Recent developments in premetric classical electrodynamics[*, †]

Friedrich W. Hehl[‡]
Inst. for Theor. Physics, University of Cologne,
50923 Köln, Germany, and
Dept. of Physics and Astronomy, University of Missouri-Columbia,
Columbia, MO 65211, USA

Yakov Itin[§]
Institute of Mathematics, Hebrew University of Jerusalem and
Jerusalem College of Engineering, Jerusalem 91904, Israel

Yuri N. Obukhov[¶]
Inst. for Theor. Physics, University of Cologne,
50923 Köln, Germany, and
Dept. of Theor. Physics, Moscow State University,
117234 Moscow, Russia

Abstract

Classical electrodynamics can be based on the conservation laws of electric charge and magnetic flux. Both laws are independent of the metric and the linear connection of spacetime. Within the framework of such a *premetric* electrodynamics — provided a *local* and *linear* constitutive law of the vacuum is added — the propagation of electromagnetic waves in the geometric-optics limit can be studied. The wave vectors of the wave fronts obey a quartic extended Fresnel equation. If one forbids birefringence in vacuum, the light cone emerges and Maxwell-Lorentz vacuum electrodynamics can be recovered. If

[*] Invited lecture given at the 3rd Summer School in Modern Mathematical Physics, 20-31 August 2004, Zlatibor, Serbia and Montenegro.

[†] Work supported by the Deutsche Forschungsgemeinschaft (DFG), Bonn, project HE 528/20-1.

[‡] e-mail address: `hehl@thp.uni-koeln.de`.

[§] e-mail address: `itin@math.huji.ac.il`.

[¶] e-mail address: `yo@thp.uni-koeln.de`.

minimal coupling of electrodynamics to gravity is assumed, then only the gravitational potential, i.e., the metric of spacetime, emerges in the constitutive law. We discuss recent results within this general framework.

PACS: 03.50.De, 04.20.Cv, 42.15.-i

Keywords: Premetric classical electrodynamics, axiomatics, conservation of charge, conservation of flux, magnetic charge, propagation of light, Fresnel equation, birefringence, light cone, coupling to gravity

1 Introduction

Premetric classical electrodynamics has been consistently formulated in 2003 in the book of Hehl and Obukhov [1]. Related developments can be found in the recent books of Lindell [2] and Russer [3] and the articles of Delphenich [4, 5]. One of the basic ideas is to formulate electrodynamics in such a way that the metric of spacetime (that is, the gravitational potential of Einstein's theory of gravity) doesn't enter the fundamental laws of electrodynamics. The practicability of such an approach has been shown in detail in [1], see also the earlier work of Toupin & Truesdell [6, 7], Post [8, 9], Kiehn et al. [10], and Kovetz [11]. In this lecture, we concentrate on the post 2003 development, for earlier results we refer to the extended bibliography in [1].

After the publication of [1], premetric electrodynamics turned out to be a lively and active subject. Mainly the following aspects were developed (we order them roughly chronologically):

1. Inclusion of *magnetic charges* in a consistent and metric-free way by Kaiser [12] and by Hehl and Obukhov [13], earlier work had been done by Edelen [14].

2. Application of the premetric formalism to the *Quantum Hall Effect* and prediction of the independence of the QHE on an external gravitational by Hehl, Obukhov, and Rosenow [15].

3. Possible *non-minimal coupling* of gravity to electromagnetism, in particular to the torsion field, by Solanki, Preuss, and Haugan [16, 17], see also Hehl and Obukhov [18], and, subsequently, by using the formalism of [1], by Rubilar et al. [19] and by Itin et al. [20, 21].

4. Derivation of the *signature of the metric* of spacetime from the Lenz rule (connected with Faraday's induction law) by Itin et al. [22].

5. The Lorentz-violating *CFJ-electrodynamics,* see Carroll, Field, and Jackiw [23], was reformulated in the premetric formalism by Itin [24]. It enhances the transparency of the CFJ-approach.

6. New derivation by Lämmerzahl et al. [25], see also [26], of the metric from linear premetric electrodynamics: We forbid *birefringence* in vacuum.

7. New insight in the *physical dimensions* of electrodynamics and in the time variability of the vacuum impedance and the fine structure constant by Tobar [27] and Hehl & Obukhov [28].

8. Effect of the *skewon* field on light propagation in an quite detailed article by Obukhov et al. [29].

9. Relation between *axion* electrodynamics, see Wilczek [30], and the so-called Post constraint, see Hehl and Obukhov [31]; compare the perfect electromagnetic conductor (PEMC) of Lindell and Sihvola [32, 33]. Scattering of electromagnetic fields at the PEMC.

10. Symbol and *hyperbolic polynomials* of the wave equation and its relation to premetric electrodynamics, as discussed by Beig [34].

11. *Nonlinear* electrodynamics by Delphenich [35], see also [36, 37].

We will now integrate these topics into a short systematic review of premetric electrodynamics.

2 Axiom 1: Electric charge conservation

We base electrodynamics on *electric charge conservation.* The conservation of electric charge was already recognized as fundamental law during the time of Franklin (around 1750) well before Coulomb discovered his force law in 1785. Nowadays, at a time, at which one can catch single electrons and single protons in traps and can *count* them individually, we are more sure than ever that electric charge conservation is a valid fundamental law of nature. Therefore matter carries as a *primary quality* something called electric charge which only occurs in positive or negative units of an elementary charge e (or, in the case of quarks, in 1/3th of it) and which can be counted in principle. Thus it is justified to introduce the physical dimension of charge q as a new and independent concept.

Mathematically, the first axiom is most conveniently formulated in terms of exterior calculus with the help of the 3-form J of electrical current. Its integral over an arbitrary 3-dimensional domain describes the total charge contained therein. It can be determined by counting the particles carrying an elementary charge. The latter can be taken as a unit of charge, in the sense of the theory of dimensions. The conservation of the current reads, in exterior calculus,

$$dJ = 0. \tag{1}$$

This law is metric-independent since it is based on a *counting* procedure for elementary charges. Given a spacetime foliation, we can 1+3 decompose the current $J = -j \wedge d\sigma + \rho$ into the 2-form of the electric current density j and the 3-form ρ of the electric charge density. Then the charge conservation law assumes a more familiar form of the continuity equation

$$\dot{\rho} + \underline{d}\, j = 0\,, \tag{2}$$

where the dot denotes the time derivative (Lie derivative with respect to a suitable vector field) and $\underline{d}$ the 3-dimensional exterior derivative. Both (1) and (2) can be equivalently formulated in the integral form, see [1] for details.

Because of the first axiom and according to a theorem of de Rham, we can represent the conserved electric current as an exterior derivative (a generalized "divergence"), i.e.,

$$J = dH. \tag{3}$$

In this way, we naturally introduce the 2-form H of the *electromagnetic excitation*. Certainly, this quantity is not uniquely determined from (3) since the transformation $H \to H + d\psi$ evidently preserves the form of (3). However, this arbitrariness is eventually fixed in the actual measurements performed with the help of ideal conductors and superconductors, which means that the electromagnetic excitation does have a direct *operational* significance, see Raith [38].

The $(1+3)$-decomposition of H is obtained similarly to the decomposition of the current J, thereby introducing the 2-form $\mathcal{D}$ of the electric excitation (historical name: "dielectric displacement") and the 1-form $\mathcal{H}$ of the magnetic excitation ("magnetic field"):

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D}. \tag{4}$$

Inserting (4) as well as the $1+3$ decomposition of the current into (3), we recover the pair of the 3-dimensional inhomogeneous Maxwell equations

$$dH = J \quad \begin{cases} \qquad \underline{d}\,\mathcal{D} = \rho\,, \\ -\dot{\mathcal{D}} + \underline{d}\,\mathcal{H} = j\,. \end{cases} \tag{5}$$

Charge conservation is a law that is valid in macro- as well as in microphysics. If charge conservation was violated, we could introduce a 4-form N (one component only!) according to $dJ = N$. Lämmerzahl, Macias, and Müller [39] studied a seemingly more restricted class of models violating charge conservation. Such models can be used as test theories for experiments on the checking of charge conservation.

3 Axiom 2: Lorentz force density

With charge conservation alone, we arrived at the inhomogeneous Maxwell equations (3). Now we need some more input for deriving the homogeneous Maxwell equations. In the purely electric case with a test charge q, we have in terms of components

$$F_a \sim q\, E_a\,, \tag{6}$$

with F as force and E as electric field covector. Generalizing (6), the simplest relativistic ansatz for defining the electromagnetic field reads:

$$\text{force density} \quad \sim \quad \text{field strength} \times \text{charge current density}\,. \tag{7}$$

We know from Lagrangian mechanics that the *force* $\sim \partial L/\partial x^i$ is represented by a *covector* with the absolute dimension of action h (here h is *not* the Planck constant but rather only denotes its *dimension*). Accordingly, with the covectorial force density f_α, the ansatz (7) can be made more precise. The *Lorentz force density* acting on a charge density is encoded into Axiom 2:

$$f_\alpha = (e_\alpha \rfloor F) \wedge J\,. \tag{8}$$

Here e_α is an arbitrary, i.e., anholonomic frame or tetrad, a basis of the tangent space, with $\alpha = 0, 1, 2, 3$, and $\rfloor$ denotes the interior product (contraction). The axiom (8) should be read as an operational procedure for defining the electromagnetic field strength 2-form F in terms of the force density f_α, known from mechanics, and the current density J, known from charge conservation. The $1+3$ decomposition of F reads

$$F = E \wedge d\sigma + B\,, \tag{9}$$

thus defining the electric and the magnetic field strengths E and B (a differential 1-form and the 2-form, respectively).

4 Axiom 3: Magnetic flux conservation

Let us recall that the (four dimensional) electric current 3-form J decomposes as $J = -j \wedge d\sigma + \rho$. Accordingly, $\int_{C_3} \rho$ represents the electric charge, ρ the electric charge density, and j the corresponding current density in three dimensions. Analogously — apart from the fact that J is a 3-form and F a 2-form — $\int_{C_2} B$ represents the magnetic flux and B the magnetic flux density. Consequently, E has to be interpreted as the current of the magnetic flux density. We could call F the (four dimensional) current of magnetic flux.

Why isn't that conventionally presented in textbooks in this way? There seem to be two reasons: (i) In textbooks the 4-dimensional point of view is only a supplementary structure which is used eventually in order to crown the development of electromagnetism, whereas we take, in the approach presented, the 4-dimensional view on electromagnetism as fundamental and indispensable right from the beginning. (ii) A "charge" density is assumed intuitively to be a 3-form, i.e., one has to integrate the charge density over a 3-volume in order to find a net charge. This is a widespread prejudice. However, also a 2-form like B, if integrated over a 2-dimensional surface, can be considered to be a "charge" (here the magnetic flux) and, accordingly, a 1-form like E as the corresponding "current density" (here the current of the magnetic flux density). Therefore, Axiom 1 in local form, $dJ = 0$, as electric charge conservation, has as analog — in local form —

$$dF = 0 \quad \begin{cases} \quad\;\; \underline{d}\,B &= 0\,, \\ \dot{B} + \underline{d}\,E &= 0\,, \end{cases} \tag{10}$$

the law of magnetic flux conservation. Since the Faraday induction law and the sourcelessness of B are consequences of (10), this axiom has a firm experimental underpinning. Note that the induction law has the form of a continuity equation for the "charge density" B.

Nevertheless, similar as with the possible violation of electric charge conservation, there have been numerous attempts to postulate the existence of magnetic charge according to

$$dF = K \quad \begin{cases} \underline{d}\,B &= \rho_{\rm mg}\,, \\ \dot{B} + \underline{d}\,E &= j_{\rm mg}\,, \end{cases} \tag{11}$$

with the magnetic current density $K = -j_{\rm mg} \wedge d\sigma + \rho_{\rm mg}$.

It has been shown by Edelen [14], and more completely by Kaiser [12] and two of us [13], that the law (11) can be made to fit into the premetric approach, provided electric charge conservation is fulfilled.

Clearly, if besides the electric charge $\rho_{\rm el}$, a magnetic charge $\rho_{\rm mg}$ is present, we have to expect that the electromagnetic field exerts a force on it. The absolute dimension $[J] \times [F]$, which enters the Lorentz force density (8), is *electric charge* $\times$ *magnetic flux* $=$ *action*. The same dimension can be found for the expression $[K] \times [H]$, namely $[K] =$ *magnetic flux* and $[H] =$ *electric charge*. We recognize here a certain reciprocity between electricity and magnetisms. The revised Axiom 2, taking into account the linearity of electrodynamics, would then read

$$f_\alpha = (e_\alpha \rfloor F) \wedge J - (e_\alpha \rfloor H) \wedge K\,. \tag{12}$$

The necessity of the minus sign will be explained in the next section. There it will also be shown that the expression for the energy-momentum current of the electromagnetic field remains the same, with or without magnetic charge, a quite remarkable fact that enables us to revise the premetric axiomatics in such a way as to accommodate the possible existence of magnetic charge.

However, we shouldn't forget that all experiments done so far attest to the absence of magnetic charge. Hence the experimental situation leaves no doubt about the absence of magnetic charge in nature. Even more so, theoretical reasons make such a structure also implausible: The modified Axiom 2, see (12), could no longer be read as a clear cut operational definition of the electromagnetic field strength F. It would mix with H in a rather confusing way.

In the 19th and the first half of the 20th century, in spite of Ampère's hypothesis of the electric origin of magnetic effects — which today is no longer a hypothesis but rather an experimental fact — magnetostatics was built up in analogy to electrostatics. The force on the fictitious magnetic charges were then given by $\rho_{\rm mg} \wedge \mathcal{H}$. This coincides with the leading piece of the last term of the right-hand-side of (12). This is the reason why $\mathcal{H}$ was called the magnetic field strength in analogy to the electric field strength E, with $\rho_{\rm el} \wedge E$ as Lorentz force density, even though the magnetic charges were assumed to be fictitious. We recognize here once more that the premetric formalism induces transparency into electrodynamics.

5 Axiom 4: Energy-momentum current

The field equations of electrodynamics $dH = J$ and $dF = 0$ have been set up. Now we have to turn to the energy-momentum distribution in the electromagnetic field. Since the Lorentz force density (8) [or (12)] determines the relation between the electromagnetic field and *mechanics,* this formula must be the starting point for a discussion of energy and momentum. If we introduce for the magnetic piece in (12) a factor β, to be determined later, then

$$f_\alpha = (e_\alpha \rfloor F) \wedge J - \beta\, (e_\alpha \rfloor H) \wedge K\,. \tag{13}$$

For $K = 0$, we recover the case of the absence of magnetic charges. We assume magnetic charges and substitute the Maxwell equations into (13):

$$f_\alpha = (e_\alpha \rfloor F) \wedge dH - \beta\, (e_\alpha \rfloor H) \wedge dF\,. \tag{14}$$

Note, however, that this equation is also valid for ordinary electrodynamics (without magnetic charges) since under those conditions $dF = 0$. In other words, (14) is valid in both cases, with or without magnetic charges. The same will be true for the subsequent formulas.

We integrate partially both terms in (14), see [1]:

$$f_\alpha = d\big[\beta\, F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\big] - \beta\, F \wedge d(e_\alpha \rfloor H) + H \wedge d(e_\alpha \rfloor F)\,. \tag{15}$$

The expression under the exterior derivative has already the desired form. We recall the main formula for the Lie derivative of an arbitrary form Φ, namely $\pounds_{e_\alpha}\Phi = d(e_\alpha \rfloor \Phi) + e_\alpha \rfloor (d\Phi)$, see Frankel [40]. This allows us to transform the second part on the right-hand-side of (15):

$$\begin{aligned} f_\alpha = d\big[\beta\, F \wedge (e_\alpha \rfloor H) &\;-\; H \wedge (e_\alpha \rfloor F)\big] \\ -\beta\, F \wedge (\pounds_{e_\alpha} H) &\;+\; H \wedge (\pounds_{e_\alpha} F) \\ +\beta\, F \wedge e_\alpha \rfloor (dH) &\;-\; H \wedge e_\alpha \rfloor (dF)\,. \end{aligned} \tag{16}$$

The last line can be put into the form

$$+\beta\, e_\alpha \rfloor [F \wedge dH] - \beta\, (e_\alpha \rfloor F) \wedge dH - e_\alpha \rfloor [H \wedge dF] + (e_\alpha \rfloor H) \wedge dF\,. \tag{17}$$

The expressions in the square brackets are 5-forms and vanish. Two terms are left over, and we find

$$\begin{aligned} f_\alpha = d\big[\beta\, F \wedge (e_\alpha \rfloor H) &\;-\; H \wedge (e_\alpha \rfloor F)\big] \\ -\beta\, F \wedge (\pounds_{e_\alpha} H) &\;+\; H \wedge (\pounds_{e_\alpha} F) \\ -\beta\, (e_\alpha \rfloor F) \wedge dH &\;+\; (e_\alpha \rfloor H) \wedge dF\,. \end{aligned} \tag{18}$$

With the help of (14), the last line can be rewritten as

$$-\beta\, f_\alpha + (1 - \beta^2)(e_\alpha \rfloor H) \wedge dF\,. \tag{19}$$

Thus, (18) reads

$$\begin{aligned}(1+\beta)\, f_\alpha = d\big[\beta\, F\wedge(e_\alpha\rfloor H) \quad &- \quad H\wedge(e_\alpha\rfloor F)\big]\\ -\beta\, F\wedge(\pounds_{e_\alpha}H) \quad &+ \quad H\wedge(\pounds_{e_\alpha}F)\\ +(1-\beta^2)(e_\alpha\rfloor H)\wedge dF\,. &\end{aligned} \tag{20}$$

As we mentioned above, also this formula is correct with and without magnetic charges as long as electric charge conservation $dJ = 0$ is taken for granted. *Without* magnetic charge the last term drops out because of $dF = 0$. Then the choice of $\beta = 1$ yields the conventional energy-momentum current of Minkowski. *With* magnetic charge the last term disturbs the whole set up. We cannot find a reasonable energy-momentum current in this case unless we put $\beta = \pm 1$. However, $\beta = -1$ would trivialize (18) to a mathematical identity and the Lorentz force density would be lost. Hence $\beta = +1$ is the only reasonable option. In other words, we choose $\beta = 1$ in (13) thus arriving at what we displayed already in (12).

Consequently, in both cases, we arrive at the same relation

$$f_\alpha = d\,{}^{\rm k}\Sigma_\alpha + X_\alpha\,, \tag{21}$$

with the *kinematic energy-momentum* 3-form of the electromagnetic field,

$${}^{\rm k}\Sigma_\alpha := \frac{1}{2}\left[F\wedge(e_\alpha\rfloor H) - H\wedge(e_\alpha\rfloor F)\right] \tag{22}$$

and the and the force density 4-form

$$X_\alpha := -\frac{1}{2}\left(F\wedge\pounds_{e_\alpha}H - H\wedge\pounds_{e_\alpha}F\right)\,. \tag{23}$$

For the derivation of the energy-momentum current, we could alternatively require (21) right from the beginning. Then the first line in (20) is reasonable because of the exterior derivative, the second line has to emerge because of the Lie derivative (in a special frame it is zero). Accordingly, the third line has to be zero.

The energy-momentum current (22) in premetric exterior calculus has also been derived by Lindell & Jancewicz [41], Segev [42], and Kaiser [12], see also the modified derivation by Itin et al. [20, 22].

The energy-momentum localization, as specified in (22), represents our Axiom 4. The current ${}^{\rm k}\Sigma_\alpha$ is electric/magnetic reciprocal under the substitutions $H \to \zeta F$, $F \to -\frac{1}{\zeta}\, H$, with an arbitrary pseudo-scalar function ζ, as discussed in detail in [1]. There it is also shown that X_α vanishes in specific cases making then ${}^{\rm k}\Sigma_\alpha$ to a conserved quantity.

6 Premetric electrodynamics and the Maxwell equations

Until now, all our considerations are generally covariant and metric-free and connection-free. They are valid in flat Minkowskian and in curved Riemannian spacetime, that is, in special relativity theory (SR) and in general relativity theory (GR), and even in a spacetime possibly carrying torsion and/or nonmetricity. Therefore Maxwell's equations in the form of (5) and (10) represent the optimal formulation of the fundamental laws of classical electrodynamics. Nonminimal couplings, which induce additional fundamental constants, will be discussed in Sec.12. In order to complete the theory, we need a relation between H and F, which we will discuss in the next section. However, before that we will have a look at dimensional analysis, today a largely neglected subject.

In accordance with the general dimensional analysis of Schouten and Dorgelo, see [43], and, in particular, of Post [8], the *absolute* dimension of J is that of a charge: $[J] = q$ (since the integral of the current over a 3-dimensional spatial domain yields the total charge). Since the exterior derivative is dimensionless, $[d] = 1$, and since the electromagnetic excitation is given by the inhomogeneous Maxwell equation (3), we conclude that the absolute dimension of the excitation $[H] = q$. Then, by means of (4), the absolute dimensions of the electric and magnetic excitations turn out to be $[\mathcal{D}] = q$ and $[\mathcal{H}] = q/t$. The *relative* dimensions are those of their frame components, $[\mathcal{H}_a] = q/(t\,\ell)$ and $[\mathcal{D}_{ab}] = q/\ell^2$, with the spatial indices $a, b = 1, 2, 3$. These are the "physical" dimensions known to physicists and engineers. Furthermore, denoting the physical dimension of an action by h, the Lorentz force equation (8) shows that the absolute dimension of the electromagnetic field strength 2-form F is $[F] = h/q = \phi$, that is, action/charge or magnetic flux ϕ, see [1]. Then by (9), the absolute dimensions of the electric and magnetic fields E and B are $[E] = \phi/t$ and $[B] = \phi$, respectively, and the relative dimensions $[E_a] = \phi/(t\,\ell)$ and $[B_{ab}] = \phi/\ell^2$.

Our analysis does not make use of the metric. Moreover, it is generally covariant and as such valid in particular in GR and SR. Furthermore it is valid on a spacetime manifold of arbitrary dimension. And, on top of that, our considerations do not depend on any particular choice of the system of physical units. Whatever your favorite system of units may be, our results will apply to it. In short: Our dimensional analysis so far is *premetric, generally covariant, dimensionally independent, and valid for any system of units.*

Quite remarkably, in $1+2$ spacetime dimensions we can develop the complete premetric phenomenological theory of the quantum Hall effect (QHE) that applies to the an electron gas system confined to a 2-dimensional plane [15]. The crucial observation is that in $1+2$ dimensions, the current is a 2-form, and the natural constitutive relation, assuming isotropy, is then

$$J = -\,\sigma_{\mathrm{H}}\,F. \tag{24}$$

From the Maxwell equations (3) and (10) we find that σ_{H} is constant, and the above dimensional analysis shows that it has the dimension of conductance. The additional microscopic study [15] supports these conclusions and ultimately identifies $1/\sigma_{\rm H}$ with the von Klitzing constant $R_{\rm K}$.

Since such a phenomenological scheme is premetric, the gravitational field (neither metric nor linear connection) ever shows up in this theory. This leads to the prediction that the QHE does not "feel" gravity. In particular, this means that the linear Hall resistance should remain constant in an arbitrary noninertial frame and in any gravitational field. An experimental verification of this prediction is highly desirable.

In contrast to the (1 + 2)-dimensional Hall electrodynamics, the electrodynamical theory in 1 + 3 spacetime dimensions normally incorporates the metric of a flat or a curved spacetime via the constitutive relation $H = H(F)$ between the excitation and the field strength. In particular, the standard Maxwell-Lorentz electrodynamics arises when we assume

$$H = \lambda_0 \,{}^\star F\,. \tag{25}$$

Here the (4-dimensional) Hodge star $\star$ is defined by the spacetime metric g_{ij} which in Cartesian coordinates reads $g_{ij} = \mathrm{diag}(c^2, -1, -1, -1)$. The above dimensional analysis shows that $\Omega_0 := 1/\lambda_0$ has the dimension of resistance. Usually, it is called the vacuum impedance.

The Maxwell equations (3) and (10), together with the Maxwell-Lorentz spacetime relation (25), constitute the foundations of classical electrodynamics:

$$d\,{}^\star F = \Omega_0\, J\,, \qquad dF = 0\,. \tag{26}$$

These laws, in the classical domain, are assumed to be of *universal validity*. Only if vacuum polarization effects of quantum electrodynamics are taken care of or if hypothetical nonlocal terms emerge due to huge accelerations, the spacetime relation $H = H(F)$ can pick up corrections yielding a *nonlinear* law.

We would like to stress that the presence of *two* fundamental constants in the Maxwell-Lorentz theory, namely c *and* Ω_0, is, in our opinion, very much underestimated. A simple application can be given in the context of of the possible variation of fundamental physical constants. The possibility of time and space variations of the fundamental constants is discussed in the literature both from an experimental and a theoretical point of view, see [44, 45, 46], for example. Of particular interest are certain indications that the fine structure constant may slowly change on a cosmological time scale. Some authors, see Peres [47, 48], for example, related some experimental evidence of the variability of the fine structure "constant" to the change of the speed of light $c = c(t, x^a)$. However, a closer look at the definition of the fine structure constant

$$\alpha_{\rm f} = \frac{e^2}{2\varepsilon_0\, c\, \mathrm{h}} = \frac{e^2}{2\,\mathrm{h}\,\lambda_0} = \frac{\Omega_0}{2R_{\rm K}} \tag{27}$$

shows, see Tobar [27] and [28], that it is explicitly given in terms of the ratio of two resistances — vacuum impedance Ω_0 and von Klitzing constant $R_{\rm K}$

(the quantum Hall resistance). Note that the speed of light c *disappeared completely!* Tobar [27] demonstrated explicitly that (27) doesn't depend on the system of units. This is inbuilt in our formalism right from its axiomatic beginnings [1].

In other words, the formula (27) demonstrates that of the two fundamental constants of electrodynamics, which appear naturally in Maxwell-Lorentz electrodynamics (see the previous paragraph), it is the vacuum impedance that enters the fine structure constant and *not* the speed of light. Accordingly, a variation of the fine structure constant $\alpha_{\mathrm{f}} = \alpha_{\mathrm{f}}(t)$ would force us to conclude that most probably $\lambda_0 = \lambda_0(t)$. A similar formalism was actually developed by Bekenstein [49], although he inclines to a different physical interpretation of a variable electron charge e. Since a variable e and/or h would yield to charge and/or flux violation, at least one Maxwell equation had to be given up. Since we consider this undesirable, we opt for $\lambda_0 = \lambda_0(t)$.

7 Axiom 5: Local and linear spacetime relation

A local and linear spacetime relation appears to be a reasonable physical assumption in the general setting of the axiomatic approach. Then, the electromagnetic excitation and the field strength are related by the local and linear constitutive law

$$H_{ij} = \frac{1}{2}\,\kappa_{ij}{}^{kl}\,F_{kl}\,. \tag{28}$$

The constitutive tensor κ has 36 independent components. One can decompose this object into its irreducible pieces. Obviously, within the premetric framework, contraction is the only tool for such a decomposition. Following Post [9], we can define the contracted tensor of type $[^1_1]$

$$\kappa_i{}^k := \kappa_{il}{}^{kl}\,, \tag{29}$$

with 16 independent components. The second contraction yields the pseudoscalar function

$$\kappa := \kappa_k{}^k = \kappa_{kl}{}^{kl}\,. \tag{30}$$

The traceless piece

$$\not\kappa_i{}^k := \kappa_i{}^k - \frac{1}{4}\,\kappa\,\delta_i^k \tag{31}$$

has 15 independent components. These pieces can now be subtracted out from the original constitutive tensor. Then,

$$\begin{aligned} \kappa_{ij}{}^{kl} &= {}^{(1)}\kappa_{ij}{}^{kl} + {}^{(2)}\kappa_{ij}{}^{kl} + {}^{(3)}\kappa_{ij}{}^{kl} & (32)\\ &= {}^{(1)}\kappa_{ij}{}^{kl} + 2\,\not\kappa_{[i}{}^{[k}\,\delta_{j]}^{l]} + \frac{1}{6}\,\kappa\,\delta_{[i}^{k}\delta_{j]}^{l}. & (33) \end{aligned}$$

By construction, ${}^{(1)}\kappa_{ij}{}^{kl}$ is the totally traceless part of the constitutive tensor:

$$ {}^{(1)}\kappa_{il}{}^{kl} = 0. \tag{34} $$

Thus, we split κ according to $36 = 20 + 15 + 1$, and the $[{}^2_2]$ tensor ${}^{(1)}\kappa_{ij}{}^{kl}$ is subject to the 16 constraints (34) and carries $20 = 36 - 16$ components.

One may call ${}^{(1)}\kappa_{ij}{}^{kl}$ the principal or the metric-dilaton part of the constitutive law. Without such a term, electromagnetic waves are ruled out, see [1, 37]. We further identify the two other irreducible parts with a *skewon* and an *axion* field, respectively. Conventionally, the skewon and the axion fields are introduced by

$$ \not S_i{}^j = -\frac{1}{2}\not\kappa_i{}^j, \qquad \alpha = \frac{1}{12}\,\kappa. \tag{35} $$

The standard Maxwell-Lorentz electrodynamics (25) arises when both skewon and axion vanish, whereas

$$ {}^{(1)}\kappa_{ij}{}^{kl} = \lambda_0\,\eta_{ij}{}^{kl}. \tag{36} $$

Here $\eta_{ijkl} := \sqrt{-g}\,\hat{\epsilon}_{ijkl}$ and $\eta_{ij}{}^{kl} = \eta_{ijmn}\,g^{mk}g^{nl}$.
Along with the original κ-tensor, it is convenient to introduce an alternative representation of the constitutive tensor, see Post [8]:

$$ \chi^{ijkl} := \frac{1}{2}\,\epsilon^{ijmn}\,\kappa_{mn}{}^{kl}. \tag{37} $$

Substituting (33) into (37), we find the decomposition

$$ \chi^{ijkl} = {}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}\,, \tag{38} $$

again with principal, skewon, and axion pieces. They are determined by

$$ {}^{(1)}\chi^{ijkl} = \frac{1}{2}\,\epsilon^{ijmn}\;{}^{(1)}\kappa_{mn}{}^{kl}, \tag{39} $$

$$ {}^{(2)}\chi^{ijkl} = \frac{1}{2}\,\epsilon^{ijmn}\;{}^{(2)}\kappa_{mn}{}^{kl} = -\,\epsilon^{ijm[k}\not\kappa_m{}^{l]}, \tag{40} $$

$$ {}^{(3)}\chi^{ijkl} = \frac{1}{2}\,\epsilon^{ijmn}\;{}^{(3)}\kappa_{mn}{}^{kl} = \frac{1}{12}\,\epsilon^{ijkl}\,\kappa. \tag{41} $$

Using the S-identity and the K-identity derived in [37], we can verify that ${}^{(2)}\chi$ is *skew-symmetric* under the exchange of the first and the second index pair, whereas ${}^{(1)}\chi$ is *symmetric*:

$$ {}^{(2)}\chi^{ijkl} = -\,{}^{(2)}\chi^{klij}, \qquad {}^{(1)}\chi^{ijkl} = {}^{(1)}\chi^{klij}. \tag{42} $$

Performing a $(1+3)$-decomposition of covariant electrodynamics [1], we can write H and F as column 6-vectors with the components built from the magnetic and electric excitation 3-vectors $\mathcal{H}_a, \mathcal{D}^a$ and electric and magnetic field strengths E_a, B^a, respectively. Then the linear spacetime relation (28) reads:

$$\begin{pmatrix} \mathcal{H}_a \\ \mathcal{D}^a \end{pmatrix} = \begin{pmatrix} \mathcal{C}^b{}_a & \mathcal{B}_{ba} \\ \mathcal{A}^{ba} & \mathcal{D}_b{}^a \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix} . \tag{43}$$

Here the constitutive tensor is conveniently represented by the 6×6 matrix

$$\kappa_I{}^K = \begin{pmatrix} \mathcal{C}^b{}_a & \mathcal{B}_{ba} \\ \mathcal{A}^{ba} & \mathcal{D}_b{}^a \end{pmatrix} , \qquad \chi^{IK} = \begin{pmatrix} \mathcal{B}_{ab} & \mathcal{D}_a{}^b \\ \mathcal{C}^a{}_b & \mathcal{A}^{ab} \end{pmatrix} . \tag{44}$$

The constitutive 3×3 matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are constructed from the components of the original constitutive tensor as

$$\mathcal{A}^{ba} := \chi^{0a0b} , \qquad \mathcal{B}_{ba} := \frac{1}{4}\, \hat{\epsilon}_{acd}\, \hat{\epsilon}_{bef}\, \chi^{cdef} , \tag{45}$$

$$\mathcal{C}^a{}_b := \frac{1}{2}\, \hat{\epsilon}_{bcd}\, \chi^{cd0a} , \qquad \mathcal{D}_a{}^b := \frac{1}{2}\, \hat{\epsilon}_{acd}\, \chi^{0bcd} . \tag{46}$$

If we resolve with respect to χ, we find the inverse formulas

$$\chi^{0a0b} = \mathcal{A}^{ba} , \qquad \chi^{abcd} = \epsilon^{abe}\, \epsilon^{cdf}\, \mathcal{B}_{fe} , \tag{47}$$

$$\chi^{0abc} = \epsilon^{bcd}\, \mathcal{D}_d{}^a , \qquad \chi^{ab0c} = \epsilon^{abd}\, \mathcal{C}^c{}_d . \tag{48}$$

The contributions of the principal, the skewon, and the axion parts to the above constitutive 3-matrices can be written explicitly as

$$\mathcal{A}^{ab} = -\varepsilon^{ab} - \epsilon^{abc}\, \not{S}_c{}^0 , \tag{49}$$

$$\mathcal{B}_{ab} = \mu^{-1}_{ab} + \hat{\epsilon}_{abc}\, \not{S}_0{}^c , \tag{50}$$

$$\mathcal{C}^a{}_b = \gamma^a{}_b - (\not{S}_b{}^a - \delta^a_b\, \not{S}_c{}^c) + \alpha\, \delta^a_b , \tag{51}$$

$$\mathcal{D}_a{}^b = \gamma^b{}_a + (\not{S}_a{}^b - \delta^b_a\, \not{S}_c{}^c) + \alpha\, \delta^b_a . \tag{52}$$

The set of the symmetric matrices $\varepsilon^{ab} = \varepsilon^{ba}$ and $\mu^{-1}_{ab} = \mu^{-1}_{ba}$ together with the traceless matrix $\gamma^a{}_b$ (i.e., $\gamma^c{}_c = 0$) comprise the principal part ${}^{(1)}\chi^{ijkl}$ of the constitutive tensor. Usually, ε^{ab} is called *permittivity* tensor and μ^{-1}_{ab} *reciprocal permeability* tensor ("impermeability" tensor), since they describe the polarization and the magnetization of a medium, respectively. The magnetoelectric cross-term $\gamma^a{}_b$ is related to the Fresnel-Fizeau effects. The skewon contributions in (49) and (50) are responsible for the electric and magnetic Faraday effects, respectively, whereas skewon terms in (51) and (52) describe optical activity. The particular case of the spatially isotropic skewon, $\not{S}_a{}^b = \frac{s}{2}\, \delta^b_a$, $\not{S}_0{}^c = 0$, $\not{S}_c{}^0 = 0$, was studied first by Nieves and Pal [50], who treated s as a third fundamental constant along with the vacuum impedance Ω_0 and the speed of light c.

8 Light propagation in vacuum

Birefringence effects are usually studied in the geometric optics approximation. It is equivalent to the Hadamard approach, where one studies the propagation of a discontinuity in the first derivative of the electromagnetic field. The basic notions are then the fields of the *wave* covector and the *ray* vector that encode the information about the propagation of a wave in a spacetime with a general constitutive relation.

The crucial observation about the surface of discontinuity S (defined locally by a function Φ such that $\Phi = const$ on S) is that across S the geometric Hadamard conditions are satisfied for the components of the electromagnetic field and their derivatives: $[F_{ij}] = 0$, $[\partial_i F_{jk}] = q_i\, f_{jk}$, $[H_{ij}] = 0$, $[\partial_i H_{jk}] = q_i\, h_{jk}$. Here $q_i := \partial_i \Phi$ is the wave covector. Then using the Maxwell equations (3) and (10) and the constitutive law (28), we find a system of algebraic equations for the jump functions:

$$\chi^{ijkl}\, q_j\, f_{kl} = 0\,, \qquad \epsilon^{ijkl}\, q_j\, f_{kl} = 0\,. \tag{53}$$

Solving the last equation in (53) by means of $f_{ij} = q_i a_j - q_j a_i$, which yields the gauge invariance $a_i \rightarrow a_i + q_i$, we are able to reduce $(53)_1$ eventually to

$$\chi^{ijkl}\, q_j q_k a_l = 0\,. \tag{54}$$

This algebraic system, with the mentioned inbuilt gauge invariance, has a nontrivial solution for a_i only if the determinant of the matrix on the left hand side vanishes. After removing the gauge freedom in the determinant [1], we arrive at the 4-dimensionally invariant *extended Fresnel equation*

$$\mathcal{G}^{ijkl}(\chi)\, q_i q_j q_k q_l = 0\,, \tag{55}$$

with the fourth rank Tamm-Rubilar (TR) tensor density of weight +1 defined by

$$\mathcal{G}^{ijkl}(\chi) := \frac{1}{4!}\,\hat{\epsilon}_{mnpq}\,\hat{\epsilon}_{rstu}\,\chi^{mnr(i}\,\chi^{j|ps|k}\,\chi^{l)qtu}\,. \tag{56}$$

It is totally symmetric, $\mathcal{G}^{ijkl}(\chi) = \mathcal{G}^{(ijkl)}(\chi)$, and thus has 35 independent components.

Different irreducible parts of the constitutive tensor (33) contribute differently to the general Fresnel equation. A straightforward analysis [1, 37] shows that the axion piece drops out completely from the TR-tensor, whereas the two remaining irreducible parts of the constitutive tensor contributes to (56) as follows:

$$\mathcal{G}^{ijkl}(\chi) = \mathcal{G}^{ijkl}({}^{(1)}\chi) + {}^{(1)}\chi^{m(i|n|j}\,\not{S}_m{}^k\,\not{S}_n{}^{l)}\,. \tag{57}$$

Birefringence (or double refraction) is the direct physical consequence of the Fresnel equation. In this case the quartic wave covector surface (55) reduces to the pair of second order light-cones. However, the influence of a skewon field on the Fresnel wave surface is qualitatively different: The characteristic sign of the skewon is the emergence of the specific *holes in the quartic Fresnel surfaces* that correspond to the directions in space along which the wave propagation is damped out completely [29]. This effect is in complete agreement with our earlier conclusion on the dissipative nature of the skewon field [1, 37].

9 No birefringence and the light cone

It has been shown recently [25] that taking the linear spacetime relation for granted, one ends up at a Riemannian lightcone provided one forbids birefringence in vacuum, see also [51]. For this the covariant equation (55) is expanded for the zeroth component q_0 of the 4-wave covector $q = (q_0, q_a)$:

$$M_0\, q_0^4 + M_1\, q_0^3 + M_2\, q_0^2 + M_3\, q_0 + M_4 = 0\,, \tag{58}$$

where the coefficients M_i are homogeneous functions of degree i in the spatial components q_a. Due to Ferrari (1545), the four solutions of the quartic equation (58) can be written as

$$q^{\uparrow}_{0(1)} = \sqrt{\alpha} + \sqrt{\beta + \frac{\gamma}{\sqrt{\alpha}}} - \delta\,, \qquad q^{\uparrow}_{0(2)} = \sqrt{\alpha} - \sqrt{\beta + \frac{\gamma}{\sqrt{\alpha}}} - \delta\,, \tag{59}$$

$$q^{\downarrow}_{0(1)} = -\sqrt{\alpha} + \sqrt{\beta - \frac{\gamma}{\sqrt{\alpha}}} - \delta\,, \qquad q^{\downarrow}_{0(2)} = -\sqrt{\alpha} - \sqrt{\beta - \frac{\gamma}{\sqrt{\alpha}}} - \delta\,, \tag{60}$$

where $\alpha, \beta, \gamma, \delta$ depend on the coefficients M_i [25].
Vanishing birefringence means that there is only one future and only one past directing light cone. There are two possibilities, namely

$$q^{\uparrow}_{0(1)} = q^{\uparrow}_{0(2)}\,, \qquad q^{\downarrow}_{0(1)} = q^{\downarrow}_{0(2)}\,, \qquad \text{i.e.,} \qquad \beta = \gamma = 0\,, \tag{61}$$

$$q^{\uparrow}_{0(1)} = q^{\downarrow}_{0(1)}\,, \qquad q^{\uparrow}_{0(2)} = q^{\downarrow}_{0(2)}\,, \qquad \text{i.e.,} \qquad \alpha = \gamma = 0\,. \tag{62}$$

Accordingly, the quartic wave surface in these cases reads

$$[(q_0 - q_0^{\uparrow})(q_0 - q_0^{\downarrow})]^2 = 0\,, \tag{63}$$

or, explicitly, dropping the square,

$$g^{ij} q_i q_j := q_0^2 + \frac{1}{2}\frac{M^a}{M} q_0 q_a + \frac{1}{8}\Big(4\frac{M^{ab}}{M} - \frac{M^a M^b}{M^2}\Big) q_a q_b = 0\,, \tag{64}$$

where M^a, M^{ab} are constructed in terms of the components of χ^{ijkl}, see [1]. Since this relation describes an unique light cone, it should be understood, up to a scalar factor, as an operational definition of a Riemannian metric.

From the condition of the existence of a unique solution (or from hyperbolicity, see Beig [34]), eq.(64) has to possess two real solutions for any given q_a. As a consequence, the signature of the metric is Lorentzian. Accordingly, the signature of the metric is a consequence of the existence of a unique real solution of the Maxwell equations in a future causal cone for arbitrary sources with compact support.

An alternative treatment of the correlation between the linear constitutive relations (28) and the signature is given by two of us in [22]. In order to provide a physical interpretation of the 4-dimensional quantities, we construct their $(1+3)$-decompositions with a number of free sign factors. For the current, we take

$$J = i_{\mathrm{T}}\, j \wedge d\sigma + i_{\mathrm{S}}\, \rho \tag{65}$$

and for the electromagnetic field

$$H = h_{\mathrm{T}}\, \mathcal{H} \wedge d\sigma + h_{\mathrm{S}}\, \mathcal{D}\,, \qquad F = f_{\mathrm{T}}\, E \wedge d\sigma + f_{\mathrm{S}}\, B\,. \tag{66}$$

We introduced here the **T**ime and **S**pace factors $i_{\mathrm{T}}, i_{\mathrm{S}}, h_{\mathrm{T}}, h_{\mathrm{S}} f_{\mathrm{T}},, f_{\mathrm{S}} = \pm 1$. For all possible signatures of the 4-dimensional metric, we obtain the expressions for the electric and magnetic energy densities that correspond to (22). The metric of a Lorentzian type turns out to be related to a positive electromagnetic energy density. This result does not depend on the values of the sign factors.

We analyzed the $(1+3)$-decompositions of the field equations, and we derived which sign factors are conventional and which do depend on the signature. We find that the electric charge has two possible signs for all signatures. For all signatures, we determine the features of the interactions between charges and between currents. Only for a metric with a Lorentzian signature we have ordinary Maxwell-Lorentz electrodynamics. In particular, it yields attraction between opposite charges and repulsion between charges of the same sign: this is *Dufay's law.* Also the magnetic force has a correct sign. In particular, it is responsible for the pulling of a ferromagnetic core into a solenoid independently of the direction of the current, in accordance with *Lenz's rule.* That is, we show in the metric-free approach that positive electromagnetic energy density together with the correct signs in Dufay's and Lenz's rules correspond to a Lorentzian signature of the metric.

10 Axion electrodynamics and the CFJ model

When the skewon is trivial, $\not S_i{}^j = 0$, the structure of the electromagnetic theory simplifies greatly. If, furthermore, the principal part has the form (36), we end up with the linear constitutive law $H = \lambda_0 {}^\star F + \alpha F$. This framework is called axion (Maxwell-Lorentz) electrodynamics, see Ni [52, 53, 54] and Wilczek [30], e.g.:

$$\lambda_0\, d^\star F + (d\alpha) \wedge F = J\,, \qquad dF = 0\,. \tag{67}$$

It is as if the current J picked up an additional piece depending on the gradient of the axion field. In tensor calculus, we have for the inhomogeneous Maxwell equation $\lambda_0\, \partial_j(\sqrt{-g}\, F^{ij}) + \epsilon^{ijkl}(\partial_j\alpha)F_{kl} = \check{J}^i$, with $\check{J}^i = \epsilon^{ijkl}J_{jkl}/6$.

As a degenerate special case, we can also consider the pure ("stand-alone") axion field with ${}^{(1)}\kappa_{ij}{}^{kl} = {}^{(2)}\kappa_{ij}{}^{kl} = 0$. Then,

$$H = \alpha\, F \qquad \text{or} \qquad \begin{cases} \mathcal{H} = -\alpha\, E\,, \\ \mathcal{D} = \quad \alpha\, B\,, \end{cases} \tag{68}$$

and the Maxwell equations read

$$(d\alpha)\wedge F = J \qquad \text{and} \qquad dF = 0\,. \tag{69}$$

This is a special case of axion electrodynamics, namely (67) with $\lambda_0 = 0$. Historically, the first person to discuss (and to reject) a constant pure axion field was Schrödinger [55], p.25, penultimate paragraph, and, as a non-constant field, Dicke [56]. The framework (68),(69) in fact corresponds to Tellegen's *gyrator* [57, 58] and to Lindell & Sihvola's *perfect electromagnetic conductor* (PEMC) [32, 33].

A further specialization of the axion electrodynamics is possible when the covector $\nu := d\alpha = \nu_i\, dx^i$ has constant components. In the cosmological context, this yields the Lorentz-violating *CFJ-electrodynamics,* see Carroll, Field, and Jackiw [23].

Even though in the extended Fresnel equation (55) the birefringence effect is independent of the axion field, such a behavior was discovered in [23]. In order to resolve this problem, we consider the standard wave ansatz

$$F = f\, e^{i\varphi}\,, \tag{70}$$

where i is the imaginary unit and $\varphi = \varphi(x^k)$, while f is a constant 2-form. We denote the wave covector as $q = d\varphi = q_i\, dx^i$. For the ansatz (70), the components of the excitation 2-form for the CFJ case become

$$\check{H}^{kl} = \chi^{klmn}\, f_{mn}\, e^{i\varphi}\,, \tag{71}$$

with $\check{H}^{kl} := \epsilon^{klmn}H_{mn}/2$. In contrast to the Hadamard method, the amplitude of $\check{H}^{kl}$ is not a constant, even if f_{mn}, the amplitude of F_{mn}, is a constant. Substituting (70) and (71) into the field equations and putting the current J to zero, we obtain a system of 8 linear equations

$$\epsilon^{ijkl}q_j f_{kl} = 0\,, \qquad \Big(\chi^{ijkl}q_j - i(\partial_j\chi^{ijkl})\Big)f_{kl} = 0 \tag{72}$$

for 6 independent variables f_{kl}. For the special case appearing in the CFJ-model, the constitutive tensor involves a principal part ${}^{(1)}\chi^{ijkl}$ like in conventional vacuum electrodynamics and the variable axion part ${}^{(3)}\chi^{ijkl} =$

$\alpha(x^m)\epsilon^{ijkl}$. Following the procedure given in [1], we rewrite this equation in the covariant form

$$\mathcal{G}^{ijkl}(\chi)\, q_i q_j q_k q_l - \chi^{ijkl}(\partial_i\alpha)(\partial_l\alpha)\, q_j q_k = 0\,. \tag{73}$$

Substituting here the CFJ constitutive tensor, we obtain ($\nu_i = \partial_i\alpha$)

$$(q_i q^i)^2 - (\nu_i q^i)^2 + (\nu_i \nu^i)(q_j q^j) = 0\,. \tag{74}$$

Finally, if we choose $\alpha = \mu\tau$, with μ as a constant and τ as proper time, we find $\nu_i = (\mu, 0, 0, 0)$ and, together with $q_i = (\omega, \mathbf{q})$,

$$(\omega^2 - \mathbf{q}^2)^2 - \mu^2\mathbf{q}^2 = 0\,, \tag{75}$$

which coincides with the CFJ dispersion law [23].

In fact we have here two different types of the birefringence effects: (i) The premetric birefringence is generated by the algebraic structure of the constitutive tensor. (ii) The CFJ birefringence is generated by derivatives of the constitutive tensor.

11 Axiom 6: Splitting of the electric current

In order to discuss the electrodynamics of continuous media, we need some further input. The crucial point is as follows: The total current density is the sum of the two contributions originating " from the inside" of the medium (which is interpreted as a bound or material charge) and " from the outside" (which is free or external charge):

$$J = J^{\rm mat} + J^{\rm ext}. \tag{76}$$

Here, the bound electric current inside matter is denoted by *mat* and the external current by *ext*. The same notational scheme is also applied to the excitation H, so we have $H^{\rm mat}$ and $H^{\rm ext}$.

Bound charges and bound currents are inherent characteristics of matter determined by the medium itself. They only emerge *inside* the medium. In contrast, free charges and free currents in general appear outside and inside matter. They can be prepared for a specific purpose by a suitable experimental arrangement (a beam of charged particles, say, and scatter them at the medium), or we could study the reaction of a medium in response to a prescribed configuration of charges and currents, $J^{\rm ext}$.

Furthermore, we assume that the charge bound by matter fulfills the usual charge conservation law separately:

$$d\, J^{\rm mat} = 0. \tag{77}$$

We call (76) together with (77) *Axiom 6*. It specifies the properties of the classical material medium. In view of the relation $dJ = 0$, resulting from the first axiom, the assumption (77) means that there is no physical exchange (or conversion) between the bound and free charges. Although the sixth axiom certainly does not exhaust all possible types of material media, it is valid for a sufficiently wide class of media.

Analogously to the Maxwell equation (3), which was derived from the conservation law (1), we introduce by means of (77) the excitation $H^{\rm mat}$ as a “potential” for the bound current:

$$J^{\rm mat} = d\, H^{\rm mat}\,. \tag{78}$$

The $(1+3)$-decomposition, following the pattern of (4), yields

$$H^{\rm mat} = -\mathcal{H}^{\rm mat} \wedge d\sigma + \mathcal{D}^{\rm mat}. \tag{79}$$

The conventional names for these newly introduced excitations are *polarization* 2-form P and *magnetization* 1-form M, i.e.,

$$\mathcal{D}^{\rm mat} \equiv -\, P\,, \qquad \mathcal{H}^{\rm mat} \equiv M\,. \tag{80}$$

The minus sign is chosen in accordance with the usual behavior of paramagnetic matter. Then, by using this in (77), we find, in analogy to the inhomogeneous Maxwell equations (5),

$$-\, \underline{d}\, P = \rho^{\rm mat}\,, \qquad \underline{d}\, M + \dot{P} = j^{\rm mat}. \tag{81}$$

The identifications (80) are only true up to an exact form. However, the uniqueness is guaranteed if we require $\mathcal{D}^{\rm mat} = 0$ for $E = 0$ and $\mathcal{H}^{\rm mat} = 0$ for $B = 0$, see [1].

In order to finalize the scheme, we define the *external excitation*

$$\mathbb{H} := H - H^{\rm mat} \quad \begin{cases} [1995/01/05v2.2eAMSfontdefinitions]\mathfrak{D} := \mathcal{D} - \mathcal{D}^{\rm mat} = \mathcal{D} + P \\ \mathfrak{H} := \mathcal{H} - \mathcal{H}^{\rm mat} = \mathcal{H} - M \end{cases} \tag{82}$$

The external excitation $\mathbb{H} = (\mathfrak{H}, \mathfrak{D})$ can be understood as an auxiliary quantity. When we differentiate (82) and eliminate dH and $dH^{\rm mat}$ by (3) and (78), respectively, we find, making use of (76), the *inhomogeneous* Maxwell equation for matter:

$$d\mathbb{H} = J^{\rm ext} \quad \begin{cases} \underline{d}\,\mathfrak{D} = \rho^{\rm ext}\,, \\ \underline{d}\,\mathfrak{H} - \dot{\mathfrak{D}} = j^{\rm ext}. \end{cases} \tag{83}$$

In Maxwell-Lorentz electrodynamics, we obtain from (82) and the universal spacetime relation (25) the expressions

$$\mathfrak{D} \;=\; \varepsilon_0\, \underline{{}^*}E \,+\, P(E,B)\,, \tag{84}$$

$$\mathfrak{H} \;=\; \frac{1}{\mu_0}\, \underline{{}^*}B - M(B,E)\,. \tag{85}$$

The polarization $P(E, B)$ is a functional of the electromagnetic field strengths E and B. In general, it can depend also on the temperature T and possibly on other thermodynamic variables specifying the material continuum under consideration; similar remarks apply to the magnetization $M(B, E)$. The system (83) looks similar to the Maxwell equations (5). However, the equations in (83) refer only to the external fields and sources. We stress that the *homogeneous* Maxwell equation *remains valid* in its original form.

12 Coupling of electrodynamics to gravity

The coupling between the electromagnetic and the gravitational field is an age-old problem. It is already related to the first observable prediction of GR about the bending of light rays of stars in the gravitational field of the Sun. The electromagnetic and gravitational effects are of rather different orders of magnitude. However, the increasing precision of modern experimental techniques gives rise to the hope that the appropriate form of the coupling can soon be determined.

In particular, we have in this context two independent but closely related problems:

(i) How does the gravitational field of a massive source depend on its electric charge?

(ii) How does the electromagnetic field of a charged massive source change when the gravity is "switched on"?

In most cases, the coupling between two fields can be represented by a specific term in the total action. This term has to respect the *diffeomorphism* invariance related to gravity as well as the *gauge* invariance of electrodynamics. Moreover, it is reasonable to require the main facts of both theories (the conservation of energy-momentum, of electric charge, and of magnetic flux) to be preserved in the modified model. Although, coordinate and gauge invariance strongly restrict the variety of admissible coupling terms, we have still an infinity set of possibilities. Indeed, we can always construct a polynomial of a chosen coupling term. This new term can also serve as an admissible additional piece of the action. Hence we will restrict ourselves to the admissible *parity conserving* coupling terms of the lowest order, see also [18].

Our main result will be that *all* such lowest order modifications of the standard Einstein-Maxwell system are completely embedded in the axiomatic approach to electrodynamics.

12.1 Coupling of electrodynamics to Einsteinian gravity

In the framework of GR, the coupling between the electromagnetic field and gravity is managed by the electromagnetic action itself

$$S(g, F) = -\frac{\lambda_0}{2}\int {}^\star F \wedge F = -\frac{\lambda_0}{4}\int F_{ij}F^{ij}\sqrt{-g}\,d^4x\,. \tag{86}$$

Here $g = g_{ij}\, dx^i \otimes dx^j = o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta$ is the metric tensor, whereas the 2-form $F = F_{ij} dx^i \wedge dx^j/2 = F_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta/2$ is the electromagnetic field strength. The typical form of the coupling Lagrangian (86) is $L(g, F) \sim (g^2 \cdot F^2)$. Here and later, we use the notation $(\_ \cdot \_)$ for a summation that is evaluated by contracting the indices.

If one adds to (86) the actions of the gravitational and the matter fields, then variation with respect to the metric yields the Einstein field equation (without cosmological constant)[1]

$$\mathrm{Ric}_{ij} - \frac{1}{2}\, R g_{ij} = \frac{8\pi G}{c^3} ({}^{(\mathtt{em})}T_{ij} + {}^{(\mathtt{mat})}T_{ij})\,, \tag{87}$$

with the Ricci tensor $\mathrm{Ric}_{ij} := R_{kij}{}^k$ and the curvature scalar $R := g^{ij}\mathrm{Ric}_{ij}$. The electromagnetic and the material energy-momentum tensors act as sources of the gravitational field.

The Reissner-Nordström solution of the Einstein-Maxwell equations

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2\,, \tag{88}$$

with $m = GM/c^2$ and $q^2 = GQ^2/(4\pi\varepsilon_0 c^4)$, describes a mass M with an electric charge Q. The electromagnetic field strength F is involved in (86) only algebraically, that is, without derivatives. Hence the inhomogeneous Maxwell equation can only be derived by a variation taken with respect to the potential A.

Let us turn then to a discussion of the potential. In the conventional textbook approach, see Landau and Lifshitz [59] and Misner, Thorne, and Wheeler [60], the *components* A_i of the electromagnetic potential 1-form $A = A_i\, dx^i$ are taken as field variables. The recipe for the "minimal" transition from SR to GR is then the "comma goes to semicolon rule" [60]; the comma "," denotes the partial, the semicolon ";" the covariant derivative. For the *components* F_{ij} of the field strength F we find then

$$F_{ij} = A_{j;i} - A_{i;j} = A_{j,i} - A_{i,j}\,, \tag{89}$$

since the Levi-Civita connection is symmetric. The relation between the components F_{ij} and A_i is then eventually recognized as independent of the metric structure. Needless to say that such a procedure is highly coordinate dependent.

In contrast, in our axiomatic approach the potential 1-form A, as (coordinate independent) *geometrical object*, is the field variable of our choice.

[1] In exterior calculus, we vary with respect to the orthonormal coframe ϑ^α and find $\frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} = \frac{8\pi G}{c^3}({}^{(\mathtt{em})}\Sigma_\alpha + {}^{(\mathtt{mat})}\Sigma_\alpha)$. Here $\eta_{\alpha\beta\gamma} = {}^\star(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$, $R^{\alpha\beta}$ is the curvature 2-form of the Riemannian spacetime, and G Newton's gravitational constant. An energy-momentum 3-form translates into an energy-momentum tensor according to $\Sigma_\alpha = T_\alpha{}^\beta \eta_\beta$, with $\eta_b = {}^\star\vartheta_\beta$.

From magnetic flux conservation we find directly

$$F = dA\,, \tag{90}$$

without messing around with covariant derivatives. In this coordinate-independent way we recognize right away that there is no chance for the metric to intervene in $F = dA$. Clearly, a coordinate *in*dependent procedure is to be preferred against a coordinate dependent one.

The conventional story then continues as follows: We "semicolonize" the Maxwell equations in order to make them fit for survival in the curved pseudo-Riemannian spacetime of GR,

$$\begin{cases} F_{ij;k} + F_{jk;i} + F_{ki;j} = 0\,, \\ F^{ij}{}_{;j} = \Omega_0\, J^i\,. \end{cases} \tag{91}$$

In the first equation the metric tensor drops out, again because of the symmetry of the Levi-Civita connection:

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0\,. \tag{92}$$

It is not sensitive to the gravity field of the source. The second equation can be rewritten as

$$(\sqrt{-g}g^{i[k}g^{l]j}F_{kl})_{,j} = \Omega_0\,\sqrt{-g}J^i = \Omega_0\,\check{J}^i\,. \tag{93}$$

Hence the gravitational field is involved in the inhomogeneous Maxwell equation only via the metric dependent expression within the parenthesis and via the determinant of the metric tensor. Incidentally, since for the Reissner-Nordström solution the determinant is the same as for a flat manifold, the electromagnetic field is not sensitive to the mass of the point charge.

In our axiomatic approach, see (26), we arrive directly at

$$\begin{cases} dF = 0\,, \\ d\,{}^\star F = \Omega_0\, J\,, \end{cases} \tag{94}$$

without discussing covariant derivatives. The Hodge star is known to depend only on the conformal part of the metric, see Frankel [40]. If we identify the currents according to $J^i = \sqrt{-g}\epsilon^{ijkl}J_{jkl}/6$, here $J = J_{ijk}\,dx^i \wedge dx^j \wedge dx^k/6$ is the current 3-form, then the endresults of both procedures coincide; compare (92),(93) with (94).

The Maxwell equations in tensor notation, if altered in the presence of the gravitational field, can be rearranged by a change of the basic notation. In accordance with (25), the electromagnetic excitation is linearly related to the field strength:

$$\check{H}^{ij} = \frac{1}{2}\,\chi^{ijkl}F_{kl}\,. \tag{95}$$

The constitutive tensor reads

$$\chi^{ijkl}(g) = \lambda_0 \sqrt{-g}(g^{ik}g^{jl} - g^{il}g^{jk})\,. \tag{96}$$

Then the electromagnetic action (86) takes the form

$$S(g,F) \;=\; -\frac{1}{4}\int F_{ij}\check{H}^{ij}d^4x = -\frac{1}{2}\int F\wedge H\,, \tag{97}$$

whereas the field equations in tensor analytical and in exterior form notations become, respectively,

$$\begin{cases} F_{ij,k} + F_{jk,i} + F_{ki,j} = 0\,, \\ \check{H}^{ij}{}_{,j} = \check{J}^i\,, \end{cases} \qquad \text{or} \qquad \begin{cases} dF = 0\,, \\ dH = J\,. \end{cases} \tag{98}$$

Consequently, in the Einstein-Maxwell system, the coupling is completely determined by the constitutive tensor (96). This coupling is referred to as *minimal coupling.* Technically, the minimal coupling is provided in the conventional coordinate dependent textbook approach by the substitution of the partial by the covariant derivatives (taken with respect to the Levi-Civita connection) and in the axiomatic exterior calculus approach by Hodge staring the field strength F.

Summing up we can state that the *minimal coupling* between electrodynamics and gravity is provided by (26) [or (98) with (95)]. The knowledge about the Riemannian structure of spacetime is fed into the Hodge star operator via the metric. In SR a flat metric appears, in GR a curved one.

Nonminimal coupling

If for some reason we believe that the minimal coupling between electrodynamics and gravity is insufficient and has to be generalized, then we have to modify the action (86) by adding so-called nonminimal terms. Still, we have to respect diffeomorphism and gauge invariance. Riemannian geometry is characterized by the curvature tensor $R_{ijk}{}^{l}$ and its traces Ric_{ij} and R. Terms proportional to the electromagnetic potential are forbidden because of gauge invariance. The term linear in F vanishes as a product of a symmetric and an antisymmetric tensors. Consequently the lowest order nonminimal Lagrangian reads

$$L(F,R) = \alpha_1 F_{ij}F^{kl}R^{ij}{}_{kl} + \alpha_2 F_{ik}F^{jk}\mathrm{Ric}^i{}_j + \alpha_3 F_{ij}F^{ij}R\,. \tag{99}$$

Let us recall several arguments in favor of a such non-minimal couplings:

(i) For the values $\alpha_1 = \alpha_3 = -4\alpha_2$, eq.(99) can be recovered from the 5-dimensional Gauss-Bonnet action [61, 62, 63, 64].

(ii) Another special set of parameters in (99)

$$\alpha_1 = -\frac{e^2\lambda_{\mathrm{c}}^2}{720\pi\mathrm{h}}\,, \quad \alpha_2 = \frac{13e^2\lambda_{\mathrm{c}}^2}{720\pi\mathrm{h}}\,, \quad \alpha_3 = -\frac{e^2\lambda_{\mathrm{c}}^2}{288\pi\mathrm{h}}\,, \tag{100}$$

were derived in the one-loop approximation of quantum electrodynamics (QED) [65, 66]. Here e is the elementary charge, $\mathrm{h} = 2\pi\hbar$ Planck's constant, and $\lambda_{\mathrm{c}} = \hbar/(mc)$ the Compton wavelength of the electron.

(iii) The Lagrangian (99) was used as a basis for models with a variable speed of light [67, 68, 69]. Such an effect is present in all models, except for the cases $\alpha_1 = \alpha_2 = 0$.

The constitutive tensor corresponding to the Lagrangian $L(F, g) + L(F, R)$ depends on the metric and the curvature tensor and its contractions:

$$\begin{aligned}\chi^{ijkl}(g,R) \;=\; & (1-4\alpha_3 R)(g^{ik}g^{jl} - g^{il}g^{jk}) - 4\alpha_1 R^{ijkl} \\ & - \alpha_2(g^{ik}\mathrm{Ric}^{jl} - g^{il}\mathrm{Ric}^{jk} + g^{jl}\mathrm{Ric}^{ik} - g^{jk}\mathrm{Ric}^{il})\,. \end{aligned} \tag{101}$$

Since (101) is derived from a Lagrangian, its skewon part (40) is zero. Also the completely antisymmetric axion part (41) vanishes due to the symmetries of R_{ijkl}. Thus, in (101) only the principal part ${}^{(1)}\chi^{ijkl}(g,R)$ is left over. Substituting (101) into the 3×3 constitutive matrices (45),(46), we obtain $\mathcal{A} = \mathcal{A}^T$, $\mathcal{B} = \mathcal{B}^T$, $\mathcal{C} = \mathcal{D}^T$. The α^1 and α^2 terms yield birefringence in general.

12.2 Maxwell's field coupled to Einstein-Cartan gravity

In Einstein-Cartan gravity, see Blagojević [70], e.g., the spacetime geometry is of the Riemann-Cartan type and as such on two fundamental structures — the metric tensor g_{ij} and the connection $\Gamma_{ij}{}^k$. These quantities satisfy the metricity condition

$$g_{ij;k} = g_{ij,k} - \Gamma_{ki}{}^l g_{lj} - \Gamma_{kj}{}^l g_{il} = 0\,. \tag{102}$$

If the connection is not symmetric, then the torsion tensor $T_{ij}{}^k$ is involved. In holonomic coordinates, it is defined by $T_{ij}{}^k = 2\Gamma_{[ij]}{}^k$. In exterior calculus, we have the 2-form $T^\alpha = D\vartheta^\alpha = T_{ij}{}^\alpha\, dx^i \wedge dx^j/2$, with the exterior covariant derivative D. Accordingly, a possible interaction of torsion with the electromagnetic field is of special interest [71, 72, 73, 74, 75, 76, 17, 18, 19].

The *minimal* way of coupling the electromagnetic field with Einstein-Cartan gravity works the same way as with Einstein gravity. Since neither the exterior derivative d nor the Hodge star * can feel torsion directly, we have again the minimal equations (26):

$$d\,{}^\star F = \Omega_0\, J\,, \qquad dF = 0\,. \tag{103}$$

They will do the job.

The conventional coordinate dependent textbook approach led to numerous misunderstandings. If one applies the semicolon rule to the special relativistic formula $F_{ij} = A_{j,i} - A_{i,j}$, then one arrives at

$$A_{j;i} - A_{i;j} = A_{j,i} - A_{i,j} - T_{ij}{}^k A_k = F_{ij} - T_{ij}{}^k A_k\,. \tag{104}$$

This expression is diffeomorphism invariant. However, it is *not* gauge invariant under $A_i \to A_i + \partial_i \phi$. Consequently, it has to be *rejected.* As we saw in the last paragraph, the axiomatic exterior calculus approach leaves the definition $F = dA$ intact, and there is no need, nor is it allowed, to entertain in covariant derivatives in this context.

Accordingly, the *free* Maxwell Lagrangian is added to the gravitational Lagrangian with the unchanged field strength. Nevertheless, in an exact solution of a gravity-Maxwell system, the torsion may depend on the electrical charge, as is exemplified by the Reissner-Nordström solution with torsion in the framework of a Poincaré gauge theory [77]. In other words, torsion is influenced by the electric charge *in*directly via the field equations. What in the Lagrangian looks like "no interaction at all," still yields an effective "minimal" interaction.

Nonminimal coupling

We are now looking for a complete family of nonminimally coupled Maxwell-torsion Lagrangians that are diffeomorphism and gauge invariant. Because of gauge invariance, the potential A_i must not appear in the action. Moreover, all expressions have to contain an even number of indices in order, if contracted, to yield a scalar. The expressions linear in the field strength are of the form $\big(F \cdot (g^2 \cdot T^2)\big)$. Although such terms yield a family of diffeomorphism and gauge invariant Lagrangians, they have to be rejected from a physical point of view. Indeed, on the level of the field equation, the Lagrangians linear in F admit the existence of a global electromagnetic field even without charges. Such a modification of classical electrodynamics seems to be unwarranted.

Hence the lowest order addenda to the Lagrangian are quadratic in F_{ij} and quadratic in torsion $T_{ij}{}^k$, i.e., of the typical form $F^2 \cdot g^3 \cdot T^2$. The contractions can be performed in the following three ways:

(i) All indices of the F-pair and of the T-pair are contracted separately,

$$\boxed{(F^2 \cdot g^2)(g \cdot T^2)\,,}$$

or, explicitly:

$$
\begin{aligned}
{}^{(1)}L(F,T) &= F_{ij}F^{ij}\, T_{mnk}T^{mnk}\,, \quad {}^{(2)}L(F,T) = F_{ij}F^{ij}\, T_{mnk}T^{nkm}\,,\\
{}^{(3)}L(F,T) &= F_{ij}F^{ij}\, T_{mn}{}^{n}T^{mk}{}_{k}\,.
\end{aligned}
\tag{105}
$$

(ii) Two free indices of the F-pair are contracted with two free indices of the T-pair, i.e.,

$$\boxed{(F^2 \cdot g)\cdot(g^2 \cdot T^2)\,.}$$

Such terms are

$$
\begin{aligned}
{}^{(4)}L(F,T) &= F_{im}F_j{}^{m}\, T^{inj}T_{nk}{}^{k}\,, \quad {}^{(5)}L(F,T) = F_{im}F_j{}^{m}\, T^{ikn}T^{j}{}_{kn}\,,\\
{}^{(6)}L(F,T) &= F_{im}F_j{}^{m}\, T^{ikn}T^{j}{}_{nk}\,, \quad {}^{(7)}L(F,T) = F_{im}F_j{}^{m}\, T^{ikn}T_{kn}{}^{j}\,,\\
{}^{(8)}L(F,T) &= F_{im}F_j{}^{m}\, T^{ik}{}_{k}T^{jn}{}_{n}\,.
\end{aligned}
\tag{106}
$$

(iii) The F-pair and the T-pair have four free indices:

$$\boxed{(F^2)\cdot(g^3\cdot T^2)\,,}$$

$$\begin{aligned}
{}^{(9)}L(F,T) &= F_{ij}F_{kl}\,T^{im}{}_{m}T^{jkl}\,, & {}^{(10)}L(F,T) &= F_{ij}F_{kl}\,T^{im}{}_{m}T^{klj}\,,\\
{}^{(11)}L(F,T) &= F_{ij}F_{kl}\,T_{m}{}^{ij}T^{mkl}\,, & {}^{(12)}L(F,T) &= F_{ij}F_{kl}\,T_{m}{}^{ik}T^{mlj}\,,\\
{}^{(13)}L(F,T) &= F_{ij}F_{kl}\,T_{m}{}^{il}T^{mjk}\,, & {}^{(14)}L(F,T) &= F_{ij}F_{kl}\,T^{mij}T^{kl}{}_{m}\,,\\
{}^{(15)}L(F,T) &= F_{ij}F_{kl}\,T^{mik}T^{lj}{}_{m}\,, & {}^{(16)}L(F,T) &= F_{ij}F_{kl}\,T^{ijm}T^{kl}{}_{m}\,,\\
{}^{(17)}L(F,T) &= F_{ij}F_{kl}\,T^{ikm}T^{lj}{}_{m}\,. & & \qquad (107)
\end{aligned}$$

Summing up, the general torsion Lagrangian reads

$$\frac{1}{\ell^2}\,L(F,T) = -\frac{1}{8}\sum_{i=1}^{17}\beta_i\,{}^{(i)}L(F,T)\,, \tag{108}$$

with the dimensionless constants β_i.
A special Lagrangian of type (108), namely

$${}^{\star}(T_\alpha\wedge F)\,T^\alpha\wedge F = \frac{1}{4}\,\left({}^{(1)}L - 4\,{}^{(5)}L + {}^{(16)}L\right)\,{}^{\star}1 \tag{109}$$

was considered recently [16, 17] in the context of a cosmological test of Einstein's equivalence principle.
Since we start with a modified Lagrangian, a skewon piece does not occur in the corresponding constitutive tensor. Thus

$$\chi^{ijkl}(g,T) = \ell^2\Big[\sum_{k\cdots q}\Big(T_{mnp}T_{qrs}\Big)\Big]^{[ij][kl]} = \widetilde{\chi}^{ijkl}(g,T) + \check{\chi}^{ijkl}(g,T)\,, \tag{110}$$

where $\widetilde{\chi}^{ijkl}(g,T)$ is the modification of the principal part, while the axion part takes the form

$$\check{\chi}^{ijkl}(g,T) = \ell^2\Big[\sum_{m\cdots s}\Big(T_{mnp}T_{qrs}\Big)\Big]^{[ijkl]}\,. \tag{111}$$

Birefringence induced by torsion. For the Lagrangians (105)–(107) we find the following effects on $\widetilde{\chi}^{ijkl}$ and on the light cone (we put $\ell = 1$):

(i) The Lagrangians (105) yield

$$\widetilde{\chi}^{ijkl}(g,T) = S(g^{ik}g^{jl} - g^{il}g^{jk}) = 2S g^{i[k}g^{l]j}\,, \tag{112}$$

where S is a scalar function quadratic in torsion. Therefore, the Tamm-Rubilar tensor density changes only by a conformal factor and the light cone is preserved.

(ii) For (106), we introduce the abbreviation $S^{ij} = S^{ji} := \left[g^2 \cdot T^2\right]^{(ij)}$. The axion field is absent. Thus,

$$\widetilde{\chi}^{ijkl}(g,T) = g^{[i|[k}\, S^{l]|j]}\,. \tag{113}$$

The 3×3 constitutive matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ read:

$$\begin{aligned} A^{ab}(g,T) &:= \widetilde{\chi}^{0b0a}(g,T) = \frac{1}{4}(g^{00}S^{ab} + g^{ab}S^{00})\,, \\ B_{ab}(g,T) &:= \frac{1}{4}\epsilon_{bcd}\epsilon_{aef}\,\widetilde{\chi}^{cdef}(g,T) = \frac{1}{4}\left(g_{ab}S_c{}^c - S_{ab}\right)\,, \\ C^a{}_b(g,T) &:= \frac{1}{2}\epsilon_{bcd}\,\widetilde{\chi}^{cd0a}(g,T) = \frac{1}{4}\epsilon^a{}_{bc}\,S^{0c}. \end{aligned} \tag{114}$$

These matrices, with $C^a{}_b(g,T) = D_b{}^a(g,T)$, obey

$$A(g,T) = A^{\mathrm{T}}(g,T), \quad B(g,T) = B^{\mathrm{T}}(g,T), \quad C(g,T) = D^{\mathrm{T}}(g,T) \tag{115}$$

(T = transposed). These relations, together with the closure condition for $\widetilde{\chi}^{ijkl}(g,T)$, guarantee the uniqueness of the light cone [1]. Thus birefringence does not emerges in this group of models either.

(iii) For the Lagrangians from the third group (107), birefringence is a generic property. This was shown for (109) in the case of spherically symmetric torsion [17, 19].

Axion field induced by torsion. The axion, that is, a pseudo-scalar field, was extensively studied in different contexts of field theory. Its quantized version is believed to provide a solution to the strong CP problem of quantum chromodynamics (QCD). The emergence of axions is a general phenomenon in superstring theory. The axion as a classical field also appears in various discussions of the equivalence principle in gravitational physics and in inflationary models. In the context discussed in this paper, the axion field is introduced in (41) with $(35)_2$ as one irreducible piece of the electromagnetic constitutive tensor of spacetime. According to (111), we may imagine that in the present case the axion field is induced by the nonminimal coupling of the Maxwell field to the torsion of spacetime. Explicitly, we find the following:

(i) The Lagrangians (105) yield

$$\check{\chi}^{ijkl}(g,T) = \ell^2 g^{[ik}g^{jl]}(T \cdot T) = 0\,. \tag{116}$$

(ii) The Lagrangians (106) yield

$$\check{\chi}^{ijkl}(g,T) = \ell^2 g^{[jl}(T \cdot T)^{ik]} = 0\,. \tag{117}$$

(iii) For those of the third group (107), the general form of the axion field $\alpha = \epsilon_{ijkl}\, \check{\chi}^{ijkl}(g,T)/4!$ is

$$\alpha = \ell^2 \epsilon_{ijkl} \Big(\alpha_1 T^{ijm} T^{kl}{}_m + \alpha_2 T^{ijk} T^{lm}{}_m \Big) \,, \tag{118}$$

where α_1, α_2 are free dimensionless parameters, which are linear combinations of the β's. For the special case (109), the axion is extracted from (118) by putting $\alpha_1 = 1, \alpha_2 = 0$.

If in vacuum an axion field α emerges, then the Lagrangian picks up an additional piece $\sim \alpha\, F \wedge F$, see [1]. Accordingly, the Maxwell equations are those displayed in (67). Only a non-constant axion field contributes. The coupling of the Maxwell field to a non-constant axion field, in the case of a plane electromagnetic wave, amounts to a rotation of the polarization vector of the wave, see [78], i.e., the axion field induces an *optical activity*.

12.3 Maxwell's field coupled to metric-affine gravity

Metric-affine gravity (MAG) is based on a spacetime geometry with completely independent metric and connection [79]. Thus, in addition to the torsion tensor, the nonmetricity tensor emerges

$$Q_{kij} = -g_{ij;k} = -g_{ij,k} + \Gamma_{ki}{}^l g_{lj} + \Gamma_{kj}{}^l g_{il} \,. \tag{119}$$

In exterior calculus, we have the nonmetricity 1-form $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, with the decomposition $Q_{\alpha\beta} = Q_{i\alpha\beta} dx^i$. In this framework, we can look for a possible interaction of the nonmetricity with the electromagnetic field. As in Einstein and Einstein-Cartan gravity, the minimal coupling remains untouched, that is, we have again $\lambda_0\, d\,{}^\star F = J$, $dF = 0$.

In *nonminimal coupling,* however, there emerge two additional types of terms, namely $F^2 \cdot g^5 \cdot Q^2$ and $F^2 \cdot g^4 \cdot Q \cdot T$. These coupling terms can be expanded in the following ways:

(i) All indices of the F-pair and of the Q-pair (or of the TQ-pair) are contracted separately:

$$\boxed{(F^2 \cdot g^2)(g^3 \cdot Q^2)\,,} \qquad \text{or} \qquad \boxed{(F^2 \cdot g^2)(g^2 \cdot Q \cdot T)\,.}$$

Examples of such terms can be easily constructed:

$${}^{(1)}L(F,Q) = F_{ij} F^{ij}\, Q_{mnk} Q^{mnk} \,, \quad {}^{(1)}L(F,Q,T) = F_{ij} F^{ij}\, Q_{mnk} T^{mnk} \,. \tag{120}$$

The corresponding constitutive tensors are of the form (S is a scalar)

$$\chi^{ijkl}(g,Q) = (g^{ik} g^{jl} - g^{il} g^{jk}) S \,. \tag{121}$$

Certainly the axion field and the birefringence effect are absent in this group of models.

(ii) Two free indices of the F-pair are contracted with two free indices of the Q-pair (or TQ-pair), i.e.,

$$\boxed{(F^2 \cdot g) \cdot (g^4 \cdot Q^2)\,,} \qquad \text{or} \qquad \boxed{(F^2 \cdot g) \cdot (g^3 \cdot Q \cdot T)\,.}$$

Examples of such Lagrangians are

$$^{(2)}L(F,Q) = F_{ik}F_j{}^k\, Q^m{}_{mn}Q^{ijn}\,, \quad ^{(2)}L(F,T,Q) = F_{ik}F_j{}^k\, T_{mn}{}^n Q^{imj}\,. \tag{122}$$

The corresponding constitutive tensor is of the form

$$\chi^{ijkl}(g,Q) = g^{[i|[k}\, S^{l]|j]}\,, \tag{123}$$

where $S^{ij} = (g^4 \cdot Q^2)^{ij}$ or $S^{ij} = (g^3 \cdot Q \cdot T)^{ij}$. This tensor is necessary symmetric since it multiplies the symmetric tensor $F_{ik}F_j{}^k$. Similarly to the non-minimal coupling to torsion (113-115), the birefringence effect is absent in this group of models. Comparing to (117) we see that also the axion part of this constitutive tensor is zero.

(iii) The F-pair and the Q-pair (or TQ-pair) have four free indices:

$$\boxed{(F^2) \cdot (g^5 \cdot Q^2)\,,} \qquad \text{or} \qquad \boxed{(F^2) \cdot (g^4 \cdot T \cdot Q)\,.}$$

For instance,

$$^{(3)}L(F,Q) = F_{ij}F_{kl}\, Q^{ik}{}_m Q^{jlm}\,, \quad ^{(3)}L(F,T,Q) = F_{ij}F_{kl}\, T^{im}{}_m Q^{kl}{}_i\,. \tag{124}$$

Possessing an axion field and birefringence are generic properties in this group of models.

In MAG, all the non-minimal coupling terms described above are of the same quadratic order in the connection. Therefore, additional Lagrangians depending on curvature, torsion, and nonmetricity have in general to be considered together.

13 Discussion

Although Maxwell's electrodynamics is a firmly established classical field theory, it is still open to new developments along many lines. Furthermore, the study of the fundamental structures of Maxwell's theory can serve as a natural starting point for non-abelian modifications with applications in high-energy physics and gravity. In particular, modern string and brane theories rely heavily on classical field-theoretic structures. In addition, there are important classical problems still awaiting for their solutions, such as the quantization of the electric charge, the existence of magnetic monopoles, and the value of the coupling constant, as well as a number of new open issues: (i) The physical place and the role of the premetric partners of the standard electromagnetic field, such as the axion, the skewon, and the dilaton fields. (ii) The appropriate form of the coupling between the electromagnetic and the gravitational field (also taking into account the possible gauge-theoretic extensions of standard GR, such as Poincaré gravity and, more generally, metric-affine gravity). (iii) The description of a high-energy electromagnetic field in a material medium.

In this lecture we gave an overview of the recent developments of the axiomatic premetric approach based on the conservation of electric charge and of magnetic flux as well as on some additional inputs such as the structure of the energy-momentum. These facts are well established theoretically and tested experimentally. The field variables and the corresponding field equations are then straightforwardly derivable from the basic axioms when certain natural restrictions on the topology of spacetime are assumed. The resulting construction is actually a topological one, in the sense that it does not depend on a specific geometry of the underlying spacetime manifold. Only the additional notion of a constitutive relation brings in the information on the specific structure of the geometry of spacetime. The open problems, indicated above, are directly linked to the investigation of the properties of the constitutive relation. Going beyond the linear and local constitutive law is, in fact, a step which takes into account the quantum nature of the actual matter sources. The corresponding dynamics of the electromagnetic field becomes highly nontrivial. A good recent analysis of the earlier nonlinear electrodynamical models can be found in [35]; specific results on the birefringence in such theories were obtained in [36].

Acknowledgments: One of the authors would like to thank Branko Dragovich, Milutin Blagojević, and Djordje Šijački for the invitation to give lectures in Zlatibor and for their hospitality. He also thanks Milutin Blagojević for many interesting discussions.

References

[1] F.W. Hehl and Yu.N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser: Boston, MA, 2003).

[2] I.V. Lindell, *Differential Forms in Electromagnetics* (IEEE Press: Piscataway, NJ, and Wiley-Interscience, 2004).

[3] P. Russer, *Electromagnetics, Microwave Circuit and Antenna Design for Communications Engingeering* (Artech House: Boston, 2003).

[4] D.H. Delphenich, *On the axioms of topological electromagnetism,* Ann. Phys. (Leipzig) **14** (2005) issue number 6, to be published; updated version of arXiv.org/hep-th/0311256.

[5] D.H. Delphenich, *Complex geometry and pre-metric electromagnetism,* arXiv.org/gr-qc/0412048.

[6] C. Truesdell and R.A. Toupin, *The classical field theories*, in: *Handbuch der Physik*, Vol. III/1, S. Flügge ed. (Springer: Berlin, 1960) pp. 226-793.

[7] R.A. Toupin, *Elasticity and electro-magnetics*, in: *Non-Linear Continuum Theories, C.I.M.E. Conference, Bressanone, Italy 1965.* C. Truesdell and G. Grioli coordinators. Pp.203-342.

[8] E.J. Post, *Formal Structure of Electromagnetics – General Covariance and Electromagnetics* (North Holland: Amsterdam, 1962, and Dover: Mineola, New York, 1997).

[9] E.J. Post, *The constitutive map and some of its ramifications*, Ann. Phys. (NY) **71** (1972) 497-518.

[10] R.M. Kiehn, G.P. Kiehn, and J.B. Roberds, *Parity and time-reversal symmetry breaking, singular solutions, and Fresnel surfaces*, *Phys. Rev.* **A43** (1991) 5665-5671.

[11] A. Kovetz, *Electromagnetic Theory* (Oxford University Press: Oxford, 2000).

[12] G. Kaiser, *Energy-momentum conservation in pre-metric electrodynamics with magnetic charges*, J. Phys. **A37** (2004) 7163-7168.

[13] F.W. Hehl and Y.N. Obukhov, *Electric/magnetic reciprocity in premetric electrodynamics with and without magnetic charge, and the complex electromagnetic field,* Phys. Lett. **A323** (2004) 169-175; arXiv.org/physics/0401083.

[14] D.G.B. Edelen, *A metric free electrodynamics with electric and magnetic charges,* Ann. Phys. (NY) **112** (1978) 366-400.

[15] F.W. Hehl, Yu.N. Obukhov and B. Rosenow, *Is the Quantum Hall Effect influenced by the gravitational field? Phys. Rev. Lett.* **93** (2004) 096804 (4 pages); arXiv.org/cond-mat/0310281.

[16] S.K. Solanki, O. Preuss, M.P. Haugan, et al., *Solar constraints on new couplings between electromagnetism and gravity*, Phys. Rev. **D69** (2004) 062001 (11 pages).

[17] O. Preuss, M.P. Haugan, S.K. Solanki and S. Jordan, *An astronomical search for evidence of new physics: Limits on gravity-induced birefringence from the magnetic white dwarf RE J0317-853,* Phys. Rev. **D70** (2004) 067101 (4 pages); arXiv.org/gr-qc/0405068.

[18] F.W. Hehl and Yu.N. Obukhov, *How does the electromagnetic field couple to gravity, in particular to metric, nonmetricity, torsion, and curvature?* In: *Gyros, Clocks, Interferometers...: Testing Relativistic Gravity in Space.* C. Lämmerzahl et al., eds. Lecture Notes in Physics Vol. **562** (Springer: Berlin, 2001) pp. 479-504; arXiv.org/gr-qc/0001010.

[19] G.F. Rubilar, Yu.N. Obukhov and F.W. Hehl, *Torsion nonminimally coupled to the electromagnetic field and birefringence.* Class. Quant. Grav. **20** (2003) L185-L191; arXiv.org/gr-qc/0305049.

[20] Y. Itin, *Noether currents and charges for Maxwell-like Lagrangians*, J. Phys. A **36** (2003) 8867-8884; arXiv.org/math-ph/0307003.

[21] Y. Itin and F.W. Hehl, *Maxwell's field coupled nonminimally to quadratic torsion: Axion and birefringence.* Phys. Rev. **D68** (2003) 127701 (4 pages); arXiv.org/gr-qc/0307063.

[22] Y. Itin and F.W. Hehl, *Is the Lorentz signature of the metric of spacetime electromagnetic in origin?* Ann. Phys. (NY) **312** (2004) 60-83; arXiv.org/gr-qc/0401016.

[23] S.M. Carroll, G.B. Field, and R. Jackiw, *Limits on a Lorentz- and parity-violating modification of electrodynamics*, Phys. Rev. **D41** (1990) 1231-1240.

[24] Y. Itin, *Caroll-Field-Jackiw electrodynamics in the pre-metric framework,* Phys. Rev. **D70** (2004) 025012 (6 pages); arXiv.org/hep-th/0403023.

[25] C. Lämmerzahl and F. W. Hehl, *Riemannian light cone from vanishing birefringence in premetric vacuum electrodynamics,* Phys. Rev. **D70** (2004) 105022 (7 pages); arXiv.org/gr-qc/0409072.

[26] F.W. Hehl and Yu.N. Obukhov, *To consider the electromagnetic field as fundamental, and the metric only as a subsidiary field,* Foundations of Physics, accepted for publication (2004); arXiv.org/physics/0404101.

[27] M.E. Tobar, *Global representation of the fine structure constant and its variation*, Metrologia, to be published (2005); arXiv.org/hep-ph/0306230v5 (original version June 2003).

[28] F.W. Hehl and Yu.N. Obukhov, *Dimensions and units in electrodynamics,* General Relativity Gravitation J., in print (to appear in April 2005); arXiv.org/physics/0407022.

[29] Y.N. Obukhov and F.W. Hehl, *Possible skewon effects on light propagation,* Phys. Rev. **D70** (2004) 125015 (14 pages); arXiv.org/physics/0409155.

[30] F. Wilczek, *Two applications of axion electrodynamics*, Phys. Rev. Lett. **58** (1987) 1799-1802.

[31] F.W. Hehl and Y.N. Obukhov, *Linear media in classical electrodynamics and the Post constraint,* Phys. Lett. **A334** (2005) 249-259; arXiv.org/physics/0411038.

[32] I.V. Lindell and A.H. Sihvola, *Perfect electromagnetic conductor* (6 pages) Preprint September 2004, submitted to JEMWA.

[33] I.V. Lindell and A.H. Sihvola, *Transformation method for problems involving perfect electromagnetic conductor (PEMC) structures* (13 pages) Preprint October 2004, submitted to IEEE Trans. Antennas Propag.

[34] R. Beig, *Concepts of Hyperbolicity and Relativistic Continuum Mechanics,* arXiv.org/gr-qc/0411092.

[35] D.H. Delphenich, *Nonlinear electrodynamics and QED,* arXiv.org/hep-th/0309108.

[36] Yu.N. Obukhov and G.F. Rubilar, *Fresnel analysis of wave propagation in nonlinear electrodynamics*, Phys. Rev. **D66** (2002) 024042 (11 pages).

[37] F.W. Hehl, Yu.N. Obukhov, and G.F. Rubilar, *On a possible new type of a T odd skewon field linked to electromagnetism.* In: Developments in Mathematical and Experimental Physics, A. Macias, F. Uribe, and E. Diaz, eds. Volume A: Cosmology and Gravitation (Kluwer Academic/Plenum Publishers: New York, 2002) pp.241-256; arXiv.org/gr-qc/0203096.

[38] W. Raith, ed., *Bergmann-Schaefer, Lehrbuch der Experimentalphysik, Vol.2, Elektromagnetismus*, new 9th ed. (de Gruyter: Berlin, to be published 2005/06).

[39] C. Lämmerzahl, A. Macias, and H. Müller, *Lorentz invariance violation and charge (non)conservation: A general theoretical frame for extensions of the Maxwell equations,* Phys. Rev. **D71** (2005) 025007 (15 pages).

[40] T. Frankel, *The Geometry of Physics* – An Introduction (Cambridge University Press: Cambridge, 1997, 1999).

[41] I.V. Lindell and B. Jancewicz, *Maxwell stress dyadic in differential-form formalism,* IEE Proc.-Sci. Meas. Technol. **147** (2000) 19-26.

[42] R. Segev, *Metric-independen analysis of the stress-energy tensor,* J. Math. Phys. **43** (2002) 3220-3231.

[43] J.A. Schouten, *Tensor Analysis for Physicists,* 2nd ed. reprinted (Dover: Mineola, New York 1989). In this book a review can be found on the theory of physical dimensions due to Dorgelo and Schouten.

[44] S.G. Karshenboim, *Search for possible variation of the fine structure constant* (33 pages), `physics/0311080`.

[45] S.G. Karshenboim and E. Peik, eds., *Astrophysics, Clocks and Fundamental Constants, Lecture Notes in Physics*, **648** (346 pages) (Springer: Berlin, 2004).

[46] J.-P. Uzan, *The fundamental constants and their variation: observational and theoretical status*, *Rev. Mod. Phys.* **75** (2003) 403–455.

[47] A. Peres, *Variability of fundamental constants*, *Int. J. Mod. Phys.* **12** (2003) 1751–1754.

[48] A. Peres, *The speed of light need not be constant* (3 pages), `gr-qc/0210066`.

[49] J.D. Bekenstein, *Fine-structure constant variability, equivalence principle, and cosmology*, *Phys. Rev.* **D66** (2002) 123514 (17 pages).

[50] J.F. Nieves and P.B. Pal, *The third electromagnetic constant of an isotropic medium*, Am. J. Phys. **62** (1994) 207-216.

[51] C. Lämmerzahl, *The geometry of matter fields.* In: V. de Sabbata and J. Audretsch (eds.), *Quantum Mechanics in Curved Space–Time*, NATO ASI series, series B, physics Vol. 230. Plenum Press, New York (1990) p. 23.

[52] W.-T. Ni, *A non-metric theory of gravity.* Dept. Physics, Montana State University, Bozeman. Preprint December 1973. The paper is available via `http://gravity5.phys.nthu.edu.tw/webpage/ article4/index.html`

[53] W.-T. Ni, *Equivalence principles and electromagnetism*, *Phys. Rev. Lett.* **38** (1977) 301-304.

[54] W.-T. Ni, *Equivalence principles and precision experiments.* In: *Precision Measurement and Fundamental Constants II,* B.N. Taylor, W.D. Phillips, eds. Nat. Bur. Stand. (US) Spec. Publ. 617, US Government Printing Office, Washington, DC (1984).

[55] E. Schrödinger, *Space-Time Structure* (Cambridge University Press: Cambridge, 1954).

[56] R.H. Dicke, *The theoretical significance of experimental relativity*, (Gordon and Breach: New York, 1964).

[57] B.D.H. Tellegen, *The gyrator, a new electric network element,* Philips Res. Rep. **3** (1948) 81–101.

[58] B.D.H. Tellegen, *The gyrator, an electric network element,* Philips Technical Review **18** (1956/57) 120–124. Reprinted in H.B.G. Casimir and S. Gradstein (eds.) *An Anthology of Philips Research.* Philips' Gloeilampenfabrieken, Eindhoven (1966) pp.186–190.

[59] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Vol.2 of *Course of Theoretical Physics*, transl. from the Russian (Pergamon: Oxford, 1962).

[60] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman: San Francisco, 1973).

[61] G.W. Horndeski, *Conservation of charge and the Einstein-Maxwell field equations*, J. Math. Phys. **17** (1976) 1980–1987.

[62] G.W. Horndeski and J. Wainwright, *Energy momentum tensor of the electromagnetic field*, Phys. Rev. **D16** (1977) 1691–1701.

[63] H. A. Buchdahl, *On a Lagrangian for nonminimally coupled gravitational and electromagnetic fields,* J. Phys. **A12** (1979) 1037–1043.

[64] F. Mueller-Hoissen and R. Sippel, *Spherically symmetric solutions of the nonminimally coupled Einstein-Maxwell equations,* Class. Quant. Grav. **5** (1988) 1473–1488.

[65] I.T. Drummond and S.J. Hathrell, *QED vacuum polarization in a background gravitational field and its effect on the velocity of photons,* Phys. Rev. **D22** (1980) 343–355.

[66] R.D. Daniels and G.M. Shore, *'Faster than light' photons and charged black holes,* Nucl. Phys. **B425** (1994) 634–650; arXiv.org/hep-th/9310114.

[67] M. Novello, L.A.R. Oliveira and J.M. Salim, *Is the number of photons conserved in a expanding Universe?,* Class. Quant. Grav. **7** (1990) 51–65.

[68] R. Lafrance and R.C. Myers, *Gravity's rainbow,* Phys. Rev. **D51** (1995) 2584–2590; arXiv.org/hep-th/9411018.

[69] P. Teyssandier, *Variation of the speed of light due to non-minimal coupling between electromagnetism and gravity* (12 pages), arXiv.org/gr-qc/0303081 (March 2003).

[70] M. Blagojević, Gravitation and Gauge Symmetries (IOP Publishing: Bristol, 2002).

[71] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester, *General relativity with spin and torsion: Foundations and prospects*, Rev. Mod. Phys. **48** (1976) 393–416 and refs. given.

[72] I.M. Benn, T. Dereli, and R.W. Tucker, *Gauge field interactions in spaces with arbitrary torsion*, Phys. Lett. **B96** (1980) 100-104.

[73] R.T. Hammond, *Gauge Invariant Electromagnetic Coupling With Torsion Potential,* Gen. Rel. Grav. **23** (1991) 1195–1203.

[74] R.A. Puntigam, C. Lämmerzahl, and F.W. Hehl, *Maxwell's theory on a post-Riemannian spacetime and the equivalence principle,* Class. Quant. Grav. **14** (1997) 1347–1356; arXiv.org/gr-qc/9607023.

[75] I.L. Shapiro, *Physical aspects of the space-time torsion,* Phys. Rept. **357** (2001) 113–213; arXiv.org/hep-th/0103093.

[76] R. T. Hammond, *Torsion Gravity,* Repts. Prog. Phys. **65** (2002) 599–649.

[77] F. Gronwald and F. W. Hehl, *On the gauge aspects of gravity*, in: *Proc. Int. School of Cosm. & Gravit.* 14th Course: Quantum Gravity, held May 1995 in Erice, Italy. Proceedings. Erice, May 1995. P.G. Bergmann et al. (eds.). World Scientific, Singapore (1996) pp. 148–198; arXiv.org/gr-qc/9602013.

[78] M. Haugan and C. Lämmerzahl, *On the experimental foundations of the Maxwell equations*, Ann. Phys. (Leipzig) **9** (Special Issue, 2000) SI-119–SI-124; arXiv.org/gr-qc/0002075.

[79] F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne'eman, *Metric affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance,* Phys. Rept. **258** (1995) 1–171.