# Mathematics and physics of consistency of the Born rule with normal probability distribution

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**Abstract.** Newtonian and Scrödinger dynamics can be formulated in a physically meaningful way within the same Hilbert space framework. The resulting unexpected relation between classical and quantum motions goes beyond the results provided by the Ehrenfest theorem. The normal probability distribution and the Born rule turn out to be related. A dynamical mechanism responsible for the latter formula is proposed and applied to measurements of macroscopic and microscopic systems.

## 1. Introduction
In recent papers [1] and [2] an important new connection between classical and quantum dynamics was derived. The starting point was a realization of classical and quantum mechanics on an equal footing within the same Hilbert space framework, and identification of observables with vector fields on the sphere of normalized states. This resulted in a physically meaningful interpretation of the components of the velocity of state. Newtonian dynamics was shown to be the Schrödinger dynamics of a system whose state is constrained to the classical phase space submanifold in the Hilbert space of states. This also resulted in a formula relating the normal probability distribution and the Born rule, and interpretation of quantum collapse in terms of diffusion of the state on the projective space of states. Simply put, the classical space and classical phase space of a system of particles can be identified with a submanifold of the space of states of the corresponding quantum system. When the system is constrained to the submanifold, it behaves classically. Otherwise, it behaves quantum-mechanically. The velocity of the state at any point of the classical space submanifold can be decomposed into the classical (velocity, acceleration) and non-classical (phase velocity, spreading) components. The curvature of the sphere of states is determined from the canonical commutation relations.

In this paper, we continue to explore the implications of the proposed geometric framework. It has been known since Einstein that the thermal motion of molecules in a liquid results in the Brownian motion of pollen grain in the liquid, with probability distribution of the position of the grain satisfying the diffusion equation. It is shown here that the state of a microscopic particle exposed to the same random potential as the one experienced by the pollen grain is equally likely to get displaced in any direction tangent to the projective space of states. The relationship between the normal probability distribution and the Born rule that was established earlier signifies, then, that the probability density for the state to reach a particular point in the space of states is given by the Born rule. A diffusion equation for the motion of the state in these conditions is then obtained.

In this model, the role of the measuring device in measurements on macroscopic and microscopic particles is similar. In both cases, the device is designed to record values of the measured quantity and is responsible for a noise and the resulting distribution of these values. The effect of the noise can be modeled by a diffusion equation. In the case of a measurement on a macroscopic particle, the diffusion equation is the familiar equation on the classical space $\mathbb{R}^3$. In the case of a measurement on a microscopic particle, the equation describes the random motion of the state of the particle and represents an extension of the diffusion equation from the classical space onto the space of states. Under the diffusion, the state of a microscopic particle can reach the classical space submanifold in the space of states and trigger a detector. The probability of reaching a particular point of the submanifold is given by the Born rule.

To make the paper self-contained, we begin with a review of the results reported in [1] and [2].

## 2. The classical mechanics in the language of quantum states

Experience shows that macroscopic bodies have a well-defined position in space at any time. In quantum mechanics the state of a spinless particle with a known position $\mathbf{a} \in \mathbb{R}^3$ is described by the Dirac delta function $\delta_{\mathbf{a}}^3(\mathbf{x}) = \delta^3(\mathbf{x} - \mathbf{a})$. The map $\omega : \mathbf{a} \longrightarrow \delta_{\mathbf{a}}^3$ provides a one-to-one correspondence between points $\mathbf{a} \in \mathbb{R}^3$ and state "functions" $\delta_{\mathbf{a}}^3$. This allows us to describe points in $\mathbb{R}^3$ in functional terms and identify the set $\mathbb{R}^3$ with the set $M_3$ of all delta functions in the space of state functions of the particle.

The common Hilbert space $L_2(\mathbb{R}^3)$ of state functions of a particle does not contain delta functions. By writing the inner product of functions $\varphi, \psi \in L_2(\mathbb{R}^3)$ as

$$(\varphi, \psi)_{L_2} = \int \delta^3(\mathbf{x} - \mathbf{y}) \varphi(\mathbf{x}) \overline{\psi}(\mathbf{y}) d^3\mathbf{x} d^3\mathbf{y} \tag{1}$$

and approximating the kernel $\delta^3(\mathbf{x} - \mathbf{y})$ with a Gaussian function, one obtains a new inner product in $L_2(\mathbb{R}^3)$

$$(\varphi, \psi)_{\mathbf{H}} = \int e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}} \varphi(\mathbf{x}) \overline{\psi}(\mathbf{y}) d^3\mathbf{x} d^3\mathbf{y}. \tag{2}$$

The Hilbert space $\mathbf{H}$ obtained by completing $L_2(\mathbb{R}^3)$ with respect to this inner product contains delta functions and their derivatives. In particular,

$$\int e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}} \delta^3(\mathbf{x} - \mathbf{a}) \delta^3(\mathbf{y} - \mathbf{a}) d^3\mathbf{x} d^3\mathbf{y} = 1. \tag{3}$$

It follows that the set $M_3$ of all delta functions $\delta_{\mathbf{a}}^3(\mathbf{x})$ with $\mathbf{a} \in \mathbb{R}^3$ form a submanifold of the unit sphere in the Hilbert space $\mathbf{H}$, diffeomorphic to $\mathbb{R}^3$.

The kernel $\delta^3(\mathbf{x} - \mathbf{y})$ of the metric on $L_2(\mathbb{R}^3)$ is analogous to the Kronecker delta $\delta_{ik}$, representing Euclidean metric in orthogonal coordinates. The "skewed" kernel $e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}$ of the metric on $\mathbf{H}$ is then analogous to the Euclidean metric represented in linear coordinates with skewed axes by a constant non-diagonal matrix $g_{ik}$.

The map $\rho_\sigma : \mathbf{H} \longrightarrow L_2(\mathbb{R}^3)$ that relates $L_2$ and $\mathbf{H}$-representations is given by the Gaussian kernel

$$\rho_\sigma(\mathbf{x}, \mathbf{y}) = \left( \frac{1}{2\pi\sigma^2} \right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\sigma^2}}. \tag{4}$$

In fact, multiplying the operators (integrating the product of the corresponding kernels) one can see that

$$k(\mathbf{x}, \mathbf{y}) = (\rho_\sigma^* \rho_\sigma)(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}, \tag{5}$$

which is consistent with (2). The map $\rho_\sigma$ transforms delta functions $\delta_{\mathbf{a}}^3$ to Gaussian functions $\widetilde{\delta}_{\mathbf{a}}^3 = \rho_\sigma(\delta_{\mathbf{a}}^3)$, centered at $\mathbf{a}$. The image $M_3^\sigma$ of $M_3$ under $\rho_\sigma$ is an embedded submanifold of the unit sphere in $L_2(\mathbb{R}^3)$ made of the functions $\widetilde{\delta}_{\mathbf{a}}^3$. The map $\omega_\sigma = \rho_\sigma \circ \omega : \mathbb{R}^3 \longrightarrow M_3^\sigma$ is a diffeomorphism. Here $\omega$ is the same as before. In what follows, the obtained realizations will be used interchangeably.

Let $\mathbf{r} = \mathbf{a}(t)$ be a path with values in $\mathbb{R}^3$ and let $\varphi = \delta_{\mathbf{a}(t)}^3$ be the corresponding path in $M_3$. It is easy to see that the norm $\left\|\frac{d\varphi}{dt}\right\|_H^2$ of the velocity in the space $\mathbf{H}$ is given by

$$\left\|\frac{d\varphi}{dt}\right\|_H^2 = \left.\frac{\partial^2 k(\mathbf{x}, \mathbf{y})}{\partial x^i \partial y^k}\right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}} \frac{d\mathbf{a}^i}{dt} \frac{d\mathbf{a}^k}{dt}. \tag{6}$$

Here $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8\sigma^2}}$ as in (5), so that

$$\left.\frac{\partial^2 k(\mathbf{x}, \mathbf{y})}{\partial x^i \partial y^k}\right|_{\mathbf{x}=\mathbf{y}=\mathbf{a}} = \frac{1}{4\sigma^2}\delta_{ik}, \tag{7}$$

where $\delta_{ik}$ is the Kronecker delta symbol. Assuming now that the distance in $\mathbb{R}^3$ is measured in the units of $2\sigma$, we obtain

$$\left\|\frac{d\varphi}{dt}\right\|_H = \left\|\frac{d\mathbf{a}}{dt}\right\|_{\mathbb{R}^3}. \tag{8}$$

It follows that the map $\omega : \mathbb{R}^3 \longrightarrow \mathbf{H}$ is an isometric embedding. Furthermore, the set $M_3$ is complete in $\mathbf{H}$ so that there is no vector in $\mathbf{H}$ orthogonal to all of $M_3$. By defining the operations of addition $\oplus$ and multiplication by a scalar $\lambda \odot$ via $\omega(\mathbf{a}) \oplus \omega(\mathbf{b}) = \omega(\mathbf{a}+\mathbf{b})$ and $\lambda \odot \omega(\mathbf{a}) = \omega(\lambda \mathbf{a})$ with $\omega$ as before, we obtain $M_3$ as a vector space isomorphic to the Euclidean space $\mathbb{R}^3$.

The projection of velocity and acceleration of the state $\delta_{\mathbf{a}(t)}^3$ onto the Euclidean space $M_3$ yields correct Newtonian velocity and acceleration of the classical particle:

$$\left(\frac{d}{dt}\delta_{\mathbf{a}}^3(\mathbf{x}), -\frac{\partial}{\partial x^i}\delta_{\mathbf{a}}^3(\mathbf{x})\right)_{\mathbf{H}} = \frac{da^i}{dt} \tag{9}$$

and

$$\left(\frac{d^2}{dt^2}\delta_{\mathbf{a}}^3(\mathbf{x}), -\frac{\partial}{\partial x^i}\delta_{\mathbf{a}}^3(\mathbf{x})\right)_{\mathbf{H}} = \frac{d^2 a^i}{dt^2}. \tag{10}$$

The Newtonian dynamics of the classical particle can be derived from the principle of least action for the action functional $S$ on paths in $\mathbf{H}$, defined by

$$\int k(\mathbf{x}, \mathbf{y})\left[\frac{m}{2}\frac{d\varphi_t(\mathbf{x})}{dt}\frac{d\overline{\varphi}_t(\mathbf{y})}{dt} - V(\mathbf{x})\varphi_t(\mathbf{x})\overline{\varphi}_t(\mathbf{y})\right]d^3\mathbf{x}\,d^3\mathbf{y}\,dt, \tag{11}$$

where $m$ is the mass of the particle, $V$ is the potential and $k(\mathbf{x}, \mathbf{y}) = e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}$ (same as in (5) with $2\sigma = 1$; see (8)). Namely, under the constraint $\varphi_t(\mathbf{x}) = \delta^3(\mathbf{x} - \mathbf{a}(t))$ the action (11) becomes

$$S = \int \left[\frac{m}{2}\left(\frac{d\mathbf{a}}{dt}\right)^2 - V(\mathbf{a})\right]dt, \tag{12}$$

which is the classical action functional for the particle.

This shows that a classical particle can be considered a constrained dynamical system with the state $\varphi$ of the particle and the velocity $\frac{d\varphi}{dt}$ of the state as dynamical variables. A similar

realization exists for mechanical systems consisting of any number of classical particles. For example, the map $\omega \otimes \omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbf{H} \otimes \mathbf{H}$, $\omega \otimes \omega(\mathbf{a}, \mathbf{b}) = \delta_{\mathbf{a}}^3 \otimes \delta_{\mathbf{b}}^3$ identifies the configuration space $\mathbb{R}^3 \times \mathbb{R}^3$ of a two particle system with the embedded submanifold $M_6 = \omega \otimes \omega(\mathbb{R}^3 \times \mathbb{R}^3)$ of the Hilbert space $\mathbf{H} \otimes \mathbf{H}$. Consider a path $(\mathbf{a}(t), \mathbf{b}(t))$ in $\mathbb{R}^3 \times \mathbb{R}^3$ and the corresponding path $\delta_{\mathbf{a}(t)}^3 \otimes \delta_{\mathbf{b}(t)}^3$ with values in $M_6$. For any $t$, the vectors $\frac{d}{dt}\delta_{\mathbf{a}(t)}^3 \otimes \delta_{\mathbf{b}(t)}^3$ and $\delta_{\mathbf{a}(t)}^3 \otimes \frac{d}{dt}\delta_{\mathbf{b}(t)}^3$ are tangent to $M_6$ at the point $\delta_{\mathbf{a}(t)}^3 \otimes \delta_{\mathbf{b}(t)}^3$ and orthogonal in $\mathbf{H} \otimes \mathbf{H}$. The space $M_6$ with the induced metric is isometric to the direct product $\mathbb{R}^3 \times \mathbb{R}^3$ with the natural Euclidean metric. Projection of velocity and acceleration of the state $\varphi(t) = \delta_{\mathbf{a}(t)}^3 \otimes \delta_{\mathbf{b}(t)}^3$ onto the basis vectors $\left(-\frac{\partial}{\partial x^i}\delta_{\mathbf{a}(t)}^3\right) \otimes \delta_{\mathbf{b}(t)}^3$ and $\delta_{\mathbf{a}(t)}^3 \otimes \left(-\frac{\partial}{\partial x^k}\delta_{\mathbf{b}(t)}^3\right)$ yields the velocity and acceleration of the particles by means of the formulas similar to (9) and (10).

## 3. Observables as vector fields

Quantum observables can be identified with vector fields on the space of states. Given a self-adjoint operator $\widehat{A}$ on a Hilbert space $L_2$ of square-integrable functions (it could in particular be the tensor product space of a many body problem) one can introduce the associated linear vector field $A_\varphi$ on $L_2$ by

$$A_\varphi = -i\widehat{A}\varphi. \tag{13}$$

If $D$ is the domain of the operator $\widehat{A}$, then $A_\varphi$ maps $D$ into the vector space $L_2$. Because $\widehat{A}$ is self-adjoint, the field $A_\varphi$, being restricted to the sphere $S^{L_2}$ of unit normalized states, is tangent to the sphere. The commutator of observables and the commutator (Lie bracket) of the corresponding vector fields are related in a simple way:

$$[A_\varphi, B_\varphi] = [\widehat{A}, \widehat{B}]\varphi. \tag{14}$$

Furthermore, a Hilbert metric on the space of states yields a Riemannian metric on the sphere. For this consider the realization $L_{2\mathbb{R}}$ of the Hilbert space $L_2$, i.e., the real vector space of pairs $X = (\mathrm{Re}\psi, \mathrm{Im}\psi)$ with $\psi$ in $L_2$. If $\xi, \eta$ are vector fields on $S^{L_2}$, define a Riemannian metric $G_\varphi : T_{\mathbb{R}\varphi}S^{L_2} \times T_{\mathbb{R}\varphi}S^{L_2} \longrightarrow \mathbb{R}$ on the sphere by

$$G_\varphi(X, Y) = \mathrm{Re}(\xi, \eta). \tag{15}$$

Here $X = (\mathrm{Re}\xi, \mathrm{Im}\xi)$, $Y = (\mathrm{Re}\eta, \mathrm{Im}\eta)$ and $(\xi, \eta)$ denotes the $L_2$-inner product of $\xi, \eta$.

The Riemannian metric on $S^{L_2}$ yields a Riemannian (Fubini-Study) metric on the projective space $CP^{L_2}$, which is the base of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$. For this an arbitrary tangent vector $X \in T_{R\varphi}S^{L_2}$ is decomposed into two components: tangent and orthogonal to the fibre $\{\varphi\}$ through $\varphi$ (i.e., to the plane $C^1$ containing the circle $S^1 = \{\varphi\}$). The differential $d\pi$ maps the tangential component to zero-vector. The orthogonal component of $X$ can be then identified with $d\pi(X)$. If two vectors $X, Y$ are orthogonal to the fibre $\{\varphi\}$, the inner product of $d\pi(X)$ and $d\pi(Y)$ in the Fubini-Study metric is equal to the inner product of $X$ and $Y$ in the metric $G_\varphi$:

$$(d\pi(X), d\pi(Y))_{FS} = G_\varphi(X, Y). \tag{16}$$

The resulting metrics can be used to find physically meaningful components of vector fields $A_\varphi$ associated with observables. Since $A_\varphi$ is tangent to $S^{L_2}$, it can be decomposed into components tangent and orthogonal to the fibre $\{\varphi\}$ of the fibre bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$. These components have a simple physical meaning, justifying the use of the projective space $CP^{L_2}$. From

$$\overline{A} \equiv (\varphi, \widehat{A}\varphi) = (-i\varphi, -i\widehat{A}\varphi), \tag{17}$$

one can see that the expected value of an observable $\widehat{A}$ in state $\varphi$ is the projection of the vector $-i\widehat{A}\varphi \in T_\varphi S^{L_2}$ onto the fibre $\{\varphi\}$. Because

$$(\varphi, \widehat{A}^2\varphi) = (\widehat{A}\varphi, \widehat{A}\varphi) = (-i\widehat{A}\varphi, -i\widehat{A}\varphi), \tag{18}$$

the term $(\varphi, \widehat{A}^2\varphi)$ is the norm of the vector $-i\widehat{A}\varphi$ squared. The vector $-i\widehat{A}_\perp\varphi = -i\widehat{A}\varphi - (-i\overline{A}\varphi)$ associated with the operator $\widehat{A} - \overline{A}I$ is orthogonal to the fibre $\{\varphi\}$. Accordingly, the variance

$$\Delta A^2 = (\varphi, (\widehat{A} - \overline{A}I)^2\varphi) = (\varphi, \widehat{A}_\perp^2\varphi) = (-i\widehat{A}_\perp\varphi, -i\widehat{A}_\perp\varphi) \tag{19}$$

is the norm squared of the component $-i\widehat{A}_\perp\varphi$. Recall that the image of this vector under $d\pi$ can be identified with the vector itself. It follows that the norm of $-i\widehat{A}_\perp\varphi$ in the Fubini-Study metric coincides with its norm in the Riemannian metric on $S^{L_2}$ and in the original $L_2$-metric.

The Schrödinger equation

$$\frac{d\varphi}{dt} = -i\widehat{h}\varphi \tag{20}$$

is an equation for the integral curves of the vector field $-i\widehat{h}\varphi$ on the sphere $S^{L_2}$. Let's decompose $-i\widehat{h}\varphi$ onto the components parallel and orthogonal to the fibre. The parallel component of $\frac{d\varphi}{dt}$ is numerically

$$\mathrm{Re}(-i\varphi, -i\widehat{h}\varphi) = \overline{E}, \tag{21}$$

i.e., the expected value of the energy. The decomposition of the velocity vector $\frac{d\varphi}{dt}$ into the parallel and orthogonal components is then given by

$$\frac{d\varphi}{dt} = -i\overline{E}\varphi + -i(\widehat{h} - \overline{E})\varphi = -i\overline{E}\varphi - i\widehat{h}_\perp\varphi. \tag{22}$$

The orthogonal component of the velocity $\frac{d\varphi}{dt}$ is equal to $-i\widehat{h}_\perp\varphi$. From this and equation (19) we conclude that: *The velocity of evolution of state in the projective space is equal to the uncertainty of energy.* Equation (22) also demonstrates that the physical state is driven by the operator $\widehat{h}_\perp$, associated with the uncertainty in energy rather than the energy itself.

The uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2}\left|\left(\varphi, [\widehat{A}, \widehat{B}]\varphi\right)\right| \tag{23}$$

follows geometrically from the comparison of areas of rectangle $A_{|XY|}$ and parallelogram $A_{XY}$ formed by vectors $X = -i\widehat{A}_\perp\varphi$ and $Y = -i\widehat{B}_\perp\varphi$:

$$A_{|XY|} \geq A_{XY}. \tag{24}$$

There is also an uncertainty identity, [3]:

$$\Delta A^2 \Delta B^2 = A_{XY}^2 + G_\varphi^2(X, Y). \tag{25}$$

## 4. Components of the velocity of state under the Schrödinger evolution

From (22) we know that for any state $\varphi \in S^{L_2}$ the velocity of state $\frac{d\varphi}{dt}$ in the Schrödinger equation can be decomposed onto the components parallel and orthogonal to the fibre $\{\varphi\}$ of the bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$:

$$\frac{d\varphi}{dt} = -i\overline{E}\varphi - i\widehat{h}_\perp\varphi. \tag{26}$$

The norm of the parallel component $-i\overline{E}\varphi$ is the expected value of energy $\overline{E}$. It represents the phase velocity of state. The norm of the orthogonal component $-i\widehat{h}_\perp\varphi$ is equal to the

uncertainty of energy $\Delta E$ on the state $\varphi$. It represents the velocity of motion of the fibre $\{\varphi\}$. In particular, from (26) it follows that under the Schrödinger evolution the speed of evolution of state in the projective space is equal to the uncertainty in energy.

The orthogonal component $-i\widehat{h}_\perp \varphi$ of the velocity can be further decomposed into physically meaningful components. Suppose that at $t = 0$, a microscopic particle is prepared in the state $\varphi_0 \equiv \varphi_{\mathbf{a},\mathbf{p}}$ given by

$$\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}}, \tag{27}$$

where $\sigma$ is the same as in (4) and $\mathbf{p} = m\mathbf{v}_0$ with $\mathbf{v}_0$ being the initial group-velocity of the packet. The set $M_{3,3}^\sigma$ of all initial states $\varphi_{\mathbf{a},\mathbf{p}}$ given by (27) form a 6-dimensional embedded submanifold in $L_2(\mathbb{R}^3)$. Consider the set of all fibres of the bundle $\pi : S^{L_2} \longrightarrow CP^{L_2}$ through the points of $M_{3,3}^\sigma$. The resulting bundle $\pi : M_{3,3}^\sigma \times S^1 \longrightarrow M_{3,3}^\sigma$ identifies $M_{3,3}^\sigma$ with a submanifold of $CP^{L_2}$, denoted by the same symbol. The map $\Omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow M_{3,3}^\sigma$,

$$\Omega(\mathbf{a},\mathbf{p}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{a})^2}{4\sigma^2}} e^{i\frac{\mathbf{p}(\mathbf{x}-\mathbf{a})}{\hbar}} \tag{28}$$

is a diffeomorphism from the classical phase space of the particle onto $M_{3,3}^\sigma$. For $\Omega = re^{i\theta}$, where $r$ is the modulus and $\theta$ is the argument of $\Omega$, the vectors $\frac{\partial r}{\partial a^\alpha}e^{i\theta}$ and $i\frac{\partial\theta}{\partial p^\beta}re^{i\theta}$ are orthogonal to the fibre $\{\varphi_0\}$ at the point $\varphi_0$ in $L_2(\mathbb{R}^3)$. These vectors can be then identified with vectors tangent to $M_{3,3}^\sigma$, considered as a submanifold of $CP^{L_2}$. They form a basis in the tangent space $T_{\{\varphi_0\}}M_{3,3}^\sigma$. Furthermore, the induced Riemannian metric is the usual Fubini-Study metric on $CP^{L_2}$, restricted to $M_{3,3}^\sigma$.

For any path $\{\varphi\} = \{\varphi_\tau\}$ with values in $M_{3,3}^\sigma \subset CP^{L_2}$ the norm of velocity vector $\frac{d\{\varphi\}}{d\tau}$ in the Fubini-Study metric is given by

$$\left\|\frac{d\{\varphi\}}{d\tau}\right\|_{FS}^2 = \frac{1}{4\sigma^2}\left\|\frac{d\mathbf{a}}{d\tau}\right\|_{\mathbb{R}^3}^2 + \frac{\sigma^2}{\hbar^2}\left\|\frac{d\mathbf{p}}{d\tau}\right\|_{\mathbb{R}^3}^2. \tag{29}$$

It follows that under a proper choice of units, the map $\Omega$ is an isometry that identifies the Euclidean phase space $\mathbb{R}^3 \times \mathbb{R}^3$ of the particle with the submanifold $M_{3,3}^\sigma \subset CP^{L_2}$ furnished with the induced metric. The map $\Omega$ is an extension to the phase space of the isometric embedding $\omega_\sigma = \rho_\sigma \circ \omega$ of the space $\mathbb{R}^3$, introduced in the section entitled "The classical mechanics in the language of quantum states".

The obtained embedding of the classical phase space into the space of quantum states is physically meaningful. To see this let us first project the orthogonal component $-\frac{i}{\hbar}\widehat{h}_\perp \varphi$ of the velocity $\frac{d\varphi}{dt}$ onto vectors tangent to the curves of constant values of $\mathbf{p}$ and $\mathbf{a}$ (classical space and momentum space components) in the projective manifold $M_{3,3}^\sigma$. Calculation of the projection of the velocity $\frac{d\varphi}{dt}$ onto the unit vector $-\widehat{\frac{\partial r}{\partial a^\alpha}}e^{i\theta}$ (i.e., the classical space component of $\frac{d\varphi}{dt}$) for an arbitrary Hamiltonian of the form $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$ yields

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, -\widehat{\frac{\partial r}{\partial a^\alpha}}e^{i\theta}\right)\bigg|_{t=0} = \left(\frac{dr}{dt}, -\widehat{\frac{\partial r}{\partial a^\alpha}}\right)\bigg|_{t=0} = \frac{v_0^\alpha}{2\sigma}. \tag{30}$$

Calculation of the projection of velocity $\frac{d\varphi}{dt}$ onto the unit vector $i\widehat{\frac{\partial\theta}{\partial p^\alpha}}\varphi$ (momentum space component) gives

$$\mathrm{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{\partial\theta}{\partial p^\alpha}}\varphi\right)\bigg|_{t=0} = \frac{mw^\alpha\sigma}{\hbar}, \tag{31}$$

where

$$mw^\alpha = -\left.\frac{\partial V(\mathbf{x})}{\partial x^\alpha}\right|_{\mathbf{x}=\mathbf{x}_0} \tag{32}$$

and $\sigma$ is assumed to be small enough for the linear approximation of $V(\mathbf{x})$ to be valid within intervals of length $\sigma$.

The velocity $\frac{d\varphi}{dt}$ also contains component due to the change in $\sigma$ (spreading), which is orthogonal to the fibre $\{\varphi\}$ and the phase space $M_{3,3}^\sigma$, and is equal to

$$\left.\text{Re}\left(\frac{d\varphi}{dt}, i\widehat{\frac{d\varphi}{d\sigma}}\right)\right|_{t=0} = \frac{\sqrt{2}\hbar}{8\sigma^2 m}. \tag{33}$$

Calculation of the norm of $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ at $t = 0$ gives

$$\left\|\frac{d\varphi}{dt}\right\|^2 = \frac{\overline{E}^2}{\hbar^2} + \frac{\mathbf{v}_0^2}{4\sigma^2} + \frac{m^2\mathbf{w}^2\sigma^2}{\hbar^2} + \frac{\hbar^2}{32\sigma^4 m^2}, \tag{34}$$

which is the sum of squares of the found components. This completes a decomposition of the velocity of state at any point $\varphi_{\mathbf{a},\mathbf{p}} \in M_{3,3}^\sigma$.

For a closed system the norm of $\frac{d\varphi}{dt} = \frac{i}{\hbar}\widehat{h}\varphi$ is preserved in time. For a system in a stationary state, this amounts to conservation of energy. In fact, in this case $\varphi_t(\mathbf{x}) = \psi(\mathbf{x})e^{-\frac{iEt}{\hbar}}$, which is a motion along the phase circle, and

$$\left\|\frac{d\varphi}{dt}\right\|^2 = \frac{E^2}{\hbar^2}. \tag{35}$$

For a closed system in any initial state the norm of the phase component (expected energy) and orthogonal component (energy uncertainty) of the velocity $\frac{d\varphi}{dt}$ are both preserved.

From (30) and (31) and a simple consistency check, showing that the rate of change of the projection in (30) is given by acceleration $\mathbf{w}$, one can see that the phase space components of the velocity of state $\frac{d\varphi}{dt}$ assume correct classical values at any point $\varphi_{\mathbf{a},\mathbf{p}} \in M_{3,3}^\sigma$. This remains true for the time dependent potentials as well. The immediate consequence of this and the linear nature of the Schrödinger equation is that under the Schrödinger evolution with the Hamiltonian $\widehat{h} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x},t)$, the state constrained to $M_{3,3}^\sigma \subset CP^{L_2}$ moves like a point in the phase space representing a particle in Newtonian dynamics. More generally, Newtonian dynamics of $n$ particles is the Schrödinger dynamics of $n$-particle quantum system whose state is constrained to the phase-space submanifold $M_{3n,3n}^\sigma$ of the projective space for $L_2(\mathbb{R}^3) \otimes \dots \otimes L_2(\mathbb{R}^3)$, formed by tensor product states $\varphi_1 \otimes \dots \otimes \varphi_n$ with $\varphi_k$ of the form (27). On the contrary, there exists a unique extension of the Newtonian dynamics on the classical phase space submanifold to a unitary dynamics in the full Hilbert space [2].

Note again that the velocity and acceleration terms in (34) are orthogonal to the fibre $\{\varphi_{\mathbf{a},\mathbf{p}}\}$ of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$, showing that these Newtonian variables have to do with the motion in the projective space $CP^{L_2}$. The velocity of spreading is orthogonal to the fibre and to the phase space submanifold $M_{3,3}^\sigma$.

## 5. The Born rule and the normal probability distribution
The isometric embedding of the classical space $M_3^\sigma$ into the space of states $L_2(\mathbb{R}^3)$ results in a relationship between distances in $\mathbb{R}^3$ and in the projective space $CP^{L_2}$. The distance between two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is $\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^3}$. Under the embedding of the classical space into the space

of states, the variable $\mathbf{a}$ is represented by the state $\tilde{\delta}_{\mathbf{a}}^3$. The set of states $\tilde{\delta}_{\mathbf{a}}^3$ form a submanifold $M_3^\sigma$ in the Hilbert spaces of states $L_2(\mathbb{R}^3)$, which is "twisted" in $L_2(\mathbb{R}^3)$. It belongs to the sphere $S^{L_2}$ and spans all dimensions of $L_2(\mathbb{R}^3)$. The distance between the states $\tilde{\delta}_{\mathbf{a}}^3$, $\tilde{\delta}_{\mathbf{b}}^3$ on the sphere $S^{L_2}$ or in the projective space $CP^{L_2}$ is not equal to $\|\mathbf{a} - \mathbf{b}\|_{\mathbb{R}^3}$. In fact, the former distance measures length of a geodesic between the states while the latter is obtained using the same metric on the space of states, but applied along a geodesic in the twisted manifold $M_3^\sigma$. The precise relation between the two distances is given by

$$e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{4\sigma^2}} = \cos^2 \theta(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3), \tag{36}$$

where $\theta$ is the Fubini-Study distance between states in $CP^{L_2}$. The distance $\theta$ in the projective space of states $CP^{L_2}$ appears here for a good reason: the fibres of the fibration $\pi : S^{L_2} \longrightarrow CP^{L_2}$ through the points of the classical space $M_3^\sigma$ are orthogonal to this space. This is why the distance in $M_3^\sigma$ can be expressed in terms of the distance in $CP^{L_2}$. Despite the non-trivial geometry contained in (36), the formula itself is easy to derive. The left hand side is the result of integration in $|(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3)|^2$. On the other hand, the same expression is equal to the right side of (36) by definition of the Fubini-Study metric.

The relation (36) has an immediate implication onto the form of probability distributions of random variables over $M_3^\sigma$ and $CP^{L_2}$. In particular, consider a random variable over $CP^{L_2}$. Suppose that the restricted random variable defined over $M_3^\sigma = \mathbb{R}^3$ is distributed normally on $\mathbb{R}^3$. Then the direction-independent probability of transition between arbitrary quantum states must satisfy the Born rule. So, *The normal distribution law on $M_3^\sigma$ implies the Born rule on $CP^{L_2}$. Conversely, the Born rule implies the normal distribution law for states in $M_3^\sigma$.*

The fact that the Born rule implies the normal distribution on $M_3^\sigma$ is straightforward. According to the Born rule, the probability density $f_{\mathbf{a},\sigma}(\mathbf{b})$ to find the particle in a state $\tilde{\delta}_{\mathbf{a}}^3$ at a point $\mathbf{b}$ is equal to

$$f_{\mathbf{a},\sigma}(\mathbf{b}) = |\widetilde{\delta_{\mathbf{a}}^3}(\mathbf{b})|^2 = |(\widetilde{\delta_{\mathbf{a}}^3}, \delta_{\mathbf{b}}^3)|^2 = \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} e^{-\frac{(\mathbf{a}-\mathbf{b})^2}{2\sigma^2}}, \tag{37}$$

which is the normal distribution function. It follows that on the elements of $M_3^\sigma$, the Born rule *is* the rule of normal distribution.

The Born rule on $M_3^\sigma$ can be also written in term of the probability $P(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3)$ of transition between the states $\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3$ in $M_3^\sigma$:

$$P(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3) = |(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3)|^2. \tag{38}$$

Assuming $\widetilde{\delta_{\mathbf{b}}^3}$ is sufficiently sharp, the formulas (37) and (38) mean the same thing. In fact,

$$|(\widetilde{\delta_{\mathbf{a}}^3}, \widetilde{\delta_{\mathbf{b}}^3})|^2 = f_{\mathbf{a},\sqrt{2}\sigma}(\mathbf{b})(\Delta x)^3, \tag{39}$$

where $f_{\mathbf{a},\sqrt{2}\sigma}$ is the normal distribution function with variance $\sqrt{2}\sigma$ and $\Delta x = \sqrt{4\pi\sigma^2}$. This relates the probability in (38) to the normal probability density in (37) and identifies $P(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3)$ with the probability of finding the macroscopic particle near the point $\mathbf{b}$.

Conversely, suppose we have a rule for probability of transition between states in $CP^{L_2}$ which gives the normal distribution law for the states in $M_3^\sigma$ and depends only on the distance between states. Let's show that this must be the Born rule. In fact, the Fubini-Study distance between the states $\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3$ takes on all values from 0 to $\pi/2$, which is the largest possible distance between points in $CP^{L_2}$. By assumption, the probability $P(\varphi, \psi)$ of transition between any states $\varphi$ and

$\psi$ depends only on the Fubini-Study distance $\theta(\pi(\varphi), \pi(\psi))$ between the states. Given arbitrary states $\varphi, \psi \in S^{L_2}$, let then $\tilde{\delta}_{\mathbf{a}}^3$, $\tilde{\delta}_{\mathbf{b}}^3$ be two states in $M_\sigma^3$, such that

$$\theta(\pi(\varphi), \pi(\psi)) = \theta(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3). \tag{40}$$

It then follows that

$$P(\varphi, \psi) = P(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3) = \cos^2 \theta(\tilde{\delta}_{\mathbf{a}}^3, \tilde{\delta}_{\mathbf{b}}^3) = \cos^2 \theta(\pi(\varphi), \pi(\psi)), \tag{41}$$

which yields the Born rule for arbitrary states. This proves the claim and completes a review of [1] and [2].

## 6. Measurements on macroscopic and microscopic particles

We are now in a position to compare the process of measurement in the classical and quantum physics. First of all, the classical space and phase space are now submanifolds in the Hilbert space of states. This allows us to use the same language when analyzing both types of measurement. Second, the Newtonian dynamics is now a restriction of the Schrödinger dynamics to the classical phase space submanifold. Conversely, the Schrödinger dynamics is a unique unitary extension of the Newtonian dynamics from the classical phase space to the Hilbert space. This allows us to begin with a model of measurement satisfying Newton laws and extend it to a model consistent with the rules of quantum mechanics. Finally, the normal probability law is the restriction of the Born rule to the classical space submanifold. Conversely, the Born rule is the unique isotropic extension of the normal probability law from the classical space to the space of states. In particular, a classical model of measurement with a normal distribution of the measured quantity should lead us to a model consistent with the Schrödinger dynamics and the Born rule.

Measurements performed on a macroscopic particle satisfy generically the normal distribution law for the measured observable. This is consistent with the central limit theorem and indicates that a specific way in which the observable was measured is not important. To be specific, consider measurements of position of a particle. A common way of finding the position of a macroscopic particle is to expose it to light of sufficiently short wavelength and to observe the scattered photons. Due to the unknown path of the incident photons, multiple scattering events on the particle, random change in position of the particle, etc., the process of observation can be described by the diffusion equation with the observed position of the particle experiencing Brownian motion from an initial point during the time of observation. This results in the normal distribution of observed position of the particle.

The ability to describe measurement of position of a macroscopic particle as a diffusion seems to be a general feature of measurements in the macro-world, independent of a particular measurement set up. The averaging process making the central limit theorem applicable and leading to the normal distribution of the position random variable can be seen as the result of random hits experienced by the particle from the surrounding particles participating in the measurement. These random hits are equally likely to come from any direction, independent of the initial position of the particle, leading to Brownian motion and the validity of the diffusion equation for the probability density of the position random variable for the particle.

It is claimed now that at any time $t$ the initial state $\psi$ of a microscopic particle undergoing a position measurement is equally likely to shift in any direction in the tangent space $T_{\{\psi\}} CP^{L_2}$. Suppose, for example, that the particle is exposed to a stream of photons of sufficiently high frequency and number density. The scattered photons are then observed to determine the position of the particle. The field of photons in the experiment will be treated classically, as a fluctuating potential in a region surrounding the source. The assumptions about the potential that will be made will determine to what extent the derived results can be applied to different experiments of measuring position of a microscopic particle.

Recall that the space $M_{3,3}^\sigma$ is complete in $L_2(\mathbb{R}^3)$. Consider the subset of $M_{3,3}^\sigma$ formed by the states

$$\varphi_{\mathbf{mn}}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/4} e^{-\frac{(\mathbf{x}-\alpha\mathbf{n})^2}{4\sigma^2}} e^{i\frac{\beta\mathbf{mx}}{\hbar}}, \tag{42}$$

where $\alpha = \sqrt{2\pi}\sigma$, $\beta = \frac{h}{\sqrt{2\pi}\sigma}$ and $\mathbf{m}, \mathbf{n}$ take values on the lattice $\mathbb{Z}^3 \times \mathbb{Z}^3$ of points with integer coordinates in $\mathbb{R}^3 \times \mathbb{R}^3$. The set of functions (42) is known to be also complete in $L_2(\mathbb{R}^3)$. Any state in $L_2(\mathbb{R}^3)$ can be then represented by a linear combination of states $\varphi_{\mathbf{mn}}$. (For $\alpha\beta < h$ the system of functions $\varphi_{\mathbf{mn}}$ is called the Gabor or Weil-Heisenberg frame.) In particular, the initial state $\psi$ of the particle can be represented by a sum

$$\psi = \sum_{\mathbf{m},\mathbf{n}} C_{\mathbf{mn}} \varphi_{\mathbf{mn}}. \tag{43}$$

The set $M_3^\sigma$ is also complete in $L_2(\mathbb{R}^3)$. Here too there exist countable subsets of $M_3^\sigma$ that are complete in $L_2(\mathbb{R}^3)$. Moreover, an arbitrary initial state $\psi$ in $L_2(\mathbb{R}^3)$ can be approximated as well as necessary by a finite discrete sum

$$\psi \approx \sum_{\mathbf{n}} C_{\mathbf{n}} \widetilde{\delta}_{\mathbf{a}-\gamma\mathbf{n}}^3, \tag{44}$$

where $\mathbf{a}$ is arbitrary, $\mathbf{n} \in \mathbb{Z}^3$, and the value of $\gamma > 0$ together with the number of terms in the sum depend on $\psi$ and the needed approximation. Taking $\gamma$ sufficiently small, let's partition the space $\mathbb{R}^3$ into the cubes of edge $\gamma$ centered at the lattice points $\gamma\mathbf{n}$ and consider the indicator function $1_{\mathbf{n}}$ for each cube. The potential $\widehat{V}$ can be written as a sum $\sum_{\mathbf{n}} 1_{\mathbf{n}} \widehat{V}_{\mathbf{n}}$. The components $\widehat{V}_{\mathbf{n}}$ for different $\mathbf{n}$ can be assumed to be independent, identically distributed random variables. In the case of position measurement by scattering photons off the particle, the components $\widehat{V}_{\mathbf{n}}$ can be associated with a single photon at time $t$.

For simplicity, let's neglect the kinetic energy term in the Hamiltonian $\widehat{h}$. Let us denote the solution of the Shrödinger equation with the initial state $\psi$ by $\chi(t)$ and let us write $\chi(t) = e^{-\frac{i\overline{V}t}{\hbar}} \psi(t)$, where $\overline{V} = (\widehat{V}\psi, \psi)$ and $\psi(0) = \psi$. We then have at $t = 0$:

$$\frac{d\psi}{dt} = -\frac{i}{\hbar} \widehat{V}_\perp \psi, \tag{45}$$

where $\widehat{V}_\perp = \widehat{V} - \overline{V}$, as before. This equation gives the velocity of the state $\chi(t)$ in the projective space $CP^{L_2}$ at $t = 0$. To prove that under the action of $\widehat{V}_\perp$ all directions of velocity of state in $T_\psi CP^{L_2}$ are equally likely, consider the components of the velocity in the basis $-i\widetilde{\delta}_{\mathbf{m}}^3 \equiv -i\widetilde{\delta}_{\mathbf{a}-\gamma\mathbf{m}}^3$

$$\left(\frac{d\psi}{dt}, -i\widetilde{\delta}_{\mathbf{m}}^3\right) = \frac{1}{\hbar}(\widehat{V}_\perp \psi, \widetilde{\delta}_{\mathbf{m}}^3). \tag{46}$$

For any given potential and a given $\psi$, the form in (46) is a function of the distance between the points $\psi$ and $\widetilde{\delta}_{\mathbf{m}}^3$ in the Fubini-Study metric and, possibly, of the vector $\eta \in T_\psi CP^{L_2}$ of a fixed norm, tangent to the geodesic from $\{\psi\}$ to $\widetilde{\delta}_{\mathbf{m}}^3$ in $CP^{L_2}$. We have $\widehat{V} = \sum_{\mathbf{n}} 1_{\mathbf{n}} \widehat{V}_{\mathbf{n}}$, where each $\widehat{V}_{\mathbf{n}}$ is a random variable. Accordingly, (46) defines a random variable for each $\mathbf{m}$. Dividing these variables by $|(\psi, \widetilde{\delta}_{\mathbf{m}}^3)|$ we obtain a new set of random variables

$$\frac{\frac{1}{\hbar}(\widehat{V}_\perp \psi, \widetilde{\delta}_{\mathbf{m}}^3)}{|(\psi, \widetilde{\delta}_{\mathbf{m}}^3)|}. \tag{47}$$

Because $|(\psi, \widetilde{\delta}^3_{\mathbf{m}})|$ depends only on the distance between $\{\psi\}$ and $\widetilde{\delta}^3_{\mathbf{m}}$, the probability distributions of the random variables given by (46) and (47) are either both dependent or both independent of $\eta$. Provided the potential does not change much within each cube, up to a constant phase factor, the expression (47) is equal to

$$\tfrac{1}{\hbar}(V_{\mathbf{m}} - \overline{V}). \tag{48}$$

From the decomposition (44), the near-orthogonality of the functions $\widetilde{\delta}^3_{\mathbf{m}}$ and the definition of $\overline{V}$, we have

$$\overline{V} = \sum_{\mathbf{n}} V_{\mathbf{n}} |C_{\mathbf{n}}|^2. \tag{49}$$

Because $\sum |C_{\mathbf{n}}|^2 = 1$, the mean value of the random variable in (48) is zero:

$$E(V_{\mathbf{m}} - \overline{V}) = E(V_{\mathbf{m}}) - E(V_{\mathbf{m}}) \sum_{\mathbf{n}} |C_{\mathbf{n}}|^2 = 0. \tag{50}$$

As discussed, the random variables $V_{\mathbf{m}}$ at different cubes, i.e., for different values of $\mathbf{m}$ can be considered independent and identically distributed. It follows that the probability distributions of the random variables $V_{\mathbf{m}} - \overline{V}$ have a zero mean and are identical for all values of $\mathbf{m}$. With the help of the central limit theorem one can also claim that these distributions are normal. So the random vector with components (47) has an isotropic multivariate Gaussian distribution (the covariance matrix is proportional to the identity).

The standard deviation $\Delta V$ for the distribution of $V_{\mathbf{n}}$ in time satisfies the uncertainty relation $\Delta V \tau > h$, where $\tau$ is the time interval of observation. Therefore, the phase $-\frac{1}{\hbar}\overline{V}\tau$ in the phase factor of $\chi(t)$ should be considered uniformly distributed on $[0, 2\pi)$ so that any value of the phase factor is equally likely. It follows that the components of $\frac{d\chi}{dt}$ may have arbitrary constant phase factors at any given time and so the complex random vector made of these components has an isotropic normal circularly symmetric distribution. Accordingly, the vector $\frac{d\chi}{dt}\tau$ is equally likely to point in any direction in $T_\psi CP^{L_2}$ at $t = 0$. To make this result valid at an arbitrary moment of time, it remains to assume that the distribution of potentials is stationary and that changes in potentials over time are independent random variables.

Under a measurement of position of a macroscopic particle the observed particle is exposed to a random potential that is responsible for the normal distribution of the position random variable. We now see that the state of a microscopic particle undergoing a similar measurement and exposed to the same potential will experience a random motion on the sphere of states under which any direction $\eta$ of displacement of the state is equally likely. From the section entitled "The Born Rule and the Normal Probability Distribution" we know that the normal distribution on $M_3^\sigma$ and equal probability of any direction of displacement of the state result in the Born rule. That is, under the random potential produced by the measuring device the state $\psi$ of the measured microscopic particle performs a random walk on the sphere of states and the probability for the state of reaching a neighborhood of any point $\varphi$ on the sphere is given by the Born rule: $P(\varphi, \psi) = |(\varphi, \psi)|^2$.

Given the lack of Lebesgue measure on an infinite-dimensional Hilbert space, one may wonder how the state would have any chance of reaching a neighborhood of a given point in $S^{L_2}$. However, a realistic measuring device occupies a finite volume in the classical space. So the potential created by it can only affect a region $Q \subset \mathbb{R}^3$ of a finite volume $V$. The initial state $\psi$ of the particle can be split onto the state $\psi_Q = \psi|_Q$ with support in $Q$ (restriction of $\psi$ to $Q$) and the leftover state $\chi = \psi - \psi_Q$. The state $\chi$ is not going to change under the potential and does not participate in the measurement (the probability for it of reaching a detector in $Q$ is zero). Furthermore, possible group-velocity $\mathbf{v}_g$ of the measured particle in the given potential is also

bounded. The corresponding momentum $m\mathbf{v}_g$ of the particle belongs then to a bounded region $P \subset \mathbb{R}^3$. Therefore the state $\psi_Q$ of the particle is limited to a superposition of states in the region $P \times Q$ in the phase space $\mathbb{R}^3 \times \mathbb{R}^3 = M_{3,3}^\sigma$. But there are only finitely many elements of the Weil-Heisenberg frame in such a region. Therefore, under the measurement the state $\psi_Q$ evolves in a finite-dimensional subspace $V_{P \times Q}$ of the Hilbert space $L_2(\mathbb{R}^3)$. In particular, the Lebesgue volume of a neighborhood $Q_\mathbf{a}$ of any point $\tilde{\delta}_\mathbf{a}^3$ (the state of a particle with a known position) in $V_{P \times Q}$ is well defined. Accordingly, the state has a non-vanishing probability of reaching $Q_\mathbf{a}$ and the relative probabilities of reaching neighborhoods of different points are given by the Born rule.

## 7. The motion of a state under a measurement

Let's now look into details of the motion of a state under a measurement. Note that in the non-relativistic quantum mechanics the particle, and, therefore, its state in a single particle Hilbert space, cannot disappear or get created. The unitary property of evolution means that the state can only evolve along the unit sphere in the space of states $L_2(\mathbb{R}^3)$. To express this conservation of states in the case of observation of position of the particle, consider the density of states functional $\rho_t[\varphi; \psi_Q]$. This functional measures the number of states that belong to a neighborhood of a point $\varphi$ on the sphere of states $S^{V_{P \times Q}} \subset V_{P \times Q}$ at time $t$ for an ensemble of particles, prepared in the initial state $\psi_Q$ each. Under the isometric embedding $\omega : \mathbb{R}^3 \longrightarrow M_3^\sigma \subset L_2(\mathbb{R}^3)$ the states in $M_3^\sigma$ are identified with (positions of) particles. So the density of states functional $\rho_t[\varphi; \psi_Q]$ must be an extension of the density of macroscopic particles $\rho_t(\mathbf{a}; \mathbf{b})$ on $\mathbb{R}^3$. Here $\mathbf{b}$ is the initial position of the particle. In other words, $\rho_t(\mathbf{a}; \mathbf{b}) = \rho_t[\tilde{\delta}_\mathbf{a}^3; \tilde{\delta}_\mathbf{b}^3]$.

In the case of macroscopic particles the conservation of the number of particles is expressed in differential form by the continuity equation. For instance, if $\rho_t(\mathbf{a}; \mathbf{b})$ is the density at a point $\mathbf{a} \in \mathbb{R}^3$ of an ensemble of Brownian particles with initial position $\mathbf{b}$ and $\mathbf{j}_t(\mathbf{a}; \mathbf{b})$ is the current density at $\mathbf{a}$ of the particles, then

$$\frac{\partial \rho_t(\mathbf{a}; \mathbf{b})}{\partial t} + \nabla \mathbf{j}_t(\mathbf{a}; \mathbf{b}) = 0. \tag{51}$$

The conservation of states of an ensemble of microscopic particles is expressed by the continuity equation that follows from the Schrödinger dynamics. This is the same equation (51) with

$$\rho_t = |\psi|^2, \quad \text{and} \quad \mathbf{j}_t = \frac{i\hbar}{2m}(\psi \nabla \overline{\psi} - \overline{\psi} \nabla \psi). \tag{52}$$

For the states $\psi \in M_{3,3}^\sigma$ we obtain

$$\mathbf{j}_t = \frac{\mathbf{p}}{m}|\psi|^2 = \mathbf{v}\rho_t. \tag{53}$$

Because the restriction of Schrödinger evolution to $M_{3,3}^\sigma$ is the corresponding Newtonian evolution, the function $\rho_t$ in (53) must be the density of particles. This density was denoted earlier by $\rho_t(\mathbf{a}; \mathbf{b})$. It gives the number of Brownian particles that start at $\mathbf{b}$ and by the time $t$ reach a neighborhood of a point $\mathbf{a}$. The relation $\rho_t(\mathbf{a}; \mathbf{b}) = \rho_t[\tilde{\delta}_\mathbf{a}^3; \tilde{\delta}_\mathbf{b}^3]$ tells us that $\rho_t$ in (52) must be then the density of states $\rho_t[\tilde{\delta}_\mathbf{a}^3; \psi]$. It gives the number of particles in initial state $\psi$ found under the measurement at time $t$ in a state $\tilde{\delta}_\mathbf{a}^3$ (i.e., on a neighborhood of the point $\mathbf{a}$ in $\mathbb{R}^3$).

In the integral form the conservation of states in $L_2(\mathbb{R}^3)$ can be written in the following form:

$$\rho_{t+\tau}[\varphi; \psi] = \int \rho_t[\varphi + \eta; \psi]\gamma[\eta]D\eta, \tag{54}$$

where $\gamma[\eta]$ is the probability functional of the variation $\eta$ in the state $\varphi$ and integration goes over all variations $\eta$ such that $\varphi + \eta \in S^{L_2}$. When the state of the particle is constrained to $M_3^{\mathscr{G}} = \mathbb{R}^3$ this equation must imply the usual diffusion on $\mathbb{R}^3$. The restriction of (54) to $M_3^{\mathscr{G}}$ means that $\varphi = \tilde{\delta}_{\mathbf{a}}^3$ and $\eta = \tilde{\delta}_{\mathbf{a}+\epsilon}^3 - \tilde{\delta}_{\mathbf{a}}^3$, where $\epsilon$ is a displacement vector in $\mathbb{R}^3$. As we already know, the function $\rho_t[\tilde{\delta}_{\mathbf{a}}^3; \tilde{\delta}_{\mathbf{b}}^3] = \rho_t(\mathbf{a}; \mathbf{b})$ is the usual density of particles in space. Let us substitute this into (54), replace $\gamma[\eta]$ with the corresponding probability density function $\gamma(\epsilon) \equiv \gamma[\tilde{\delta}_{\mathbf{a}+\epsilon}^3 - \tilde{\delta}_{\mathbf{a}}^3]$ and integrate over the space $\mathbb{R}^3$ of all possible vectors $\epsilon$. As in the Einstein derivation of the Brownian motion, assume that $\gamma(\epsilon)$ is the same for all $\mathbf{a}$ and independent of the direction of $\epsilon$ (space symmetry). Therefore, the terms $\int \epsilon^k \gamma(\epsilon) d\epsilon$ and $\int \epsilon^k \epsilon^l \gamma(\epsilon) d\epsilon$ with $k \neq l$ vanish. It follows by the Einstein derivation that

$$\frac{\partial \rho_t(\mathbf{a}; \mathbf{b})}{\partial t} = k \Delta \rho_t(\mathbf{a}; \mathbf{b}), \tag{55}$$

where $k = \frac{1}{2\tau} \int \epsilon^2 \gamma(\epsilon) d\epsilon$ is a constant.

The diffusion equation (55) describes the dynamics of an ensemble of particles in the classical space $M_3^{\mathscr{G}}$. If initially all particles in the ensemble are at the origin, then the density of the particles at a point $x \in \mathbb{R}$ (one dimensional case) at time $t$ is given by

$$\rho_t(x; 0) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}. \tag{56}$$

The path of a Brownian particle under random hits from the surrounding particles is a particular path of the state $\psi$ in the space of states under the corresponding random potential. Since the distribution of displacement of $\psi$ is isotropic, the distribution (56) can be expressed in terms of the Fubini-Study distance between states. From (36) we have

$$e^{-\frac{x^2}{4\sigma^2}} = \cos^2 \theta, \tag{57}$$

where $\theta$ is the Fubini-Study distance between the state $\tilde{\delta}_x(u) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-\frac{(u-x)^2}{4\sigma^2}}$ and the like state centered at $x = 0$. Therefore,

$$x^2 = -4\sigma^2 \ln(\cos^2 \theta). \tag{58}$$

Equating the probability density for the Brownian particle initially at the origin to be found at time $t$ at the point $x$ with the probability density for transition between the corresponding states (see (39)), we have

$$\frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}} = \frac{1}{\sqrt{4\pi\sigma^2}} \rho_t(\theta), \tag{59}$$

where by (36) the function $\rho_t(\theta)$ is equal to $\cos^2 \theta$ for $t = \frac{\sigma^2}{k}$. Denoting this value by $\tau$ and substituting (58) into (59), we have

$$\rho_t(\theta) = \sqrt{\frac{\tau}{t}} \left(\cos^2 \theta\right)^{\frac{\tau}{t}}. \tag{60}$$

As $t$ increases, $\left(\cos^2 \theta\right)^{\tau/t}$ approaches 1 for all $\theta \in [0, \pi/2)$, while $\rho_t(\pi/2) = 0$. Note that the integral of $\rho_t$ with respect to $dx = \frac{dx}{d\theta} d\theta$ is, of course, 1.

From (58) we have

$$\left(\frac{d\theta}{dx}\right)^2 = -\frac{1}{2\sigma^2} \ln(\cos \theta) \cdot \cot^2 \theta \tag{61}$$

and

$$\frac{d^2\theta}{dx^2} = \frac{\cot\theta}{4\sigma^2}\left[1 + 2\ln(\cos\theta)\cdot\csc^2\theta\right].\tag{62}$$

This yields the second derivative $\frac{d^2}{dx^2}$ in the the form of the following operator $\Delta_\theta$:

$$\Delta_\theta = \frac{\cot\theta}{4\sigma^2}\left[1 - 2\ln(\cos\theta)\cdot\frac{d}{d\theta}\left(\cot\theta\frac{d}{d\theta}\right)\right].\tag{63}$$

The diffusion equation takes the form

$$\frac{\partial\rho_t(\theta)}{\partial t} = k\Delta_\theta\rho_t(\theta),\tag{64}$$

where $\rho_t(\theta) = \rho_t[\varphi;\psi]$ with $\theta$ being the Fubini-Study distance between the initial state $\psi$ and the variable point $\varphi$. The corresponding fundamental solution is the function in (60).

The parameter $\tau = \frac{\sigma^2}{k}$ should be interpreted as the time interval of observation. It is the time from the beginning of the diffusion process of the initial state $\psi$ to the moment when the state has reached the end point $\varphi$ (the moment of observation). Of course, this time may vary in different trials. However, the same is true of the time interval of observation of a Brownian particle in a particular experiment. By the central limit theorem the normal distribution on the left side of (59) is correct for the averaged time interval of observation $\tau$. Because the left hand side of (59) is valid, the density (60) is accurate as well, giving the Born rule at $t = \tau$. One can say that a specific measuring device has an associated time parameter $\tau$ and diffusion coefficient $k$, and, therefore, the variance of the normal distribution of the position $\sigma^2 = k\tau$. The value of $\sigma$ is then used in the isometric embedding $\omega: \mathbf{a} \longrightarrow \widetilde{\delta}_{\mathbf{a}}^3$, giving the relationship (57) and leading to the validity of the Born rule at the time of observation.

Note that if the diffusion of a state takes place but an observation is not made, the state continues its random walk on the sphere of states. Because of this additional (Schrödinger) evolution, the Born rule for any future observation has to be applied to the evolved state, rather than the original state $\psi$.

## 8. Collapse of a quantum state

It was shown that under the action of a random potential typically experienced by a Brownian particle, the state of a microscopic particle is equally likely to fluctuate in any direction tangent to the projective space of states. This, together with the normal distribution of position of the Brownian particle, signifies the validity of the Born rule for the motion of state. The motion satisfies the diffusion equation (64), whose fundamental solution is (60). It was shown that the state under the random potential evolves in the finite-dimensional subspace of the space of states. As a result, there is a non-vanishing probability for the initial state $\psi$ of reaching a neighborhood of any non-orthogonal state $\varphi$.

The presented process of transition between states is very different from what is usually understood by the collapse. The fact that a noise may lead to random fluctuation of state is rather simple and goes against of what one normally tries to achieve when explaining collapse under a measurement. The collapse models utilize various ad hoc additions to Schrödinger equation with the goal of explaining why the state under the resulting stochastic process "concentrates" to an eigenstate of the measured observable (usually, position or energy) [6]-[17]. Instead, it is argued here that under a generic measurement an ensemble of states in the same initial position $\psi$ "diffuses" isotropically into the space of states. Whenever a particular state in the ensemble reaches a neighborhood of an eigenstate of the measured observable, we say that the "collapse" has occurred. In this case the measuring device can record the value of the measured physical quantity.

The role of the measuring device in this scenario is reduced to initiating the diffusion(creating a "noise") and to registering a particular location of the diffused state. For instance, the "noise" in the position measuring device under consideration is due to the stream of photons. The device then registers the state reaching a point in $M_3^\sigma$. In a similar way, a momentum measuring device registers the states that reach under the diffusion the eigen-manifold of the momentum operator (which is the image of $M_3^\sigma$ under the Fourier transformation). Note also the similarity in the role of measuring devices in quantum and classical mechanics: in both cases the devices are designed to measure a particular physical quantity and inadvertently create a "noise", which results in a distribution of values of the measured quantity.

It follows, in particular, that the measuring device in quantum mechanics is not responsible for creating a basis into which the state is to be expanded. If several measuring devices are present, they are not "fighting" for the basis. When the eigen-manifolds of the corresponding observables don't overlap, only one of them can "click" for the measured particle as the state can reach only one of the eigen-manifolds at a time.

What does it all say about measurement of position of macroscopic and microscopic particles? During the period of observation of position of a macroscopic particle, the position is a random variable with normal distribution. Normally, observation happens during a short enough interval of time and the spread of the probability density is sufficiently small. A particular value of position variable during the observation is simply a realization of one of the possible outcomes. The change in observed position of the particle can be equivalently thought of as either a stochastic process $\mathbf{b}_t$ with values in $\mathbb{R}^3$ or a process $\tilde{\delta}^3_{\mathbf{b},t}$ with values in $M_3^\sigma$. The advantage of the latter representation is that the position random variable $\tilde{\delta}^3_{\mathbf{b}}$ gives both, the position of the particle in $M_3^\sigma = \mathbb{R}^3$ and the probability density to find it in a different location $\mathbf{a}$ (in the state $\tilde{\delta}^3_{\mathbf{a}}$), due to uncontrollable interactions with the surroundings under the observation.

Measuring position of a microscopic particle has, in essence, a very similar nature. Under observation the state $\psi$ is a random variable with values in the space of states $CP^{L_2}$. To measure position is to observe the state on the submanifold $M_3^\sigma$ or $M_{3,3}^\sigma$ in $CP^{L_2}$. In this case, the random variable $\psi$ assumes one of the values $\tilde{\delta}^3_{\mathbf{a}}$, with the uniquely defined probability density compatible with the normal density in the space $\mathbb{R}^3$. This probability density is given by the Born rule. Here too the random variable $\psi$ gives both, the position of the state of the particle in $CP^{L_2}$ and the probability density to find the particle in a different state $\tilde{\delta}^3_{\mathbf{a}}$.

So the difference between the measurements is two-fold. First, under a measurement the state $\psi$ of a microscopic particle is a random variable over the entire space of states $CP^{L_2}$ and not just over the submanifold $M_3^\sigma$. Second, unless $\psi$ is already constrained to $M_3^\sigma$ (the case which would mimic the measurement of position of a macroscopic particle), to measure position is to observe the state that "diffused" enough to reach the submanifold $M_3^\sigma$. To put it differently, the measuring device is not where the initial state was. Assuming the state has reached $M_3^\sigma$, the probability density of reaching a particular point in $M_3^\sigma$ is given, as we saw, by the Born rule.

We don't use the term collapse of position random variable when measuring position of a macroscopic particle. Likewise, there seems to be no physics in the term collapse of the state of a microscopic particle. Instead, due to the diffusion of state, there is a probability density to find the particle in various locations on $CP^{L_2}$. In particular, the state may reach the space manifold $M_3^\sigma = \mathbb{R}^3$. If that happens and we have detectors spread over the space, then one of them clicks. If the detector at a point $\mathbf{a} \in \mathbb{R}^3$ clicks, that means the state is at the point $\tilde{\delta}^3_{\mathbf{a}} \in CP^{L_2}$ (that is, the state *is* $\tilde{\delta}^3_{\mathbf{a}}$). The number of clicks at different points $\mathbf{a}$ when experiment is repeated is given by the Born rule. The state is not a "cloud" in $\mathbb{R}^3$ that shrinks to a point under observation. Rather, the state is a point in $CP^{L_2}$ which may or may not be on $\mathbb{R}^3 = M_3^\sigma$. When the detector clicks we know that the state is on $M_3^\sigma$.

Note once again that there is no need in any new mechanism of "collapse" in the model. An observation is not about a "concentration" of state and the stochastic process initiated by

the observation is in agreement with the conventional Schrödinger equation with a randomly fluctuating potential ("noise"). The origin of the potential depends on the type of measuring device or properties of the environment, capable of "measuring" the system. Fluctuation of the potential can be traced back to thermal motion of molecules, atomic vibrations in solids, vibrational and rotational molecular motion, and the surrounding fields.

## 9. Summary

The dynamics of a classical $n$-particle mechanical system can be identified with the Schrödinger dynamics constrained to the classical phase space submanifold $M^\sigma_{3n,3n}$ in the space of states. Conversely, there is a unique unitary time evolution on the space of states of a quantum system that yields Newtonian dynamics when constrained to the classical phase space. This results in a tight, previously unnoticed relationship between classical and quantum physics. In particular, under a measurement of the position of a macroscopic particle, the position random variable generically obeys the normal distribution law. The normal distribution law for the position variable implies the Born rule for transitions between arbitrary quantum states. Therefore, any classical (i.e., based on Newtonian dynamics) model of measurement of a macroscopic particle that predicts the normal distribution of the position random variable extends in a unique way to the corresponding quantum (i.e., satisfying Schrödinger dynamics) model that enforces the Born rule for probability of transition between states. The central limit theorem makes it easy for the outcomes of a measurement of a classical system to satisfy the normal distribution law. It follows that the Born rule in measurements of a quantum system is as generic as the normal distribution law in classical measurements.

In this paper, the proved relationship between classical and quantum concepts is taken to mean that physical laws, which govern the behavior of macroscopic and microscopic bodies, are fundamentally the same. For instance, there exists a unique extension of the classical Brownian motion from the classical space submanifold $M^\sigma_3$ to the space of states $CP^{L_2}$ of the particle. Because the Brownian motion can model the process of measurement in classical physics, its unique extension is taken to be adequate for the description of measurements on microscopic systems.

With this understood, the process of measurement on a quantum system can be described in terms of a diffusion of state of the measured system in which the state has equal probability of being displaced in any direction in the space of states $CP^{L_2}$. The role of the measuring device is reduced to creating a "noise" that triggers the diffusion in $CP^{L_2}$ and in recording the diffused state when it reaches a particular region in $CP^{L_2}$. The conclusion is that the so-called collapse of the wave function in the framework is not about an instantaneous "concentration" of state near an eigenstate of the measured observable. Instead, it is about diffusion on the space of states under interaction with the measuring device and the environment. The "collapse" to an eigenstate of an observable happens when the state under the diffusion reaches the eigen-manifold of that observable. In the case of position measurements, the state must reach the classical space or phase space submanifolds in $CP^{L_2}$. Due to the enormous amount of collisions between a macroscopic body and the particles in the surroundings, position of the body is constantly measured. As a result, the diffusion process for macroscopic bodies can trivialize, which may explain why they remain in the classical space and, therefore, have a definite position.

We see that macroscopic and microscopic particles may not be so different after all. The only important distinction is that microscopic systems within the proposed framework live in the space of states while their macroscopic counterparts live in the classical space submanifold. Since our own life happens primarily in the macro-world, it is difficult for us to understand the infinite-dimensional quantum world around us. As soon as the classical-space-centered point of view is extended to its Hilbert-space-centered counterpart, the new, clearer view of the classical-quantum relationship emerges.

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## References

 [1] Kryukov A 2017 *J. Math. Phys.* **52** 082103
 [2] Kryukov A 2018 *J. Math. Phys.* **59** 052103
 [3] Kryukov A 2007 *Phys. Lett.* A **370** 419
 [4] Sanz A and Miret-Artés S 2012 *Dynamics of Open Classical Systems* in: Lecture Notes in Physics vol 850 (Berlin: Springer) p 47
 [5] Percival I 1998 *Quantum State Diffusion* (Cambridge: Cambridge University Press)
 [6] Pearle P 1976 *Phys. Rev.* D **13** 857
 [7] Pearle P 1999 *Collapse Models* in: Lecture Notes in Physics vol 526 (Berlin: Springer) p 195
 [8] Ghirardi G, Rimini A and Weber T 1986 *Phys. Rev.* D **34** 470
 [9] Ghirardi G, Pearle P and Rimini A 1990 *Phys. Rev.* A **42** 78
[10] Diósi L 1989 *Phys. Rev.* D **40** 1165
[11] Adler S 2002 *J. Phys.* A **35** 841
[12] Adler S 2004 *Quantum theory as an emergent phenomenon* (Cambridge: Cambridge University Press)
[13] Adler S, Brody D, Brun T and Hughston L 2001 *J. Phys.* A **34** 8795
[14] Adler S and Brun T 2001 *J. Phys.* A **34** 4797
[15] Adler S and Horwitz L 2000 *J. Math. Phys.* **41** 2485
[16] Hughston L 1996 *Proc. Roy. Soc. London* A 953
[17] Bassi A, Lochan K, Satin S, Singh T and Ulbricht H 2013 *Rev. Mod. Phys.* **85** 471
[18] Joos E, Giulini D, Kiefer C, Kupsch J and Stamatescu I 2003 *Decoherence and the appearance of a classical world in quantum theory* (Berlin: Springer)