Pion masses

From Fig. 1, note that the force components in the y- and z-planes cancel and the net force is only in the x-direction for the negative and positive charge densities, where the negative q_d charges move slightly towards the proton center and then back towards the just separated, polarized positive cloud, causing an oscillation with frequency ω. A volume element perpendicular to the x-direction is $dV = \pi(y^2 + z^2)dx$ and the force on the negative q_d cloud in the neutron is

$$dF_x^- = \frac{\rho^- dV ke}{r^2} = -\frac{e}{2\pi a^3}\frac{ke}{r^2}\pi(y^2+z^2)dx \tag{7}$$

$$= -\frac{ke^2}{2a^3}\left(\frac{4ax - x^2 - 3a^2}{4ax - 3a^2}\right)dx \tag{8}$$

where we used Eqs. 3 and 4 and k is the Coulomb constant. The net force on the negative cloud in the neutron from the proton electric field is then

$$F_x^- = -\frac{ke^2}{2a^3}\int_a^{3a} \frac{4ax - x^2 - 3a^2}{4ax - 3a^2}dx \tag{9}$$

$$= -\frac{ke^2}{16\,a^2}\left(5 - \frac{9}{8}\ln 9\right) \tag{10}$$

Next assume the fluctuations of the negatively charged cloud in the neutron can be described by

$$x(t) = x_0 e^{-i\omega t} \tag{11}$$

$$F_x^- = m_d \frac{d^2x}{dt^2} = -m_d\,\omega^2 x \tag{12}$$

where m_d is the bare mass of the down quark q_d. Next use Eqs. 10 and 12 to solve for x:

$$x = \frac{ke^2}{a^2}\frac{1}{16\,m_d\,\omega^2}\left(5 - \frac{9}{8}\ln 9\right) \tag{13}$$

The polarization is given by $P^- = \rho^- x$ and the relative dielectric constant ε_r by

$$\varepsilon_r = 1 + \frac{P^-}{\varepsilon_0\,E_{ave}}$$

$$= 1 - \frac{e}{2\pi a^3}\frac{ke^2}{a^2}\frac{1}{16\,m_d\,\omega}\left(5 - \frac{9}{8}\ln 9\right)\frac{1}{\varepsilon_0\,E_{ave}}$$

where E_{ave} is the average electric field in the neutron from the nearby proton. The average electric field can be approximated by

$$\int \frac{ke}{r^2}dx = ke\int_a^{3a}\frac{dx}{4ax - 3a^2} \tag{14}$$

$$= \frac{ke}{4a}\ln 9 = 2a\,E_{ave} \tag{15}$$

$$E_{ave} = \frac{1}{2}\ln 9 \cdot \frac{ke}{(2a)^2} \tag{16}$$

where we have taken the average electric field from the proton located at the center of the neutron ($r = 2a$). The dielectric constant then becomes

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \tag{17}$$

from where we can identify the plasma frequency ω_p^- of the negative q_d cloud in the neutron as

$$(\omega_p^-)^2 = \frac{ke^2}{a}\frac{1}{a^2 m_d}\left(\frac{5}{\ln 9} - \frac{9}{8}\right) \tag{18}$$

Note that the $(\omega_p^-)^2$ is proportional to $1/m_d$, which is analogous to the resonance frequency $\omega^2 \propto 1/m$ of the simple system of a mass m attached to a spring in one dimension.

The Yukawa potential is

$$V_Y = -g\,\frac{1}{r}\,e^{-k_s r}$$

where g is a scaling constant and $1/k_s$ is the approximate range for the nucleon-nucleon interaction. In Yukawa's original theory, $k_s = m_\pi c/\hbar$ and m_π is the pion mass and c is the speed of light. If we assume that the plasma frequency is identified with this range, then we have

$$(k_s^-)^2 = \frac{(\omega_p^-)^2}{c^2} \tag{19}$$

$$m_{\pi^-} = \frac{k_s^- \hbar}{c} \tag{20}$$

from which we arrive at an approximate expression for the π^- pion mass

$$m_{\pi^-} \approx \frac{\hbar}{c}\left(\frac{ke^2}{a}\frac{1}{m_d c^2}\frac{1}{a^2}\left(\frac{5}{\ln 9} - \frac{9}{8}\right)\right)^{1/2} \tag{21}$$

$$\approx 152\ MeV/c^2 \tag{22}$$

using $m_d \approx 4.67\,MeV/c^2$ and $a = 0.8409\,fm$ [5]. The experimental value is $139.57\,MeV/c^2$ [6], which is about 9% different from the approximate expression in Eq. 21. While measurements of the proton charge radius has been fluctuating some in recent years, we used the most recent measurements above. However using a proton radius of $a = 0.8751\,fm$ based on other experiments (see for example [7]), we obtain for the charged pion mass $143\,MeV/c^2$, which only differs by 2.9% from measurements.

The density fluctuations of the q_d cloud in the neutron is oscillating back and forth between the slightly polarized positive cloud in the neutron and the positive field from the proton. In addition, the masses of the quark combinations up quark + anti-down quark is the same as the down quark + anti-up quark. Hence one would expect that the positive density fluctuations of the q_u cloud would be of the same frequency. Therefore we assume

$$\omega_p^- \approx \omega_p^+ \tag{23}$$

which would imply that

$$m_{\pi^-} \approx m_{\pi^+} \tag{24}$$

4 Analysis of the Neutral Pion mass

From the quark model, we know that the charged pions consist of either $u\overline{d}$ for π^+ and $d\overline{u}$ for π^- (assuming that u is the up quark and d is the down quark and that $\overline{d}$ and $\overline{u}$ are their respective anti-quarks). Note that the masses of the quarks and anti-quarks are the same, hence $m_d = m_{\overline{d}}$ and $m_u = m_{\overline{u}}$. However, the masses differ for $m_d \neq m_u$.

The neutral pion π^0 consists of a combination of $d\overline{d}$ and $u\overline{u}$ quarks, where these two quark pair masses are quite different ($m_d m_d \neq m_u m_u \neq m_d m_u$). Therefore, we would expect that the fluctuation frequencies involving a combination of $d\overline{d}$ and $u\overline{u}$ would differ from a pair of either $u\overline{d}$ or $d\overline{u}$, since

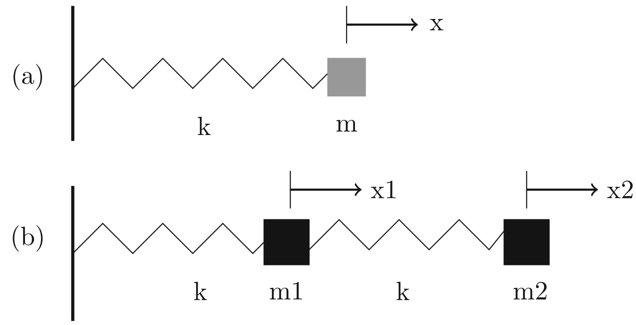

Fig. 2 Analog models of intra-nucleon charged density fluctuations for **(a)** the charged pions and **(b)** the neutral pion

in the latter case the masses of the positive charged pion or negative charged pion would be the same. The assumption of this article, is that the proton electric field has induced a polarization of the positive and negative charge clouds in the neutron, while it was assumed that the proton's charge $+e$ remained fixed at $r = 0$. In reality, the dynamics of the neutron quarks are likely to induce electric field fluctuations for the quarks of the proton as well. Hence, in this manner a neutral charged exchange meson could mediate a residual strong force interaction just like a charged pion, but with a different fundamental frequency.

Consider a simple analogous model, where two coupled masses are attached to two springs and a wall, as in Fig. 2. The spring constants are the same, but the masses are different. The previous analysis for the masses of the positive and negative charged pions are similar to the model in Fig. 2a, while the model for the neutral pion is assumed similar to the coupled system in Fig. 2b. The k in Fig. 2 is the spring constant.

The equations of motion for 2(b) in one dimension are

$$m_1\frac{d^2x_1}{dt^2} = -kx_1 - k(x_1 - x_2) \tag{25}$$

$$m_2\frac{d^2x_2}{dt^2} = -k(x_2 - x_1) \tag{26}$$

Assume $x_1 = A\sin(\omega t + \delta)$ and $x_2 = B\sin(\omega t + \delta)$, which yields a simple solution for the resonance frequency ω

$$\omega^2 = k\left(\frac{1}{m_1} + \frac{1}{2\,m_2}\right) \pm k\left(\frac{1}{m_1^2} + \frac{1}{4\,m_2^2}\right)^{1/2} \tag{27}$$

Next identify $m_1 = m_d^2$ and $m_2 = m_u^2$ (where $m_u \approx 2.16\,MeV/c^2$) [5], an effective mass can be estimated approximately as (choose the negative sign)

$$\frac{1}{m_{eff}^2} = \left(\frac{1}{m_d^2} + \frac{1}{2\,m_u^2}\right) - \left(\frac{1}{m_d^4} + \frac{1}{4\,m_u^4}\right)^{1/2} \tag{28}$$

Based on this analogous system, define an effective mass as a combination of $u\overline{u}$ and $d\overline{d}$ masses and insert it into Eq. 21 to yield

$$m_{\pi^0} \approx \frac{\hbar}{c}\left(\frac{ke^2}{a}\frac{1}{m_{eff}c^2}\frac{1}{a^2}\left(\frac{5}{\ln 9}-\frac{9}{8}\right)\right)^{1/2} \quad (29)$$

$$\approx 143.6\, MeV/c^2 \quad (30)$$

which is 6.6% different from the experimental value (134.98 MeV/c^2 [6]) of the mass of the π^0. If we again use the earlier proton charge radius of 0.8751 fm, we obtain a neutral prion mass of 135.15 MeV/c^2, which only differs by 0.5% from neutral pion mass measurements.

5 Estimates of the Deuteron electric moments

Electric Dipole Moments (EDM) of subatomic particles provide a tool for probing CP violating physics beyond the standard model. Since we have explored a simple phenomenological model here of the neutron-proton hadron, we could use it to also obtain an estimate of the EDM of the deuteron. Individually the EDM for the neutron has been limited via experiments to be $d_n < 0.18 * 10^{-25} e\,cm$ and similarly the EDM for the proton is limited to be $d_p < 2.1 * 10^{-25} e\,cm$. If we assume that the EDM for the deuteron is approximately $d_d \approx d_n + d_p$, one would infer that the EDM for the deuteron is also zero. However, if we assume that the proton induces a small polarization in the neutron (when adjacent in a deuteron) and a similar induced small polarization in the proton, then from our model the small displacement is approximately

$$x_d = \frac{ke^2}{a^2}\frac{1}{16\, m_d\, (\omega_p^-)^2}\left(5-\frac{9}{8}\ln 9\right) = \frac{ln(3)}{8}\, a \quad (31)$$

where we have used Eq. 13 and replaced ω^2 with the estimated plasma frequency $(\omega_p^-)^2$ as derived earlier (Eq. 18).

In Fig. 3, we have aligned the neutron (in gray) above the proton along the x-axis. The point s is far away from the deuteron, hence $r >> a$, where a is the radius of each nucleon as before and r is the vector from the center of the deuteron to the point s. The negative charge cloud in the neutron (N) is centered at $x = a - x_d/2$, while the positive charge cloud is centered at $x = a + x_d/2$, both along the positive x-axis. The small displacement x_d between charged clouds in the neutron is assumed to be caused by the nearby proton. From a far distance at s, it appears as if the negative charged cloud is centered at the open circle in the neutron, while the positive charge cloud is centered at the filled circle. The open circle will be treated as if all the negative charge in the neutron is located at this point, with a charge of $-2\,e/3$, while the filled circle will have a charge of $+2\,e/3$. The distances from each of these points to the point s are labeled r_1 and r_2 respectively. Note that if there is a small charge separation in the neutron, then this would form a perfect dipole (when adjacent to a proton).

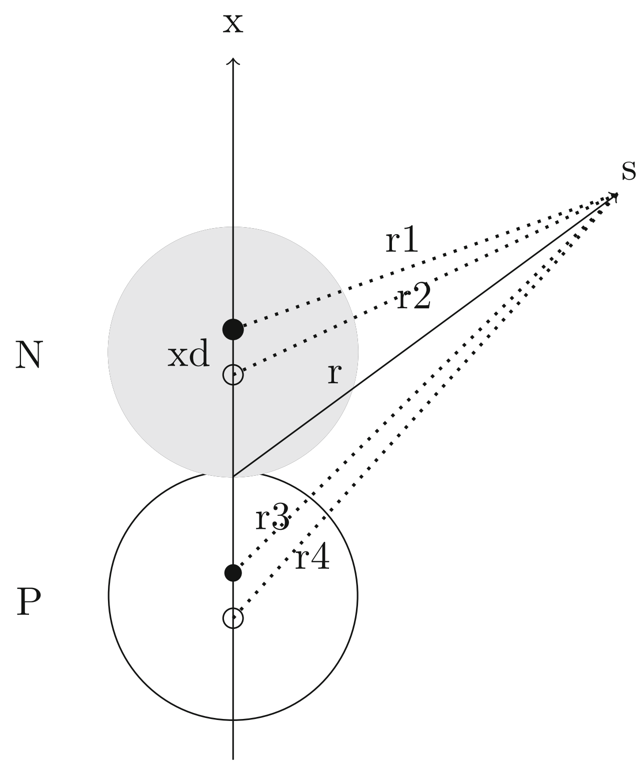

Fig. 3 Estimating the moments of the deuteron, where the neutron (N) is placed above the proton (P) and the point s is far away, such that $r >> a >> xd$.

Next, we assume that the polarized clouds in the neutron induce a small polarization in the proton (P) as well and that to a distant observer, the positive charged cloud is centered at $-a + x_3$ (filled circle) and that negative charged cloud is centered at $-a - x_4$ (open circle). Similarly to the case for the neutron, we will treat the positive charge as if it was all located at the filled circle with a charge of $+4\,e/3$ and the negative charge located at the open circle with a charge of $-e/3$. This is then an unbalanced dipole (charges differ at the end points) and the small displacements x_3 and x_4 are not equal, however since each displacement is very small compared to a and to r, we will assume for now that they can be approximated by $x_3 \approx x_4 \approx x_d/2$. The vectors from each charge point in the proton to the point s are denoted r_3 and r_4.

The angle between the x-axis and the vector r is θ and since the point s is far from the deuteron, we will assume for simplicity, that all angles from the x-axis to each vector r_i are approximately the same as θ. Next the potential $V(s)$ at the point s is given by

$$4\pi\epsilon_0\, V(s) = \frac{2\,e/3}{r_1} + \frac{-2\,e/3}{r_2} + \frac{4\,e/3}{r_3} + \frac{-e/3}{r_4} \tag{32}$$

Each of the r_i are similar in nature, hence for example for r_1, we have

$$r_1 = \left((a+x_1)^2 + r^2 - 2\,(a+x_1)\,r\,cos(\theta) \right)^{-1/2} \tag{33}$$

$$= \frac{1}{r}\left(1 + \left(\frac{a+x_1}{r}\right)^2 - 2\left(\frac{a+x_1}{r}\right) cos(\theta) \right)^{-1/2} \tag{34}$$

$$= \frac{1}{r}\left(P_0(cos(\theta)) + P_1(cos(\theta))\left(\frac{a+x_1}{r}\right) + \right. \tag{35}$$

$$\left. P_2(cos(\theta))\left(\frac{a+x_1}{r}\right)^2 + ... \right) \tag{36}$$

where $x_1 = x_d/2$ and the $P_i(cos(\theta))$ are the Legendre Polynomials. We expand each of the r_i and collect terms to obtain (keeping only the first three terms)

$$4\pi\epsilon_0\, V(s) \approx \frac{e}{r} + \frac{e}{r^2}\, cos(\theta)\,(3x_1 - a) + \tag{37}$$

$$\frac{e\,a}{r^3}\,(3\,cos^2(\theta) - 1)\left(\frac{a}{2} - \frac{x_1}{3}\right) \tag{38}$$

Note that the first term is just the net charge *e* as expected for the deuteron. Setting $\theta = 0$ for the dipole term, would yield an estimate for the electric dipole moment (p_d) for the deuteron

$$p_d \approx e\,(3\,x_1 - a) = -6.7 * 10^{-14} e\,cm \tag{39}$$

Similarly for the quadrupole term, if we select $\theta = \pi/2$, the quadrupole moment (Q_d) for the deuteron is estimated as

$$|Q_d| \approx e\,a\left(\frac{a}{2} - \frac{x_1}{3}\right) = 0.34\,e\,fm^2 \tag{40}$$

The experimental value for the quadrupole moment of the deuteron is $Q_d = 0.2859\,e\,fm^2$, which differs by about 18% from the estimate provided here.

6 Conclusion

These simple toy models provide closed expressions for the masses of the pions, but are ignoring strong force effects, other than the fact that the quarks are confined within their respective nucleons. Moreover, the GMOR [8] relation from QCD indicates that the $m_\pi^2 = (m_d + m_u)B$, where B is the quark condensate. Hence, from QCD's perspective, the bare masses of the quarks may go to zero, while in the simple model explored here, it would not work, since expressions for the pion masses are proportional to $1/m_d$.

It is surprising that such a simple model agrees relatively well with experimental values of the pion masses and the quadrupole moment of the deuteron, especially since the strong force are orders of magnitude larger than the electromagnetic (EM) force. One reason could be that the strong force peaks at a distance of about $1\,fm$ and falls off exponentially at larger distances and become approximately equal to the EM force at about $1.75\,fm$. The distance between the two centers of the nucleons in the deuteron are about $1.7\,fm$ and the strong force continues to fall exponentially beyond this separation (note also that it is the residual strong force that binds the neutron to the proton, where the strong force also must contain the quarks within each nucleon). Moreover, in calculating ω_p^2, we used the ratio of Polarizaton P^- over the electric field E_{ave}. It is possible that this ratio cancels out potential strong force terms that could appear in the estimates for the polarization and the average field.

Data Availibility Statement This manuscript has no associated data. [Author's comment: No data sharing since no data sets were generated. Hence no change to the paper.]

Code Availability Statement This manuscript has no associated code/software. [Author's comment: No code sharing, since no software was generated. Hence no change to the paper.]

Declaration

Conflict of interest The author has no conflicts to disclose.

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