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New interpretation of Kaluza-Klein compactification

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Abstract. Utilizing Jordan algebras and superalgebras, we focus on the significant role of lattices associated with discrete subgroups of the non-compact groups of the magic square. These lattices are not just a part of the process, but they play a profound role in reformulating conformal field theories and forming superstring theories, thereby pointing to a more comprehensive theory connected with a chiral 27-dimensional lattice, which is an extension of Conway's lattice. We also highlight our newly discovered 16-dimensional algebra, which resembles Hurwitz's algebras and its connection to superstrings, octonionic color algebras, and Galois Fields.

1 Introduction: Visible spaces and their visible groups.

In recent publications, we exposed the new algebras [1, 2] based on Galois Fields, showing their connection to superstring theories and octonionic multiplication rules. For further clearance of the many ideas presented in this note, we refer the reader to a list of publications[3]-[14] In what follows, we first describe visible spaces and their visible groups, then describe invisible groups as dark halos around visible ones. This will be a new interpretation of the Kaluza-Klein compactification, an idea emerging from supergravity in the $D = 11$ case and its compactification to $D = 4$. The additional seven dimensions can be compacted to S^7 with the associated group $O(7)$, where S^7 is the visible 7-dimensional space and $O(8)$ is the visible group of S^7.

For the compactification to a seven-dimensional torus, the visible group will be$[U(1)]^7$, which is the abelian Cartan subgroup of E_7, which becomes the dark group of a 7-dimensional flat space (the torus is locally flat). This is similar to the visible disk of the galaxy (T^7 with group $O(7)$ of rotations) surrounded by the dark halo of invisible matter, symbolizing the generators of E_7 associated with the root systems in a similar way one obtains E_8, E_9, E_{10} by compactifying to $D = 3$, $D = 2$, and $D = 1$; The dark group treatment made into string theory with $E_8 \times E_8$ or $O(32)$ compactification of $D = 26$ string to $D = 10$.

2 E_{10} as the Dark Group of space-time

If instead of $D = 11$ we start from $D = 10$, then the whole $9 + 1$ dimensional space can be regarded as the Cartan torus of E_{10}, the hyperbolic extension of E_8. This corresponds to a space-time with some periodic boundary conditions, the period being associated with some quantum theoretical cutoff. The visible group of this $9 + 1$ dimensional space-time would be $O(9, 1)$, but regarded as a torus; it would lie in the Cartan disk of the invisible E_{10}.

If we start from $D = 26 = (25 + 1)$ space-time (dimension of the string), then torus compactification to $D = 10$ would give the root lattices of $E_8 \times E_8$ or $O(32)$ as the dark (internal) group of the 16 compactified small dimensions.

Now we compactify the remaining 10 dimensions to a torus with large spatial periods and large or (∞) time period and identify it with the visible part of E_{10} with visible Lorentz group $O(9, 1)$. Hence the groups of the visible compactification $O(9, 1) \times O(8) \times O(8)$, or $O(9, 1) \times O(16)$ are lifted to $E_{10} \times E_8 \times E_8$ or $E_{10} \times O(32)$

with space-time group $O(9,1)$ sitting in dark external group E_{10}, and $O(8)\times O(8)$ or $O(16)$ internal groups sitting in $E_8 \times E_8$ or $O(32)$ dark internal group.

External and internal groups are supported by their 10 dimensional Lorentzian and 16 dimensional Euclidean lattices. They are embedded into a $25+1$ Lorentzian lattice $L(25,1)$. This is the Conway-Sloane lattice[15]. It corresponds to a new Lorentzian extension of the hyperbolic group E_{10}. Cartan generators can be introduced to give the structure of a new infinite Lie algebra with a rank 26 generalization (called E_∞ by Conway and Sloane). It has sublattices $E_{10}\times E_8 \times E_8$, $E_{10}\times O(32)$, $E_{10}\times O(16)\times O(16)$, i.e. the groups of known heterotic string theories. Hence, a new string theory could be associated with E_∞, fusing external and internal symmetries together. Note that E_∞ also contains the Leech lattice (not a group!) and the Monster.

At this point, it may be helpful to summarize some known results (mostly due to Kac[16]) about hyperbolic groups:

Characterized by Dynkin diagrams such that if a point is erased, the remaining diagram is either an extended Dynkin diagram (associated with affine algebra $\hat{G}$) or a Dynkin diagram of a Lie group. They stop at rank 10 corresponding to the hyperbolic extension of E_8. Among those (the strictly hyperbolic groups) are those which are not $\hat{G}$, but give Lie group G when one point is erased. They stop at rank 4.

Extensions which correspond to even Lorentzian lattices and are not hyperbolic groups stop at rank 26 (the Conway-Sloane lattice) [$8k+2$, families with $k=1,2,3$]. Again we find the critical dimensions 4, 10, 26 for hyperbolic and Lorentzian algebras.

3 Construction of the Root Lattices of E_8, $E_9 = \hat{E}_8$ and E_{10}:

We note the following results:

(1) $E_8 \times \mathcal{L}(E_8)$ is the lattice generated by integer octonions. The root system of E_8 is the regular solid inscribed in S^7 by the 240 integer octonions of unit length (multiplied by $\sqrt{2}$ in the conventional normalization of roots). $E_8 \times E_8$ is a pair of two integer octonions.

(2) E_9 and E_{10} have root systems represented by 2×2 exceptional Jordan (hermitian octonionic) matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ light-like vector added for E_9, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ light like vectors added for E_{10}. This is the new result.

Now, let us look at 1×1 matrices (Division algebras): For real integers, the root lattice is $SU(2)\sim O(3)$. For complex integers its $SU(2)\times SU(2) \sim O(4)$. For quaternionic integers it's $O(8)$ and for the octonionic ones, it's E_8.

Definition of integer elements:

$$\omega + \bar{\omega} = 2Sc\, \omega, \quad \omega\bar{\omega} = N(\omega). \tag{1}$$

Hence

$$\omega^2 - (2Sc\, \omega)\omega + N(\omega) = 0 \tag{2}$$

is satisfied by both quaternions and octonions. Further, ω is an integer if both coefficients $2Sc\, \omega$ and $N(\omega)$ are integers and if ω's form a ring (under addition and multiplication). An integer quaternion or octonion can have half-odd integer components and integer ones.

For $\mathcal{C}$: $z = x+iy$ if x and y are both half-odd integers or one of them is a half-odd integer, $|z|^2$ can not be an integer. Hence, Gaussian integers have the same form $m+in$.

Given two roots, one generates a third root by Weyl reflection (the first root is reflected with respect to the hyperplane perpendicular to the second root).

Let x be one root, a the second root, then a combined with x through a Weyl reflection giving $x^{'}$ such that

$$x^{'} = -a\bar{x}a \tag{3}$$

with $|a| = |x| = 1$. Special case will be if $a = x$, then $x^{'} = -x$. We note that

$$N(x^{'}) = N(a)^2\, N(x) \tag{4}$$

so that $|x^{'}| = 1$. Also, if x and a are integer, so is $x^{'}$. Above formula is valid for $\mathcal{C}$, $\mathcal{H}$ and $\mathcal{O}$.

If $|a| \neq |x|$, then the formula is

$$x^{'} = -a\bar{x}(\bar{a})^{-1} = -\frac{a\bar{x}a}{|a|^2} = x - 2(a,x)\frac{a}{(a,a)} \tag{5}$$

$$(a,x) = Sc\,(a\bar{x}) = a_\alpha x^\alpha, \quad (a,a) = |a|^2 = a_\alpha a^\alpha. \tag{6}$$

In this form it is Weyl's reflection formula.

4 Application to Root Spaces of $O(4)$ and the $O(8)$

As a prelude, consider the $O(4)$ root space: There are 4 roots lying on the unit circle S^1. In the Gauss plane, they can be represented by $\pm 1, \pm i$. These are of the form $y = m + in$, (m, n integers, called Gaussian integers) subject to the constraint $|y|^2 = y\bar{y} = 1$, giving 4 Gaussian integers on the unit circle. We generalize to $O(8)$ by replacing Gaussian integers with quaternionic integers (due to Frobenius and Hurwitz). A quaternion q satisfies the equation

$$q^2 - 2(Sc\, q)q + N(q) = 0 \tag{7}$$

where $q = q_0 + \vec{e} \cdot \vec{q}$, $\bar{q} = q_0 - \vec{e} \cdot \vec{q}$, $2Sc\, q = q + \bar{g} = 2q_0$, and $N(q) = |q|^2 = q\bar{q}$.

Quaternionic integers s satisfy

$$q^2 + aq + b = 0 \tag{8}$$

where a and b are integers and they form a ring, like ordinary integers. Hence q can have integer or half odd integer components. Roots are eigenvalues of the quaternionic operator $H = H_4 + j_1 H_1 + j_2 H_2 + j_3 H_3$. If we restrict $N(q) = 1$, selecting integers that lie on S^3 given by $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, we find solutions: 8 roots $(\pm 1, \pm e_1, \pm e_2, \pm e_3)$, and $2^4 = 16$ roots $\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$, a total of 24 roots, closed under Weyl reflections $x^{'} = -a\bar{x}a$. Hence $O(8)$ roots are quaternionic integer points of S^3.

5 New algebras relating superstrings and Galois Fields

Existence of Pythagorean triangles, the triples of positive integers (x, y, z) solving $x^2 + y^2 = z^2$ may be transformed to a question about the existence of complex numbers $x + iy$ such that $(x + iy)(x - iy) = z^2$. If in such a triple, integers x, y have only 1 as a common division (that they are relatively prime) and are of different parity (a case when all x, y, z have no common division > 1) then it is possible to check $x + iy, x - iy$ considered as elements of the bigger ring $Z[i]$ of the Gaussian integers $a + ib$, where $a, b \in Z$ also have only 1 as a common divisor up to the sign which here can be ± 1 as well as $\pm i$. We see that it must exist two integers m and n such that $x + iy = (m + n)^2$ giving $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$. Assuming $m > n$ and m, n are relatively prime of different parities we get all the Pythagorean triples and each triple only once. The only (not motivated) assumption is that Gaussian integers behave like integers and have unique factorization in prime factors.

In fact, if we want to find all positive integer triples such that $x^3 + y^3 = z^3$, we can also try to factorize

$$z^3 - y^3 = (z - y)((z - \epsilon y)(z - \epsilon^2 y) = x^3 \tag{9}$$

where $\epsilon = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{-3}}{2}$ where we have $\epsilon^3 = 1$ and $\epsilon \neq 1$, this last factorization can be considered as a product of a ring $Z[\epsilon]$ of so-called Eisenstein integers [17]-[19]. It is also a ring with unique factorization into primes of this ring. We want to mention that no positive integers exist in solving this famous Fermat's cubic equation[20]. Extending these ideas into quaternionic and octonionic cases may lead to deeper connections with string theories in a way parallel to the above description and a possible step toward the solution of cubic equation.

We are now ready to show Galois field connections based on diagonal matrices Q forming groups having entries of identity and Pauli matrix σ_3 and define them as follows:

Using 2×2 identity matrix I and the Pauli spin matrix σ_3 we write down the following Q matrices $Q_0, Q_1, ..., Q_7$ having real diagonal entries as follows:

$$Q_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad Q_1 = \begin{pmatrix} I & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} I & 0 \\ 0 & -\sigma_3 \end{pmatrix} \quad Q_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{10}$$

$$Q_4 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, Q_6 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & I \end{pmatrix}, \quad Q_7 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -I \end{pmatrix}, \tag{11}$$

These matrices multiply in the same manner as octonions: For example, $Q_1 Q_2 = Q_3$, $Q_2 Q_4 = Q_6$, etc., and their associator

$$[Q_\alpha, Q_\beta, Q_\gamma] = 0 \tag{12}$$

only for combinations $\alpha\beta\gamma = 123, 246, 435, 367, 651, 572, 714$.

Also adding seven more Q's to the above ones, namely $Q_8 = -Q_7$, $Q_9 = -Q_6$, $Q_{10} = -Q_5$, $Q_{11} = -Q_4$, $Q_{12} = -Q_3$ we complete the $FG(16)$ algebra.

We now describe the first eight 8×8 diagonal matrices resembling octonion algebra. Being diagonal matrices, they commute and associate. Denoting 3×3 unit matrix by I, we write them as follows:

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

$$Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{14}$$

$$Q_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

These matrices define a set of reflection operations in eight-dimensional Euclidean space. For example, Q_7 is the parity operator in R^8. They close under the matrix multiplication with the rules

$$Q_n^2 = Q_0, \quad Q_0 Q_\mu = Q_\mu, \quad Q_\mu Q_\nu = \delta_{\mu\nu} Q_0 + |\epsilon_{\mu\nu\alpha}| Q_\alpha \tag{16}$$

and their triple product

$$Q_\alpha Q_\beta Q_\gamma = \delta_{(\alpha\beta} Q_{\gamma)} + |\epsilon_{\alpha\beta\gamma}| Q_0 + |\psi_{\alpha\beta\gamma\mu}| Q_\mu \tag{17}$$

Thus, their matrix multiplication rules are determined by the structure constants of octonion products. Since an octonion can be regarded as a pair of quaternions

$x = q + e_7 p$. Denoting $Q_n x$ by $x_{(n)}$, we can express the reflections in the following way:

$$x_{(0)} = q + e_7 p = q + \bar{p} e_7 \qquad x_{(1)} = q - e_7 \bar{p} = q - p e_7 \tag{18}$$

$$x_{(2)} = q + e_7 \bar{p} = q + p e_7 \qquad x_{(3)} = q - e_7 p = q - \bar{p} e_7 \tag{19}$$

$$x_{(4)} = \bar{q} + e_7 \bar{p} = \bar{q} + p e_7 \qquad x_{(5)} = \bar{q} - e_7 \bar{p} = \bar{q} - p e_7 \tag{20}$$

$$x_{(6)} = \bar{q} + e_7 p = \bar{q} + \bar{p} e_7 \qquad x_{(7)} = \bar{q} - e_7 p = \bar{q} - \bar{p} e_7 \tag{21}$$

Thus, we see that Q_n generates eight possible constructions of an octonion from a pair of quaternions.

6 The E_8 root space and the complete root system of E_8

Integer octonion ω satisfies

$$\omega^2 + a\omega + b = 0, \tag{22}$$

where a and b are ordinary integers (Dickson[22], Coxeter[23]).

$$a = -(\omega + \bar{\omega}) = -2Sc\,\omega, \qquad b = N(\omega) = \omega\bar{\omega}. \tag{23}$$

Hence, integer octonions have also integer or half-odd integer components $\omega_1, \ldots \omega_8$.

Consider unit norm integer octonions

$$\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2 + \omega_8^2 = 1 \tag{24}$$

This is the equation of S^7. Thus we are selecting points on the unit S^7 that are integer octonions. Let

$$\begin{array}{llll} l_7 = \frac{1}{2}(1 + e_7) & l_1 = \frac{1}{2}(e_1 + e_4) & l_2 = \frac{1}{2}(e_2 + e_5) & l_3 = \frac{1}{2}(e_3 + e_6) \\ l_8 = \frac{1}{2}(1 - e_7) & l_4 = \frac{1}{2}(e_1 - e_4) & l_5 = \frac{1}{2}(e_2 - e_5) & l_6 = \frac{1}{2}(e_3 - e_6) \end{array} \tag{25}$$

Note that these are related to the split octonion units([21])

$$u_0 = \frac{1 + ie_7}{2}, \quad u_1 = \frac{e_1 + ie_4}{2}, \quad u_2 = \frac{e_2 + ie_5}{2}, \quad u_3 = \frac{e_3 + ie_6}{2}, \tag{26}$$

and their conjugates.

Consider expressions $\rho_{rs} = \pm l_r \pm l_s$ $(r \neq s)$, $|\rho_{rs}| = 1$. There are 112 such roots (roots of $O(16)$), also

$$\frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6 \pm l_7 \pm l_8) \tag{27}$$

with any odd number of minus signs. There are 128 such roots associated with $E_8/SO(16)$. We have total of 240 roots of E_8. Complex octonions (products of Gaussian unit integers with octonion unit integers) give roots of $E_8 \times E_8$. Note that l's also multiply as the Q's according to octonion multiplication rules.

Another grouping of the roots with respect to nine l's (adjoining $l_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3)$) gives the decomposition of E_8 with respect to $E_6 \times SU(3)$.

The root space of $E_8 \times E_8$ has unique properties (self dualities of the Cartan matrix) directly connected with its octonionic structure used in heterotic string theory.

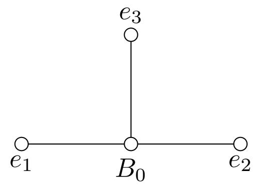

Fig. 1. The $O(8)$ Diagram

Here e_1, e_2 and e_3 are the quaternionic imaginary units or quaternionic subgroup of octonions. $B_0 = \frac{1}{2}(1 - e_1 - e_2 - e_3)$. Next we look at the extended $O(8)$ diagram. We will designate it as $\hat{O}(8)$.

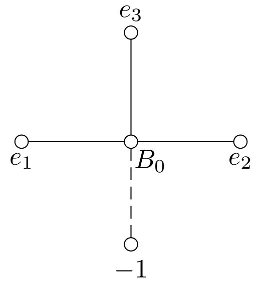

Fig. 2. The $\hat{O}(8)$ Diagram

The extended diagram has a four-fold symmetry associated with a discrete subgroup of the norm group $O(4)$ of the quaternion units $1, e_1, e_2, e_3$. It is the diagram of the affine extension $\hat{O}(8)$ of $O(8)$, or $O(8)$ current algebra on S^1. Next we look at the E_6 and the extended E_6 diagrams.

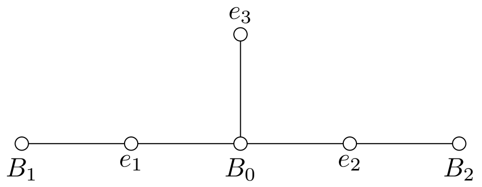

Fig. 3. The E_6 Diagram

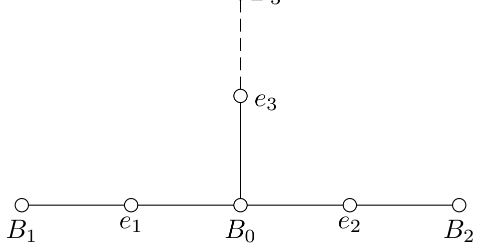

Fig. 4. The extended E_6 Diagram

This diagram has three-fold symmetry, with $B_1 = -\frac{1}{2}(1 + e_1 + e_5 - e_6)$ and $B_2 = -\frac{1}{2}(1 + e_2 + e_6 - e_4)$

Using the split octonion units u_1, u_2, u_3 above, we write

$\vec{u}$ is the (3) representation of $SU(3)$ which is the subgroup of the automorphism group G_2 of octonions that leaves e_7 invariant.

$$U\,u = u^{'}, \qquad U\,U^{\dagger} = 1, \tag{28}$$

$U \in SU(3)$ is a complex unitary 3×3 matrix. A special form is

$$U = e^{-\frac{2\pi i}{3}\left(\frac{\lambda_2-\lambda_5+\lambda_7}{\sqrt{3}}\right)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{29}$$

which leads to

$B_1 \to B_3, B_2 \to B_1$ and $B_3 \to B_2$.

The E_7 diagram looks like

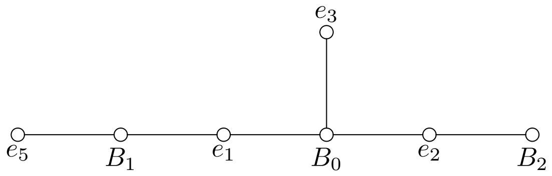

Fig. 5. The E_7 Dynkin Diagram

The E_8 Dynkin diagram is shown below. By Weyl reflections the eight principal roots generate all the 240 roots of E_8.

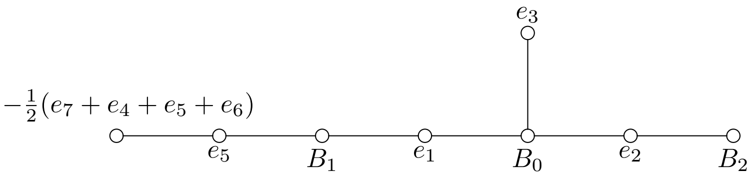

Fig. 6. The E_8 Dynkin Diagram

The extended E_8 Dynkin diagram is shown below. It will be referred as $\hat{E}_8$ or affine extension of E_8 (also called E_9).

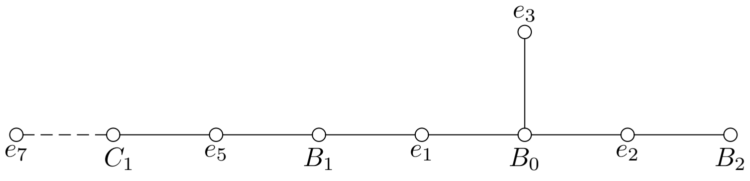

Fig. 7. The Affine Extension of E_8, (or E_9) Dynkin Diagram

Before we examine root spaces more closely, let's examine the Jordan algebras of 2×2 matrices and the continuous and discrete groups associated with them (J_2):

Consider a 2×2 hermitian matrix of the form

$$A = A^{\dagger} = \begin{pmatrix} \alpha_+ & a \\ \bar{a} & \alpha_- \end{pmatrix}, \quad \alpha_{\pm} \in \mathcal{R} \tag{30}$$

with $a \in \mathcal{R}, \mathcal{C}, \mathcal{H}$ or $\mathcal{O}$ (or $(J_2^1), (J_2^2), (J_2^4)$, or (J_2^8). Commutative product (Jordan multiplication) defined by

$$A.B = B.A = \frac{1}{2}\{AB\} = \frac{1}{2}(AB + BA). \tag{31}$$

Letting $A.B = C$, the associator is

$$[A, B, C] = (A.B).C - A.(B.C) \tag{32}$$

leads to the Jordan identity

$$[A, B, A^2] = 0 \tag{33}$$

and ensures power associativity. For example

$$(A^3)^2 = (A^2)^3 = A^6 \tag{34}$$

unambigously, and

$$A^2.A^4 = A^6, \tag{35}$$

etc. We also note that

$$(A.A^2).A^3 = A.(A^2.A^3) = A^6 \tag{36}$$

etc. The equations (31) and (33) are also satisfied by 3×3 hermitian matrices over all the Hurwitz algebras (octonionic case being called exceptional) giving the algebras $J_3^1, J_3^2, J_3^4, J_3^8$. They are also satisfied by $n \times n$ hermitian matrices over the three associative algebras $\mathcal{R}, \mathcal{C}$ and $\mathcal{H}$. The multiplicative norm for the Jordan algebras is defined by

$$N(J) = Det\ J. \tag{37}$$

Now, let us look at the integer elements for 2×2 Jordan algebras. Consider the 2×2 hermitian matrix (30), where we replace $\alpha_\pm$ by α_1 and α_2. The multiplicative norm of A is

$$||A|| = |Det\ A|, \quad A\bar{A} + I\ Det\ A = 0 \tag{38}$$

where

$$\bar{A} = \begin{pmatrix} \alpha_2 & -a \\ -\bar{a} & \alpha_1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{39}$$

The trace form is

$$\alpha = \alpha_1 + \alpha_2 = Tr\ A = Tr\ \bar{A}. \tag{40}$$

We have

$$A + \bar{A} = I\ Tr\ A. \tag{41}$$

Multiplying by A we have

$$A^2 - A\ Tr\ A + I\ Det\ A = 0. \tag{42}$$

A is an integer element when $Tr\ A$ and $Det\ A$ are integers. Thus $\alpha_1 + \alpha_2$ and $\alpha_1\alpha_2 - |a|^2$ are integers; a is a real integer, gaussian integer, quaternionic or octonionic integer. When α_1, α_2 and the components of a are real, A represents a vector in the $(m + 1, 1)$ dimensional Minkowski space (like the momentum vector) which is a representation of $O(m + 1, 1)$ with $m = 1, 2, 4$ and 8.

An integer matric is a representation of the infinite discrete subgroup $\Gamma(m)$ of the Hurwitz algebra related to Lorentz group $O(m + 1, 1)$. For $m = 1$, Γ is the modular group $SL(2, Z)$. For $m = 2$, $\Gamma(2)$ is the complex modular (or Kleinian group). In the case of $m = 4$ it is the discrete subgroup of $SL(2, \mathcal{H})$ or $O(5, 1)$. Finally, for $m = 8$ it is a discrete subgroup of $O(9, 1)$.

If we fix the norm (determinant) to be unity, the integer hermitian 2×2 matrix N satisfies

$$N^2 - N\ Tr\ N + I = 0. \tag{43}$$

$$N = \begin{pmatrix} \nu_1 & n \\ -\bar{n} & \nu_2 \end{pmatrix}, \tag{44}$$

with $\nu_1 + \nu_2$ being an integer, and $\nu_1\nu_2 - n\bar{n} = \pm 1$. Hence ν_1 and ν_2 are integers and n is an integer element of Hurwitz algebras.

We have different cases: *Firstcase:* $\nu_1\nu_2 = 0$, with $\nu_1 + \nu_2 \neq 0$. Hence $\nu_1 = 0$ or $\nu_2 = 0$. We can choose $||N|| = n\bar{n} \geq 0$, $n\bar{n} = 1$. Hence, n is an integer Hurwitz element of norm 1. It represents a root vector of $O(3), O(4), O(8)$ or E_8. ν_1 (or ν_2) is an integer. Then we have infinite solutions of (43). ν_1 can be associated with the winding number on a circle S^1, so that $N = \begin{pmatrix} \nu_1 & n \\ \bar{n} & 0 \end{pmatrix}$, $\quad (n\bar{n} = 1), \quad \nu_1 \in Z$ represents the loop algebras $\widetilde{O}(3), \widetilde{O}(4), \widetilde{O}(8)$, or $\widetilde{E}_8 = E_9$.

The root system of $\widetilde{O}(m+1,1)$ is obtained from the root system of $O(m)$ represented by the unit discrete transverse vector $\begin{pmatrix} 0 & n \\ \bar{n} & 0 \end{pmatrix}$ by adjoining the longitudinal vector $\begin{pmatrix} \nu_1 & 0 \\ 0 & 0 \end{pmatrix}$ where $\nu_1 = \nu_0 + \nu_{m+1}$ and $\nu_2 = \nu_0 - \nu_{m+1} = 0$.

second case: The 2×2 discrete matrix of norm

$$\begin{pmatrix} \nu_1 & n \\ \bar{n} & \nu 2 \end{pmatrix} \tag{45}$$

with $n\bar{n} - \nu_1\nu_2 = 1$ (unit integer), and $\nu_1 + \nu_2 = integer$ satisfies (43) and there are an infinite number of solutions with $n\bar{n} \geq 1$.

These correspond to the root systems of the hyperbolic extensions of $O(m)$ with $m = 1, 2, 4$ or 8. For $m = 8$ we have the infinite parameter group called E_{10}.

7 Groups associated with 2×2 continious and discrete Jordan algebras

Let

$$X = X^\dagger \in J_2^{(i)}, \quad X^{'} = L\, X\, L^\dagger \tag{46}$$

for $i = 1, 2, 4$. $Det\ L = 1$ and

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{47}$$

where the elements belong to $\mathcal{R}, \mathcal{C}$ or $\mathcal{H}$. Then $L \in SL(2, \mathcal{R})$ which is isomorphic to $SO(2,1)$, or $L \in SL(2, \mathcal{C})$ which is isomorphic to $SO(3,1)$, or $L \in SL(2, \mathcal{H})$ which is isomorphic to $SO(5,1)$. The isomorphism groups are the lorentz groups $SO(i+1, 1)$.

In the octonionic case, the linear action of the group on X must also include associators. The ones that are not included in left and right multiplication are in the G_2 automorphism group of the octonion algebra. Dimension of G_2 is 14. A 2×2 unimodular octonionic matrix has $4 \times 8 - 1 = 31$ parameters. The unimodular linear action on $X \in J_2^8$ has $31 + 14 = 45$ parameters, which is $SO(9,1) \sim$ "$SL(2, \mathcal{O})$" (meaning linear action).

Now let us look at the *discrete case*: If X is a discrete element of J_2^i, then a discrete subgroup of L (Lorentz group) will transform it into another discrete element $X^{'}$. Then, for J_2^i, $i = 1, 2, 4$, so that in above L, a, b, c, d are integers $Z(i = 1)$, Gaussian integers $Z_{\mathcal{C}} = m + in$ $(i = 2)$ or quaternionic integers $Z_{\mathcal{H}}(i = 4)$. These are the real modular group $\Gamma \subset SL(2, \mathcal{R})$, the complex modular group $\Gamma_{\mathcal{C}} \subset SL(2, \mathcal{C})$, or the qaternionic modular group $\Gamma_{\mathcal{H}} \subset SL(2, \mathcal{H})$.

In the octonionic case, there is also a discrete subgroup Γ_ω of $SO(9,1)$ that one can regard as a subgroup of "$SL(2, \mathcal{O})$". Hence we have the sequence of modular groups $SL(2, Z^{(i)})$, $i = 1, 2, 4, 8$.

8 Root space of E_9 (affine extension of E_8)

Let

$$X = \begin{pmatrix} x_+ & \xi \\ \bar{\xi} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_+ & \eta \\ \bar{\eta} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} z_+ & \zeta \\ \bar{\zeta} & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{48}$$

Then

$$\frac{1}{2}\{X\rho, Y\rho\} = Z\rho \tag{49}$$

or

$$Z = \frac{1}{2}(X\rho Y + Y\rho X) = \{X\rho Y\} = (X.\rho).Y + X.(\rho.Y) - (X.Y).\rho \tag{50}$$

This is Jacobson's isotopic (with respect to the hermitian metric ρ) Jordan product. It is also valid for 3×3 matrices.

$$-X\bar{X} = -Det\ X = \xi\bar{\xi} \tag{51}$$

is the same as the norm of the transverse part. If x_+ any integer X forms a modular group. If ξ is an octonionic integer then X generates the root system of the affine infinite group (Kac-Moody) E_9 obtained by adjoining to E_8 the light like orthogonal vector

$$\begin{pmatrix} x_+ & 0 \\ 0 & 0 \end{pmatrix} \tag{52}$$

with x_+ an integer.

9 The E_{10} Diagram and Root Space of E_{10}

The principal roots now generate an infinite number of roots associated with an infinite algebra (hyperbolic extension of E_8). Its infinite subalgebra is E_9 (called affine extension of E_8). Root diagram is shown below:

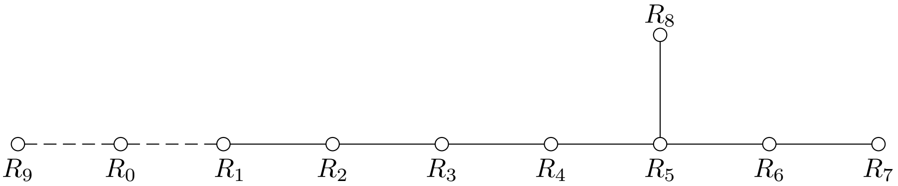

Fig. 8. The E_{10} Diagram

Here

$$R_a = \begin{pmatrix} 0 & \rho_a \\ \bar{\rho}_a & 0 \end{pmatrix} \qquad a = 1, \dots, 8 \tag{53}$$

$$R_0 = \begin{pmatrix} 1 & \rho_0 \\ \bar{\rho}_0 & 0 \end{pmatrix}, \quad R_9 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{54}$$

The roots are R_A, $A = 0, 1, \dots, 8, 9$; ρ_0 and ρ_a are integer octonions. The scalar product is given by

$$(R_A, R_B) = \frac{1}{2} Tr(R_A.R_B) \tag{55}$$

so that

$$(R_a, R_b) = Sc(\rho_a \bar{\rho}_b), \quad (R_0, R_a) = Sc(\rho_0 \bar{\rho}_a), \tag{56}$$

$$(R_9, R_a) = 0, \quad (R_9, R_0) = -\frac{1}{2}, \quad (R_a, R_b) = Sc(\rho_a \bar{\rho}_b). \tag{57}$$

$Sc(\rho_a, \bar{\rho}_b)$ is either $-\frac{1}{2}$ or 0. If $(R_A, R_B) = -\frac{1}{2}$ then A and B are joined; If $(R_A, R_B) = 0$ then A and B are not joined. The ρ_0 and ρ_a are the principal roots of the extended E_8 diagram. They are given by:

$$\begin{aligned}
\rho_0 = e_7, \quad \rho_1 = -\tfrac{1}{2}(e_7 + e_4 + e_5 + e_6), \quad \rho_2 = e_5, \\
\rho_3 = B_1 = -\tfrac{1}{2}(1 + e_1 + e_5 - e_6), \quad \rho_4 = e_1, \\
\rho_5 = B_0 = \tfrac{1}{2}(1 - e_1 - e_2 - e_3), \quad \rho_6 = e_2, \\
\rho_7 = B_2 = -\tfrac{1}{2}(1 + e_2 + e_6 - e_4), \quad \rho_8 = e_3.
\end{aligned} \tag{58}$$

10 Integer elements for 3×3 Jordan algebras

Consider

$$J = \begin{pmatrix} \alpha_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \bar{a}_1 & \alpha_3 \end{pmatrix} = J^\dagger. \tag{59}$$

Define the Freudenthal product by

$$J \times J = J^{-1} \; Det \; J. \tag{60}$$

We have the formula that is valid even for $Det \; J = 0$:

$$J \times J = J^2 - J \; Tr \; J - \frac{1}{2} I (Tr \; J^2 - Tr \; J \;\; Tr \; J) \tag{61}$$

where I is the 3×3 identity matrix. Also

$$J.(J \times J) = I \; Det \; J. \tag{62}$$

We also have

$$Tr \; J \times J = -\frac{1}{2}(Tr \; J^2 - Tr \; J \;\; Tr \; J) \tag{63}$$

so that J satisfies the secular equation

$$J^3 - J^2\, Tr\ J + J\, Tr(J \times J) - I\ Det\ J = 0. \tag{64}$$

J will be an integer Jordan matrix if $Tr\ J$, $Tr\ J \times J$ and $Det\ J$ are integers. If $|Det\ J| = 1$ we have again infinite number of solutions representing infinite number of roots (representations of generalizations of the modular group $SL(3, Z)$).

Let us now define Jacobson triple product for Jordan algebras:

$$\{A\ B\ C\} = (A.B).C + A.(B.C) - (A.C).B \tag{65}$$

For ordinary matrices over reals or complex numbers, one finds

$$\{A\ B\ C\} = \frac{1}{2}(A\ B\ C + C\ B\ A). \tag{66}$$

which is obviously hermitian if A, B, and $C$are hermitian. Special case when $C = A$

$$\{A\ B\ A\} = A\ B\ A = 2(A.B).A - (A.A).B. \tag{67}$$

Hence

$$Det\{A\ B\ A\} = (Det\ A)^2\ Det\ B. \tag{68}$$

The formula is true for all Jordan algebras including the exceptional Jordan algebra J_3^8.

Weyl reflections for discrete elements of 2×2 and 3×3 Jordan algebras:
Let A such that A^{-1} exists ($Det\ A \neq 0$). Consider elements A and X of J_2 or J_3. Let

$$X^{'} = -\lambda\{A\ \bar{X}\ \bar{A}^{-1}\}, \qquad (\bar{X} = X^{-1}\ Det\ X). \tag{69}$$

We have $\lambda = 1$ for 2×2 matrix, and $\lambda = \frac{Det\ A}{Det\ X}$ for 3×3 matrix.

$$|Det\ X^{'}| = |Det\ A|\ |Det\ X|\ |Det\ A|^{-1} = Det\ X \tag{70}$$

so that if $|Det\ X| = 1$, $|Det\ X^{'}| = 1$.

If A and X are integral elements of J_2 and J_3 with unit determinant, they correspond to lattice elements. Then $X^{'}$ is also a lattice element. The special case when $A = X$. Then $X^{'} = -X$ is also a lattice element. Furthermore,

$$X^{'} = X - 2(A^{-1}.X).A \tag{71}$$

which is seen to be a generalization of the Weyl reflection formula

$$B\ \bar{A} + A\ \bar{B} = I\ Tr(A\ \bar{B}) \tag{72}$$

and

$$A^{'} = -\frac{B\ \bar{A}\ B}{B\ \bar{B}} = A - \frac{B}{B\ \bar{B}}\ Tr\ (A\ \bar{B}) = -\{B\ \bar{A}\ \bar{B}^{-1}\}. \tag{73}$$

11 Embedding of $E_{10} \times E_8 \times E_8$ in the 3×3 integer lattice

We write the 3×3 integer lattice associated with the Conway-Sloane lattice and reduce it with respect to the 2×2 integer E_{10} lattice:

$$\begin{pmatrix} k_+ & \gamma & \alpha \\ \bar{\gamma} & k_- & \beta \\ \bar{\alpha} & \bar{\beta} & \ell \end{pmatrix}. \tag{74}$$

Here α, β and γ are integer octonions, and k_+, k_- and ℓ are real integers. We note that the E_{10} lattice sits in the upper left 2×2 part, i.e., in $\begin{pmatrix} k_+ & \gamma \\ \bar{\gamma} & k_- \end{pmatrix}$

which also gives us the quantized space-time momenta in $(9+1)$ Minkowski space-time dimensions; γ corresponds to E_8 lattice lifting $O(8)$ helicity group; α and β each contain the E_8 internal symmetry groups. $(\alpha \quad \beta)$ part corresponds to $E_8 \times E_8$ lattice (or $O(32)$); ℓ is the lattice of $SL(2, \mathcal{R})$. We note that when $|\alpha| = 1$, and $|\beta| = 1$ give $E_8 \times E_8$. If α is restricted to 112 roots of $O(16)$, then $(\alpha \quad \beta)$ give $O(16) \times O(16)$. If $\alpha\,\bar{\alpha} + \beta\,\bar{\beta} = 1$ and α and β restricted, we have $O(32)$ roots. Generally speaking

$$\begin{pmatrix} 0 & \gamma & \alpha \\ \bar{\gamma} & 0 & \beta \\ \bar{\alpha} & \bar{\beta} & 0 \end{pmatrix} \tag{75}$$

can represent other Niemeier lattices like E_8^3, $E_8 \times O(32)$, etc., or the Leech lattice.

12 Use of Jordan matrices for the heterotic string

To obtain the heterotic string one starts from two 26 dimensional vectors that arise in closed string theory. Let X_μ^L be left moving and X_μ^R be the right moving that are both solutions of the closed string equations of motion. X_μ^R is compactified as above, while X_μ^L is turned into a superstring by turning 16 of its components X_a^L into the 16 components of a Weyl-Majorana spinor in $D = 10$. This can be done because $X_a^L(\tau, \sigma)$ is a two-dimensional field theory by a process of fermionization. In the end one has the 10 dimensional bosonic vectors X_α^L, X_α^R $(\alpha = 1, \ldots, 10)$, the spinor in $D = 10$: ψ_A $(A = 1, \ldots, 16)$ and the group G_r describing an internal local gauge symmetry. $G_r = E_8 \times E_8$ gives the best phenomenology.

Further compactification of $10 - 4 = 6$ dimensions (on Calabi-Yau surfaces or orbifolds) breaks one of the E_8 groups further to E_6 which is known to be one of the possible grand unified theories that include $SO(10)$ and $SU(5)$, thus making contact with particle phenomenology. There remain fundamental unsolved problems in the scenario, namely the principle for the right compactification (true vacuum), and the cosmological constant problem ($\Lambda = 0$ after supersymmetry breaking).

In the present approach X_μ^L and X_μ^R are identified with the components X^L and X^R of the 27 and $\bar{27}$ dimensional representation of $E_{6,-26}$. The 27th dimension is interpreted as the bosonic ghost dimension introduced by Siegel for a covariant quantization of the 26 dimensional string. $E_{6,-26}$ is a real form of E_6.

Then $X^{L,R}$ have the form

$$X = \begin{pmatrix} V & \psi \\ \psi^\dagger & v \end{pmatrix} \tag{76}$$

where V is a 2×2 matrix representing the 10 dimensional vectors V_L and V_R, ψ is a 2×1 matrix representing the 16 dimensional spinor for X_L and the roots of $E_8 \times E_8$ or $O(32)$ for X_R, and v the ghost variable.

Now, a new possibility arises: If V is also compactified on a torus of large radii, V can be interpreted as being associated with a group. The quantized momenta P_α $(\alpha = 1, \ldots, 10)$ would then be identified with the Cartan torus of the hyperbolic infinite group E_{10}. One then obtains the root lattice of $E_{10} \times E_8 \times E_8$ or $E_{10} \times O(32)$. These are sub lattices of a $26 + 1$ (Conway-Sloane) lattice or a $26 + 2$ lattice associated with a new infinite group $\tilde{G}$, shown to exist by Conway as the minimal extension of E_{10}.

This points out to the possibility of a new string theory as a representation of $\tilde{G}$ which would have as special solutions the 3 known heterotic strings ($E_8 \times E_8$, $O(32)$ and $O(16) \times O(16)$) obtained through various breakings of $\hat{G}$ into its subgroup that contains $E_{10} = \tilde{E}_8$.

The 4 dimensional theory could be obtained by breaking $\tilde{E}_8$ into $\tilde{O}_4$ which contains the lorentz group $SL(2, \mathcal{C})$ as a subgroup.

Thus, E_{10} acquires new importance if its Cartan subgroup is identified with the $(9+1)$ Minkowski space-time with periodic conditions.

Space time was previously associated with $O(3,1)$ or the coset $O(4,2)/Inh.O(3,1) \times Dilatations$. Now, there is the possibility of regarding it as the Cartan subgroup of the hyperbolic group $\tilde{O}(4)$ with rank 4.

This new space is nested in the $(9+1)$ space-time that is identified with the Cartan subgroup of $\tilde{E}_8 = E_{10}$ (hyperbolic extension of E_8) which also appears in the internal sector twice.

13 Conclusions and Prospects

A new point of view emerges. E_{10} is interpreted as the dark halo groups of the visible $(9+1)$ dimensional space-time, allowing it to be integrated with the internal symmetry groups $E_8 \times E_8$, $O(32)$ and $O(16) \times O(16)$ of the heterotic string.

Representations of the Lorentzian root lattice of E_{10} by integer 2×2 octonionic Jordan matrices that also represent quantized momenta in periodic space-time are developed.

Root system of the Conway-Sloane lattice as integer 3×3 exceptional Jordan matrix which has $E_{10} \times E_8 \times E_8$, etc. as sublattices. Associated root diagrams were worked out

Expression of the Weyl reflections for E_{10} and the Conway-Sloane lattice by means of Jordan products.

Prospects for writing right-moving string and left-moving superstring Lagrangians by means of Jordan matrices $J_{L,R}(z)$ in 27 and $\bar{2}7$ representations of $E_{6,-26}$ and as cosets G/H conformal fields ($G = E_7$, $H = E_6 \times U(1)$).

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