Wall-crossing and orientations for invariants counting coherent sheaves on CY fourfolds

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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2021

Contents

I would like to dedicate this thesis to my mother Olga Uljanova, who passed away during my first year in Oxford. Without my grandparents Ludmila Invannikova and Peter Bojko, I would not be where I am now, so my biggest thanks goes to them. I will also forever be grateful for the support of Else Novac, who made every day of the past few years better. Finally, I would like to thank Richard Koschuch, who encouraged me to pursue my academic interests.

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I declare that the work contained in this thesis has not been submitted towards any other degree or qualification at the University of Oxford or at any other university or institution.

Abstract

Borisov–Joyce [23] and Oh–Thomas [141] defined virtual invariants counting sheaves on Calabi–Yau fourfolds. Similarly to Donaldson invariants [47], these depend on existence and choice of orientations on moduli spaces of coherent sheaves. The first part of the thesis addresses this question for quasi-projective Calabi–Yau fourfolds, generalizing the work of Cao–Gross–Joyce [30]. The orientations on compactly supported perfect complexes are expressed in terms of a pull-back of gauge-theoretic ones which live on the classifying space $\mathcal{C}_X^{\mathrm{cs}}$ of compactly supported K-theory. The proof relies on a choice of a compactification, which allows us to directly obtain orientability of moduli spaces of stable pairs. In the second part of the thesis, we study the conjectural wall-crossing formulae of Gross–Joyce–Tanaka [76]. We begin, by addressing the conjecture of Cao–Kool [32], which expresses the virtual integrals of a tautological line bundle $L^{[n]}$ on the Hilbert scheme of points $\mathrm{Hilb}^n(X)$ in terms of the MacMahon function. We also obtain a prediction for the K-theoretic refinement of this invariant proposed by Nekrasov [135], which coincides with the expectations from the result for $\mathbb{C}^4$. Studying the invariants further, we find a universal transformation relating them to integrals on Hilbert schemes of points for elliptic surfaces. To understand this, we recover the previously known results for Quot-schemes on elliptic surfaces using similar wall-crossing arguments. We will further study this in [18] to recover and generalize the full result of Arbesfeld–Johnson–Lim–Oprea–Pandharipande [5] for surfaces including divisor contributions.

Acknowledgments

The author wishes to express his gratitude to Dominic Joyce for supervising this thesis and showing patience in the face of my immeasurable stubbornness.

We also thank Noah Arbesfeld, Chris Brav, Yalong Cao, Simon Donaldson, Mathieu Florence, Michel van Garrel, Jacob Gross, Drew Johnson, Martijn Kool, Edwin Kutas, Sven Meinhardt, Sergej Monavari, Jeongseok Oh, Dragos Oprea, Rahul Pandharipande, Hector Papoulias, Yuuji Tanaka, Richard Thomas, Bertrand Toën, Markus Upmeier and Andrzej Weber.

The author was supported by a Clarendon scholarship of the University of Oxford.

Chapter 1

Introduction

We choose to begin our journey of enumerative geometry with Donaldson [47], who introduced his famous invariants counting ASD connections on 4-manifolds and used them to restrict the possible intersection forms on the middle cohomology. This required solving the compactness, orientability and transversality questions of the moduli space. On complex surfaces one can instead use algebraic methods to count coherent sheaves as in Mochizuki [134], Tyurin [171] and Göttsche–Nakajima–Yoshioka [71], which gives a different approach to the compactification problem and the orientations are natural. In complex three dimension Thomas [163] defined *holomorphic Casson* invariants which are more commonly known as *DT-invariants* and were further generalized by Joyce–Song [96] and Kontsevich–Soibelman [106]. There are two different ways of thinking about these:

1. When counting ideal sheaves of curves or –a more refined approach– *stable pairs*, these are conjecturally (Maulik–Nekrasov–Okounkov–Pandharipande [127], [128], Pandharipande–Thomas [144]) related to Gromov-Witten invariants counting stable maps of curves [73], [173]. Moreover, both theories are well-defined for any smooth 3-fold.

2. Counting motivic invariants of general semi-stable sheaves and studying wall-crossing as in Joyce–Song [96] and Kontsevich–Soibelman [106], which is restricted to Calabi–Yau geometries. The wall-crossing formulae are expressed in terms of Ringel–Hall (Lie) algebras [96, Thm. 3.14].

Note that we are restricting ourselves to these 2 points, but the subject is vast and has many other connections, some of which the author is aware of and some not. In pursuit of obtaining a similar framework for Calabi–Yau fourfolds Cao–Leung [37] and Borisov–Joyce [23] defined DT_4-type invariants counting coherent sheaves. The former construction works for moduli spaces of vector bundles and smooth moduli spaces, while the latter relied on derived differential geometry. A fully algebraic approach has been developed by Oh–Thomas [141] which is expected to be equivalent to [23] (minus a minor technical detail). These invariants have already been studied by multiple authors [32, 33, 38, 39, 34, 35, 41, 42, 36, 40]. Unlike their lower-dimensional algebraic counterparts, they again depend on existence and choice of orientation. However, they do inherit the two ways of thinking about them:

1. Counting stable pairs is conjecturally ([38, 39, 41]) related to Gromov–Witten invariants of Klemm–Pandharipande [103].

2. Invariants counting semi-stable sheaves conjecturally satisfy universal wall-crossing formulae (Gross–Joyce–Tanaka [76]). The Ringel–Hall Lie algebras of [96] are replaced by Lie algebras associated to natural vertex algebras on homology constructed by Joyce [91].

We are again willingly ignoring some other points of view (see Diaconescu–Sheshmani–Yau [45]). In this thesis, we plan to address two questions which are related to the above:

- Orientability of moduli spaces has been proven for projective Calabi–Yau fourfolds by Cao–Gross–Joyce [30]. We adapt their techniques of stabilization of holomorphic and complex vector bundles to the non-compact setting. The main difficulty in doing so comes from the moduli stack of perfect complexes (see Toën–Vaquié [168]) on a quasi-projective X classifies only compactly supported perfect complexes. As a result, there is no natural map from holomorphic vector bundles to this stack. Instead we use compactification and excision arguments, developing the "algebraic excision" and excision for complex elliptic symbols generalizing the commonly used one of Donaldson [47], Donaldson–Kronheimer [49, §7.1] and Atiyah–Singer [8] (see also Upmeier [172]). Comparing the two excisions, we obtain the result.

- We follow the guidelines laid out by Gross–Joyce–Tanaka [76, §4.4], defining the moduli stack of pairs and conjecturing a wall-crossing formula for Joyce–Song stable pairs (see [96, §5.4]). Applying it to the setting of Hilbert schemes of points, we reduce the proof of conjecture of Cao–Kool [32] to the wall-crossing conjecture. A K-theoretic enrichment of this is the *Nekrasov genus* introduced by Nekrasov [135] and studied by Cao–Kool–Monavari [35]. We show (relying on the wall-crossing conjecture) that it takes the expected form in compact geometries. A surprising consequence of the computations is a direct correspondence between generating series of certain integrals on Hilbert schemes of elliptic surfaces and those on compact Calabi–Yau fourfolds. We explain this correspondence via wall-crossing for Quot-schemes in the final section of

Chapter 4.

The contents of the chapters are as follows. In Chapter 2, we review the different approaches to moduli problems of sheaves and complexes on varieties ending in a short summary of facts we will need about *derived stacks* and *higher stacks* of perfect complexes. We review the definition of orientations on -2-*shifted moduli stacks* as given in Borisov–Joyce [23], which only require the duality on the *cotangent complex* $\mathbb{L}_{\boldsymbol{\mathcal{M}}} \cong \mathbb{L}^\vee_{\boldsymbol{\mathcal{M}}}[-2]$ of a *derived stack* $\boldsymbol{\mathcal{M}}$ to construct a natural *orientation* $\mathbb{Z}_2$*-bundle* $O^\omega \to \mathcal{M}$, where $\mathcal{M}$ is the associated higher stack. For a quasi-projective Calabi–Yau fourfold, we have the stack $\mathcal{M}_X$ of compactly supported perfect complexes with the corresponding bundle $O^\omega \to \mathcal{M}_X$. The necessity of existence of trivializations $\mathbb{Z}_2 \cong O^\omega$ is explained in §2.3, where we discuss the bare minimum about the Oh-Thomas [141] construction of *virtual fundamental classes* and recall their *virtual Riemann–Roch* formula relating virtual K-theoretic invariants and cohomological invariants.

Both Chapter 3 and Chapter 4 have already appeared as preprints [19], [20]. In Chapter 3, we use the term *Calabi–Yau 4-fold* to denote a pair (X, Ω), where X is a smooth quasi-projective variety and Ω is a nowhere vanishing section of the canonical bundle of X. Our goal is to prove orientability of the moduli stack $\mathcal{M}_X$ of compactly supported perfect complexes on X. Our first approach is through *spin compactifications*. Spin-structures on a projective complex manifold X are equivalent to the choice of a square root Θ of the canonical line bundle $\Theta^2 \cong K_X$. Assuming existence of such Θ, we construct an orientation $O^\Theta \to \mathcal{M}_X$ over the moduli stack of perfect complexes on X. We check a straightforward generalization of Cao–Gross–

Joyce [30], which allows us to prove orientability for certain examples including the total space of $K_Y \to Y$, where Y is a 3-fold and K_Y its canonical bundle. While we do not give a counterexample to this method, we expect that it would not be applicable for a general toric Calabi–Yau with a simple example in 2-dimensions (see Example 3.1.13). To remedy this we use *algebraic excision* in §3.1.3, to construct a $\mathbb{Z}_2$-bundle on $O^{\bowtie} \to \mathcal{M}^{\mathrm{sp}}_{Y,D}$, where Y is a compactification of X, such that $Y \backslash X = D$ is a strictly normal crossing divisor. The stack classifies perfect complexes on two copies of Y identified on the divisor D and $O^{\bowtie}$ depends on some additional data $\bowtie$. As such, it admits an open embedding of $\mathcal{M}_X \hookrightarrow \mathcal{M}_Y \times_{\mathcal{M}_D} \mathcal{M}_Y$. The rest of the chapter is dedicated to proving that $O^{\bowtie}$ is trivializable and relating it to a gauge-theoretic orientation bundle. This is obtained first by generalizing the excision principle to work for complex elliptic pseudo-differential operators and then comparing it by hand to the algebraic geometric construction. We obtain two consequences of this:

Theorem 1.0.1 (Thm. 3.1.20)**.** *Let (X,Ω) be a quasi-projective Calabi–Yau fourfold, then the $\mathbb{Z}_2$-bundle $O^\omega \to \mathcal{M}_X$ is trivializable. Let $\mathcal{C}^{\mathrm{cs}}_X = \mathrm{Map}_{C^0}\big((X^+,+),(BU\times\mathbb{Z},0)\big)$ be the topological space of pointed maps, where X^+ is the one point compactification of X and $0 \in BU\times\mathbb{Z}$ the identity. For a choice of a compactification $i_X : X \to Y$ with a strictly normal crossing divisor $D = Y\backslash X$, there exists a canonical $\mathbb{Z}_2$-bundle $O^{\mathrm{cs}} \to \mathcal{C}^{\mathrm{cs}}_X$ and a natural map*

$$\Gamma^{\mathrm{cs}} : (\mathcal{M}_X)^{\mathrm{top}} \longrightarrow \mathcal{C}^{\mathrm{cs}}_X\,,$$

where $(-)^{\mathrm{top}}$ is the topological realization functor of Blanc [16], such that there is a *canonical isomorphism*

$$\mathfrak{I} : \Gamma^{\mathrm{cs}\,*}(O^{\mathrm{cs}}) \xrightarrow{\sim} O^{\omega} \,.$$

In particular, if $\alpha \in K^0_{\mathrm{cs}}(X)$ is a compactly supported K-theory class and M_α a moduli scheme of stable perfect complexes with class α, the above gives orientations on M_α which only differ by a global minus sign for fixed compactification Y. It is important to note that the constructions used in the proof of this statement depend very much on the choice of the compactifications, and it would be interesting if this additional requirement could be removed (see Remark 3.4.5).

If X is quasi-projective and $\mathcal{M}$ is a moduli stack of pairs of the form $\mathcal{O}_X \to F$, where F is compactly supported, we need a modification, because the complex is not compactly supported. Let $\mathcal{E} \to \mathcal{M}$ be the universal family, then for fixed compactification $X \subset Y$ there is a natural isomorphism from Definition 3.1.21:

$$\det\Big(\big(\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{E},\mathcal{E})\big)_0\Big) \cong \det{}^*\Big(\big(\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{E},\mathcal{E})\big)_0\Big)\,,$$

where $(-)_0$ denotes the trace-less part and we use the notation $\underline{\mathrm{Hom}}_Z(E,F)$ for two perfect complexes E, F on $X \times Z$ to denote $\pi_{2*}(E^\vee \otimes F)$, where $\pi_2 : X \times Z \to Z$ is the projection. To this there is an associated $\mathbb{Z}_2$-bundle $O^0 \to \mathcal{M}$. We can now state the result.

Theorem 1.0.2 (Thm. 3.1.22)**.** *Let $i : X \to Y$ be a compactification with $Y \backslash X = D$ strictly normal crossing. Let $\eta : \mathcal{M} \to \mathcal{M}_X \times_{\mathcal{M}_D} \mathcal{M}_Y$ be given by $[E] \to, [\bar{E}, \mathcal{O}_Y]$, where $\bar{E}$ is the extension by a structure sheaf to the divisor, then there is a canonical isomorphism*

$$\eta^*(O^{\bowtie}) \cong O^0\,.$$

In particular, O^0 is trivializable.

Consequentially, if all stable pairs parameterized by $\mathcal{M}$ are of a fixed class $[\![\mathcal{O}_X]\!]+\alpha$, where $\alpha \in K^0_{\mathrm{cs}}(X)$, this determines unique orientations up to a global sign for a fixed compactification Y of X. After fixing a compactification $X \subset Y$, we can also study compatibility of orientations under direct sum of perfect complexes. Note that $(\mathcal{M}_X)^{\mathrm{top}}$ and $\mathcal{C}^{\mathrm{cs}}_X$ are H-spaces (see §3.2.2) with the binary operation $\mu : (\mathcal{M}_X)^{\mathrm{top}} \times (\mathcal{M}_X)^{\mathrm{top}} \to (\mathcal{M}_X)^{\mathrm{top}}$. The main results are summarized in:

Theorem 1.0.3 (Theorem 3.4.4)**.** *Let $\mathcal{C}^{\mathrm{cs}}_\alpha$ denote the connected component of $\mathcal{C}^{\mathrm{cs}}_X$ corresponding to $\alpha \in K^0_{\mathrm{cs}}(X) = \pi_0(\mathcal{C}^{\mathrm{cs}}_X)$ and $O^{\mathrm{cs}}_\alpha = O^{\mathrm{cs}}|_{\mathcal{C}^{\mathrm{cs}}_\alpha}$. There is a canonical isomorphism $\phi^\omega : O^\omega \boxtimes O^\omega \to \mu^*(O^\omega)$ such that for fixed choices of trivialization o^{cs}_α of O^{cs}_α, we have*

$$\phi^\omega\Big(\mathfrak{I}\big(\Gamma^* o^{\mathrm{cs}}_\alpha\big) \boxtimes \mathfrak{I}\big(\Gamma^* o^{\mathrm{cs}}_\beta\big)\Big) \cong \epsilon_{\alpha,\beta}\mathfrak{I}\big(\Gamma^* o^{\mathrm{cs}}_{\alpha+\beta}\big)\,.$$

where the $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ satisfy $\epsilon_{\beta,\alpha} = (-1)^{\bar{\chi}(\alpha,\alpha)\bar{\chi}(\beta,\beta)+\bar{\chi}(\alpha,\beta)}\epsilon_{\alpha,\beta}$ and $\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma} = \epsilon_{\beta,\gamma}\epsilon_{\alpha,\beta+\gamma}$ for all $\alpha,\beta,\gamma \in K^0_{\mathrm{cs}}(X)$

In Chapter 4, we will use the conjectural wall-crossing along the lines of Gross–Joyce–Tanaka [76, §4.4] as stated in a precise form in Conjecture 4.2.10 and apply it to Hilbert schemes of points. Recall that the *Hilbert scheme of points* $\mathrm{Hilb}^n(X)$ on X is the moduli space of ideal sheaves of length n. Their complex *virtual dimension* is equal to n and we need degree n insertions to obtain invariants. We now summarize results which follow assuming Conjecture 4.2.10 holds.

One natural insertion on $\mathrm{Hilb}^n(X)$ studied by Cao–Kool [32], Cao–Qu [40] for

Calabi–Yau fourfolds is the top Chern class $c_n(L^{[n]})$ of the vector bundle

$$L^{[n]} = \pi_{2*}\big(\mathcal{F}_n \otimes \pi_X^*(L)\big)\,, \tag{1.0.1}$$

where $X \xleftarrow{\pi_X} X \times \mathrm{Hilb}^n(X) \xrightarrow{\pi_2} \mathrm{Hilb}^n(X)$ are the projections and $\mathcal{O} \to \mathcal{F}_n$ is the universal complex on $X \times \mathrm{Hilb}^n(X)$.

For the generating series of invariants

$$I(L;q) = 1 + \sum_{n>0} I_n(L)q^n = 1 + \sum_{n>0} \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} c^n(L^{[n]})q^n \tag{1.0.2}$$

Cao–Kool [32] conjecture the following:

Conjecture 1.0.4 (Cao–Kool [32])**.** *Let X be a projective Calabi–Yau fourfold and L a line bundle on X then*

$$I(L;q) = M(-q)^{c_1(L)\cdot c_3(X)} \tag{1.0.3}$$

for some choice of orientations. Here $M(q) = \prod_{i=1}^{\infty}(1-(q)^i)^{-i}$ is the MacMahon function.

Cao–Qu [40] prove that if $L = \mathcal{O}(D)$ for a smooth connected divisor $D \subset X$, then this conjecture holds for some choices of orientations. We use their result in Theorem 4.3.1 to reduce Conjecture 1.0.4 for any line bundle L to Conjecture 4.2.10. The wall-crossing conjecture also implies that the orientations for which this holds are independent of L and can be expressed in terms of compatibility under direct sums of orientations as in [76, Thm. 2.27] and Theorem 3.4.6. We call these orientations *point-canonical*. We then go on to study many new invariants that have not been considered for compact Calabi–Yau 4-folds which we hope will give directions for new research. We address here the three main consequences:

1. K-theoretic invariants for Calabi–Yau 3-folds using *twisted virtual structure sheaves* were introduced by Nekrasov–Okounkov [136] to study the correspondence between DT_3 invariants and curve-counting in CY 5-folds. The idea is to think of the usual virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ of Fantechi–Göttsche [51] as a Dolbeault operator on the moduli space of sheaves. The twisted virtual structure sheaf is obtained by tensoring with a square root of the virtual canonical line bundle and similarly to (3.1.7) is meant to represent the Dirac operator. Oh–Thomas [141, §6] give us a twisted virtual structure sheaf $\hat{\mathcal{O}}^{\mathrm{vir}}$ on $\mathrm{Hilb}^n(X)$. Note that in the four-fold case the twist is necessary for the object to be independent of the choices made in the construction. On $\mathbb{C}^4$, Nekrasov [135] and Nekrasov–Piazzalunga [137] define the invariants

$$\chi\big(\hat{\mathcal{O}}^{\mathrm{vir}} \otimes \Lambda_1^{\bullet} L_{y^{-1}}^{[n]} \otimes \det{}^{-\frac{1}{2}}(L_{y^{-1}}^{[n]})\big)\,,$$

where L is trivial line bundle with weight y^{-1} of an extra $\mathbb{C}^*$ action and $\Lambda_y^{\bullet} V = \sum_{i\geq 0}(-y)^i \Lambda^i V$ for a vector bundle V. Their conjectured formula for the generating series related to counting solid partitions was recently proven by Kool–Rennemo [107] and can be expressed as

$$K(L_{y^{-1}};q) = \mathrm{Exp}\bigg[\chi\Big(\mathbb{C}^4, q\frac{(T\mathbb{C}^4 - T^*\mathbb{C}^4)(L_{y^{-1}}^{\frac{1}{2}} - L_{y^{-1}}^{-\frac{1}{2}})}{(1-qL_{y^{-1}}^{\frac{1}{2}})(1-qL_{y^{-1}}^{-\frac{1}{2}})}\Big)\bigg]\,, \tag{1.0.4}$$

where χ means the equivariant Euler characteristic and Exp[-] is the *plethystic exponential* (see Theorem 4.5.5). As all results thus far have been restricted to the equivariant toric setting, we study these invariants for compact CY4. Let us set the notation

$$\hat{\chi}^{\mathrm{vir}}(V) = \chi(\hat{\mathcal{O}}^{\mathrm{vir}} \otimes V)\,. \tag{1.0.5}$$

For $\alpha_i \in K^0(X)$ define

$$\begin{aligned} \mathscr{N}_y(\alpha_i^{[n]}) &= \Lambda^\bullet_{y^{-1}}\big(\alpha_i^{[n]}\big) \otimes \det{}^{-\frac{1}{2}}\big(\alpha_i^{[n]} \cdot y^{-1}\big)\,, \\ K(\vec{\alpha}, \vec{y}; q) &= 1 + \sum_{n>0} q^n \hat{\chi}^{\mathrm{vir}}\big(\mathrm{Hilb}^n(X), \mathscr{N}_{y_1}(\alpha_1^{[n]}) \otimes \cdots \otimes \mathscr{N}_{y_N}(\alpha_N^{[n]})\big)\,. \end{aligned} \tag{1.0.6}$$

Our motivation for studying these is to understand what the relation between local and global geometries are. Most importantly, we show in Theorem 4.5.5 assuming Conjecture 4.2.10 that when $N = 1$ and $\mathrm{rk}(\alpha) = 1$, we obtain for point-canonical orientations

$$K(\alpha, y; q) = \mathrm{Exp}\left[\chi\Big(X, q\frac{(TX - T^*X)(\alpha^{\frac{1}{2}}y^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}y^{-\frac{1}{2}})}{(1 - q\alpha^{\frac{1}{2}}y^{\frac{1}{2}})(1 - q\alpha^{-\frac{1}{2}}y^{-\frac{1}{2}})}\Big)\right], \tag{1.0.7}$$

which is the conjectured formula of Nekrasov, when one replaces X with $\mathbb{C}^4$ and α with $\mathcal{O}_{\mathbb{C}^4}$. Motivated by Cao–Kool–Monavari [35, Prop. 1.15], we also show that the coefficients of (1.0.6) are integers whenever the sum of ranks of α_i is odd and $\frac{c_1(\alpha_i)}{2} \in H^2(X, \mathbb{Z})$.

2. For a surface S and a line bundle $L \to S$ the Segre series

$$R(S, L; q) = \int_{\mathrm{Hilb}^n(S)} s_{2n}\big(L^{[n]}\big) q^n$$

appeared in Tyurin [171] in relation to Donaldson invariants on complex surfaces.

Its precise form was conjectured by Lehn [114]. This was proven by Marian–Oprea–Pandharipande [124] for K3 surfaces and the general case in [126]. For any rank, these invariants have been considered by Marian–Oprea–Pandharipande [125], because of their relation to Verlinde numbers and strange duality (see Johnson [89]).

Let $\vec{\alpha} = (\alpha_1, \dots \alpha_N)$ for $\alpha_1, \dots \alpha_N \in G^0(X)$ and $\vec{t} = (t_1, \dots, t_N)$, $\vec{a} = (a_1, \dots, a_N) = \big(\mathrm{rk}(\alpha_1), \dots, \mathrm{rk}(\alpha_N)\big)$. We define the *$DT_4$-Segre series* for Calabi–Yau 4-folds by

$$R(\vec{\alpha}, \vec{t}; q) = 1 + \sum_{n>0} q^n \int_{\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}}} s_{t_1}\big(\alpha_1^{[n]}\big) \dots s_{t_N}\big(\alpha_N^{[n]}\big)\,. \tag{1.0.8}$$

The corresponding series for virtual fundamental classes of Quot-schemes on surfaces were studied by Oprea–Pandharipande [143]. When $N = 1$, we will use

$$R(\alpha; q) = 1 + \sum_{n>0} q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} s_n\big(\alpha^{[n]}\big)\,.$$

Firstly, we define the following universal transformation

$$U\big(f(q)\big) = \prod_{n>0}\prod_{k=1}^{n} f(-e^{\frac{2\pi i k}{n}} q)^{-n}\,.$$

Recall then that the *Fuss-Catalan numbers* and their generating series defined by Fuss [62] are given by

$$C_{n,a} = \frac{1}{an+1}\binom{an+1}{n}, \qquad \mathscr{B}_a(q) = \sum_{n\geq 0} C_{n,a} q^n\,. \tag{1.0.9}$$

Theorem 1.0.5 (4.5.1)**.** *If Conjecture 4.2.10 holds, then for point-canonical orienta-*

tions we have

$$R(\vec{\alpha},\vec{t};q) = U\Big[(1+t_1z)^{c_1(\alpha_1)\cdot c_3(X)}\cdots(1+t_Nz)^{c_1(\alpha_N)\cdot c_3(X)}\Big]\,,$$

where z *is the unique solutions to* $z(1+t_1z)^{a_1}\cdots(1+t_Nz)^{a_N}=q$. *Moreover, we have the explicit expression*

$$R(\alpha;q) = \begin{cases} U\big[\mathscr{B}_{a+1}(-q)^{-c_1(\alpha)\cdot c_3(X)}\big] & \text{for } a\geq 0 \\ \\ U\big[\mathscr{B}_{-a}(q)^{c_1(\alpha)\cdot c_3(X)}\big] & \text{for } a<0\,. \end{cases}$$

One of the most notable properties of the Segre series on surfaces is their correspondence with the *Verlinde series* $V(S,\alpha;q) = 1+\sum_{n>0}q^n\chi\Big(\det\big(\alpha^{[n]}\big)\Big)$ motivated by strange duality as in Johnson [89]. An explicit formulation was given by Marian–Oprea–Pandharipande [126] as a change of variables $z=f(q)$, $w=g(q)$ giving

$$V(S,\alpha;z) = R(S,-\alpha;w)\,. \tag{1.0.10}$$

This was developed further in [125]. For virtual classes of Quot-schemes this duality appeared in Arbesfeld et al [5].

In the case of Calabi–Yau 4-folds, we define *DT_4-Verlinde numbers* for each $\alpha\in K^0(X)$ by

$$V(\alpha;q) = 1+\sum_{n>0}q^n\chi^{\mathrm{vir}}\Big(\mathrm{Hilb}^n(X),\det\big(\alpha^{[n]}\big)E^{\frac{1}{2}}\Big)\,,$$

where $E=\det\big(\mathcal{O}_X^{[n]}\big)$, and the square-root is taken in rational K-theory. Assuming

Conjecture 4.2.10, the resulting duality on 4-folds stated in Theorem 4.5.12 is

$$V(\alpha; q) = R(\alpha; -q)\,.$$

Note that this will hold for <u>any</u> choice of orientations.

3. We discuss here a more general result, linking all invariants discussed above to those for elliptic surfaces and elliptic curves by a direct computation. Recall that for a surface S or a curve C, $\mathrm{Quot}_{S/C}(\mathbb{C}^N, n)$ parameterize surjective morphisms $\mathbb{C}^N \otimes \mathcal{O}_{S/C} \to F$, where F is a zero-dimensional sheaf with $\chi(F) = n$. The virtual fundamental classes constructed by Marian–Oprea–Pandharipande [124, Lem. 1.1] and Marian–Oprea [123] also fit into the wall-crossing frame-work as stable pairs, and we will be able to recover the results of Arbesfeld et al [5]*, Lim [118], Oprea–Pandharipande [143] in our future work [18] via wall-crossing. In fact, our result there will be more general, because it allows us to integrate any insertion of topological nature while the above authors only consider multiplicative genera of tautological classes. We give a small excerpt in the last section from the above related to elliptic curves and surfaces which explains the following (see also Theorem 4.5.14 for the analogous statement for K-theoretic invariants): Let S be an elliptic surface and f, h be multiplicative genera (see §4.4.2 or for a standard reference in relation to Hilbert schemes [50]). There is a universal series $A(q)$ depending on f, h and $a = \mathrm{rk}(\beta)$, such that

$$1 + \sum_{n>0} q^n \int_{\left[\mathrm{Quot}(S,\mathbb{C}^1,n)\right]^{\mathrm{vir}}} f(\beta^{[n]})h\big(T^{\mathrm{vir}}\big) = A(q)^{c_1(\beta)\cdot c_1(S)}\,, \tag{1.0.11}$$

*Here we expect to futher include contributions from divisors captured by Seiberg–Witten invariants.

where T^{vir} denotes the virtual tangent bundle. Assuming Conjecture 4.2.10, we show in Theorem 4.5.14 that for $\alpha \in K^0(X)$ with $\mathrm{rk}(\alpha) = a$ and $h(z) = g(z)g(-z)$ we have in terms of the same universal series $A(q)$:

$$1 + \sum_{n>0} q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} f(\alpha^{[n]}) g\big(T^{\mathrm{vir}}_{\mathrm{Hilb}^n(X)}\big) = U\big(A(q)^{c_1(\alpha)\cdot c_3(X)}\big)\,. \tag{1.0.12}$$

One can further relate these invariants to integrals over symmetric products of elliptic curves by geometric arguments as in Oprea–Pandharipande [143] or by studying wall-crossing for elliptic surfaces and curves.

1.0.1 Future research

The above results lead to many interesting open questions that the author would like to focus on in the future some of which include:

(Q1): Does there exist a degeneration argument along the lines of Levine–Pandharipande [116] using the technology of Li–Wu [117] or Maulik–Ranganathan [130] explaining the relation between (1.0.4) and (1.0.7).

(Q2): Is there a geometric interpretation of the Segre–Verlinde duality for Calabi–Yau fourfolds as in Johnson [89]?

(Q3): What is the geometric interpretation if any of the universal transformation U comparing the invariants in (1.0.11) and (1.0.12)?

Chapter 2

Background

2.1 Connections

We begin by a quick review of gauge theory motivating the results that follow. For more details see e.g. Donaldson–Kronheimer [49] or Joyce [92]. Let X be a smooth connected manifold of dimension n. Let $\pi : P \to X$ be a principal G bundle for a connected Lie group G with the Lie algebra $\mathfrak{g}$ and the right action $\tau : P \times G \to P$. A connection ∇ on P has the following two equivalent definitions:

1. As a rank n distribution ∇ on P that is invariant under the right action of G on P, and $\pi_* : TP \to \pi^*(TX)$ induces an isomorphism between the distribution and $\pi^*(TX)$.

2. As a $\mathfrak{g}$-valued one form $\omega_\nabla \in \Omega^1(P, \mathfrak{g})$, such that for each $p \in P$ and under the identification of the tangent space $T_p(\pi^{-1}(\pi(p))$ of the fiber at p with $\mathfrak{g}$, the action of ω_∇ restricted to this space is the identity. One also requires that its pullback under the right action of $g \in G$ is given by $\tau(-, g)^*\omega_\nabla = \mathrm{ad}_{g^{-1}} \circ \omega_\nabla$.

The second formulation identifies after fixing a connection ∇_0 the space of connections with $\Gamma^\infty(T^*X \otimes \mathrm{ad}(P))$, where $\mathrm{ad}(P) = \mathfrak{g} \times_G P$ is the associated bundle to the adjoint action $\mathrm{ad} : G \to \mathrm{End}(\mathfrak{g})$. Recall that *Gauge group* is defined by $\mathcal{G}_P = \{\phi \in \mathrm{Diff}(P) :$ (2.1.1) are commutative$\}$:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P \\ \downarrow & & \downarrow \\ X & = & X \end{array}, \qquad \begin{array}{ccc} P \times G & \xrightarrow{\phi\times\mathrm{id}} & P \times G \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_1} \\ P & \xrightarrow{\phi} & P\,. \end{array} \tag{2.1.1}$$

The first description of connections gives us a natural action of the gauge group $\mathcal{G}$ of P on $\mathcal{A}_P$, where $\mathcal{A}_P$ is the space of connections. Then the *irreducible* connections have stabilizer group $Z(G) \subset \mathcal{G}$ and we denote their set by $\mathcal{A}_P^{\mathrm{irr}}$ and one considers the space $B_P^{\mathrm{irr}} = \mathcal{A}_P^{\mathrm{irr}} / (\mathcal{G}_P / Z(G))$.

Definition 2.1.1. We will use the notation Ω^i to denote real i-forms on X, $\mathcal{A}^i$, $\mathcal{A}^{p,q}$ to denote complex i and (p,q)-forms. For each connection $\nabla_P \in \mathcal{A}_P$ let $\nabla_{\mathrm{ad}(P)}$ be the associated connection on $\mathrm{ad}(P)$ and for a differential operators $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ between vector-bundles E_0, E_1 let $D^{\nabla_{\mathrm{ad}(P)}} : \Gamma^\infty(E_0 \otimes \mathrm{ad}(P)) \to \Gamma^\infty(E_1 \otimes \mathrm{ad}(P))$ be the twisted operator (see Donaldson–Kronheimer [49, §2.1], Cao–Gross–Joyce [30, Def. 1.2], Joyce–Upmeier [98, Def. 2.4]).

We now give a short motivation for the definition of invariants counting coherent sheaves on Calabi–Yau 4-folds.

Definition 2.1.2. A *Calabi–Yau fourfold* is a pair (X, Ω), where X is a smooth complex quasi-projective four-fold and Ω a trivialization of its canonical bundle.

For given (X, Ω) we have anti-linear maps:

$$\# : \mathcal{A}^{0,k} \to \mathcal{A}^{0,4-k}\,, \qquad \#^2 = 1\,,$$
$$\alpha \wedge \#\beta = (-1)^{\lceil q/2 \rceil} \langle \alpha, \beta \rangle \bar{\Omega}\,, \quad \alpha \in \mathcal{A}^{0,k}, \beta \in \mathcal{A}^{0,k}\,, \tag{2.1.2}$$

where $\langle -, - \rangle$ is the induced metric on forms. This induces a splitting $\mathcal{A}^{0,2} = \mathcal{A}^{0,2}_+ \oplus \mathcal{A}^{0,2}_-$ into real subspaces giving an elliptic complex (see Atiyah–Singer [8, §7]) for each connection ∇_P :

$$0 \to \mathcal{A}^{0,0} \otimes_{\mathbb{R}} \mathrm{ad}(P) \xrightarrow{\bar{\partial}^{\nabla_{\mathrm{ad(P)}}}} \mathcal{A}^{0,1} \otimes_{\mathbb{R}} \mathrm{ad}(P) \xrightarrow{\bar{\partial}_+^{\nabla_{ad(P)}}} \mathcal{A}^{0,2}_+ \otimes_{\mathbb{R}} \mathrm{ad}(P) \to 0 \tag{2.1.3}$$

When we view X as a compact complex manifold with the data (X, J, g, ω), where J is the complex structure, g Kähler metric and ω the symplectic form. If $\mathrm{Hol}(g) = SU(4)$, connections satisfying

$$F^{0,2}_+ = \bar{\partial}_+^{\nabla_{\mathrm{ad}(P)}} \circ \bar{\partial}^{\nabla_{\mathrm{ad}(P)}} = 0\,, \quad F \wedge \omega = 0 \tag{2.1.4}$$

are called $SU(4)$-*instantons* and their moduli spaces were studied by Cao–Leung [37] and Donaldson–Thomas [48]. While the Hermitian Yang–Mills equations

$$F^{0,2} = \bar{\partial}^{\nabla_{\mathrm{ad}(P)}} \circ \bar{\partial}^{\nabla_{\mathrm{ad}(P)}} = 0\,, \qquad F \wedge \omega = 0$$

are overdetermined, $SU(4)$-instantons give rise to virtual fundamental classes assuming some compactness: Recall that *Spin*(7) is the group of transformations on $\mathbb{R}^8$

preserving the 4-form:

$$\Omega_0 = dx_{1234} + dx_{1278} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467}\,,$$
$$- dx_{2358} - dx_{2369} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678}\,,$$

where $dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. There is a natural embedding $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$ which makes (X, J, g, ω) into a *Spin*(7) manifold (see [92, §10.6]). On a Spin(7) manifold, there is a natural splitting $\Lambda^2 T^*X = \Lambda^2_7 T^*X \oplus \Lambda^2_{21} T^*X$. Denoting $\pi_7 : \Lambda^2 T^*X \to \Lambda^2_7 T^*X$ the projection, this gives rise to the elliptic equation

$$\pi_7 \circ F = 0\,,$$

which coincides with the one in (2.1.4) when X is Calabi–Yau. The set of connections $B_P^{\mathrm{Spin}(7)} \subset B_P^{\mathrm{irr}}$ then forms a derived manifold (see e.g. Joyce [94]) and carries a virtual cobordism class constructed using orientations from Theorem 3.2.9 if compact.

We only used this opportunity to set some notation and abandon this view-point for that of coherent sheaves and perfect complexes for the following reason: If on top of (2.1.4), one additionally requires that the $\mathrm{ad}(P)$-valued $(0,2)$-form $F^{0,2} = \partial^{\nabla_{\mathrm{ad}(P)}} \circ \partial^{\nabla_{\mathrm{ad}(P)}}$ satisfies the topological condition

$$\int_X \mathrm{tr}(F^{0,2} \wedge F^{0,2})\Omega = 0 \quad \text{then} \quad F^{0,2}_+ = 0 \iff F^{0,2} = 0\,,$$

so if G is $U(n)$ and E is the bundle associated to P via the natural representation

$\mathbb{C}^n$, it will be holomorphic. In this case, we have the isomorphisms

$$H^1((2.1.3)) \cong \mathrm{Ext}^1(E,E)\,, \qquad H^2((2.1.3)) \cong \mathrm{Ext}^2(E,E)_+\,,$$

where $\mathrm{Ext}^2(E,E)_+ \subset \mathrm{Ext}^2(E,E)$ is a real subspace with respect to the real structure (2.1.2) which descends to cohomologies.

2.2 Moduli spaces of coherent sheaves and perfect complexes

In the previous section, we have motivated working with holomorphic vector bundles. This subsection is dedicated to developing the language of moduli stacks on the side of algebraic geometry that will be used later. For background on sheaves and complexes, we recommend Hartshorne [79], Huybrechts [86] and Gelfand–Manin [67]. Recall first that we have the following fully faithful inclusions of categories

$$\mathrm{Vect}(\mathrm{X}) \subset \mathrm{Coh}(X) \subset D^b\big(\mathrm{Coh}(X)\big)\,, \qquad \mathrm{Coh}_{\mathrm{cs}}(X) \subset D^b\big(\mathrm{Coh}_{\mathrm{cs}}(X)\big)\,, \tag{2.2.1}$$

where the latter describes compactly supported coherent sheaves and complexes of coherent sheaves with compactly supported cohomologies. To find answers to the corresponding moduli problems, we need the language of stacks, higher stacks and derived stacks. From now on we work always over $\mathbb{C}$.

- The 2-category of Artin stacks **ArtSt** consists of 2-functors

$$\mathbf{Aff} \to \mathbf{Grp}\,,$$

where **Aff** is the 2-category of affine schemes and **Grp** is the category of groupoids. These have to additionally satisfy a *descent condition* with respect to the étale topology and have a *smooth atlas*. See for example Olsson [142].

- Some foundational work on derived stacks has been done by Toën and Vezzosi [170, 167, 169], in the setting of model categories (see Hovey [85] and Hirschhorn [82]) and Lurie [120] in the setting of ∞-categories (see Lurie [122], [121], Toën [166], Gaitsgory–Rozenblyum [65], Gaitsgory [63]). The latter has become the standard approach nowadays. Model categories then serve as a direct way of constructing ∞-categories. Roughly speaking an ∞-category is a category with morphisms forming simplicial sets

$$\mathrm{Hom}(S,T) \in \mathrm{Ob}(\mathbf{sSet})\,,$$

where the composition

$$\mathrm{Hom}_{\mathbf{sSet}}(S,T) \times \mathrm{Hom}_{\mathbf{sSet}}(T,U) \longrightarrow \mathrm{Hom}_{\mathbf{sSet}}(S,U)$$

is a simplicial map. An example is the category **Sset** itself, which one uses to model $\mathbf{Grp}_\infty$ the ∞-*groupoids* and **sComm** the *simplicial commutative* $\mathbb{C}$-*algebras*, which are the *simplicial objects* over the category of $\mathbb{C}$-algebras. Let **dAff** be the ∞-category of derived affine schemes (opposite of **sComm**), then the ∞-category of derived stacks **dSt** consists of ∞-functors

$$\mathbf{dAff} \longrightarrow \mathbf{Grp}_\infty\,.$$

On morphisms, these act as simplicial maps. The objects of **dSt** need to satisfy a *hyper-descent* condition with respect to *étale hyper-coverings* (see Toën–Vezzosi [170, Def. 3.4.8], Lurie [122, §6.5.4]). To make things locally more computable, Brav–Bussi–Joyce [25], Borisov–Joyce [23] use the equivalence to *connective commutative differential graded algebras* (see [25, Def. 2.1], Schwede–Shipley [155])

$$\mathbf{sComm} \longrightarrow \mathbf{cdga}^{\leq 0}\,.$$

- Analogously, replacing **dAff** by the category of affine schemes **Aff** one obtains the ∞-category of **hSt** of *higher stacks* as ∞-functors

$$\mathbf{Aff} \longrightarrow \mathbf{Grp}_\infty\,,$$

  Moreover, there is an inclusion $i : \mathbf{Aff} \to \mathbf{dAff}$ of affine schemes which induces a truncation ∞-functor as its Quillen left adjoint $t_0 : \mathbf{dSt} \to \mathbf{hSt}$ (or modulo homotopy, simply left adjoint). In particular, we have the map $i_{\boldsymbol{S}} : t_0(\boldsymbol{S}) \to \boldsymbol{S}$, which we will use in general to restrict bundles and complexes on $\boldsymbol{S}$ to $t_0(\boldsymbol{S})$.

The non-derived cases in (2.2.1) have solutions to the moduli problems in Artin stacks (see Laumon–Moret-Bailly [111, §2.4.4, 3.4.4 & 4.6.2]):

$$\mathfrak{M}^{\mathrm{vb}} \subset \mathfrak{M}_X\,, \quad \mathfrak{M}_{\mathrm{cs}}$$

One needs to do extra work for the second pair, because complexes in derived categories do not satisfy the descent condition. For this, one needs the notion of *dg-categories* (for background on dg-categories see Keller [101] and Toën [165]). Morphisms of objects $\mathrm{Hom}(A, B)$ in a dg-category $\mathcal{T}$ form complexes of vector bundles and their composition

$$\mathrm{Hom}_{\mathcal{T}}(A, B) \otimes \mathrm{Hom}_{\mathcal{T}}(B, C) \to \mathrm{Hom}_{\mathcal{T}}(A, C)$$

is a morphism of complexes. Toën [165] introduces homotopy theory of dg-categories, which is then used in Toën and Vaquié [168] to define for $\mathcal{T}$ its associated moduli stack as a derived stack $\boldsymbol{\mathcal{M}}_{\mathcal{T}}$. This defines a functor from the homotopy category of dg-categories to the homotopy category of derived stacks

$$\boldsymbol{\mathcal{M}}_{(-)} : \mathrm{Ho}(\mathbf{dg\text{ - }Cat})^{\mathrm{op}} \to \mathrm{Ho}(\mathbf{dSt}).$$

Composing with $\mathrm{Ho}(t_0) : \mathrm{Ho}(\mathbf{dSt}) \to \mathrm{Ho}(\mathbf{hSt})$ mapping to the homotopy category of higher stacks. We denote the composition by $\mathcal{M}_{(-)}$.

The idea is now to replace $D^b(\mathrm{QCoh}(X))$ with its *dg-enrichment* $L_{\mathrm{Qcoh}}(X)$ satisfying (i) it has the same objects as $D^b\big(\mathrm{QCoh}(X)\big)$, (ii) for any two complexes, we have $H^0(\mathrm{Hom}_{L_{\mathrm{QCoh}(X)}}(E, F)) = \mathrm{Hom}_{D^b(\mathrm{Coh}(X))}(E, F)$.

Then forming $L_{\mathrm{pe}}(X) \subset L_{\mathrm{Qcoh}}(X)$ containing perfect complexes one uses the above to construct

$$\boldsymbol{\mathcal{M}}_X = \boldsymbol{\mathcal{M}}_{L_{\mathrm{pe}}(X)}\,, \quad \mathcal{M}_X = \mathcal{M}_{L_{\mathrm{pe}}}(X)\,.$$

These classify objects E in $D^b\mathrm{QCoh}(X)$ for which $\underline{\mathrm{Hom}}(F, E)$ is perfect for all perfect

F as explained in [168], [26, Ex. 3.7] . In particular, if X

- is smooth then $\mathcal{M}_X$ classifies compactly supported perfect complexes. This is standard and can be shown by the local to global spectral sequence (see e.g. Huybrechts [86, proof of Lemma 3.9]).
- it is smooth and proper, then it classifies all perfect complexes and can be expressed as a mapping stack (see [168, p. 60])

$$\boldsymbol{\mathcal{M}}_X = \mathbf{Map}_{\mathbf{dSt}}(X, \mathbf{Perf}_{\mathbb{C}})\,, \quad \mathcal{M}_X = \mathrm{Map}_{\mathbf{hSt}}(X, \mathrm{Perf}_{\mathbb{C}})\,, \tag{2.2.2}$$

  where $\mathbf{Perf}_{\mathbb{C}}$ is the derived stack of perfect dg-modules/complexes over $\mathbb{C}$ as defined in [167, Definition 1.3.7.5] and $\mathrm{Perf}_{\mathbb{C}} = t_0(\mathbf{Perf}_{\mathbb{C}})$.
- is not smooth, then it can classify objects which are not perfect. As an example, one can take $X = \mathrm{Spec}(k[x]/\langle x^2\rangle)$ and its complex k concentrated in degree 0.

(Co)tangent complexes on $\mathcal{M}_X$ give an important tool for studying deformation-obstruction theory of moduli problems. Certain derived stacks $\boldsymbol{S}$ called *locally geometric locally of finite representation* admit perfect cotangent complexes $\mathbb{L}_{\boldsymbol{S}}$ (see [167, Cor. 2.2.3.3], [168]). When X is smooth and proper, $\mathcal{M}_X$ was shown to satisfy this property by Toën–Vaquié [168, Cor. 3.29]. The same argument does not apply when X is not proper, however perfect cotangent complexes still exist by [26, §3.4/p.2]*.

For (X, Ω) CY4, let $\mathbb{T} \to \boldsymbol{\mathcal{M}}_X$ be the tangent complex and $\mathbb{L} \cong \mathbb{T}^\vee$ the cotangent

*The author is aware of a different straightforward argument, where one embeds $X \to Y$ for some smooth Y which induces an open embedding $\mathcal{M}_X \to \mathcal{M}_Y$. Then the (co)tangent complex on $\mathcal{M}_X$ is the pull-back of the on on $\mathcal{M}_Y$.

complex. The relation to obstruction theory is a result of the isomorphism

$$\mathbb{T}|_{[E]} \cong \underline{\mathrm{Hom}}(E, E)[1]\,,$$

for each $\mathbb{C}$-point $[E]$, which was shown by Brav–Dyckerhoff [26, p. 3.21] and Toën–Vaquié [168, Cor. 3.17]. To replace the duality (2.1.2) for $SU(4)$-instantons by the corresponding notion on $\boldsymbol{\mathcal{M}}_X$, we recall -2-*shifted symplectic structures* introduced by Pantev–Toën–Vaquié–Vezzosi [145]. The usual single complex of differential forms on a smooth manifold is replaced by the double complex $(\Lambda^\bullet \mathbb{L}_X[\bullet], d, d_{\mathrm{dR}})$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d_{\mathrm{dR}}} & & \uparrow{\scriptstyle d_{\mathrm{dr}}} & & \uparrow{\scriptstyle d_{\mathrm{dr}}} & & \\
\Lambda^3\mathbb{L} & \xrightarrow{d} & \Lambda^3\mathbb{L}_X[1] & \xrightarrow{d} & \Lambda^3\mathbb{L}[2] & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d_{\mathrm{dR}}} & & \uparrow{\scriptstyle d_{\mathrm{dr}}} & & \uparrow{\scriptstyle d_{\mathrm{dr}}} & & \\
\Lambda^2\mathbb{L} & \xrightarrow{d} & \Lambda^2\mathbb{L}_X[1] & \xrightarrow{d} & \Lambda^2\mathbb{L}[2] & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d_{\mathrm{dR}}} & & \uparrow{\scriptstyle d_{\mathrm{dr}}} & & \uparrow{\scriptstyle d_{\mathrm{dr}}} & & \\
\mathbb{L} & \xrightarrow{d} & \mathbb{L}[1] & \xrightarrow{d} & \mathbb{L}[2] & \longrightarrow & \cdots
\end{array}$$

A 2-*form of degree* n is an element in $[\omega_0] \in H^n(\Lambda^2\mathbb{L}_X)$. A *closed 2-form of degree* n is an element in $[\omega] \in H^n(\bigoplus_{k\geq 0} \Lambda^{2+k}\mathbb{L}[k])$ and it has its *underlying* 2-form $[\omega_0]$ given by projection inducing a morphism

$$\theta^\omega : \mathbb{L} \longrightarrow \mathbb{T}[-n]\,. \tag{2.2.3}$$

When $n = -2$ and θ^ω is an isomorphism then $[\omega]$ is -2-*shifted symplectic.* For a quasi-projective (X, Ω), -2-shifted symplectic structures were constructed by Brav–Dyckerhoff [26, Prop. 5.3] and [26, Thm, 5.5 (1)] extending the work of Pantev–

Toën–Vaquié–Vezzosi [145] for projective Calabi–Yau fourfolds.

Definition 2.2.1 (see also Borisov–Joyce [23, Def. 3.26])**.** In [153], Schürg–Toën–Vezzosi construct a perfect determinant map $\mathbf{det} : \mathbf{Perf}_{\mathbb{C}} \to \mathbf{Pic}$. For a perfect complex $C \in L_{\mathrm{pe}}$ on a derived stack $\boldsymbol{S}$ corresponding to a map $\mathbf{u} : \boldsymbol{S} \to \mathbf{Perf}_{\mathbb{C}}$, we denote by $\det(C)$ the line bundle corresponding to the composition $\mathbf{det} \circ \mathbf{u}$ and $\Lambda_{\underline{0}} = \det(\mathbb{L})$. Then (2.2.3) induces the isomorphism $i^\omega : \Lambda_{\underline{0}} \cong \Lambda^*_{\underline{0}}$ and an *orientation* is a choice of isomorphism

$$o : \Lambda_{\underline{0}} \cong \mathcal{O}_{\boldsymbol{M}} \qquad o^2 = \mathrm{ad}(i^\omega)\,,$$

where $\mathrm{ad}(i^\omega) : \Lambda^2_{\underline{0}} \cong i^\omega$ denotes the adjoint isomorphism to i^ω.

2.3 DT_4 invariants

Existence and the choice of orientations in Definition 2.2.1 is vital for defining invariants counting sheaves and complexes on a Calabi–Yau 4-fold and so are the −2-shifted symplectic structures. For this section, we assume that $H^i(\mathcal{O}_X) = 0$ for $i = 1, 2, 3$, and use the notation $G^0(S)$ to denote the Grothendieck group of coherent sheaves on S for any scheme S. When (X, Ω) is compact, one can obtain this restriction by requiring that for a choice of metric g on X its holonomy group is $SU(4)$, then we are back in the setting of §2.1. These virtual fundamental classes were originally defined by Borisov–Joyce [23] and Cao–Leung [37]. We choose to follow the more recent algebraic geometric approach of Oh–Thomas [141] as it offers multiple additional techniques including the construction of a virtual structure sheaf appearing in

Chapter 4. A different approach was also given by Kiem–Park [102] which also allows the definition of reduced invariants.

To talk about virtual fundamental classes, let us briefly review stability conditions. There are many notions of stability conditions: see Rudakov [152], Joyce [93], Bridgeland [27]† and Toda [164, §2]. The following is one of the most general definitions:

Definition 2.3.1. Let $\mathcal{A}$ be an abelian category, $K_0(\mathcal{A}) \to K(\mathcal{A})$ a quotient of its Grothendieck group and $C(\mathcal{A})$ the image of $[\![\cdot]\!] : \mathrm{Ob}(\mathcal{A})\backslash\{0\} \to K(\mathcal{A})$, such that $0 \notin C(\mathcal{A})$. A *stability condition* on $\mathcal{A}$ is a triple $(\tau, T, \leq)$, such that $(T, \leq)$ is a totally ordered set and $\tau : C(\mathcal{A}) \to T$ is a map satisfying the *see-saw condition*: For each $\alpha, \beta, \gamma : \gamma = \alpha + \beta$

$$\tau(\alpha) < \tau(\gamma) < \tau(\beta) \quad \text{or} \quad \tau(\alpha) > \tau(\gamma) > \tau(\beta)\,, \quad \text{or} \quad \tau(\alpha) = \tau(\beta) = \tau(\gamma)\,.$$

A non-zero object E is *semi-stable* when for a non-zero $F \subset E$ we have $\tau([\![F]\!]) \leq \tau([\![E]\!])$ and *stable* if $F \subsetneq E$ implies $\tau([\![F]\!]) < \tau([\![E]\!])$.

We only recall here the notion of Gieseker stability as in Huybrechts–Lehn [86, Def. 1.2.4]. That is the only notion necessary to make sense of Conjecture 4.2.10 except for pair stability, which we recall there. Fixing an ample line bundle $\mathcal{O}(1)$, the *reduced Hilbert polynomial* of a non-zero sheaf F is the unique polynomial p_F with leading coefficient 1 satisfying $\alpha_F p_F(n) = \chi\big(F(n)\big)$, for $n \gg 0, \alpha_F \in \mathbb{Q}$. Then a sheaf F is said to be

- *Gieseker semistable* if it is pure and for each non-zero $E \subset F$, $p_E(n) \leq p_F(n)$,

†Here also see B.–Dimitrov [21].

$n \gg 0$

- *Gieseker stable* if it is pure and for each non-zero $E \subsetneq F$, $p_E(n) < p_F(n)$, $n \gg 0$.

In terms of Definition 2.3.1, this can be interpreted as fixing $C\big(\mathrm{Coh}(X)\big) \subset K\big(\mathrm{Coh}(X)\big)$, where the latter is the numerical Grothendieck group, and defining the map $p_{\{E\}} : C\big(\mathrm{Coh}(X)\big) \to \mathbb{Q}[t]$ with the order given by

$$p \leq q \iff \deg(p) > \deg(q) \quad \text{or} \quad \big(\deg(p) = \deg(q) \quad \text{and} \quad p(n) \leq q(n) \quad \text{for}\, n \gg 0\big)\,.$$

Recall now that for a proper scheme S and its (truncated) *cotangent complex* $\mathbb{L}_S \to S$ (see e.g. Illusie [88], Behrend–Fantechi [13] or Battistella–Carocci–Manolache [11]) a *perfect obstruction theory* is a map from a two term perfect complex $\mathbb{F} \to \mathbb{L}_S$, such that $H^0(\mathbb{F}) \to H^0(\mathbb{L}_S)$ is an isomorphism and $H^{-1}(\mathbb{F}) \to H^{-1}(\mathbb{L}_S)$ is surjective. Then Behrend–Fantechi [13] construct the virtual fundamental class

$$[S]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(S)\,, \qquad \mathrm{vd} = \mathrm{rk}(\mathbb{F})\,.$$

Example 2.3.2. Let S be a projective surface and consider the variety $\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)$ of 1-dimensional quotients $\mathbb{C}^N \otimes \mathcal{O}_S \to F$ for $\chi(F) = n$ and $[F] = \beta$, then it has a natural perfect obstruction theory constructed by Marian–Oprea–Pandharipande [124, Lem. 1.1] given by $\mathbb{F} = \Big(\tau_{[0,1]}\underline{\mathrm{Hom}}_{\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)}(\mathcal{I},\mathcal{F})\Big)^{\vee}$, where

$$\mathcal{I} = (\mathbb{C}^N \otimes \mathcal{O}_X \to \mathcal{F}) \to S \times \mathrm{Quot}(S, \mathbb{C}^N, n)$$
$$\mathrm{vd}\Big(\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)\Big) = \beta^2 + Nn$$

as can be seen by Hirzebruch–Riemann–Roch.

Moving on to Calabi–Yau fourfolds, one needs some adaptations. Naively, for a given projective moduli scheme of Gieseker stable sheaves the deformation at E corresponds to $\mathrm{Ext}^1(E,E)$ and obstruction to $\mathrm{Ext}^2(E,E)$. However, similarly to $SU(4)$-instantons, this does not lead to well defined virtual fundamental classes because then $\mathrm{vd}|_E = \mathrm{Ext}^1(E,E) - \mathrm{Ext}^2(E,E)$ depends on E. The approach proposed by Borisov–Joyce [23] and Cao–Leung [37] inspired by (2.1.3) requires taking

$$\mathrm{Ext}^1(E,E)\,, \qquad \mathrm{Ext}^2(E,E)_+\,,$$

where $\mathrm{Ext}^2(E,E)_+$ is a real subspace with respect to the algebraic Serre-duality $\mathrm{Ext}^2(E,E) \cong \mathrm{Ext}^2(E,E)^*$. The construction of Oh–Thomas [141] is claimed to be equivalent to those above (as promised in [141, p. 6] and to appear in [140]) but instead it relies on taking an isotropic subspace $\mathrm{Ext}^2(E,E)_q$ with respect to the pairing $q : \mathrm{Ext}^2(E,E) \times \mathrm{Ext}^2(E,E) \to \mathbb{C}$.

This leads to the question of orientations. In the language of Borisov–Joyce [23], orientations are continuous choices of orientations on the real space $\mathrm{Ext}^2(E,E)_+$, while in the algebraic setting of Oh–Thomas [141] there are two families of isotropic subspaces related by $SO(\mathrm{ext}^2(E,E),\mathbb{C})$. The resulting classes are dependent on these choices. We now describe more thoroughly the latter construction, which requires $\chi(\alpha,\alpha) \in 2\mathbb{Z}$.

Let $\mathcal{E}$ be the twisted universal sheaf (see Caldararu [29]) on $X\times M$, then Huybrechts–

Thomas [87] construct the Atiyah class

$$\mathrm{At}: \mathbb{E} := \tau_{[-2,0]}\Big(\underline{\mathrm{Hom}}_M(\mathcal{E},\mathcal{E})[3]\Big) \longrightarrow \mathbb{L}\,.$$

By Grothendieck–Verdier duality [78, VII, 3.4(c)] there is an isomorphism $\mathbb{E} \cong \mathbb{E}^\vee[2]$. Oh–Thomas [141, Proposition 4.1] show that it admits a self dual, locally free resolution

$$(T \to E \to T^*) \longrightarrow \mathbb{E}\,,$$

where E is an $O(n,\mathbb{C})$ bundle. If orientations of Definition 2.2.1 exist, E reduces to an $SO(n,\mathbb{C})$-bundle and Oh–Thomas [141] construct

$$[M]^{\mathrm{vir}} = H_{2-\chi(\alpha,\alpha)}\big(M, \mathbb{Z}[2^{-1}]\big)\,, \qquad \hat{\mathcal{O}}^{\mathrm{vir}} \in G_0\big(M, \mathbb{Z}[2^{-1}]\big)\,. \tag{2.3.1}$$

The latter is the *twisted virtual structure sheaf*, and both $[M]^{\mathrm{vir}}$ and $\hat{\mathcal{O}}^{\mathrm{vir}}$ depend on an additional choice of orientation. It is important to note that the construction relies heavily for now on existence of -2-shifted symplectic structures, unless one is in a setting where everything can be made explicit. The point of view of real (derived) manifolds of Borisov–Joyce [23] is in some way more natural and the author thinks of the Oh-Thomas construction as being less fundamental but more approachable and computable. This is especially clear from the following:

Theorem 2.3.3 (Oh–Thomas [141, Thm. 6.1])**.** *Let M be projective with a fixed choice of orientation, $V \in G^0\big(M\big)$, then*

$$\hat{\chi}^{\mathrm{vir}}\big(V\big) = \int_{[M]^{\mathrm{vir}}} \sqrt{\mathrm{Td}}(\mathbb{E})\mathrm{ch}(V)\,. \tag{2.3.2}$$

Proof. This is just Theorem [141, Theorem 6.1] stated in terms of $\hat{\chi}^{\mathrm{vir}}(-)$ using the notation 1.0.5. $\square$

Recall that for a real vector bundle E on a manifold M its $\hat{A}$-genus satisfies $\hat{A}(E) = \sqrt{Td}(E \otimes \mathbb{C})$. This tells us that $\hat{\mathcal{O}}^{\mathrm{vir}}$ may be thought of as the Dirac operator on $M_\alpha(\tau)$ via the Atiyah–Singer Index Theorem [3]. We will see in Theorem 4.5.5 that this is slightly misleading as in general (2.3.2) will not be an integer and a correction by a square-root of a "tautological" determinant line bundle will be necessary.

Chapter 3

Orientations for DT invariants on quasi-projective Calabi–Yau fourfolds

As explained in §2.3, orientability of moduli spaces is crucial for studying invariants on Calabi–Yau 4-folds. Our goal in this chapter was to prove orientability in its full generality in hopes that it would be useful for degeneration arguments similar to those by Levin–Pandharipande [116] and Maulik–Pandharipande-Thomas [129]. We hope that eventually this will also find application when studying invariants for general toric manifolds. As of writing the thesis, the author is not aware of a published result proving orientability in the local cases apart from Diaconescu–Sheshmani–Yau [45]. For compact X with $\mathrm{H}^{\mathrm{odd}}(X) = 0$ and moduli spaces of stable coherent sheaves this was originally addressed by Cao–Leung [31]. It was then extended by Cao–Gross–Joyce [30] to all compact geometries and perfect complexes. One standard well known result in the local case can be found in Oh–Thomas [141, §7] or Diaconescu–

Sheshmani–Yau [45], where one considers the total bundle of the canonical line bundle $K_Y \to Y$ for a Fano 3-fold Y. More recently, Kool–Rennemo [107] constructed explicit orientations for Quot-schemes of points on $\mathbb{C}^4$ by embedding it into a non-commutative Quot-scheme.

We begin this chapter by discussing a straight-forward generalization of Cao–Gross–Joyce [30] to spin geometries. This proves orientability in Definition 2.3 for X which admit compactifications with spin structures. As we do not expect these to always exist, we approach the problem differently in 3.1.3 and obtain a result without additional restrictions on X. This chapter can be found in a slightly different but equivalent form in the author's previous work [19].

3.1 Orientation bundles on moduli stacks of perfect complexes

3.1.1 Twisted virtual canonical bundles

If X is proper, we can use the description of $\mathcal{M}_X$ as a mapping stack to construct a universal complex on $X \times \mathcal{M}_X$: If

$$u : X \times \mathcal{M}_X \to \mathrm{Perf}_{\mathbb{C}} \tag{3.1.1}$$

is the canonical morphisms for $\mathcal{M}_X$ as a mapping stack, and $\mathcal{U}_0$ is the universal complex on $\mathrm{Perf}_{\mathbb{C}}$ used by Pantev–Toën–Vaquié–Vezzosi in [145], then one defines the universal complex $\mathcal{U}_X = u^*(\mathcal{U}_0)$ on $X \times \mathcal{M}_X$. When X is quasi-projective and not necessarily proper, we need a different construction of the universal structure sheaf.

Definition 3.1.1.

$$\xi_Y = \mathcal{M}_{(i_X^*)} : \mathcal{M}_X \to \mathcal{M}_Y \tag{3.1.2}$$

be the image of the pullback $i_X^* : L_{\mathrm{pe}}(Y) \to L_{\mathrm{pe}}(Y)$, then it acts on Spec$(A)$-points by the right adjoint of $\big(i_X \times \mathrm{id}_{\mathrm{Spec}(A)}\big)^*$ as follows from its construction in Toën–Vaquié [168, §3.1] and therefore by the pushforward $(i_X \times \mathrm{id}_{\mathrm{Spec}(A)})_*$ of compactly supported families of perfect complexes on X. We define $\mathcal{U}_X \to X \times \mathcal{M}_X$ by

$$\mathcal{U}_X = \xi_Y^*(\mathcal{U}_Y)\,.$$

It is independent of the choice of a compactification*.

When $Z = \prod_{i\in I} Z_i$, we will use $\pi_{I'} : Z \to \prod_{i\in I'} Z_i$ for $I' \subset I$ to denote the projection to I' components of the product. We use this also for general fiber products. Let Y now be any a quasi-projective smooth four-fold, L a coherent sheaf on Y and $\mathcal{U}_Y \in \mathrm{L}_{\mathrm{pe}}(Y \times \mathcal{M}_Y)$ its universal complex compactly supported in Y. We define

$$\mathcal{E}\mathrm{xt}_L = \pi_{2,3\,*}(\pi_{1,2}^*\mathcal{U}_Y^\vee \otimes \pi_{1,3}^*\mathcal{U}_Y \otimes \pi_1^* L)\,, \quad \mathbb{P}_L = \Delta_{\mathcal{M}_Y}^*\mathcal{E}\mathrm{xt}_L\,. \tag{3.1.3}$$

As pushforward along $\pi_{2,3} : Y \times \mathcal{M}_Y \times \mathcal{M}_Y \to \mathcal{M}_Y \times \mathcal{M}_Y$ maps (compactly supported) perfect complexes in Y to perfect complexes, it has a right adjoint $\pi_{2,3}^!$ by Lurie's adjoint functor theorem Gaitsgory–Rozenblyum [64, Thm. 2.5.4, §1.1.2], Lurie [122, Cor. 5.5.2.9]. Moreover, $\pi_{2,3}^! = \pi_{2,3}^*(-)\otimes K_Y[4]$, which gives us the usual Serre duality in families

$$\mathcal{E}\mathrm{xt}_L \cong \sigma^*(\mathcal{E}\mathrm{xt}^\vee_{(K_Y\otimes L^\vee)})[-4]\,, \tag{3.1.4}$$

*Simply choose a common compactification $Y \leftarrow Y'' \to Y'$ and compare the resulting universal sheaves where $\sigma : \mathcal{M}_Y \times \mathcal{M}_Y \to \mathcal{M}_Y \times \mathcal{M}_Y$ is the map interchanging the factors.

Definition 3.1.2. Let Y be smooth and and L a coherent sheaf on Y, then as (3.1.3) are perfect, we construct the *L-twisted virtual canonical bundle*

$$\Sigma_L = \det(\mathcal{E}\mathrm{xt}_L)\,, \quad \Lambda_L = \det(\mathbb{P}_L)\,.$$

Moreover, Σ_L, Λ_L are *$\mathbb{Z}_2$-graded* with *degree* given by a map $\deg(\Lambda_L) : \mathcal{M}_X \to \mathbb{Z}_2$, such that

$$\deg(\Sigma_L)|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} \equiv \chi(\alpha, \beta \cdot L) \pmod 2\,, \quad \deg(\Lambda_L)|_{\mathcal{M}_\alpha} \equiv \chi(\alpha, \alpha \cdot L) \pmod 2\,.$$

where $\alpha, \beta \in K^0(X)$ and $\mathcal{M}_\alpha$ is the stack of complexes with class $[\![E]\!] = \alpha$. See Definition 3.2.11 for more details.

From the duality (3.1.4), we obtain the isomorphisms

$$\Sigma_L \cong \sigma^*(\Sigma_{K_X \otimes L^\vee})^*\,, \quad \theta_L : \mathbb{P}_L \longrightarrow \mathbb{P}^\vee_{(K_X \otimes L^\vee)}[-4]\,, \quad i_L : \Lambda_L \longrightarrow \Lambda^*_{(K_X \otimes L^\vee)}\,. \quad (3.1.5)$$

Let us recall some definitions: Let $(-)^{\mathrm{an}} : \mathbf{Aff} \to \mathbf{Top}$ be the functor mapping an affine scheme over $\mathrm{Spec}(\mathbb{C})$ to its analytification. The category **Top** is **Top**-enriched, moreover **Top** can serve as a model for the ∞-category of ∞-groupoids. Thus Blanc [16] defines

$$(-)^{\mathrm{top}} : \mathrm{Ho}(\mathbf{hSt}) \longrightarrow \mathrm{Ho}(\mathbf{Top})\,, \quad (3.1.6)$$

which practically by definition satisfies $X^{\mathrm{top}} = X^{\mathrm{an}}$ up to homotopy equivalence for any $\mathbb{C}$-scheme X and commutes with homotopy colimits [16, Prop. 3.2.1].

We also generalize the contents of Definition 2.2.1

Definition 3.1.3. Let $L \to X$ be a complex line bundle with an isomorphism $\tau : L \to L^*$. Then we define the *square root* $\mathbb{Z}_2$*-bundle associated with* τ denoted by O^τ. This bundle is given by the sheaf of its sections in the respective topology:

$$O^\tau(U) = \{o : L|_U \xrightarrow{\sim} \underline{\mathbb{C}}_U : o \otimes o = \mathrm{ad}(\tau)\}\,.$$

3.1.2 Spin orientability

When X is a compact Calabi–Yau fourfold, Cao–Gross–Joyce [30, Theorem 1.15] prove that $O^\omega \to \mathcal{M}_X$ is trivializable. One could generalize their result by replacing the requirement of X being Calabi–Yau by a weaker one.

Definition 3.1.4. Let X be a smooth projective variety and K_X its canonical divisor class. A divisor class Θ, such that $2\Theta = K_X$ is called a *theta characteristic*. We say that (X, Θ) for a given choice of a theta characteristic Θ is *spin*. For a given (X, Θ), we will use the notation $K_{\mathcal{M}_X} = \Lambda_\Theta$. We then have by (3.1.5) the isomorphism

$$i^{\mathcal{S}} : K_{\mathcal{M}_X} \xrightarrow{\sim} K^*_{\mathcal{M}_X}$$

and by Definition 3.1.3 the associated $\mathbb{Z}_2$-bundle $O^{\mathcal{S}} \to \mathcal{M}_{\mathcal{X}}$.

Remark 3.1.5. A choice of Θ is equivalent to a choice of spin structure on X^{an} (see Atiyah [10, Proposition 3.2]).

Definition 3.1.6. Let Z be a projective variety over $\mathbb{C}$. Let $\mathcal{M}^Z$ be the mapping stack from (2.2.2). Let $u_Z : Z \times \mathcal{M}^Z \to \mathrm{Perf}_{\mathbb{C}}$ be the canonical map. Applying $(-)^{\mathrm{top}}$

and using Blanc [16, §4.2], we obtain $(u_Z)^{\mathrm{top}} : Z^{\mathrm{an}} \times (\mathcal{M}^Z)^{\mathrm{top}} \to BU \times \mathbb{Z}$. This gives us

$$\Gamma_Z : (M^Z)^{\mathrm{top}} \longrightarrow \mathrm{Map}_{C^0}(Z^{\mathrm{an}}, BU \times \mathbb{Z}) .$$

For any topological space T we use the notation $\mathcal{C}_T = \mathrm{Map}_{C^0}(T, BU \times \mathbb{Z})$.

Proposition 3.1.7. *Let (X, Θ) be spin with the orientation bundle $O^{\mathcal{S}} \to \mathcal{M}_X$. Let $\Gamma_X : (\mathcal{M}_X)^{\mathrm{top}} \to \mathcal{C}_X$ be as in Definition 3.1.6 and apply $(-)^{\mathrm{top}}$ to obtain a $\mathbb{Z}_2$-bundle $(O^{\mathcal{S}})^{\mathrm{top}} \to (\mathcal{M}_X)^{\mathrm{top}}$. There is a canonical isomorphism of $\mathbb{Z}_2$-bundles*

$$(O^{\mathcal{S}})^{\mathrm{top}} \cong \Gamma_X^*(O_{\mathcal{C}}^{\not{D}_+}) ,$$

where $O_{\mathcal{C}}^{\not{D}_+} \to \mathcal{C}_X$ is the $\mathbb{Z}_2$-bundle from Joyce–Tanaka–Upmeier [97, Definition 2.22] applied to the positive Dirac operator $\not{D}_+ : S_+ \to S_-$ as in Cao–Gross–Joyce [30, Theorem 1.11]. In particular, $O^{\mathcal{S}} \to \mathcal{M}_X$ is trivializable by the aforementioned theorem.

Proof. This is a simple generalization of the proof of [30, Theorem 1.15] relying on the fact that Theorem 3.2.9 requires X^{an} to be a spin manifold to trivialize the orientation bundle on $\mathcal{B}_X$. We only discuss the corresponding real structure on the differential geometric side replacing [30, Definition 3.24]. We have the pairing $\wedge^S : (\mathcal{A}^{0,k} \otimes \Theta) \otimes (\mathcal{A}^{0,4-k} \otimes \Theta) \to \mathcal{A}^{4,4}$, and the corresponding *spin Hodge star* $\star^S : \mathcal{A}^{0,q} \otimes \Theta \to \mathcal{A}^{0,n-q} \otimes \Theta$

$$(\beta \otimes t) \wedge^S \star_k^S(\alpha \otimes s) = \langle \beta \otimes s, \alpha \otimes t \rangle \bar{\Omega} \qquad \alpha, \beta \in A^{0,k},\ s, t \in \Gamma^\infty(\Omega) ,$$

where $\Omega \in \mathcal{A}^{4,4}$ is the volume form. As a result, we have the real structures: $\#_1^S : \mathcal{A}^{0,\mathrm{even}} \otimes \Theta \to \mathcal{A}^{0,\mathrm{even}} \otimes \Theta$ and $\#_2^S : \mathcal{A}^{0,\mathrm{odd}} \otimes \Theta \to \mathcal{A}^{0,\mathrm{odd}} \otimes \Theta$, where again $\#_1^S|_{\mathcal{A}^{0,2q} \otimes \Theta} =$

$(-1)^q\star^S$ and $\#_2^S|_{\mathcal{A}^{0,2q+1}\otimes\Theta} = (-1)^{q+1}\star^S$. The Dolbeault operator commutes with these $D_\Theta \circ \#_1^S = \#_2^S \circ D_\Theta$ and its real part is the positive Dirac operator $\not{D} : S_+ \to S_-$ by Friedrich [59, §3.4]. As twisting by connections only corresponds to tensoring symbols of operators by identity, this extends also to real structures on $\det(D^{\nabla_{\mathrm{End}(E)}})$. □

Remark 3.1.8. Note that one can also state the equivalent of Cao–Gross–Joyce [30, Theorem 1.15(c)], expressing the comparison of orientations under direct sums on $\mathcal{M}_X$ in terms of the comparison on $\mathcal{C}_X$.

Suppose that X is Calabi–Yau and that there exists Y smooth with an open embedding $X \hookrightarrow Y$, where Y is spin. We say that Y is a *spin compactification* of X. We now state the weaker result about orientability for a non-compact Calabi–Yau fourfold. Recall, that we have the map ξ_Y from (3.1.2).

Corollary 3.1.9. *Let X be a Calabi–Yau fourfold, and let Y be a spin compactification of X with a choice of Θ and an isomorphism $\phi : \mathcal{O}_X \xrightarrow{\sim} \Theta|_X$, then there exists an induced isomorphism of $\mathbb{Z}_2$ bundles on $\mathcal{M}_X$:*

$$O^\omega \cong \xi_Y^*(O^\Theta)\,. \tag{3.1.7}$$

In particular, $\mathcal{M}_X$ is orientable.

Proof. Let E be a perfect complex on X with a proper support, then the $\mathbb{Z}_2$ torsors at $[E]$ of both of the above $\mathbb{Z}_2$-bundles are given by

$$\{o_E : \det\big(\underline{\mathrm{Hom}}(E,E)\big) \xrightarrow{\sim} \mathbb{C} \text{ s.t. } o_E \otimes o_E = \mathrm{ad}(i^\omega)|_{[E]}\}$$

where i^ω is the Serre duality, and we used the isomorphism $E \otimes \Theta \cong E$ induced by ϕ.

Thus we have a natural identification of both $\mathbb{Z}_2$-bundles in families. By Proposition 3.1.7, the right hand side of (3.1.7) is trivializable, so the second statement follows. $\square$

Remark 3.1.10. Let Y be a spin compactification of X and $Y \backslash X = D$ be a divisor. Let $D = \bigcup_{i=1}^N D_i$ be its decomposition into irreducible components. If we can write the canonical divisor class of Y as $K_Y = \sum_{i=1}^N a_i D_i$, where $a_i \equiv 0 \pmod 2$, then one can take

$$\Theta = \sum_{i=1}^N \frac{a_i}{2} D_i$$

as the square root. After choosing a meromorphic section $\bar{\Omega}^{\frac{1}{2}}$ of Θ with poles and zeros on D, one obtains an isomorphism $\phi : \mathcal{O}_X \xrightarrow{\sim} \Theta|_X$. Then the condition of Corollary 3.1.9 is satisfied.

Example 3.1.11. The simplest example is $\mathbb{C}^4$. While its natural compactification $\mathbb{P}^4$ is not spin, one can choose to compactify it as $\mathbb{P}^1 \times \mathbb{P}^3$ or $(\mathbb{P}^1)^{\times 4}$ which are both spin, both of which satisfy the property in Remark 3.1.10 by choosing $\pi_1^*\big(\mathcal{O}(-1)\big) \otimes \pi_2^*\big(\mathcal{O}(-2)\big)$ and $\bigotimes_{i=1}^4 \pi_i^*\big(\mathcal{O}(-1)\big)$ as the square roots of K_Y.

Example 3.1.12. Let S be a smooth projective variety $0 \leq \dim_{\mathbb{C}}(S) = k \leq 4$ and let $E \to S$ be a vector bundle, s.t. $\det(E) = K_S$. Then $X = \mathrm{Tot}(E \to S)$ is Calabi–Yau. Taking its smooth compactification $Y = \mathbb{P}(E \oplus \mathcal{O}_S)$ with the divisor at infinity $D = \mathbb{P}(E) \subset \mathbb{P}(E \oplus \mathcal{O}_S)$, one can show that $K_Y = -(\mathrm{rk}(E) + 1)D$. If $\mathrm{rk}(E) \in 2\mathbb{Z} + 1$, we see that we can choose $\Theta = \mathcal{O}_Y(-\frac{(\mathrm{rk}(E)+1)}{2} D)$ which satisfies the property of Remark 3.1.10. Then if $\mathrm{rk}(E) + k = 4$, this is an example of Corollary 3.1.9, when $\mathrm{rk}(E) = 1, 3$.

If $X = \mathrm{Tot}(L_1 \oplus L_2 \to S)$ for a smooth projective surface S and its line bundles L_1, L_2, s.t. $L_1 L_2 = K_S$, then the spin compactification can be obtained as $\mathbb{P}(L_1 \oplus$

$\mathcal{O}_S) \times_S \mathbb{P}(L_2 \oplus \mathcal{O}_S)$.

Example 3.1.13. Suppose we have a toric variety X (see Fulton [60], Cox [44]) given by a fan in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Suppose it is smooth and it contains the natural cone spanned by $(e_i)_{i=1}^n$. Define the hyperplanes

$$H_i = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{j=1}^n x_j = i\}.$$

Then X is Calabi–Yau if and only if all the primitive vectors of rays of the fan lie in H_1 and all the cones are spanned by a basis. A simple generalization of this well known statement shows that X is spin if and only all the primitive vectors lie in $H_{\text{odd}} = \bigcup_{i \in 2\mathbb{Z}+1} H_i$. Starting from a toric Calabi–Yau X, one can compactify X to a projective smooth toric variety Y by adding divisors corresponding to primitive vectors. In general, we will not have spin compactifications: Consider the fan in $\mathbb{R}^2$ with more than 3 primitive vectors in H_1, then any compactification will be consecutive blow ups of a Hirzebruch surface at points, with at least one blow up.

A common way of constructing Calabi–Yau manifolds is by removing anti-canonical divisors from a Fano manifold. To further illustrate the scarceness of spin-compactifications in even dimension, we study the classification of toric projective Fano fourfolds by Batyrev [12]. Using the condition described above, we can show that there exist only 4 smooth toric Fano fourfolds with a spin structure. These are $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(2))$, $\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ and $4\mathbb{P}^1$ corresponding the the polytopes B_2, B_4, D_{12} and L_8 respectively. Note that there are 123 smooth projective toric Fano fourfolds in total.

The last observation motivated the entire rest of the chapter.

3.1.3 Orientation for a non-compact Calabi–Yau via algebraic excision principle

Let (X, Ω) be a Calabi–Yau fourfold, then we fix a compactification Y with $D = Y \backslash X$ a *strictly normal crossing divisor*, i.e. it is a union of smooth divisors with transversal intersections (any k-fold intersection is in particular smooth). By Hironaka [81, Main Theorem 1], Bierstone–Milman [15] there exists a compactification with a normal crossing divisor by embedding into a projective space and taking resolutions. The additional strictness condition can be obtained, by subsequent blow-ups of all $1 \leq k \leq 4$ intersections. Consider the triples (E, F, ϕ), where $E, F \in L_{\mathrm{pe}}(Y)$ and $\phi : E|_D \xrightarrow{\sim} F|_D$. We will take the difference of the determinants $\det\big(\underline{\mathrm{Hom}}(E, E)\big)$ and $\det\big(\underline{\mathrm{Hom}}((F, F)\big)$ and cancel the contributions which live purely on the divisor. One could think of this as an algebraic version of the excision principle defined for complex operators in §3.2.3. Let us now make the described method more rigorous.

Let X, Y and D be as in the paragraph above, then we can write D as the union

$$D = \bigcup_{i=1}^{N} D_i \tag{3.1.8}$$

where each D_i is a smooth divisor. We require Ω to be algebraic, then there exists a unique meromorphic section $\bar{\Omega}$ of K_Y, s.t. $\bar{\Omega}|_X = \Omega$. The poles and zeroes of $\bar{\Omega}$ express K_Y uniquely in the following form $K_Y = \sum_{i=1}^{N} a_i D_i$, where $a_i \in \mathbb{Z}$. We may write for the canonical line bundle:

$$K_Y = \bigotimes_{i=1}^{N} \mathcal{O}(k_i D_i) = \bigotimes_{i=1}^{N} \mathcal{O}(\mathrm{sgn}(k_i) D_i)^{\otimes |k_i|} \,. \tag{3.1.9}$$

Let N_D be the free lattice spanned by the divisors D_i which we from now on denote by the elements $e_i \in N_D$. For a line bundle $L = \otimes_{i=1}^N \mathcal{O}(a_i D_i)$ we write $L_{\underline{a}}$, where $\underline{a} = (a_1, \dots, a_N)$. We will also use the notation $L_{\underline{k}} = K_Y$. Then for a non-zero global section s_i of $\mathcal{O}(D_i)$ one has the usual exact sequence

$$0 \longrightarrow L_{\underline{a}} \xrightarrow{\cdot s_i} L_{\underline{a}+e_i} \longrightarrow L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i} \longrightarrow 0\,.$$

As all the operations used to define $\mathcal{E}\mathrm{xt}_{\underline{a}} = \mathcal{E}\mathrm{xt}_{L_{\underline{a}}}$ and $\mathbb{P}_{\underline{a}} = \mathbb{P}_{L_{\underline{a}}}$ in Definition 3.1.2 are derived, we obtain distinguished triangles

$$\begin{gathered} \mathcal{E}\mathrm{xt}_{\underline{a}} \longrightarrow \mathcal{E}\mathrm{xt}_{\underline{a}+e_i} \longrightarrow \mathcal{E}\mathrm{xt}_{L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i}} \xrightarrow{[1]} \mathcal{E}\mathrm{xt}_{\underline{a}}[1]\,, \\ \mathbb{P}_{\underline{a}} \longrightarrow \mathbb{P}_{\underline{a}+e_i} \longrightarrow \mathbb{P}_{L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i}} \xrightarrow{[1]} \mathbb{P}_{\underline{a}}[1]\,. \end{gathered} \tag{3.1.10}$$

By (2.2.2) both $\mathcal{M}_Y$ and $\mathcal{M}_{D_i}$ can be expressed as mapping stacks $\mathrm{Map}\big(Y, \mathrm{Perf}_{\mathbb{C}}\big)$ and $\mathrm{Map}\big(D_i, \mathrm{Perf}_{\mathbb{C}}\big)$, respectively. Let $\mathrm{inc}_{D_i} : D_i \to Y$ be the inclusion, then we denote by $\rho_i : \mathcal{M}_Y \to \mathcal{M}_{D_i}$ the morphisms induced by the pullback $(\mathrm{inc}_{D_i})^* : L_{\mathrm{pe}}(Y) \to L_{\mathrm{pe}}(D_i)$. For each divisor D_i we set $L_{\underline{a}}|_{D_i} = L_{\underline{a},i}$ and

$$\mathcal{E}\mathrm{xt}_{\underline{a},i} = \pi_{2,3\,*}(\pi_{1,2}^* \mathcal{U}_{D_i}^\vee \otimes \pi_{1,3}^* \mathcal{U}_{D_i} \otimes \pi_1^* L_{\underline{a},i})\,, \qquad \mathbb{P}_{\underline{a},i} = \Delta^* \mathcal{E}\mathrm{xt}_{\underline{a},i}\,,$$

Lemma 3.1.14. *We have the isomorphism*

$$\begin{gathered} \mathcal{E}\mathrm{xt}_{L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i}} \cong (\rho_i \times \rho_i)^* (\mathcal{E}\mathrm{xt}_{\underline{a}+e_i,i})\,, \\ \mathbb{P}_{L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i}} \cong \rho_i^* (\mathbb{P}_{\underline{a}+e_i,i})\,. \end{gathered}$$

where we use the same notation for the complexes $\mathbb{P}$ on $\mathcal{M}_Y$ and $\mathcal{M}_{D_i}$.

Proof. For universal complex $\mathcal{U}_Y, \mathcal{U}_{D_i}$ on $Y \times \mathcal{M}_Y$, $D_i \times \mathcal{M}_{D_i}$ we have $\mathcal{U}_Y|_{D_i \times \mathcal{M}_Y} = (\mathrm{id}_{D_i} \times \rho_i)^* \mathcal{U}_{D_i}$ as follows from the commutative diagram

$$\begin{array}{ccc} Y \times \mathcal{M}_Y & \longrightarrow & \mathrm{Perf}_{\mathbb{C}} \\ {\scriptstyle \mathrm{inc}_{D_i} \times \mathrm{id}_{\mathcal{M}_Y}} \uparrow & & \uparrow \\ D_i \times \mathcal{M}_Y & \xrightarrow{\mathrm{id}_{D_i} \times \rho_i} & D_i \times \mathcal{M}_{D_i} \,. \end{array}$$

For the dual we also have $\mathcal{U}_Y^\vee|_{D_i \times \mathcal{M}_Y} = (\mathrm{id}_{D_i} \times \rho_i)^* \mathcal{U}_{D_i}^\vee$. Thus we have the following equivalences

$$\begin{aligned} & \mathcal{E}\mathrm{xt}_{L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i}} \\ =& \pi_{2,3*}\big(\pi_{1,2}^*(\mathcal{U}^\vee) \otimes \pi_{1,3}^*(\mathcal{U}) \otimes \pi_1^*(L_{\underline{a}+e_i} \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i})\big) \\ \cong& \pi_{2,3*}\Big((\mathrm{inc}_{D_i} \times \mathrm{id}_{\mathcal{M}_Y \times \mathcal{M}_Y})_*\big(\mathcal{U}^\vee \otimes \mathcal{U} \otimes \pi_1^* L_{\underline{a}+e_i}\big)|_{D_i \times \mathcal{M}_Y \times \mathcal{M}_Y}\Big) \\ \cong& \pi_{2,3*}\Big((\mathrm{inc}_{D_i} \times \mathrm{id}_{\mathcal{M}_Y \times \mathcal{M}_Y})_* \circ (\mathrm{id}_{D_i} \times \rho_i \times \rho_i)^*(\pi_{1,2}^*(\mathcal{U}_{D_i}^\vee) \otimes \pi_{1,3}^*(\mathcal{U}_{D_i}) \otimes \pi_1^* L_{\underline{a}+e_i,i})\Big) \\ \cong& \pi_{2,3*}\Big((\mathrm{id}_{D_i} \times \rho_i \times \rho_i)^*(\pi_{1,2}^*(\mathcal{U}_{D_i}^\vee) \otimes \pi_{1,3}^*(\mathcal{U}_{D_i})) \otimes \pi_1^*(L_{\underline{a}+e_i,i})\Big) \\ \cong& (\rho_i \times \rho_i)^*(\mathcal{E}\mathrm{xt}_{\underline{a}+e_i,i})\,, \end{aligned}$$

the first isomorphism is the projection formula [63, Lem. 3.2.4] and the last step follows from the base change isomorphism $\pi_{2,3*} \circ (\mathrm{id}_{D_i} \times \rho_i)^* \cong (\rho_i \times \rho_i)^* \circ \pi_{2,3*}$ using Gaitsgory [63, Prop. 2.2.2]† and that the diagram

$$\begin{array}{ccccc} D \times \mathcal{M}_X \times \mathcal{M}_X & \longrightarrow & D \times M_D \times M_D & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ M_X \times M_X & \longrightarrow & M_D \times M_D & \longrightarrow & * \end{array}$$

†These references are stated for derived stacks. So we should work with derived stacks until we construct the isomorphisms in Definition 3.1.16, which we can then restrict by §2.2.

consists of Cartesian diagrams by the pasting law in ∞-categories. The second formula follows using $(\rho_i \times \rho_i) \circ \Delta_{\mathcal{M}_Y} = \Delta_{\mathcal{M}_{D_i}} \circ \rho_i : \mathcal{M}_X \to \mathcal{M}_{D_i} \times \mathcal{M}_{D_i}$. $\square$

After taking determinants of (3.1.10), we obtain the isomorphisms

$$\Sigma_{\underline{a}+e_i} \cong \Sigma_{\underline{a}} \otimes \rho_i^* \Sigma_{\underline{a}+e_i,i} \,, \quad \Lambda_{\underline{a}+e_i} \cong \Lambda_{\underline{a}} \otimes \left(\rho_i^* \Lambda_{\underline{a}+e_i,i}\right), \tag{3.1.11}$$

where we omit writing L. We have the maps $i_{D_i} : D_i \to Y$, $i_D : i_D \to Y$ inducing

$$\begin{aligned} \rho_i : &\mathcal{M}_Y \longrightarrow \mathcal{M}_{D_i} \,, \qquad \rho_D : \mathcal{M}_Y \longrightarrow \mathcal{M}^D \,, \\ \mathcal{M}_D^{\mathrm{sp}} &:= \prod_{i=1}^N \mathcal{M}_{D_i} \quad \text{and} \quad \rho = \prod_{i=1}^N \rho_i : \mathcal{M}_Y \longrightarrow \mathcal{M}_D^{\mathrm{sp}} \end{aligned}$$

Note that we have the obvious map $\mathcal{M}^D \to \mathcal{M}_{D_i}$ induced by the inclusion $D_i \hookrightarrow D$. This gives

$$\mathrm{sp} : \mathcal{M}_Y \times_{\mathcal{M}^D} \mathcal{M}_Y = \mathcal{M}_{Y,D} \longrightarrow \mathcal{M}_Y \times_{\mathcal{M}_D^{\mathrm{sp}}} \mathcal{M}_Y = \mathcal{M}_{Y,D}^{\mathrm{sp}} \,. \tag{3.1.12}$$

Definition 3.1.15. For given X, Y as above let $\bar{\Omega}$ be a meromorphic section of K_Y restricting to Ω. Let $\mathfrak{ord}$ denote the decomposition of D into irreducible components as in (3.1.8), which also specifies their order, such that there exist $0 \le N_1 \le N_2 \le N$, such that $a_i = 0$ for $0 < i \le N_1$, $a_i > 0$ for $N_1 < i \le N_2$ and $a_i < 0$ for $N_2 < i \le N$, where a_i are the coefficients from (3.1.9). For the construction, we may assume $N_1 = 0$. We define *extension data* as the following ordered collection of sections

$$\bowtie = \Big((s_{i,k})_{\substack{i\in\{1,\dots N_2\} \\ 1\le k\le a_i}}, (t_{j,l})_{\substack{j\in\{N_2+1,\dots N\} \\ 1\le l\le -a_j}} \Big) \,, \quad s_{i,k} : \mathcal{O}_Y \to \mathcal{O}_Y(D_i) \,, \quad t_{j,l} : \mathcal{O}_Y \to \mathcal{O}_Y(D_j) \,.$$

such that $\prod_{\substack{i\in\{1,\dots N_2\}\\ 1\le k\le a_i}} s_{i,k} \prod_{\substack{j\in\{N_2+1,\dots,N\}\\ 1\le l\le -a_j}} (t_{j,l})^{-1} = \Omega$ and $s_{i,k}$, $t_{j,k}$ are holomorphic with zeros only on D_i, resp. D_j.

This leads to a definition of a new $\mathbb{Z}_2$-bundle:

Definition 3.1.16. On $\mathcal{M}^{\mathrm{sp}}_{Y,D}$ we have the line bundle

$$\mathcal{L}_{Y,D} = \pi_1^*\Lambda_{\underline{0}} \otimes (\pi_2^*\Lambda_{\underline{0}})^*\,, \tag{3.1.13}$$

where $\mathcal{M}_Y \xleftarrow{\pi_1} \mathcal{M}^{\mathrm{sp}}_{Y,D} \xrightarrow{\pi_2} \mathcal{M}_Y$ are the natural projections.

For a fixed choice $\bowtie$, there is a natural isomorphism

$$\begin{aligned}\vartheta^{\mathrm{sp}}_{\bowtie} : \mathcal{L}_{Y,D} &\cong \pi_1^*\Lambda_{\underline{0}} \otimes \pi_1^*\circ\rho^*(\Lambda_D)\otimes \pi_1^*\circ\rho^*(\Lambda_D)^*\otimes(\pi_2^*\Lambda_{\underline{0}})^* \\ &\cong \pi_1^*\Lambda_{\underline{k}}\otimes\pi_2^*(\Lambda_{\underline{k}})^* \cong \pi_1^*(\Lambda_{\underline{0}})^*\otimes\pi_2^*(\Lambda_{\underline{0}}) \cong \mathcal{L}^*_{Y,D}\,,\end{aligned} \tag{3.1.14}$$

Here $\Lambda_D \to \mathcal{M}^{\mathrm{sp}}_D$ are line bundles and we used the commutativity of

$$\begin{array}{ccc}\mathcal{M}^{\mathrm{sp}}_{Y,D} & \xrightarrow{\pi_2} & \mathcal{M}_Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \rho} \\ \mathcal{M}_Y & \xrightarrow{\rho} & \mathcal{M}^{\mathrm{sp}}_D\end{array}$$

in the first step. The bundles $\Lambda_D = \Lambda^*_{D,-}\Lambda_{D,+}$ appear as the result of using chosen $s_{i,k}$ to construct isomorphism (3.1.11) for the first N_2 divisors, then $\Lambda_{D,-}$ is obtained from using $t_{j,k}^{-1}$ and (3.1.11). Thus we will have the expressions:

$$\begin{aligned}\Lambda_{D,+} =&\Lambda_{\underline{k}-\sum_{i=1}^{N_2-1} a_i e_i+(a_{N_2}-1)e_{N_2},N_2} \cdots \otimes \Lambda_{(\underline{k}-\sum_{i=1}^{N_2-1} a_i e_i),N_2} \otimes \cdots \\ &\otimes \Lambda_{\underline{k}-(a-1)e_1,1} \otimes \ldots \otimes \Lambda_{\underline{k},1} \\ \Lambda_{D,-} =&\Lambda_{\underline{k}-\sum_{i=1}^{N_2} a_i e_i,N_2+1} \otimes \ldots\ldots \otimes \Lambda_{-e_N,N} .\end{aligned}$$

The second to last step uses (3.1.5). We define the $\mathbb{Z}_2$-bundles by using Definition 3.1.3:

$$\begin{aligned}\vartheta_{\bowtie}^{\mathrm{sp}} : \mathcal{L}_{Y,D} \to&(\mathcal{L}_{Y,D})^* , \qquad O_{\mathrm{sp}}^{\bowtie} \to \mathcal{M}_{Y,D}^{\mathrm{sp}} , \\ &O^{\bowtie} = \mathrm{sp}^*\big(O_{\mathrm{sp}}^{\bowtie}\big)\end{aligned} \tag{3.1.15}$$

where $O_{\mathrm{sp}}^{\bowtie}$ associated to $\vartheta_{\bowtie}^{\mathrm{sp}}$.

The important property of the $\mathbb{Z}_2$-bundle $O^{\bowtie}$ is that it is going to allow us to use index theoretic excision on the side of gauge theory to prove its triviality. One should think of the triples $[E, F, \phi]$ which are the spec(A)-points in $\mathcal{M}_{Y,D}$ as similar objects to the relative pairs in [172, Definition 2.5] with identification given in some neighborhood of the divisor D. The $\mathbb{Z}_2$-bundle $O^{\bowtie}$ only cares about the behavior of the complexes in X.

Definition 3.1.17. Recall that from Definition 3.1.6 we have the maps Γ_Y : $(\mathcal{M}_Y)^{\mathrm{top}} \to \mathcal{C}_Y$ and $\Gamma_D : (\mathcal{M}^D)^{\mathrm{top}} \to \mathcal{C}_D$, We define Γ as the composition

$$\begin{aligned}&(\mathcal{M}_{Y,D})^{\mathrm{top}} \longrightarrow (\mathcal{M}_Y)^{\mathrm{top}} \times^h_{(\mathcal{M}^D)^{\mathrm{top}}} (\mathcal{M}_Y)^{\mathrm{top}} \\ &\longrightarrow \mathcal{C}_Y \times^h_{\mathcal{C}_D} \mathcal{C}_Y \simeq \mathcal{C}_Y \times_{\mathcal{C}_D} \mathcal{C}_Y = \mathcal{C}_{Y,D} .\end{aligned} \tag{3.1.16}$$

The first map is induced by the homotopy commutative diagram obtained from applying $(-)^{\text{top}}$ to the Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_{Y,D} & \longrightarrow & \mathcal{M}_Y \\ \downarrow & & \downarrow \\ \mathcal{M}_Y & \longrightarrow & \mathcal{M}^D \end{array} .$$

The second map uses homotopy commutativity of

$$\begin{array}{ccccc} (\mathcal{M}_Y)^{\text{top}} & \longrightarrow & (\mathcal{M}^D)^{\text{top}} & \longleftarrow & (\mathcal{M}_Y)^{\text{top}} \\ \downarrow{\scriptstyle \Gamma_Y} & & \downarrow{\scriptstyle \Gamma_D} & & \downarrow{\scriptstyle \Gamma_Y} \\ \mathcal{C}_Y & \longrightarrow & \mathcal{C}_D & \longleftarrow & \mathcal{C}_Y \end{array} .$$

The final homotopy equivalence is the result of the map $(\text{inc}_D)^{\text{an}} : (D)^{\text{an}} \to Y^{\text{an}}$ being a cofibration for the standard model structure on **Top**. The map $\mathcal{C}_Y \to \mathcal{C}_D$ is a fibration so the homotopy fiber-product is given by the strict fiber-product up to homotopy equivalences.

We now state the theorem which follows from Proposition 3.3.15 below and is the main tool in proving orientability of $\mathcal{M}_X$.

Theorem 3.1.18. *For X, Y and D fix $\mathfrak{ord}$ and the extension data $\bowtie$ as in Definition 3.1.16, then the $\mathbb{Z}_2$-bundle*

$$O^{\bowtie} \to \mathcal{M}_{Y,D} \tag{3.1.17}$$

is trivializable. Let $D_O^{\mathcal{C}} \to \mathcal{C}_{Y,D}$ be the trivializable $\mathbb{Z}_2$-bundle from (3.3.1)*, then there exists a canonical isomorphism*

$$\mathfrak{I}^{\bowtie} : \Gamma^*(D_O^{\mathcal{C}}) \cong (O^{\bowtie})^{\text{top}} . \tag{3.1.18}$$

We now reinterpret this result to apply it to the orientation bundle of interest $O^\omega \to \mathcal{M}_X$.

Definition 3.1.19. Let $\zeta : \mathcal{M}_X \to \mathcal{M}_{Y,D}$ be the open embedding of stacks mapping $[E] \mapsto ([i_{X\,*}E], 0)$. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_X^{\mathrm{top}} & \xrightarrow{\xi_Y^{\mathrm{top}}} & \mathcal{M}_{Y,D} \\ \big\downarrow{\scriptstyle \Gamma^{\mathrm{cs}}} & & \big\downarrow{\scriptstyle \Gamma} \\ C_X^{\mathrm{cs}} & \xrightarrow{i_{\mathrm{cs}}} & \mathcal{C}_{Y,D}\,, \end{array} \tag{3.1.19}$$

where

$$\kappa^{\mathrm{cs}} : \mathcal{C}_X^{\mathrm{cs}} = \mathcal{C}_Y \times_{\mathcal{C}_D} \{0\} \hookrightarrow \mathcal{C}_{Y,D},$$

and $\mathcal{C}_Y \times_{\mathcal{C}_D} \{0\} = \mathrm{Map}_{C^0}\big((X^+, +), (BU \times \mathbb{Z}, 0)\big)$. The space $C_X^{\mathrm{cs}} = \mathrm{Map}_{C^0}\big((X^+, +), (BU \times \mathbb{Z}, 0)\big)$ is the classifying space of compactly supported K-theory on X (see Spanier [159], Ranicki–Roe [149, §2], May [132, Chapter 21]): $\pi_0(C_X^{\mathrm{cs}}) := K^0_{\mathrm{cs}}(X)$. We define

$$O^{\mathrm{cs}} := (\kappa^{\mathrm{cs}}\,)^*(D_O^{\mathcal{C}})\,. \tag{3.1.20}$$

The following is the first important consequence of Theorem 3.1.18 and leads to the construction of virtual fundamental classes using Borisov–Joyce [23] or Oh–Thomas [141] using the -2-shifted symplectic structures of §2.2. It also gives preferred choices of orientations at fixed points up to a global sign when defining/computing invariants using localization as in [141], [37] for moduli spaces of compactly supported sheaves M_α with a fixed K-theory class $\alpha \in K^0_{\mathrm{cs}}$ and for a given compactification Y.

Theorem 3.1.20. *Let (X, Ω) be a Calabi–Yau fourfold then $\mathbb{Z}_2$-bundle $O^\omega \to \mathcal{M}_X$*

is trivializable. Moreover, for a fixed choice of embedding $X \to Y$, with $D = Y \backslash X$ strictly normal crossing, there exists a canonical isomorphism

$$\mathfrak{I} : (\Gamma^{\mathrm{cs}}_X)^*(O^{\mathrm{cs}}) \cong (O^{\omega})^{\mathrm{top}} .$$

Proof. We prove this in 3 steps:

1. We have a natural isomorphism $\xi_Y^*(O^{\bowtie}) \cong O^{\omega}$: Consider a spec$(A)$-point $[E] \in \mathcal{M}_X$, $\tilde{E} = (\mathrm{id} \times i_X)_*(E)$, then at the corresponding spec(A)-point $([\tilde{E}], 0) \in \mathcal{M}_{Y,D}$, the isomorphism $\vartheta_{\bowtie}$ is given by

$$\det(\underline{\mathrm{Hom}}(\tilde{E}, \tilde{E})) \otimes \mathbb{C} \cong \det{}^*(\underline{\mathrm{Hom}}(\tilde{E}, \tilde{E} \otimes K_Y)) \cong \det{}^*(\underline{\mathrm{Hom}}(\tilde{E}, \tilde{E})) ,$$

where we use that $\Lambda_L|_{[0]} \cong \mathbb{C}$, the first isomorphism is Serre duality and the second one is the composition of isomorphisms induced by $E \xrightarrow{s_{i,k}} E(D_i)$ and $E(-D_j) \xrightarrow{t_{j,l}} E$. As E is compactly supported in X, these isomorphisms compose into $E \xrightarrow{\Omega} E \otimes K_Y$ by the assumption on $\bowtie$. Therefore $\vartheta_{\bowtie}|_{([\tilde{E}],0)}$ coincides with $i^{\omega}|_{[E]}$ and their associated $\mathbb{Z}_2$-bundles are identified.

2. By Lemma 3.3.8, we know O^{cs}, O^{ω} are independent of choice of $\bowtie$. We define a family of $\mathfrak{I}(\bowtie)$ for fixed $\mathfrak{ord}$:

$$\mathfrak{I}(\bowtie) : (\Gamma^{\mathrm{cs}}_Y)^*(O^{\mathrm{cs}}) \overset{(3.1.19)}{\cong} (\zeta^{\mathrm{top}})^* \circ \Gamma^*(D^{\mathcal{C}}_O) \cong (\zeta^{\mathrm{top}})^* \circ (O^{\bowtie})^{\mathrm{top}} \overset{1.}{\cong} (O^{\omega})^{\mathrm{top}} .$$

Any two choices of $s_{i,k}$ differ by $\mathbb{C}^*$ (and same holds for $t_{j,l}$). Therefore the set of $\bowtie$ corresponds to $(\mathbb{C}^*)^{\sum_i^N |a_i| - 1}$ which is connected and $\mathfrak{I}(\bowtie)$ does not depend on $\bowtie$.

3. For simplicity, let us assume we only have two different divisors D_1, D_2. We

then have the isomorphism obtained from applying (3.1.11) twice

$$\pi_1^*\big(\Lambda_{e_1+e_2}\big) \cong \pi_1^*(\Lambda_{\underline{0}})(\rho_2 \circ \pi_1)^*\Lambda_{e_2,2}(\rho_1 \circ \pi_1)^*\Lambda_{e_1+e_2,1}\,, \tag{3.1.21}$$

$$\pi_1^*\big(\Lambda_{e_1+e_2}\big) \cong \pi_1^*(\Lambda_{\underline{0}})(\rho_1 \circ \pi_1)^*\Lambda_{e_1,1}(\rho_2 \circ \pi_1)^*\Lambda_{e_1+e_2,2}\,, \tag{3.1.22}$$

$$\pi_2^*\big(\Lambda_{e_1+e_2}\big) \cong \pi_2^*(\Lambda_{\underline{0}})(\rho_2 \circ \pi_2)^*\Lambda_{e_2,2}(\rho_1 \circ \pi_2)^*\Lambda_{e_1+e_2,1}\,, \tag{3.1.23}$$

$$\pi_2^*\big(\Lambda_{e_1+e_2}\big) \cong \pi_2^*(\Lambda_{\underline{0}})(\rho_1 \circ \pi_2)^*\Lambda_{e_1,1}(\rho_2 \circ \pi_2)^*\Lambda_{e_1+e_2,2}\,. \tag{3.1.24}$$

To show that there is no difference between the chosen two orders, we use the commutative diagram, where all rows and columns and rows fit into distinguished triangles:

$$\begin{array}{ccccc}
\mathbb{P} & \longrightarrow & \mathbb{P}_{e_1} & \longrightarrow & \rho_1^*\big(\mathbb{P}_{e_1,1}\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}_{e_2} & \longrightarrow & \mathbb{P}_{e_1+e_2} & \longrightarrow & \rho_1^*\big(\mathbb{P}_{e_1+e_2,1}\big) \\
\downarrow & & \downarrow & & \downarrow \\
\rho_2^*\big(\mathbb{P}_{e_2,2}\big) & \longrightarrow & \rho_2^*(\mathbb{P}_{e_1+e_2,2}) & \longrightarrow & \rho_{1,2}^*\big(\mathbb{P}_{e_1+e_2,1,2}\big)
\end{array}$$

We used in the bottom right corner term the restriction $\rho_{1,2} : \mathcal{M}_Y \to \mathcal{M}_{D_1 \cap D_2}$ and $\mathbb{P}_{e_1+e_2,1,2} = \mathbb{P}_{\mathcal{O}(D_1+D_2)|_{D_1\cap D_2}}$. Taking determinants of all four corner terms of the diagram (see Knudsen–Mumford [104, Prop. 1]) and pulling back by π_1, we get both (3.1.21) and (3.1.22) where the latter comes with $(-1)^{\deg\left((\rho_{D_2}\circ\pi_1)^*\Lambda_{\mathcal{O}(D_1|_{D_1})}\right)\deg\left((\rho_1\circ\pi_1)^*\Lambda_{\mathcal{O}(D_2|_{D_2})}\right)}$. This holds also for (3.1.23), (3.1.24). By commutativity of the diagram, we see that choosing the step (3.1.21), (3.1.23) or (3.1.22), (3.1.24) we obtain the same as the signs cancel. As this permutes any two divisors, we obtain independence in the general case. From Lemma 3.3.8, Proposition 3.3.7 and Proposition 3.3.15, $D_O^{\mathcal{C}}$ and $\mathfrak{I}$ are independent of the order. Note that this should be all considered under pull-back by $\mathrm{sp} : \mathcal{M}_{Y,D} \to \mathcal{M}_{Y,D}^{\mathrm{sp}}$ to have natural isomorphism independent of choices on the smooth intersection $D_1 \cap D_2$:

$$\mathrm{sp}^*\big(\pi^* \circ \rho_{1,2}^*(\mathbb{P}_{e_1+e_2,1,2})\big) \cong \mathrm{sp}^*\big(\pi_2^* \circ \rho_{1,2}^*(\mathbb{P}_{e_1+e_2,1,2})\big)\,.$$

□

Let us discuss another straight-forward consequence of the framework used in Theorem 3.1.18. For (X, Ω) a quasi-projective Calabi–Yau fourfold, let M be a moduli scheme of stable pairs $\mathcal{O}_X \to F$ where F is compactly supported (see [39, 36, 42, 96, 164]) or ideal sheaves of proper subvarieties. To make sense out of Serre duality, generalizing the approach in Kool–Thomas [108, §3] and Maulik–Pandharipande–Thomas [129, §3.2], we choose a compactification Y as in Theorem 3.1.18.

Definition 3.1.21. Let $\mathcal{E} = (\mathcal{O}_{X\times M} \to \mathcal{F}) \to X \times M$ be the universal perfect complex on M. Using the inclusion $i_X : X \to Y$ we obtain the universal sheaf $\bar{\mathcal{E}} = \big(\mathcal{O}_M \to (i_X \times \mathrm{id}_M)_*(\mathcal{F})\big) \to Y \times M$. We have the following isomorphism, where $(-)_0$ denotes the trace-less part:

$$\begin{aligned} i_M^\omega : \det\big(\underline{\mathrm{Hom}}_M(\mathcal{E},\mathcal{E})_0\big) &\cong \det\big(\underline{\mathrm{Hom}}_M(\bar{\mathcal{E}},\bar{\mathcal{E}})_0\big) \cong \det{}^*\big(\underline{\mathrm{Hom}}_M(\bar{\mathcal{E}},\bar{\mathcal{E}}\otimes L_{\underline{k}})_0\big) \\ &\overset{\kappa}{\cong} \det{}^*\big(\underline{\mathrm{Hom}}_M(\bar{\mathcal{E}},\bar{\mathcal{E}})_0\big) \cong \det{}^*\big(\underline{\mathrm{Hom}}_M(\mathcal{E},\mathcal{E})_0\big)\,, \end{aligned}$$

where κ is constructed using the isomorphisms

$$\underline{\mathrm{Hom}}_M(\bar{\mathcal{E}},\bar{\mathcal{E}}\otimes L_{\underline{a}})_0 \cong \underline{\mathrm{Hom}}_M(\bar{\mathcal{E}},\bar{\mathcal{E}}\otimes L_{\underline{a}-e_i})_0 \tag{3.1.25}$$

in each step determined by $\bowtie$ as in Definition 3.1.16. The orientation bundle $O_M^\omega \to$

M is defined as the square root $\mathbb{Z}_2$-bundle associated with i^{ω}_M.

Let $\bar{\mathcal{M}}$ be a moduli stack of stable pairs or ideal sheaves on Y of subvarieties with proper support in X with the projection $\pi_{\mathbb{G}_m} : \bar{\mathcal{M}} \to M$ which is a $[*/\mathbb{G}_m]$ torsor. We have an inclusion $\eta : \bar{\mathcal{M}} \to \mathcal{M}_{Y,D}$ given on spec(A)-points by mapping $[\bar{\mathcal{E}}] \mapsto ([\bar{\mathcal{E}}, \mathcal{O}_Y])$.

The following result leads to the construction of virtual fundamental classes when M is compact (assuming one believes the existence of shifted symplectic structures on pairs as in Preygel [147] or Bussi [28]) and preferred choices of orientations up to a global sign at fixed points when using localization for a fixed K-theory class $[\![\mathcal{O}_X]\!] + \alpha$ for $\alpha \in K^0_{\mathrm{cs}}(X)$ and a choice of compactification Y.

Theorem 3.1.22. *Let (X, Ω) be a quasi-projective Calabi–Yau fourfold and let Y be its compactification as in Theorem 3.1.18. Let $O^{\omega}_M \to M$ be the orientation bundle from Definition 3.1.21 for M a moduli scheme of stable pairs or ideal sheaves of proper subschemes of X. There is a canonical isomorphism of $\mathbb{Z}_2$-bundles*

$$\pi^*_{\mathbb{G}_m}(O^{\omega}_M) \cong \eta^*(O^{\bowtie}) \,.$$

In particular, $O^{\omega}_M \to M$ is trivializable.

Proof. The universal perfect complex $\mathcal{E}_{\bar{\mathcal{M}}}$ on $\bar{\mathcal{M}}$ is given by $(\mathrm{id}_Y \times \pi_{\mathbb{G}_m})^*(\bar{\mathcal{E}})$. We have:

$$\gamma : \det\big(\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}(\mathcal{E}_{\bar{\mathcal{M}}}, \mathcal{E}_{\bar{\mathcal{M}}})_0\big) \cong \det\big(\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}(\mathcal{E}_{\bar{\mathcal{M}}}, \mathcal{E}_{\bar{\mathcal{M}}})\big)\det{}^*\big(\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}(\mathcal{O}, \mathcal{O})\big) \cong \eta^*(\mathcal{L}_{Y,D}) \,,$$

such that the following diagram of isomorphism commutes:

$$\begin{array}{ccc}
\eta^*(\mathcal{L}_{Y,D}) & \xrightarrow[\eta^*(\vartheta^{\bowtie})]{\sim} & \eta^*(\mathcal{L}_{Y,D})^* \\
\sim\big\uparrow\gamma & & \gamma^{-*}\big\downarrow\sim \\
\det\big(\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}})_0\big) & \xrightarrow[\pi^*_{\mathbb{G}_m}(i^{\omega}_M)]{\sim} & \det^*\big(\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}})_0\big)^*
\end{array}$$

which follows from the commutativity of

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}}\otimes L_{\underline{a}-e_i}\big)_0 & \longrightarrow & \underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}}\otimes L_{\underline{a}-e_i}\big) & \xrightarrow{\mathrm{tr}} & \underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{O},\mathcal{O}\otimes L_{\underline{a}-e_i}\big) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}}\otimes L_{\underline{a}}\big)_0 & \longrightarrow & \underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}}\otimes L_{\underline{a}}\big) & \xrightarrow{\mathrm{tr}} & \underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{O},\mathcal{O}\otimes L_{\underline{a}}\big) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{E}_{\bar{\mathcal{M}}},\mathcal{E}_{\bar{\mathcal{M}}}\otimes L_{\underline{a},i}\big) & \xrightarrow{\mathrm{tr}} & \underline{\mathrm{Hom}}_{\bar{\mathcal{M}}}\big(\mathcal{O},\mathcal{O}\otimes L_{\underline{a},i}\big)
\end{array}$$

in each step (3.1.25). As a result, the $\mathbb{Z}_2$-bundles associated to these are canonically isomorphic and we apply Theorem 3.1.18. □

3.2 Some technical tools

In this section, we review and develop further the necessary language for working with orientations. This includes developing an excision principle for complex determinant line bundles generalizing the work of Upmeier [172], Donaldson [47], [49] and Atiyah–Singer [8].

3.2.1 Topological stacks

The definition of a topological stack follows at first the standard definition of stacks over the standard site of topological spaces. It can be found together with all basic results in Noohi [138] and Metzler [133], the homotopy theory of topological stacks is developed by Noohi in [139]. For each groupoid of topological spaces $[G \rightrightarrows X]$, one defines a prestack $\lfloor X/G \rfloor$, such that the objects of the groupoid $\lfloor X/G \rfloor(W)$ correspond to the continuous maps $W \to X$ for any $W \in \mathrm{Ob}(\mathbf{Top})$. The morphisms between $\alpha : W \to X$ and $\beta : W \to X$ correspond to $\lambda : W \to G$ which are mapped respectively to α and β under the two maps $G \rightrightarrows X$. One also defines $[X/G]$ as the stack associated to this prestack. The following result makes working with topological stacks much easier.

Proposition 3.2.1 (Noohi [138, p.26]). *Every topological stack $\mathcal{X}$ has the form of an associated stack $[X/G]$ for some topological groupoid $[G \rightrightarrows X]$. The canonical map $X \to [X/G]$ gives a chart of $\mathcal{X}$. Conversely $[X/G]$ associated to any groupoid is a topological stack.*

Remark 3.2.2. The definition of a topological stack given in [138] is more complicated and depends on the choice of a class of morphisms called local fibrations (LF). Instead, we are using Noohi's definition of topological stacks from [139] which corresponds to pretopological stacks in [138].

In [139], Noohi proposes a homotopy theory for a class of topological stacks called *hoparacompact*. Let $\mathbf{tSt_{hp}}$ denote the 2-category of hoparacompact topological stacks. A *classifying space* of $\mathcal{X}$ in $\mathbf{tSt_{hp}}$ is a topological space $X = \mathcal{X}^{cla}$ with a representable map $\pi^{\mathrm{cla}} : X \to \mathcal{X}$ such that for any $T \to \mathcal{X}$, where T is a topological space, its base change $T \times_{\mathcal{X}} X \to T$ is a weak homotopy equivalence.

In [139, §8.1], Noohi provides a functorial construction of the classifying space $\mathcal{X}^{\mathrm{cla}}$ for every hoparacompact topological stack $\mathcal{X}$, such that the resulting space is paracompact. In fact, [139, Corollary 8.9] states that the functor

$$(-)^{\mathrm{cla}} : \mathrm{Ho}(\mathbf{tSt_{hp}}) \longrightarrow \mathrm{Ho}(\mathbf{pTop})$$

is an equivalence of categories, where **pTop** denotes the category of paracompact topological spaces. Note that as, we are interested in comparing $\mathbb{Z}_2$-bundles, it is enough to restrict to finite CW complexes and weak homotopy equivalences are replaced by usual ones avoiding the question of ghost maps.

3.2.2 Moduli stack of connections and their $\mathbb{Z}_2$-graded H-principal $\mathbb{Z}_2$-bundles

Let X be a smooth connected manifold of dimension n and $\pi : P \to X$ be a principal G bundle for a connected Lie group G with the Lie algebra $\mathfrak{g}$. Recall from §2.1 that we have the action of $\mathcal{G}_P$ on $\mathcal{A}_P$. This action will be continuous and the spaces are paracompact because they are infinity CW-complexes, so we get a hoparacompact stack $\mathcal{B}_P = [\mathcal{A}_P/\mathcal{G}_P]$.

If X is a compact spin Kähler fourfold, let S_+, S_- denote the positive and negative spinor bundles and $\not{D} : S_+ \to S_-$ the positive Dirac operator, then for each connection $\nabla_P \in \mathcal{A}_P$ one can define the *twisted Dirac operator*

$$\not{D}^{\nabla_{\mathrm{ad}(P)}} : \Gamma^\infty(\mathrm{ad}(P) \otimes S_+) \to \Gamma^\infty(\mathrm{ad}(P) \otimes S_-)\,. \tag{3.2.1}$$

This induces an $\mathcal{A}_P$ family of real elliptic operators and therefore gives by §3.2.3 a real line bundle $\det^{\not{D}}_P \to \mathcal{A}_P$. Because $\mathcal{G}_P$ maps the kernel of (3.2.1) to the kernel and same for the cokernels, this $\mathbb{R}$-bundle is $\mathcal{G}_p$ equivariant and descends to an $\mathbb{R}$-bundle on $\mathcal{B}_P$. The orientation bundle of which we denote by $O^{\not{D}}_P \to \mathcal{B}_P$. One takes the unions over all isomorphism classes of $U(n)$-bundles for all n:

$$\mathcal{B}_X = \coprod_{[P]} \mathcal{B}_P \,, \qquad O^{\not{D}} = \coprod_{[P]} O^{\not{D}}_P \,. \tag{3.2.2}$$

These are the *orientation bundles* of Joyce–Tanaka–Upmeier [97] and Cao–Gross–Joyce [30]. For the proof of Theorem 3.1.18, we will rely on the properties of special principal $\mathbb{Z}_2$-bundles under homotopy-theoretic group completion of H-spaces. For background on H-spaces, see Hatcher [80, §3.C], May–Ponto [132, §9.2] and Cao-Gross–Joyce [30, §3.1]. Recall that an *(admissible) H-space* is a triple (X, e_X, μ_X) of a topological space X, its point $e_x \in X$ and a continuous map $\mu_X : X \times X \to X$ is called an *H-space*, if it induces a commutative monoid in Ho(**Top**). An admissible H-space X is *group-like* if $\pi_0(X)$ is a group. Note that there are many ways how to include higher homotopies into the theory of H-spaces. For A^n-spaces see Stasheff [160] and [161]. For E^∞-spaces see May [131], for Γ-spaces see Segal [157]. While E^∞-spaces and Γ-spaces are roughly the same, A^∞ spaces do not require commutativity. All our spaces fit into these frameworks which by [97, Example 2.19] give us additional control over the $\mathbb{Z}_2$-bundles on them. One also defines H-maps as the obvious maps in the category of H-spaces. We use the notion of *homotopy-theoretic group completions* from May [131, §1]. One has the following universality result for homotopy theoretic group completion, that we will use throughout.

Proposition 3.2.3 (Caruso–Cohen–May–Taylor [43, Proposition 1.2])**.** *Let $f : X \to Y$ be a homotopy-theoretic group-completion. If $\pi_0(X)$ contains a countable cofinal* *sequence, then for each weak H-map $g : X \to Z$, where Z is group-like, there exists a weak H-map $g' : Y \to Z$ unique up to weak homotopy equivalence, such that $g' \circ f$ is weakly homotopy equivalent to g.*

Note that weak H-maps correspond to relaxing the commutativity to hold only up to weak homotopy equivalences. We will again not differentiate between the two. Let us now merge the definition of $\mathbb{Z}_2$-graded commutativity with the notion of H-principal $\mathbb{Z}_2$-bundles of Cao–Gross–Joyce [30].

Definition 3.2.4. Let (X, e_X, μ_X) be an H-space. A $\mathbb{Z}_2$-bundle $O \to X$ together with a continuous map $\deg(O) : X \to \mathbb{Z}_2$ satisfying

$$\deg(O) \circ \mu(x, y) = \deg(O)(x) + \deg(O)(y)$$

is a $\mathbb{Z}_2$*-graded* $\mathbb{Z}_2$*-bundle.* If O_1, O_2 are $\mathbb{Z}_2$-graded then the isomorphism

$$O_1 \otimes_{\mathbb{Z}_2} O_2 \cong O_2 \otimes_{\mathbb{Z}_2} O_1 \,.$$

differs by the sign $(-1)^{\det(O_1)\deg(O_2)}$ from the naive one. Moreover, $O_1 \otimes_{\mathbb{Z}_2} O_2$ has grading $\deg(O_1) + \deg(O_2)$. A pullback of a $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundle, is naturally $\mathbb{Z}_2$-graded. An isomorphism of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundles has to preserve the grading. A *weak H-principal* $\mathbb{Z}_2$*-graded* $\mathbb{Z}_2$*-bundle* on X is a $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundle $P \to X$, such that there exists an isomorphism p of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundles on $X \times X$

$$p : P \boxtimes_{\mathbb{Z}_2} P \to \mu_X^*(P) \,.$$

A $\mathbb{Z}_2$*-graded strong H-principal* $\mathbb{Z}_2$*-bundle* on X is a pair (Q, q) of a trivializable $\mathbb{Z}_2$

graded $\mathbb{Z}_2$-bundle $Q \to X$ and an isomorphism of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$ bundles on $X \times X$

$$q : Q \boxtimes_{\mathbb{Z}_2} Q \to \mu_X^*(Q)\,,$$

such that under the homotopy $h : \mu_X \circ (\mathrm{id}_X \times \mu_X) \simeq \mu_X \circ (\mu_X \times \mathrm{id}_X)$ the following two isomorphisms of the $\mathbb{Z}_2$-bundles on $X \times X \times X$ are identified

$$(\mathrm{id}_X \times \mu_X)^*(q) \circ (\mathrm{id} \times q) : Q \boxtimes_{\mathbb{Z}_2} Q \boxtimes_{\mathbb{Z}_2} Q \to \big(\mu_X \circ (\mathrm{id}_X \times \mu_X)\big)^* Q$$

and

$$(\mu_X \times \mathrm{id}_X)^*(q) \circ (q \times \mathrm{id}) : Q \boxtimes_{\mathbb{Z}_2} Q \boxtimes_{\mathbb{Z}_2} Q \to \big(\mu_X \circ (\mu_X \times \mathrm{id}_X)\big)^* Q\,.$$

The isomorphism $i : (P, p) \to (Q, q)$ of $\mathbb{Z}_2$-graded strong H-principal $\mathbb{Z}_2$-bundles has to solve $\mu_X^* i \circ p = q \circ (i \boxtimes i)$.

Pullbacks of $\mathbb{Z}_2$-graded H-principal $\mathbb{Z}_2$-bundles under H-maps are naturally $\mathbb{Z}_2$-graded H-principal. Including the $\mathbb{Z}_2$-gradedness, we obtain a minor modification of Cao–Gross–Joyce [30, Proposition 3.5].

Proposition 3.2.5. *Let $f : X \to Y$ be a homotopy-theoretic group completion of H-spaces, then for*

(i) a $\mathbb{Z}_2$-graded weak H-principal $\mathbb{Z}_2$-bundle $P \to X$, there exists a unique $\mathbb{Z}_2$-graded weak H-principal $\mathbb{Z}_2$-bundle $P' \to Y$ such that $f^*(P')$ is isomorphic to P.

(ii) a $\mathbb{Z}_2$-graded strong H-principal $\mathbb{Z}_2$-bundle (Q, q) on X, there exists a unique $\mathbb{Z}_2$-graded strong H-principal $\mathbb{Z}_2$-bundle (Q', q') on Y unique up to a canonical isomorphism, such that $(f^*Q', (f \times f)^*q')$ is isomorphic to (Q, q).

Proof. Without the $\mathbb{Z}_2$-graded condition the result is stated in Cao–Gross–Joyce [30, Proposition 3.5]. Then as $\deg(P)$ respectively $\deg(Q)$ can be viewed as additive maps $\pi_0(X) \to \mathbb{Z}_2$ and $\pi_0(Y)$ is a group-completion, there exist unique extensions of the grading. □

We often suppress the maps μ_X and e_X for an H-space X, we also write Q instead of (Q, q) for a strong H-principal $\mathbb{Z}_2$-bundle when q is understood.

Lemma 3.2.6. *Let $O_1, O_2 \to X$ be $\mathbb{Z}_2$-graded strong (resp. weak) H-principal $\mathbb{Z}_2$-bundles. Then $O_1 \otimes_{\mathbb{Z}_2} O_2$ is a $\mathbb{Z}_2$-graded strong (resp. weak) H-principal $\mathbb{Z}_2$-bundle.*

Proof. Let $q_i : O_i \boxtimes O_i \to \mu_X^*(O_i)$ be the isomorphisms from Definition 3.2.4. Then we define

$$\begin{aligned} q :&(O_1 \otimes_{\mathbb{Z}_2} O_2) \boxtimes_{\mathbb{Z}_2} (O_1 \otimes_{\mathbb{Z}_2} O_2) \overset{\text{Def 3.2.4}}{\cong} (O_1 \boxtimes_{\mathbb{Z}_2} O_1) \otimes_{\mathbb{Z}_2} (O_2 \boxtimes_{\mathbb{Z}_2} O_2) \\ &\overset{p_1 \otimes p_2}{\cong} \mu_X^*(O_1) \otimes_{\mathbb{Z}_2} \mu_X^*(O_2) \cong \mu^*(O_1 \otimes_{\mathbb{Z}_2} O_2)\,. \end{aligned}$$

Notice that we get an extra sign $(-1)^{\deg(\pi_1^*(O_2))\deg(\pi_2^*(O_1))}$. To check associativity 3.2.4, we need commutativity of

$$\begin{array}{ccc} (O_1 \otimes O_2) \boxtimes (O_1 \otimes O_2) \boxtimes (O_1 \otimes O_2) & \xrightarrow{(-1)^{\deg(\pi_2^*(O_2))\deg(\pi_3^*(O_1))}} & (O_1 \otimes O_2) \boxtimes \mu_X^*(O_1 \otimes O_2) \\ \Big\downarrow {\scriptstyle (-1)^{\deg(\pi_1^*(O_2)\deg(\pi_2^*(O_1))}} & & \Big\downarrow {\scriptstyle (-1)^{\deg\left(\mu_X^*(\pi_1^*(O_2) \otimes \pi_2^*(O_2)\right)\deg\left(\pi_2^*(O_1)\right)}} \\ \mu_X^*(O_1 \otimes O_2) \boxtimes_{\mathbb{Z}_2} (O_1 \otimes O_2) & \xrightarrow{(-1)^{\deg\left(\mu_X^*(\pi_2^*(O_1) \boxtimes \pi_3 * (O_1)\right)\deg\left(\pi_1^*(O_2)\right)}} & \begin{array}{c} (\mu_X \times \mathrm{id}_X)^* \circ \mu_X^*(O_1 \otimes O_2) \cong \\ \circ(\mathrm{id}_X \times \mu_X)^* \circ \mu_X^*(O_1 \otimes O_2)\,.O_2) \end{array} \end{array}$$

Without the extra signs, it would be commutative because O_i are strong H-principal.

To check the signs note that going down and right, resp. right and down we get

$$
\begin{aligned}
&(-1)^{\deg(\pi_2^*(O_2)\deg(\pi_3^*)(O_1)+(\deg\pi_2^*(O_1)\deg+(\pi_3^*)(O_1))\deg\left(\pi_1^*(O_1)\right)} \\
&= (-1)^{\deg(\deg(\pi_2^*(O_2)\deg(\pi_3^*)(O_1)+(\deg\pi_1^*(O_2)\deg+(\pi_2^*)(O_1))\deg\left(\pi_1^*(O_2)\right)} .
\end{aligned}
$$

□

With the $\mathbb{Z}_2$-grading we need to distinguish between duals of strong H-principal $\mathbb{Z}_2$-bundles.

Definition 3.2.7. Let (O, p) be a strong H-principal $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundle. Its *dual* (O^*, p^*) will be defined to be a strong H-principal $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundle, such that as $\mathbb{Z}_2$-bundles $O^* = O$ and the isomorphism

$$
p^* : O^* \boxtimes_{\mathbb{Z}_2} O^* \xrightarrow{\sim} \mu_X^*(O^*) ,
$$

is given by $p^* = (-1)^{\deg(\pi_1^*(O))\deg(\pi_2^*(O))}p$, where π_1, π_2 are the projections $X \times X \to X$.

Example 3.2.8. An example of an H-space is the topological space $(\mathcal{B}_X)^{\mathrm{cla}}$, where the multiplication $\mu_{\mathcal{B}_X} : \mathcal{B}_X \times \mathcal{B}_X \to \mathcal{B}_X$ is given by mapping $([\nabla_P], [\nabla_Q] \mapsto [\nabla_P \oplus \nabla_Q])$, and we take $(\mu_{\mathcal{B}_X})^{\mathrm{cla}} : (\mathcal{B}_X)^{\mathrm{cla}} \times (\mathcal{B}_X)^{\mathrm{cla}} \to (\mathcal{B}_X)^{\mathrm{cla}}$. It is $\mathbb{Z}_2$-graded (see [172, 175] for the corresponding grading of real determinant line bundles) in the following sense: Let $[\nabla_P] \in \mathcal{B}_X$, then

$$
\deg(O^{\not{D}_+})([\nabla_P]) = \chi^{\not{D}}(E, E) , \tag{3.2.3}
$$

where E is the $\mathbb{C}^n$ vector bundle associated to the $U(n)$-bundle P and $\chi^{\not{D}}(E, E) = \mathrm{ind}\left(\not{D}^{\nabla_{\mathrm{End}(E)}}\right)$ is the complex index from Definition 3.2.11.

We will need the following formulation of [30, Thm. 1.11]:

Theorem 3.2.9 (Cao–Gross–Joyce [30, Thm 1.11])**.** *Let X be a compact spin manifold of dimension 8, then the $\mathbb{Z}_2$-bundle $O^{\not{D}_+} \to \mathcal{B}_X$ are $\mathbb{Z}_2$-graded strong H-principal $\mathbb{Z}_2$-bundles.*

3.2.3 Complex excision

Pseudo-differential operators over $\mathbb{R}^n$ are explained in Hörmander [84]. For background on pseudo-differential operators on manifolds, we recommend Lawson–Michelson [113, §3.3], Atiyah–Singer [8, §5], Donaldson–Kronheimer [49, p. 7.1.1], and Upmeier [172, Appendix A]. We will not review the definition due to its highly analytic nature, as we do not use it explicitly. The excision principle for differential operators was initiated by Seeley [156] and used by Atiyah–Singer [8]. Its refinement to excision for $\mathbb{Z}_2$-bundles was applied by Donaldson [47], Donaldson–Kronheimer [49] and categorified by Upmeier [172]. We use these ideas and extend them to complex determinant line bundles. In author's [19, §3.6, §3.7] this is done slightly differently and in more detail.

From now on we will be assuming that all real bundles come with a choice of a metric and all complex vector bundles with a choice of a hermitian metric. Note that the spaces of metrics are convex and therefore contractible. When we use convex, we automatically mean non-empty.

Let X be a manifold, $E, F \to X$ complex vector bundles, $P : \Gamma^\infty_{\mathrm{cs}}(E) \to \Gamma^\infty(F)$ pseudo-differential operator of degree m, then its symbol $\sigma(P) : \pi^*(E) \to \pi^*(F)$, where $\pi : T^*X \to X$ is the projection map, is a homogeneous of degree m on each fiber of T^*X linear homomorphism. One says that P is *elliptic*, when its symbol $\sigma(P)$

is an isomorphism outside of the zero section $X \subset T^*X$.

We will be working with continuous families of symbols and pseudo-differential operators as defined in [8, p. 491] or as in Upmeier [172, Appendix]. For a topological space M, we denote the corresponding set of elliptic pseudo-differential M-families by $\Psi_m(E, F; M)$ and the elliptic symbol M-families by $S_m(E, F; M)$ with the map

$$\sigma : \Psi_m(E, F; M) \to S_m(E, F; M)\,. \tag{3.2.4}$$

It is compatible with respect to addition, scalar multiplication, composition and taking duals (see [8, §5] for details). It is standard to restrict to degree 0 operators and symbols using

$$\begin{array}{ccc} P & \xrightarrow{\quad\sigma\quad} & \sigma(P) \\ \big\downarrow{\scriptstyle \Psi_0} & & \big\downarrow{\scriptstyle S_0} \\ (1+PP^*)^{-\frac{1}{2}}P & \xrightarrow{\quad\sigma\quad} & (\sigma(P)\sigma(P)^*)^{-\frac{1}{2}}\sigma(P) \end{array}.$$

If X is compact then each $P \in \Psi_m(E, F; M)$ gives an M-family of Fredholm operators between Hilbert spaces containing $\Gamma^\infty_{\mathrm{cs}}(E)$ and $\Gamma^\infty(E)$ such that $\ker(P)$ and $\mathrm{coker}(P)$ lie in $\Gamma^\infty(E_0)$ and $\Gamma^\infty(E_1)$ respectively.

Let P be a continuous Y-family of Fredholm operators $P_y : H_0 \to H_1$ for each y, where H_i are Hilbert spaces. Determinant line bundle $\det(P) \to Y$ of P is defined in Zinger [175] using stabilization (in this case one only needs the H_i to be Banach spaces) and in Upmeier [172, Definition 3.4], Freed [54] or Quillen [148]. We will use the conventions from [172, Definition 3.4].

Definition 3.2.10 (Phillips [146])**.** Let $P : H \to H$ be a self adjoint Fredholm operator on the Hilbert space H. The *essential spectrum* $\mathrm{spec}_{\mathrm{ess}}(P)$ is the set $\lambda \in \mathbb{R}$, such that $P - \lambda\mathrm{Id}$ is not Fredholm. We denote by $\mathrm{spec}(P)$ the spectrum of P. For each $\mu > 0$, such that $\pm\mu \notin \operatorname{spec}(P)$ and $(-\mu,\mu) \cap \operatorname{spec}_{\mathrm{ess}}(P) = \emptyset$, one defines $V_{(-\mu,\mu)}(P) \subset H$ as the subspace of eigenspaces of P for eigenvalues $-\mu < \lambda < \mu$. If P is positive semi-definite, we will also write $V_{[0,\mu)}(P)$. If P is skew adjoint, we will also denote the set of its eigenvalues by $\operatorname{spec}(P)$ (note that $\operatorname{spec}(P) = i\operatorname{spec}(-iP)$).

For a Y family of self adjoint Fredholm operators, one can choose $\mathfrak{U} \subset Y$ sufficiently small and μ from Definition 3.2.10, such that $V_{(-\mu,\mu)}(P)$ becomes a vector bundle on $\mathfrak{U}$. This can be used to define topology on the union of determinant lines

$$\det(P_y) = \det(P_y) \otimes \det(P_y^*)^*\,, y \in Y\,.$$

as in [172, Definition 3.4].

Definition 3.2.11. The bundle $\det(P)$ is $\mathbb{Z}_2$-graded with degree $\operatorname{ind}(P)$, where $\operatorname{ind}(P) = \dim(\operatorname{Ker}(P_y)) - \dim(\operatorname{Ker}(P_y^*)) = \operatorname{ind}(P_y)$. If we have two Y-families P_1 and P_2, then the isomorphism

$$\det(P_1) \otimes \det(P_2) \cong \det(P_2) \otimes \det(P_1) \tag{3.2.5}$$

differs from the naive one by the sign $(-1)^{\operatorname{ind}(P_1)\operatorname{ind}(P_2)}$.

We have the "inverse" of (3.2.4)

$$S_0(E,F,M) \ni p \longmapsto P \in S_0\big(E,F,M \times \sigma^{-1}(p)\big)$$

Which we use to abuse the notation

$$\begin{array}{ccccc} \det(p) & & \det(P_0) & & \det(P) \\ \downarrow & = & \downarrow & & \downarrow \\ Y & & * \times Y & \xrightarrow{i_{P_0}\times \mathrm{id}} & \sigma^{-1}(p) \times Y \end{array}$$

Here $\det(P_0) = (i_{P_0} \times \mathrm{id})^* \det(P)$. Note that as $\sigma^{-1}(p)$ is convex ([172, Theorem 4.6]), for two different choices $P_0, P_1 \in \sigma^{-1}(P)$ we have natural isomorphisms $\det(P_0) \cong \det(P_1)$. Therefore

Lemma 3.2.12. *The complex line bundle* $\det(p)$ *is well-defined up to natural choices of isomorphisms.*

The following lemma is meant for book-keeping purposes.

Lemma 3.2.13. *Let* $p_i \in S_m(E_i, F_i; M)$ *for* $i = 1, 2$ *and* $q \in S_m(E, F, Y \times I)$*.*

(i) (Functoriality.) If $\mu_E : E_1 \to E_2$*,* $\mu_F : F_1 \to F_2$ *are isomorphisms such that*

$$\begin{array}{ccc} \pi^*(E_1) & \xrightarrow{p_1} & \pi^*(F_1) \\ \downarrow{\scriptstyle \pi^*(\mu_E)} & & \downarrow{\scriptstyle \pi^*(\mu_F)} \\ \pi^*(E_2)\pi^* & \xrightarrow{p_2} & \pi^*(F_2) \end{array} \tag{3.2.6}$$

commutes, then there is a natural isomorphism $\det(p_1) \to \det(p_2)$ *.*

(ii) (Direct sums.) There is a natural isomorphism

$$\det(p_1 \oplus p_2) \longrightarrow \det(p_1)\det(p_2)\,. \tag{3.2.7}$$

(iii) (Adjoints.) There is a natural isomorphism

$$\det((p_1)^*) \longrightarrow \det{}^*(p_1)\,. \tag{3.2.8}$$

(iv) *(Triviality.) If $p_1 = \pi^*(\mu)$ for some isomorphism $\mu : E_1 \to F_1$, then there is a natural isomorphism*

$$\det(p^+) \longrightarrow \mathbb{C}\,. \tag{3.2.9}$$

(v) *(Transport.) There is a natural isomorphism* $\det(q)|_{Y\times\{0\}} \cong \det(q)|_{Y\times\{1\}}$*. such that for $q_i \in S_m(E_i, F_i, Y \times I)$ we have the commutative diagram*

$$\begin{array}{ccc}
\det(q_1 \oplus q_2)|_{Y\times\{0\}} & \xrightarrow{(v)} & \det(q_1 \oplus q_2)|_{Y\times\{1\}} \\
\downarrow{\scriptstyle (ii)} & & \downarrow{\scriptstyle (ii)} \\
\det(q_1)|_{Y\times\{0\}} \otimes \det(q_2)|_{Y\times\{0\}} & \xrightarrow{(v)\otimes(v)} & \det(q_1)|_{Y\times\{1\}} \otimes \det(q_2)|_{Y\times\{1\}}
\end{array}$$

Proof. For

(i) make a natural choice of a pair $(P_1, P_2) \in \sigma^{-1}(p_1) \times \sigma^{-1}(p_2)$ commuting with μ_E, μ_F and apply [172, Proposition 3.5 (i)].

(ii) make a natural choice of any $P_1 \times P_2 \in \sigma^{-1}(p_1) \times \sigma^{-1}(p_2)$ in loc cit.

(iii) make a natural choice $P_1 \in \sigma^{-1}(p_1)$ in loc cit.

(iv) make the choice $P_1 = \mu$ in loc cit.

(v) make a contractible choice of $Q_0 \in \sigma^{-1}(q)$, then we have natural isomorphism τ such that $\tau_t : \det(Q_0)|_{Y\times\{0\}} \cong \det(Q_0)|_{Y\times\{t\}}$ for all $t \in I$ and $\tau_0 = \mathrm{id}$, then consider the one for $t = 1$. The commutativity of the diagram follows immediately from the definition.

□

The following definition is the main reason, why we introduced the above concepts.

Definition 3.2.14. Let E_i, F_i be vector bundles on compact manifold X and $p_i \in S_0(E_i, F_i; Y)$. Let $U, V \subset X$ be open, $U \cup V = X$ and $\mu_E : E_1|_U \to E_2|_U$, $\mu_F : F_1|_U \to F_2|_U$ isomorphism, such that

$$\begin{array}{ccc} \pi^*(E_1|_U) & \xrightarrow{p_1|_{T^*U}} & \pi^*(F_1|_U) \\ \Big\downarrow{\scriptstyle \pi^*(\mu_E)} & & \Big\downarrow{\scriptstyle \pi^*(\mu_F)} \\ \pi^*(E_2|_U) & \xrightarrow{p_2|_{T^*U}} & \pi^*(F_2|_U) \end{array} \tag{3.2.10}$$

commutes. Choose a function $\chi \in C^\infty_{\text{cpt}}(V, [0,1])$ with $\chi|_{X \setminus U} = 1$. Then we obtain that:

$$t \in I \longmapsto (p_1, p_2, \mu_E, \mu_F)^\chi_t = \begin{pmatrix} (1 - t + t\chi)p_1 & t(1-\chi)\pi^*\mu_F^* \\ t(1-\chi)\pi^*\mu_E & -(1-t+t\chi)(p_2)^* \end{pmatrix} \tag{3.2.11}$$

is elliptic.

The following result might appear deceptively obvious, but the usual I^2-family argument does not go through.

Lemma 3.2.15. *Let X be compact, E_i, F_i, complex vector bundles on X and $p_i \in S_0(E_i, F_i; Y)$ with isomorphism $\mu_E : E_1 \to E_2$, $\mu_F : F_1 \to F_2$, satisfying* (3.2.10) *on X then we have the commutativity up to contractible choices*

$$\begin{array}{ccc} \det(p_1)\det(p_2^*) & \xrightarrow{\text{Prop. 3.2.13 (ii)}} & \det\big((p_1, p_2, \mu_E, \mu_F)^0_0\big) \\ \Big\downarrow{\scriptstyle \text{Prop. 3.2.13 (iii) + (i)}} & & \Big\downarrow{\scriptstyle \text{Prop. 3.2.13 (v)}} \\ \mathbb{C} & \xleftarrow[\text{Prop. 3.2.13 (iv)}]{} & \det\big((p_1, p_2, \mu_E, \mu_F)^0_1\big) \end{array}$$

Proof. Choose $(P_1, P_2) \in \sigma^{-1}(p_1) \times \sigma^{-1}(p_2)$ commuting with μ_E, μ_F and construct

$$\Psi_t = \begin{pmatrix} (1-t)P_1 & t\mu_F^* \\ t\mu_E & -(1-t)P_2^* \end{pmatrix} \in \sigma^{-1}\big((p_1, p_2, \mu_E, \mu_F)_t^0\big)\,.$$

By composing Ψ_t with $\left(\begin{smallmatrix} 0 & \mu_E^{-1} \\ (\mu_F^*)^{-1} & 0 \end{smallmatrix}\right)$ we obtain

$$\tilde{\Psi}_t = \begin{pmatrix} t\,\mathrm{id} & -(1-t)P^* \\ (1-t)P & t\,\mathrm{id} \end{pmatrix} : E_1 \oplus F_2 \longrightarrow E_1 \oplus F_2\,.$$

Let $\nu \in \mathbb{R}^{>0}$ and $\mathfrak{U} \subset Y$ be chosen sufficiently small as in Upmeier [172, Definition 3.4], such that $V_{[0,\nu)}(\tilde{\Psi}_0^*\tilde{\Psi}_0)$ is a vector bundle.

Notice that $\tilde{\Psi}_t^*\tilde{\Psi}_t = \tilde{\Psi}_t\tilde{\Psi}_t^*$. Moreover, by spectral theorem each non-zero eigenvalue $\lambda^2 \in (0,\nu)$ of $\tilde{\Psi}_0^*\tilde{\Psi}_0$ has multiplicity $2k$ for some positive integer k and then $\tilde{\Psi}_0$ has eigenvalues $i\lambda$, $-i\lambda$ each of multiplicity k in its set of eigenvalues $\mathrm{spec}(\tilde{\Psi}_0)$. The eigenvectors of $\tilde{\Psi}_t^*\tilde{\Psi}_t$ remain the same, but corresponding eigenvalues are $\lambda^2(1-t)^2+t^2$. We therefore define $\nu(t) = \nu(1-t)^2 + t^2$ and we have a natural isomorphism

$$V_{[0,\nu)}(\tilde{\Psi}_0^*\tilde{\Psi}_0) \cong V_{[0,\nu(t))}(\tilde{\Psi}_t^*\Psi_t) \tag{3.2.12}$$

given by the identity for all $t \in I$ (here one extends to $t = 1$ by considering the same finite set of eigenvectors which now have eigenvalue 1), which gives a continuous isomorphism of vector bundles on $\mathfrak{U} \times I$ and restricts to identity for $t = 0$. The isomorphisms of determinant line bundles is then given by

$$\alpha_\nu(t) : \det(\Psi_0) \overset{[172,\text{ Def. }3.4]}{\cong} \det(V_{[0,\nu)}(\Psi_0^*\Psi_0))\det{}^*(V_{[0,\nu)}(\Psi_0\Psi_0^*))$$
$$\overset{(3.2.12)}{\cong} \det(V_{[0,\nu(t))}(\Psi_t^*\Psi_t))\det{}^*(V_{[0,\nu(t))}(\Psi_t\Psi_t^*)) \overset{[172,\text{ Def. }3.4]}{\cong} \det(\Psi_t)\,.$$

We see that this is a representative of the transport Prop. 3.2.13 (v) because it restricts to identity at $t = 1$. To see that this isomorphism is independent of ν, we can restrict to a single point $y \in Y$. Let $\nu' > \nu > 0$, then for $\Psi_0(y)$ choose its diagonalization when restricted to $V_{[0,\nu')}(\Psi_0^*\Psi_0)$. From looking at the definition [172, Definition 3.4] it is then easy to see that

$$\alpha_{\nu'}(t) = \prod_{\substack{\mu\in\mathrm{spec}(\Psi_0)\\ \nu<|\mu|^2<\nu'}} \frac{(1-t)+\mu^{-1}t}{[(1-t)^2+|\mu|^{-2}t^2]^{\frac{1}{2}}}\alpha_\nu(t)\,.$$

As each $\mu = i\lambda$ comes with its conjugate of the same multiplicity, the factor is equal to one. Let $\alpha' : \det(\Psi_0) \cong \det(P)\det(P^*) \cong \mathbb{C}$ be isomorphism combining (3.2.6),(3.2.7) and (3.2.8), then it can be checked in the same way that

$$\alpha_\nu(1) = \prod_{\substack{\mu\in\mathrm{Spec}(\Psi_0)\\ 0<|\mu|<\nu}} \frac{|\mu|^2}{\mu}\alpha'\,,$$

where the factor again becomes one. By covering Y by such sets $\mathfrak{U}_i$ and choosing appropriate ν_i, we can glue the isomorphisms on $\mathfrak{U}_i \times I$, because they coincide on the overlaps $(\mathfrak{U}_i \cap \mathfrak{U}_j) \times I$. Composing $\alpha(t) : \det(\tilde{\Psi}_0) \to \det(\tilde{\Psi}_t)$ with $\begin{pmatrix} 0 & \mu_E^{-1} \\ (\mu_F^*)^{-1} & 0 \end{pmatrix}$, we obtain Prop. 3.2.13 (v) and the commutativity of the diagram. □

Remark 3.2.16. Note that when $p_i \in S_0(E_i, F_i; Y)$ have a real structure and

μ_E, μ_F preserve it, then there exists a natural $\mathbb{Z}_2$-bundle $\mathrm{or}(p_1, p_2, \mu_E, \mu_F)^\chi \subset \det((p_1, p_2, \mu_E, \mu_F)^\chi$ as in Donaldson–Kronheimer [49, §7.1.1] or Upmeier [172] The transport isomorphism of Proposition 3.2.13 (v) for the $Y \times I$ family along $\mathrm{or}(p_1, p_2, \mu_E, \mu_F)$ is canonical, because it is the standard transport along fibers $\mathbb{Z}_2$.

Our main object of study are going to be twisted Dirac operators and Dolbeault operators. Let X be a manifold, P a $U(n)$-principal bundle, V_n a representation of $U(n)$ and E the associated vector bundle, then for a given connection ∇_P on P and its associated connection ∇_E, the twisted operator D^{∇_E} has the degree 0 symbol

$$S_0\big(\sigma(D)\big) \otimes \mathrm{id}_{\pi^*\left(E\right)} =: \sigma_E(D)\,.$$

If $\Phi : V \to W$ is an isomorphism of vector bundles, we will also write $\Phi = \mathrm{id} \otimes \Phi : E \otimes V \to E \otimes W$.

Let us now formulate the excision isomorphism for complex operators in the form we will need in 3.3.3. This generalizes [172, Thm. 2.10] to complex determinant line bundles. Moreover, for real operators it is slightly more general then [172, Thm. 2.13] in that, we do not require a framing of bundles, but isomorphisms in [172, Thm. 2.13(b)]. This would already follow from [172, Thm. 2.10], but we obtain it as a consequence of Remark 3.2.16. Note that we also do not require the isomorphisms below to be unitary, as this is not necessary for the operators in (3.2.11) to be elliptic.

Definition 3.2.17. Let X_i be compact, E_i, F_i, vector bundles on X_i for $i = 1, 2$ and $D_i : \Gamma^\infty(E_i) \to \Gamma^\infty(F_i)$ complex/real elliptic differential operators. Moreover, let $S_i, T_i \subset X_i$ open, such that $S_i \cup T_i = X_i$ and $I_S : S_1 \to S_2$ an isomorphism. We then denote by

$$\begin{array}{ccc} \sigma_{V_1}(D_1) & \overset{\Phi_1}{\dashrightarrow} & \sigma_{W_1}(D_1) \\ {\scriptstyle \xi_V} \downarrow & & \downarrow {\scriptstyle \xi_W} \\ \sigma_{V_2}(D_2) & \overset{\Phi_2}{\dashrightarrow} & \sigma_{W_2}(D_2) \end{array} \tag{3.2.13}$$

the collection of isomorphisms $\Phi_i : V_i|_{T_i} \xrightarrow{\sim} W_i|_{T_i}$, $\xi_V : I_V^*(V_2|_{S_2}) \xrightarrow{\sim} V_1|_{S_1}$, $\xi_W : I_S^*(W_2|_{S_2}) \xrightarrow{\sim} W_1|_{S_1}$ satisfying $\xi_W \circ \Phi_1 = I_S^*(\Phi_2) \circ \xi_V$ for families of vector bundles $V_i, W_i \to X_i$.

Lemma 3.2.18. *For the data given by* (3.2.13) *and a compact subsets K_i, s.t. $X_i \backslash K_i \subset T_i$ identified by I_S,we have natural isomorphisms in families*

$$\Xi(D_i, \xi_{V/W}, \Phi_i) : \det\big(\sigma_{V_1}(D_1)\big)\det{}^*\big(\sigma_{W_1}(D_1)\big) \xrightarrow{\sim} \det\big(\sigma_{V_2}(D_2)\big)\det{}^*\big(\sigma_{W_2}(D_2)\big)\,,$$

such that for another set of data

$$\begin{array}{ccc} \sigma_{V_1'}(D_1) & \overset{\Phi_1'}{\dashrightarrow} & \sigma_{W_1'}(D_1) \\ {\scriptstyle \xi_{V'}} \downarrow & & \downarrow {\scriptstyle \xi_{W'}} \\ \sigma_{V_2'}(D_2) & \underset{\Phi_2'}{\dashrightarrow} & \sigma_{W_2'}(D_2) \end{array}$$

for the same S_i, T_i, K the diagram is commutative up to natural isotopies

$$\begin{array}{ccc} \det\big(\sigma_{V_1\oplus V_1'}(D_1)\big)\det{}^*\big(\sigma_{W_1\oplus W_1'}(D_1)\big) & \xrightarrow{\Xi(D_i,\xi_{V\oplus V'/W\oplus W'},\Phi\oplus\Phi')} & \det\big(\sigma_{V_2\oplus V_2'}(D_2)\big)\det{}^*\big(\sigma_{W_2\oplus W_2'}(D_2)\big) \\ \downarrow {\scriptstyle (3.2.7),\,(3.2.8)} & & \downarrow {\scriptstyle (3.2.7),\,(3.2.8)} \\ \begin{array}{c}\det\big(\sigma_{V_1}(D_1)\big)\det\big(\sigma_{V_1'}(D_1)\big) \\ \det{}^*\big(\sigma_{W_1'}(D_1)\big)\det{}^*\big(\sigma_{W_1}(D_1)\big)\end{array} & \xrightarrow{\Xi(D_i,\xi_{V/W},\Phi)\ \Xi^*(D_i,\xi_{V'/W'},\Phi')} & \begin{array}{c}\det\big(\sigma_{V_2}(D_2)\big)\det\big(\sigma_{V_2'}(D_2)\big) \\ \det{}^*\big(\sigma_{W_2'}(D_2)\big)\det{}^*\big(\sigma_{W_2}(D_2)\big)\end{array} \end{array}$$

Moreover, if $S_i = X_i$, then $\Xi(D_i, \xi_{V/W}, \Phi_i) = ((3.2.6))^{-1} \circ ((3.2.6))$.

Proof. The following is standard, and we simply lift it to complex determinant line bundles. Making a contractible choice of $\chi_i \in C^\infty_{\mathrm{cs}}(S_i)$, $\chi_i|_{K_i} = 1$ identified under I_V, the composition of the following isomorphisms gives $\Xi(D_i, \xi_{V/W}, \Phi_i)$:

$$\begin{aligned}
\det\bigl(\sigma_{V_1}) \otimes \det(\sigma_{W_1}\bigr)^* &\overset{3.2.13(ii),(i)}{\cong} \det\bigl((\sigma_{V_1}(D_1), \sigma_{W_1}(D_1), \Phi_1, \Phi_1)^{\chi_1}_0\bigr) \\
&\overset{3.2.13(v)}{\cong} \det\bigl((\sigma_{V_1}(D_1), \sigma_{W_1}(D_1), \Phi_1, \Phi_1)^{\chi_1}_1\bigr) \\
&\overset{*}{\cong} \det\bigl((\sigma_{V_2}(D_2), \sigma_{W_2}(D_2), \Phi_2, \Phi_2)^{\chi_2}_1\bigr) \\
&\overset{3.2.13(v)}{\cong} \det\bigl((\sigma_{V_2}(D_2), \sigma_{W_2}(D_2), \Phi_2, \Phi_2)^{\chi_2}_0\bigr) \\
&\cong \det\bigl(\sigma_{V_2}(D_2))\det^*(\sigma_{W_2(D_1)}\bigr)\,,
\end{aligned}$$

where for the step $*$, we are making a contractible choice of $P_i \in \sigma^{-1}\bigl((\sigma_{V_i}(D_i), \sigma_{W_i}(D_i), \Phi_i, \Phi_i)^{\chi}_1\bigr)$ supported representatives in S_i of the two symbols on both sides as in Upmeier [172, Thm. A.6] identified by the isomorphism $\xi_{V'}, \xi_{W'}$ and using that

$$\begin{aligned}
\ker(P_i) &\in \Gamma^\infty_{\mathrm{cs}}\bigl(S_i, (E_i \otimes V_i) \oplus (F_i \otimes W_i)\bigr)\,, \\
\operatorname{coker}(P_i) &\in \Gamma^\infty_{\mathrm{cs}}\bigl(S_i, (F_i \otimes V_i) \oplus (E_i \otimes W_i)\bigr)\,.
\end{aligned}$$

The second statement follows from the compatibility under direct sums in 3.2.13(v). The final statement is just Lemma 3.2.15. □

3.3 Proof of Theorem 3.1.18

We construct here a double $\tilde{Y}$ for our manifold X, such that the "compactly supported" orientation on X can be identified with the one on $\tilde{Y}$. We use homotopy theoretic group completion to reduce the problem to trivializing the orientation $\mathbb{Z}_2$-bundles on the moduli space of pairs of vector bundles generated by global sections identified on the normal crossing divisor. Then we express the isomorphism $\vartheta_{\bowtie}$ from Definition 3.1.16 using purely vector bundles in §3.3.3. We then construct the isotopy between the two different real structures to obtain an isomorphism of $\mathbb{Z}_2$-bundles by hand. The final result of this section is contained in Proposition 3.3.11 and Proposition 3.3.15.

3.3.1 Relative framing on the double

Here we construct the double of a non-compact X, such that it can be used in §3.3.2 to define orientations back on moduli spaces over X.

Definition 3.3.1. Let X be a non-compact spin manifold $\dim_{\mathbb{R}}(X) = n$. Let $K \subset X$ be a compact subset. Choose a smooth exhaustion function $d : X \to [0, \infty)$ Then by Sard's theorem for a generic $c > \max\{d(x) : x \in K\}$ the set $U = \{x \in X \,|\, d(x) \leq c\}$ is a manifold with the boundary $\partial U = \{U \in X \,|\, d(g) \leq c\}$. Normalizing the gradient $\mathrm{grad}(d_g)$ restricted to ∂U, we obtain a normal vector field ν to ∂U. Let V be the tubular neighborhood of ∂U in X, then it is diffeomorphic to $(-1, 1) \times \partial U$ and is a collar. We define $\tilde{Y} := U \cup_{\partial U} (-U)$, where $-U$ denotes a copy of U with negative orientation. Then $\tilde{Y}$ admits a natural spin-structure which restricts to the original one on U (see for example [69, p. 193]). Since we do not need it here explicitly, we do not give its description.

Let $T = X \backslash K$, where K is compact, then define $\tilde{T} = (\bar{T} \cap U) \cup (-U)$ (see Figure 3.3.1).

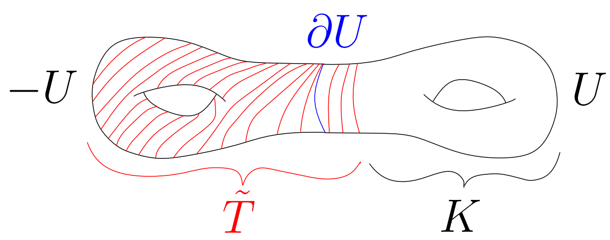

Figure 3.3.1: Spin manifold $\tilde{Y}$ and the subset $\tilde{T} \subset \tilde{Y}$.

Let $P, Q \to \tilde{Y}$, be two $U(n)$, bundles, such that there exists an isomorphism $P|_{\tilde{T}} \cong Q|_{\tilde{T}}$. We define now the moduli stack of pairs of connections on principal bundles identified on $\tilde{T}$.

Definition 3.3.2. Consider the space $\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}}$, where $\mathcal{G}_{P,Q,\tilde{T}}$ is the set of smooth isomorphisms $\tilde{\phi} : P|_{\tilde{T}} \to Q|_{\tilde{T}}$. Let $\mathcal{G}_P \times \mathcal{G}_Q$ be the product of gauge groups. We have a natural action

$$
\begin{aligned}
(\mathcal{G}_P \times \mathcal{G}_Q) \times (\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}}) &\to \mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} \\
(\gamma_P, \gamma_Q, \nabla_P, \nabla_Q, \tilde{\phi}) &\mapsto (\gamma_P(\nabla_P), \gamma_Q(\nabla_Q), \gamma_Q \circ \tilde{\phi} \circ (\gamma_P)^{-1}) \,.
\end{aligned}
$$

We denote the quotient stack by $\mathcal{B}_{P,Q,\tilde{T}} = [\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} / \mathcal{G}_P \times \mathcal{G}_Q]$. Let us define the union

$$
\mathcal{B}_{\tilde{Y},\tilde{T}} = \bigcup_{\substack{[P],[Q]: \\ [P|_{\tilde{T}}]=[Q|_{\tilde{T}}]}} \mathcal{B}_{P,Q,\tilde{T}} \,,
$$

where we chose representatives P, Q for the isomorphism classes.

There exist natural maps $\mathcal{B}_{\tilde{Y}} \xleftarrow{p_1} \mathcal{B}_{\tilde{Y},\tilde{T}} \xrightarrow{p_2} \mathcal{B}_{\tilde{Y}}$ induced by $\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} \to \mathcal{A}_Q$ and $\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} \to \mathcal{A}_P$. Let $O^{\not{D}_+} \to \mathcal{B}_{\tilde{Y}}$ be the $\mathbb{Z}_2$-bundles from (3.2.2), then we define

$$
D_O(\tilde{Y}) = p_1^*(O^{\not{D}_+}) \boxtimes_{\mathbb{Z}_2} p_2^*((O^{\not{D}_+})^*) \,, \tag{3.3.1}
$$

where $(O^{\not{D}_+})^*$ is from Definition 3.2.8. Let us now construct an explicit representative $(\mathcal{B}_{\tilde{Y},\tilde{T}})^{\mathrm{cla}}$.

Definition 3.3.3. Let P and Q be $U(n)$-bundles on $\tilde{Y}$ isomorphic on $\tilde{T}$. Consider the following two quotient stacks

$$\mathcal{P}_Q = \left[\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} \times P / \mathcal{G}_P \times \mathcal{G}_Q\right],$$
$$\mathcal{Q}_P = \left[\mathcal{A}_P \times \mathcal{A}_Q \times \mathcal{G}_{P,Q,\tilde{T}} \times Q / \mathcal{G}_P \times \mathcal{G}_Q\right],$$

which are $U(n)$-bundles on $\tilde{Y} \times \mathcal{B}_{P,Q,\tilde{T}}$. We have a natural isomorphism $\tau_{P,Q}$: $\mathcal{P}_Q|_{\tilde{T}\times\mathcal{B}_{P,Q,\tilde{T}}} \to \mathcal{Q}_P|_{\tilde{T}\times\mathcal{B}_{P,Q,\tilde{T}}}$ given by $[\nabla_P, \nabla_Q, \tilde{\phi}, p] \mapsto [\nabla_P, \nabla_Q, \tilde{\phi}, \tilde{\phi}(p)]$. After taking appropriate unions, we obtain bundles $\mathcal{P}_1, \mathcal{P}_2 \to \tilde{Y} \times \mathcal{B}_{\tilde{Y},\tilde{T}}$ with an isomorphism $\mathcal{P}_1|_{\tilde{T}\times\mathcal{B}_{\tilde{Y},\tilde{T}}} \cong \mathcal{P}_2|_{\tilde{T}\times\mathcal{B}_{\tilde{Y},\tilde{T}}}$. Pulling $\mathcal{P}_i$ back to $\tilde{Y}\times(\mathcal{B}_{\tilde{Y},\tilde{T}})^{\mathrm{cla}}$, we obtain $\mathcal{P}_i^{\mathrm{cla}}$ fiber bundles, which are $U(n)$-bundles on each connected components for some $n \geq 0$. Together with the isomorphism τ^{cla}, these induce two maps

$$\mathfrak{p}_1, \mathfrak{p}_2 : \tilde{Y} \times \mathcal{B}_{\tilde{Y},\tilde{T}} \longrightarrow \bigsqcup_{n\geq 0} BU(n),$$

with a unique (up to contractible choices) homotopy $H_{\mathfrak{p}} : \tilde{T}\times\mathcal{B}_{\tilde{Y},\tilde{T}}\times I \to \bigsqcup_{n\geq 0} BU(n)$ between $\mathfrak{p}_1$ and $\mathfrak{p}_2$ restricted to $\tilde{T} \times \mathcal{B}_{\tilde{Y},\tilde{T}}$. We obtain the following homotopy commutative diagram

$$\begin{array}{ccc} (\mathcal{B}_{\tilde{Y}} \times_{\mathcal{B}_{\tilde{T}}} \mathcal{B}_{\tilde{Y}})^{\mathrm{cla}} & \longrightarrow & \mathrm{Map}_{\mathcal{C}^0}(\tilde{Y}, \bigsqcup_{n\geq 0} BU(n)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}^0}(\tilde{Y}, \bigsqcup_{n\geq 0} BU(n)) & \longrightarrow & \mathrm{Map}_{\mathcal{C}^0}(\tilde{T}, \bigsqcup_{n\geq 0} BU(n)) . \end{array} \tag{3.3.2}$$

This induces a map $\mathcal{B}_{\tilde{Y},\tilde{T}} \to V_{\tilde{Y}} \times^h_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_{\tilde{Y}}$, where we use the notation

$$\mathcal{V}_Z = \mathrm{Map}_{\mathcal{C}^0}(Z, \bigsqcup_{n\geq 0} BU(n))$$

for each topological space Z. If $\bar{T} \hookrightarrow X$ is a neighborhood deformation retract pair then so is $\tilde{T} \hookrightarrow Y$. It is then a cofibration in **Top** and the left vertical and lower horizontal arrow of (3.3.2) are fibrations in **Top**. This implies that the natural map

$$\mathcal{V}_{\tilde{Y},\tilde{T}} = \mathcal{V}_{\tilde{Y}} \times_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_{\tilde{Y}} \longrightarrow \mathcal{V}_{\tilde{Y}} \times^h_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_{\tilde{Y}}$$

is a homotopy equivalence. By homotopy inverting, we construct $\mathfrak{R} : (\mathcal{B}_{\tilde{Y},\tilde{T}})^{\mathrm{cla}} \to \mathcal{V}_{\tilde{Y},\tilde{T}}$. It can be shown by following the arguments of Atiyah–Jones [7], Singer [158], Donaldson [49, Prop 5.1.4] and Atiyah–-Bott [6] that this is a weak homotopy equivalence. We therefore have the natural $\mathbb{Z}_2$-bundle $(D_O(\tilde{Y}))^{\mathrm{cla}} \to \mathcal{V}_{\tilde{Y},\tilde{T}}$.

We summarize some obvious statements about the above constructions. There is a natural map $u_n : BU(n) \to BU \times \mathbb{Z}$, such that $\pi_2 \circ u_n = n$. This induces maps $\mathcal{V}_Z \to \mathcal{C}_Z$ which are homotopy theoretic group completions for any Z. In particular we have a natural map $\tilde{\Omega} : \mathcal{V}_{\tilde{Y},\tilde{T}} \to \mathcal{C}_{\tilde{Y},\tilde{T}} := \mathcal{C}_{\tilde{Y}} \times_{\mathcal{C}_{\tilde{T}}} \mathcal{C}_{\tilde{Y}}$.

Lemma 3.3.4. *The spaces* $(\mathcal{B}_{\tilde{Y},\tilde{T}})^{\mathrm{cla}}$, $\mathcal{V}_{\tilde{Y},\tilde{T}}$ *are H-spaces. The maps* $(p_1)^{\mathrm{cla}}, (p_2)^{\mathrm{cla}}, \mathfrak{R}$ *are H-maps. In particular,* $(D_O(\tilde{Y}))^{\mathrm{cla}} \to (\mathcal{B}_{\tilde{Y},\tilde{T}})^{\mathrm{cla}} \simeq \mathcal{V}_{\tilde{Y},\tilde{T}}$ *is a* $\mathbb{Z}_2$*-graded strong H-principal* $\mathbb{Z}_2$*-bundle and there exists a unique* $\mathbb{Z}_2$*-graded strong H-principal* $\mathbb{Z}_2$*-bundle* $D^{\mathcal{C}}_O(\tilde{Y}) \to \mathcal{C}_{\tilde{Y},\tilde{T}}$ *up to canonical isomorphisms, such that there is a canonical isomorphism*

$$\mathfrak{q}^*(D^{\mathcal{C}}_O(\tilde{Y})) \cong D_O(\tilde{Y}),$$

where $\mathfrak{q} : \mathcal{V}_{\tilde{Y},\tilde{T}} \to \mathcal{C}_{\tilde{Y},\tilde{T}}$ *is the homotopy theoretic group completion.*

Proof. The last statement follows using Proposition 3.2.5 (ii). □

3.3.2 Homotopy commutative diagram of H-spaces

We use the definitions of moduli spaces of vector bundles generated by global section of Friedlander–Walker [57] used by Cao-Gross–Joyce [30, Definition 3.18]. For definition of Ind-schemes see for example Gaitsgory–Rozenblyum [64], for general treatment of indization of categories see Kashiwara-Shapira [100, §6]. For Z a scheme over $\mathbb{C}$, this moduli space is defined as the mapping Ind-scheme :

$$\mathcal{T}_Z = \mathrm{Map}_{\mathbf{IndSch}}(Z, \mathrm{Gr}(\mathbb{C}^\infty))\,,$$

where **IndSch** is the category of Ind-schemes over $\mathbb{C}$ and we view $\mathrm{Gr}(\mathbb{C}^\infty)$ as an object in this category.

Definition 3.3.5. Induced by the embedding of schemes $D \hookrightarrow Y$, we obtain a map $\rho_D^{\mathrm{vb}} : \mathcal{T}_Y \to \mathcal{T}_D$. We can construct the fiber-product in Ind-schemes $\mathcal{T}_{Y,D}$. There is a natural map $\Omega_Y : \mathcal{T}_Y \to \mathcal{M}_Y$ given by composing with the natural $\mathrm{Gr}(\mathbb{C}^\infty) \to \mathrm{Perf}_{\mathbb{C}}$. Together with the map $\Omega_D^{\mathrm{ag}} : \mathcal{T}_D \to \mathcal{M}^D$ constructed in the same way, we obtain a homotopy commutative diagram of higher stacks:

$$\begin{array}{ccccc}
\mathcal{T}_Y & \xrightarrow{\rho_D^{\mathrm{vb}}} & \mathcal{T}_D & \xleftarrow[\rho_D^{\mathrm{vb}}]{} & \mathcal{T}_Y \\
\downarrow{\scriptstyle \Omega_Y^{\mathrm{ag}}} & & \downarrow{\scriptstyle \Omega_D^{\mathrm{ag}}} & & \downarrow{\scriptstyle \Omega_Y^{\mathrm{ag}}} \\
\mathcal{M}_Y & \xrightarrow{\rho_D} & \mathcal{M}^D & \xleftarrow[\rho_D]{} & \mathcal{M}_Y\,,
\end{array}$$

which induces $\Omega^{\mathrm{ag}} : \mathcal{T}_{Y,D} = \mathcal{T}_Y \times_{\mathcal{T}_D} \mathcal{T}_Y \to \mathcal{M}_{Y,D}$.

As $(-)^{\mathrm{top}}$ commutes with homotopy colimits by Blanc [16, Prop. 3.7] for an Ind-scheme $\mathcal{S}$ considered as a higher stack represented by the sequence of closed embeddings of finite type schemes $S_0 \to S_1 \to S_2 \to \ldots$, its topological realization $(\mathcal{S})^{\mathrm{top}}$ is the co-limit in the **Top** of the sequence $S_0^{\mathrm{an}} \to S_1^{\mathrm{an}} \to S_2^{\mathrm{an}} \to \ldots$, because the maps are closed embeddings of CW-complexes and thus cofibrations. Using that filtered co-limits commute with finite limits, we can express $\mathcal{T}_{Y,D}$ as the filtered co-limit of $\mathcal{T}_Y^p \times_{\mathcal{T}_D^p} \mathcal{T}_Y^p$, where $\mathcal{T}_Z^p = \mathrm{Map}_{\mathbf{Sch}}(Z, \mathrm{Gr}(\mathbb{C}^p))$ for any scheme Z. From this, it also follows that

$$(\mathcal{T}_{Y,D})^{\mathrm{top}} = \varinjlim_{p\to\infty} (\mathcal{T}_Y^p)^{\mathrm{an}} \times_{(\mathcal{T}_D^p)^{\mathrm{an}}} (\mathcal{T}_Y^p)^{\mathrm{an}} = \mathcal{T}_Y^{\mathrm{an}} \times_{\mathcal{T}_D^{\mathrm{an}}} \mathcal{T}_Y^{\mathrm{an}} .$$

We have therefore constructed a map

$$\Omega^{\mathrm{top}} : \mathcal{T}_Y^{\mathrm{an}} \times_{\mathcal{T}_D^{\mathrm{an}}} \mathcal{T}_Y^{\mathrm{an}} \to (\mathcal{M}_Y \times_{\mathcal{M}^D} \mathcal{M}_Y)^{\mathrm{top}} . \tag{3.3.3}$$

The following is a non-trivial modification of [30, Prop. 3.22], [74, Prop. 4.5] to the case of $\mathcal{M}_{Y,D}$. We use in the proof the language of spectra (see Strickland [162] and Lewis–May [72]). We only use that the infinite loop space functor $\Omega^\infty : \mathbf{Sp} \to \mathrm{Top}$ preserves homotopy equivalences, where **Sp** is the category of topological spectra.

Proposition 3.3.6. *The map* $\Omega^{\mathrm{top}} : \mathcal{T}_Y^{\mathrm{an}} \times_{\mathcal{T}_D^{\mathrm{an}}} \mathcal{T}_Y^{\mathrm{an}} \to (\mathcal{M}_{Y,D})^{\mathrm{top}}$ *is a homotopy theoretic group completion of H-spaces.*

Proof. Let us recall that in a symmetric closed monoidal category $\mathcal{C}$ with the internal hom functor $\mathrm{Map}_{\mathcal{C}}(-,-)$ the contravariant functor $C \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ maps co-limits to limits. Thus push-outs are mapped to pullbacks because the homotopy category of higher stacks is symmetric closed monoidal as shown by Toën–Vezzosi in [170, Theorem 1.0.4].

The following diagram

$$\begin{array}{ccc} D & \xrightarrow{i_D} & Y \\ \downarrow{\scriptstyle i_D} & & \\ Y & & \end{array} \tag{3.3.4}$$

has a push-out $Y \cup_D Y$ in the category of schemes over $\mathbb{C}$ using that i_D is a closed embedding and Schwede [154, Corollary 3.7]. Moreover, the result of Ferrand [53, §6.3] tells us that $Y \cup_D Y$ is projective.

We conclude that there are natural isomorphisms

$$\mathcal{M}_{Y,D} \cong \mathrm{Map}_{\mathbf{HSt}}(Y \cup_D Y, \mathrm{Perf}_{\mathbb{C}}) = \mathcal{M}^{Y \cup_D Y}\,,$$

$$\mathcal{T}_{Y,D} \cong \mathrm{Map}_{\mathbf{IndSch}}(Y \cup_D Y, \mathrm{Gr}^{\infty}(\mathbb{C})) = \mathcal{T}_{Y \cup_D Y}\,.$$

In fact, under these isomorphisms, the map Ω from Definition 3.3.5 corresponds to the natural map $\Omega_{Y \cup_D Y} : \mathcal{T}_{Y \cup_D Y} \to \mathcal{M}^{Y \cup_D Y}$.

For a quasi-projective variety Z over $\mathbb{C}$, Friedlander–Walker define in [57, Definition 2.9] the space $\mathcal{K}^{\mathrm{semi}}(Z)$ as the infinity loop space $\Omega^{\infty}\mathcal{T}^{\mathrm{an}}_Z$, where they use that $\mathcal{T}^{\mathrm{an}}$ is an E_∞-space. Therefore there is a map $\mathcal{T}^{\mathrm{an}}_Z \to \mathcal{K}^{\mathrm{semi}}(Z)$, which is a homotopy theoretic group completion by [119, p. 6.4] and [112, §2]. For a dg-category $\mathcal{D}$ over $\mathbb{C}$, Blanc [16, Definition 4.1] defines the connective semi-topological K-theory $\tilde{\mathbf{K}}^{\mathrm{st}}(\mathcal{D})$ in the category $\mathbf{Sp}$. Moreover, in [16, Theorem 4.21], he constructs an equivalence between the $\tilde{\mathbf{K}}^{\mathrm{st}}(\mathcal{D})$ and the spectrum of the topological realization of the higher moduli stack of perfect modules of $\mathcal{D}$‡. This induces a homotopy equivalence

‡This moduli stack is denoted in Blanc [16] by $\mathcal{M}^{\mathcal{D}}$. Unlike the moduli stacks in Toën–Vaquié, it classifies only perfect dg-modules over $\mathcal{D}$ and not the pseudo-perfect ones. For the case $\mathcal{D} = L_{\mathrm{pe}}(Y)$ it therefore coincides with the mapping stack $\mathcal{M}^Y$. When Y is projective and smooth, we already

$\Omega^\infty \tilde{\mathbf{K}}^{\mathrm{st}}(\mathrm{Perf}(Z)) \to (\mathcal{M}^Z)^{\mathrm{top}}$ of H-spaces. In [4, Theorem 2.3], Antieau–Heller prove existence of a natural homotopy equivalence between the H-spaces $\Omega^\infty \tilde{\mathbf{K}}^{\mathrm{st}}(\mathrm{Perf}(Z))$ and $\mathcal{K}^{\mathrm{semi}}(Z)$. The composition

$$\mathcal{T}_Z^{\mathrm{an}} \longrightarrow \mathcal{K}^{\mathrm{semi}}(Z) \longrightarrow \Omega^\infty \tilde{\mathbf{K}}^{\mathrm{st}}(\mathrm{Perf}(Z)) \longrightarrow (\mathcal{M}^Z)^{\mathrm{top}}$$

for $Z = Y \cup_D Y$ is homotopy equivalent to $\Omega^{\mathrm{top}}_{Y\cup_D Y}$. We have thus shown that Ω^{top} is a homotopy theoretic group-completion. □

We now make $(O^{\bowtie})^{\mathrm{top}} \to (\mathcal{M}_{Y,D})^{\mathrm{top}}$ into a weak H-principal $\mathbb{Z}_2$-bundle with respect to the binary operation μ on $(\mathcal{M}_{Y,D})^{\mathrm{top}}$ which is determined by

$$\begin{array}{ccccc} & & \mathcal{M}_Y \times \mathcal{M}_Y & \xrightarrow{\mu_{\mathcal{M}_Y}} & \mathcal{M}_Y \\ & \nearrow^{\pi_{1,3}} & \downarrow{\scriptstyle \rho_D\times\rho_D} & & \downarrow{\scriptstyle \rho_D} \\ \mathcal{M}_{Y,D} \times \mathcal{M}_{Y,D} & & \mathcal{M}^D \times \mathcal{M}^D & \xrightarrow{\mu_{\mathcal{M}^D}} & \mathcal{M}^D \\ & \searrow_{\pi_{2,4}} & \uparrow{\scriptstyle \rho_D\times\rho_D} & & \uparrow{\scriptstyle \rho_D} \\ & & \mathcal{M}_Y \times \mathcal{M}_Y & \xrightarrow{\mu_{\mathcal{M}_Y}} & \mathcal{M}_Y\,. \end{array} \tag{3.3.5}$$

It can be checked to be commutative and associative in $\mathbf{Ho(HSta)}_{\mathbb{C}}$. In fact, as $\mathcal{M}_{Y,D}$ is a homotopy fiber product of Γ-objects, it is itself one in $\mathbf{HSta}_{\mathbb{C}}$ (see Bousfield–Friedlander [24, §3] for definition of Γ-objects in model categories and Blanc [16, p. 45] for the construction in this case). Let us set some notation. Let us set some notation. For any $\underline{a}$, $\underline{b}$, we have the isomorphisms

$$\pi_{1,3}^*(\Sigma_{\underline{a}}) \otimes \pi_{2,4}^*(\Sigma_{\underline{a}})^* \cong \pi_{1,3}^*(\Sigma_{\underline{b}}) \otimes \pi_{2,4}^*(\Sigma_{\underline{b}})^*$$

by similar construction as in (3.1.14). In particular, for fixed $\bowtie$ we obtain the iso-

know that $\mathcal{M}^Y$ and $\mathcal{M}_Y$ are equivalent.

morphism

$$\sigma_{\bowtie} : \pi_{1,3}^*(\Sigma_{\underline{k}}) \otimes \pi_{2,4}^*(\Sigma_{\underline{k}})^* \xrightarrow{\sim} \pi_{1,3}^*(\Sigma_{\underline{0}}) \otimes \pi_{2,4}^*(\Sigma_{\underline{0}})^* \,. \tag{3.3.6}$$

Proposition 3.3.7. *Let $O^{\bowtie} \to \mathcal{M}_{Y,D}$ be the $\mathbb{Z}_2$-bundle from Definition 3.1.16, then there exists an isomorphism $\phi^{\bowtie} : O^{\bowtie} \boxtimes_{\mathbb{Z}_2} O^{\bowtie} \to \mu_{\mathcal{M}}^*(O^{\bowtie})$ depending on $\bowtie$ but independent of $\mathfrak{ord}$. Moreover, we have*

$$(\mathrm{id}_{\mathcal{M}_{Y,D}} \times \mu_{\mathcal{M}})^*(\phi^{\bowtie})(\mathrm{id} \times \phi^{\bowtie}) = (\mu_{\mathcal{M}} \times \mathrm{id}_{\mathcal{M}_{Y,D}})^*(\phi^{\bowtie}) \circ (\phi^{\bowtie} \times \mathrm{id}) \,.$$

Proof. First note, that we have the isomorphism

$$\mu_{\mathcal{M}_Y}^*(\Lambda_{\underline{0}}) \cong \pi_1^*\Lambda_{\underline{0}} \otimes \pi_{1,2}^*\Sigma_{\underline{0}} \otimes \pi_2^*\Lambda_{\underline{0}} \otimes \pi_{1,2}^* \circ \sigma^*\Sigma_{\underline{0}} \,.$$

Using this together with (3.3.5) we obtain the following commutative diagram

$$
\begin{array}{ccc}
 & & \pi_1^*\Lambda_{\underline{0}} \otimes \pi_3^*\Lambda_{\underline{0}} \otimes \pi_4^*\Lambda_{\underline{0}}^* \otimes \pi_2^*\Lambda_{\underline{0}}^* \\
 & & \downarrow \\
 & & \begin{matrix} \pi_1^*\Lambda_{\underline{0}}\otimes\pi_{1,3}^*\Sigma_{\underline{0}}\otimes\pi_{1,3}^*\Sigma_{\underline{k}}^*\otimes\pi_3^*\Lambda_{\underline{0}} \\ \otimes\pi_4^*\Lambda_{\underline{0}}^*\otimes\pi_{2,4}^*\Sigma_{\underline{0}}^*\otimes\pi_{2,4}^*\Sigma_{\underline{k}}\otimes\pi_2^*\Lambda_{\underline{0}}^* \end{matrix} \\
 & & \downarrow \\
\mu^*(\pi_1^*\Lambda_{\underline{0}} \otimes \pi_2^*\Lambda_{\underline{0}}^*) & \longrightarrow & \begin{matrix} \pi_1^*\Lambda_{\underline{0}}\otimes\pi_{1,3}^*\Sigma_{\underline{0}}\otimes\pi_{1,3}^*\circ\sigma^*\Sigma_{\underline{0}}\otimes\pi_3^*(\Lambda_{\underline{0}}) \\ \otimes\pi_4^*\Lambda_{\underline{0}}^*\otimes\pi_{2,4}^*\circ\sigma^*\Sigma_{\underline{0}}^*\otimes\pi_{2,4}^*\Sigma_{\underline{0}}\otimes\pi_2^*\Lambda_{\underline{0}}^* \end{matrix} \\
\Big\downarrow & & \downarrow \\
\Big\downarrow & & \begin{matrix} \pi_1^*\Lambda_{\underline{k}}^*\otimes\pi_{1,3}^*\circ\sigma^*\Sigma_{\underline{k}}^*\otimes\pi_{1,3}^*\Sigma_{\underline{k}}^*\otimes\pi_3^*(\Lambda_{\underline{k}}^*) \\ \otimes\pi_4^*\Lambda_{\underline{k}}\otimes\pi_{2,4}^*\Sigma_{\underline{k}}\otimes\pi_{2,4}^*\circ\sigma^*\Sigma_{\underline{k}}^*\otimes\pi_2^*\Lambda_{\underline{k}} \end{matrix} \\
\Big\downarrow & & \downarrow \\
\mu^*(\pi_1^*\Lambda_{\underline{0}}^* \otimes \pi_2^*\Lambda_{\underline{0}}) & \longrightarrow & \begin{matrix} \pi_1^*\Lambda_{\underline{0}}^*\otimes\pi_{1,3}^*\Sigma_{\underline{0}}^*\otimes\pi_{1,3}^*\circ\sigma^*\Sigma_{\underline{0}}^*\otimes\pi_3^*\Lambda_{\underline{0}}^* \\ \otimes\pi_4^*\Lambda_{\underline{0}}\otimes\pi_{2,4}^*\Sigma_{\underline{0}}\otimes\pi_{2,4}^*\circ\sigma^*\Sigma_{\underline{0}}\otimes\pi_2^*\Lambda_{\underline{0}} \end{matrix} \\
 & & \downarrow \\
 & & \begin{matrix} \pi_1^*\Lambda_{\underline{0}}^*\otimes\pi_{1,3}^*\Sigma_{\underline{0}}^*\otimes\pi_{1,3}^*\Sigma_{\underline{k}}\otimes\pi_3^*\Lambda_{\underline{0}}^* \\ \otimes\pi_4^*\Lambda_{\underline{0}}\otimes\pi_{2,4}^*\Sigma_{\underline{0}}\otimes\pi_{2,4}^*\Sigma_{\underline{k}}^*\otimes\otimes\pi_2\Lambda_{\underline{0}}^* \end{matrix} \\
 & & \downarrow \\
 & & \pi_1^*\Lambda_{\underline{0}}^* \otimes \pi_3^*\Lambda_{\underline{0}}^* \otimes \pi_4^*\Lambda_{\underline{0}} \otimes \pi_2^*\Lambda_{\underline{0}}
\end{array}
$$

Where the left vertical arrow is $\mu^*(\vartheta_{\bowtie})$ and the composition of all arrows on the right is $\pi_{1,2}^*(\vartheta_{\bowtie}) \otimes \pi_{3,4}^*(\vartheta_{\bowtie})$ by generalization of the arguments in Cao–Gross–Joyce [30, p. 43]. To construct arrows on the right, we use multiple times Serre duality and (3.3.6). This is what induces the isomorphism $\phi^{\bowtie}$. Note that we need to permute $\pi_2^*\Lambda_{\underline{0}}$ through $\pi_3^*\Lambda_{\underline{0}}$ and $\pi_4^*\Lambda_{\underline{0}}^*$ on both ends, giving the extra sign

$$(-1)^{\deg(\pi_2^*\Lambda_{\underline{0}})\left(\deg(\pi_3^*\Lambda_{\underline{0}})+\deg(\pi_4^*\Lambda_{\underline{0}})\right)} \tag{3.3.7}$$

for the isomorphism of $\mathbb{Z}_2$-bundles. Checking the associativity of the isomorphism combines the ideas of the proof of associativity in Lemma 3.2.6 and the ones used in the diagram above. The independence of $\mathfrak{ord}$ follows by the same arguments as used in 3. of the proof of Theorem 3.1.18. □

Using the notation from Definition 3.3.3, we have an obvious map

$$\Lambda : \mathcal{T}_Y^{\mathrm{an}} \times_{\mathcal{T}_D^{\mathrm{an}}} \mathcal{T}_Y^{\mathrm{an}} \longrightarrow \mathcal{V}_{Y,D} \tag{3.3.8}$$

which corresponds to the inclusion of holomorphic maps into the continuous maps to $\mathrm{Gr}(\mathbb{C}^\infty)^{\mathrm{an}}$. This map is continuous (see Friedlander–Walker [58]). Let $\bar{T}_i \supset D_i$ be closed tubular neighborhoods for $i = 1, \ldots, N$. One can construct homotopy retracts H_i of $\bar{T}_i$ to D_i which can be extended to $\tilde{H}_i : I \times Y \to Y$, such that $\tilde{H}_i|_{I\times\bar{T}_i} = H_i$ and $\tilde{H}_i(t,-)|_{Y\backslash(1+\epsilon_i)T_i} = \mathrm{id}_{Y\backslash(1+\epsilon_i)T_i}$, where $(1+\epsilon_i)T_i$ denotes some tubular neighborhood containing $\bar{T}_i$. We concatenate them to get $\tilde{H}$, $H = \tilde{H}|_{\bar{T}\times I}$. Using that D has locally analytically the form $\mathbb{C}^{4-k} \times \{(z_1, \ldots, z_k) \in \mathbb{C}^k : z_1 \cdot \ldots \cdot z_k = 0\}$, one can assume that $\tilde{H}(t,T) \subset T$ and $\tilde{H}(t,D) = D$[§]. The pullback along $\tilde{H}(1,-)$ and $H(1,-)$ induces homotopy equivalences

$$\Upsilon : \mathcal{V}_{Y,D} \longrightarrow \mathcal{V}_{Y,\bar{T}}\,, \qquad \Upsilon_{\mathcal{C}} : \mathcal{C}_{Y,D} \longrightarrow \mathcal{C}_{Y,\bar{T}}\,, \tag{3.3.9}$$

which we use from now on to identify the spaces. As $X \subset Y$ is Calabi–Yau, choosing $K = X\backslash T$, where T is the interior of $\bar{T}$, we construct spin $\tilde{Y}$ as in Definition 3.3.1. Define

$$G_{\tilde{Y}} : \mathcal{V}_{Y,\bar{T}} \longrightarrow \mathcal{V}_{\tilde{Y},\tilde{\bar{T}}}\,, \qquad G_{\tilde{Y}}^{\mathcal{C}} : \mathcal{C}_{Y,\bar{T}} \longrightarrow \mathcal{C}_{\tilde{Y},\tilde{\bar{T}}}\,, \tag{3.3.10}$$

[§]One can construct this by taking a splitting of $0 \to TD_i \to TY \to ND_i \to 0$, taking geodesic flow in the normal direction for all D_i. Then around each intersection projecting the flow to be parallel to each of the other divisors.

by $G_{\tilde{Y}}(m_1, m_2) = (\tilde{m}_1, \tilde{m}_2)$ for each $(m_1, m_2) \in \mathcal{V}_Y \times_{\mathcal{V}_{\bar{T}}} \mathcal{V}_Y$ such that

$$\tilde{m}_1|_U = m_1|_U\,, \quad \tilde{m}_1|_{-U} = m_1|_U\,, \qquad \tilde{m}_2|_U = m_2|_U\,, \quad \tilde{m}_2|_{-U} = m_1|_U\,. \tag{3.3.11}$$

Which gives us $\mathbb{Z}_2$-bundles:

$$D_O := \Upsilon^* \circ G^*_{\tilde{Y}}(D_O(\tilde{Y}))\,, \quad D_O^{\mathcal{C}} \longrightarrow \mathcal{C}_{Y,D}\,. \tag{3.3.12}$$

Lemma 3.3.8. *Let E, F, ϕ be smooth vector bundles and $\phi : E|_D \to F|_D$ be an isomorphism, smooth on each D_i. Then there exists a contractible choice of isomorphism $\bar{\Phi} : E|_{\bar{T}} \to F|_{\bar{T}}$, $\Phi_i : E|_{\bar{T}_i} \to F|_{\bar{T}_i}$ such that $\Phi_i|_{D_i} = \phi|_{D_i}$ and Φ_i can be deformed into $\bar{\Phi}$ along isomorphism. Moreover, the map* (3.3.10) *corresponds to*

$$[E, F, \phi] \longmapsto [E, F, \bar{\Phi}]\,.$$

The $\mathbb{Z}_2$-graded strong H-principal $\mathbb{Z}_2$-bundles D_O and $D_O^{\mathcal{C}}$ are independent of the choices made.

Proof. The isomorphism $H^*(E|_D) \cong E|_{\bar{T}}$ can be constructed by parallel transport along a contractible choice of partial connections in the I direction (see e.g. Lang [110, §IV.1]) which are piece-wise smooth. Doing the same for F gives us $\Phi_i : F|_{\bar{T}} \cong H^*(F|_D) \cong H^*(E|_D) \cong E_{\bar{T}}$. As, we can re-parameterize the order using an I^N-family of homotopies, it will be independent of it. Moreover, each Φ_i is defined by $F|_{\bar{T}_i} \cong H_i^*(F|_{D_i}) \cong H_i^*(E|_{D_i}) \cong E|_{\bar{T}_i}$, which can be deformed to $\bar{\Phi}$ along the transport. The choices of splittings $0 \to TD_i \to TY \to ND_i \to 0$,where ND_i is the normal bundle are contractible and so is the choice of metric for geodesic flow.

Different choices of sizes of these neighborhoods correspond to a choice of some small $\epsilon_i > 0$. For each choice of the data above, the $\mathbb{Z}_2$-bundle $D_O \to \mathcal{V}_{Y,D}$ is independent of the choices made during the construction of $\tilde{Y}$ in Definition 3.3.1. For this let $(\tilde{Y}_1, \tilde{T}_1)$, $(\tilde{Y}_2, \tilde{T}_2)$ be two pairs constructed using Definition 3.3.1. Recall that this corresponds to fixing two different sets $U_{1,2} \supset K$ with a boundary. A family $Z \to \mathcal{V}_{Y,D}$ gives $Z \xrightarrow{z_1} \mathcal{V}_{\tilde{Y}_1, \tilde{T}_1}$, $Z \xrightarrow{z_2} \mathcal{V}_{\tilde{Y}_2, \tilde{T}_2}$ Which can be interpreted as the following diagram of (families) of vector bundles:

$$\begin{array}{ccc} \tilde{E}_1 & \xrightarrow{\tilde{\Phi}_1 \text{ on } \tilde{T}_1} & \tilde{F}_1 \\ {\scriptstyle \text{id on } U_1 \cap U_2} \Big\downarrow & & \Big\downarrow {\scriptstyle \text{id on } U_1 \cap U_2} \\ \tilde{E}_2 & \xrightarrow[\tilde{\phi}_2 \text{ on } \tilde{T}_2]{} & \tilde{F}_2 \end{array},$$

Induced by [172, Thm. 2.10], we have the isomorphism $z_1^*\big(D_O(\tilde{Y}_1)\big) \cong \operatorname{or}\big(\not{D}^{\nabla_{\mathrm{ad}(\tilde{P}_1)}}\big)\operatorname{or}\big(\not{D}^{\nabla_{\mathrm{ad}(\tilde{Q}_1)}}\big)^* \cong \operatorname{or}\big(\not{D}^{\nabla_{\mathrm{ad}(\tilde{P}_2)}}\big)\operatorname{or}\big(\not{D}^{\nabla_{\mathrm{ad}(\tilde{Q}_2)}}\big)^* \cong z_2^*\big(D_O(\tilde{Y}_1)\big)$, where $\tilde{P}_i$, $\tilde{Q}_i$ are the unitary frame bundle for $\tilde{E}_i$, $\tilde{F}_i$. The fact that it is an isomorphism of strong H-principal Z_2-bundles follows from compatibility under sums in Theorem [172, Thm.2.10 (iii)] and the natural orientations of $\operatorname{or}(\not{D}^{\nabla_{(\tilde{P}_1 \times_{\tilde{Y}} \tilde{Q}_1)_{U(n)\times U(m)} \bar{\mathbb{C}}^n \otimes \mathbb{C}^m}})$ and $\operatorname{or}(\not{D}^{\nabla_{(\tilde{P}_2 \times_{\tilde{Y}} \tilde{Q}_2)_{U(n)\times U(m)} \bar{\mathbb{C}}^n \otimes \mathbb{C}^m}})$ as in [97, Ex. 2.11] compatible under excision as they are determined by the complex structures which are identified.

$\square$

From now on, we will not distinguish between Ind-schemes and their analytifications. Note that by Proposition 3.2.3, we have the commutative diagram

$$\begin{array}{ccccccc} \mathcal{T}_{Y,D} & \xrightarrow{\Lambda} & \mathcal{V}_{Y,D} & \xrightarrow{\Upsilon} & \mathcal{V}_{Y,\bar{T}} & \xrightarrow{G_{\tilde{Y}}} & \mathcal{V}_{\tilde{Y},\tilde{T}} \\ \downarrow{\scriptstyle (\Omega^{\mathrm{ag}})^{\mathrm{top}}} & & \downarrow{\scriptstyle \Omega} & & \downarrow & & \downarrow{\scriptstyle \tilde{\Omega}} \\ (\mathcal{M}_{Y,D})^{\mathrm{top}} & \xrightarrow{\Gamma} & \mathcal{C}_{Y,D} & \xrightarrow{\Upsilon_{\mathcal{C}}} & \mathcal{C}_{Y,\bar{T}} & \xrightarrow{G^{\mathcal{C}}_{\tilde{Y}}} & \mathcal{C}_{\tilde{Y},\tilde{T}} \end{array} . \tag{3.3.13}$$

The map Γ was expressed explicitly in Definition 3.1.17 and all the vertical arrows are homotopy theoretic group completions.

3.3.3 Comparing excisions

The results of this section have been also obtained in the author's work [19] by different but equivalent means. We begin by defining a new set of differential geometric line bundles on $\mathcal{T}_{Y,D} \times \mathcal{T}_{Y,D}$. Let $D = \bar{\partial} + \bar{\partial}^* : \Gamma^\infty(\mathcal{A}^{0,\mathrm{even}}) \to \Gamma^\infty(\mathcal{A}^{0,\mathrm{odd}})$, then we define

$$\hat{\Sigma}^{\mathrm{dg}}_{\underline{a},P,Q} \longrightarrow \mathcal{A}_P \times \mathcal{A}_Q$$

given by a complex line $\det(D^{\nabla_{\mathrm{Hom}(E,F\otimes L_{\underline{a}})}})$ at each point (∇_P, ∇_Q), where E, F are the associated complex vector bundles to P, Q and $\nabla_{\mathrm{Hom}(E,F)}$ is the induced connection on $E^* \otimes F$. This descends to a line bundle on $\hat{\Sigma}^{\mathrm{dg}}_{\underline{a},P,Q} \to \mathcal{B}_P \times \mathcal{B}_Q$. Taking the union over isomorphism classes $[P]$, $[Q]$, we obtain a line bundle

$$\hat{\Sigma}^{\mathrm{dg}}_{\underline{a}} \longrightarrow \mathcal{B}_Y \times \mathcal{B}_Y \, .$$

Using the natural map $\Lambda \times \Lambda : \mathcal{T}_Y \times \mathcal{T}_Y \to \mathcal{V}_Y \times \mathcal{V}_Y \simeq (\mathcal{B}_Y)^{\mathrm{cla}} \times (\mathcal{B}_Y)^{\mathrm{cla}}$, we can pull back these bundles to obtain

$$\Sigma^{\mathrm{dg}}_{\underline{a}} \longrightarrow \mathcal{T}_Y \times \mathcal{T}_Y \quad \text{and} \quad \Lambda^{\mathrm{dg}}_{\underline{a}} = \Delta^*(\Sigma^{\mathrm{dg}}_{\underline{a}}). \tag{3.3.14}$$

Lemma 3.3.9. *After a choice of* $\bowtie$ *there exist natural isomorphisms*

$$\begin{aligned}\sigma_{\bowtie} :& \pi^*_{1,3}(\Sigma^{\mathrm{dg}}_{\underline{k}}) \otimes \pi^*_{2,4}(\Sigma^{\mathrm{dg}}_{\underline{k}})^* \xrightarrow{\sim} \pi^*_{1,3}(\Sigma^{\mathrm{dg}}_{\underline{0}}) \otimes \pi^*_{2,4}(\Sigma^{\mathrm{dg}}_{\underline{0}})^*\,, \\ \tau_{\bowtie} :& \pi^*_{1}(\Lambda^{\mathrm{dg}}_{\underline{k}}) \otimes \pi^*_{2}(\Lambda^{\mathrm{dg}}_{\underline{k}})^* \xrightarrow{\sim} \pi^*_{1}(\Lambda^{\mathrm{dg}}_{\underline{0}}) \otimes \pi^*_{2}(\Lambda^{\mathrm{dg}}_{\underline{0}})^*\,.\end{aligned}$$

Moreover, we have the isomorphisms

$$\#_{\Sigma^{\mathrm{dg}}_{\underline{a}}} : \Sigma^{\mathrm{dg}}_{\underline{a}} \xrightarrow{\sim} \sigma^*(\Sigma^{\mathrm{dg}}_{\underline{k}-\underline{a}})^*\,, \quad \#_{\underline{a}} : \Lambda^{\mathrm{dg}}_{\underline{a}} \xrightarrow{\sim} (\Lambda^{\mathrm{dg}}_{\underline{k}-\underline{a}})^*\,. \tag{3.3.15}$$

Proof. The following construction works in families due to the work done in §3.2.3, so we restrict ourselves to a point $(p, q) = ([E_1, F_1, \phi_1], [E_2, F_2, \phi_2])$, where $\phi_{1/2,i} : E_{1/2}|_{D_i} \to F_{1/2}|_{D_i}$ are isomorphism. We also set the notation

$$V_{\underline{a}} = \mathrm{End}(E_1, F_1 \otimes L_{\underline{a}}) \quad \text{and} \quad W_{\underline{a}} = \mathrm{End}(E_2, F_2 \otimes L_{\underline{a}})\,. \tag{3.3.16}$$

Using $\bar{\Phi}_{\underline{a}}$ to denote the isomorphism $V_{\underline{a}}|_T \to W_{\underline{a}}|_T$ constructed in Lemma 3.3.8. We then have the data

$$\begin{array}{ccc} \sigma_{V_{\underline{k}}} & \xrightarrow{i\Phi_{\underline{k}}} & \sigma_{W_{\underline{k}}} \\ {\scriptstyle \Omega^{-1}}\downarrow & & \downarrow{\scriptstyle \Omega^{-1}} \\ \sigma_{V_{\underline{0}}} & \xrightarrow[i\Phi_{\underline{0}}]{} & \sigma_{W_{\underline{0}}} \end{array}$$

using that Ω is invertible outside of D. We therefore obtain the isomorphism $\sigma_{\bowtie}$

from Lemma 3.2.18. Then we have the standard definition of the Hodge star $\star$: $\Gamma^\infty(\mathcal{A}^{p,q}) \to \Gamma^\infty(\mathcal{A}^{4-p,4-q})$, given by $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \Omega \wedge \bar{\Omega}$, where $\langle -, - \rangle$ is the hermitian metric on forms. We define then the anti-linear maps

$$\begin{aligned} \#_1 : \mathcal{A}^{0,\text{even}} \to \mathcal{A}^{0,\text{even}} \otimes K_X\,, \qquad & \#_2 : \mathcal{A}^{0,\text{odd}} \to \mathcal{A}^{0,\text{odd}} \otimes K_X \\ \#_1^{\text{op}} : \mathcal{A}^{0,\text{even}} \otimes K_X \to \mathcal{A}^{0,\text{even}}\,, \qquad & \#_2^{\text{op}} : \mathcal{A}^{0,\text{odd}} \otimes K_X \to \mathcal{A}^{0,\text{odd}}\,, \end{aligned} \tag{3.3.17}$$

by $\#_1|_{\mathcal{A}^{0,2q}} = (-1)^q \star, \#_2|_{\mathcal{A}^{0,2q+1}} = (-1)^{q+1}\star, \#_1^{\text{op}}|_{\mathcal{A}^{4,2q}} = (-1)^q\star, \#_2^{\text{op}}|_{\mathcal{A}^{4,2q+1}} = (-1)^{q+1}\star$. These solve $\#_i^{\text{op}} \circ \#_i = \text{id}$ and $\#_i \circ \#_i^{\text{op}} = \text{id}$. We have the commutativity relations $D_{K_X} \circ \#_1 = \#_2 \circ D$ and $\#_2^{\text{op}} \circ D_{K_X} = D \circ \#_1^{\text{op}}$ and obtain the isomorphisms $\det(\sigma_{V_{\underline{a}}}) \cong \overline{\det(\sigma_{V_{\underline{k}-\underline{a}}})} \cong \det(\sigma_{V_{\underline{k}-\underline{a}}})^*$. where the second isomorphism on both lines uses the hermitian metrics on forms which descend to a hermitian metric on the determinant. □

Definition 3.3.10. Let $\mathcal{U}^{\text{vb}} \to Y \times \mathcal{T}_Y$ be the universal vector bundle generated by global sections. We define

$$\begin{aligned} \mathcal{E}\text{xt}^{\text{vb}}_{\underline{a}} &= \pi_{1,2\,*}(\pi^*_{1,3}\mathcal{U}^*_{\text{vb}} \otimes \pi^*_{1,3}\mathcal{U}_{\text{vb}} \otimes \pi^*_1(L_{\underline{a}}))\,, \qquad & \mathbb{P}^{\text{vb}}_{\underline{a}} &= \Delta^*\mathcal{E}\text{xt}^{\text{vb}}_{\underline{a}} \\ \Sigma^{\text{vb}}_{\underline{a}} &= \det(\mathcal{E}\text{xt}^{\text{vb}}_{\underline{a}})\,, & \Lambda^{\text{vb}}_{\underline{a}} &= \Delta^*\big(\det(\Sigma^{\text{vb}}_{\underline{a}})\big) \end{aligned}$$

We then have the isomorphisms

$$\begin{aligned} \sigma^{\text{vb}}_{\bowtie} &= (\Omega^{\text{ag}})^*(\sigma_{\bowtie}) : \pi^*_{1,3}(\Sigma^{\text{vb}}_{\underline{k}}) \otimes \pi^*_{2,4}(\Sigma^{\text{vb}}_{\underline{k}})^* \xrightarrow{\sim} \pi^*_{1,3}(\Sigma^{\text{vb}}_{\underline{0}}) \otimes \pi^*_{2,4}(\Sigma^{\text{vb}}_{\underline{0}})^*\,, \\ \tau^{\text{vb}}_{\bowtie} &= \Delta^*(\sigma^{\text{vb}}_{\bowtie}) : \pi^*_1(\Lambda^{\text{vb}}_{\underline{k}}) \otimes \pi^*_2(\Lambda^{\text{vb}}_{\underline{k}})^* \xrightarrow{\sim} \pi^*_1(\Lambda^{\text{vb}}_{\underline{0}}) \otimes \pi^*_2(\Lambda^{\text{vb}}_{\underline{0}})^*\,. \end{aligned}$$

In particular, we have the $\mathbb{Z}_2$-bundle $O^{\bowtie}_{\text{vb}} \to \mathcal{T}_{Y,D}$, such that naturally $O^{\bowtie}_{\text{vb}} \cong (\Omega^{\text{ag}})^* O^{\bowtie}$.

The following proposition is the result of trying to develop a more general framework of relating compactly supported coherent sheaves to compactly supported pseudo-differential operators using cohesive modules of Block [17] and Yu [174].

Proposition 3.3.11. *There are natural isomorphisms $\kappa^{\text{a,d}}_{\underline{a}} : \Sigma^{\text{vb}}_{\underline{a}} \cong \Sigma^{\text{dg}}_{\underline{a}}$ such that the the following diagram commutes up to natural isotopies:*

$$\begin{array}{ccc} \pi_{1,3}^*(\Sigma^{\text{vb}}_{\underline{k}}) \otimes \pi_{2,4}^*(\Sigma^{\text{vb}}_{\underline{k}})^* & \xrightarrow{\sigma^{\text{vb}}_{\bowtie}} & \pi_{1,3}^*(\Sigma^{\text{vb}}_{\underline{0}}) \otimes \pi_{2,4}^*(\Sigma^{\text{vb}}_{\underline{0}})^* \\ {\scriptstyle \pi_{1,3}^*(\kappa^{\text{a,d}}_{\underline{k}}) \otimes \pi_{2,4}^*(\kappa^{\text{a,d}}_{\underline{k}})^{-*}} \downarrow & & \downarrow {\scriptstyle \pi_{1,3}^*(\kappa^{\text{a,d}}_{\underline{0}}) \otimes \pi_{2,4}^*(\kappa^{\text{a,d}}_{\underline{0}})^{-*}} \\ \pi_{1,3}^*(\Sigma^{\text{dg}}_{\underline{k}}) \otimes \pi_{2,4}^*(\Sigma^{\text{dg}}_{\underline{k}})^* & \xrightarrow{\sigma^{dg}_{\bowtie}} & \pi_{1,3}^*(\Sigma^{\text{dg}}_{\underline{0}}) \otimes \pi_{2,4}^*(\Sigma^{\text{dg}}_{\underline{0}})^* , \end{array} \tag{3.3.18}$$

$$\begin{array}{ccc} \pi_1^*(\Lambda^{\text{vb}}_{\underline{k}}) \otimes \pi_2^*(\Lambda^{\text{vb}}_{\underline{k}})^* & \xrightarrow{\tau^{\text{vb}}_{\bowtie}} & \pi_1^*(\Lambda^{\text{vb}}_{\underline{0}}) \otimes \pi_2^*(\Lambda^{\text{vb}}_{\underline{0}})^* \\ \downarrow & & \downarrow \\ \pi_1^*(\Lambda^{\text{dg}}_{\underline{k}}) \otimes \pi_2^*(\Lambda^{\text{dg}}_{\underline{k}})^* & \xrightarrow{\tau^{dg}_{\bowtie}} & \pi_1^*(\Lambda^{\text{dg}}_{\underline{0}}) \otimes \pi_2^*(\Lambda^{\text{dg}}_{\underline{0}})^* . \end{array}$$

Proof. We examine up close the definitions of each object involved and show that up to natural isotopies in families the diagram commutes. We begin therefore with the definition of $\tau^{a,d}_{\underline{a}}$. We restrict again to a point $(p,q) = ([E_1, F_1, \phi_1], [E_2, F_2, \phi_2])$ as it can be shown by using the arguments of §3.2.3, [30, Prop. 3.25] that our methods work in families. We also use

$$V_{\underline{a},i} := V_{\underline{a}}|_{D_i}\,, \qquad W_{\underline{a},i} := W_{\underline{a}}|_{D_i} \qquad \phi_{\underline{a},i} : V_{\underline{a},i} \xrightarrow{\sim} W_{\underline{a},i}$$

$$\Phi_{\underline{a},i} : V_{\underline{a}}|_{T_i} \xrightarrow{\sim} W_{\underline{a}}|_{T_i}\,, \qquad \Phi_{\underline{a}} : V_{\underline{a}}|_T \xrightarrow{\sim} W_{\underline{a}}|_T$$

Let $E \to Y$ be a holomorphic vector bundle, then $R\Gamma^\bullet(E) = \Gamma(E \otimes \mathcal{A}^{0,\bullet})$. where the differential is given by $\bar{\partial}_E = \bar{\partial}^{\nabla_E}$. Here ∇_E is the corresponding Chern connection. Let $D_E = \bar{\partial}_E + \bar{\partial}_E^* : \Gamma(E \otimes \mathcal{A}^{0,\mathrm{even}}) \to \Gamma(E \otimes \mathcal{A}^{0,\mathrm{odd}})$, then Hodge theory gives us the natural isomorphisms $\det(D_E) \cong \det\big(R\Gamma^\bullet(E)\big)$ after making a contractible choice of metric on E. Continuing to use the notation from Lemma 3.3.9, we obtain the isomorphisms

$$\kappa_{\underline{a}}^{\mathrm{a,d}}|_p : \Sigma_{\underline{a}}^{\mathrm{vb}}|_p \cong \det\big(R\Gamma^\bullet(V_{\underline{a}})\big) \cong \det\big(D_{V_{\underline{a}}}\big) \cong \det(D^{\nabla_{V_{\underline{a}}}}) \cong \Sigma_{\underline{a}}^{\mathrm{ag}}|_p$$

generalizing those of Cao–Gross–Joyce [30, Prop. 3.25], Cao–Leung [31, Thm. 2.2], Joyce–Upmeier [98, p. 38]. Recall now that at (p,q) the isomorphism $\pi_{1,3}^*(\Sigma_{\underline{a}}) \otimes \pi_{2,4}^*(\Sigma_{\underline{a}}^*) \cong \pi_{1,3}^*(\Sigma_{\underline{a}-e_i}) \otimes \pi_{2,4}^*(\Sigma_{\underline{a}-e_i}^*)$ is given by

$$\begin{aligned}\det\big(R\Gamma^\bullet(V_{\underline{a}})\big) \otimes \det\big(R\Gamma^\bullet(W_{\underline{a}})\big)^* &\cong \det\big(R\Gamma^\bullet(V_{\underline{a}-e_i})\big) \otimes \det\big(R\Gamma^\bullet(V_{\underline{a},i})\big)\\ \otimes \det\big(R\Gamma^\bullet(W_{\underline{a},i})\big)^* \otimes \det\big(R\Gamma^\bullet(W_{\underline{a}-e_i})\big)^* &\cong \det\big(R\Gamma^\bullet(V_{\underline{a}-e_i})\big) \otimes \det\big(R\Gamma^\bullet(W_{\underline{a}-e_i})\big)^*,\end{aligned} \tag{3.3.19}$$

where we are using the short exact sequences $0 \to V_{\underline{a}-e_i} \to V_{\underline{a}} \to V_{\underline{a}|_{D_i}} \to 0$ and $0 \to W_{\underline{a}-e_i} \to W_{\underline{a}} \to W_{\underline{a}|_{D_i}} \to 0$. We have the exact complex

$$V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \xrightarrow{(f_{V_{\underline{a}}},\, f_{W_{\underline{a}}})} K_{\underline{a},i} \xrightarrow{\begin{pmatrix} p_{V_{\underline{a}}} \\ p_{W_{\underline{a}}}\end{pmatrix}} V_{\underline{a}} \oplus W_{\underline{a}} \xrightarrow{(r_{V_{\underline{a}}},\, -\phi_{\underline{a},i} \circ r_{W_{\underline{a}}})} V_{\underline{a},i} \to 0\,, \tag{3.3.20}$$

Here $\rho_{V_{\underline{a}}/W_{\underline{a}}}$ are restrictions, $f_{V_{\underline{a}}/W_{\underline{a}}}$ the factors of inclusion and $p_{V_{\underline{a}}/W_{\underline{a}}} \circ f_{V_{\underline{a}}/W_{\underline{a}}} = s_i$ for the section $s_i : \mathcal{O} \xrightarrow{s_i} \mathcal{O}(D_i)$. Moreover, $K_{\underline{a},i} := \ker\big(r_{V_{\underline{a}}} - \phi_{\underline{a},i} \circ r_{W_{\underline{a}}}\big)$ is locally free, because $\mathcal{V}_{\underline{a}}|_{D_i}$ has homological dimension 1. This holds also for the corresponding family on $Y \times \mathcal{T}_{Y,D} \times \mathcal{T}_{Y,D}$ by the same argument. The following is a simple consequence of the construction.

Lemma 3.3.12. *We have the quasi-isomorphisms*

$$\begin{array}{ccc} V_{\underline{a}} \oplus W_{\underline{a}} & \xrightarrow{(f_{V_{\underline{a}}}, f_{W_{\underline{a}}})} & K_{\underline{a},i} \\ \downarrow{\scriptstyle \pi_{V_{\underline{a}-e_i}}} & & \downarrow{\scriptstyle p_{V_{\underline{a}}}} \\ V_{\underline{a}-e_i} & \xrightarrow{s_i} & V_{\underline{a}} \end{array}, \qquad \begin{array}{ccc} V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} & \xrightarrow{(f_{V_{\underline{a}}}, f_{W_{\underline{a}}})} & K_{\underline{a},i} \\ \downarrow{\scriptstyle \pi_{W_{\underline{a}-e_i}}} & & \downarrow{\scriptstyle p_{W_{\underline{a}}}} \\ W_{\underline{a}-e_i} & \xrightarrow{s_i} & W_{\underline{a}} \end{array}.$$

Using $C_{\underline{a}}$ to denote both upper cones and $C_{V_{\underline{a}}}$, $C_{W_{\underline{a}}}$ to denote the lower cones, this gives a commutative diagram $\begin{array}{ccc} C_{\underline{a}} & \longrightarrow & C_{V_{\underline{a}}} \\ \downarrow & & \downarrow \\ C_{W_{\underline{a}}} & \longrightarrow & W_{\underline{a},i} \end{array}$ *of quasi-isomorphisms.*

Therefore (3.3.19) becomes

$$\begin{aligned} &\det\big(\mathrm{R}\Gamma^\bullet(V_{\underline{a}})\big) \otimes \det{}^*\big(R\Gamma^\bullet(V_{\underline{a}-e_i})\big) \otimes \det(R\Gamma^\bullet(W_{\underline{a}-e_i})) \otimes \det{}^*(R\Gamma^\bullet(W_{\underline{a}})\big) \\ &\cong \det\big(R\Gamma^\bullet\big(C_{\underline{a}}\big)\big) \otimes \det{}^*\Big(R\Gamma^\bullet\big(C_{\underline{a}}\big)\Big) \cong \mathbb{C}\,. \end{aligned}$$

Using compatibility with respect to different filtrations discussed in [104, p. 22] and that a dual of an evaluation is a coevaluation in the monoidal category of line bundles together with checking the correct signs one can show that this is expressed as

$$\begin{aligned} &\det(R\Gamma^\bullet(V_{\underline{a}})) \otimes \det{}^*(R\Gamma^\bullet(V_{\underline{a}-e_i})) \otimes \det(R\Gamma^\bullet(W_{\underline{a}-e_i}) \otimes \det{}^*\big(R\Gamma^\bullet\big(W_{\underline{a}}\big)\big) \\ &\cong \frac{\det\Big(R\Gamma^\bullet\Big(V_{\underline{a}}\Big)\Big) \otimes \det{}^*\Big(R\Gamma^\bullet(V_{\underline{a}-e_i})\Big) \otimes \det{}^*\Big(R\Gamma^\bullet(K_{\underline{a},i})\Big) \otimes \det\Big(R\Gamma^\bullet(V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i})\Big)}{\det{}^*\Big(R\Gamma^\bullet(V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i})\Big) \otimes \det\Big(R\Gamma^\bullet(K_{\underline{a},i})\Big) \otimes \det\Big(R\Gamma^\bullet(W_{\underline{a}-e_i})\Big) \otimes \det{}^*\Big(R\Gamma^\bullet(W_{\underline{a}})\Big)} \cong \mathbb{C}\,, \end{aligned} \tag{3.3.21}$$

where the last isomorphism is the consequence of the following short exact sequences:

$$\begin{aligned} 0 &\longrightarrow V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \longrightarrow K_{\underline{a},i} \oplus V_{\underline{a}-e_i} \longrightarrow V_{\underline{a}} \longrightarrow 0\,, \\ 0 &\longrightarrow V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \longrightarrow K_{\underline{a},i} \oplus W_{\underline{a}-e_i} \longrightarrow W_{\underline{a}} \longrightarrow 0\,. \end{aligned} \tag{3.3.22}$$

Let $\bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3$ be the holomorphic structures on each of the three terms of the first sequence, then choosing its splitting we can define $\bar{\partial}_2^t = \begin{pmatrix} \bar{\partial}_1 & t\bar{\partial}_{(1,1)} \\ 0 & \bar{\partial}_3 \end{pmatrix}$ and deforming to $t = 0$ the sequence splits. This gives us the following diagram commuting up to isotopy:

$$\begin{array}{ccc} \det\big(R\Gamma^\bullet(V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i})\big)\det(R\Gamma^\bullet(V_{\underline{a}})) & \xrightarrow{[104,\text{ Cor. } 2]} & \det\big(R\Gamma^\bullet(K_{\underline{a},i} \oplus V_{\underline{a}-e_i})\big) \\ \downarrow & & \downarrow \\ \det\big(D_{V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \oplus V_{\underline{a}}}\big) & \xrightarrow{\quad * \quad} & \det\big(D_{K_{\underline{a},i} \oplus V_{\underline{a}-e_i}}\big)\,, \end{array}$$

(and a similar one for the second sequence) where $*$ corresponds to the isomorphism (3.2.6) for

$$\begin{pmatrix} f_{V_{\underline{a}}} & f_{W_{\underline{a}}} & p^*_{V_{\underline{a}}} \\ 0 & \mathrm{id} & s_i^* \end{pmatrix} : V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \oplus V_{\underline{a}} \longrightarrow K_{\underline{a},i} \oplus V_{\underline{a}-e_i}\,. \tag{3.3.23}$$

In Lemma 3.3.14 below, we show that there exists a natural isomorphism $\Phi_{K_{\underline{a},i}} : K_{\underline{a},i}|_{T_i} \to K_{\underline{a},i}|_{T_i}$, such that all the diagrams below satisfy the conditions in Definition 3.2.17. We therefore get

$$\begin{array}{ccc} \sigma_{V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \oplus V_{\underline{a}}} & \xrightarrow{\begin{pmatrix} 0 & i\Phi^{-1}_{\underline{a}-e_i,i} & 0 \\ i\Phi_{\underline{a}-e_i,i} & 0 & 0 \\ 0 & 0 & i\Phi_{\underline{a},i} \end{pmatrix}} & \sigma_{V_{\underline{a}-e_i} \oplus W_{\underline{a}-e_i} \oplus W_{\underline{a}}} \\ \Big\downarrow {\scriptstyle \begin{pmatrix} f_{V_{\underline{a}}} & f_{W_{\underline{a}}} & p^*_{V_{\underline{a}}} \\ 0 & \mathrm{id} & s_i^* \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} f_{V_{\underline{a}}} & f_{W_{\underline{a}}} & p^*_{V_{\underline{a}}} \\ 0 & \mathrm{id} & s_i^* \end{pmatrix}} \\ \sigma_{K_{\underline{a},i} \oplus V_{\underline{a}-e_i}} & \xrightarrow[\begin{pmatrix} i\Phi_{K_{\underline{a},i}} & 0 \\ 0 & i\Phi_{\underline{a}-e_i} \end{pmatrix}]{} & \sigma_{K_{\underline{a},i} \oplus W_{\underline{a}-e_i}} \end{array}$$

We may restrict (3.3.23) to $X \backslash (1-\epsilon)\bar{T}_i$ because we already cover T_i by the other isomorphisms in the diagram. We choose the compact set $K_i = (1-\epsilon/2)T_i$. Then deform $t \mapsto \left(\begin{smallmatrix} f_{V_{\underline{a}}} & f_{W_{\underline{a}}} & tp^*_{V_{\underline{a}}} \\ 0 & t\mathrm{id} & s_i^* \end{smallmatrix}\right)$ as these are now isomorphisms in $X\backslash(1-\epsilon)T_i$. Moreover, rotating

$$t \mapsto \begin{pmatrix} \sin(t)\,\mathrm{id} & i\cos(t)\,\Phi^{-1}_{\underline{a}-e_i,i} & 0 \\ i\cos(t)\Phi_{\underline{a}-e_i,i} & \sin(t)\,\mathrm{id} & 0 \\ 0 & 0 & \Phi_{\underline{a},i} \end{pmatrix}, \begin{pmatrix} i\cos(t)\,\Phi_{K_{\underline{a},i}} + \sin(t)\,\mathrm{id} & 0 \\ 0 & \Phi_{\underline{a}-e_i} \end{pmatrix}$$

we obtain the separate two diagrams

$$\begin{array}{ccc} \sigma_{V_{\underline{a}-e_i}\oplus W_{\underline{a}-e_i}} & \overset{\left(\begin{smallmatrix}\mathrm{id} & 0 \\ 0 & \mathrm{id}\end{smallmatrix}\right)}{\dashrightarrow} & \sigma_{V_{\underline{a}-e_i}\oplus W_{\underline{a}-e_i}} \\ {\scriptstyle (f_{V_{\underline{a}}}\ f_{W_{\underline{a}}})}\downarrow & & \downarrow{\scriptstyle (f_{V_{\underline{a}}}\ f_{W_{\underline{a}}})} \\ \sigma_{K_{\underline{a},i}} & \underset{\mathrm{id}}{\dashrightarrow} & \sigma_{K_{\underline{a},i}} \end{array}, \quad \begin{array}{ccc} \sigma_{V_{\underline{a}-e_i}} & \overset{\Phi_{\underline{a}-e_i,i}}{\dashrightarrow} & \sigma_{W_{\underline{a}-e_i}} \\ {\scriptstyle s_i^*}\downarrow & & \downarrow{\scriptstyle s_i^*} \\ \sigma_{V_{\underline{a}}} & \underset{\Phi_{\underline{a},i}}{\dashrightarrow} & \sigma_{W_{\underline{a}}} \end{array}$$

In the left diagram we can extend the identities to all of Y, so by Lemma 3.2.18, we showed that (3.3.21) coincides with the adjoint of Ξ from Lemma 3.2.18 for the right diagram. Using this for each step $\underline{k} = \sum_i k_i e_i$, deforming the isomorphisms $\Phi_{\underline{a},i}$ on T_i into $\Phi_{\underline{a}}$ on T and taking $K = \bigcap_{i=1}^N K_i$, we obtain (3.3.18) for the data $\bowtie$ because $\prod (s_{i,k})^* \prod (t_{j,k})^{-*} = \Omega^*$. The second diagram in (3.3.18) is obtained by pulling back along $\Delta : T_{Y,D} \to T_{Y,D} \times T_{Y,D}$. □

Remark 3.3.13. We could replace in (3.3.18) the labels $\underline{k}$, $\underline{0}$ by arbitrary $\underline{a}$, $\underline{b}$.

Lemma 3.3.14. *There exists a natural isomorphism $\Phi_{K_{\underline{a},i}} : K_{\underline{a},i}|_{T_i} \to K_{\underline{a},i}|_{T_i}$, such that $i\cos(t)\Phi_{K_{\underline{a},i}} + \sin(t)\mathrm{id}_{K_{\underline{a},i}}$ are invertible for all t and such that all diagrams used in the proof of Proposition 3.3.11 satisfy the condition of Definition 3.2.17.*

Proof. Using the octahedral axiom, one can show that there are short exact sequences

$0 \to V_{\underline{a}-e_i} \xrightarrow{f_{V_{\underline{a}}}} K_{\underline{a},i} \xrightarrow{p_{W_{\underline{a}}}} W_{\underline{a}} \to 0$ and $0 \to W_{\underline{a}-e_i} \xrightarrow{f_{W_{\underline{a}}}} K_{\underline{a},i} \xrightarrow{p_{V_{\underline{a}}}} V_{\underline{a}} \to 0$. We obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{\underline{a}-e_i}|_{D_i} & \longrightarrow & K_{\underline{a},i}|_{D_i} & \longrightarrow & W_{\underline{a}}|_{D_i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -\phi_{\underline{a}-e_i,i}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi_{\underline{a},i}} & & \\ 0 & \longrightarrow & W_{\underline{a}-e_i}|_{D_i} & \longrightarrow & K_{\underline{a},i}|_{D_i} & \longrightarrow & V_{\underline{a}}|_{D_i} & \longrightarrow & 0 \end{array}$$

as can be seen from the following diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V_{\underline{a}-e_i,i} & \longrightarrow & K_{\underline{a},i}|_{D_i} & \longrightarrow & V_{\underline{a},i} \oplus W_{\underline{a},i} & \longrightarrow & V_{\underline{a},i} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle -\phi_{\underline{a}-e_i,i}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\phi_{\underline{a},i}} & & \\ 0 & \longrightarrow & W_{\underline{a}-e_i,i} & \longrightarrow & K_{\underline{a},i}|_{D_i} & \longrightarrow & V_{\underline{a},i} \oplus W_{\underline{a},i} & \longrightarrow & W_{\underline{a},i} & \longrightarrow & 0 \end{array}$$

induced by restricting

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\underline{a},i}|_{D_i} & \longrightarrow & V_{\underline{a},i} \oplus W_{\underline{a},i} & \longrightarrow & V_{\underline{a},i} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\phi_{\underline{a},i}} & & \\ 0 & \longrightarrow & K_{\underline{a},i}|_{D_i} & \longrightarrow & V_{\underline{a},i} \oplus W_{\underline{a},i} & \longrightarrow & W_{\underline{a},i} & \longrightarrow & 0 \end{array}$$

to the divisor. Choosing a splitting of the first exact sequence in T_i. we obtain $(f_{V_{\underline{a}}}, f_{W_{\underline{a}}}) = \left(\begin{smallmatrix} \mathrm{id} & -\Phi'_{\underline{a}-e_i,i} \\ 0 & s_i \end{smallmatrix}\right)$, where $\Phi'_{\underline{a}-e_i,i}|_{D_i} = \phi_{\underline{a}-e_i,i}$, so we can take $\Phi_{\underline{a}-e_i,i} = \Phi'_{\underline{a}-e_i,i}$. This induces also the splitting of the second sequence in T_i and we can define the isomorphism by

$$\Phi_{K_{\underline{a},i}} : K_{\underline{a},i}|_{T_i} \cong W_{\underline{a}-e_i}|_{T_i} \oplus V_{\underline{a}}|_{T_i} \xrightarrow{\left(\begin{smallmatrix} \Phi_{\underline{a}-e_i} & 0 \\ 0 & \Phi_{\underline{a}} \end{smallmatrix}\right)} V_{\underline{a}-e_i}|_{T_i} \oplus W_{\underline{a}} \cong K_{\underline{a},i}|_{T_i}\,.$$

In this splitting, we then have the invertible isomorphisms $i\cos(t)\Phi_{K_{\underline{a},i}} + \sin(t)\mathrm{id} = \left(\begin{smallmatrix} ie^{it}\Phi_{\underline{a}-e_i} & 0 \\ s_i & ie^{-it}\Phi_{\underline{a}} \end{smallmatrix}\right)$ where we used $\mathrm{id}_{K_{s_i}} = (f_{V_{\underline{a}}}, f_{W_{\underline{a}}}) \circ (f_{V_{\underline{a}}}, f_{W_{\underline{a}}})^{-1} = \left(\begin{smallmatrix} -\Phi_i^{-1} & 0 \\ s_i & \Phi_i \end{smallmatrix}\right)$ and one can check directly, these satisfy the necessary commutativity for all steps needed in Proposition 3.3.11. □

Proposition 3.3.15. *There are natural isomorphisms* $\Lambda^*(D_O) \cong (\Omega^{\mathrm{ag}})^*(O^{\bowtie})$ *of strong H-principal* $\mathbb{Z}_2$*-bundles independent of* $\mathfrak{ord}$.

Proof. Let $O^{\bowtie}_{\mathrm{dg}}$ be the $\mathbb{Z}_2$-bundle associated to

$$\vartheta^{\mathrm{dg}}_{\bowtie} = \left(\pi_1^*(\#_{\underline{k}}) \otimes \pi_2^*(\#_{\underline{k}})^{-1}\right) \circ (\tau^{\mathrm{dg}}_{\bowtie})^{-1} : \pi_1^*(\Lambda^{\mathrm{dg}}_{\underline{0}}) \otimes \pi_2^*(\Lambda^{\mathrm{dg}}_{\underline{0}})^* \to \pi_1^*(\Lambda^{\mathrm{dg}}_{\underline{0}})^* \otimes \pi_2^*(\Lambda^{\mathrm{dg}}_{\underline{0}})\,,$$

then it is a strong H-principal $\mathbb{Z}_2$-bundle by similar arguments to 3.3.7 using Lemma 3.3.9 and part 2 of Lemma 3.2.16, where we obtain similarly as in (3.3.7) additional signs $(-1)^{\deg(\pi_2^*\Lambda^{\mathrm{dg}}_{\underline{0}})\left(\deg(\pi_3^*\Lambda^{\mathrm{dg}}_{\underline{0}})+\deg(\pi_4^*\Lambda^{\mathrm{dg}}_{\underline{0}})\right)}$.

We have the isomorphisms of strong H-principal $\mathbb{Z}_2$-bundles

$$\Lambda^*(O^{\bowtie}) \cong O^{\bowtie}_{\mathrm{dg}} \cong D_O\,,$$

where the first isomorphism is constructed by applying Proposition 3.3.11 and Lemma 3.2.18 and is clearly strong H-principal. For the second one, let us again restrict ourselves to the point (p,q) as in the proof of Proposition 3.3.18. Note that for all t we have isomorphisms:

$$\begin{array}{ccc}
\det(\sigma_{V_{\underline{0}}},\sigma_{W_{\underline{0}}},\bar{\Phi},\bar{\Phi})^t_\chi & \xrightarrow{\text{Prop. 3.2.13(v)}} & \det(\sigma_{V_{\underline{0}}},\sigma_{W_{\underline{0}}},\bar{\Phi},\bar{\Phi})^1_\chi \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_1^*(\Omega)\otimes\pi_2^*(\Omega^{-*})} \\
\det(\sigma_{V_{\underline{0}}},\sigma_{W_{\underline{0}}},\bar{\Phi},\bar{\Phi})^t_\chi & \xrightarrow{\text{Prop. 3.2.13(v)}} & \det(\sigma_{V_{\underline{k}}},\sigma_{W_{\underline{k}}},\bar{\Phi},\bar{\Phi})^1_\chi \\
\Big\downarrow{\scriptstyle (3.3.17)} & & \Big\downarrow{\scriptstyle (3.3.17)} \\
\det^*(\sigma_{V_{\underline{0}}},\sigma_{W_{\underline{0}}},\bar{\Phi},\bar{\Phi})^t_\chi & \xrightarrow{\text{Prop. 3.2.13(v)}} & \det^*(\sigma_{V_{\underline{0}}},\sigma_{W_{\underline{0}}},\bar{\Phi},\bar{\Phi})^1_\chi
\end{array}$$

The composition of the vertical arrows on the left is precisely $\vartheta^{\mathrm{dg}}_{\bowtie}$, while the associated $\mathbb{Z}_2$-torsor to the real structure corresponding to vertical arrows on the right is by excision as in Lemma 3.3.8 isomorphic to $D_O|_{(p,q)}$ using that $\bar{\Phi}$ are unitary and (3.3.17) restrict outside of ϵT to (2.1.2). As the diagram is commutative for all t, we obtain the required isomorphisms $D_O|_{(p,q)} \cong O^{\bowtie}_{\mathrm{dg}}$. It is an isomorphisms of strong H-principal $\mathbb{Z}_2$-bundles by the compatibility under direct sums of Lemma 3.2.16 and using the arguments of [30, Prop. 3.25] (see also proof of Lemma 3.3.8) together with paying attention to the signs above.

The independence of any choice of $\mathfrak{ord}$ can then be shown because diagram (3.3.18) commutes and both $\sigma^{\mathrm{vb}}_{\bowtie}$ and $\sigma^{\mathrm{dg}}_{\bowtie}$ are independent (see proof of Theorem 3.1.20). We only sketch the idea, as the precise formulation is comparably more tedious than the proof of Proposition 3.3.11: one uses excision on the diagram (3.1.3) using the common resolutions of Lemma 3.3.12, where the top right corner of (3.1.3) has resolution $V_{\underline{0}} \oplus W_{\underline{0}} \to K_{e_2,2}$, bottom left $V_{\underline{0}} \oplus W_{\underline{0}} \to K_{e_1,1}$ and the bottom right one

$$(V_{\underline{0}} \oplus W_{\underline{0}} \to K_{e_1,1} \oplus V_{e_2} \oplus W_{e_2} \to K_{e_1+e_2,1}) \cong (V_{\underline{0}} \oplus W_{\underline{0}} \to K_{e_2,2} \oplus V_{e_1} \oplus W_{e_1} \to K_{e_1+e_2,2})\,.$$

The automorphism on $K_{e_1,1}|_{T_1}$, $K_{e_2,2}|_{T_2}$, $K_{e_1+e_2,1}|_{T_1}$, $K_{e_1+e_2,2}|_{T_i}$ are then the ones constructed in Lemma 3.3.14 and one follows the arguments of Proposition 3.3.11 to remove contributions of all K's. $\square$

Theorem 3.1.18 now follows from the above corollary together with applying Proposition 3.2.5 (i) and then (ii).

3.4 Orientation groups for non-compact Calabi–Yau fourfolds

In this final section, we describe the behavior of orientations under direct sums. We recall the notion of orientation group from [97] and formulate the equivalent version of [97, Theorem 2.27] for the non-compact setting, where we replace K-theory with compactly supported K-theory. For background on compactly supported cohomology theories, see Spanier [159], Ranicki–Roe [149, §2]. From the algebraic point of view, see the discussion in Joyce–Song [96, §6.7] and Fulton [61, §18.1].

3.4.1 Orientation on compactly supported K-theory

In (3.3.12), we define the strong H-principal $\mathbb{Z}_2$-bundle $D_O^{\mathcal{C}} \to \mathcal{C}_{Y,D}$. We first describe its commutativity rules as in [97, Definition 2.22].

Definition 3.4.1. Let $\mu_{\mathcal{C}} : \mathcal{C}_{Y,D} \times \mathcal{C}_{Y,D} \to \mathcal{C}_{Y,D}, \mu_{\rm cs} : \mathcal{C}^{\rm cs} \times \mathcal{C}^{\rm cs} \to \mathcal{C}^{\rm cs}$ be the binary maps and

$$\tau : D_O^{\mathcal{C}} \boxtimes D_O^{\mathcal{C}} \longrightarrow \mu_{\mathcal{C}}^*(D_O^{\mathcal{C}})\,, \quad \tau^{\rm cs} : O^{\rm cs} \boxtimes O^{\rm cs} \longrightarrow \mu_{\rm cs}^*(O^{\rm cs}) \tag{3.4.1}$$

be the isomorphisms of $\mathbb{Z}_2$-bundles on $\mathcal{C}_{Y,D} \times \mathcal{C}_{Y,D}$, which make $(D_O^{\mathcal{C}}, \tau)$ into a strong H-principal $\mathbb{Z}_2$-bundle and $\tau^{\rm cs}$ a restriction of τ.

We recall the notion of Euler-form as defined in Joyce–Tanaka–Upmeier [97, Definition 2.20] for real elliptic differential operators.

Definition 3.4.2. Let X be a smooth compact manifold, E_0, E_1 vector bundles on X and $P : E_0 \to E_1$ a real or complex elliptic differential operator. Let $E, F \to X$

be complex vector bundles, the Euler form $\chi_P : K^0(X) \times K^0(X) \to \mathbb{Z}$ is defined by

$$\chi_P(\llbracket E \rrbracket, \llbracket F \rrbracket) = \mathrm{ind}_{\mathbb{C}}(\sigma(P) \otimes \mathrm{id}_{\pi^*(\mathrm{Hom}(E,F))})$$

together with bi-additivity of χ_P. We used the notation from §3.2.3 for symbols of operators. If X is spin and $P = \not{D}_+$, we write $\chi_X^{\mathbb{R}} := \chi_{\not{D}}$. Similarly, if X is a complex manifold and $D = \bar{\partial} + \bar{\partial}^* : A^{0,\mathrm{even}} \to A^{0,\mathrm{odd}}$ is the Dolbeault operator, then we use $\chi_X := \chi_D$.

Recall that we have a comparison map $c : G_0(X) \to K^0(X)$, where $G_0(X)$ is the Grothendieck group associated to $D^b\mathrm{Coh}(X)$. Let $\chi_X^{\mathrm{ag}} : K_0(X) \times K_0(X) \to \mathbb{Z}$ be defined by $\chi_X^{\mathrm{ag}}(E,F) = \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{C}}(\mathrm{Ext}^i(E,F))$, then when X is smooth , we have $\chi_X \circ (c \times c) = \chi_X^{\mathrm{ag}}$.

Proposition 3.4.3. *Let $i_1, i_2 : Y \to Y \cup_D Y$ be the inclusions of the two copies of Y and $\delta(\alpha,\beta) \in K^0(Y \cup_D Y)$ denote a K-theory class, such that $i_1^*(\delta(\alpha,\beta)) = \alpha$ and $i_2^*(\delta(\alpha,\beta)) = \beta$. We have the bijection $\pi_0(\mathcal{C}_{Y,D}) = K^0(Y \cup_D Y)$. Let $\mathcal{C}_{\delta(\alpha,\beta)}$ be the components corresponding to $\delta(\alpha,\beta)$ and $D_O^{\mathcal{C}}|_{\delta(\alpha,\beta)}$ the restriction of $D_O^{\mathcal{C}}$ to it. Suppose a choice of trivialization $o_{\delta(\alpha,\beta)} : \mathbb{Z}_2 \to D_O^{\mathcal{C}}|_{\delta(\alpha,\beta)}$ is given for each $\delta(\alpha,\beta) \in K^0(Y \cup_D Y)$, then define $\epsilon_{\delta(\alpha_1,\beta_1),\delta(\alpha_2,\beta_2)} \in \{-1,1\}$ by*

$$\tau\big(o_{\delta(\alpha_1,\beta_1)} \boxtimes_{\mathbb{Z}_2} o_{\delta(\alpha_2,\beta_2)}\big) = \epsilon_{\delta(\alpha_1,\beta_1),\delta(\alpha_2,\beta_2)} o_{\delta(\alpha_1,\beta_1)+\delta(\alpha_2,\beta_2)}$$

These signs satisfy:

$$\epsilon_{\delta(\alpha_2,\beta_2),\delta(\alpha_1,\beta_1)} = (-1)^{\left(\chi_Y(\alpha_1,\alpha_1)-\chi_Y(\beta_1,\beta_1)\right)\left(\chi_Y(\alpha_2,\alpha_2)-\chi_Y(\beta_2,\beta_2)\right)+\chi_Y(\alpha_1,\alpha_2)-\chi_Y(\beta_1,\beta_2)} \epsilon_{\delta(\alpha_1,\beta_1),\delta(\alpha_2,\beta_2)}\,.$$

$$\epsilon_{\delta(\alpha_1,\beta_1),\delta(\alpha_2,\beta_2)}\epsilon_{\delta(\alpha_1+\alpha_2,\beta_1+\beta_2),\delta(\alpha_3,\beta_3)} = \epsilon_{\delta(\alpha_2,\beta_2),\delta(\alpha_3,\beta_3)}\epsilon_{\delta(\alpha_2+\alpha_3,\beta_2+\beta_3),\delta(\alpha_1,\beta_1)}\,. \tag{3.4.2}$$

Let $(M_{\gamma(\alpha,\beta)})^{\mathrm{top}} = \Gamma^{-1}(\mathcal{C}_{\gamma(\alpha,\beta)})$, $(O^{\bowtie}_{\gamma(\alpha,\beta)})^{\mathrm{top}} = (O^{\bowtie})^{\mathrm{top}}|_{(M_{\gamma(\alpha,\beta)})^{\mathrm{top}}}$ *and* $o^{\mathrm{ag}}_{\gamma(\alpha,\beta)} = \mathfrak{I}^{\bowtie}\big(\Gamma^*(o_{\gamma(\alpha,\beta)})\big)$ *the trivializations of* $(O^{\bowtie}_{\gamma(\alpha,\beta)})^{\mathrm{top}}$ *obtained using* $\mathfrak{I}^{\bowtie}$ *from* (3.1.18)*. Let* $\phi^{\bowtie}$ *be from Proposition 3.3.7, then it satisfies*

$$\phi^{\bowtie}\big(o^{\mathrm{ag}}_{\delta(\alpha_1,\beta_1)} \boxtimes_{\mathbb{Z}_2} o^{\mathrm{ag}}_{\delta(\alpha_2,\beta_2)}\big) = \epsilon_{\delta(\alpha_1,\beta_1),\delta(\alpha_2,\beta_2)} o^{\mathrm{ag}}_{\delta(\alpha_1,\beta_1)+\delta(\alpha_2,\beta_2)}\,.$$

Proof. Recall the definition of $\tilde{Y}$, $\tilde{T}$ from Definition 3.3.1. One can express $D^{\mathcal{C}}_{O}(\tilde{Y})$ as a product of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-bundles $p_1^*(O^{\not{D}_{\tilde{Y}}}_{\mathcal{C}}) \otimes p_2^*(O^{\not{D}_{\tilde{Y}}}_{\mathcal{C}})^*$ obtained by Proposition 3.2.5 from $O^{\not{D}_{\tilde{Y}}}$ in Example 3.2.8, where $\mathcal{C}_{\tilde{Y}} \xleftarrow{p_1} \mathcal{C}_{\tilde{Y}} \times_{\mathcal{C}_D} \mathcal{C}_{\tilde{Y}} \xrightarrow{p_2} \mathcal{C}_{\tilde{Y}}$ are the projections and $\deg\big(O^{\not{D}_{\tilde{Y}}}_{\mathcal{C}}\big)|_{\mathcal{C}_{\tilde{\alpha}}} = \chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\alpha},\tilde{\alpha})$. Using Definition 3.2.8 and Lemma 3.2.6 together with Joyce–Tanaka–Upmeier [97, p. 2.26], a simple computation shows that for each $\tilde{\gamma}_i(\tilde{\alpha}_i,\tilde{\beta}_i)$, which under inclusion $\tilde{\iota}_{1,2} : \tilde{Y} \to \tilde{Y} \cup_{\tilde{T}} \tilde{Y}$ restrict to $\tilde{\alpha}_i, \tilde{\beta}_i$ respectively, we have the formula

$$\tilde{\tau}_{\tilde{\gamma}_2(\tilde{\alpha}_2,\beta_2),\tilde{\gamma}_1(\tilde{\alpha}_1,\tilde{\beta}_1)} = (-1)^{\left(\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\alpha}_1,\tilde{\alpha}_1)-\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\beta}_1,\tilde{\beta}_1)\right)\left(\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\alpha}_2,\tilde{\alpha}_2)-\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\beta}_2,\tilde{\beta}_2)\right)+\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\alpha}_1,\tilde{\alpha}_2)-\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\beta}_1,\tilde{\beta}_2)} \tilde{\tau}_{\tilde{\gamma}_1(\tilde{\alpha}_1,\tilde{\beta}_1),\tilde{\gamma}_2(\tilde{\alpha}_2,\tilde{\beta}_2)} \tag{3.4.3}$$

Two points $[E^{\pm},F^{\pm},\phi^{\pm}]$ of $\mathcal{V}_Y \times_{\mathcal{V}_{\tilde{T}}} \mathcal{V}_Y$ map to $[\tilde{E}^{\pm},\tilde{F}^{\pm},\tilde{\phi}^{\pm}] \in \mathcal{V}_{\tilde{Y},\tilde{T}}$, as described in

(3.3.11). Using excision on index and Definition 3.4.2, it follows that

$$\chi^{\mathbb{R}}_{\tilde{Y}}([\![\tilde{E}^+]\!],[\![\tilde{E}^-]\!])-\chi^{\mathbb{R}}_{\tilde{Y}}([\![\tilde{F}^+]\!],[\![\tilde{F}^-]\!])=\chi_Y([\![E^+]\!],[\![E^-]\!])-\chi_Y([\![F^+]\!],[\![F^-]\!])\,.$$

Using (3.3.13) and biadditivity of χ, we obtain

$$\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\alpha}_1,\tilde{\alpha}_2)-\chi^{\mathbb{R}}_{\tilde{Y}}(\tilde{\beta}_1,\tilde{\beta}_2)=\chi_Y(\alpha_1,\alpha_2)-\chi_Y(\beta_1,\beta_2)\,,$$

where $\tilde{\alpha}_i,\tilde{\beta}_i$ are K-theory classes glued from α_i,β_i as in (3.3.11), from which we obtain

$$\begin{aligned}\tau_{\delta(\alpha_2,\beta_2),\delta(\alpha_1,\beta_1)}=&(-1)^{\big(\chi_Y(\alpha_1,\alpha_1)-\chi_Y(\beta_1,\beta_1)\big)\big(\chi_Y(\alpha_2,\alpha_2)-\chi_Y(\beta_2,\beta_2)\big)+\chi_Y(\alpha_1,\alpha_2)-\chi_Y(\beta_1,\beta_2)}\\&\tau_{\delta(\alpha_1,\beta_1),\delta(\alpha_2,\beta_2)}\,.\end{aligned}$$

This leads to (3.4.2) by using that $D^{\mathcal{C}}_O$ is strong H-principal. To conclude the final statement of the proposition, one applies Proposition 3.3.15. □

Recall from Definition 3.1.19 that we have the map $\Gamma^{\mathrm{cs}}:(\mathcal{M}_X)^{\mathrm{top}}\to\mathcal{C}^{\mathrm{cs}}_X$. There exists a compactly supported Chern character which is an isomorphism

$$\mathrm{ch}_{\mathrm{cs}}:K^*_{\mathrm{cs}}(X)\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow H^*_{\mathrm{cs}}(X,\mathbb{Q}) \tag{3.4.4}$$

of $\mathbb{Z}_2$-graded rings. We also have the Euler form on $H^{\mathrm{even}}_{\mathrm{cs}}(X,\mathbb{Q})$:

$$\begin{aligned}&\bar{\chi}:H^{\mathrm{even}}_{\mathrm{cs}}(X,\mathbb{Q})\times H^{\mathrm{even}}_{\mathrm{cs}}(X,\mathbb{Q})\longrightarrow\mathbb{Q}\\&\bar{\chi}(a,b)=\deg(a^\vee\cdot b\cdot\mathrm{td}(TX))_4\,.\end{aligned} \tag{3.4.5}$$

Combining (3.4.4) and (3.4.5), one gets $\bar{\chi} : K^0_{\mathrm{cs}}(X) \times K^0_{\mathrm{cs}}(X) \to \mathbb{Z}$. Note that, we have $\bar{\chi}(\alpha, \beta) = \chi_Y(\bar{\alpha}, \bar{\beta})$, where for a class $\alpha \in K^0_{\mathrm{cs}}(X)$, $\bar{\alpha}$ denotes the class of $K^0(Y)$ extended trivially to D.

Using that O^ω is strong H-principal by the same arguments as in the compact case (see Cao–Gross–Joyce [30, Lemma 3.13]), we have the isomorphisms $\phi^\omega : O^\omega \boxtimes_{\mathbb{Z}_2} O^\omega \to \mu^*(O^\omega)$. We can now state the main result of comparison of signs under sums in non-compact Calabi–Yau fourfolds.

Theorem 3.4.4. *Let* $\mathcal{C}^{\mathrm{cs}}_\alpha$ *denote the connected component of* $\mathcal{C}^{\mathrm{cs}}_X$ *corresponding to* $\alpha \in K^0_{\mathrm{cs}}(X) = \pi_0(\mathcal{C}^{\mathrm{cs}}_X)$ *and* $O^{\mathrm{cs}}_\alpha = O^{\mathrm{cs}}|_{\mathcal{C}^{\mathrm{cs}}_\alpha}$. *Let* $(\mathcal{M}_\alpha)^{\mathrm{top}} = (\Gamma^{\mathrm{cs}})^{-1}(\mathcal{C}^{\mathrm{cs}}_\alpha)$. *After fixing choices of trivializations* o^{cs}_α *of* O^{cs}_α, *we define* $\epsilon_{\alpha,\beta}$

$$\phi^\omega\Big(\mathfrak{I}\big(\Gamma^* o^{\mathrm{cs}}_\alpha\big) \boxtimes \mathfrak{I}\big(\Gamma^* o^{\mathrm{cs}}_\beta\big)\Big) \cong \epsilon_{\alpha,\beta}\mathfrak{I}\big(\Gamma^* o^{\mathrm{cs}}_{\alpha+\beta}\big)\,.$$

If one moreover fixes the preferred choice of o^{cs}_0, *such that*

$$\tau^{\mathrm{cs}}(o^{\mathrm{cs}}_0 \boxtimes o^{\mathrm{cs}}_0) = o^{\mathrm{cs}}_0\,, \tag{3.4.6}$$

then $\epsilon : (\alpha, \beta) \mapsto \epsilon_{\alpha,\beta} \in \{\pm 1\}$ *is up to equivalences the unique group 2-cocycle satisfying*

$$\epsilon_{\alpha,\beta} = (-1)^{\bar{\chi}(\alpha,\alpha)\bar{\chi}(\beta,\beta)+\bar{\chi}(\alpha,\beta)}\epsilon_{\beta,\alpha} \tag{3.4.7}$$

Proof. Recall from the proof of Theorem 3.1.18 that we have the isomorphism $\xi^*_Y\big(O^{\bowtie}\big) \cong O^\omega$ which is by constructions in Proposition 3.3.7 strong H-principal as can be checked directly by comparing $\mathbb{Z}_2$-torsors at each $\mathbb{C}$-point $[i_*(E), 0]$.

By Proposition 3.4.3 after setting β_1 and β_2 equal to 0 and $\alpha_1 = \bar{\alpha}$, $\alpha_2 = \bar{\beta}$ it then follows that, we have (3.4.7) together with

$$\epsilon_{\alpha,0} = \epsilon_{0,\alpha} = 1\,, \qquad \epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma} = \epsilon_{\beta,\gamma}\epsilon_{\alpha,\beta+\gamma} \tag{3.4.8}$$

which it precisely the cocycle condition. The first condition follows from (3.4.6). □

Remark 3.4.5. If the need arises to show independence of compactification, one can relate two compactifications $Y_1 \leftarrow \tilde{Y} \rightarrow Y_2$ by a common one obtained as a blow up of the closure of $X \hookrightarrow Y_1 \times Y_2$. Then one could use [150, Thm. 1.2] comparing Hodge cohomologies for locally free sheaves under blow up in hopes of showing independence of the isomorphism $\mathfrak{I}$.

We now discuss the orientation group from Joyce–Tanaka–Upmeier [97, Definition 2.26] applied to $K^0_{\mathrm{cs}}(X)$ instead of $K^0(X)$ in our non-compact setting. The *compactly supported orientation group* is defined as

$$\Omega_{\mathrm{cs}}(X) = \{(\alpha, o^{\mathrm{cs}}_\alpha) : \alpha \in K^0_{\mathrm{cs}}(X), o^{\mathrm{cs}}_\alpha \text{ orientation on } \mathcal{C}^{\mathrm{cs}}_\alpha\}\,.$$

The multiplication is given by $(\alpha, o^{\mathrm{cs}}_\alpha) \star (\beta, o^{\mathrm{cs}}_\beta) = \big(\alpha+\beta, \tau^{\mathrm{cs}}_{\alpha,\beta}(o^{\mathrm{cs}}_\alpha \boxtimes_{\mathbb{Z}_2} o^{\mathrm{cs}}_\beta)\big)$. The resulting group is the unique group extension $0 \to \mathbb{Z}_2 \to \Omega_{\mathrm{cs}}(X) \to K^0_{\mathrm{cs}}(X) \to 0$ for the group 2-cycle ϵ of Theorem 3.4.4. Choices of orientations induce a splitting $K^0_{\mathrm{cs}} \to \Omega_{\mathrm{cs}}(X)$ as sets. This fixed $\epsilon : K^0_{\mathrm{cs}}(X) \times K^0_{\mathrm{cs}}(X) \to \mathbb{Z}_2$ in Theorem 3.4.4. Let us describe the method used in [97, Thm. 2.27] for extending orientation. Choosing generators of $K^0_{\mathrm{cs}}(X)$, one obtains

$$K^0_{\mathrm{cs}}(X) \cong \mathbb{Z}^r \times \prod_{k=1}^{p} \mathbb{Z}_{m_k} \times \prod_{j=1}^{q} \mathbb{Z}_{2^{p_j}}\,, \tag{3.4.9}$$

where $m_k > 2$ odd and $p_j > 0$. Fixing a choice of isomorphism (3.4.9), choose orientation on each $C^{\mathrm{cs}}_{\alpha_i}$, $\alpha_i = (0,\dots,0,1,,0\dots,0)$, where 1 is in position i. Use τ^{cs} to obtain orientations for all $\alpha \in K^0_{\mathrm{cs}}(X)$ by adding generators going from left to right in the form $(a_1,\dots,a_p,(b_j)_{j=1}^q,(c_k)_{k=1}^p)$ and using in each step $o^{\mathrm{cs}}_{\alpha'+g} = \tau^{\mathrm{cs}}(o^{\mathrm{cs}}_{\alpha'} \boxtimes o^{\mathrm{cs}}_g)$, where g is a generator. As a result, one obtains the splitting:

$$\mathrm{Or}(\mathfrak{o}) : \Omega_{\mathrm{cs}}(X) \cong K^0_{\mathrm{cs}}(X) \times \{-1,1\} \cong \mathbb{Z}^r \times \prod_{k=1}^q \mathbb{Z}_{m_k} \times \prod_{j=1}^p \mathbb{Z}_{2^{p_j}} \times \{-1,1\}, \quad (3.4.10)$$

where $\mathfrak{o}$ is the set of orientation on C^{cs}_α for the chosen generators α. Let $\bar{\chi}_{ij} := \bar{\chi}(\alpha_i,\alpha_j)$. The next result replaces K-theory by compactly supported K-theory in Joyce–Tanaka–Upmeier [97, Thm. 2.27] and considers the $\mathbb{Z}_2$-bundle O^{cs} we constructed. We mention this also because in (ii) it describes the rule for obtaining the signs $\epsilon_{\alpha,\beta}$ which will be useful in Chapter 4.

Theorem 3.4.6 ([97, Thm. 2.27])**.** *Let $Or(\mathfrak{o})$ be the isomorphism* (3.4.10) *for a given choice of orientations $\mathfrak{o}$ on generators corresponding to the isomorphism* (3.4.9)*. Let T_2 be the 2-torsion subgroup of $K^0_{\mathrm{cs}}(X)$. Then:*

(i) Define the map $\xi : T_2 \to \mathbb{Z}_2$ as $\Xi(\gamma) = \epsilon_{\gamma,\gamma}$, Then it is a group homomorphism.

(ii) Using $\mathrm{Or}(\mathfrak{o})$ *from* (3.4.10) *to identify $\Omega^{\mathrm{cs}}(X)$ with $\mathbb{Z}^r \times \prod_{k=1}^p \mathbb{Z}_{m_k} \times \prod_{j=1}^q \mathbb{Z}_{2^{p_j}} \times \{-1,1\}$ the induced group structure on the latter becomes*

$$\begin{aligned}&\Big(a_1,\dots,a_r,(b_j)_{j=1}^p,(c_k)_{k=1}^q,o\Big) \star \Big(a'_1,\dots,a'_r,(b'_j)_{j=1}^p,(c'_k)_{k=1}^q,o'\Big)\\ &= \Big(a_1+a'_1,\dots,a_r+a'_r,(b_j-b'_j)_{j=1}^p,(c_k+c'_k)_{k=1}^q,\\ &(-1)^{\sum_{1\le h<i\le r}(\bar{\chi}_{hi}+\bar{\chi}_{hh}\bar{\chi}_{ii})a'_h a_i}\Xi(\gamma)o\cdot o'\Big),\end{aligned}$$

where $\gamma = (0, \ldots, 0, (0)_{k=1}^q, (\tilde{c}_j)_{j=1}^q)$ *and*

$$\tilde{c}_j = \left\lfloor \frac{\bar{c}_j + \bar{c}'_j}{\overline{c_j + c'_j}} \right\rfloor 2^{p_j - 1}$$

for the unique representatives $0 \leq \bar{c}_j, \bar{c}'_j, \overline{c_j + c'_j} < 2^{p_j}$.

Chapter 4

Wall-crossing for CY fourfolds

In this chapter we study DT4-type invariants for compact Calabi–Yau fourfolds using the wall-crossing ideas sketched in Gross–Joyce–Tanaka [76, §4.4]. Unlike their [76, Conjecture 4.11], we need the category of pairs and wall-crossing formulae for Joyce–Song pairs, which we formulate in Conjecture 4.2.10. Using the recent result of Cao–Qu [40], we show that the conjecture of Cao–Kool [32] follows from ours. As we investigate other implications of the wall-crossing formulae, we find a general expression for tautological integrals on Hilbert schemes of points. Their relation to Hilbert schemes on surfaces is then discussed in §4.5.4. Most of the contents of this chapter can be found in the author's previous work [20]. Moreover, similar methods will be used to recover the results of Arbesfeld–Johnson–Lim–Oprea–Pandharipande [5], Lim [118], Oprea–Pandharipande [143] in author's future work [18]. We only consider the simplest case of elliptic surfaces and curves in the last section to recover the correspondence of 4.5.14.

4.1 Vertex algebras in algebraic geometry and topology

In physics, 2-dimensional conformal field theories are distinguished from all other dimensions by their richness. Among many other applications they can be used to capture the dynamics of a string in a curved space-time. Vertex algebras defined by Borcherds [22] for the purpose of tackling the monstrous moonshine conjecture and studying representations of Kac–Moody algebras express the chiral part of a conformal field theory and the *state to field correspondence* for the excited states of a string. In this section, we recall their definition and a particular example useful to us. We recall Joyce's construction of these in algebraic geometry and formulate an alternative topological construction. We continue working from this more topological point of view as it is better suited to our setting.

4.1.1 Vertex algebras

Let us recall first the definition of vertex algebras focusing on *graded super-lattice vertex algebra*. For background literature, we recommend [22, 99, 55, 56, 115, 66], with Borcherds [22] being most concise.

Definition 4.1.1. A $\mathbb{Z}$-*graded vertex algebra* over a field $\mathbb{Q}$ is a collection of data $(V_*, T, |0\rangle, Y)$, where V_* is a $\mathbb{Z}$-graded vector space, $T : V_* \to V_{*+2}$ is graded linear, $|0\rangle \in V_0$, $Y : V_* \to \mathrm{End}(V_*)[\![z, z^{-1}]\!]$ is graded linear after setting $\deg(z) = -2$, satisfying the following: Let $u, v, w \in V_*$, then

i. We always have $Y(u, z)v \in V_*(\!(z)\!)$,

ii. $Y(|0\rangle, z)v = v$,

iii. $Y(v, z)|0\rangle = e^{zT}v$,

iv. Let $\delta(z) = \sum_{n\in\mathbb{Z}} z^n \in \mathbb{Q}[\![z, z^{-1}]\!]$

$$z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y\big(Y(u, z_0)v, z_2\big)w = z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y(u, z_1)Y(v, z_2)w$$
$$- (-1)^{\deg(u)\deg(v)} z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y(v, z_2)Y(u, z_1)w\,.$$

By Borcherds [22], the graded vector-space $V_{*+2}/T(V_*)$ carries a graded Lie algebra structure determined by

$$[\bar{u}, \bar{v}] = [z^{-1}]Y(u, z)v\,. \tag{4.1.1}$$

Let $A^\pm$ be abelian groups and $\chi^\pm : A^\pm \times A^\pm \to \mathbb{Z}$ be symmetric, resp. anti-symmetric bi-additive maps. Let us denote $\mathfrak{h}^\pm = A^\pm \otimes_{\mathbb{Z}} \mathbb{Q}$ and fix a basis of $B^\pm$ of $\mathfrak{h}^\pm$. For (A^+, χ^+) and a choice of a group 2-cocycle $\epsilon : A^+ \times A^+ \to \mathbb{Z}_2$ satisfying

$$\epsilon_{\alpha,\beta} = (-1)^{\chi^+(\alpha,\beta)+\chi^+(\alpha,\alpha)\chi^+(\beta,\beta)}\epsilon_{\beta,\alpha}\,, \quad \forall\, \alpha, \beta \in A^+(X) \tag{4.1.2}$$

there is a natural graded vertex algebra on

$$\mathbb{Q}[A^+] \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}[u_{v,i}, v \in B^+, i > 0] \cong \mathbb{Q}[A^+] \otimes_{\mathbb{Q}} \mathrm{Sym}\big(\mathfrak{h}^+ \otimes t^2\mathbb{Q}[t^2]\big)\,, \tag{4.1.3}$$

where the isomorphism takes $u_{v,i} \mapsto v \otimes t^{2i}$ and t is of degree 1. This vertex algebra is called the *generalized lattice vertex algebra* (see [115, §6.4], [99, §5.4]). For given

(A^{-},χ^{-}), Abe [1] describes a natural $\mathbb{Z}$-graded vertex algebra on

$$\Lambda_{\mathbb{Q}}[u_{v,i}, v\in B^{-}, i>0] \cong \Lambda\big(\mathfrak{h}^{-}\otimes t\mathbb{Q}[t^2]\big)\,, \tag{4.1.4}$$

where the isomorphism is given by $u_{v,i}\mapsto v\otimes t^{2i-1}$ and t degree 1. Suppose we have vertex algebras $(V_*, T_V, |0\rangle_V, Y_V)$ and $(W_*, T_W, |0\rangle_W, Y_W)$, then there is a graded Vertex algebra on their tensor product, with state to field correspondence

$$Y_{V_*\otimes W_*}(v\otimes w, z)(u\otimes t) = (-1)^{\deg(u)\deg(w)}Y_{V_*}(v,z)u\,Y_{W_*}(w,z)t\,.$$

The super lattice vertex algebra for $(A^{+}\oplus A^{-},\chi^{\bullet})$ is then given by the tensor product of (4.1.3) and (4.1.4).

From the definition of the super-lattice vertex algebra $(V_*, T, |0\rangle, Y)$ associated to $(A_{+}\oplus A_{-},\chi^{\bullet})$ we can easily deduce the fields on the generators of:

$$\begin{aligned} V_* &\cong \mathbb{Q}[A_+]\otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![u_{v,i}, v\in B, i>0]\!] \\ &\cong \mathbb{Q}[A_+]\otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}\big(A_+\otimes_{\mathbb{Z}} t^2\mathbb{Q}[t^2]\big)\otimes_{\mathbb{Q}}\Lambda_{\mathbb{Q}}\big(A_-\otimes_{\mathbb{Z}} t\mathbb{Q}[t^2]\big)\,. \end{aligned}$$

Let $\alpha\in A_+$, such that $\alpha=\sum_{v\in B_+}\alpha_v v$. We use e^{α} to denote the corresponding element in $\mathbb{Q}[A_+]$. For $K \in V_*$, we have

$$\begin{aligned} Y(e^0\otimes u_{v,1},z)e^\beta\otimes K &= e^\beta\otimes\Big\{\sum_{k>0}u_{v,k}\cdot Kz^{k-1} \\ &\quad +\sum_{k>0}\sum_{w\in B}k\chi^\bullet(v,w)\frac{dK}{du_{w,k}}z^{-k-1}+\chi^\bullet(v,\beta)z^{-1}\Big\}, \\ Y(e^\alpha\otimes 1,z)e^\beta\otimes K &= \epsilon_{\alpha,\beta}z^{\chi^+(\alpha,\beta)}e^{\alpha+\beta}\otimes\exp\Big[\sum_{k>0}\sum_{v\in B_+}\frac{\alpha_v}{k}u_{v,k}z^k\Big] \\ &\quad \exp\Big[-\sum_{k>0}\sum_{v\in B_+}\chi^+(\alpha,v)\frac{d}{du_{v,k}}z^{-k}\Big]K\,. \end{aligned} \tag{4.1.5}$$

Note that by the reconstruction lemma [115, Thm. 5.7.1], Ben-Zvi [55, Thm. 4.4.1] and Kac [99, Thm. 4.5] these formulae are sufficient for determining all fields.

4.1.2 Axioms of vertex algebras on homology

For a higher stack $\mathcal{S}$, we denote by $H_*(\mathcal{S}) = H_*(\mathcal{S}^{\mathrm{top}})$, $H^*(\mathcal{S}) = H^*(\mathcal{S}^{\mathrm{top}})$ its Betti (co)homology as in Joyce [91], Gross [74]. Note that we will always treat $H_*(T,\mathbb{Q})$ as a direct sum and $H^*(T,\mathbb{Q})$ as a product over all degrees. Following May–Ponto [132, §24.1] define the topological K-theory of $\mathcal{S}$ to be

$$K^0(\mathcal{S}) = [\mathcal{S}^{\mathrm{top}}, BU\times\mathbb{Z}]\,,$$

where $[X,Y] = \pi_0\big(\mathrm{Map}_{C^0}(X,Y)\big)$. For any $\mathcal{E}$ in $L_{\mathrm{pe}}(\mathcal{S})$ there is a unique map $\phi_{\mathcal{E}}$: $\mathcal{S}\to\mathrm{Perf}_{\mathbb{C}}$ in $\mathbf{Ho}(\mathbf{HSt})$. Using Blanc [16, §4.1], this gives

$$[\![\mathcal{E}]\!] : \mathcal{S}^{\mathrm{top}} \longrightarrow BU\times\mathbb{Z}\,.$$

in **Ho**(**Top**). We then have a well defined map assigning to each perfect complex $\mathcal{E}$ its class $[\![\mathcal{E}]\!] \in K^0(\mathcal{S})$.

The cohomology of $BU \times \mathbb{Z}$ is given by

$$H^*(BU \times \mathbb{Z}) \cong \mathbb{Q}[\mathbb{Z}] \otimes_{\mathbb{Q}} \mathbb{Q}[\![\beta_1, \beta_2, \ldots]\!] \,,$$

where $\beta_i = \mathrm{ch}_i(\mathfrak{U})$ and $\mathfrak{U}$ is the universal K-theory class. Similarly to [91], we define $\mathrm{ch}_i(\mathcal{E}) = [\![\mathcal{E}]\!]^*(\beta_i)$ and the Chern classes by the Newton identities for symmetric polynomials:

$$\sum_{n\geq 0} c_n(\mathcal{E})q^n = \exp\Big[\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\mathrm{ch}_n(\mathcal{E})q^n\Big] \,. \tag{4.1.6}$$

As $BU \times \mathbb{Z}$ is a ring space [132, §4.1], the set $K^0(\mathcal{S})$ carries a natural ring structure. Moreover, by similar arguments as in [132, §4.1], one also has a map $(-)^\vee : BU\times\mathbb{Z} \to BU\times\mathbb{Z}$ inducing a map $(-)^\vee : K^0(\mathcal{S}) \to K^0(\mathcal{S})$. When $\mathcal{S}$ is replaced with a compact CW-complex X, this becomes the standard K-theory $K^0(X)$ and $(-)^\vee$ corresponds to taking duals.

Definition 4.1.2 (Joyce [91])**.** Let $(\mathcal{A}, K(\mathcal{A}), \mathcal{M}, \Phi, \mu, \Theta, \epsilon)$ be data satisfying:

- $\mathcal{A}$ is an abelian category or derived category.
- Let $K_0(\mathcal{A}) \to K(\mathcal{A})$ be a map of abelian groups. For each $E \in \mathrm{Ob}(\mathcal{A})$ denote $[\![E]\!] \in K(\mathcal{A})$ the image of its class.
- $\mathcal{M}$ a moduli stack of objects in $\mathcal{A}$ with an action $\Phi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ corresponding to multiplication by $\lambda\mathrm{id}$ of $\mathrm{Aut}(E)$ for any $E \in \mathrm{Ob}(\mathcal{A})$ and a map $\mu : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ corresponding to direct sum.

- For each $\alpha \in K(\mathcal{A})$, $\mathcal{M}_\alpha$ is an open and closed substack of objects $[\![E]\!] = \alpha$.
- $\Theta \in L_{\mathrm{pe}}(\mathcal{M} \times \mathcal{M})$ satisfying $\sigma^*(\Theta) \cong \Theta^\vee[2n]$ for some $n \in \mathbb{Z}$ where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ interchanges factors and

$$(\mu \times \mathrm{id}_{\mathcal{M}})^*(\Theta) \cong \pi_{13}^*(\Theta) \oplus \pi_{23}^*(\Theta)\,, \qquad (\mathrm{id}_{\mathcal{M}} \times \mu)^*(\Theta) \cong \pi_{12}^*(\Theta) \oplus \pi_{13}^*(\Theta)$$
$$(\Phi \times \mathrm{id}_{\mathcal{M}})^*(\Theta) \cong \mathcal{V}_1 \boxtimes \Theta\,, \qquad (\mathrm{id}_{\mathcal{M}} \times \Phi)^*(\Theta) \cong \mathcal{V}_1^* \boxtimes \Theta\,, \qquad (4.1.7)$$

  where $\mathcal{V}_1$ is the universal line bundle on $[*/\mathbb{G}_m]$. One also writes $\Theta_{\alpha,\beta} = \Theta|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta}$ and $\chi(\alpha,\beta) = \mathrm{rk}(\Theta_{\alpha,\beta})$, where $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is a bi-additive symmetric form.

- A group 2-cocycle $\epsilon : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}_2$ satisfying (4.1.2) with respect to $\chi^+ = \chi$.

Let $\hat{H}_*(\mathcal{M})$ be the homology with shifted grading given by $\hat{H}_n(\mathcal{M}_\alpha) = H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$ for each $\alpha \in K(X)$, then using the above data one constructs a vertex algebra $(\hat{H}_*(\mathcal{M}), |0\rangle, e^{zT}, Y)$ over the $\mathbb{Q}$ vector space $\hat{H}_*(\mathcal{M})$. It is defined by:

- $|0\rangle = 0_*(*)$, where $0 : * \to \mathcal{M}$ is the inclusion of the zero object,
- $T(u) = \Phi_*(t \boxtimes u)$ for all $u \in \hat{H}_*(\mathcal{M})$ where $t \in H_2([*/\mathbb{G}_m]) = H_2(\mathbb{CP}^\infty)$ is the generator of homology given by inclusion $\mathbb{CP}^1 \subset \mathbb{CP}^\infty$.
- The state to field correspondence Y is given by

$$Y(u,z)v = \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)} \mu_*^{\mathrm{top}}(e^{zT} \otimes \mathrm{id})\big((u \boxtimes v) \cap c_{z^{-1}}(\Theta_{\alpha,\beta})\big)\,.$$

  for all $u \in \hat{H}_a(\mathcal{M}_\alpha)$, $v \in \hat{H}_b(\mathcal{M}_\beta)$.

The following definition is familiar to experts and can be extracted from a more general formula for generalized complex cohomology theories in Gross [75, Prop. 5.3.8].

Definition 4.1.3. Let $(\mathcal{C}, \mu, 0)$ be an H-space with a $\mathbb{CP}^\infty$ action $\Phi : \mathbb{CP}^\infty \times \mathcal{C} \to \mathcal{C}$ which is an H-map. Let $\theta \in K^0(\mathcal{C}) = [\mathcal{C}, BU \times \mathbb{Z}]$ be a K-theory class satisfying $\sigma^*(\theta) = \theta^\vee$ and

$$(\mu \times \mathrm{id}_\mathcal{C})^*(\theta) = \pi_{13}^*(\theta) + \pi_{23}^*(\theta)\,, \qquad (\mathrm{id}_\mathcal{C} \times \mu)^*(\theta) = \pi_{12}^*(\theta) + \pi_{13}^*(\theta)\,,$$
$$(\Phi \times \mathrm{id}_\mathcal{C})^*(\theta) = V_1 \boxtimes \theta\,, \qquad (\mathrm{id}_\mathcal{C} \times \Phi)^*(\theta) = V_1^* \boxtimes \theta\,,$$

where $V_1 \to \mathbb{CP}^\infty$ is the universal line bundle.

Let $\pi_0(\mathcal{C}) \to K$ be a morphism of commutative monoids. Denote $\mathcal{C}_\alpha$ to be the open and closed subset of $\mathcal{C}$ which is the union of connected components of $\mathcal{C}$ mapped to $\alpha \in K$. We write again $\theta_{\alpha,\beta} = \theta|_{\mathcal{C}_\alpha \times \mathcal{C}_\beta}$, and $\chi(\alpha, \beta) = \mathrm{rk}(\theta_{\alpha,\beta})$ must be a symmetric bi-additive form on K. Let $\epsilon : K \times K \to \{-1, 1\}$ satisfying (4.1.2), (3.4.8) be a group 2-cocycle and $\hat{H}_a(\mathcal{C}_\alpha) = H_{a-\chi(\alpha,\alpha)}(\mathcal{C}_\alpha)$. Then we denote by $(\hat{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y)$ the vertex algebra on the graded $\mathbb{Q}$-vector space $\hat{H}_*(\mathcal{C})$ defined for the data $(\mathcal{C}, K(\mathcal{C}), \Phi, \mu, 0, \theta, \epsilon)$ by

- $|0\rangle = 0_*(*)$ and $T(u) = \Phi_*(t \boxtimes u)$ as before
- the state to field correspondence Y is given by

$$Y(u, z)v = \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)} \mu_*(e^{zT} \otimes \mathrm{id})\big((u \boxtimes v) \cap c_{z^{-1}}(\theta_{\alpha,\beta})\big)\,,$$

  for all $u \in \hat{H}_a(\mathcal{C}_\alpha)$, $v \in \hat{H}_b(\mathcal{C}_\beta)$.

Remark 4.1.4. We can assign to $(\mathcal{A}, K(\mathcal{A}), \mathcal{M}, \Phi, \mu, \Theta, \epsilon)$ the data

$$(\mathcal{M}^{\mathrm{top}}, C_0(\mathcal{A}), \Phi^{\mathrm{top}}, \mu^{\mathrm{top}}, 0^{\mathrm{top}}, \theta, \epsilon)$$

from Definition 4.1.3, where $C_0(\mathcal{A}) \subset K(A)$ is the cone of all $[\![E]\!] \in K(A)$, Φ^{top}, μ^{top}, 0^{top} are maps in $\mathbf{Ho}(\mathbf{Top})$ and $\theta := [\![\Theta]\!]$. The two vertex algebras obtained on $\hat{H}_*(\mathcal{M})$ are clearly the same.

The wall-crossing formulae in Joyce [95], Gross–Joyce–Tanaka [76] are expressed in terms of a Lie algebra defined by Borcherds [22]. Let $(\hat{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y)$ be the vertex algebra from Definition 4.1.3 and define

$$\Pi_{*+2} : \hat{H}_{*+2}(\mathcal{C}) \longrightarrow \check{H}_*(\mathcal{C}) = \hat{H}_{*+2}(\mathcal{C})/T\big(\hat{H}_*(\mathcal{C})\big)\,,$$

then, by (4.1.1), this has a natural Lie algebra structure.

4.1.3 Insertions

To compute invariants using the homology classes of Conjecture 4.2.10, we need to consider elements in the dual of $\check{H}_0(\mathcal{M})$ or $\check{H}_0(\mathcal{N}_{q,n})$ (see Definition 4.2.3). We do so in the algebraic topological language, as it is more general and is closer to the computations that follow.

Definition 4.1.5. Let $(\mathcal{C}, K(\mathcal{C}), \Phi, \mu, 0, \theta, \epsilon)$ be the data as in Definition 4.1.3, then a *weight 0 insertion* is a cohomology class $I \in H^*(\mathcal{C})$ satisfying

$$\Phi^*(I) = 1 \boxtimes I\,.$$

Lemma 4.1.6. *Let $I \in H^*(\mathcal{C})$ be a weight 0 insertion, then $I_m \in H^m(\mathcal{C})$ induces a well defined map $\check{I}_{-2+\chi(\alpha,\alpha)+m} : \check{H}_{-2+\chi(\alpha,\alpha)+m}(\mathcal{C}_\alpha) \to \mathbb{Q}$ for all $m \geq 0$.*

Proof. Suppose that we have $V, V' \in H_m(\mathcal{C})$ such that $V - V' = D(W)$ for $W \in H_{m-2}(\mathcal{C})$. As $D(W) = \Phi_*(t \boxtimes W)$, using the push-pull formula in (co)homology we see

$$
\begin{aligned}
D(W) \cap I_m = \Phi_*(t \boxtimes W) \cap I_m &= \Phi_*\big(t \boxtimes W \cap \Phi^*(I_m)\big) \\
= \Phi_*\big((t \boxtimes W) \cap (1 \boxtimes I_m)\big) &= \Phi_*\big(t \boxtimes (W \cap I_m)\big) = 0\,.
\end{aligned}
$$

Integrating cohomology class $1 \in H_0(\mathcal{C})$ on both sides shows that $I_m(V - V') = 0$. ☐

Let $[M] \in \check{H}_m(\mathcal{C})$ and I a weight zero insertion. Then we will use the notation

$$
\int_{[M]} I = \check{I}_m([M])\,. \tag{4.1.8}
$$

Example 4.1.7. Suppose that $\mathcal{J} \in K^0(\mathcal{C} \times \mathcal{C})$ satisfies

$$
(\Phi \times \mathrm{id}_{\mathcal{C}})^*(\mathcal{J}) = \mathcal{V}_1^* \boxtimes \mathcal{J}\,, \qquad (\mathrm{id}_{\mathcal{C}} \times \Phi)^*(\mathcal{J}) = \mathcal{V}_1 \boxtimes \mathcal{J}\,,
$$

then $\mathcal{I} = \Delta^*(\mathcal{J})$ satisfies $\Phi^*(\mathcal{I}) = 1 \boxtimes \mathcal{I}$. In particular if $p(x_1 t, x_2 t^2, \dots)$ is a power series in infinitely many variables then $I = p(\mathrm{ch}_1(\mathcal{I}), \mathrm{ch}_2(\mathcal{I}), \dots)$ is a weight zero insertion.

Often times insertions behave well under direct sums. In the algebraic setting the following definition has been stated more generally in [76, Definition 2.11].

Definition 4.1.8. Let $(\hat{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y)$ be a vertex algebra for the data in Definition 4.1.3. Let $F \to \mathcal{C} \times \mathcal{C}$ be a vector bundle satisfying

$$(\mu \times \mathrm{id}_{\mathcal{C}})^*(F) \cong \pi_{13}^*(F) \oplus \pi_{23}^*(F)\,, \qquad (\mathrm{id}_{\mathcal{C}} \times \mu)^*(F) \cong \pi_{12}^*(F) \oplus \pi_{13}^*(F)$$
$$(\Phi \times \mathrm{id}_{\mathcal{C}})^*(F) \cong V_1^* \boxtimes F\,, \qquad (\mathrm{id}_{\mathcal{C}} \times \Phi)^*(F) \cong V_1 \boxtimes F\,, \tag{4.1.9}$$

such that $\xi(\alpha, \beta) := \mathrm{rk}(F|_{\mathcal{C}_\alpha \times \mathcal{C}_\beta})$ is constant for all $\alpha, \beta \in K(\mathcal{C})$. Then define the *$F$-twisted vertex algebra* associated to $(\hat{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y)$ to be the vertex algebra given by Definition 4.1.3 for the data $(\mathcal{C}, K(\mathcal{C}), \Phi, \mu, 0, \theta^F, \epsilon^\xi)$, where

$$\theta^F = \theta + [\![F^*]\!] + [\![\sigma^*(F)]\!]\,,$$
$$\epsilon^\xi_{\alpha,\beta} = (-1)^{\xi(\alpha,\beta)}\epsilon_{\alpha,\beta} \qquad \forall\ \alpha, \beta \in K(\mathcal{C})\,.$$

We denote this vertex algebra by $(\tilde{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y^F)$.

One can then conclude by the same arguments as in the proof of [76, Thm. 2.12](see Joyce [91]) that:

Proposition 4.1.9. *In the situation of Definition 4.1.8 let $E = \Delta^*(F)$ and consider the morphism of graded $\mathbb{Q}$-vector spaces $(-) \cap c_\xi(E) : \hat{H}_*(\mathcal{C}) \to \tilde{H}_*(\mathcal{C})$, such that on $\hat{H}_*(\mathcal{C}_\alpha)$ it acts by $u \mapsto u \cap c_{\xi(\alpha,\alpha)}(E)$. Then it induces a morphism of vertex algebras*

$$(-) \cap c_\xi(E) : (\hat{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y) \longrightarrow (\tilde{H}_*(\mathcal{C}), |0\rangle, e^{zT}, Y^F)\,. \tag{4.1.10}$$

Moreover, let $[-,-]$ be the Lie bracket on $\check{H}_*(\mathcal{C})$ and $[-,-]^F$ the Lie bracket on

$\mathring{H}_*(\mathcal{C}) = \tilde{H}_{*+2}(\mathcal{C})/T\big(\tilde{H}_*(\mathcal{C})\big)$. *Then* (4.1.10) *induces a well-defined map of Lie algebras*

$$(-)\cap c_\xi(E) : (\check{H}_*(\mathcal{C}), [-,-]) \longrightarrow (\mathring{H}_*(\mathcal{C}), [-,-]^F)\,.$$

4.2 Wall-crossing for pairs

After fixing a particular choice of orientations, we begin by constructing an explicit vertex algebra on the moduli stack of pairs and relating it to the vertex algebra of *topological pairs*. We then conjecture a wall-crossing formula for *Joyce–Song stable pairs* in the homology of the previous vertex algebras. The final subsection serves the purpose of giving an explicit description of vertex algebras of topological pairs.

4.2.1 Point-canonical orientations

Recall from Theorem 3.4.4 or Cao–Gross–Joyce [30, Thm. 1.15] that there are unique cocycles $\epsilon : K^0(X)\times K^0(X) \to \mathbb{Z}_2$ after fixing trivializations o_α of O_α for each $\alpha \in K^0(X)$, where we omit $(-)^{\mathrm{cs}}$ as we are working purely within the compact setting. We do so by using generators and the trivialization of $\mathrm{Or}(\mathfrak{o})$ of (3.4.10). For our purposes, we will have a preferred set of generators.

Let us now fix orientations o_α for $\alpha = N[\![\mathcal{O}_X]\!] + np$. For this we set p to be the K-theory class of a sky-scraper sheaf at some point $x\in X$. Let M_p denote the moduli scheme of sheaves of class p. There is an isomorphism $M_p = X$ and Cao–Leung [37, Proposition 7.17]) showed that

$$[M_p]^{\mathrm{vir}} = \pm \mathrm{Pd}(c_3(X)) \in H_2(X)\,, \tag{4.2.1}$$

where Pd(−) denotes the Poincare dual. There is a natural map $m_p : M_p \to \mathcal{M}_X$ giving $\Gamma \circ m_p^{\text{top}} : M_p^{\text{top}} \to \mathcal{C}_p$. Similarly, the point moduli space $\{\mathcal{O}_X\} = *$ comes with a natural map $i_{\mathcal{O}_X} : \{\mathcal{O}_X\} \to \mathcal{M}_X$ and carries a natural virtual fundamental class $1 \in H_0(*) \cong \mathbb{Z}$.

Definition 4.2.1. In (3.4.6), choose the order of generators of $K^0(X)$ such that $[\![\mathcal{O}_X]\!] < p < g$ for any other generator $g \in G$. We fix orientation o_g for all $g \in G$, such that $(\Gamma \circ m_p^{\text{top}})^*(o_p)$ induces the Oh-Thomas/ Borisov–Joyce virtual fundamental class $[M_p]^{\text{vir}} = \text{Pd}(c_3(X))$ and $o_{[\![\mathcal{O}_X]\!]}$ induces the virtual fundamental class $[\{\mathcal{O}_X\}]^{\text{vir}} = 1 \in H_0(\text{pt})$. We will denote these choices of orientations o_p^{can} and $o_{[\![\mathcal{O}_X]\!]}^{\text{can}}$ respectively. By the construction in Theorem 3.4.6, these determine orientations for all $\alpha \in K^0(X)$.

Remark 4.2.2. We will see that this is the right choice of orientations for working with Hilbert schemes of points. On the other hand, when working with stable pairs for 1-dimensional sheaves as in [41], the author checked that the correct orientations to recover the wall-crossing formula in [41, Conj. 0.2] are obtained by fixing the order of generators such that $p < (\beta, 1) < [\![\mathcal{O}_X]\!]$, where $(\beta, 1)$ denotes the K-theory class such that $\text{ch}(\beta, 1) = (\beta, 1) \in H^6(X) \oplus H^8(X)$.

4.2.2 Vertex algebras over pairs

In this section, a Calabi–Yau fourfold (X, Ω) must additionally satisfy $H^i(\mathcal{O}_X) = 0$ for $i = 1, 2, 3$. We now construct the vertex algebra on the auxiliary category of pairs and its topological analog.

Definition 4.2.3. Let X be a Calabi–Yau fourfold and $\mathcal{A} = \text{Coh}(X)$. Fix a choice of an ample divisor H and let τ denote the Gieseker stability condition with respect to H. Then let $\mathcal{A}_q$ be a full abelian subcategory of $\mathcal{A}$ with objects the zero sheaf and τ-semistable sheaves with reduced Hilbert polynomial q. Define $\mathcal{B}_q$ to be the abelian category of triples (E, V, ϕ), where $E \in \mathrm{Ob}(\mathcal{A}_q)$, $V \in \mathrm{Vect}_{\mathbb{C}}$ and $\phi : V \otimes \mathcal{O}_X(-n) \to E$. The morphisms are pairs $(f, g) : (E, V, \phi) \to (E', V', \phi')$ where $f : E \to E'$ and $g : V \to V'$ satisfy $\phi' \circ g \otimes \mathrm{id}_{\mathcal{O}_X(-n)} = f \circ \phi$. The moduli stack $\mathcal{N}_q$ of $\mathcal{B}_q$ is Artin by [96, Lem. 13.2]. Moreover, consider the full exact subcategory $\mathcal{B}_{q,n}$ of objects (E, V, ϕ), such that $H^i\big(E(n)\big) = 0$ for $i > 0$ and the corresponding open substack $\mathcal{N}_{q,n}$, where the openness follows from [79, Thm. 12.8.].

Let

$$\lambda : K_0(\mathcal{A}) \longrightarrow K^0(X) \tag{4.2.2}$$

be induced by the usual comparison map. We define $C(\mathcal{A}_{q,n}) \subset K_0(\mathcal{A})$ to be the cone of $[\![E]\!]$ for non-zero $E \in \mathrm{Ob}(\mathcal{A}_{q,n})$. Let $\mathcal{C}_0(\mathcal{B}_{q,n}) = (C(\mathcal{A}_{q,n}) \sqcup \{0\}) \times (\mathbb{N} \sqcup \{0\})$, then for all $(\alpha, d) \in \mathcal{C}_0(\mathcal{B}_{q,n})$ define $\mathcal{N}^{\alpha,d}_{q,n}$ as follows:

- If $(\alpha, d) \in C(\mathcal{A}_{q,n}) \times \mathbb{N}$ then $\mathcal{N}^{\alpha,d}_{q,n}$ is the total space of a vector bundle $\pi_{\alpha,d}$: $\pi_{2*}\big(\mathcal{F}^{\alpha}_{q,n}\big) \boxtimes \mathcal{V}^*_d \to \mathcal{M}^{\alpha}_{q,n} \times [*/\mathrm{GL}(d, \mathbb{C})]$. Here

$$\mathcal{F}^{\alpha}_{q,n} = \pi_1^*\big(\mathcal{O}_X(n)\big) \otimes \mathcal{E}^{\alpha}_{q,n}\,, \tag{4.2.3}$$

  where $\mathcal{E}^{\alpha}_{q,n}$ is the universal sheaf over $X \times \mathcal{M}^{\alpha}_{q,n}$ and $\mathcal{M}^{\alpha}_{q,n}$ the moduli stack of τ-semistable objects E with $p_E = q$ and $[\![E]\!] = \alpha$. We also use $\mathcal{V}_d$ to denote the universal vector bundle of rank d,

- $\mathcal{N}^{\alpha,0}_{q,n} = \mathcal{M}^{\alpha}_{q,n}$, $\mathcal{N}^{0,d}_{q,n} = [*/\mathrm{GL}(d, \mathbb{C})]$ and $\mathcal{N}^{0,0}_{q,n} = *$.

Then we have

$$\mathcal{N}_{q,n} = \bigsqcup_{(\alpha,d)\in C_0(\mathcal{B}_{q,n})} \mathcal{N}_{q,n}^{\alpha,d}\,.$$

We now describe the remaining ingredients needed in Definition 4.1.2. For a perfect complex/K-theory/cohomology class κ on $Z_i \times Z_j$, we use the notation $(\kappa)_{i,j}$ to denote $\pi_{i,j}^*(\kappa)$, where

$$\pi_{i,j} : \prod_{k\in I} Z_k \longrightarrow Z_i \times Z_j$$

is a projection to the i, j components.

Definition 4.2.4. We have a natural action $\Phi_{\mathcal{N}_{q,n}} : [*/\mathbb{G}_m] \times \mathcal{N}_{q,n}^{\alpha,d} \to \mathcal{N}_{q,n}^{\alpha,d}$ which is a lift of the diagonal $[*/\mathbb{G}_m]$ action on the base $\mathcal{M}_{q,n}^{\alpha} \times [*/\mathrm{Gl}(d,\mathbb{C})]$ to the total space. We define the map of monoids

$$K(\Omega) : C_0(\mathcal{B}_{q,n}) \xrightarrow{(\lambda,\mathrm{id})} K^0(X) \times \mathbb{Z}\,. \tag{4.2.4}$$

Let $\Theta_{\alpha,\beta} = \underline{\mathrm{Hom}}_{\mathcal{M}_{q,n}^{\alpha}\times\mathcal{M}_{q,n}^{\beta}}(\mathcal{E}_{q,n}^{\alpha}, \mathcal{E}_{q,n}^{\beta})^\vee$. Let $\mathcal{F}_{q,n}^{\alpha}$ be as in (4.2.3). We define $\Theta^{\mathrm{pa}}_{(\alpha_1,d_1),(\alpha_2,d_2)} \in L_{\mathrm{pe}}(\mathcal{N}_{q,n}^{\alpha_1,d_1} \times \mathcal{N}_{q,n}^{\alpha_2,d_2})$ for all $(\alpha_i, d_i) \in C_0(\mathcal{N}_{q,n})$ by

$$\begin{aligned}\Theta^{\mathrm{pa}}_{(\alpha_1,d_1),(\alpha_2,d_2)} =&(\pi_{\alpha_1,d_1} \times \pi_{\alpha_2,d_2})^* \\ &\Big\{(\Theta_{\alpha_1,\alpha_2})_{1,3} \oplus \Big((V_{d_1} \boxtimes V_{d_2}^*)^{\oplus 2}\Big)_{2,4} \oplus \Big(\mathcal{V}_{d_1} \boxtimes \pi_{2*}(\mathcal{F}_{q,n}^{\alpha_2})^\vee\Big)_{2,3}[1] \\ &\oplus \Big(\pi_{2*}(\mathcal{F}_{q,n}^{\alpha_1}) \boxtimes \mathcal{V}_{d_2}^*\Big)_{1,4}[1]\Big\}\,. \end{aligned} \tag{4.2.5}$$

The perfect complex Θ^{pa} on $\mathcal{N}_{q,n} \times \mathcal{N}_{q,n}$ is defined to have the restriction to $\mathcal{N}_{q,n}^{\alpha_1,d_1} \times$

$\mathcal{N}_{q,n}^{\alpha_2,d_2}$ given by (4.2.5). The corresponding bi-additive symmetric form is given by

$$\chi^{\mathrm{pa}}\big((\alpha,d_1),(\alpha_2,d_2)\big) = \mathrm{rk}\big(\Theta^{\mathrm{pa}}_{(\alpha_1,d_1),(\alpha_2,d_2)}\big)$$
$$= \chi(\alpha_1,\alpha_2) + 2d_1d_2 - d_1\big(\chi(\alpha_2(n))\big) - d_2\chi\big(\alpha_1(n)\big)\,.$$

The signs $\epsilon^{\mathrm{pa}}_{(\alpha_1.d_1),(\alpha_2.d_2)}$ are defined by:

$$\epsilon^{\mathrm{pa}}_{(\alpha_1.d_1),(\alpha_2.d_2)} = \epsilon_{\lambda(\alpha_1 - d_1[\![\mathcal{O}_X(n)]\!]),\lambda(\alpha_2-d_2[\![\mathcal{O}_X(n)]\!])}\,. \tag{4.2.6}$$

where ϵ is from Theorem 3.4.4.

We show that the conditions of Definition 4.1.2 are satisfied (only) in K-theory, i.e.

$$\big((\mathcal{N}_{q,n})^{\mathrm{top}}, C_0(\mathcal{B}_{q,n}), \mu^{\mathrm{top}}_{\mathcal{N}_{q,n}}, \mu^{\mathrm{top}}_{\mathcal{N}_{q,n}}, 0^{\mathrm{top}}, [\![\Theta^{\mathrm{pa}}]\!], \epsilon^{\mathrm{pa}}\big) \tag{4.2.7}$$

satisfy assumptions of Definition 4.1.3.

Lemma 4.2.5. *The data* (4.2.7) *satisfies the conditions in 4.1.3. Denote by* $(\hat{H}(\mathcal{N}_{q,n}), |0\rangle, e^{zT}, Y)$ *the corresponding vertex algebra.*

Proof. To show that $[\![\Theta^{\mathrm{pa}}]\!]$ satisfies $\sigma^*([\![\Theta^{\mathrm{pa}}]\!]) = [\![\Theta^{\mathrm{pa}}]\!]^\vee$ we note that

$$\sigma^*_{\alpha_1,\alpha_2}(\Theta_{\alpha_2,\alpha_1}) \cong \Theta^\vee_{\alpha_1,\alpha_2}[-4]\,,$$
$$\sigma^*_{(\alpha_1,d_1),(\alpha_2,d_2)}\big(V_{d_2}\boxtimes V^*_{d_1}\big)_{2,4} = (V^*_{d_1}\boxtimes V_{d_2})_{2,4} = (V_{d_1}\boxtimes V^*_{d_2})^*\,,$$
$$\sigma^*_{(\alpha_1,d_1),(\alpha_2,d_2)}\big((V_{d_2}\boxtimes \pi_{2*}(\mathcal{F}^{\alpha_1}_{q,n})^\vee)_{2,3}\big) = (\pi_{2*}(\mathcal{F}^{\alpha_1}_{q,n})^\vee\boxtimes V_{d_2})_{1,4} = (\pi_{2*}(\mathcal{F}^{\alpha_1}_{q,n})\boxtimes V^*_{d_2})^\vee_{1,4}\,.$$

The rest of the properties for $[\![\Theta^{\mathrm{pa}}]\!]$ follow immediately, because V_d and $\pi_{2*}(\mathcal{F}^{\alpha,d}_{q,n})$ are weight 1 (see Joyce [91]) with respect to the $[*/\mathbb{G}_m]$ action and they are additive under sums. The signs $\epsilon_{(\alpha_1,d_1),(\alpha_2,d_2)}$ satisfy (4.1.7) for χ^{pa} because the map $\tau : K(\mathcal{A}_{q,n})\times\mathbb{Z} \to K(X)$ given by $\tau(\alpha, d) = \lambda(\alpha) - d[\![\mathcal{O}_X(-n)]\!]$ is a group homomorphism satisfying $\chi \circ (\tau \times \tau) = \chi^{\mathrm{pa}}$. The latter statement uses that $\chi(\mathcal{O}_X) = 2$. □

We use the map $\Sigma : \mathcal{N}_{q,n} \to \mathcal{M}_X \times \mathrm{Perf}_{\mathbb{C}}$ where $\mathcal{M}_X = \mathcal{M}_{L_{\mathrm{pe}}(X)}$. For each $(\alpha, d) \in C_0(\mathcal{N}_{q,n})$ the restriction $\Sigma_{(\alpha,d)} = \Sigma|_{\mathcal{N}^{\alpha,d}_{q,n}}$ can be expressed as $\Sigma_{(\alpha,d)} = (\iota^{\alpha}_{q,n} \times \iota_d) \circ \pi_{\alpha,d}$, where $\iota^{\alpha}_{q,n} : \mathcal{M}^{\alpha}_{q,n} \to \mathcal{M}_X$ and $\iota_d : [*/\mathrm{GL}(d,\mathbb{C})] \to \mathrm{Perf}_{\mathbb{C}}$ are the inclusions. As $\pi_{\alpha,d} : \mathcal{N}^{\alpha,d}_{q,n} \to \mathcal{M}^{\alpha}_{q,n} \times [*/\mathrm{GL}(d,\mathbb{C})]$ is an $\mathbb{A}^1$-homotopy equivalence we do not lose any information.

While there is an explicit description of $H_*(\mathcal{M}_X)$ (see Gross [74]) in terms of the semi-topological K-theory groups $K^*_{\mathrm{sst}}(X)$ of Friedlander–Walker [57], we will not use it because these can be complicated for general Calabi–Yau fourfolds. Instead we transfer the problem into completely topological setting using

$$\Omega = (\Gamma \times \mathrm{id}) \circ \Sigma^{\mathrm{top}} : \mathcal{N}^{\mathrm{top}}_{q,n} \longrightarrow \mathcal{M}^{\mathrm{top}}_X \times BU \times \mathbb{Z} \longrightarrow \mathcal{C}_X \times BU \times \mathbb{Z}\,, \tag{4.2.8}$$

where Γ is from Definition 3.1.17. This will induce a morphism of vertex algebras when using the correct data on $\mathcal{P}_X := \mathcal{C}_X \times BU \times \mathbb{Z}$. Denote by $\mathfrak{U}$ and $\mathfrak{E}$ the universal K-theory classes on $BU \times \mathbb{Z}$, respectively $X \times \mathcal{C}_X$. We will also use the notation $\mathfrak{F}_n = \pi_1^*([\![\mathcal{O}_X(n)]\!]) \cdot \mathfrak{E}$.

Definition 4.2.6. Define $\theta_{\mathcal{P}} \in K^0(\mathcal{P}_X \times \mathcal{P}_X)$ by

$$\theta_{\mathcal{P}} = (\theta_{\mathcal{C}})_{1,3} + 2(\mathfrak{U} \boxtimes \mathfrak{U}^{\vee})_{2,4} - \big(\mathfrak{U} \boxtimes \pi_{2\,*}(\mathfrak{F}_n)^{\vee}\big)_{2,3} - \big(\pi_{2\,*}(\mathfrak{F}_n) \boxtimes \mathfrak{U}^{\vee}\big)_{1,4}\,,$$

where $\theta_{\mathcal{C}} = \pi_{2,3\,*}\big(\pi_{1,2}^*(\mathfrak{E}) \cdot \pi_{1,3}^*(\mathfrak{E})^{\vee}\big)$.

Let $\Phi_{\mathcal{P}}$ be given by the diagonal action* on $\mathcal{C}_X \times (BU \times \mathbb{Z})$. We use the natural H-space structure $(\mathcal{P}_X, \mu, 0)$. Choosing $K(\mathcal{P}_X) = K^0(X) \times \mathbb{Z}$ we set for all $(\alpha_i, d_i) \in K(X) \times \mathbb{Z}$:

$$\begin{aligned} \tilde{\chi}\big((\alpha_1, d_1), (\alpha_2, d_2)\big) &= \chi(\alpha_1, \alpha_2) + 2d_1 d_2 - d_1 \chi\big(\alpha_1(n)\big) - d_2 \chi\big(\alpha_2(n)\big)\,, \\ \tilde{\epsilon}_{(\alpha_1,d_1),(\alpha_2,d_2)} &= \epsilon_{\alpha_1 - d_1 [\![\mathcal{O}_X(n)]\!], \alpha_2 - d_2 [\![\mathcal{O}_X(n)]\!]}\,, \end{aligned} \tag{4.2.9}$$

where ϵ is from Theorem 3.4.4. We construct the vertex algebra $(\hat{H}_*(\mathcal{P}_X), |0\rangle, e^{zT}, Y)$.

Proposition 4.2.7. *The map $\Omega_* : H_*(\mathcal{N}_{q,n}) \to H_*(\mathcal{P}_X)$ induces a morphism of graded vertex algebras $(\hat{H}_*(\mathcal{N}_{q,n}), |0\rangle, e^{zT}, Y) \to (\hat{H}_*(\mathcal{P}_X), |0\rangle, e^{zT}, Y)$. It gives a morphism of Lie algebras*

$$\bar{\Omega}_* : (\check{H}_*(\mathcal{N}_{q,n}), [-,-]) \longrightarrow (\check{H}_*(\mathcal{P}_X), [-,-])\,.$$

Proof. We use this opportunity to check that conditions of Definition 4.1.3 are satisfied. Using arguments from the proof of Lemma 4.2.5 and Gross [75, Prop. 5.3.12], we reduce it to showing that $\sigma^*(\theta_{\mathcal{C}}) = \theta_{\mathcal{C}}^\vee$. Recall that we have the natural homotopy theoretic group completion $\gamma : \mathcal{V}_X \to \mathcal{C}_X$. Using universality of the group-completion from Proposition 3.2.3, we restrict it to showing

$$\sigma^*(\gamma^*(\theta_{\mathcal{C}})) = \gamma^*(\theta_{\mathcal{C}})^\vee. \tag{4.2.10}$$

Two compact families $K, L \to \mathcal{V}_X$ correspond to two families of vector bundles V_K, V_L which we can assume to be smooth along X so we choose partial connections

*Using the left-multiplication on $U(n)$ by $U(1)$ we get the action of $\mathbb{CP}^\infty$ on $BU(n)$. Taking a union over all n we get a $\mathbb{CP}^\infty$ action on $\bigsqcup_n BU(n)$. As $\bigsqcup BU(n) \to BU \times \mathbb{Z}$ is a homotopy theoretic group completion, using [43, Proposition 1.2] we can extend to an action on $BU \times \mathbb{Z}$

$\nabla_{V_K}, \nabla_{V_L}$ in the X direction for both of them. The pullback of the class $\gamma^*(\theta_{\mathcal{C}})$ to $K\times L$ is the index of the family of operators $(\bar\partial+\bar\partial^*)^{\nabla_{V_L^*\otimes V_K}} : \Gamma^\infty(\mathcal{A}^{0,\mathrm{even}}\otimes V_L^*\otimes V_K) \to \Gamma^\infty(\mathcal{A}^{0,\mathrm{even}}\otimes V_K^*\otimes V_L)$. Using Serre duality, we have the formula $\mathrm{ind}\Big((\bar\partial+\bar\partial^*)^{\nabla_{V_L^*\otimes V_K}}\Big) = \mathrm{ind}\Big((\bar\partial+\bar\partial^*)^{\nabla_{V_L\otimes V_K^*}}\Big) \in K^0(K\times L)$, where $\mathrm{ind}(-)$ is the family index of Atiyah–Singer [9]. This is precisely (4.2.10) by the family index theorem [3, §3.1].

To show that Ω_* induces a morphism of vertex algebras we note that in $\mathbf{Ho}(\mathbf{Top})$, $\Omega : (\mathcal{N}_{q,n})^{\mathrm{top}} \to \mathcal{P}_X$ is a morphism of monoids with $\mathbb{CP}^\infty$ action. The pullback $\Omega^*(\theta_{\mathcal{P}})$ is equal to $[\![\Theta^{\mathrm{pa}}]\!]$ by construction and arguments in the proofs of [74, Prop. 5.12, Lem. 6.2]. By considering the action of Ω on connected components, we get precisely $K(\Omega) : C_0(\mathcal{B}_{q,n}) \to K^0(X)\times\mathbb{Z}$ from (4.2.4) which satisfies

$$\tilde\chi \circ \big(K(\Omega)\times K(\Omega)\big) = \chi^{\mathrm{pa}} : C_0(\mathcal{B}_{q,n}) \times C_0(\mathcal{B}_{q,n}) \longrightarrow \mathbb{Z}\,,$$
$$\tilde\epsilon_{K(\Omega)(\alpha_1,d_1),K(\Omega)(\alpha_2,d_2)} = \epsilon^{\mathrm{pa}}_{(\alpha_1,d_1),(\alpha_2,d_2)}$$

for the same choices of $\epsilon_{\alpha,\beta}$ in (4.2.6) and (4.2.9). Therefore $\Omega_* : \hat H_*(\mathcal{N}_{q,n}) \to \hat H_*(\mathcal{P}_X)$ is a degree 0 graded morphism compatible with the vertex algebra structure. □

Remark 4.2.8. We will only restrict to the case when n=0, as we will be working with 0-dimensional sheaves only in the following sections.

4.2.3 Wall-crossing conjecture for Calabi–Yau fourfolds

In this subsection, we conjecture the wall-crossing formula for Joyce–Song stable pairs, following Joyce–Song [96, §5.4], Joyce [95]. For the abelian category of coherent sheaves the conjecture has been stated by Gross–Joyce–Tanaka [76, Conj. 4.11]. Before this, we recall some background from [91].

Consider an Artin stack $\mathcal{M}_{\backslash 0} = \mathcal{M}\backslash 0$, where $\mathcal{M}$ as in Definition 4.1.2. Using rigidification as in Abramovich–Olsson–Vistoli [2] and Romagny [151], one defines $\mathcal{M}^{\mathrm{pl}}_{\backslash 0} = \mathcal{M}_{\backslash 0} \,/\!\!/ [*/\mathbb{G}_m]$. One can define a shifted grading on $\check{H}_*(\mathcal{M}^{\mathrm{pl}}_{\backslash 0}) = H_{*+2-\chi(\alpha,\alpha)}(\mathcal{M}^{\mathrm{pl}}_{\backslash 0})$ such that the projection $\Pi^{\mathrm{pl}} : \mathcal{M}_{\backslash 0} \to \mathcal{M}^{\mathrm{pl}}_{\backslash 0}$ induces a map of graded $\mathbb{Q}$-vector spaces, Joyce [91] proves that it factors $\hat{H}_{*+2}(\mathcal{M}_{\backslash 0}) \xrightarrow{\Pi} \check{H}_*(\mathcal{M}_{\backslash 0}) \xrightarrow{\check{\Pi}_*} \check{H}_{*+2}(\mathcal{M}^{\mathrm{pl}}_{\backslash 0})$, such that $\check{\Pi}_0 : \check{H}_0(\mathcal{M}_{\backslash 0}) \to \check{H}_0(\mathcal{M}^{\mathrm{pl}}_{\backslash 0})$ is an isomorphism. If τ is a stability condition on $\mathcal{A}$ from Definition 2.3.1 and $0 \neq \alpha \in K(\mathcal{A})$, then let $M^{\mathrm{st}}_\alpha(\tau)$ denote the moduli scheme of τ-stable objects in class α and $\mathcal{M}^{\mathrm{st}}_\alpha(\tau) \subset \mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$ the finite type stacks of τ-stable and τ-semistable objects. There is a natural open embedding $i^{\mathrm{st}}_\alpha : M^{\mathrm{st}}_\alpha(\tau) \hookrightarrow \mathcal{M}^{\mathrm{pl}}_{\backslash 0}$. In particular, if $[M^{\mathrm{st}}_\alpha(\tau)]^{\mathrm{vir}} \in H_*\big(M^{\mathrm{st}}_\alpha(\tau)\big)$ is defined, then we write $[M^{\mathrm{st}}_\alpha(\tau)]_{\mathrm{vir}} = i^{\mathrm{st}}_{\alpha *}\big([M^{\mathrm{st}}_\alpha(\tau)]^{\mathrm{vir}}\big) \in H_*\big(\mathcal{M}^{\mathrm{pl}}_{\backslash 0}\big)$.

Let now X be a CY fourfold, $\mathcal{A} = \mathrm{Coh}(X)$ and τ a Gieseker stability, then in the case that $\mathcal{M}^{\mathrm{st}}_\alpha(\tau) = \mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$, Oh–Thomas [141] and Borisov–Joyce [23] construct virtual fundamental classes $[M^{\mathrm{st}}_\alpha(\tau)]^{\mathrm{vir}} \in H_{2-\chi(\alpha,\alpha)}(M^{\mathrm{st}}_\alpha(\tau))$. Thus we have $[M^{\mathrm{st}}_\alpha(\tau)]_{\mathrm{vir}} \in \hat{H}_0(\mathcal{M}^{\mathrm{pl}}_{\backslash 0})$. We lift it to an element $\big(\check{\Pi}_0\big)^{-1}\big([M^{\mathrm{st}}_\alpha(\tau)]_{\mathrm{vir}}\big)$ which we also denote by $[M^{\mathrm{st}}_\alpha(\tau)]_{\mathrm{vir}}$.

For $\mathcal{A} = \mathrm{Coh}(X)$ we now fix the data from Definition 4.1.2.

Definition 4.2.9. Define $(\mathcal{A}, K(\mathcal{A}), \Phi, \mu, \Theta, \epsilon)$ as follows:

- $\lambda : K_0(\mathcal{A}) \xrightarrow{\lambda} K^0(X) =: K(\mathcal{A})$, $\Theta = \mathcal{E}\mathrm{xt}^\vee_{\mathcal{O}_X}$ from (3.1.3).
- For $\alpha, \beta \in K(\mathcal{A})$ define $\epsilon_{\alpha,\beta} = \epsilon_{\lambda(\alpha),\lambda(\beta)}$ using the orientations from Theorem 3.4.4.

Moreover, use the fixed orientations above to construct Oh–Thomas/ Borisov–Joyce classes $[M^{\mathrm{st}}_\alpha(\tau)]^{\mathrm{vir}} \in H_*(M^{\mathrm{st}}_\alpha(\tau))$ for all α,τ, such that $\mathcal{M}^{\mathrm{st}}_\alpha(\tau) = \mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$. Let τ^{pa} denote the Joyce–Song stability condition on pairs (see Joyce–Song [96, §5.4]) and $N^{\mathrm{st}}_{(\alpha,1)}(\tau^{\mathrm{pa}})$ the moduli scheme of Joyce–Song stable pairs, then [23], [141] still give us $\left[N^{\mathrm{st}}_{(\alpha,1)}(\tau^{\mathrm{pa}})\right]^{\mathrm{vir}} \in H_{2-\chi^{\mathrm{pa}}((\alpha,1),(\alpha,1))}(N^{\mathrm{st}}_{(\alpha,1)}(\tau^{\mathrm{pa}}))$. The chosen orientation is again used to determine orientations of $\left[N^{\mathrm{st}}_{(\alpha,1)}(\tau^{\mathrm{pa}})\right]^{\mathrm{vir}}$ under the inclusion $N^{\mathrm{st}}_{(\alpha,1)} \to \mathcal{M}_X$.

Conjecture 4.2.10. *Let τ be a Gieseker stability, then there are unique classes $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}} \in \check{H}_0(\mathcal{M}_\alpha)$ for all $\alpha \in K(\mathcal{A})$ satisfying:*

i. If $\mathcal{M}^{\mathrm{ss}}_\alpha(\tau) = \mathcal{M}^{\mathrm{st}}_\alpha(\tau)$ then $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}} = [M^{\mathrm{st}}_\alpha(\tau)]_{\mathrm{vir}}$.

ii. If $\tilde{\tau}$ is another Gieseker stability condition then these classes satisfy the wall-crossing formula [76, eq. (4.1)] in $\check{H}_*(\mathcal{M})$. If $\mathcal{M}^{\mathrm{ss}}_\beta(\tau) = \mathcal{M}^{\mathrm{ss}}_\beta(\tilde{\tau})$ then $[\mathcal{M}^{\mathrm{ss}}_\beta(\tau)]_{\mathrm{inv}} = [\mathcal{M}^{\mathrm{ss}}_\beta(\tilde{\tau})]_{\mathrm{inv}}$.

iii. If $\tau(\alpha) = q$, we have the formula in $\check{H}_*(\mathcal{N}_{q,n})$:

$$[N^{\mathrm{st}}_{(\alpha,1)}(\tau^{\mathrm{pa}})]_{\mathrm{vir}} = \sum_{\substack{k\geq 1,\alpha_1,\dots,\alpha_k\in C(\mathcal{A})\\ \alpha_1+\dots+\alpha_k=\alpha,\tau(\alpha)=\tau(\alpha_i)}} \frac{(-1)^k}{k!}\big[\big[\dots\big[[\mathcal{N}_{(0,1)}]_{\mathrm{inv}}, [\mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau)]_{\mathrm{inv}}\big],\dots\big], [\mathcal{M}^{\mathrm{ss}}_{\alpha_k}(\tau)]_{\mathrm{inv}}\big],$$

where $[\mathcal{N}_{(0,1)}]_{\mathrm{inv}} \in \check{H}_0(\mathcal{N}^{0,1}_{q,n}) \cong \mathbb{Z}$ is the generator determined by orientation on $\mathcal{C}_{[\![\mathcal{O}_X]\!]}$, and $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}} = [\mathcal{M}^{\mathrm{ss}}_{(0,\alpha)}(\tau^{\mathrm{pa}})]_{\mathrm{inv}}$.

One can compute well-defined invariants using $[M^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}}$ and weight 0 insertions from 4.1.5. We will use the map $\Omega_* : H_*(\mathcal{N}_q) \to H_*(\mathcal{P}_X)$ and give explicit formulae for $[M^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}}$ in the cases we study.

4.2.4 Explicit vertex algebra of topological pairs

We give here an explicit description of the vertex algebra $(\hat{H}_*(\mathcal{P}_X), |0\rangle, e^{zT}, Y)$ which will apply some of the work of Joyce [91] and Gross [74]. We also set some notations, conventions and write down some useful identities. In the following, X is a connected smooth projective variety of dimension n unless specified.

Definition 4.2.11. Let us write $(0,1)$ for the generator of $\mathbb{Z}$ in $K^0(X)\oplus\mathbb{Z}$. Let $\mathrm{ch}: K^0(X)\otimes\mathbb{Q}\oplus K^1(X)\otimes\mathbb{Q}\to H^*(X)$ be the Chern character. For each $0\leq i\leq 2n$ choose a subset $B_i\subset K^0(X)\otimes\mathbb{Q}\oplus K^1(X)\otimes\mathbb{Q}$ such that $\mathrm{ch}(B_i)$ is a basis of $H^i(X)$. We take $B_0=\{[\![\mathcal{O}_X]\!]\}$ and $B_{2n}=\{p\}$. Then we write $B=\bigsqcup_i B_i$ and $\mathbb{B}=B\sqcup\{(0,1)\}$. Let $K_*(X)$ denote the topological K-homology of X. Let $\mathrm{ch}^\vee: K_*(X)\otimes\mathbb{Q}\to H_*(X)$ be defined by commutativity of the following:

$$\begin{array}{ccc} K_*(X)\otimes\mathbb{Q}\otimes_{\mathbb{Q}} K^*(X)\otimes\mathbb{Q} & \xrightarrow{\mathrm{ch}^\vee\otimes\mathrm{ch}} & H_*(X)\otimes H^*(X) \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\quad\mathrm{id}\quad} & \mathbb{Q} \end{array},$$

then choose $B^\vee\subset K_*(X)\otimes\mathbb{Q}$ such that $B^\vee$ is a dual basis of B, we also write $\mathbb{B}^\vee=B^\vee\sqcup\{(0,1)\}$, where $(0,1)$ is the natural generator of $\mathbb{Z}$ in $K_0(X)\oplus K_1(X)\oplus\mathbb{Z}$. The dual of $\sigma\in\mathbb{B}$ will be denoted by $\sigma^\vee\in\mathbb{B}^\vee$. For each $\sigma\in\mathbb{B}$, $(\alpha,d)\in K^0(X)\times\mathbb{Z}$ and $i\geq 0$ we define

$$e^{(\alpha,d)}\otimes\mu_{\sigma,i}=\mathrm{ch}_i(\mathfrak{E}_{(\alpha,d)}/\sigma^\vee)\,, \tag{4.2.11}$$

using the slant product $K^i(Y\times Z)\times K_j(Y)\to K^{i-j}(Z)$. We have a natural inclusion $\iota_{\mathcal{C},\mathcal{P}}:\mathcal{C}_X\to\mathcal{P}_X$: $x\mapsto(x,1,0)\in\mathcal{C}_X\times BU\times\mathbb{Z}$, so we identify $H_*(\mathcal{C}_X)$ with the image of $(\iota_{\mathcal{C},\mathcal{P}})_*$, which in turn corresponds to $H_*(\mathcal{C}_X)\boxtimes 1\subset H_*(\mathcal{C}_X)\boxtimes H_*(BU\times\mathbb{Z})=H_*(\mathcal{P}_X)$. The universal K-theory class $\mathfrak{E}_{\mathcal{P}}$ on $(X\sqcup *)\times(\mathcal{P}_X)$ restricts to $\mathfrak{E}\boxtimes 1$ on

$(X \times \mathcal{C}_X) \times BU \times \mathbb{Z}$ and $1 \boxtimes \mathfrak{U}$ on $* \times \mathcal{C}_X \times (BU \times \mathbb{Z})$.

The next proposition follows by the arguments in the proof of Gross [74, Thm. 4.15] and the remark below it.

Proposition 4.2.12. *The cohomology ring $H^*(\mathcal{P}_X)$ is generated by $\{e^{(\alpha,d)} \otimes \mu_{\sigma,i}\}_{(\alpha,d)\in K^0(X)\times\mathbb{Z},\sigma\in\mathbb{B},i\geq 1}$. Moreover, there is a natural isomorphism of rings*

$$H^*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X) \oplus \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![\mu_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\!] \,. \tag{4.2.12}$$

From now on, when we compute explicitly with $H^*(\mathcal{P}_X)$, we replace it using this isomorphism. The dual of (4.2.12) gives us an isomorphism

$$H_*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\!] \,, \tag{4.2.13}$$

where we use the normalization

$$e^{(\alpha,d)} \otimes \prod_{\substack{\sigma\in\mathbb{B}\\ i\geq 1}} \mu_{\sigma,i}^{m_{\sigma,i}} \Big(e^{(\beta,e)} \otimes \prod_{\substack{\tau\in\mathbb{B}\\ j\geq 1}} u_{\tau,j}^{n_{\tau,j}} \Big) = \begin{cases} \prod_{\substack{v\in\Lambda\\ i\geq 1}} \frac{m_{\sigma,i}!}{(i-1)!^{m_{\sigma,i}}} & \text{if } {\scriptstyle (\alpha,d)=(\beta,e), m_{\sigma,i}=n_{\sigma,i} \atop \scriptstyle \forall \sigma\in\mathbb{B}, i\geq 1} \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.14}$$

We will be using the following simple result in computations later.

Lemma 4.2.13. *Let $f(tx_\sigma, t^2x_\sigma, \ldots)$ be a power-series, then for any set of coefficients $a_{\sigma,j}$ we have*

$$\begin{aligned} &e^{(\alpha,d)} \otimes \exp\Big(\sum_{\substack{j>0\\ \tau\in\mathbb{B}}} a_{\tau,j}\mu_{\tau,j}q^j \Big) \Big(e^{(\beta,e)} \otimes f(u_{\sigma,1}, u_{\sigma,2}, \ldots) \Big) \\ &= \delta_{\alpha,\beta}\delta_{d,e} f(a_{\sigma,1}q, a_{\sigma,2}q^2, \ldots, \frac{a_{\sigma,k}}{(k-1)!}q^k, \ldots) \,. \end{aligned}$$

Proof. Notice that acting with $e^{(\alpha,d)} \otimes \prod_{\substack{i\geq 1\\ \sigma\in\mathbb{B}}} \mu_{\sigma,i}^{m_{\sigma,i}}$ corresponds to acting with $\delta_{\alpha,\beta}\delta_{d,e}\prod_{\substack{i\geq 1\\ \sigma\in\mathbb{B}}}\Big(\frac{1}{(i-1)!}\frac{d}{du_{\sigma,i}}\Big)^{m_{\sigma,i}}$ and then evaluating at $u_{\sigma,i}=0$. As a result we obtain

$$
\begin{aligned}
&e^{(\alpha,d)}\otimes \exp\Big(\sum_{i\geq 1} a_{\sigma,j}\frac{d}{du_{\sigma,j}}q^j\Big)\Big(e^{(\beta,e)}\otimes f(u_{\sigma,1},u_{\sigma,2},\dots)\Big)|_{u_{\sigma,i}=0}\\
&=\delta_{\alpha,\beta}\delta_{d,e}f(a_{\sigma,1}q,a_{\sigma,2}q^2,\dots,\frac{a_{\sigma,k}}{(k-1)!}q^k,\dots)\,,
\end{aligned}
$$

by a standard computation. □

When $\sigma=(v,0)$ or $\sigma=(0,1)$ we will shorten the notation to

$$
\mu_{\sigma,i}=\mu_{v,i}\,,\quad u_{\sigma,i}=u_{v,i}\quad\text{or}\quad \mu_{\sigma,i}=\beta_i\,,\quad u_{\sigma,i}=b_i.
$$

Setting $\beta_i=0$, $b_i=0$ and only considering $\mathbb{Q}[K^0(X)]\subset\mathbb{Q}[K^0(X)\oplus\mathbb{Z}]$ gives us the (co)homology of $H^*(\mathcal{C}_X)$, $H_*(\mathcal{C}_X)$ up to a canonical isomorphism. Using this notation we can write

$$
\mathrm{ch}(\mathfrak{E}_\alpha)=\sum_{v\in B}\mathrm{ch}(v)\boxtimes(\sum_{i\geq 0}e^\alpha\otimes\mu_{v,i})\,. \tag{4.2.15}
$$

Let X now be a CY fourfold. The following theorem is the topological version of [74, Thm. 1.1], [91, Thm. **] extending it also to pairs.

Proposition 4.2.14. *Let $\mathbb{Q}[K^0(X)\times\mathbb{Z}]\otimes_{\mathbb{Q}}\mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i},\sigma\in\mathbb{B},i>0]\!]$ be the generalized super-lattice vertex algebra associated to $\big((K^0(X)\oplus\mathbb{Z})\oplus K^1(X),\tilde{\chi}^\bullet\big)$, where $\tilde{\chi}^\bullet=\tilde{\chi}\oplus\chi^-$ for $\tilde{\chi}$ from* (4.2.9) *and*

$$
\chi^-:K^1(X)\times K^1(X)\longrightarrow\mathbb{Z}\,,\quad \chi^-(\alpha,\beta)=\int_X\mathrm{ch}(\alpha)^\vee\mathrm{ch}(\beta)\mathrm{Td}(X)\,. \tag{4.2.16}
$$

Then the isomorphism (4.2.13) *induces a graded isomorphism of vertex algebras*

$$\hat{H}^*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\!]\,,$$

if the same signs $\tilde{\epsilon}_{(\alpha,d),(\beta,e)}$ *from* (4.2.9) *are used for constructing the vertex algebras on both sides. On the right hand side the degrees are given by*

$$\deg\Big(e^{(\alpha,d)} \otimes \prod_{\sigma\in B_{\mathrm{even}}\sqcup\{(0,1)\}\,,i>0} u_{\sigma,i}^{m_{\sigma,i}} \otimes \prod_{\substack{v\in B_{\mathrm{odd}}\\ j>0}} u_{v,j}^{m_{v,j}}\Big)$$
$$= \sum_{\substack{\sigma\in B_{\mathrm{even}}\sqcup\{(0,1)\}\\ i>0}} m_{\sigma,i}2i + \sum_{\substack{v\in B_{\mathrm{odd}}\\ j>0}} m_{v,j}(2j-1) - \tilde{\chi}\Big((\alpha,d),(\alpha,d)\Big)\,.$$

Proof. The proof is nearly identical to [74, Thm. 1.1], [91, Thm. **]. We need an explicit expression for $\mathrm{ch}_k(\theta_{\mathcal{P}})$ replacing Proposition [74, Prop. 5.2] and a similar result in [91, Thm. **] for quivers. This is given in Lemma 4.4.6. □

Before we move on to the applications, let us write down some identities we will need later on. From now on, we always fix a point-canonical orientation of Definition 4.2.1, the associated signs of $\epsilon_{\alpha,\beta}$ of Theorem 3.4.4 and the corresponding $\tilde{\epsilon}_{(\alpha,d),(\beta,e)}$ from (4.2.9).

Lemma 4.2.15. *Consider the vertex algebra* $(\hat{H}_*(\mathcal{P}_X), |0\rangle, e^{zT}, Y)$*, then*

i. $\mathrm{rk}(\mathfrak{E}_{\alpha,d}/\sigma^\vee) = (\alpha,d)(\sigma^\vee)$

ii. Let $v_1,\dots v_k \in B_{\text{even}}$ and $i_1,\dots,i_k \geq 1$, then

$$T\big(e^{(\alpha,d)}\otimes u_{v_1,i_1}\cdots u_{v_k,i_k}\big) = e^{(\alpha,d)}\sum_{\sigma\in B_{\text{even}}\sqcup\{(0,1)\}}(\alpha,d)(\sigma^\vee)u_{\sigma,1}u_{v_1,i_1}\cdots u_{v_k,i_k}$$
$$+\sum_{l=1}^k i_l u_{v_1,i_1}\cdots u_{v_l,i_{l-1}}u_{v_l,i_l+1}u_{v_l,i_{l+1}}\cdots u_{v_k,i_k}\,,$$

iii. For all $k,l,M,N\geq 0$ we have $\tilde{\epsilon}_{(kp,N),(lp,M)} = (-1)^{Mk}$.

Proof. i. To see this, we use functoriality of the slant product:

$$\begin{aligned}&\mathrm{rk}(\mathfrak{E}_{(\alpha,d)}/\sigma^\vee) = \mathrm{rk}\big(i^*_{c,b}(\mathfrak{E}_{(\alpha,d)}/\sigma^\vee)\big)\\ &= \mathrm{rk}\big((\mathrm{id}\times i_{c,b})^*(\mathfrak{E}_{\alpha,d})/\sigma^\vee\big) = 1\boxtimes(\alpha,d)/\sigma^\vee = (\alpha,d)(\sigma^\vee)\,,\end{aligned}$$

where $i_{c,b}$ is an inclusion of a point into $\mathcal{P}_{(\alpha,d)}$. The second statement is a generalization of [74, Lemma 5.5] using i. A similar formula has been shown in [91] for quivers. The last statement follows from Theorem 3.4.6 together with Definition 4.2.1 and (4.2.9). □

We will often avoid specifying the connected component where the (co)homology class sits by simply omitting $e^{(\alpha,d)}$, e^α where it is obvious.

4.3 Cao–Kool conjecture

After reformulating Conjecture 1.0.4 in terms of the vertex algebra of pairs, we compute (assuming Conjecture 4.2.10) the virtual fundamental classes of semistable 0-dimensional sheaves viewed as elements of $H_*(\mathcal{C}_X)$ by wall-crossing in the vertex algebra of Definition 4.3.4 and using the result of Cao–Qu [40, Theorem 1.2]. By wall-crossing back we prove Conjecture 1.0.4.

4.3.1 L-twisted vertex algebras

For a Calabi–Yau fourfold X let $\mathrm{Hilb}^n(X)$ be the Hilbert scheme of n points and $\big[\mathrm{Hilb}^n(X)\big]^{\mathrm{vir}} \in H_{2n}\big(\mathrm{Hilb}^n(X)\big)$ the virtual fundamental class defined by Oh-Thomas [141, Thm. 4.6] using the orientations in Definition 4.2.1. We consider the vector bundle $L^{[n]} \to \mathrm{Hilb}^n(X)$ given by (1.0.1). The real rank of $L^{[n]}$ is $2n$, so Cao–Kool [32] define

$$I_n(L) = \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} c^n\big(L^{[n]}\big)\,. \tag{4.3.1}$$

The proof of Conjecture 1.0.4 will be given at the end of subsection 4.3.2 in the following form.

Theorem 4.3.1. *Let X be a smooth projective Calabi–Yau fourfold for which Conjecture 4.2.10 holds, and L a line bundle on X. Then*

$$I(L;q) = 1 + \sum_{n=1}^{\infty} I_n(L)q^n = M(-q)^{c_1(L)\cdot c_3(X)}$$

for the point-canonical orientations of Definition 4.2.1.

For the invariants $I_n(L)$ this is equivalent to

$$\begin{aligned} I_n(L) &= \sum_{k\geq 1} d_k(n) I(L)^k\,, \text{ where} \\ d_k(n) &= \sum_{\substack{n_1,\dots,n_k \\ \sum_i n_i = n}} \frac{1}{k!}\prod_{i=1}^{k}\sum_{l|n_i}(-1)^{n_i}\frac{n_i}{l^2}, \quad I(L) = c_1(L)\cdot c_3(X)\,. \end{aligned} \tag{4.3.2}$$

Let us interpret this in the language of §4.2.3. Take $\mathcal{A}_q = \mathcal{A}_0$ to be the abelian category of sheaves with 0-dimensional support. Let $\mathcal{B}_q = \mathcal{B}_0$ be the corresponding category of pairs from Definition 4.2.3 and $\mathcal{N}_0$ its moduli stack with $n = 0$. We have the identification $\mathrm{Hilb}^n(X) = N^{\mathrm{st}}_{(np,1)}(\tau^{\mathrm{pa}})$, noting that $p(F) = 1$ for any zero-dimensional sheaf F. This gives us

$$\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}} = \big[N^{\mathrm{st}}_{(np,1)}(\tau^{\mathrm{pa}})\big]_{\mathrm{vir}}$$

by part i. of Conjecture 4.2.10.

As $\mathrm{Hilb}^n(X)$ carries a universal family $\mathscr{I}_n \to X\times \mathrm{Hilb}^n(X)$, there exists a natural lift of the open embedding $\iota_n^{\mathrm{pl}} : \mathrm{Hilb}^n(X) \to \mathcal{N}_0^{\mathrm{pl}}$

$$\begin{array}{ccc} & & \mathcal{N}_0 \\ & \overset{\iota_n}{\nearrow} & \big\downarrow \Pi^{\mathrm{pl}} \\ \mathrm{Hilb}^n(X) & \xrightarrow{\iota_n^{\mathrm{pl}}} & \mathcal{N}_0^{\mathrm{pl}} \end{array} \quad . \tag{4.3.3}$$

We use ι_n to express (4.3.1) in terms of insertions on $\mathcal{N}_0$.

Definition 4.3.2. Using the notation from Definition 4.2.4, for all $(np, d) \in C_0(\mathcal{B}_0)$ we will write $\mathcal{N}_{n,d} = \mathcal{N}_{np,d}$. Then define $\mathcal{L}^{[n_1,n_2]}_{d_1,d_2} \to \mathcal{N}_{n_1,d_1}\times\mathcal{N}_{n_2,d_2}$ by

$$\mathcal{L}^{[n_1,n_2]}_{d_1,d_2} = (\pi_{n_1,d_1}\times\pi_{n_2,d_2})^*\Big(\mathcal{V}^*_{d_1}\boxtimes \pi_{2*}\big(\pi_X^*(L)\otimes\mathcal{E}_0\big)\Big)_{2,3}\,,$$

where $\mathcal{E}_0$ is the universal sheaf on $\mathcal{M}_0$ the moduli stack of $\mathcal{A}_0$. It is a vector bundle of rank d_1n_2. We define

$$\mathcal{L}^{[-,-]}|_{\mathcal{N}_{n_1,d_1}\times\mathcal{N}_{n_2,d_2}} = \mathcal{L}^{[n_1,n_2]}_{d_1,d_2}\,.$$

Set $\mathcal{L} = \Delta^*\big(\mathcal{L}^{[-,-]}\big)$. From Example 4.1.7, we know that $c^i(\mathcal{L})$ is a weight 0 insertion and from definition it follows that $\iota_n^*(\mathcal{L}) = L^{[n]}$. Using this together with $\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}} = \Big(\Pi^{\mathrm{pl}} \circ \iota_n\Big)_*\Big(\big[\mathrm{Hilb}^n(X)\big]^{\mathrm{vir}}\Big)$, we see that

$$I_n(L) = \int_{[\mathrm{Hilb}^n(X)]_{\mathrm{vir}}} c^n(\mathcal{L})\,. \tag{4.3.4}$$

The following is clear from the construction.

Lemma 4.3.3. *The vector bundle $\mathcal{L}^{[-,-]} \to \mathcal{N}_0 \times \mathcal{N}_0$ satisfies the conditions of Definition 4.1.8. Let $(\tilde{H}_*(\mathcal{N}_0), |0\rangle, e^{zT}, Y^L)$ be the $\mathcal{L}^{[-,-]}$-twisted vertex algebra, $\big(\mathring{H}_*(\mathcal{N}_0), [-,-]^L\big)$ the associated Lie algebra. By Proposition 4.1.9 we have the morphism $(-) \cap c^{\mathrm{top}}\big(\mathcal{L}\big) : \big(\check{H}_*(\mathcal{N}_0), [-,-]\big) \to \big(\mathring{H}_*(\mathcal{N}_0), [-,-]^L\big)$.*

We construct its topological counterpart.

Definition 4.3.4. Define the data $\big(\mathcal{P}_X, K(\mathcal{P}_X), \Phi_{\mathcal{P}}, \mu_{\mathcal{P}}, 0, \theta^L_{\mathcal{P}}, \tilde{\epsilon}^L\big)$ as follows:

- $K(\mathcal{P}_X) = K^0(X) \times \mathbb{Z}$.
- Set $\mathfrak{L} = \pi_{2*}\big(\pi_X^*(L) \otimes \mathfrak{E}\big) \in K^0(\mathcal{C}_X)$. Then on $\mathcal{P}_X \times \mathcal{P}_X$ we define

$$\theta^L_{\mathcal{P}} = (\theta)_{1,3} + 2(\mathfrak{U} \boxtimes \mathfrak{U}^\vee)_{2,4} - \Big(\mathfrak{U} \boxtimes \big(\pi_{2*}(\mathfrak{E}) - \mathfrak{L}\big)^\vee\Big)_{2,3} - \Big(\big(\pi_{2*}(\mathfrak{E}) - \mathfrak{L}\big) \boxtimes \mathfrak{U}^\vee\Big)_{1,4}.$$

- The symmetric form $\tilde{\chi}^L : (K^0(X) \times \mathbb{Z}) \times (K^0(X) \times \mathbb{Z}) \to \mathbb{Z}$ is given by

$$\begin{aligned}\tilde{\chi}^L\big((\alpha,d),(\beta,e)\big) &= \mathrm{rk}\big(\theta^L_{(\alpha,d),(\beta,e)}\big)\\ &= \chi(\alpha,\beta) + 2de - d\big(\chi(\beta) - \chi(\beta\cdot L)\big) - e\big(\chi(\alpha) - \chi(\alpha \cdot L)\big)\,.\end{aligned} \tag{4.3.5}$$

- The signs are defined by

$$\tilde{\epsilon}^L_{(\alpha,d),(\beta,e)} = (-1)^{d\chi(L\cdot\beta)}\tilde{\epsilon}_{(\alpha,d),(\beta,e)}\,, \tag{4.3.6}$$

  in terms of $\tilde{\epsilon}_{\alpha,\beta}$ from (4.2.9).

We denote by $(\tilde{H}_*(\mathcal{P}_X), |0\rangle, e^{zT}, Y^L)$ the vertex algebra associated to this data and $(\mathring{H}_*(\mathcal{P}_X), [-,-]^L)$ the corresponding Lie algebra.

We are unable to use Proposition 4.1.9 directly because $\mathfrak{U}^\vee \boxtimes \mathfrak{L}$ is not a vector bundle. However, one can easily show the following result similar to Proposition 4.2.14 and Proposition 4.2.7.

Proposition 4.3.5. *Let* $\mathbb{Q}[K^0(X)\times\mathbb{Z}]\otimes_{\mathbb{Q}}\mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma\in\mathbb{B}, i>0]\!]$ *be the generalized super-lattice vertex algebra associated to* $\big((K^0(X)\oplus\mathbb{Z})\oplus K^1(X), (\tilde{\chi}^L)^\bullet\big)$, *where* $(\tilde{\chi}^L)^\bullet = \tilde{\chi}^L\oplus\chi^-$ *for* $\tilde{\chi}^L$ *from* (4.3.5) *and* χ^- *from* (4.2.16). *The isomorphism* (4.2.13) *induces an isomorphism of graded vertex algebras*

$$\tilde{H}^*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X)\times\mathbb{Z}]\otimes_{\mathbb{Q}}\mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma\in\mathbb{B}, i>0]\!]\,,$$

if the same signs $\tilde{\epsilon}^L_{(\alpha,d),(\beta,e)}$ *from* (4.3.6) *are used for constructing the vertex algebras on both sides. On the right hand side the degrees are given by*

$$\begin{aligned}&\deg\Big(e^{(\alpha,d)}\otimes\prod_{\sigma\in B_{\mathrm{even}}\sqcup\{(0,1)\},i>0}u_{\sigma,i}^{m_{\sigma,i}}\otimes\prod_{\substack{v\in B_{\mathrm{odd}}\\ j>0}}u_{v,j}^{m_{v,j}}\Big)\\ &=\sum_{\substack{\sigma\in B_{\mathrm{even}}\sqcup\{(0,1)\}\\ i>0}}m_{\sigma,i}2i+\sum_{\substack{v\in B_{\mathrm{odd}}\\ j>0}}m_{v,j}(2j-1)-\tilde{\chi}^L\Big((\alpha,d),(\alpha,d)\Big)\,.\end{aligned}$$

The map $\Omega_* : H_*(\mathcal{N}_q) \to H_*(\mathcal{P}_X)$ *induces a morphism of graded vertex algebras* $(\tilde{H}_*(\mathcal{N}_q), |0\rangle, e^{zT}, Y^L) \to (\tilde{H}_*(\mathcal{P}_X), |0\rangle, e^{zT}, Y^L)$ *and of graded Lie algebras*

$$\bar{\Omega}_* : \big(\mathring{H}_*(\mathcal{N}_q), [-,-]^L\big) \longrightarrow \big(\mathring{H}_*(\mathcal{P}_X), [-,-]^L\big)\,.$$

Proof. Using Lemma 4.4.6 for $\alpha = [\![L]\!]$, we see that

$$\mathrm{ch}_k(\mathfrak{U}^\vee \boxtimes \mathfrak{L}) = \sum_{\substack{v\in B_{\mathrm{even}}\\ j=l+k}} (-1)^l \chi(L^*, v)\beta_l \boxtimes \mu_{v,k}\,.$$

Using $\chi(L^*, v) = \chi(v \cdot L)$, one can prove the first part of the theorem by following the proof of [74, Thm. 1.1] or [91, Thm. **]. To show the second part, note that $(\Omega\times\Omega)^*(\mathfrak{U}^\vee \boxtimes \mathfrak{L}) = \mathcal{L}^{[-,-]}$ and

$$\xi^L\big((n_1 p, d_1), (n_2 p, d_2)\big) := \mathrm{rk}\Big(\mathcal{L}^{[n_1,n_2]}_{d_1,d_2}\Big) = d_1 n_2 = d_1\chi(n_2 p\cdot L)\,.$$

The statement then follows from Definition 4.1.8 and Definition 4.3.4 by the same arguments as in the proof of Proposition 4.2.7. □

This completes the following diagram of morphisms of Lie algebras:

$$\begin{array}{ccc} \big(\check{H}_*(\mathcal{N}_q), [-,-]\big) & \xrightarrow{\bar{\Omega}_*} & \check{H}_*(\mathcal{P}_X, [-,-]) \\ {\scriptstyle \cap c^{\mathrm{top}}(\mathcal{L})}\big\downarrow & & \\ \big(\mathring{H}_*(\mathcal{N}_q), [-,-]^L\big) & \xrightarrow{\bar{\Omega}_*} & \mathring{H}_*(\mathcal{P}_X, [-,-]^L) \end{array}\,.$$

4.3.2 Computing virtual fundamental cycles of 0-dimensional sheaves

Applying part iii. of Conjecture 4.2.10 to $\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}}$ we obtain in $\check{H}_*(\mathcal{N}_0)$

$$\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}} = \sum_{\substack{k\geq 1, n_1,\dots,n_k>0\\ n_1+\dots+n_k=n}} \frac{(-1)^k}{k!}\big[\big[\dots\big[[\mathcal{M}_{(1,0)}]_{\mathrm{inv}}, [\mathcal{M}_{n_1p}]_{\mathrm{inv}}\big],\dots\big], [\mathcal{M}_{n_kp}]_{\mathrm{inv}}\big]\,, \tag{4.3.7}$$

where we used part ii. of Conjecture 4.2.10 to conclude that $[\mathcal{M}_{n_ip}]_{\mathrm{inv}} = [\mathcal{M}^{\mathrm{ss}}_{n_ip}(\tau)]_{\mathrm{inv}}$ are independent of stability conditions. To make the notation simpler, we write

$$\mathscr{H}_n = \bar{\Omega}_*\Big(\big[\mathrm{Hilb}^n(X)\big]_{\mathrm{vir}}\Big)\,, \quad \text{and} \quad \mathscr{M}_{np} = \bar{\Omega}_*\big([\mathcal{M}_{np}]_{\mathrm{inv}}\big)\,.$$

Using Lemma 4.3.3, we can apply Proposition 4.1.9 together with (4.3.7) to get

$$I_n(L)1_{(n,1)} = \sum_{\substack{k\geq 1, n_1,\dots,n_k>0\\ n_1+\dots+n_k=n}} \frac{(-1)^k}{k!}\big[\big[\dots\big[[\mathcal{M}_{(1,0)}]_{\mathrm{inv}}, [\mathcal{M}_{n_1p}]_{\mathrm{inv}}\big]^L,\dots\big]^L, [\mathcal{M}_{n_kp}]_{\mathrm{inv}}\big]^L\,, \tag{4.3.8}$$

where $1_{(n,1)} \in \check{H}_0(\mathcal{N}_{(n,1)})$ denotes the natural generator.

Applying Proposition 4.2.7 to (4.3.7), we get a wall-crossing formula in $\check{H}_*(\mathcal{P}_X)$

$$\mathscr{H}_n = \sum_{\substack{k\geq 1, n_1,\dots,n_k>0\\ n_1+\dots+n_k=n}} \frac{(-1)^k}{k!}\big[\big[\dots\big[e^{(0,1)}\otimes 1, \mathscr{M}_{n_1p}\big],\dots\big], \mathscr{M}_{n_kp}\big]\,. \tag{4.3.9}$$

Applying Proposition 4.3.5 to (4.3.8) we obtain a wall-crossing formula in $\mathring{H}_*(\mathcal{P}_X)$:

$$I_n(L)e^{(np,1)}\otimes 1 = \sum_{\substack{k\geq 1, n_1,\dots,n_k>0\\ n_1+\dots+n_k=n}} \frac{(-1)^k}{k!}\big[\big[\dots\big[e^{(0,1)}\otimes 1, \mathscr{M}_{n_1p}\big]^L,\dots\big]^L, \mathscr{M}_{n_kp}\big]^L\,. \tag{4.3.10}$$

Let $\Omega_{np} = \Omega|_{(\mathcal{N}_{(np,0)})^{\mathrm{top}}} : (\mathcal{N}_{(np,0)})^{\mathrm{top}} \to \mathcal{C}_X \subset \mathcal{P}_X$, then we will describe the image of $H_2(\mathcal{N}_{(np,0)})$ under $(\Omega_{np})_*$. For this note that it follows from (4.2.13) that there is a natural isomorphism for all $\alpha \in K^0(X)$

$$H_2(\mathcal{C}_\alpha) \cong H^{\mathrm{even}}(X) \oplus \Lambda^2 H^{\mathrm{odd}}(X)\,. \tag{4.3.11}$$

Lemma 4.3.6. *The image of* $(\Omega_{np})_* : H_2(\mathcal{N}_{(np,0)}) \to H_2(\mathcal{C}_{np})$ *is contained in* $H^6(X) \oplus H^8(X)$ *under the isomorphism* (4.3.11).

Proof. We show that $\Omega^*_{(np,0)}(e^{np} \otimes \mu_{v,i}) = 0$ whenever $v \notin B_6 \cup B_8$. Then for any class $U \in H_*(\mathcal{N}_{(np,0)})$ we get

$$e^{np} \otimes \mu_{v,1}\big((\Omega_{np})_*(U)\big) = \Omega^*_{np}(e^{np} \otimes \mu_{v,1})(U) = 0$$

for $v \in B_{\mathrm{even}} \backslash (B_6 \cup B_8)$ and

$$e^{np} \otimes \mu_{v,1}\mu_{w,1}\big((\Omega_{np})_*(U)\big) = \Omega^*_{np}(e^{np} \otimes \mu_{v,1}\mu_{w,1})(U) = 0$$

for $v, w \in B_{\mathrm{odd}}$. The conclusion then follows from (4.2.14).

The K-theory class $[\![\mathcal{E}_{np}]\!]$ of the universal sheaf of points on $\mathcal{N}_{(np,0)}$ is given by $(\mathrm{id}_X \times \Omega_{np})^*(\mathfrak{E}_{np})$. Then from (4.2.15) we see

$$\mathrm{ch}(\mathcal{E}_{np}) = \sum_{\substack{v \in B \\ i \geq 0}} v \boxtimes \Omega^*_{np}(e^{np} \otimes \mu_{v,i})\,. \tag{4.3.12}$$

We also know that $\mathrm{ch}_i(\mathcal{E}_{np}) = 0$ for $i < 4$ by dimension arguments. Using that X is a CY, we have $H^7(X)$ =0. We thus only need to consider $v \in B_j$ for $j < 6$. Then from looking at (4.3.12) we see $v \boxtimes \Omega^*_{np}(e^{np} \otimes \mu_{v,1}) = 0$ because it is in degree $2+j<8$ or $1+j<8$ and B is a basis. Therefore $\Omega^*_{np}(e^{np} \otimes \mu_{v,1}) = 0$. $\square$

Notice that we can write

$$\mathscr{M}_{np} = e^{(np,1)} \otimes 1 \cdot \mathscr{N}_{np} + \mathbb{Q}T(e^{(np,1)} \otimes 1). \tag{4.3.13}$$

Proposition 4.3.7. *We can choose $\mathscr{N}_{np}$ from* (4.3.13)*, such that for some $a_v(n) \in \mathbb{Q}$, $v \in B_6$ we have*

$$\mathscr{N}_{np} = \sum_{v \in B_6} a_v(n) u_{v,1} \,.$$

Proof. As $\mathscr{N}_{np} = (\Omega_{np})_*\big([\mathcal{M}_{np}]_{\mathrm{inv}}\big)$, by Lemma 4.3.6, we have

$$\mathscr{N}_{np} = \sum_{v \in B_6 \cup B_8} a_v(n) u_{v,1} \,.$$

From Lemma 4.2.15, we see that $T(e^{np} \otimes 1) = e^{np} \otimes n u_{p,1}$. Therefore, we get $H^8(X) = T\big(H_0(\mathcal{C}_{np})\big)$ which concludes the proof. $\square$

Let $\mathrm{Amp}(X) \subset H^2(X)$ be the image of the ample cone under the natural map $A^1(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$. Let us choose B_2 such that its elements are $c_1(L)$ for very ample line bundles L. This is possible: We assumed $H^2(\mathcal{O}_X) = 0$ for X a CY fourfold, so $H^2(X) = H^{1,1}(X)$. Thus every element in $H^2(X)$ is obtained as $\frac{m}{n}[D]$ for an algebraic divisor $D \subset X$ and $m, n \in \mathbb{Z}$. On the other hand $[D] + n[H]$ is very ample if $n \gg 0$ and H very ample so $[D] = ([D] + n[H]) - n[H]$, where both terms are very ample.

Using the Poincaré pairing on $H^2(X) \times H^6(X) \to \mathbb{Q}$ we choose a basis B_6 of $H^6(X)$ which is dual to B_2.

Lemma 4.3.8. *For each line bundle L such that $c_1(L) \in B_2$ and $c_1(L) \cdot c_3(X) \neq 0$, there exist unique orientations $o_n(L)$ on $\mathrm{Hilb}^n(X)$ such that Theorem 1.0.4 holds for L.*

Proof. If L is very ample, since $\dim(X) > 1$, Bertini's theorem [79, Thm. 8.18] tells us that there exists a smooth connected divisor D such that $L = \mathcal{O}_X(D)$. The lemma then follows from

Theorem 4.3.9 (Cao–Qu [40, Thm. 1.2])**.** *Conjecture 1.0.4 holds for any X and $L \cong \mathcal{O}_X(D)$ for a smooth connected divisor D.*

The uniqueness of $o_n(L)$ in the case $c_1(L) \cdot c_3(X) \neq 0$ follows because changing orientations changes the sign as $\mathrm{Hilb}^n(X)$ is connected for all n and X by Hartshorne [77]. □

Let us denote $o(n)$ the orientations on $\mathrm{Hilb}^n(X)$ induced by the point-canonical orientations. We will see that the orientations $o_n(L) = o(n)$ for all L with $c_1(L) \cdot c_3(X) \neq 0$.

Theorem 4.3.10. *If Conjecture 4.2.10 holds for X, then the following is true:*

i. For all L from Lemma 4.3.8 with $c_1(L) \cdot c_3(X) \neq 0$ the orientation $o_n(L)$ coincide with the ones obtained from the point-canonical orientations in Definition 4.2.1.

ii. Let $\mathscr{N}(q) = \sum_{n>0} e^{np} \otimes \mathscr{N}_{np}\, q^n$ be the generating series, then we can express its exponential as

$$\exp\big(\mathscr{N}(q)\big) = M(e^p q)^{\left(\sum_{v \in B_6} c_3(X)_v u_{v,1}\right)}, \tag{4.3.14}$$

where $c_3(X)_v = c_3(X)\big(\mathrm{ch}(v^\vee)\big)$. *Equivalently, we can write this as*

$$\mathscr{N}_{np} = \sum_{l|n} \frac{n}{l^2} \sum_{v \in B_6} c_3(X)_v u_{v,1}\,. \tag{4.3.15}$$

Proof. We prove the theorem by induction on n. We begin by giving an explicit formula for the brackets in (4.3.10). Using (4.1.5) together with Lemma 4.2.15 iii., we have:

$$\begin{aligned} &Y^L(e^{(mp,1)} \otimes 1, z)(e^{(np,0)} \otimes \mathscr{N}_{np}) \\ &= (-1)^n e^{\left((m+n)p,1\right)} \exp\left[\sum_{j>0} \frac{b_j + my_j}{j} z^j\right]\left[1 - z^{-1}\sum_{v \in B_{\mathrm{even}}} \tilde{\chi}^L((mp,1),(v,0))\frac{d}{du_{v,1}}\right]\mathscr{N}_{np}\,. \end{aligned}$$

Using (4.3.5) together with Proposition 4.3.7 we get the following after taking $[z^{-1}](-)$ of the last formula:

$$[e^{(mp,1)} \otimes 1, e^{(np,0)} \otimes \mathscr{N}_{np}]^L = -(-1)^n e^{\left((m+n)p,1\right)} \otimes \sum_{v \in B_6} \int_X c_1(L)\mathrm{ch}(v) a_v(n)\,. \tag{4.3.16}$$

Let L_1 be such that $c_1(L_1) \in B_2$ and $v_1 \in B_6$ its dual. For now let us not fix the orientation $o_p = o_p^{\mathrm{can}}$, but fix $o_{[\![\mathcal{O}_X]\!]} = o^{\mathrm{can}}_{[\![\mathcal{O}_X]\!]}$ and use the rest of Definition 4.2.1.

For $n = 1$, we can choose o_p so that $o(1) = o_1(L_1)$, then $I_1(L_1) = -I(L_1)$. Using (4.3.16) together with Lemma 4.3.8 and (4.3.2), we get $\int_X c_1(L_1)\mathrm{ch}(v_1)a_{v_1}(1) = I(L_1)$. Therefore $a_{v_1}(1) = c_3(X)_{v_1}$. Suppose that L_2 is a line bundle with $c_1(L_2) \in B_2$ different from $c_1(L_1)$ and $I(L_i) \neq 0$ for $i = 1, 2$. If $o_1(L_2) = -o_1(L_1)$, then $a_{v_2}(1) = -c_3(X)_{v_2}$ for $v_2 \in B_6$ the dual of $c_1(L_2)$ and this contradicts Lemma 4.3.8: For any

$A, B \in \mathbb{Z}_{>0}$ we know that $L = L_1^{\otimes A} \otimes L_2^{\otimes B}$ is very ample, then from (4.3.16) we get

$$-\Big[AI(L_1) + BI(L_2)\Big] = I_1(L) = -AI(L_1) + BI(L_2)$$

which can not be true for all A, B. For any $v \in B_6$, we then have $a_{v_1}(1) = c_3(X)_{v_1}$.

This shows i. and ii. for $n = 1$ except $o_p = o_p^{\mathrm{can}}$. Let us now assume that i. and ii. hold for all $1 \leq k \leq n$ except $o_p = o_p^{\mathrm{can}}$. If $o_{n+1}(L_1) = -o(n+1)$ then $I_{n+1}(L_1) = -[q^{n+1}]\{M(-q)^{c_1(L_1)\cdot c_3(X)}\}$. Using the assumption together with (4.3.16) we get using the notation of (4.3.2)

$$\sum_{\substack{k>1, n_1,\dots,n_k>0\\ n_1+\cdots+n_k=n+1}} \frac{(-1)^k}{k!}\Big[\Big[\dots\big[e^{(0,1)}\otimes 1, \mathscr{M}_{n_1 p}\big]^{L_1}, \dots\Big]^{L_1}, \mathscr{M}_{n_k p}\Big]^{L_1} = \sum_{k>0} d_k(n+1) I(L_1)^k\,.$$

Subtracting this from $I_{n+1}(L_1)$ and using (4.3.2) expresses $a_v(n+1)$ as

$$a_v(n+1) = -d_1(n+1)I(L_1) - 2d_2(n+1)I(L_1)^2 - \cdots - 2d_{n+1}(n+1)I(L_1)^{n+1}\,,$$

Let $L = L_1^{\otimes N}$ for $N > 0$, then wall-crossing and using (4.3.16) with (4.3.10) gives

$$I_{n+1}(L) = \sum_{k=2}^{n+1} d_k(n+1)\big(NI(L_1)\big)^k - d_1(n+1)NI(L_1) - 2N\sum_{k=2}^{n+1} d_k(n+1)I(L_1)^k\,.$$

By comparing the coefficients of different powers of N with $\pm I_{n+1}(L)$, we obtain a contradiction. This also shows $o_{n+1}(L_1) = o(n+1) = o_{n+1}(L_2)$ for any L_2 with

$I(L_2) \neq 0$. Assuming ii. holds for coefficients $k < n$ and using (4.3.8),(4.3.2) gives us

$$\begin{aligned}(-1)^{n+1} a_{v_1}(n+1) &= \sum_{k\geq 1} d_k(n+1) I(L_1)^k - \sum_{k>1} d_k(n+1) I(L_1)^k \\ &= (-1)^{n+1} \sum_{l|n+1} \frac{n+1}{l^2} c_3(X)_{v_1} \,.\end{aligned}$$

This holds for all $L_i \in B_2$ with their duals $v_i \in B_6$, so part ii. follows as we obtain (4.3.15).

To finish the proof of part i., we only need to show that $o_p = o_p^{\mathrm{can}}$. For this, choose L such that $I(L) \neq 0$. Using Lemma 4.4.6, we see $c_1(\mathfrak{L}) = \sum_{v\in B_{\mathrm{even}}} \chi(L^\vee, v)\mu_{v,1}$. Using $X = M_p$ and (4.2.1), we see that $\int_{[M_p]^{\mathrm{vir}}} c_1(L) = \pm I(L)$. By (4.3.13) this is equal to

$$\int_{\mathscr{M}_p} \sum_{v\in B_{\mathrm{even}}} \chi(L^\vee, v)\mu_{v,1} \,, \qquad \mathscr{M}_p = \sum_{v\in B_6} c_3(X)_v u_{v,1} + c_p u_{p,1} \,,$$

which gives $I(L) + c_p$. As c_p does not depend on L it has to be 0. Therefore for the invariants to coincide, we need $o_p = o_p^{\mathrm{can}}$. □

Remark 4.3.11. Changing orientation $o_p \mapsto -o_p$ changes $o_{np} \mapsto (-1)^n o_{np}$, so if the classes $[\mathcal{M}_{np}]_{\mathrm{inv}}$ were constructed using Borisov–Joyce [23] or Oh–Thomas [141], then we would get

$$\mathscr{N}_{np} = (-1)^n \sum_{l|n} \frac{n}{l^2} \sum_{v\in\Lambda} c_3(X)_v u_{v,1}$$

However, as these are obtained indirectly through wall-crossing, we should check this is satisfied. Choosing o_p such that $o(1) = -o_1(L_1)$ in the proof of Theorem 4.3.10

does indeed give this formula. Similarly, switching to $-o^{\mathrm{can}}_{[\![\mathcal{O}_X]\!]}$ does not change the result as it should not.

The following is shown just for completeness, as we will prove a much more general statement for tautological insertions using any K-theory class in §4.5.1

Proof of Theorem 4.3.1 Using (4.3.16), (4.3.15) and (4.3.10) we obtain for any line bundle L that (4.3.2) holds. □

4.4 Virtual classes of Hilbert schemes of points and invariants

In this section, we use the result of Theorem 4.3.10. One could think of $e^{np} \otimes \mathscr{N}_{np} \in \check{H}_2(\mathcal{C}_X)$ and $\mathscr{H}_n \in \check{H}_{2n}(\mathcal{P}_X)$ as explicit invariants already. We use wall-crossing from (4.3.9) to compute $\mathscr{H}_n$ and then consider insertions, which can be expressed in the form $\exp\big[F(\mu_{v,k})\big]$, where $F(\mu_{v,k})$ is linear in $\mu_{v,k}$. After obtaining a general formula for the corresponding invariants, we apply it to multiplicative genera of tautological classes and virtual tangent bundles showing that they fit into this class. Thus we obtain an explicit expression for these, which will be used in the following section to compute new invariants.

4.4.1 Virtual fundamental cycle of Hilbert schemes

The following could be viewed as the main result of this chapter.

Theorem 4.4.1. *If Conjecture 4.2.10 holds, then the generating series* $\mathscr{H}(q) = 1 + \sum_{n>0} \frac{\mathscr{H}_n}{e^{(np,1)}} q^n$ *for point-canonical orientations is given by*

$$\mathscr{H}(q) = \exp\left[\sum_{n>0}\sum_{l|n, v\in B_6} (-1)^n \frac{n}{l^2} c_3(X)_v [z^n]\Big\{U_v(z)\exp\Big[\sum_{j>0}\frac{ny_j}{j}z^j\Big]\Big\}q^n\right], \quad (4.4.1)$$

where we fix the notation $y_j = u_{p,j}$ *and* $U_v(z) = \sum_{k>0} u_{v,k} z^k$.

Remark 4.4.2. Notice that the only $u_{\sigma,k}$ that appear in (4.4.1) are for $\sigma = (v, 0)$, $v \in B_6 \cup B_8 =: B_{6,8}$. We may therefore assume $K^1(X) = 0$ from now on. As there is no contribution of b_j, this is the unique representation of $\mathscr{H}_n$ without terms with b_j as can be seen from Lemma 4.2.15. Using (4.3.3), we have a class $\tilde{\mathscr{H}_n} = \Omega_* \circ \iota_{n*}\Big(\big[\mathrm{Hilb}^n(X)\big]^{\mathrm{vir}}\Big)$ which satisfies $\Pi_0(\tilde{\mathscr{H}_n}) = \mathscr{H}_n$. There will also be no terms containing b_j in $\tilde{\mathscr{H}_n}$, Thus $[z^n]\big(\mathcal{H}(q)\big)$ describe this canonical lift.

Proof. We begin by using the reconstruction lemma for vertex algebras to write the field $Y(e^{(np,0)} \otimes u_{v,1}) =: Y(u_{v,1}, z)Y(e^{(np,0)} \otimes 1, z) :$, where $: - :$ denotes the normal ordered product for fields of vertex algebras (see [115, §3.8], [55, Def. 2.2.2], [99, §3.1]) which acts on $e^{(mp,1)} \otimes U$ as

$$\begin{aligned}
&: Y(u_{v,1}, z)Y(e^{(np,0)} \otimes 1, z) : (e^{(mp,1)} \otimes U) = \\
&(-1)^n z^{-n} e^{((n+m)p,1)} \otimes \Big\{\big(\sum_{k>0} u_{v,k} z^{k-1}\big)\exp\Big[\sum_{i>0}\frac{ny_i}{i}z^i\Big]\exp\Big[-n\sum_{i>0}\frac{d}{du_{[\![\mathcal{O}_X]\!],i}}z^i\Big] \\
&\exp\Big[n\sum_{i>0}\frac{d}{db_i}z^i\Big] + \exp\Big[\sum_{i>0}\frac{ny_i}{i}z^i\Big]\exp\Big[-n\sum_{i>0}\frac{d}{du_{[\![\mathcal{O}_X]\!],i}}z^i\Big] \\
&\exp\Big[n\sum_{i>0}\frac{d}{db_i}z^i\Big]\Big[\tilde{\chi}((v,0),(mp,1))z^{-1} + \sum_{k>0,\sigma\in\mathbb{B}} k\tilde{\chi}((v,0),\sigma)\frac{d}{du_{\sigma,k}}z^{-k-1}\Big]\Big\}U \quad (4.4.2)
\end{aligned}$$

Where we used (4.2.9) to get $\tilde{\chi}((np,0),\sigma) = n$ if $\sigma = ([\![\mathcal{O}_X]\!], 0)$, $\tilde{\chi}((np,0),\sigma) = -n$ if

$\sigma = (0,1)$ and 0 otherwise together with part iii. of Lemma 4.2.15.

We claim that for any $r > 0$, $n_1, \dots, n_r > 0$ and $\sum_{i=1}^r n_i = n$, we have the following :

$$\begin{aligned}&\big[e^{(n_1p,0)} \otimes N(n_1p), \big[e^{(n_2p,0)} \otimes N(n_2p), \big[\dots, \big[e^{(n_rp,0)} \otimes N(n_rp), e^{(0,1)} \otimes 1\big]\dots\big]\big]\\&= e^{(np,1)} \otimes \prod_{i=1}^r \mathcal{H}_{n_i}\end{aligned}$$

in $\check{H}(\mathcal{P}_X)$, where

$$\mathcal{H}_n = \sum_{l|n, v\in B_6} (-1)^n \frac{n}{l^2} c_3(X)_v [z^n]\Big\{U_v(z)\exp\Big[\sum_{j>0} \frac{ny_j}{j} z^j\Big]\Big\}.$$

We show this by induction on r, where for $r = 0$ it is obvious. Assuming that it holds for $r-1$, we need to compute

$$\begin{aligned}&[e^{(n_1p,0)} \otimes N(n_1p), e^{((n-n_1)p,1)} \otimes \prod_{i=2}^r \mathcal{H}^{n_i}]\\&= [z^{-1}]\Big\{ : Y(u_{v,1}, z) Y(e^{(n_1p,0)} \otimes 1, z) : e^{((n-n_1)p,1)} \otimes \prod_{i=2}^r \mathcal{H}^{n_i}\Big\}\end{aligned}$$

From Remark 4.4.2, we see that we can replace all $\exp\Big[n\sum_{i>0}\frac{d}{db_i}\Big]$ and $\exp\Big[-n\sum \frac{d}{du_{[\![\mathcal{O}_X]\!],i}}\Big]$ by 1 in (4.4.2). The second term under the curly bracket in (4.4.2) vanishes, because it contains $\tilde{\chi}((v,0),(mp,1))z^{-1} = \chi(n,mp)z^{-1} - \chi(v)z^{-1}$ where the result is zero for degree reasons and because $\mathrm{td}_1(X) = 0$. In the term with $\sum_{k>0,\sigma\in\mathbb{B}} k\tilde{\chi}((v,0),\sigma)\frac{d}{du_{\sigma,k}}z^{-k-1}$ the sum can be taken over all $\sigma = (w,0)$, $w \in B_{6,8}$ by Remark 4.4.2 so it vanishes because $\chi(v,w) = 0$ whenever $v, w \in B_{6,8}$. We are therefore left with

$$
\begin{aligned}
&[z^{-1}]\Big\{(-1)^{n_1}z^{-n_1}e^{(n_1p,1)}\otimes\big(\sum_{k>0}u_{v,k}z^{k-1}\big)\exp\Big[\sum_{i>0}\frac{y_i}{i}z^i\Big]\Big\}\prod_{i=2}^{r}\mathcal{H}^{n_i}\\
&=(-1)^n e^{(n_1p,1)}[z^{n_1}]\Big\{U_v(z)\exp\Big[\sum_{j>0}\frac{n_1y_j}{j}z^j\Big]\Big\}\prod_{i=1}^{r}\mathcal{H}^{n_i}\,.
\end{aligned}
$$

Multiplying with the coefficients $\sum_{l|n}\frac{n}{l^2}\sum_{v\in B_6}c_3(X)_v$ of $\mu_{v,1}$ in (4.3.15) and summing over all $v\in B_6$, we obtain the result as we are able to rewrite (4.3.9) by reordering the terms keeping track of the signs as

$$
\mathscr{H}_n=\sum_{\substack{k\geq 1,n_1,\ldots,n_k\\ \sum n_i=n}}\frac{1}{k!}\big[e^{(n_1p,0)}\otimes N(n_1p),\big[\ldots,\big[e^{(n_kp,0)}\otimes N(n_kp),e^{(0,1)}\otimes 1\big]\ldots\big]\big]\,.
$$

□

We now describe a general formula for integrating topological insertions over $\mathscr{H}_n$ which will be applied in the following to examples.

Proposition 4.4.3. *Let $\mathscr{A}\subset K^0(X)\backslash\{0\}$ be a finite subset. For each $\alpha\in\mathscr{A}$ let us have some exponential generating series*

$$
A_\alpha(z,\mathfrak{p})=\sum_{n\geq 0}a_\alpha(n,\mathfrak{p})\frac{z^n}{n!}\,,
$$

where $\mathfrak{p}=(p_1,p_2,\ldots)$ are additional variables and $\alpha(n,\mathfrak{p})\in\mathbb{Q}[\![\mathfrak{p}]\!]$, s.t. $a_\alpha(0,0)=0$. If $\mathcal{I}_n\in H^*\big(\mathrm{Hilb}^n(X)\big)$ is such that $\mathcal{I}_n=(\Omega\circ\iota_n)^*(\mathcal{T})$ for a weight 0 insertion $\mathcal{T}\in$

$H^*(\mathcal{P}_X)$, where

$$\int_{\mathscr{H}_n} \mathcal{T} = \int_{\mathscr{H}_n} \exp\Big[\sum_{\alpha\in\mathscr{A}} \sum_{\substack{k\geq 0\\ v\in B_{6,8}}} a_\alpha(k,\mathfrak{p})\chi(\alpha^\vee, v)\mu_{v,k}\Big],$$

then the generating series $\mathrm{Inv}(q) = 1 + \sum_{n>0}\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} \mathcal{I}_n q^n$ *is given by*

$$\begin{aligned}\prod_{\alpha\in\mathscr{A}} \exp\Big\{ \sum_{n>0}(-1)^n \sum_{l|n} \frac{n}{l^2}[z^n]\Big[z\frac{d}{dz}\big(A_\alpha(z,\mathfrak{p}) - A_\alpha(0,\mathfrak{p})\big)\\ \exp\Big(\sum_{\alpha\in\mathscr{A}} \mathrm{rk}(\alpha)A_\alpha(z,\mathfrak{p})\Big)^n\Big]q^n\Big\}^{c_1(\alpha)\cdot c_3(X)}.\end{aligned} \quad (4.4.3)$$

Proof. Using Lemma 4.2.13 to act on the homology classes $\mathscr{H}_n$ from Theorem 4.4.1, we obtain

$$\begin{aligned}&\mathrm{Inv}(q)\\ &= \exp\Big[\sum_{n>0}\sum_{l|n}(-1)^n\frac{n}{l^2}c_3(x)_v[z^n]\Big\{\sum_{k>0}\sum_{\alpha\in\mathscr{A}}\frac{a_\alpha(k,\mathfrak{p})}{(k-1)!}z^k\exp\Big[\sum_{j>0}\sum_{\alpha\in\mathscr{A}} n\frac{\mathrm{rk}(\alpha)a_\alpha(j,\mathfrak{p})}{j!}\Big]\Big\}\Big]\end{aligned}$$

which can be seen to be equal to (4.4.3). □

We get the following simple consequence of the above results.

Corollary 4.4.4. *With the notation and assumptions from Proposition 4.4.3 it follows that* $\mathrm{Inv}(q)$ *depends only on* $c_1(\alpha)\cdot c_3(X)$ *and* $\mathrm{rk}(\alpha)$ *for all* $\alpha\in\mathscr{A}$*. For more general insertions, the invariants only depend on* $\int_X c_3(X)\cdot(-) : H^2(X)\to\mathbb{Z}$*.*

Remark 4.4.5. For the classes $[\mathcal{M}_{np}]_{\mathrm{inv}} \in \check{H}_2(\mathcal{N}_X)$, we did not find any interesting non-zero invariants of the form $\int_{[\mathcal{M}_{np}]_{\mathrm{inv}}} \nu$ for some weight 0 insertion ν on $\mathcal{N}_X$. We already know that $\mathcal{L}|_{\mathcal{N}_{n,0}} = 0$. On the other hand, if one takes $T_0(\alpha) = \pi_{2*}(\pi_X^*(\alpha)\otimes$

$\mathcal{E}_0$) on $\mathcal{M}_0$ for any class $\alpha \in G^0(X)$ (see §4.5.1), we can consider its topological counterpart $\mathfrak{T}_{\mathrm{wt}=1}(\alpha) = \pi_{2*}(\pi_X^*(\alpha) \otimes \mathfrak{E})$ on $\mathcal{C}_X$ which has weight 1. Then denoting $\nu = p(\mathrm{ch}_1(\mathfrak{T}_{\mathrm{wt}=1}(\alpha)), \mathrm{ch}_2(\mathfrak{T}_{\mathrm{wt}=1}(\alpha)), \ldots)$, the integral $\int_{[\mathcal{M}_{np}]_{\mathrm{inv}}} \nu$ is not well defined as it does not satisfy Lemma 4.1.6.

Moreover, consider the complex $\mathbb{E}_0 = \pi_*\big(\underline{\mathrm{Hom}}_{\mathcal{M}_0}(\mathcal{E}_0, \mathcal{E}_0)\big)$ on $\mathcal{M}_0$, then this will be weight zero. However, taking $\nu = p(\mathrm{ch}_1(\mathbb{E}_0), \mathrm{ch}_2(\mathbb{E}_0), \ldots)$ we get

$$\int_{[\mathcal{M}_{np}]_{\mathrm{inv}}} \nu = \int_{e^{(0,np)} \otimes \mathcal{N}_{np}} p(\mathrm{ch}_1(\Delta^*(\theta)), \mathrm{ch}_2(\Delta^*(\theta), \ldots)\,,$$

which can be shown to be always zero.

4.4.2 Multiplicative genera as insertions

The main examples we want to address are *multiplicative genera* of *tautological classes* below.

For a scheme S, let $G^0(S)$ and $G_0(S)$ denote its Grothendieck groups of vector bundles and coherent sheaves respectively. We have the map $\lambda : G^0(S) \to K^0(S)$ which we often neglect to write, i.e. $\lambda(\alpha) = \alpha$. We have the Chern-character $\mathrm{ch} : G^0(S) \to A^*(S, \mathbb{Q})$ which under the natural maps to $K^0(S)$ and $H^*(S, \mathbb{Q})$ corresponds to the topological Chern-character $\mathrm{ch} : K^0(S) \to H^{\mathrm{even}}(X, \mathbb{Q})$.

Let $f(\mathfrak{p}, z) = \sum_{n\geq 0} f_n(\mathfrak{p}) z^n \in \mathbb{Q}[\![\mathfrak{p}]\!][\![z]\!]$ be a formal power-series in formal power-series of additional variables $\mathfrak{p} = (p_1, \ldots, p_k)$ with $f(0,0) = 1$, then a *multiplicative* *genus* $\mathscr{G}_{f(\mathfrak{p},\cdot)}$ of Hirzebruch [83, §4] associated to f is a group homomorphism

$$\mathscr{G}_f : G^0(X) \longrightarrow \big(A^*(X,\mathbb{Q})[\![\mathfrak{p}]\!]\big)_1 , \tag{4.4.4}$$

where $\big(A^*(X,\mathbb{Q})[\![\mathfrak{p}]\!]\big)_1$ denotes the multiplicative group of the ring $A^*(X,\mathbb{Q})[\![\mathfrak{p}]\!]$ containing power-series with constant term in $\mathfrak{p}$ and $A^0(X,\mathbb{Q})$ being 1. For each vector bundle $E \to X$ of $\mathrm{rk}(E) = a$ there is by using the splitting principles a unique factorization $c(E) = \prod_{i=1}^a (1 + x^i)$, where $x^i \in A^1(X,\mathbb{Q})$. Then $\mathscr{G}_f$ is given by

$$\mathscr{G}_f(E) = \prod_{i=1}^{a} f(\mathfrak{p}, x_i)\,.$$

Define $\Lambda^\bullet_t : G^0(S) \to \Big(G^0(S)[\![t]\!]\Big)_1$ to be a group homomorphism acting on each vector bundle E by

$$[E] \mapsto \sum_{i=0}^{\infty}[\Lambda^i E](-t)^i\,,$$

where $\Big(G^0(S)[\![t]\!]\Big)_1$ denotes the power-series in t with constant term $[\![\mathcal{O}_X]\!] \in G^0(S)$, $G0(S)$ is a group under the addition and $\Big(G^0(S)[\![t]\!]\Big)_1$ under the multiplication induced by the tensor product. We also have $\mathrm{Sym}^\bullet_t : G^0(S) \to \Big(G^0(S)[\![t]\!]\Big)_1$, where $\mathrm{Sym}^\bullet_{-t}(\alpha) = \Big(\Lambda^\bullet_t(\alpha)\Big)^{-1}$ for all $\alpha \in G^0(S)$. On $\mathrm{Hilb}^n(X)$, we will consider the classes

$$\alpha^{[n]} = \pi_{2\,*}\big(\pi_X^*(\alpha)\otimes\mathcal{F}_n\big)\,, \quad \alpha \in G^0(X)\,, \qquad T_n^{\mathrm{vir}} = \underline{\mathrm{Hom}}_{\mathrm{Hilb}^n(X)}\big(\mathscr{I}_n, \mathscr{I}_n\big)_0[1]\,, \tag{4.4.5}$$

where $\mathscr{I}_n = \big(\mathcal{O}_X \to \mathcal{F}_n\big)$ is the universal complex on $\mathrm{Hilb}^n(X)$ and $(-)_0$ denotes the trace-less part. The corresponding topological analogs are

$$\mathfrak{T}(\alpha) = \mathfrak{U}^\vee \boxtimes \pi_{2*}\big(\pi_X^*(\alpha)\otimes \mathfrak{E}\big) \qquad \text{and} \qquad -\theta_{\mathcal{P}}^\vee \in K^0(\mathcal{P}_X\times\mathcal{P}_X)\,.$$

Lemma 4.4.6. *In $H^*(\mathcal{P}_X\times\mathcal{P}_X)$ the following holds for all $\alpha\in K^0(X)$:*

$$\begin{aligned}\mathrm{ch}_j\big(\mathfrak{T}(\alpha)\big) &= \sum_{\substack{v\in B_{\mathrm{even}}\\ j=l+k}} (-1)^l\chi(\alpha^\vee,v)\beta_l\boxtimes\mu_{v,k}\,,\\ \mathrm{ch}_k(\theta_{\mathcal{P}}) &= \sum_{\substack{i+j=k\\ \sigma,\tau\in\mathbb{B}\backslash B_{\mathrm{odd}}}}(-1)^j\tilde{\chi}(\sigma,\tau)\mu_{\sigma,i}\boxtimes\mu_{\tau,j} + \sum_{\substack{i+j=k+1\\ v,w\in B_{\mathrm{odd}}}}(-1)^j\chi^-(v,w)\mu_{v,i}\boxtimes\mu_{w,j}\,.\end{aligned}$$

We also have the identity

$$\mathrm{ch}(T_n^{\mathrm{vir}}) = -(\Omega\circ\iota_n)^*\big(\mathrm{ch}(\Delta^*\theta_{\mathcal{P}}^\vee)\big)+2$$

in $H^*\big(\mathrm{Hilb}^n(X)\big)$.

Proof. Using Atiyah–Hirzebruch–Riemann–Roch [46], we get

$$\mathrm{ch}_i\big(\pi_{2*}(\pi_X^*(\alpha)\otimes\mathfrak{E})\big) = \sum_{v\in B_{\mathrm{even}}}\int_X \mathrm{ch}(\alpha)\mathrm{ch}(v)\mathrm{Td}(X)\boxtimes\mu_{v,i} = \sum_{v\in B_{\mathrm{even}}}\chi(\alpha^\vee,v)\mu_{v,i}\,.$$

Taking a product with $\mathrm{ch}(\mathfrak{U}^\vee)$ and using $\beta_j=\mathrm{ch}_j(\mathfrak{U})$, we get

$$\mathrm{ch}_j\big(\mathcal{T}(\alpha)\big) = \sum_{\substack{v\in B_{\mathrm{even}}\\ j=l+k}}(-1)^l\chi(\alpha^\vee,v)\beta_l\boxtimes\mu_{v,k}\,. \tag{4.4.6}$$

A similar explicit computation leads to the second formula. Let us therefore address the final statement.

Let $\mathscr{P} : \mathrm{Hilb}^n(X) \to \mathcal{M}_X$ map $[\mathcal{O}_X \to F]$ to $[\mathcal{O}_X[1] \oplus F]$ and $\mathcal{E}\mathrm{xt}_n = \underline{\mathrm{Hom}}_{\mathrm{Hilb}^n(X)}\big(\mathscr{I}_n, \mathscr{I}_n\big)$. We have the following $\mathbb{A}^1$-homotopy commutative diagram:

$$\begin{array}{ccc} \mathrm{Hilb}^n(X) & \xrightarrow{\mathscr{P}} & \mathcal{M}_X \\ & \searrow^{\mathcal{E}\mathrm{xt}_n} & \downarrow \mathcal{E}\mathrm{xt} \\ & & \mathrm{Perf}_{\mathbb{C}} \end{array}, \tag{4.4.7}$$

where $\mathcal{E}\mathrm{xt}, \mathcal{E}\mathrm{xt}_n$ are the maps associated to the perfect complexes of the same name. From Definition 4.2.5, we easily deduce $\iota_n^* \circ \Delta^*\big(\Theta^{\mathrm{pa}}\big) = \mathscr{P}^*\mathcal{E}\mathrm{xt}^\vee$. Taking topological realization of (4.4.7), we obtain that

$$[\![\mathcal{E}\mathrm{xt}_n]\!] = (\mathscr{P}^{\mathrm{top}})^*[\![\mathcal{E}\mathrm{xt}]\!] = \iota_n^*[\![\Delta^*\big(\Theta^{\mathrm{pa}}\big)^\vee]\!] = (\Omega \circ \iota_n)^*\big(\Delta^*(\theta_{\mathcal{P}})\big)^\vee .$$

Finally, we use $\mathrm{rk}\big((\mathcal{E}\mathrm{xt}_n)_0\big) = \mathrm{rk}(\mathcal{E}\mathrm{xt}_n) - 2$. $\square$

To simplify notation, we will not write $\mathfrak{p}$ unless necessary and use $f(\alpha^{[n]})$ and $f(T_n^{\mathrm{vir}})$ instead of the full $\mathscr{G}_f(-)$.

Lemma 4.4.7. *Let f be an invertible power-series, then*

$$\begin{aligned} \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} f(\alpha^{[n]}) &= \int_{\mathscr{H}_n} \exp\Big[\sum_{\substack{k\geq 0\\ v\in B_{6,8}}} a_\alpha(k)\chi(\alpha^\vee, v)\mu_{v,k}\Big]\,, \quad \text{where} \\ A_\alpha(z) &= \sum_{k\geq 0} a_\alpha(k)\frac{z^k}{k!} = \log\big(f(z)\big)\,, \\ \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} f(T_n^{\mathrm{vir}}) &= \int_{\mathscr{H}_n} \exp\Big[\sum_{\substack{k\geq 0\\ v\in B_{6,8}}} a_{[\![\mathcal{O}_X]\!]}(k)\chi(v)\mu_{v,k}\Big]\,, \quad \text{where} \\ A_{[\![\mathcal{O}_X]\!]}(z) &= \sum_{k\geq 0} a_{[\![\mathcal{O}_X]\!]}(k)\frac{z^k}{k!} = \log\big(f(z)f(-z)\big)\,. \end{aligned}$$

Proof. We show that in the action of $\mathrm{ch}_k(\theta_{\mathcal{P}}^\vee)$ from Lemma 4.4.6 on $\mathscr{H}_n$ only terms linear in $\mu_{v,k}$, $k > 0$ will have non-trivial contributions. In Remark 4.4.2, we set $K^1(X) = 0$, thus we only need to look at $\sum_{\substack{i+j=k \\ \sigma,\tau\in\mathbb{B}\setminus B_{\mathrm{odd}}}} (-1)^i\tilde{\chi}(\sigma,\tau)\mu_{\sigma,i}\boxtimes\mu_{\tau,j}$ and we claim it reduces to the action by

$$-\sum_{v\in B_{6,8}} \big(1+(-1)^k\big)\chi(v)\mu_{v,k} = -\big(1+(-1)^k\big)\mu_{p,k} \tag{4.4.8}$$

For $i, j > 0$ if $\sigma = (0,1)$ or $\tau = (0,1)$, then due to Remark 4.4.2 this term vanishes. If $\sigma = (v,0)$, $\tau = (w,0)$ then $v, w \in B_{6,8}$ and $\tilde{\chi}(\sigma,\tau) = \chi(v,w) = 0$. So consider the case $i = 0$, then $j = k > 0$. If $\sigma = (v,0)$, $\tau = (w,0)$ or $\tau = (0,1)$ then the term is again 0, because $\mu_{v,0} = np(v^\vee) = 0$ unless $v = p$ in which case $\chi(v,w) = 0$ for each $w \in B_{6,8}$. However, if $\sigma = (0,1)$, $\tau = (v,0)$, then $\mu_{\sigma,0} = 1$ and $\tilde{\chi}((0,1),(v,0)) = -\chi(v)$. If $j = 0$, then the same applies, thus the statement follows.

Let E be a vector bundle with $c(E) = \prod_{i=1}^a(1+x_i)$, then we write

$$f(E) = \prod_{i=1}^a f(x_i) = \exp\Big[\sum_{n>0} g_n \sum_{i=1}^a \frac{x_i}{n!}t^n\Big],$$

where $\sum_{n>0}\frac{g_n}{n!}x^n = \log\big(f(x)\big)$ and $\sum_{i=1}^a \frac{x_i}{n!} = \mathrm{ch}_i(E)$. This extends to any class $\alpha \in G^0\big(\mathrm{Hilb}^n(X),\mathbb{Q}\big)$, so we get after using Remark 4.4.2, (4.4.8) and Lemma (4.4.6) that

$$\int_{\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}}} f(\alpha^{[n]}) = \int_{\mathscr{H}_n} \exp\Big[\sum_{\substack{k\geq 0\\ v\in B_{6,8}}} g_k\chi(\alpha^\vee, v)\mu_{v,k}t^k\Big],$$

$$\int_{\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}}} f(T_n^{\mathrm{vir}}) = \int_{\mathscr{H}_n} \exp\Big[\sum_{\substack{k\geq 0\\ v\in B_{6,8}}} g_k\big(1+(-1)^k\big)\chi(v)\mu_{v,k}t^k\Big],$$

where, we use $\mu_{p,0} = n$ and $\mathrm{rk}(T_n^{\mathrm{vir}}) = 2n$. From this we immediately see $A_\alpha(z) =$

$\log\big(f(z)\big)$ and $A_{[\![\mathcal{O}_X]\!]}(z) = \log\big(f(z)f(-z)\big)$. $\square$

As an immediate Corollary of (4.4.8), we obtain the following:

Corollary 4.4.8. *For each n let $p_n(x_1 t, x_2 t^2, \ldots)$ be a formal power-series in infinitely many variables, then*

$$\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} p_n(\mathrm{ch}_1(T_n^{\mathrm{vir}}), \mathrm{ch}_2(T_n^{\mathrm{vir}}), \ldots) = 0\,.$$

Proof. We use

$$\int_{\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}}} p_n(\mathrm{ch}_1(T_n^{\mathrm{vir}}), \mathrm{ch}_2(T_n^{\mathrm{vir}}), \ldots) = \int_{\mathscr{H}_n} \tilde{p}_n(\mu_{p,1}, \mu_{p,2}, \ldots)\,,$$

where we use some new formal power-series $\tilde{p}(x_1 t, x_2 t^2\,, \ldots)$ given by (4.4.8). Because each term in (4.4.1) contains at least one factor of the form $\mu_{v,k}$ for $v \in B_6$, $k > 0$, the above integral is zero by (4.2.14). $\square$

Definition 4.4.9. Let us define the *universal transformation* U of formal power-series $U : \big(R[\![t]\!]\big)_1 \to \big(R[\![t]\!]\big)_1$ by

$$f(t) \mapsto \prod_{n>0} \prod_{k=1}^{n} f(-e^{\frac{2\pi i k}{n}} t)^{-n}\,, \tag{4.4.9}$$

for any ring R. Moreover, we will use the notation

$$\{f\}(t) = f(t)f(t)\,.$$

In fact, U is a well-defined bijection. To see this, note: $\log\Big(\prod_{k=1}^{n} f(-e^{\frac{2\pi i k}{n}} t)^{-n}\Big) = -\sum_{m=0} n^2 f_{nm} q^{nm}$ by Knuth [105, eq. (13), p. 89]. Therefore $\prod_{k=1}^{n} f(-e^{\frac{2\pi i k}{n}} t)^{-n} =$

$1 + O(t^n)$. This is precisely the condition necessary for the infinite product to be well-defined. Moreover, it maps integer valued power-series back to integer-valued ones. To construct an inverse, we can take the logarithm of (4.4.9) to get $\sum_{n>0}\sum_{m=0} n^2 f_{nm} q^{nm} = \sum_{n>0}\sum_{l|n} \frac{n^2}{l^2} f_n q^n$ where $\log\bigl(f(q)\bigr) = \sum_{n>0} f_n q^n$. This corresponds to acting with a diagonal invertible matrix on the coefficients f_n, so we have an inverse.

Example 4.4.10. Acting with U^{-1} on the MacMahon function $M(q)$, we obtain $(1+q)$.

We will need later the following generalization of the Lagrange inversion theorem:

Lemma 4.4.11. *Let $Q(t) \in R[\![t]\!]$ (with a non-vanishing constant term) and $g_i(x)$ for $i = 1, \ldots, N$ be the different solutions to*

$$\bigl(g_i(x)\bigr)^N = xQ\bigl(g_i(x)\bigr) \tag{4.4.10}$$

then for any formal power-series $\phi(t)$, $\phi(0) = 0$ we have

$$\sum_{k=1}^{N} \phi\bigl(g_i(x)\bigr) = \sum_{n>0} \frac{1}{n}[t^{nN-1}]\Bigl(\phi'(t)Q(t)^n\Bigr)x^n\,.$$

Proof. The usual Lagrange formula (see e.g. Gessel [68, Thm. 2.1.1]) tells us that for $h(x) = xQ\bigl(h(x)\bigr)$, we have $[x^n]\phi\bigl(h(x)\bigr) = \frac{1}{n}[t^{n-1}]\phi'(t)Q(t)^n \quad \text{for} \quad n > 0$. Taking the unique Newton–Puiseux series satisfying $g(x^{\frac{1}{N}}) = x^{\frac{1}{N}}Q^{\frac{1}{N}}\bigl(g(x^{\frac{1}{N}})\bigr)$ for a fixed N'th root of Q, we can write by Weierstrass preparation theorem together with the Newton–Puiseux theorem (see e.g. [90, Chap. 3.2, Chap. 5.1,] every solution of

(4.4.10) by $g_k(x) = g(e^{\frac{2\pi ki}{N}} x^{\frac{1}{N}})$. We obtain

$$\sum_{k=1}^{N} \phi\big(g_k(x)\big) = \sum_{k=1}^{N} \sum_{n>0} \frac{1}{n}[t^{n-1}]\big(\phi'(t)Q^{\frac{n}{N}}(t)\big)\Big(e^{\frac{2\pi ik}{N}} x^{\frac{1}{N}}\Big)^n$$
$$= \sum_{n>0} \frac{1}{n}[t^{nN-1}]\Big(\phi'(t)Q^n(t)\Big)x^n \,.$$

□

We prove now the main result that we will use throughout the next section.

Proposition 4.4.12. *Let* $f_0(\mathfrak{p},\cdot), f_1(\mathfrak{p},\cdot), \ldots, f_M(\mathfrak{p},\cdot)$ *be power-series with* $f(0,0) = 1$*, then define*

$$\mathrm{Inv}(\vec{f},\vec{\alpha},q) = 1 + \sum_{n>0} \int_{\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}}} f_0(T_n^{\mathrm{vir}}) f_1(\alpha_1^{[n]}) \cdots f_M(\alpha_M^{[n]}) q^n \,,$$

where $(\vec{-})$ *is meant to represent a vector, and we omit the additional variables. Then setting* $\mathrm{rk}(\alpha_i) = a_i$*, we have*

$$\mathrm{Inv}(\vec{f},\vec{a},q) = U\left\{ \left[\prod_{i=1}^{M} \frac{f_i(0)}{f_i(H(q))} \right]^{c_1(\alpha_i)\cdot c_3(X)} \right\}, \tag{4.4.11}$$

where $H(q)$ *is the unique solution for*

$$q = \frac{H}{\prod_{j=1}^{M} f_j^{a_j}(H)\,\{f_0\}(H)} \,.$$

Proof. Combining Lemma 4.4.7 with Proposition 4.4.3, we obtain

$$\mathrm{Inv}(q) = \prod_{i=1}^{M} \exp\Big\{ \sum_{n>0} \sum_{l|n} \frac{n}{l^2} (-1)^n [z^{n-1}] \Big[\frac{d}{dz} \Big(\log\big(f_i(z)\big) - \log\big(f_i(0)\big) \Big) \\ \prod_{j=1}^{M} f_j(z)^{a_j n} \{f_0\}^n(z) \Big] q^n \Big\}^{c_1(\alpha_i)\cdot c_3(X)}$$

setting $\phi = \log(f_i) - \log\big(f_i(0)\big)$, $Q = \prod_{i=1}^{M} f_i^{a_i n}\{f_0\}^n$ and using Lemma 4.4.11, this gives

$$\begin{aligned} &\prod_{i=1}^{M} \exp\Big\{ \sum_{n>0} \sum_{l|n} \frac{n^2}{l^2} [t^n] \Big[\log f_i\big(H(t)\big) - \log f_i(0) \Big] (-q)^n \Big\}^{c_1(\alpha_i)\cdot c_3(X)} \\ &= \prod_{i=1}^{M} \exp\Big\{ \sum_{n>0} n \sum_{k=1}^{n} [t^n] \Big[\log f_i\big(H(t)\big) - \log f(0) \Big] (-e^{\frac{2\pi i k}{n}} q)^n \Big\}^{c_1(\alpha_i)\cdot c_3(X)} \\ &= \prod_{i=1}^{M} \prod_{n>0} \prod_{k=1}^{n} \Big[\frac{f_i\big(H(-e^{\frac{2\pi i k}{n}} q)\big)}{f_i(0)} \Big]^{n c_1(\alpha_i)\cdot c_3(X)} = \prod_{i=1}^{M} U \Big[\frac{f_i(0)}{f_i\big(H(q)\big)} \Big]^{c_1(\alpha_i)\cdot c_3(X)}, \end{aligned}$$

where $H(q)$ is the solutions of (4.4.11). □

4.5 New invariants

We define and compute many new invariants using the formula derived in the previous section. These include tautological series, virtual Verlinde numbers and Nekrasov genera. We study their symmetries and their relation to lower-dimensional geometries. We obtain an explicit correspondence between virtual Donaldson invariants on elliptic surfaces and DT4 invariants on projective Calabi–Yau fourfolds via the universal U transformation. Note that the Segre–Verlinde correspondence among the results that follow could be traced back to already existing results of Oprea–Pandharipande [143] and Arbesfeld–Johnson–Lim–Oprea–Pandharipande [5] using Theorem 4.5.14, but as we worked these out independently we prefer to present them so. The final section is dedicated to wall-crossing for quot-schemes of elliptic surfaces and curves.

4.5.1 Segre series

Setting $f_0 = 1$ and $f_i = (1 + t_i x)^{-1}$ in Proposition 4.4.12, we obtain the *generalized* DT_4*-Segre series*

$$R(\vec{\alpha}, \vec{t}; q) = 1 + \sum_{n>0} q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} s_{t_1}(\alpha_1^{[n]}) \cdots s_{t_M}(\alpha_M^{[n]}) \,.$$

Theorem 4.5.1. *Let $\alpha_1, \ldots, \alpha_M \in G^0(X)$, $a = \mathrm{rk}(\alpha)$, then assuming Conjecture 4.2.10 for point-canonical orientations we have*

$$R(\vec{\alpha}, \vec{t}; q) = U\Big[(1 + t_1 z)^{c_1(\alpha_1)\cdot c_3(X)} \cdots (1 + t_M z)^{c_1(\alpha_M)\cdot c_3(X)}\Big]\,,$$

where z is the solution to $z(1 + t_1 z)^{a_1} \cdots (1 + t_M z)^{a_M} = q$. Moreover, we have the explicit expression:

$$R(\alpha; q) = \begin{cases} U\big[\mathscr{B}_{a+1}(-q)^{-c_1(\alpha)\cdot c_3(X)}\big] & \text{for } a \geq 0 \\ \\ U\big[\mathscr{B}_{-a}(q)^{c_1(\alpha)\cdot c_3(X)}\big] & \text{for } a < 0 \end{cases} \,. \tag{4.5.1}$$

Proof. The first statement follows immediately from Proposition 4.4.12. Specializing to the DT_4 *Segre series*

$$R(\alpha;q) = \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} s_n(\alpha^{[n]})$$

we obtain for $a = \mathrm{rk}(\alpha)$

$$R(\alpha;q) = U\big[(1+z)\big]^{c_1(\alpha)\cdot c_3(X)}, \quad \text{where} \quad q = z(1+z)^a .$$

The theorem then follows from the following lemma.

Lemma 4.5.2. *Let y be the solutions of $y(y+1)^a = q$ for $a > 0$, then*

$$\frac{1}{1+y} = \begin{cases} \mathscr{B}_{a+1}(-q) & \text{for } a \geq 0 \\ \\ \mathscr{B}_{-a}(q)^{-1} & \text{for } a < 0 \end{cases} . \tag{4.5.2}$$

Proof. We use Lemma 4.4.11. For $a \geq 0$ we change variables $z = 1/(1+y)$ this implies $g(z) := (1-z)/z^{a+1} = q$. Then the statement follows from

$$\begin{aligned} [(z-1)^{n-1}]\Big(\frac{(z-1)}{g(z)}\Big)^n &= [(z-1)^{n-1}](-1)^n\big(1+(z-1)\big)^{(a+1)n} \\ &= \frac{(-1)^n}{n}\binom{(a+1)n}{n-1} = \frac{(-1)^n}{(a+1)n+1}\binom{(a+1)n+1}{n} \\ &= (-1)^n C_{n,a} , \end{aligned}$$

where we used the notation from (1.0.9). When $a < 0$, then change variables by $(u+1) = (z+1)^{-1}$ to get $z = -u(u+1)^{-1}$ and thus $-u(u+1)^{-a-1} = q$. Then $(1+z)^{-1} = (1+u) = \mathscr{B}_{-a}(q)^{-1}$ by the above. □

Using this, we obtain (4.5.1). $\square$

4.5.2 K-theoretic insertions

In this section, we use the Oh–Thomas Riemann–Roch formula from Theorem 2.3.2 for their twisted virtual structure sheaf. Recall that $\mathrm{Td} = \mathscr{G}_f$ for $f(x) = \frac{x}{1-e^{-x}}$ which satisfies

$$\{\sqrt{f}\}(x) = \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} \,. \tag{4.5.3}$$

An immediate consequence of Corollary 4.4.8 is

Corollary 4.5.3. *For all $n > 0$, $\hat{\chi}^{\mathrm{vir}}\big(\mathrm{Hilb}^n(X)\big) = 0$.*

Nekrasov genus gives us refinements of invariants considered in §4.5.1 as

$$K_n(\vec{\alpha}, \vec{y}) = \hat{\chi}^{\mathrm{vir}}\Big(\mathscr{N}_{y_1}\big(\alpha_1^{[n]}\big) \cdots \mathscr{N}_{y_M}\big(\alpha_M^{[n]}\big)\Big)\,, \qquad K(\vec{\alpha}, \vec{y}; q) = \sum_{n>0} K_n(\vec{\alpha}, \vec{y}) q^n\,.$$

It is given by its series $\mathscr{N}_y(x) = (y^{\frac{1}{2}} e^{-\frac{x}{2}} - y^{-\frac{1}{2}} e^{\frac{x}{2}})$. Note that $\mathscr{N}_0(0) \neq 1$, but we can write it as $\mathscr{N}_y(x) = (1 - y^{-1}e^x)e^{-\frac{x}{2}} y^{\frac{1}{2}}$ and simply keep track of $y^{\frac{1}{2}}$ separately. DT_4 Segre series of §4.5.1 can be obtained as a *classical limit* of these invariants. Explicitly this means the following:

Proposition 4.5.4. *For any $\alpha_1, \ldots, \alpha_M \in G^0(X)$ with $a_j = \mathrm{rk}(\alpha_j)$ and i_j, such that $\sum_j i_j = n$, we have*

$$\lim_{y_1 \to 1^+} \cdots \lim_{y_M \to 1^+} (1 - y_1^{-1})^{i_1 - a_1 n} \cdots (1 - y_M^{-1})^{i_M - a_M n} K_n(\vec{\alpha}, \vec{y}) = (-1)^n \int_{\left[\mathrm{Hilb}^n(X)\right]^{\mathrm{vir}}} c_{i_1}\big(\alpha_1^{[n]}\big) \cdots c_{i_M}\big(\alpha_M^{[n]}\big)\,.$$

Proof. We conclude it from a more general result for any scheme S.

Define $A^{>m} = \bigoplus_{i>m} A^i(S,\mathbb{Q})$ and $A^{\leq m} = A^*(S,\mathbb{Q})/A^{>m}$. Let $\gamma \in G^0(S)$, $a = \mathrm{rk}(\gamma)$ and $k \geq 0$, then we claim that in $A^{\leq k}$ the following holds:

$$\mathscr{L}_k(\gamma) := \lim_{y\to 1^+}(1-y^{-1})^{k-a}\Big[\mathrm{ch}\Big(\Lambda^{\bullet}_{y^{-1}}\gamma \cdot \det{}^{-\frac{1}{2}}(\gamma\cdot y^{-1})\Big)\Big] = (-1)^k c_k(\gamma) \ :$$

Let $\gamma = [\![E]\!] - [\![F]\!]$ and $c(E) = \prod_{i=1}^{e}(1-x_i)$, $c(F) = \prod_{i=1}^{f}(1-z_i)$, then in $A^{\leq l}$, we have

$$\begin{aligned}\mathscr{L}_l(E) &= \lim_{\lambda\to 0^+}\Big[(1-e^{-\lambda})^{l-e}\prod_{i=1}^{e}(e^{\frac{\lambda}{2}-\frac{x_i}{2}} - e^{-\frac{\lambda}{2}+\frac{x_i}{2}})\Big] \\ &= \lim_{\lambda\to 0^+}\Big[(\lambda - O(\lambda^2))^{l-e}\prod_{i=1}^{e}((\lambda - x_i) + O\big((\lambda-x_i)^3\big)\Big] = [\lambda^{-l}]\prod_{i=1}^{e}(1-\lambda^{-1}x_i) \\ &= (-1)^l c_l([\![E]\!])\,.\end{aligned}$$

Similarly, we obtain in $A^{\leq m}$

$$\begin{aligned}&\mathscr{L}_m(-[\![F]\!]) = \lim_{\lambda\to 0^+}\Big[(1-e^{-\lambda})^{m+f}\prod_{j=1}^{f}(e^{\frac{\lambda}{2}-\frac{z_j}{2}} - e^{-\frac{\lambda}{2}+\frac{z_j}{2}})^{-1}\Big] = [\lambda^{-m}]\prod_{j=1}^{f}(1-\frac{x_i}{\lambda})^{-1} \\ &= (-1)^m c_m(-[\![F]\!])\,.\end{aligned}$$

We combine these two to obtain $\mathscr{L}_k(\gamma) = \sum_{l+m=k}\mathscr{L}_l(E)\mathscr{L}_m(-[\![F]\!]) = (-1)^k c_k(\gamma)$, where both equalities are true only in $A^{\leq k}$.

To conclude the proof, we apply this statement to each $\mathscr{N}_{y_i}\big(\alpha_i^{[n]}\big)$ separately. Then using $\sum_j i_j = n$ we see from Theorem 2.3.2 that we are integrating $c_{i_1}\big(\alpha_1^{[n]}\big)\dots c_{i_M}\big(\alpha_M^{[n]}\big)\sqrt{\mathrm{Td}_0}\big(T_n^{\mathrm{vir}}\big)$. $\square$

Only the case where $\sum_j \mathrm{rk}(\alpha_j) = 2b+1$ is interesting, because one then obtains integer invariants assuming that $c_1(\alpha_i)$ are divisible by two. Moreover, we mostly focus on $M=1$ and $\mathrm{rk}(\alpha_1)=1$ which is motivated by the work of Nekrasov [135], Nekrasov–Piazzalunga [137] and Cao–Kool–Monavari [35].

Theorem 4.5.5. *If Conjecture 4.2.10 holds, then for all $\alpha_1, \ldots, \alpha_M$ with $a_i = \mathrm{rk}(\alpha_i)$, $\sum_i a_i = 2b+1$ and point-canonical orientations, we have*

$$K(\vec{\alpha}, \vec{y}; q) = \prod_{i=1}^{M} U\left[\frac{(y_i-1)^2 u}{(y_i-u)^2}\right]^{\frac{1}{2}c_1(\alpha_i)\cdot c_3(X)}, \quad \text{where} \quad q = \frac{(u-1)u^b}{\prod_{j=1}^{M}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}u)^{a_i}}\,.$$

When $M=1$, $\alpha_1 = \alpha$, $y_1 = y$, $a_1 = 1$, then

$$K(\alpha, y; q) = \mathrm{Exp}\left[\chi\Big(X, q\frac{(TX - T^*X)(\alpha^{\frac{1}{2}}y^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}y^{-\frac{1}{2}})}{(1-q\alpha^{\frac{1}{2}}y^{\frac{1}{2}})(1-q\alpha^{-\frac{1}{2}}y^{-\frac{1}{2}})}\Big)\right],$$

where $\mathrm{Exp}[f(y,q)] = \exp\Big[\sum_{n>0}\frac{f(y^n,q^n)}{n}\Big]$. In particular, the coefficients of $K(\vec{\alpha}, \vec{y}; q)$ lie in $\mathbb{Z}[y_1^{\pm\frac{1}{2}}, \ldots, y_M^{\pm\frac{1}{2}}]$ if $\frac{c_1(\alpha_i)}{2} \in H_2(X, \mathbb{Z})$.

Proof. Using Proposition 4.4.12 together with (4.5.3) and Theorem 2.3.2, we obtain

$$K(\vec{\alpha}, \vec{y}; q) = \prod_{i=1}^{M} U\left[\frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{y_i^{\frac{1}{2}}e^{-\frac{z_i}{2}} - y_i^{\frac{1}{2}}e^{\frac{z_i}{2}}}\right]^{c_1(\alpha_i)\cdot c_3(X)}, \quad \text{where } q = \frac{e^{\frac{z_i}{2}} - e^{-\frac{z_i}{2}}}{\prod_{i=1}^{M}(y^{\frac{1}{2}}e^{-\frac{z_i}{2}} - y^{-\frac{1}{2}}e^{\frac{z_i}{2}})},$$

setting $u = e^z$ and using

$$\sqrt{\frac{(1-uy_i^{-1})^2}{(1-y_i^{-1})^2 u}} = \frac{\big(y^{\frac{1}{2}}u^{-\frac{1}{2}} - y^{-\frac{1}{2}}u^{\frac{1}{2}}\big)}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}$$

we obtain

$$K(\vec{\alpha}, \vec{y}; q) = \prod_{i=1}^{M} U\left[\frac{(y_i - 1)^2 u}{(y_i - u)^2}\right]^{\frac{1}{2}c_1(\alpha_i)\cdot c_3(X)}, \quad \text{where} \quad q = \frac{(u-1)u^b}{\prod_{j=1}^{M}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}u)^{a_j}}.$$

When $M = 1$ and $a_1 = 1$ then this gives

$$u = \frac{1 + qy^{\frac{1}{2}}}{1 + qy^{-\frac{1}{2}}}$$

which after plugging into the above formula gives rise to

$$\begin{aligned}
K(\alpha, y; q) &= U\Big[\big(1 + qy^{\frac{1}{2}}\big)\big(1 + qy^{-\frac{1}{2}}\big)\Big]^{\frac{1}{2}c_1(\alpha)\cdot c_3(X)} \\
&= \sqrt{M(qy^{\frac{1}{2}})M(qy^{-\frac{1}{2}})}^{c_1(\alpha)\cdot c_3(X)} \\
&= \mathrm{Exp}\Big[\frac{qy^{\frac{1}{2}}}{(1 - qy^{\frac{1}{2}})^2} - \frac{qy^{-\frac{1}{2}}}{(1 - qy^{-\frac{1}{2}})^2}\Big]^{\frac{1}{2}c_1(\alpha)\cdot c_3(X)} \\
&= \mathrm{Exp}\left[\chi\Big(X, q\frac{(TX - T^*X)(\alpha^{\frac{1}{2}}y^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}y^{-\frac{1}{2}})}{(1 - q\alpha^{\frac{1}{2}}y^{\frac{1}{2}})(1 - q\alpha^{-\frac{1}{2}}y^{-\frac{1}{2}})}\Big)\right],
\end{aligned}$$

where the second equality uses $M(q) = \mathrm{Exp}[\frac{q}{(1-q)^2}]$ and the last equality uses Grothendieck–Riemann–Roch. □

The following remark is the result of the search for the correct replacement for the χ_y-genus and elliptic genus of Fantechi–Göttsche, it was motivated by Cao–Kool–Monavari [35, Remark 1.19] to answer what the correct generalization of the above invariants should be. The authors of loc cit. tried the χ_y-genus, we explain why this is not the right choice.

Remark 4.5.6. On a real manifold M, a natural generalization of the $\hat{A}$ genus is the universal elliptic genus which can be computed as

$$W(M,V,q)=\int_{[M]}\hat{A}(X)\prod_{k\geq 1}\mathrm{ch}\big(\mathrm{Sym}^{\bullet}_{q^k}(TM\otimes\mathbb{C})\big)(1-q^k)^{2\mathrm{dim}(M)}\,.$$

The χ_y-genus is however defined only for complex manifold, as it needs the additional complex structure. This motivates:

Definition 4.5.7. We define the DT_4 *Witten-genus* of $M_\alpha^{\mathrm{st}}(\tau)$ by

$$W(M_\alpha^{\mathrm{st}}(\tau),V,q)=\chi(\hat{\mathcal{O}}^{\mathrm{vir}}\otimes\bigotimes_{k\geq 1}\mathrm{Sym}^{\bullet}_{q^k}(\mathbb{E}-\mathrm{rk}(\mathbb{E}))\otimes V)\,.$$

Example 4.5.8. Let M be a moduli scheme with a perfect obstruction theory $\mathbb{F}^\bullet\xrightarrow{\mathrm{At}}\mathbb{L}_M$ as in Behrend–Fantechi [13], then [141, 37, 45] consider the -2-*shifted cotangent bundle* 3-term obstruction theory $\mathbb{E}^\bullet=\mathbb{F}^\bullet\oplus(\mathbb{F}^\bullet)^\vee[2]\xrightarrow{(\mathrm{At},0)}\mathbb{L}_M$. In this situation, Oh–Thomas [141, §8] show

$$\hat{\mathcal{O}}^{\mathrm{vir}}=\mathcal{O}^{\mathrm{vir}}\sqrt{K^{\mathrm{vir}}}\,,$$

where $\mathcal{O}^{\mathrm{vir}}$ is the virtual structure sheaf of Fantechi–Göttsche [51], $K^{\mathrm{vir}}=\det(\mathbb{F}^\bullet)$ and the square root is taken in $G^0(M,\mathbb{Z}[2^{-1}])$, where it always exists (see Oh–Thomas [141, Lemma 2.1]). The term on the right hand side is in fact the *twisted virtual structure sheaf* $\hat{\mathcal{O}}^{\mathrm{vir}}_{NO}$ of Nekrasov–Okounkov [136]. If $\mathrm{rk}(\mathbb{F}^\bullet)=0$, i.e. virtual dimension of M is 0, then

$$W(M,V,q)=\chi\Big(\hat{\mathcal{O}}^{\mathrm{vir}}_{NO}\otimes\bigotimes_{k\geq 1}\mathrm{Sym}^{\bullet}_{q^k}\big(\mathbb{F}^\bullet\oplus(\mathbb{F}^\bullet)^\vee\big)\otimes V\Big)\,,$$

which is the *virtual chiral elliptic genus* of Fasola–Monavari–Ricolfi [52] motivated by the work of physicists Benini–Bonelli–Poggi–Tanzini [14]. As the assumption on rank is a bit silly, one should really work equivariantly, and we plan to return to this question as we expect to relate the recent work of Kool–Rennemo [107] with the work of Fasola–Monavari–Ricolfi [52] by dimensional reduction as in [36], [107], where it is considered only the $\hat{A}$-genus.

For Hilbert schemes, the correct object to study which generalizes Nekrasov's genus is the *Nekrasov–Witten genus*

$$W(\mathrm{Hilb}^n(X), \mathcal{N}_y(L^{[n]}), q)$$

. Using Proposition 4.4.12 the corresponding generating series can be expressed as

$$1 + \sum_{n>0} z^n W(\mathrm{Hilb}^n(X), \mathcal{N}_y(L^{[n]}), q) = U\Big[\frac{(y_i - 1)^2 u}{(y_i - u)^2}\Big],$$

where

$$z = \frac{u-1}{y^{\frac{1}{2}} - y^{-\frac{1}{2}} u} \prod_{k>0} \frac{(1-q^k u)(1-q^k u^{-1})}{(1-q^k)^2}.$$

4.5.3 Untwisted K-theoretic invariants

We propose a version of *DT_4 Verlinde numbers* for Calabi–Yau fourfolds as higher dimensional analogues of Verlinde numbers for surfaces studied in [50, 125, 70]. After computing generating series for these invariants, we obtain a simple Segre–Verlinde correspondence. In the spirit of Calabi–Yau fourfolds, they require an additional twist by a square-root of a tautological determinant line bundle.

Definition 4.5.9. Let $E = \det\big(\mathcal{O}_X^{[n]}\big)$, then the *untwisted virtual structure sheaf* is defined by

$$\mathcal{O}^{\mathrm{vir}} = \hat{\mathcal{O}}^{\mathrm{vir}} \otimes E^{\frac{1}{2}}.$$

We define the *untwisted virtual characteristic*

$$\chi^{\mathrm{vir}}\Big(\mathrm{Hilb}^n(X), A\Big) = \hat{\chi}^{\mathrm{vir}}\Big(\mathrm{Hilb}^n(X), E^{\frac{1}{2}} \otimes A\Big) = \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} \sqrt{\mathrm{Td}}\big(T_n^{\mathrm{vir}}\big)\mathrm{ch}\big(E^{\frac{1}{2}}\big)\mathrm{ch}(A)\,.$$

Clearly, this changes $A_{[\![\mathcal{O}_X]\!]}(z) = z/(e^{\frac{z}{2}} - e^{-\frac{z}{2}})$ from Lemma 4.4.7 to $A_{[\![\mathcal{O}_X]\!]}(z) = z/(1-e^{-z})$.

Definition 4.5.10. Let X be a Calabi–Yau fourfold, then its *square root DT_4 Verlinde series* are defined for all $\alpha \in G^0(X)$ by

$$V^{\frac{1}{2}}(\alpha;q) = 1 + \sum_{n>0} V_n^{\frac{1}{2}}(\alpha)q^n = 1 + \sum_{n>0} \hat{\chi}^{\mathrm{vir}}\big(\mathrm{Hilb}^n(X), \det{}^{\frac{1}{2}}(L_\alpha^{[n]}) \otimes E^a\big)q^n\,,$$

where $L_\alpha = \det(\alpha)$, $a = \mathrm{rk}(\alpha)$. The *DT_4 Verlinde series* is defined by

$$V(\alpha;q) = 1 + \sum_{n>0} V_n(\alpha)q^n = 1 + \sum_{n>0} \chi^{\mathrm{vir}}\big(\mathrm{Hilb}^n(X), \det(\alpha^{[n]})\big)q^n\,.$$

Remark 4.5.11. Just for the purpose of this remark, let us define *negative square root Verlinde series* by

$$V^{-\frac{1}{2}}(\alpha;q) = 1 + \sum_{n>0} V_n^{-\frac{1}{2}}(\alpha)q^n = 1 + \sum_{n>0} \hat{\chi}^{\mathrm{vir}}\big(\mathrm{Hilb}^n(X), \det{}^{-\frac{1}{2}}(L_\alpha^{[n]}) \otimes E^{-a}\big)q^n\,,$$

for each $\alpha \in G^0(X)$, where $a = \mathrm{rk}(\alpha)$.

1. When $\alpha = [\![V]\!]$ is a vector bundle of rank a, one can show that

$$[y^{\mp\frac{n}{2}(2a+1)}]\Big(K_n(L_\alpha \oplus E^{\oplus 2a}, y)\Big) = V_n^{\pm\frac{1}{2}}(V)\,.$$

2. From the expression $K(L, y; q) = \sqrt{M(qy^{\frac{1}{2}})M(qy^{-\frac{1}{2}})}^{c_1(\alpha)\cdot c_3(X)}$, we obtain that Nekrasov generating series decouples into the positive and negative square-root Verlinde series:

$$K(L, y; q) = V^{\frac{1}{2}}(\mu(L)y^{-1}; q)V^{-\frac{1}{2}}(\mu(L)y^{-1}; q)\,,$$

where $\mu(L) = L - \mathcal{O}_X$, as it can be written as a product of series only with positive or negative powers of $y^{\frac{1}{2}}$. Thus

$$V^{\pm\frac{1}{2}}(\mu(L); q) = M(q)^{\frac{1}{2}c_1(L)\cdot c_3(X)}\,.$$

3. By applying Proposition 4.4.12, one can show that $V(\alpha; q) = \big(V^{\frac{1}{2}}(\alpha; q)\big)^2$.

Theorem 4.5.12. *Assuming Conjecture 4.2.10 holds, we have the following Segre–Verlinde correspondence for any choice of orientations on* $\mathrm{Hilb}^n(X)$*:*

$$V(\alpha; q) = R(\alpha; -q)\,.$$

Proof. From Proposition 4.4.12 together with (4.5.3) and Definition 4.5.9, we see after setting $a = \mathrm{rk}(\alpha)$ that

$$V(\alpha; q) = U(e^z)^{c_1(\alpha)\cdot c_3(X)}\,, \quad \text{where} \quad q = \frac{(1 - e^{-z})}{e^{az}}\,.$$

Changing variables to $t = e^z - 1$ we obtain

$$V(\alpha; q) = U(1+t)^{c_1(\alpha)\cdot c_3(X)}\,, \quad \text{where} \quad q = t(t+1)^{-(a+1)}\,.$$

We therefore see from Lemma 4.5.2 that

$$V(\alpha; q) = \begin{cases} U\big[\mathscr{B}_{a+1}(q)^{-c_1(\alpha)\cdot c_3(X)}\big] & \text{for } a \geq 0 \\ \\ U\big[\mathscr{B}_{-a}(-q)^{c_1(\alpha)\cdot c_3(X)}\big] & \text{for } a < 0 \end{cases}. \tag{4.5.4}$$

Comparing with (4.5.1) concludes the proof. □

We can also study the series:

$$Z(\vec{\alpha}, \vec{k}; q) = 1 + \sum_{n>0} q^n \chi^{\mathrm{vir}}\big(\wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_M} \alpha_M^{[n]}\big)\,.$$

We show that they give rise to interesting formulae. This was motivated by investigating the rationality question as studied in [5] and their example [5, Ex. 7]

Example 4.5.13. For $\alpha \in G^0(X)$, take the series $Z(\alpha; q) = \sum_{n>0} \chi^{\mathrm{vir}}(\alpha^{[n]})$, then it can be expressed as

$$Z(\alpha; q) = \frac{\partial}{\partial y} Z(\alpha, y; q)|_{y=0}\,, \quad \text{where} \quad Z(\alpha, y; q) = 1 + \sum_{n>0} \chi^{\mathrm{vir}}(\Lambda^{\bullet}_{-y} \alpha^{[n]})\,.$$

Using Proposition 4.4.12, we have

$$Z(\alpha, y; q) = \Big[\prod_{n>0} \prod_{k=1}^{n} \frac{1 + y e^{z(-\omega_n^k q)}}{1+y}\Big]^{c_1(\alpha)\cdot c_3(X)}\,, \quad \text{where} \quad q = \frac{1 - e^{-z}}{(1 + y e^z)^a}\,.$$

After changing variables $1+u=e^z$, this gives

$$Z(\alpha,y;q)=\Big[\prod_{n>0}\prod_{k=1}^{n}\frac{1+y\big(1+u(-\omega_n^k q)\big)}{1+y}\Big]^{c_1(\alpha)\cdot c_3(X)}, \quad \text{where} \quad q=\frac{u}{(1+u)(1+y+yu)^a}$$

Acting with $\partial/\partial y$ on the last formula, using that the terms under the product are equal to 1 for $y=0$ and that the derivative $(\partial/\partial y)u$ exist we obtain from a product rule for infinite products

$$Z(\alpha;q)=c_1(\alpha)\cdot c_3(X)\sum_{n>0}\sum_{k=1}^{n}u(-\omega_n^k q) \quad \text{where} \quad u=\frac{q}{1-q}$$

We can write this as

$$Z(\alpha;q)=c_1(\alpha)\cdot c_3(X)\sum_{n>0}\frac{(-q)^n}{1-(-q)^n}=c_1(\alpha)\cdot c_3(X)S(-q),,$$

where $S(q)$ is the Lambert series as considered by Lambert [109].

4.5.4 4D-2D-1D correspondence

We obtain a one-to-one correspondence between invariants on compact CY fourfolds and elliptic surfaces.

Recall from Example 2.3.2 the virtual obstruction theory on $\mathrm{Quot}_S(\mathbb{C}^N,n)$, then when $N=1$, we have $\mathrm{Quot}_S(\mathbb{C}^1,n)=\mathrm{Hilb}^n(S)$ and the virtual fundamental classes get identified with

$$\big[\mathrm{Hilb}^n(S)\big]^{\mathrm{vir}}=\big[\mathrm{Hilb}^n(S)\big]\cap c_n(K^n_{\mathrm{Hilb}}(S)^\vee)$$

using that $\mathrm{Hilb}^n(S)$ is smooth. Here we also use $\chi^{\mathrm{vir}}(-)$ to denote the virtual Euler characteristic of Fantechi–Göttsche [51].

Theorem 4.5.14. *Let X be a Calabi–Yau fourfold, S an elliptic surface. Let $f_1, \ldots, f_M, g$ be power-series, $\alpha_{Y,1}, \ldots, \alpha_{Y,M}$ in $G^0(Y)$ for $Y = X, S$ and $\mathrm{rk}(\alpha_{Y,i}) = a_i$, then there exist universal series $A_1, \ldots, A_M$ depending on $f_i, \{g\}$ and a_i such that*

$$1 + \sum_{n>0} q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} f_1\big(\alpha_{X,1}^{[n]}\big) \cdots f_M\big(\alpha_{X,M}^{[n]}\big) g\big(T_n^{\mathrm{vir}}\big) = \prod_{i=1}^{M} U(A_i)^{c_1(\alpha_{X,i}) \cdot c_3(X)},$$

$$1 + \sum_{n>0} q^n \int_{[\mathrm{Hilb}^n(S)]^{\mathrm{vir}}} f_1\big(\alpha_{S,1}^{[n]}\big) \cdots f_M\big(\alpha_{S,M}^{[n]}\big) \{g\}\big(T_{\mathrm{Hilb}^n(S)}^{\mathrm{vir}}\big) = \prod_{i=1}^{M} A_i^{c_1(\alpha_{S,i}) \cdot c_1(S)}$$

Moreover, there are universal generating series B_i depending on f_i, a_i, such that

$$1 + \sum_{n>0} q^n \chi^{\mathrm{vir}}\Big(f_1\big(\alpha_{X,1}^{[n]}\big) \otimes \ldots \otimes f_M\big(\alpha_{X,M}^{[n]}\big)\Big) = \prod_{i=1}^{M} U(B_i)^{c_1(\alpha_{X,i}) \cdot c_3(X)},$$

$$1 + \sum_{n>0} q^n \chi^{\mathrm{vir}}\Big(f_1\big(\alpha_{S,1}^{[n]}\big) \otimes \ldots \otimes f_M\big(\alpha_{S,M}^{[n]}\big)\Big) = \prod_{i=1}^{M} B_i^{c_1(\alpha_{S,i}) \cdot c_1(S)}.$$

where we abuse the notation by thinking of $\mathscr{G}_{f_i}$ as mapping to $G^0(-) \otimes \mathbb{Q}$.

Proof. Arbesfeld–Johnson–Lim–Oprea–Pandharipande [5] prove general formulae for generating series

$$\sum_{n \in \mathbb{Z}} q^n \int_{[\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)]^{\mathrm{vir}}} f_1\big(\alpha_1^{[n]}\big) \cdots f_M\big(\alpha_M^{[n]}\big) h\big(T_{\mathrm{Hilb}^n(S)}^{\mathrm{vir}}\big).$$

When $\beta = 0$, $N = 1$ and $K_S^2 = 0$ the results of [5, §2.2 & Eq. (14)] imply

$$1 + \sum_{n>0} q^n \int_{[\mathrm{Hilb}^n(S)]^{\mathrm{vir}}} f_1\big(\alpha_1^{[n]}\big) \cdots f_M\big(\alpha_M^{[n]}\big) h\big(T^{\mathrm{vir}}_{\mathrm{Hilb}^n(S)}\big) = \prod_{i=1}^{M} \Big[\frac{f_i(0)}{f_i\big(H(q)\big)}\Big]^{c_1(\alpha_i)\cdot c_1(X)},$$

$$\text{where} \quad q = \frac{H}{\prod f_i^{a_i}(H)\, h(H)}\,.$$

Replacing h with $\{g\}$, and comparing to the result of Proposition 4.4.12, we obtain the first two formulae.

Using (4.5.3), we see that $\big[\sqrt{Td}\big]\Big(T^{\mathrm{vir}}_{\mathrm{Hilb}^n(S)}\Big) E^{\frac{1}{2}}$ contributes

$$\frac{x}{1+e^{-x}}$$

to the variable change above. This corresponds precisely to the Todd-genus $\mathrm{Td}\Big(T^{\mathrm{vir}}_{\mathrm{Hilb}^n(S)}\Big) = \frac{x}{1+e^{-x}}\Big(T^{\mathrm{vir}}_{\mathrm{Hilb}^n(S)}\Big)$. The second result for elliptic surface S then follows from the virtual Riemann–Roch of Fantechi–Göttsche [51] together with definition of $\chi^{\mathrm{vir}}(-)$ in §4.5.2. □

Remark 4.5.15. By the work of and Oprea–Pandharipande [143, Lem. 34] there is a relation between integrals over $[\mathrm{Quot}_S(\mathbb{C}^N, n)]$ and $[\mathrm{Quot}_C(\mathbb{C}^N, n)]^{\mathrm{vir}}$, where the former is a smooth moduli space of dimension nN and C is a smooth anti-canonical curve in S (if it exists). It is interesting that this gives a precise relation between the generating series of three sets of virtual invariants in 3 different dimensions. We will unify these results by applying the equivalent computations to the ones in §4.3 and §4.4 to recover the results of Arbesfeld et al [5], [118] and [143] in the author's future work [18].

Using that $[\mathrm{Hilb}^1(X)]^{\mathrm{vir}} = \mathrm{Pd}\big(c_3(x)\big)$ together with Theorem 2.3.2 and that we have natural isomorphisms $\Lambda^i(TX|_x) \cong Ext^i(\mathcal{O}_x, \mathcal{O}_x)$ which hold in a family one can show:

Corollary 4.5.16. *All of the results of this section hold* mod q^2.

4.5.5 4D-2D correspondence explained by wall-crossing

Virtual fundamental classes of Quot-schemes from example 2.3.2 have been used by Marian–Oprea–Pandharipande [124] to prove Lehn's conjecture [114] for the generating series of tautological invariants on Hilbert schemes of points. More recently their virtual fundamental classes $[\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)]^{\mathrm{vir}}$ were studied by Arbesfeld et al [5], Lim [118] and Oprea–Pandharipande [143]. Our goal here is to recover the formulae when $\beta = 0$ for an elliptic surface S to explain the relationship in Theorem 4.5.14. We only need one ingredient for this. Similarly, as in the case of Calabi–Yau 4-folds let us denote

$$I(L, q) = 1 + \sum_{n>0} q^n \int_{\left[\mathrm{Quot}_S(\mathbb{C}^1, n)\right]^{\mathrm{vir}}} c_n(L^{[n]})$$

Knowing these invariants, we will be able to determine $[\mathrm{Quot}_S(\mathbb{C}^N, n)]^{\mathrm{vir}}$ as an element in $H_{nN}(\mathcal{P}_S)$ similarly to what we obtained for four-folds. For this we will need a different definition of the the moduli stack of pairs. For simplicity we assume that $b_1(S) = 0$, but we then drop this requirement in Remark 4.5.22. In the sequel [18], we are going to obtain the entire information about virtual fundamental classes $[\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)]^{\mathrm{vir}}$ for any surface. Let us for now set up the general framework for $[\mathrm{Quot}_S(\mathbb{C}^N, n)]^{\mathrm{vir}}$ for any smooth projective surface S.

Definition 4.5.17.
- We consider this time the abelian category $\mathcal{B}_N$ of triples (E, V, ϕ), where $\phi : V \otimes \mathbb{C}^N \otimes \mathcal{O}_S \xrightarrow{\phi} F$ and F is a zero-dimensional sheaf.

- The moduli stack $\mathcal{N}_0^N$ is constructed as in Definition 4.2.3, except in the first bullet point we take the total space of $\pi_{np,d} : \pi_{2*}\big(\pi_1^*(\mathcal{O}_X)^{\oplus N} \otimes \mathcal{E}_{np}\big) \boxtimes \mathcal{V}_d^* \to \mathcal{M}_{np} \times [*/\mathrm{GL}(d,\mathbb{C})]$.

- We define $\Theta^{N,\mathrm{pa}}$ by

$$\Theta^{N,\mathrm{pa}}_{(n_1p,d_1),(n_2p,d_2)} = (\pi_{n_1p,d_1} \times \pi_{n_2p,d_2})^* \Big\{(\Theta_{n_1p,n_2p})_{1,3} \oplus \Big((\mathcal{V}_{d_1}^{\oplus N}) \boxtimes \pi_{2*}(\mathcal{E}_{n_2p})^\vee\Big)_{2,3}[1]\Big\} \qquad (4.5.5)$$

with $\Theta_{n_1p,n_2p} = \underline{\mathrm{Hom}}_{\mathcal{M}_{n_1p}\times\mathcal{M}_{n_2p}}(\mathcal{E}_{n_1p}, \mathcal{E}_{n_2p})^\vee$ and the form $\chi^{N,\mathrm{pa}}\big((n_1p,d_1),(n_2p,d_2)\big) = \mathrm{rk}\big(\Theta^{N,\mathrm{pa}}_{(n_1p,d_1),(n_2p,d_2)}\big) = -Nd_1n_2$

The rest of the data has obvious modification, which we do not mention here.

Note that working with surfaces the correct vertex algebra structure requires the symmetrization of Θ^{pa}, thus the correct data is

$$\Big((\mathcal{N}^p)^{\mathrm{top}}, \mathbb{Z}\times\mathbb{Z}, \mu^{\mathrm{top}}_{\mathcal{N}_q}, \mu^{\mathrm{top}}_{\mathcal{N}_q}, 0^{\mathrm{top}}, [\![\Theta^{N,\mathrm{pa}} + \sigma^*\big(\Theta^{N,\mathrm{pa}}\big)^\vee]\!], \epsilon^N\Big) \qquad (4.5.6)$$

where $\epsilon^N_{(n_1p,d_1),(n_2p,d_2)} = (-1)^{Nd_1n_2}$. We have again a universal family $\mathbb{C}^N \otimes \mathcal{O}_{S\times\mathrm{Quot}_S(\mathbb{C}^N,n)} \to \mathcal{F}$ giving us

$$\begin{array}{ccc} & & \mathcal{N}_0^N \\ & \overset{\iota_{n,N}}{\nearrow} & \big\downarrow \Pi^{\mathrm{pl}} \\ \mathrm{Quot}_S(\mathbb{C}^N,n) & \xrightarrow{\iota^{\mathrm{pl}}_{n,N}} & (\mathcal{N}_0^N)^{\mathrm{pl}} \end{array} .$$

and $[\mathrm{Quot}_S(\mathbb{C}^N,n)]_{\mathrm{vir}} \in H_*(\mathcal{N}_0^N)$. Notice, that there is an obvious modification of the Joyce–Song stability τ_N^{pa}, such that $\mathbb{C}^N \otimes \mathcal{O}_X \xrightarrow{\phi} F$ is τ_N-*stable* if and only if ϕ is surjective. Therefore, we again obtain

$$\mathrm{Quot}_S(\mathbb{C}^N, n) = N^{\mathrm{st}}_{(np,1)}(\tau^{\mathrm{pa}}_N)\,, \quad [\mathrm{Quot}_S(\mathbb{C}^N, n)]^{\mathrm{vir}} = [N^{\mathrm{st}}_{(np,1)}(\tau^{\mathrm{pa}}_N)]^{\mathrm{vir}}\,.$$

Once the work of Joyce [95] is complete, the following conjecture will be a consequence of a more general theorem after proving that some axioms are satisfied.

Conjecture 4.5.18. *For any smooth projective surface S, in $\check{H}_*\big(\mathcal{N}_0^N\big)$ we have for all n, N*

$$[\mathrm{Quot}_S(\mathbb{C}^N, n)]_{\mathrm{vir}} = \sum_{\substack{k>0, n_1,\dots,n_k\\ n_1+\dots+n_k=n}} \frac{(-1)^k}{k!}\big[\big[\dots\big[[\mathcal{N}_{(0,1)}]_{\mathrm{inv}}, [\mathcal{M}^{\mathrm{ss}}_{n_1p}]_{\mathrm{inv}}\big],\dots\big], [\mathcal{M}^{\mathrm{ss}}_{n_kp}]_{\mathrm{inv}}\big]$$

for some $[\mathcal{M}^{\mathrm{ss}}_{np}]_{\mathrm{inv}} \in \check{H}_2(\mathcal{N}_0^N)$.

We again construct the vertex algebra on topological pairs and the *L-twisted* vertex algebra.

Definition 4.5.19. Define the data $\big(\mathcal{P}_S, K(\mathcal{P}_S), \Phi_{\mathcal{P}_S}, \mu_{\mathcal{P}_S}, 0, \theta^L_{\mathcal{P}_S}, \tilde{\epsilon}^{L,N}\big)$, $\big(\mathcal{P}_S, K(\mathcal{P}_S), \Phi_{\mathcal{P}_S}, \mu_{\mathcal{P}_S}, 0, \theta_{\mathcal{P}_S}, \tilde{\epsilon}^{N}\big)$ as follows:

- $K(\mathcal{P}_S) = K^0(S) \times \mathbb{Z}$.
- Set $\mathfrak{L} = \pi_{2\,*}(\pi_S^*(L) \otimes \mathfrak{E}) \in K^0(\mathcal{C}_S)$. Then on $\mathcal{P}_S \times \mathcal{P}_S$ we define $\theta_{N,\mathrm{ob}} = (\theta)_{1,3} - N\Big(\mathfrak{U} \boxtimes \pi_{2\,*}(\mathfrak{E})^\vee\Big)_{2,3}$, where $\theta = \pi_{2,3\,*}\big(\pi^*_{1,2}(\mathfrak{E}) \cdot \pi^*_{1,3}(\mathfrak{E})^\vee\big)$ and

$$\begin{aligned}\theta_{\mathcal{P}_S,N} &= \theta_{N,\mathrm{ob}} + \sigma^*(\theta_{N,\mathrm{ob}})^\vee\\ \theta^L_{\mathcal{P}_S,N} &= \theta_{\mathcal{P}_S,N} + N\Big(\mathfrak{U} \boxtimes \mathfrak{L}^\vee\Big)_{2,3} + N\Big(\mathfrak{L} \boxtimes \mathfrak{U}^\vee\Big)_{1,4},\end{aligned}$$

- The symmetric forms $\tilde{\chi} : (K^0(S) \times \mathbb{Z}) \times (K^0(S) \times \mathbb{Z}) \to \mathbb{Z}$, $\tilde{\chi}^L : (K^0(S) \times \mathbb{Z}) \times (K^0(S) \times \mathbb{Z}) \to \mathbb{Z}$ are given by

$$\begin{aligned} \tilde{\chi}\big((\alpha, d), (\beta, e)\big) &= \chi(\alpha, \beta) + \chi(\beta, \alpha) - dN\chi(\beta) - eN\chi(\alpha)\,, \\ \tilde{\chi}^L\big((\alpha, d), (\beta, e)\big) &= \chi(\alpha, \beta) - dN\big(\chi(\beta) - \chi(\beta \cdot L)\big) \\ &\quad - eN\big(\chi(\alpha) - \chi(\alpha \cdot L)\big)\,. \end{aligned} \tag{4.5.7}$$

- The signs are defined by $\tilde{\epsilon}_{(\alpha,d),(\beta,e)} = (-1)^{\chi(\alpha,\beta)+Nd\chi(\beta)}$ and $\tilde{\epsilon}^L_{(\alpha,d),(\beta,e)} = (-1)^{\chi(\alpha,\beta)+Nd\left(\chi(\beta)-\chi(L\cdot\beta)\right)}$.

We denote by $(\hat{H}_*(\mathcal{P}_S), |0\rangle, e^{zT}, Y_N)$, resp. $(\tilde{H}_*(\mathcal{P}_S), |0\rangle, e^{zT}, Y_N^L)$ the vertex algebras associated to this data and $(\check{H}_*(\mathcal{P}_S), [-,-]_N)$, resp. $(\mathring{H}_*(\mathcal{P}_X), [-,-]_N^L)$ the corresponding Lie algebras. We now consider the map

$$\Omega^N = (\Gamma \times \mathrm{id}) \circ (\Sigma_N)^{\mathrm{top}} : (\mathcal{N}_0^N)^{\mathrm{top}} \to \mathcal{M}_X^{\mathrm{top}} \times BU \times \mathbb{Z} \to \mathcal{C}_X \times BU \times \mathbb{Z}\,, \tag{4.5.8}$$

where Σ_N maps $[E, V, \phi]$ to $[E, V \otimes \mathcal{O}_S]$.

Let $\mathbb{B} = B \sqcup \{(0,1)\}$, where $B = \bigsqcup_{i=1}^4 B_i$, $\mathrm{ch}(B_i)$ basis of $H^i(S)$ with $B_0 = \{[\![\mathcal{O}_S]\!]\}$, $B_4 = \{p\}$. Combining all the ideas of Chapter 4, we can state the following:

Proposition 4.5.20. *Let* $\mathbb{Q}[K^0(S) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\!]$ *be the generalized super-lattice vertex algebra associated to* $\big((K^0(S) \oplus \mathbb{Z}) \oplus K^1(S), (\tilde{\chi}^L)^\bullet\big)$, *resp.* $\big((K^0(S) \oplus \mathbb{Z}) \oplus K^1(S), (\tilde{\chi})^\bullet\big)$, *where* $(\tilde{\chi})^\bullet = \tilde{\chi} \oplus \chi^-$, $(\tilde{\chi}^L)^\bullet = \tilde{\chi}^L \oplus \chi^-$ *and*

$$\chi^- : K^1(S) \times K^1(S) \to \mathbb{Z}\,, \quad \chi^-(\alpha, \beta) = \int_S \mathrm{ch}(\alpha)^\vee \mathrm{ch}(\beta)\mathrm{Td}(S)\,.$$

The isomorphism (4.2.13) *induces an isomorphism of graded vertex algebras for all* N*:*

$$\hat{H}^*(\mathcal{P}_S) \cong \mathbb{Q}[K^0(S) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\!]\,,$$
$$\tilde{H}^*(\mathcal{P}_S) \cong \mathbb{Q}[K^0(S) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_{\mathbb{Q}}[\![u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\!]\,.$$

The map $(\Omega^N)_* : H_*(\mathcal{N}_0^N) \to H_*(\mathcal{P}_S)$ *induces morphisms of graded vertex algebras* $(\hat{H}_*(\mathcal{N}_0^N), |0\rangle, e^{zT}, Y_N) \to (\hat{H}_*(\mathcal{P}_S), |0\rangle, e^{zT}, Y_N)$*,* $(\tilde{H}_*(\mathcal{N}_0^N), |0\rangle, e^{zT}, Y_N^L) \to (\tilde{H}_*(\mathcal{P}_S), |0\rangle, e^{zT}, Y_N^L)$ *and of graded Lie algebras*

$$\bar{\Omega}_* : \left(\check{H}_*(\mathcal{N}_0^N), [-,-]_N\right) \longrightarrow \left(\check{H}_*(\mathcal{P}_S), [-,-]_N\right),$$
$$\bar{\Omega}_* : \left(\mathring{H}_*(\mathcal{N}_0^N), [-,-]_N^L\right) \longrightarrow \left(\mathring{H}_*(\mathcal{P}_S), [-,-]_N^L\right).$$

The following result replaces Theorem 4.3.10 and it is noticeably simpler due to canonical orientations. We use the notation

$$\mathscr{Q}_{N,n} = \bar{\Omega}_*^N\Big(\big[\mathrm{Quot}_S(\mathbb{C}^N, n)\big]_{\mathrm{vir}}\Big)\,, \quad \text{and} \quad \mathscr{M}_{np} = \bar{\Omega}_*^N\left([\mathcal{M}_{np}]_{\mathrm{inv}}\right).$$

Lemma 4.5.21. *Let* S *be a smooth projective surface with* $b_1(S) = 0$*. If Conjecture 4.5.18 holds, then*

$$\mathscr{M}_{np} = e^{(np,1)} \otimes 1 \cdot \mathscr{N}_{np} + \mathbb{Q}T(e^{(np,1)} \otimes 1)\,,$$

where for the series $\mathscr{N}(q) = \sum_{n>0} \mathscr{N}_{np} q^n$ *we have*

$$\exp\big(\mathscr{N}(q)\big) = \Big(1 - e^p q\Big)^{\Big(\sum_{v \in B_2} c_1(S)_v u_{v,1}\Big)}.$$

If S *is moreover elliptic, we have*

$$1 + \sum_{n>0} \frac{\mathscr{Q}_{N,n}}{e^{(np,1)}} q^n = \exp\Big[\sum_{n>0} [z^{nN-1}] \Big\{ \sum_{v \in B_2} -\frac{c_{1,v}}{n} U_v(z) \exp\Big[\sum_{k>0} \frac{n y_k}{k} z^k\Big]\Big\} q^n \Big]. \quad (4.5.9)$$

Proof. We have $[\mathrm{Quot}_S(\mathbb{C}^1, n)]^{\mathrm{vir}} \cap c_n(L^{[n]}) = [\mathrm{Hilb}^n(S)] \cap c_n\big((K^n_{\mathrm{Hilb}}(S))^\vee\big) \cap c_n\big(L^{[n]}\big) = (-1)^n [\mathrm{Hilb}^n(S)] \cap c_{2n}(K_S^{[n]} \oplus L^{[n]})$. Then by [125, eq. (18)], we see

$$I(L, q) = \Big(1 - q\Big)^{c_1(L) \cdot c_1(X)}.$$

We have again

$$[e^{(mp,1)} \otimes 1, e^{(np,0)} \otimes N_{np}]^L = -(-1)^n e^{(m+n)p,1} \otimes \sum_{v \in B_2} \int_X c_1(L) \mathrm{ch}(v) a_v(n).$$

By the same but simpler arguments as in the proof of Theorem 4.3.10, we obtain $\mathscr{N}(np) = -\frac{1}{n} \sum_{v \in B_2} c_1(S)_v u_{v,1}$. Then an analogous argument as in the proof of Theorem 4.4.1 leads to (4.5.9), where we are using $c_1^2(S) = 0$. □

Remark 4.5.22. Going through the above computation without the assumption $b_1(S) = 0$ one can check that under the projection $\Pi_{\mathrm{even}} : \check{H}_*(\mathcal{P}_S) \to \check{H}_{\mathrm{even}}(\mathcal{P}_S)$ we still obtain the same results for an elliptic surface. This is sufficient for us, because we never integrate odd cohomology classes, except when integrating polynomials in $\mathrm{ch}_k(T^{\mathrm{vir}})$, but as the only terms $\mu_{v,k}$ for $v \in B_{\mathrm{odd}}$ are given for $v \in B_3$, each such integral will contain a factor of $\chi^-(v,w)=0$ for $v,w\in B_3$.

As a consequence, we then obtain the following result which could also be extracted from Arbesfeld et al [5] for an elliptic surface.

Proposition 4.5.23. *Let S be a smooth projective elliptic surface and $f_0(\mathfrak{p},\cdot), f_1(\mathfrak{p},\cdot),\dots,f_m(\mathfrak{p},\cdot)$ be power-series with $f(0,0)=1$, then define*

$$\mathrm{Inv}_N(\vec{f},\vec{\alpha},q)=1+\sum_{n>0}\int_{\left[\mathrm{Quot}_S(\mathbb{C}^N,n)\right]^{\mathrm{vir}}} f_0\big(T^{\mathrm{vir}}\big)f_1\big(\alpha_1^{[n]}\big)\cdots f_m\big(\alpha_m^{[n]}\big)q^n\,.$$

Then setting $\mathrm{rk}(\alpha_j)=a_j$*, we have*

$$\mathrm{Inv}_N(\vec{f},\vec{a},q)=\left[\prod_{j=1}^N\prod_{i=1}^m\frac{f_i(0)}{f_i(H_j(q))}\right]^{c_1(\alpha_j)\cdot c_1(S)}\,, \tag{4.5.10}$$

where $H_j(q)$, $j=1,\dots,N$ are the different solutions for

$$q=\frac{H_j^N}{\prod_{i=1}^m f_i^{a_i}(H_j)\,f_0^N(H_j)}\,.$$

Proof. We can show again

$$\begin{aligned}&\int_{[\mathrm{Quot}_S(\mathbb{C}^N,n)]^{\mathrm{vir}}} f_0(T^{\mathrm{vir}})f_i(\alpha_i^{[n]})\dots f_m(\alpha_m^{[n]})\\ &=\int_{\mathscr{Q}_{N,n}}\exp\Big[\sum_{\substack{k>0\\ v\in B_{2,4}}}\sum_{i=1}^m a_{\alpha_i}(k)\chi(\alpha_i^\vee,v)\mu_{v,k}+Nb_k\chi(v)\mu_{v,k}\Big]\,,\end{aligned}$$

where $\sum_k\frac{a_{\alpha_i(k)}}{k!}q^k=\log\big(f_i(q)\big)$ and $\sum_{k>0}\frac{b_k}{k!}q^k=\log\big(f_0(q)\big)$. The rest then follows from Lemma 4.2.13 and 4.4.11 by a similar computation as in §4.4. ☐

Remark 4.5.24. For an elliptic curve C the quot-scheme $\mathrm{Quot}_C(\mathbb{C}^N,n)$ carries the obstruction theory $\mathbb{F} = \Big(\tau_{[0,1]}\underline{\mathrm{Hom}}_{\mathrm{Quot}_C(\mathbb{C}^N,n)}(\mathcal{I},\mathcal{F})\Big)^\vee$ constructed by Marian–Oprea [123] which is just a vector bundle of rank nN, therefore the construction of the vertex algebra is identical and the same computation applies. We leave it to the reader to check using [143, Thm. 3] that under the projection $\Pi_{\mathrm{even}} : \check{H}_*(\mathcal{P}_C) \to \check{H}_{\mathrm{even}}(\mathcal{P}_C)$ the generating series $1 + \sum_{n>0} \frac{\mathscr{Q}_{\mathcal{N},}}{e^{(np,1)}}$ is given by

$$\exp\Big[- \sum_{n>0} \frac{(-1)^n}{n} [z^{nN-1}] \Big\{ U_{[\![\mathcal{O}_C]\!]}(z) \exp\Big[\sum_{k>0} \frac{n y_k}{k} z^k \Big] \Big\} q^n \Big] .$$

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