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Fermat Metrics

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Abstract: In this paper we present a survey of Fermat metrics and their applications to stationary spacetimes. A Fermat principle for light rays is stated in this class of spacetimes and we present a variational theory for the light rays and a description of the multiple image effect. Some results on variational methods, as Ljusternik-Schnirelmann and Morse Theory are recalled, to give a description of the variational methods used. Other applications of the Fermat metrics concern the global hyperbolicity and the geodesic connectedeness and a characterization of the Sagnac effect in a stationary spacetime. Finally some possible applications to other class of spacetimes are considered.

Keywords: Finsler metrics; stationary spacetimes; Fermat principle; light rays; critical point theory

1. Introduction

In this paper we present a review of the Fermat metrics associated with standard stationary spacetime and some applications for the studying of the geometrical and physical properties of such spacetimes. The Fermat metrics are two Finsler metrics of Randers type and are so called since they allow to extend the Fermat principle of classical optics. In an isotropic medium, light rays can be characterised as critical points of the optical length functional, or equivalently as geodesics of the conformal metric obtained dividing the ambient metric by the refraction index, see for instance [1,2], where the geometric aspects of the classical Fermat principle are emphasised.

The problem to extend the Fermat principle to relativistic optics was settled soon after the formulation of General Relativity by Albert Einstein in 1915, and the first formulations of the Fermat principle were obtained by Hermann Weyl for static spacetimes, see [3] and also [4], and by Tullio Levi-Civita for stationary spacetimes, see [5]. The basic role played by light rays, modelled by light-like geodesics in the four dimensional spacetime, in the physical properties of the spacetime was confirmed by observation the deflection of light shown by Eddington during the 1919 solar eclipse. It was the first experimental validation of general relativity.

Around the same time, a new branch of the calculus of variations appeared because of the work of Morse [6] and Ljusternik–Schnirelmann [7], starting with previous results from Poincaré and Birkhoff, the so called calculus of variations in the large. The aim of the calculus of variations in the large was to relate the set of critical points of a smooth function on a finite dimensional manifold to the topological properties of the manifold itself, see [8]. The objective of these researches was to relate the number and the type of geodesics joining two points on a Riemannian manifold, or the existence of a closed geodesic on a compact Riemannian manifold, to the topology of the underlying manifold. The variational methods introduced by Morse, Ljusternik, and Schnirelmann fit well with the geodesics, because they are the critical points of the action integral on the space of curves with fixed boundary conditions on the curves. Since variational problems involving functions and curves are infinite dimensional in nature, Morse developed a shortening procedure to reduce the infinite dimensional variational problem to a finite dimensional one.

In the 1960s, with the fundamental paper by Richard Palais and Steve Smale [9,10], the bases of an infinite dimensional critical point theory were settled, where the relations

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between the critical points of a smooth function defined on a Hilbert (or more generally a Banach manifold), and the topological properties of the manifold were exploited. Palais applied its theory to develop an infinite dimensional theory for the geodesics on a Riemannian manifold. The critical point theory has been applied also to obtain deep results in the study of non-linear ordinary and partial differential equations, with applications to physics, geometry, and engineering, see [11].

On the other hand, in the 1970s, the first examples of a *gravitational lens* were detected. The gravitational lensing is a relativistic optical phenomena, due to the deflection of light, for which an observer in spacetime can detect multiple images of the same light source, as a star, at the same time instant. From the point of view of the geometry of spacetime, an observer at a point p of the spacetime is joined by multiple light-like geodesics starting by the world line γ of the source. The gravitational lens effect is still an active research field in general relativity, for instance in the study of detection of dark matter, see for instance [12] for a wide geometric and physical description of the gravitational lensing effect and to [13] for a more recent review with many historic recalls.

The multiple image effect was predicted already in the 1930s by the Swiss astronomer Zuicky, and it can be caused both by physical properties of spacetime (as the presence of a massive object), encoded in the geometric properties of the spacetime metric, but also by the topological properties of the spacetime, for instance if the spacetime has a compact Cauchy surface. If a Fermat principle holds, we can characterise light-like geodesics as critical points of a suitable functional, and apply the methods of the calculus of variations in the large to relate the multiple image effect to the topological and geometric properties of spacetime.

In this paper, we focus our attention on standard stationary spacetimes (or, in general, Lorentzian manifolds without any restriction on the dimension of the manifold), which are defined at Section 4. In this class of spacetimes, a Fermat principle can be stated in terms of the Fermat metrics associated to such manifolds. They are two Finsler metrics of Randers type F_+ and F_-, defined not on the whole spacetime, but only on its *spatial component*. We recall that Randers metrics were introduced by Randers in the paper [14] in order to define a Lagrangian for a charged particle in relativistic spacetime. We will state the following version of the Fermat principle, see Section 4. It states that the light-like geodesics on standard stationary spacetimes can be put into bijective correspondence with the (pre)geodesics of one of the Fermat metrics, whose geodesics are the critical points of the action integral for the Fermat metrics.

> *A light-like curve is a future (respectively past) pointing light-like geodesic if and only its spatial projection is a pregeodesic for the Fermat metric F_+ (respectively, F_-).*

A first version of the Fermat principle in stationary spacetimes was proven by Tullio Levi-Civita, see [5]. Other versions were proved in [15,16], while in [17] the Fermat metrics are explicitly introduced for the first time. We point out that, since the notions of light-like curve and light-like geodesics are conformally invariant, the Fermat principle above holds for conformally standard stationary spacetimes.

In this paper we present some applications of the Fermat metrics to the physical and geometric properties of a standard stationary spacetime. The paper is organised in the following way. In Section 2 we recall the main properties of Finsler metrics.

In Section 3 we recall the main properties of Lorentzian manifolds, because spacetimes of general relativity are modelled on four dimensional Lorentzian manifolds.

In Section 4, standard stationary Lorentzian manifold and the Fermat metrics associated are introduced. Moreover the Fermat principle for light rays involving the Fermat metrics is stated.

In Section 5 we present some physically relevant examples of standard stationary spacetimes and we evaluate their Fermat metrics.

In Section 6 we recall the main results on critical point theory, in particular both the Ljusternik–Schnirelmann and the Morse theories.

In Section 7 the main results on critical point theory for the geodesics on Finsler manifolds, and also Riemannian manifolds, are presented, using the infinite dimensional approach by Palais.

In Section 8 an alternative variational principle for geodesics on Randers metrics are presented. Sometimes this principle could be useful, because it involves a functional of class C^2, while in Finsler geometry the action functional is only of class $C^{1,1}$.

In Section 9 we present the variational theory for light rays joining a point (the observer) with a vertical line (the world line of the light source) in a standard stationary spacetime. By the Fermat principle, this variational theory is equivalent to the variational theory for the geodesics of one of the Fermat metrics, according the light rays are future or past pointing. We produce conditions which guarantee the multiple image effect and a proof of the odd image theorem in the gravitational lens effect. The case of manifolds with boundary is treated because many physically relevant spacetimes of general relativity have boundaries, similarly to Schwarzschid, Reissner–Nordstrom, and Kerr spacetimes.

In Section 10 we relate the global hyperbolicity of a stationary spacetime to the completeness of the Fermat metrics. This is a very interesting topic because global hyperbolicity is a crucial assumption in the study of evolution equations on a spacetime, as Einstein equations and wave maps. The global hyperbolicity depends on the global behaviour of the Fermat metrics, and this gives further merit to these Finsler metrics.

In Section 11, we study a more geometric problem, the geodesic connectedness of a standard stationary spacetime, that is the existence (and, eventually, also the multiplicity) of geodesics joining two arbitrary points. We shall present the main results on this problem and we shall also see that the Fermat metrics now play a role.

Finally, in Section 12 we give a geometric description of the *Sagnac effect* in terms of the Fermat metrics. The Sagnac effect is an interference effect caused by an asymmetric behaviour of light rays moving on a rotating platform, dependent on if the light rays move clockwise or counterclockwise. Geometrically it is a manifestation of an asymmetry, and this asymmetry is measured by the Fermat metrics. When the platform does not rotate, the light rays have the same behaviour, the Sagnac effect is null and there is no asymmetry because in this case the metric is static and the two Fermat metrics are equal to a Riemannian conformal metric. So in this case the Sagnac effect appears when the optical geometry is of Finsler type, while when the optical geometry is Riemannian, the Sagnac effect does not appear. Therefore, the description of the Sagnac effect by Finsler geometry is not a mere generalisation of a Riemannian one, because it appears only in the presence of a Finslerian optical geometry.

In Section 13 we present some new research direction on Fermat metrics.

2. Finsler Metrics

In this section we introduce the main notions and properties of Finsler geometry. The roots of Finsler geometry are already as a generalised metric in the Habilitationsvortrag of Bernhard Riemann, where a local arc length was defined as a homogeneous function of degree 1 in the fibre coordinates [18]. The main properties of Finsler geometry was developed by Finsler in his 1918 thesis [19]. Finsler geometry had a parallel development, but it was essentially considered as a generalisation of Riemannian geometry. Perhaps one of the main arguments against Finsler was the difficulty in representing tensor fields or differential operators in a free-coordinate form. The things began to change when Katok discovered a family of Finsler metrics of Randers type on the two sphere S^2 admitting exactly two closed geodesics [20], while it was conjectured that any Riemannian metric on S^2 should admit infinitely many geometrically distinct closed geodesics, as successively proved by Bangert and Franks. Therefore, Finsler geometry has its own peculiarity with respect to Riemannian geometry. Moreover, many applications of Finsler geometry to many fields of applied sciences have appeared, creating further interest in the study of Finsler geometry, see for instance [21–25]. A big step in the development of Finsler geometry was obtained by Chern and their school in the search for global operators, as the

Chern connection [26,27]. One of the motivations of this paper is to contribute to present geometrical and physical models, in which Finsler geometry has a valuable role.

In this section, any smooth manifold has to be considered as class C^∞, even if it suffices to be of class C^2 to introduce the main notions. We refer to the book of Bao-Chern-Shen [28] for the details.

Let $\mathcal{M}_0$ be a smooth manifold and let $T\mathcal{M}_0$ be the tangent bundle of $\mathcal{M}_0$. Moreover, we shall denote by $\mathbf{0}$ the zero-section of $T\mathcal{M}_0$ and by $T\mathcal{M}_0 \setminus \mathbf{0}$, the so called slit tangent bundle.

Definition 1. *A Finsler metric on a smooth manifold $\mathcal{M}_0$ is a function $F\colon T\mathcal{M}_0 \to \mathbb{R}_+$, such that*

$$F \in C^0(T\mathcal{M}_0, \mathbb{R}_+) \cap C^2(T\mathcal{M}_0 \setminus \mathbf{0}, \mathbb{R}_+); \tag{1}$$

$$F(x,v) > 0, \quad \forall (x,v) \in T\mathcal{M}_0, v \neq 0; \tag{2}$$

$$F(x, \lambda v) = \lambda F(x,v), \forall \lambda > 0; \tag{3}$$

for any $(x,v) \in T\mathcal{M}$, $v \neq 0$, the quadratic form

$$g(x,v)[w_1, w_2] = \frac{1}{2}\frac{\partial^2 (F^2)}{\partial s \partial t}(x, v + sw_1 + tw_2)_{s=0,t=0} \tag{4}$$

is positive definite on $T\mathcal{M}_0 \setminus \mathbf{0}$. It is sometimes called the fundamental tensor associated to the Finsler metric F.

The couple $(\mathcal{M}_0, F)$ is called Finsler manifold.

Definition 2. *A Finsler metric F is said to be* reversible *if it satisfies (1), (2), (4), and the following assumption:*

$$F(x, \lambda v) = |\lambda| F(x,v), \forall \lambda \in \mathbb{R}, \tag{5}$$

instead of assumption (3).

Then a generic Finsler metric is a positively homogeneous function of degree 1 in the tangent variable, while a reversible Finsler metric is absolutely homogeneous.

On a Finsler manifold, metric properties can be introduced as for Riemannian manifolds. Let $(\mathcal{M}_0, F)$ be a Finsler manifold, let $\gamma\colon [0,1] \to \mathcal{M}_0$ be a (piecewise) smooth curve, the length $L(\gamma)$ of γ with respect to the Finsler metric F is defined setting

$$L(\gamma) = \int_0^1 F(\gamma, \dot{\gamma}) ds. \tag{6}$$

We remark that the length $L(\gamma)$ of the curve γ depends on the orientation of γ, unless F is reversible.

We can define the distance $d(p,q)$ between two points p, $\mathcal{M}_0$ setting

$$d_F(p,q) = \inf\{L(\gamma), \gamma \in C(p,q)\},$$

where $C(p,q)$ is the set of the piecewise smooth curves γ, such that $\gamma(0) = p$ and $\gamma(1) = q$.

The function d_F satisfies all the properties of a distance, except for the symmetry $d_F(p,q) = d_F(q,p)$ for any couple of points p and q. The distance d_F is symmetric if the metric F is reversible. Moreover, the metric d_F induces a topology on $\mathcal{M}_0$ which coincides with the initial topology of the manifold.

The *asymmetry* of a Finsler metric F permits to introduce two different notions of completeness on a Finsler manifold, the *forward and backward* completeness, corresponding to forward and backward Cauchy sequences. Obviously, the two notions of completeness coincide if the metric F is reversible. Let $(\mathcal{M}_0, F)$ be a Finsler manifold and let $p \in \mathcal{M}_0$, for any $r > 0$, the *forward metric ball* $B^+(p,r)$ and the *inward metric ball* $B^-(p,r)$ are defined setting

$$B^+(p,r) = \{x \in \mathcal{M}_0 : d_F(p,x) < r\},$$

$$B^-(p,r) = \{x \in \mathcal{M}_0 : d_F(x,p) < r\}.$$

Obviously, the balls $B^+(p,r)$ and $B^-(p,r)$ are equal if the metric is reversible. A sequence $(x_k)_{k\in\mathbb{N}}$ is a *forward Cauchy sequence* (respectively, *backward Cauchy sequence*) if for any $\epsilon > 0$, there exists a number $\nu_\epsilon \in \mathbb{N}$, such that for $\nu_\epsilon \leq h \leq k$ we have $d_F(x_h, x_k) < \epsilon$ (respectively, $d_F(x_k, x_h) < \epsilon$). The Finsler manifold $(\mathcal{M}_0, F)$ is said to be *forward complete* (respectively, *backward complete*) if any forward (respectively, backward) Cauchy sequence is convergent. The two notions are not equivalent and there examples of Finsler manifolds which are complete only on a direction, see [28]. Of course, the two notions of completeness agree if the Finsler metric is reversible.

In this paper, we shall consider two fundamental classes of Finsler metrics, the square root of a Riemannian metric and Randers metrics. For other many interesting examples of Finsler metrics we refer to [28].

Example 1 (Riemannian metrics)**.** *Any Riemannian metric h on a manifold $\mathcal{M}_0$ induces a reversible Finsler metric F setting $F = \sqrt{h}$.*

In this paper, the most important example of Finsler metrics is given by Randers metrics

Example 2 (Randers metrics)**.** *A Randers metric on a manifold $\mathcal{M}_0$ is a Finsler metric F of the form*

$$F(x,v) = \sqrt{h(x)[v,v]} + h(x)[W(x),v], \tag{7}$$

where h is a Riemannian metric and $W(x)$ is a smooth vector field on $\mathcal{M}_0$, such that for any $x \in \mathcal{M}_0$

$$\|W(x)\| = \sqrt{h(x)[W(x),W(x)]} < 1. \tag{8}$$

Assumption (8) guarantees that F is a Finsler metric, moreover it is not reversible, unless the vector field $W(x)$ is null. Randers metrics were introduced as Gunnar Randers in the paper [14] and represent the Lagrangian of a relativistic charged particle under the action of an electromagnetic field, when h is the Minkowski metric. Another equivalent way to define a Randers metric is using differential forms. Indeed, let ω the differential form metrically equivalent to the vector field W, that is

$$\omega(x)[v] = h(x)[W(x),v],$$

then we have

$$F(x,v) = \sqrt{h(x)[v,v]} + \omega(x)[v]. \tag{9}$$

As in Riemannian geometry, we can define *geodesic curves* and they play a basic role in the study of the geometric properties of a Finsler manifold. Here we introduce geodesics using a "variational approach", for a more geometric approach, see [28]. For a smooth (or at least piecewise smooth) curve $x(s)\colon [0,1] \to \mathcal{M}_0$, the length functional $L(x)$ is defined at (6). Moreover, the *action functional* (sometimes called energy functional) $E(x)$ is defined setting

$$E(x) = \frac{1}{2}\int_0^1 F^2(x,\dot{x})ds. \tag{10}$$

The geodesics for the Finsler metric F are the curves which are stationary points of the action integral $E(x)$, with suitable boundary conditions, for instance curves joining two fixed points of $\mathcal{M}_0$, closed curves, or curves joining two fixed submanifolds of $\mathcal{M}_0$, see [17,28].

The geodesic equations for the the Finsler metric F represent the Euler–Lagrange equations for the action functional (10) and can be written as

$$D_{\dot{x}}\dot{x} = 0, \tag{11}$$

where $D_{\dot{x}}\dot{x}$ is the *Chern connection* for the Finsler metric F with reference $\dot{x}$, see [28] for the definition of the Chern connection. In local coordinates, a geodesic satisfies the system of non-linear differential equations

$$\ddot{x}^k + \Gamma^k_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j = 0,$$

with $i,j,k \in \{0,\dots dim\mathcal{M}_0\}$. Here the functions $\Gamma^k_{ij}(x,\dot{x})$ are the Christoffel symbols of the Chern connection. We point out that the Christoffel symbols for the Chern connection are not defined on the manifold $\mathcal{M}_0$, as for the Christoffel symbols of the Levi-Civita associated to a Riemannian metric, but rather on the slit tangent bundle $T\mathcal{M}_0 \setminus \mathbf{0}$.

As we shall see in Section 7, the action functional $E(x)$ is only of class $C^{1,1}$ on the Sobolev manifold of H^1 curves on the manifold $\mathcal{M}_0$, and not of class C^2 as for the action integral of a Riemannian metric.

We point out that, as in Riemannian geometry, the property of a curve to be a geodesic depends on the parameterisation of the curve itself. A definition independent on parameterisation of the curve is that of *pregeodesic curve*. A curve is a pregeodesic if it admits a reparemetrisation which is a geodesic.

Moreover, we also point out that if the Finsler metric is not reversible, the notion of geodesic depends on the orientation of the curve. In other words if a curve is a geodesic, the reverse oriented curve could be not a geodesic, too. Clearly this property depends on the *asymmetry* of the Finsler metric.

3. Lorentzian Manifolds

In this section we introduce some basic notions in Lorentzian geometry, we refer to [29–32] for the details about Lorentzian geometry and its applications to general relativity.

A Lorentzian manifold is a couple $(\mathcal{M}, g)$, where $\mathcal{M}$ is a connected smooth manifold and g is a Lorentzian metric on $\mathcal{M}$, that is g is a smooth symmetric tensor field such that for any $z \in \mathcal{M}$, the bilinear form $g(z) : T_z\mathcal{M} \times T_z\mathcal{M} \to \mathbf{R}$ is non-degenerate and has index equal to 1. In general relativity, a spacetime is described by a four dimensional Lorentzian manifold. The geometric properties of the Lorentzian manifold are related to the physical properties of the relativistic spacetime by the *Einstein equations*.

$$Ric_g - \frac{1}{2}S_g g = 8\pi T,$$

where Ric_g and S_g are, respectively, the *Ricci tensor* and the *scalar curvature* of the metric g, while the double symmetric tensor T is called the *energy-impulse* tensor and it encodes all the physical properties of the spacetime.

The bilinear form $g(z)$ is not positive definite so this property allows to introduce the so called causal character of vectors. Let $z \in \mathcal{M}$, a vector $v \in T_z\mathcal{M}$ is said to be:

- *timelike*, if $g(z)[v,v] < 0$;
- *lightlike*, if $g(z)[v,v] = 0$;
- *causal*, if $g(z)[v,v] \leq 0$ and $v \neq 0$;
- *spacelike*, if $g(z)[v,v] > 0$;

This tripartition of the tangent vectors is called *causal character of tangent vectors*. On a Lorentzian manifold a time orientation, introducing the past and the future, can be defined using a continuous time-like vector field $Y(p)$ on $\mathcal{M}$, that is $Y(p) \in T_p\mathcal{M}$ is a time-like vector tangent to $\mathcal{M}$ at p. It is well known that any non-compact Lorentzian manifold $\mathcal{M}$ admits a continuous time-like vector field, while some topological restriction is needed, namely the Euler-Poincaré characteristic must be 0, to have the existence of a continuous time-like vector field on a compact Lorentzian manifold.

Definition 3. *Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $Y(z)$ be a continuous time-like vector field on $(\mathcal{M}$. Let $p \in \mathcal{M}$ and let $v \in T_p\mathcal{M}$ be a causal vector, then v is said to be* future pointing *(with respect to the time orientation given by the vector field Y) if $g(p)[Y(p), v] < 0$, while v is said to be past pointing if $g(p)[Y(p), v] > 0$.*

We point out that the scalar product $g(p)[v, w]$ of two causal vectors, one of which is time-like, is never null, see for instance [30].

The causal character of vectors can be extended also to define the causal character of a curve on a Lorentzian manifold. Moreover, the time orientation of vectors can be extended to *causal curves*. Let $\gamma(s)\colon [0,1] \to \mathcal{M}$ be a smooth curve and denote by $\dot{\gamma}(s)$ the tangent vector of the curve γ at the point $\gamma(s)$.

Definition 4. *The smooth curve γ is said to be:*

- *Time-like if $\dot{\gamma}(s)$ is a time-like vector, that is $g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] < 0$, for any $s \in [0,1]$;*
- *Light-like if $\dot{\gamma}(s)$ is a light-like vector, that is $g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] = 0$;*
- *Causal if $\dot{\gamma}(s)$ is a causal vector, that is $g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] \leq 0$, for any $s \in [0,1]$ and $\dot{\gamma}(s) \neq 0$;*
- *Space-like if $\dot{\gamma}(s)$ is a space-like vector, that is $g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] > 0$, for any $s \in [0,1]$.*

Definition 5. *Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $Y(z)$ be a continuous time-like vector field on $\mathcal{M}$ which gives a time orientation on $(\mathcal{M}, g)$. A smooth causal curve $\gamma\colon [0,1] \to (\mathcal{M}, g)$ is said to be* future pointing *(respectively,* past pointing*), if for any $s \in [0,1]$ the causal tangent vector $\dot{\gamma}(s)$ at $\gamma(s)$ is future pointing (respectively, past pointing).*

As in Riemannian or Finslerian geometry, an important class of curves in a Lorentzian manifolds is given by geodesics.

Definition 6. *Let $(\mathcal{M}, g)$ be a Lorentzian manifold, a smooth curve $z\colon [0,1] \to \mathcal{M}$ is a geodesic if it satisfies*

$$D_s\dot{z}(s) = 0,$$

where $D_s\dot{z}(s)$ is the covariant derivative of the vector field $\dot{z}$ along z induced by the Levi-Civita connection of the Lorentzian metric g.

In local coordinates of the manifold $\mathcal{M}$, a curve z is a geodesic if, and only if, it satisfies the system of ordinary differential equations

$$\ddot{z}^k + \Gamma^k_{ij}(z)\dot{z}^i\dot{z}^j = 0$$

$i, j, k \in \{1, \ldots, \dim \mathcal{M}\}$, and the functions Γ^k_{ij} are the Christoffel symbols for the metric g.

We remark that the property of a curve to be a geodesic depends on its parameterisation. A notion independent on the parameterisation is the notion of *pregeodesic curve*. A curve $z\colon [0,1] \to \mathcal{M}$ is said to be a pregeodesic if there exists a continuous function $\lambda(s)\colon [0,1] \to \mathbb{R}$, such that

$$D_s\dot{z}(s) = \lambda(s)\dot{z}(s).$$

It is not difficult to show that the notion of pregeodesic in independent on its parameterisation. Moreover, it is not difficult to show that a non-constant curve is a pregeodesic if, and only if, all it admits is a reparemeterisation, which is a geodesic.

The causal character of a geodesic can be defined. Indeed, by the main properties of the Levi-Civita connection associated to a Lorentzian metric, it can be proved that if $z\colon [0,1] \to \mathcal{M}$ is a geodesic, then there exists a constant E_z, such that for any $s \in [0,1]$,

$$E_z = g(z(s))[\dot{z}(s), \dot{z}(s)].$$

Definition 7. *A non-constant geodesic $z\colon [0,1] \to \mathcal{M}$ on the Lorentzian manifold $(\mathcal{M}, g)$ is said to be:*
time-like, if $E_z = g(z)[\dot{z}, \dot{z}] < 0$;
light-like, if $E_z = g(z)[\dot{z}, \dot{z}] = 0$;
causal, if $E_z = g(z)[\dot{z}, \dot{z}] \leq 0$;
space-like, if $E_z = g(z)[\dot{z}, \dot{z}] > 0$.

Notice that we have not defined the causal character of non-constant geodesics, which are usually considered as space-like ones. The causal character of geodesics plays a fundamental role in general relativity. Indeed time-like geodesics represent the world lines of free falling particles for which only gravity, which is encoded in the metric g, acts. Light-like geodesics represent the world lines of light rays, while space-like geodesics have no physical meaning because traveling on a space-like geodesic, a particle should be faster than light.

One of the most important properties of geodesics is that they satisfy a variational principle. Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold, we consider the *action integral*

$$E(z) = \frac{1}{2}\int_0^1 g(z)[\dot{z}(s), \dot{z}(s)]ds, \tag{12}$$

defined on the set of C^1 (or piecewise C^1) curves $z\colon [0,1] \to \mathcal{M}$. The action integral is smooth and its first variation is given by

$$E'(z)[\zeta] = \int_0^1 g(z(s))[\dot{z}, D_s\zeta]ds, \tag{13}$$

where $\zeta\colon [0,1] \to T\mathcal{M}$ is a smooth vector field along z and $D_s\zeta$ is the covariant derivative of ζ along z. Now, let p and q two points of $\mathcal{M}$ and let $\Omega(p,q;\mathcal{M})$ be the set of the smooth curves $z\colon [0,1] \to \mathcal{M}$, such that $z(0) = p$ and $z(1) = q$. Moreover, let $T_z\Omega(p,q;\mathcal{M})$ be the set of the smooth vector fields ζ along z, such that $\zeta(0) = 0$ and $\zeta(1) = 0$. Then the following variational principle gives a characterisation of the geodesics joining p and q, see for instance [33].

Theorem 1. *A curve $z(s) \in \Omega(p,q;\mathcal{M})$ is a geodesic for the metric g joining the points p and q, if, and only if, $z(s)$ is a critical point of the action integral E on $\Omega(p,q;\mathcal{M})$, that is for any $\zeta \in T_z\Omega(p,q;\mathcal{M})$,*

$$E'(z)[\zeta] = \int_0^1 g(z(s))[\dot{z}, D_s\zeta]ds = 0. \tag{14}$$

In the language of the calculus of variations, the geodesics equation $D_s\dot{z} = 0$ is the *Euler–Lagrange* equation of the action integral. Sometimes the vector fields $\zeta \in T_z\Omega(p,q;\mathcal{M})$, that are with null boundary conditions, are called *variational vector fields*.

A similar variational principle holds for geodesics satisfying more general boundary conditions, as for geodesics with endpoints on two submanifolds of $(\mathcal{M})$. In particular, a variational principle holds for closed geodesics. Let $\Lambda(\mathcal{M})$ be the set of the smooth (or at least piecewise smooth) closed curves $z\colon [0,1] \to \mathcal{M}$, so that $z(0) = z(1)$. Moreover let $T_z\Lambda(\mathcal{M})$ be the set of the vector fields ζ along z, such that $\zeta(0) = \zeta(1)$. Then the following variational principle holds:

Theorem 2. *A closed curve $z(s) \in \Lambda(\mathcal{M})$ is a closed geodesic for the metric g, if, and only if, $z(s)$ is a critical point of the action integral E on $\Lambda(\mathcal{M})$, that is $E'(z)[\zeta] = 0$, for any $\zeta \in T_z\Lambda(\mathcal{M})$.*

In particular the constant curves are closed geodesics for the metric g, so the search of closed geodesics needs to avoid the trivial constant curves.

We now analyze the second variation of the action integral. Let $z(s) \in \Omega(p, q; \mathcal{M})$ be a geodesic for the metric g joining the points p and q, then the *second variation* $E''(z)$ of the action integral E at the geodesic z is the bilinear form

$$E''(z) \colon T_z\Omega(p, q; \mathcal{M}) \times T_z\Omega(p, q; \mathcal{M}) \to \mathbb{R}$$

is well defined and for any $\zeta, \zeta' \in T_z\Omega(p, q; \mathcal{M})$, we have

$$E''(z)[\zeta, \zeta'] = \int_0^1 [g(z)[D_s\gamma, D_s\gamma'] - g(z)[R(\dot{z}, \zeta)\dot{z}, \zeta']] ds, \tag{15}$$

where $R(\cdot, \cdot)\cdot$ denotes the Riemann tensor for the metric g.

We define now the notion of Jacobi vector field along a geodesic and of non-conjugate points for the metric g.

Definition 8. *A vector field $\zeta \in T_z\Omega(p, q; \mathcal{M})$ along a geodesic z joining two points p and q is called Jacobi field if, and only if, for any $\zeta' \in T_z\Omega(p, q; \mathcal{M})$,*

$$E''(z)[\zeta, \zeta'] = \int_0^1 [g(z)[D_s\gamma, D_s\gamma'] - g(z)[R(\dot{z}, \zeta)\dot{z}, \zeta']] ds = 0. \tag{16}$$

By an integration by parts and the fundamental lemma of calculus of variations, we have that $\zeta \in T_z\Omega(p, q; \mathcal{M})$ is a Jacobi field along z if, and only if, it satisfies the *Jacobi equation*

$$D_s^2\zeta + R(\dot{z}, \zeta)\dot{z} = 0. \tag{17}$$

In other words, the Jacobi fields along z are the variational vector fields along which satisfies the Jacobi equations. Equivalently the set of Jacobi fields along z is the kernel of $E''(z)$ on $T_z\Omega(p, q; \mathcal{M})$, so it has the structure of a vectorial subspace of $T_z\Omega(p, q; \mathcal{M})$. Since the Jacobi equations consist of a linear system of differential equations of second order, we find that the dimension of such vector space is finite, and it can be proved that it is at most $n = \dim \mathcal{M}$, see [30].

We give now the important definition of conjugate points along a geodesic, which holds both for Riemannian and Lorentzian manifolds, and more in general for any semi-Riemannian manifold, see [30].

Definition 9. *Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold (for instance Riemannian or Lorentzian) and let $z \colon [0, 1] \to \mathcal{M}$ be a geodesic for the Levi-Civita connection associated to the metric g, and set $z(0) = p$ and $z(1) = q$. A point $z(s)$, $s \in]0, 1]$ is said to be conjugate to $p = z(0)$ along z, if there exists a Jacobi field $\zeta(s) \colon [0, s] \to \mathcal{M}$ along the rez to the interval $[0, s]$, such that $\zeta(0) = 0$, $\zeta(s) = 0$.*

The multiplicity of the conjugate point $z(s)$ is the number of linearly independent Jacobi fields, such that $\zeta(0) = 0$, $\zeta(s) = 0$.

Clearly the multiplicity of a conjugate point is finite and it is equal at most to $n = \dim\mathcal{M}$. We give now the definition of *geometric index* of a geodesic.

Definition 10. *As in the definition above, the geometric index $\mu(z)$ of the geodesic z is the number of conjugate points $z(s)$, $s \in]0, 1[$, counted with their multiplicity.*

One of the most important results on geodesics on Riemannian manifold is the *Morse index theorem*, which states that the geometric index $\mu(z)$ of a Riemannian manifold is finite and it is equal to the Morse index of the geodesic as a critical point of the action functional, see for instance [34]. The Morse index theorem still holds for time-like and light-like geodesics on Lorentzian manifolds, see [31], while it does not hold for space-like, for which infinitely many conjugate points may exist.

We give now the definition of *non-degenerate geodesic* and *non-conjugate points* for a semi-Riemannian metric g.

Definition 11. *A geodesic $z \in \Omega(p, q; \mathcal{M})$ joining two points p and q in the semi-Riemannian manifold $(\mathcal{M}, g)$ is said to be non-degenerate if the point q is non-conjugate to p along z, that is the null field $\zeta = 0$ is the unique Jacobi field along z with $\zeta(0) = 0$ and $\zeta(1) = 0$, or equivalently* $\mathrm{Ker} E''(z) = \{0\}$.

Definition 12. *Two points p and q of a semi-Riemannian manifold $(\mathcal{M}, g)$ are said to be non-conjugate if any geodesic $z \in \Omega(p, q; \mathcal{M})$ joining p and q is non-degenerate, or equivalently p and q are non-conjugate along any geodesic joining them.*

From the variational point of view, the fact that p and q are non-degenerate is equivalent to say that the critical points of the action integral $E(z)$ on $\Omega(p, q; \mathcal{M})$ are non-degenerate, and this is equivalent to say that $E(z)$ is a *Morse function*.

It can be proved, using the Sard theorem, that any couple of points p and q, except for a residual set of $\mathcal{M} \times \mathcal{M}$, are non-degenerate, see [34].

4. Standard Stationary Lorentzian Manifolds and the Fermat Metrics

We introduce the class of standard stationary spacetimes and the Fermat metrics in these spacetimes. We fix a Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ and we consider on $\mathcal{M}_0$ a smooth vector field $\delta(x)$ and a smooth strictly positive function $\beta(x)\colon \mathcal{M}_0 \to \mathbf{R}$. Then the *standard stationary Lorentian manifold* defined by the data $(\mathcal{M}_0, \langle \cdot, \cdot \rangle, \delta(x), \beta(x))$ is the Lorentzian manifold $(\mathcal{M}, g)$, where $\mathcal{M}$ is the product manifold $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and the metric g is defined in the following way: for any $= (x, t) \in \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, and for any $\zeta = (\xi, \tau), \zeta' = (\xi', \tau') \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$,

$$g(z)[\zeta, \zeta'] = g(x, t)[(\xi, \tau), (\xi', \tau')] = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau - \beta(x)\tau\tau'. \quad (18)$$

It can be proven that the bilinear form $g(z)$ is non-degenerate and its index is equal to 1, so it defines a Lorentzian manifold, which we shall call the standard stationary Lorentzian manifold with initial data $(\mathcal{M}_0, \langle \cdot, \cdot, \rangle, \delta(x), \beta(x))$. The vector field is sometimes called the *shift*, while the positive function $\beta(x)$ is called the *lapse* of the standard stationary metric g, see, for instance, [33] for other details.

Definition 13. *A standard stationary Lorentzian manifold is said to be static if the shift vector $\delta(x)$ is null, so the metric tensor of a standard static Lorentzian manifold has the form:*

$$g(z)[\zeta, \zeta'] = g(x, t)[(\xi, \tau), (\xi', \tau')] = \langle \xi, \xi' \rangle - \beta(x)\tau\tau'. \quad (19)$$

Many physically relevant spacetimes of general relativity are standard static or stationary, as the Minkowski spacetime of special relativity, the Schwarzschild spacetime outside the event horizon and the Reissner–Nordstrom spacetime outside the external event horizon are static, while the Kerr spacetime outside the stationary surface is stationary. In Section 5, we shall define these spacetimes and we evaluate their Fermat metrics.

Let $z(s) = (x(s), t(s))\colon [0, 1] \to \mathcal{M}_0 \times \mathbb{R}$ be a smooth curve on a standard stationary Lorentzian manifold, we shall often call the curve $x(s)\colon [0, 1] \to \mathcal{M}_0$ the *spatial part of* $z(s)$, while we shall call $t(s)\colon [0, 1] \to \mathbb{R}$ the *temporal part of* $z(s)$.

Remark 1. *A standard stationary Lorentzian manifolds admits a canonical time orientation given by the vector field $Y(z) = Y(x, t) = (0, 1)$, with $0 \in T_x\mathcal{M}$ is the null vector and $1 \in T_t\mathbb{R} \equiv \mathbb{R}$.*

Then we can define future pointing and past pointing causal vectors and future pointing and past pointing causal curves. A causal vector $\zeta = (\xi, \tau)$ is future pointing (respectively, past pointing) if, and only if, $\tau > 0$ (respectively, $\tau < 0$). A smooth causal curve $z(s) = (x(s), t(s)) \colon [0,1] \to \mathcal{M}$ is future pointing (respectively, past pointing) if, and only if, $\dot{t}(s) > 0$ (respectively, $\dot{t}(s) < 0$), for any $s \in [0,1]$.

Remark 2. *The notion of standard stationary is a particular case of the more general notion of* stationary Lorentzian manifold. *A Lorentzian manifold $(\mathcal{M}, g)$ is said to be stationary if it admits a time-like and Killing vector field $Y(z)$. Therefore $Y(z)$ satisfies $g(z)[Y(z), Y(z)] < 0$ and the flow generated by the vector field $Y(z)$ is given by isometries at any stage. A stationary Lorentzian manifold is said to be* static *if the orthogonal distribution $Y^{\perp}$ on the tangent bundle $T\mathcal{M}$ is integrable, see for instance [30,31] for the details. In a standard stationary Lorentian manifold the time-like vector field $Y(z) = Y(x,t) = (0,1)$ is Killing, essentially because the shift $\delta(x)$ and the lapse $\beta(x)$ does not depend on the variable t (they are invariant by the flow of Y). Moreover, a standard static Lorentzian manifold is static. An example of a non-standard stationary Lorentzian manifold is the* Godel spacetime, *see [29].*

We now define the Fermat metrics associated to a standard stationary Lorentzian manifolds. They are two Finsler metrics of Randers type, which reduce to a Riemannian metric when the metric is standard static. Such metrics are related to light-like vectors, light-like curves and light-like geodesics by a version of the Fermat principle, which is an extension of the classical Fermat principle to light rays in standard stationary spacetimes.

Let $(\mathcal{M}, g)$ be a stationary spacetime, with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and the metric g is given by (18). We shall assume that all curves are defined in the interval $[0,1]$.

Let $z(s) = (x(s), t(s)) \colon [0,1] \to \mathcal{M}$ be a smooth curve and assume that $z(s)$ is a light-like curve, then $z(s)$ satisfies

$$E(z) = g(z)[\dot{z}, \dot{z}] = \langle \dot{x}, \dot{x} \rangle + 2\langle \delta(x), \dot{x} \rangle \dot{t} - \beta(x)\dot{t}^2 = 0 \tag{20}$$

We solve Equation (20) with respect to $\dot{t}$ and we obtain two solutions:

$$\dot{t}_+(s) = \langle \frac{\delta(x)}{\beta(x)}, \dot{x} \rangle + \sqrt{\frac{\langle \dot{x}, \dot{x} \rangle}{\beta(x)} + \frac{\langle \delta(x), \dot{x} \rangle^2}{\beta(x)^2}}$$

and

$$\dot{t}_-(s) = \langle \frac{\delta(x)}{\beta(x)}, \dot{x} \rangle - \sqrt{\frac{\langle \dot{x}, \dot{x} \rangle}{\beta(x)} + \frac{\langle \delta(x), \dot{x} \rangle^2}{\beta(x)^2}}$$

Consider the Riemannian conformal metric $g_0 = \langle \cdot, \cdot \rangle / \beta(x)$, then we have also

$$\dot{t}_+(s) = \sqrt{g_0(x)[\dot{x}, \dot{x}] + g_0(x)[\delta(x), \dot{x}]^2} + g_0(x)[\delta(x), \dot{x}] > 0 \tag{21}$$

and

$$\dot{t}_-(s) = -\sqrt{g_0(x)[\dot{x}, \dot{x}] + g_0(x)[\delta(x), \dot{x}]^2} + g_0(x)[\delta(x), \dot{x}] < 0 \tag{22}$$

Then, in a fixed spatial curve $x(s) \colon [0,1] \to \mathcal{M}_0$, there are exactly two light-like curves $z_+(s) = (x(s), t_+(s))$ and $z_-(s) = (x(s), t_-(s))$ on $\mathcal{M}$, having the curve $x(s)$ as a spatial component. We notice that by (21) and (22), the light-like curve z_+ is future pointing, while the light-like z_- is past pointing.

In other words, we have shown that any spatial curve $x(s) \colon [0,1] \to \mathcal{M}_0$ can be lifted up to exactly two light-like curves z_+ and z_-, being z_+ pointing to the future and z_- pointing to the past.

Therefore, we have found two Finsler metrics of Randers type $F_+, F_- \colon T\mathcal{M}_0 \to \mathbb{R}_+$, defined for any $(x,v) \in T\mathcal{M}_0$, $x \in \mathcal{M}_0$ and $v \in T_x\mathcal{M}_0$:

$$F_+(x,v) = \sqrt{g_0(x)[v,v] + g_0(x)[\delta(x),v]^2} + g_0(x)[\delta(x),v] \tag{23}$$

and

$$F_-(x,v) = \sqrt{g_0(x)[v,v] + g_0(x)[\delta(x),v]^2} - g_0(x)[\delta(x),v] \tag{24}$$

We call them the Fermat metrics associated to the standard stationary Lorentzian manifold $(\mathcal{M}, g)$ and were explicitly introduced in [17].

Now, let $z_+ = (x(s), t_+(s)) \colon [0,1] \to \mathcal{M}$ be a future pointing light-like curve, then the travel time $T_+(z)$ of z_+ is a defined setting.

$$\begin{aligned} T(z_+) &= t(1) - t(0) = \int_0^1 \dot{t}_+(s)ds \\ &= \int_0^1 \left(\sqrt{g_0(x)[\dot{x},\dot{x}] + g_0(x)[\delta(x),\dot{x}]^2} + g_0(x)[\delta(x),\dot{x}] \right) ds \\ &= \int_0^1 F_+(x,\dot{x})ds, \end{aligned} \tag{25}$$

so we have $T(z_+) = \int_0^1 F_+(x,\dot{x})ds$, that is *the travel time $T(z_+)$ is equal to the length of the spatial projection x with respect to the Finsler metric F_+.*

Analogously, let $z_- = (x(s), t_-(s)) \colon [0,1] \to \mathcal{M}$ be a past pointing light-like curve, then the travel time $T(z)$ of z_- is defined setting

$$\begin{aligned} T(z_-) &= t(1) - t(0) = \int_0^1 \dot{t}_-(s)ds \\ &= \int_0^1 \left(-\sqrt{g_0(x)[\dot{x},\dot{x}] + g_0(x)[\delta(x),\dot{x}]^2} + g_0(x)[\delta(x),\dot{x}] \right) ds \\ &= -\int_0^1 F_-(x,\dot{x})ds, \end{aligned} \tag{26}$$

so that $T(z_-) = -\int_0^1 F(x,\dot{x})ds$, that is the travel time $T(z_-)$ is equal to minus the length of the spatial projection x with respect to the Finsler metric F_-.

So we have characterised the travel time of a light-like curve $z(s) = (x(s), t(s))$ of a standard stationary Lorentzian metric as the length of its spatial part x with respect to a Finsler metric of Randers type, the metric F_+ if $z(s)$ is future pointing, the metric F_-, if $z(s)$ is past pointing.

This characterisation recalls the Fermat principle in classical geometric optics. In an isotropic medium, modelled with a smooth manifold $\mathcal{M}_0$ with Riemannian metric g_0, the light speed depends only on the position and not on the direction, and is proportional to the refraction index $n(x) > 0$ of the medium. The travel time of a curve $x(s) \colon [0,1] \to \mathcal{M}_0$ in the medium is given by

$$T(x) = \int_0^1 \frac{\sqrt{g_0(x)[\dot{x},\dot{x}]}}{n(x)} ds \equiv \int_0^1 \frac{ds}{v},$$

so the the travel time $T(x)$ is equal to the length of the curve x with respect to the conformal metric g_0/n, which is often called the *optical metric*.

Theorem 3 (Fermat principle in classical optics)**.** *The light rays joining two fixed points in the isotropic medium $\mathcal{M}_0$ with refraction index $n(x)$ are the stationary points of the travel time $T(x)$ and they are the geodesics of the optical metric g_0/n.*

Inspired by the Fermat principle in classical optics, we give the following definition

Definition 14. *Let $(\mathcal{M}, g)$ be a standard stationary spacetime, the Fermat metrics associated to $(\mathcal{M}, g)$ are the Finsler–Randers metrics defined at (23) and (24) and are often called optical metrics associated to $(\mathcal{M}, g)$.*

Remark 3. *If the standard stationary spacetime $(\mathcal{M}, g)$ is also static, the vector field δ is null and the two Fermat metrics F_+ and F_- are both equal to the reversible Finsler metric $\sqrt{g_0}$, induced by the Riemannian metric g_0, that is*

$$F_+(x, v) = F_-(x, v) = \sqrt{g_0(x)[v, v]}.$$

Therefore, in a static spacetime we have only one optical metric and it is a Riemannian metric.

We state an extension of the Fermat principle of classical optics to standard stationary Lorentzian manifolds. Let $(\mathcal{M}, g)$ be a stationary Lorentzian manifold, with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and the standard stationary metric g given by (18). Let $p = (x_0, 0) \in \mathcal{M}$ and let $\gamma(s) = (x_1, s) \colon \mathbb{R} \to \mathcal{M}$ be a vertical line which represents the world line of a stationary observer, with x_0, $x_1 \in \mathcal{M}_0$. We give a characterisation of light-like geodesics (light rays) $z(s) = (x(s), t(s)) \colon [0, 1] \to \mathbf{R}$ joining p with γ, that is $x(0) = x_0$, $x(1) = x_1$, $t(0) = 0$, while $t(1)$ is "free". The following theorem holds.

Theorem 4 (Fermat principle for future pointing light-like geodesics)**.** *A light-like curve $z_+(s) = (x_+(s), t_+(s)) \colon [0.1] \to \mathcal{M}$ is a future pointing light-like geodesic for g joining p and the curve γ if, and only if, the temporal component $t_+(s)$ satisfies (21) with $x = x_+$, and the spatial component x_+ is a pregeodesic for the Fermat metric F_+ . Moreover, the travel time $T(z_+)$ of z_+ is equal to the length $F_+(x_+)$ of x_+ with respect to F_+. Finally, the spatial part x_+ is a pregeodesic for the Fermat metric F_+ parameterised in order that $g_0(\dot{x}_+, \dot{x}_+) + g_0(\delta(x_+), \dot{x}_+)^2$ is a constant.*

An analogous result holds for past pointing light-like geodesics.

Theorem 5 (Fermat principle for past pointing light-like geodesics)**.** *A light-like curve $z_-(s) = (x_-(s), t_-(s)) \colon [0.1] \to \mathcal{M}$ is a past pointing light-like geodesic for g joining p and the curve γ if, and only if, the temporal part $t_-(s)$ satisfies (22), with $x = x_-$ and the spatial part x_- is a pregeodesic for the Fermat metric F_-. Moreover, the travel time $T(z_-)$ is equal to the minus the length $-F_-(x_-)$ of x_- with respect to F_-. Finally, the spatial part x_- is a pregeodesic for the Fermat metric F_- parameterised, in order that $g_0(\dot{x}_-, \dot{x}_-) + g_0(\delta(x_-), \dot{x}_-)^2$ is a constant.*

The previous theorems was first proved in [17], exploiting the role of the Fermat metrics in the statement. Previous result on variational characterisations of light rays for stationary spacetimes were obtained by H. Weyl in [3], who stated a Fermat principle in static spacetimes, see also [4]. A Fermat principle for light rays on stationary spacetimes was stated by Tullio Levi Civita, see [5] and the paper [15].

Remark 4. *We point out that all the previous results for the Fermat metrics and the Fermat principles hold for the class of conformally standard stationary Lorentzian manifold $\bar{g} = \alpha(x, t)g$, where g is a standard stationary metric and $\alpha \colon \mathcal{M} \to \mathbb{R}$ is a smooth and strictly positive function. This is due to the fact that the light-like curves and light-like pregeodesics are invariant by a conformal change of the metric. This fact permits to apply the results also to wider classes of Lorentzian metrics.*

Remark 5. *A Fermat principle holds for arbitrary spacetimes, and it was first stated by Kovner [35], see also [36].*

5. Some Example of Fermat Metrics

In this section, we present some examples of Fermat metrics, associated to physically relevant spacetimes of general relativity, for more details on the physical properties of these spacetimes see [29].

Example 3. *[Minkowski spacetime] The $(n+1)$- Minkowski spacetime $(\mathbb{R}^{n+1}, g_M)$ of special relativity is a static Lorentzian manifold, with*

$$g(x,t)[(\xi,\tau),(\xi',\tau')] = \langle \xi, \xi' \rangle - \tau\tau',$$

where $\langle \cdot, \cdot \rangle$ is the standard, positive definite Euclidean scalar product in $\mathbb{R}^n$. Clearly, the Minkowski spacetime $(\mathbb{R}^{n+1}, g_M)$ is a static Lorentzian manifold, and the two Fermat metrics F_+ and F_- are equal to the Euclidean metric $\langle \cdot, \cdot \rangle$.

Example 4 (Swarzschild spacetime)**.** *Let m be a positive constant, let (r, θ, ϕ) the polar coordinates in $\mathbb{R}^3$, with $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\theta \in]0, \pi[$, $\varphi \in]0, 2\pi[$, and consider the manifold $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, where*

$$\mathcal{M}_0 = \{x \in \mathbb{R}^3 : r = |x| > 2m\}.$$

The Scharzschild metric ds^2 on the manifold $\mathcal{M}$ is defined setting in the coordinates (r, θ, ϕ, t),

$$ds^2 = \frac{1}{\beta(r)} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) - \beta(r) dt^2, \tag{27}$$

where

$$\beta(r) = 1 - \frac{2m}{r}.$$

The couple $(\mathcal{M}, ds^2)$ is called the Schwarzschild spacetime outside the event horizon. From the physically point of view, the Schwarzschild spacetime represents the empty spacetime outside a spherically symmetric massive body of mass m, as a star. The number $r = 2m$ is called the Schwarzschild radius. The Swarzschild spacetime is static, so the two Fermat metrics F_+ and F_- are equal to the Riemannian conformal metric

$$g_0 = \frac{1}{\beta(r)} \left[\beta(r) dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \right],$$

which is a complete Riemannian metric on $\mathcal{M}_0$.

Example 5 (Reissner–Nördstrom spacetime)**.** *Another physical relevant example of static spacetime is the Reissner–Nördstorm spacetime, which represents the spacetime outside a spherically symmetric charged body of mass m and electric charge e. The Reissner–Nördstorm metric is defined as in (27), but now with $\beta(r)$ defined as*

$$\beta(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}.$$

Assuming that $m > e$, then the Reissner–Nördstrom spacetime is defined and is static on the manifold $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, where

$$\mathcal{M}_0 = \{x \in \mathbb{R}^3 : r = |x| > m + \sqrt{m^2 - e^2}\}.$$

Again, the two Fermat metrics are equal.

We present now a stationary and non-static spacetime, the Kerr *spacetime, for which the Fermat are defined and distinct.*

Example 6 (Kerr spacetime)**.** *The Kerr spacetime outside the stationary limit surface represents the stationary, axis-symmetric, and asymptotically flat spacetime outside a rotating massive object. Let m and a be two constant, with $m > 0$, which represent, respectively, the mass and the angular momentum of the massive body as measured from infinity. In the Boyer–Lindquist coordinates (r, θ, ϕ, t) coordinates, the Kerr metric is given by*

$$g_K = \lambda(r,\theta)\left(\frac{dr^2}{\Delta(r)} + \theta^2\right) + (r^2 + a^2)d\phi^2 - dt^2 + \frac{2mr}{\lambda(r,\theta)}(asin^2\theta d\phi - dt)^2, \tag{28}$$

where

$$\lambda(r,\theta) = r^2 + a^2 cos^2\theta,$$

and

$$\Delta(r) = r^2 - 2mr + a^2.$$

Notice that, when $a = 0$, the Kerr metric reduces to the Schwarzschild metric. Assume that $m^2 > a^2$, and consider the stationary limit surface

$$\mathcal{M}_a = \{(r,\theta,\phi) : r = m + \sqrt{m^2 - a^2cos^2\theta}\},$$

then the metric (28) is stationary on the Kerr spacetime outside the stationary limit surface

$$\mathcal{N}_a = \{(r,\theta,\phi) : r > m + \sqrt{m^2 - a^2cos^2\theta}\} \times \mathbb{R}, \tag{29}$$

and, equipped with the metric (28), is called the Kerr spacetime outside the stationary limit surface.

Now we evaluate the Fermat metrics in the spacetime outside the limit surface. Let $z(s) = (r(s), \theta(s), \phi(s), t(s)) \colon [0,1] \to \mathcal{M}_a$ be a light-like curve with image in the Kerr spacetime outside the stationary limit surface, so it satisfies:

$$0 = \lambda(r,\theta)\left(\frac{\dot{r}^2}{\Delta(r)} + \dot{\theta}^2\right) + (r^2 + a^2)\dot{\phi}^2 - \dot{t}^2 + \frac{2mr}{\lambda(r,\theta)}(asin^2\theta\dot{\phi} - \dot{t})^2.$$

Setting

$$\beta(r) = 1 - \frac{2mr}{\lambda(r,\theta)},$$

and solving with respect to $\dot{t}$ we obtain

$$\dot{t} = \frac{2marsen^2\phi}{\lambda(r,\theta)\beta(r,\theta)}\dot{\phi} \pm \sqrt{\frac{4m^2a^2r^2sin^4\phi}{\lambda^2\beta^2}\dot{\phi}^2 + \frac{\lambda/\Delta\dot{r}^2 + \lambda\dot{\theta}^2 + (r^2+a^2)sin^2\theta\dot{\phi}^2 + a^2sin^4\theta\dot{\phi}^2}{\beta}}.$$

Therefore, we obtain the two Fermat metrics for any $x = (r, \theta, \phi)$ and for any tangent vector $v = (v_r, v_\theta, v_\phi)$ to x,

$$F_+(r,\theta,\phi)[v_r, v_\theta, v_\phi] = \sqrt{A(r,\theta)v_r^2 + B(r,\theta)v_\theta^2 + C(r,\theta)v_\phi^2} + \frac{2marsen^2\theta}{\lambda(r,\theta)\beta(r,\theta)}v_\phi, \tag{30}$$

and

$$F_-(r,\theta,\phi)[v_r, v_\theta, v_\phi] = \sqrt{A(r,\theta)v_r^2 + B(r,\theta)v_\theta^2 + C(r,\theta)v_\phi^2} - \frac{2marsen^2\theta}{\lambda(r,\theta)\beta(r,\theta)}v_\phi, \tag{31}$$

where

$$A(r,\theta) = \frac{\lambda(r,\theta)}{\beta(r,\theta)\Delta(r)},$$

$$B(r,\theta) = \frac{\lambda(r,\theta)}{\beta(r,\theta)},$$

$$C(r,\theta) = \frac{4ma^2r^2sin^4\theta}{\lambda(r,\theta)^2\beta(r,\theta)^2} + \frac{(r^2+a^2)sin^2\theta}{\beta(r,\theta)} + \frac{2mra^2sin^4\theta}{\lambda(r,\theta)\beta(r,\theta)}.$$

6. A Review of Critical Point Theory

In this section, we review the main results of critical point theory that we shall apply to Finsler manifolds and light rays. The critical point theory on infinite dimensional manifolds was developed by Richard Palais and Stephen Smale in a series of papers in the 1960s, see [9,10,37,38]. Then the theory has been applied to the study of the solutions of non-linear ordinary or partial differential equations, and geometric analysis. Since, from [9], the theory has been applied to the study of geodesics on Riemannian manifolds. Successively geodesics on Finsler manifolds and Lorentzian manifolds have been studied. We refer to [11,33,39,40] for the proof of the main results on critical point theory and several applications to non-linear differential equations.

Let (X,h) be a smooth (C^3), possibly infinite dimensional Riemannian manifold, we consider a functional $f : X \to \mathbb{R}$ of class C^1. A critical point of f is a point $x \in X$, such that $f'(x) = 0$. A *critical value* of f is a real number c, such that there exists a critical point x of f, such that $f(x) = c$.

We introduce now the *Palais–Smale* (PS) compactness condition, which plays a basic role in infinite dimensional variational problems.

Definition 15. *Let $f : X \to \mathbf{R}$ be a functional of class C^1 on a Riemannian manifold (X,h), we say that f satisfies the Palais–Smale (PS) compactness condition, if any sequence $(x_k)_{k\in\mathbf{N}}$ of points of X, such that:*

(i) $\{f(x_k)\}_{k\in\mathbf{N}}$ *is bounded;*
(i) $\|\nabla f(x_k)\| \to 0$,

admits a converging subsequence. Here $\|\cdot\|$ is the norm induced on the tangent bundle TX by the Riemannian metric h on X

We state now that the theorem on the existence of a minimum point of a smooth functional satisfying the (PS) condition, see [39,40].

Theorem 6. *Let $f : (X,h) \to \mathbf{R}$ be a C^1 functional, defined on a complete Riemannian manifold (X,h), and assume that f is bounded from below and satisfies the Palais–Smale condition on X. Then the infimum is attained by f and there exists a point $x_0 \in X$, such that $f(x_0) = \inf_X f$.*

We now state a multiplicity result of critical points of the functional f, for which the topological properties of the manifold X plays a basic role. We now introduce a topological invariant, the *Ljusternik–Schnirelmann category* of a topological space, which gives a measure of how much a topological space has a rich topology. The Ljusternik–Schnirelmann category gives a lower bound on the number of critical points of a C^1 functional.

Definition 16. *Let X be a topological space and let A be a subset of X, the Ljusternik–Schnirelmann category of the set A in X, denoted by $\mathrm{cat}_X(A)$ is equal to the minimal number of closed and contractible subsets of X which cover A. If such a minimal number does not exist, we set $cat_X(A) = +\infty$. Moreover we set $\mathrm{cat}(X) = \mathrm{cat}_X(X)$.*

The following theorem holds, see [33,39,40].

Theorem 7. *Let $f : (X,h) \to \mathbf{R}$ be a C^1 functional defined on a complete Riemannian manifold (X,h), bounded from below and satisfying the Palais–Smale condition. Then the functional f has at least $\mathrm{cat}(X)$ critical points. Moreover, if $\mathrm{cat}(X) = +\infty$, then there exists a sequence x_n of critical points of f, such that $f(x_n) \to +\infty$.*

The proof of this theorem was obtained by Ljusternik and Schnirelmann at the end of the 1920s. If the manifold X is not contractible, for instance X is a Hilbert space, the previous theorem reduces to existence of a minimum point for f.

We now present the main results on Morse theory for a functional f defined on a complete Riemannian manifold, bounded from below and satisfying the Palais–Smale condition. Morse theory gives more precise estimates on the critical points of a functional, with respect to the Ljusternik–Schnirelmann theory, in particular on the number of critical points. However, in order to develop the Morse theory, we need a major regularity assumption on the functional f, namely it has to be of class C^2. Moreover, we assume that the critical points of f have finite Morse index and are non-degenerate, and sometimes we shall say that f is *a Morse function*.

Therefore, assume that the functional f is of class C^2 on the Riemannian manifold (X, h) and let x be a critical point of f, we can define the Hessian $H_f(x)$ of the functional f at the point x in the following way: denote by T_xX the tangent space at x to X, the Hessian $H_f(x)$ at x is the bilinear form $H_f(x) : T_xX \times T_xX \to \mathbb{R}$ defined setting, for any $\xi \in T_xX$,

$$H_f(x)[\xi, \xi] = \left(\frac{d^2 f(\gamma(s))}{ds^2}\right)_{s=0},$$

where $\gamma :]-\varepsilon, \varepsilon[\to X$ is a smooth curve, such that $\gamma(0) = x$, $\dot{\gamma}(0) = \xi$, and then we extend $H_f(x)$ by polarisation to any couple of tangent vectors.

The *Morse index* $m(x, f)$ of the critical point x of f is the maximal dimension of a subspace of T_xX where $H_f(x)$ is negative definite. The *augmented Morse index* $m^*(x, f)$ is defined as $m^*(x, f) = m(x, f) + \dim\ker H_f(x)$, where

$$\ker H_f(x) = \{\xi \in T_xX : H_f(x)[\xi, \xi'] = 0, \forall \xi' \in T_xX\}.$$

Clearly the Morse index and the augmented Morse index may be equal to $+\infty$ if X is an infinite dimensional Riemannian manifold. The critical point x is said *non-degenerate* if the linear operator $f''(x) : T_xX \to T_xX$ induced by $H_f(x)$ on T_xX, equipped of the Hilbert space structure $h(x)$, is an isomorphism. The functional f is said to be a *Morse function* if it is of class C^2 on the manifold (X, h) and all its critical points are non-degenerate.

We give now some recalls on algebraic topology that we need to state the main results of Morse theory, see for instance [41] for the details.

We consider a topological pair (A, B), where A is a topological space and B is a subspace of A, and we consider an algebraic field $\mathcal{K}$ a field. For any $k \in \mathbb{N}$, $H_k(A, B; \mathcal{K})$ the k-th relative homology group, with coefficients in $\mathcal{K}$, of the topological pair (A, B) are well defined. Moreover, since $\mathcal{K}$ is a field, the homology group $H^k(A, B; \mathcal{K})$ is also a vector field and its dimension $\beta_k(A, B; \mathcal{K}) \in \mathbb{N} \cup \{+\infty\}$ is called the k–th *Betti number* of (A, B) with respect to $\mathcal{K}$.

The Poincarè polynomial $\mathcal{P}(A, B; \mathcal{K})(r)$ of the pair (A, B), with respect to the field $\mathcal{K}$ is a formal series whose coefficients are positive cardinal numbers belonging to $\mathbb{N} \cup \{+\infty\}$, and it defined setting

$$\mathcal{P}(A, B; \mathcal{K})(r) = \sum_{k=0}^{\infty} \beta_k(A, B; \mathcal{K}) r^k.$$

We state now the Morse relations for a Morse functional of class C^2, and having critical points of finite index. The Morse relations link the set of the critical points of the functional to the Poincaré polynomial of the manifold with respect to the field $\mathcal{K}$. Again, the Morse relations allow to relate the set of critical points of the functional f to the topological properties of the manifold X measured by the Poincaré polynomial. As corollaries of the Morse relations we obtain the classical Morse inequalities and the total Betti number formula for the critical points of f. For the proofs see [39,40].

Theorem 8. *Let $f : X \to \mathbf{R}$ be a functional of class C^2, defined on a complete Riemannian manifold (X, h) of class C^3. Assume that f is bounded from below, satisfies the Palais–Smale (PS) condition on X. Moreover, assume that all the critical points of f are non-degenerate and have finite Morse index $m(x, f)$, and denote by $K(f)$ the set of the critical points of f.*

Then for any field $\mathcal{K}$ there exists a formal series $Q(r)$ with coefficients in $\mathbb{N} \cup \{+\infty\}$, such that

$$\sum_{x \in K(f)} r^{m(x,f)} = \mathcal{P}(X, \mathcal{K})(r) + (1 + r)Q(r). \tag{32}$$

Moreover, let for any $k \in \mathbf{N}$, $\beta_k(X; \mathcal{K})$ the k–th Betti number of the manifold X with respect to the field $\mathcal{K}$ and denote by $M(f, k)$ the number of critical points x of f, such that the Morse index $m(x, f)$ is equal to k. Then the Morse inequalities hold:

$$M(f, k) \geq \beta_k(X; \mathcal{K}). \tag{33}$$

Finally, denote by $\mathcal{B}(X, \mathcal{K})$ the total Betti number of f with respect to the field $\mathcal{K}$, that is

$$\mathcal{B}(X, \mathcal{K}) = \sum_{k=0}^{\infty} \beta_k(X, \mathcal{K}).$$

Then the total Betti number formula on the number $\sharp K(f)$ of the critical points of f holds:

$$\sharp K(f) = \mathcal{B}(X, \mathcal{K}) + 2Q(1). \tag{34}$$

We point out that, under the assumption of the previous theorem, it can be proved the set of the critical points of the functional f is countable, because nondegenerate critical points are isolated and (PS) condition holds, so that Equation (32) is well defined.

Remark 6. *We point out that in order to state the Morse relations (32), we need that the functional f is of class C^2, for instance in order to define the Morse index. In the next section we shall see that the action integral of a Finsler is only of class $C^{1,1}$, that is of class C^1 with locally Lipschitz gradient. In spite of this, we shall see that Morse relations hold also for Finsler metrics.*

7. Variational Theory for Geodesics on a Finsler Manifold

In this section, we state some results on the existence and multiplicity of geodesics joining two points on a Finsler manifold. We refer to [17] for the details of the proof using the infinite dimensional variational approach and general boundary conditions. Moreover, we state the Morse relations for geodesics joining two non-conjugate points, see [42,43]. The variational approach was already introduced by Franco Mercuri in the study of closed geodesics on a compact Finsler manifold, see [44]. An approach using the geodesic shortening flow was developed by Matthias in [45], and finally a variational approach for reversible Finsler metrics was previously developed by Kozma, Kristaly, and Varga, see [46].

We fix the interval $I = [0, 1]$. Moreover, let $n \in \mathbb{N}$, $n \geq 1$, we shall consider the Sobolev space $H^{1,2}(I, \mathbb{R}^n)$ of the absolutely continuous curves $x(s) \colon I \to \mathbb{R}^n$, such that the derivative $\dot{x}(s) \in L^2(I, \mathbb{R}^n)$, see for instance [47]. It is well known that the space $H^{1,2}(I, \mathbb{R}^n)$ has a structure of Hilbert space with norm

$$\|x\|^2 = \int_0^1 |x(s)|^2 \mathrm{d}s + \int_0^1 |\dot{x}(s)|^2 \mathrm{d}s.$$

We consider also the subspace $H_0^{1,2}(I, \mathbb{R}^n)$ of $H^{1,2}(I, \mathbb{R}^n)$ of the curves $x(s)$, such that $x(0) = x(1) = 0$. On the subspace $H_0^{1,2}(I, \mathbb{R}^n)$ an equivalent norm is given by

$$\|x\|_0^2 = \int_0^1 |\dot{x}(s)|^2 \mathrm{d}s.$$

Definition 17. *Let $\mathcal{M}$ be a smooth connected manifold of dimension $n \geq 1$, the* Sobolev manifold *$H^{1,2}(I, \mathcal{M})$ is the set of the curves $x(s) \colon I \to \mathcal{M}$, such that for any local chart (U, φ) of the manifold $\mathcal{M}$ (where $\varphi \colon U \to \varphi(U) \subset \mathbb{R}^n$ is an homeomorphism onto the open subset $\varphi(U)$ of $\mathbb{R}^n$), the curve $\varphi \circ x \colon x^{-1}(U) \to \mathbb{R}^n$ is of class $H^{1,2}(x^{-1}(U), \mathbb{R}^n)$.*

The Sobolev manifolds were first introduced in the fundamental paper in [9], and it was also proven that $H^{1,2}(I, \mathcal{M})$ is equipped of a structure of infinite dimensional Hilbert manifold modelled on the Hilbert space $H^{1,2}(I, \mathbb{R}^n)$. For any $x \in H^{1,2}(I, \mathcal{M})$, the tangent space $T_x H^{1,2}(I, \mathcal{M})$ to $H^{1,2}(I, \mathcal{M})$ at the curve x is given by

$$T_x H^{1,2}(I, \mathcal{M}) = \{\xi \in H^{1,2}(I, T\mathcal{M}), \pi \circ \xi = x\} \tag{35}$$

where $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$ and $\pi = T\mathcal{M} \to \mathcal{M}$ is its bundle projection. In other words, an element $\xi(s) \in T_x H^{1,2}(I, \mathcal{M})$ is a continuous vector field along the curve $x(s)$, such that, locally, $\xi(s)$ is of class $H^{1,2}$.

The Sobolev manifold $H^{1,2}(I, \mathcal{M})$ admits two important submanifolds for the study of variational problems of geodesics. Both submanifolds are still infinite dimensional. The first consists of curves joining two fixed points. Let p and q on $\mathcal{M}$ and set

$$\Omega^{1,2}(p, q; \mathcal{M}) = \{x(s) \in H^{1,2}(I, \mathcal{M}) : x(0) = p, x(1) = q\}. \tag{36}$$

It can be proved, see for instance [48] that $\Omega^{1,2}(p, q; \mathcal{M})$ is a smooth infinite dimensional submanifold of $H^{1,2}(I, \mathcal{M})$ and the tangent space at a point $x \in \Omega^{1,2}(p, q; \mathcal{M})$ is given by

$$T_x \Omega^{1,2}(p, q; \mathcal{M}) = \{\xi \in T_x H^{1,2}(I, \mathcal{M}) : \xi(0) = 0, \xi(1) = 0\}.$$

The second submanifold $\Lambda^{1,2}(\mathcal{M})$ consists of closed curves and it is defined as

$$\Lambda^{1,2}(\mathcal{M}) = \{x(s) \in H^{1,2}(I, \mathcal{M}) : x(0) = x(1)\}. \tag{37}$$

Additionally, $\Lambda^{1,2}(\mathcal{M})$ is an infinite submanifold of $H^{1,2}(I, \mathcal{M})$ and the tangent space at a point $x \in \Lambda^{1,2}(p, q; \mathcal{M})$ is given by

$$T_x \Lambda^{1,2}(I, \mathcal{M}) = \{\xi \in T_x H^{1,2}(I, \mathcal{M}) : \xi(0) = \xi(1)\}.$$

We have introduced the Sobolev manifolds where the variational problems are well defined for geodesics on a Riemannian manifold, or more generally on a Finsler manifold and a Lorentzian manifold. We state now some results on the existence and multiplicity for geodesics on a Finsler manifold. Moreover we state the Morse relations for the non-conjugate points. We fix a Finsler manifold $(\mathcal{M}, F)$ as in Definition 1, we consider the action integral $E(x) \colon H^{1,2}(I, \mathcal{M}) \to \mathbb{R}$,

$$E(x) = \frac{1}{2} \int_0^1 F^2(x, \dot{x}) ds. \tag{38}$$

defined on the Hilbert–Sobolev manifold of the H^1-curves on the manifold $\mathcal{M}$. The action functional E is of class $C^{1,1}$, that is, it is of class C^1 and its differential is locally Lipschitz. Moreover, if the Finsler metric F is the square root of a Riemannian metric, then $E(x)$ is of class C^2, see for instance [49].

The following variational principles hold which characterise the geodesics joining two points or the closed geodesics joining two points, see [17].

Theorem 9. *Let $(\mathcal{M}, F)$ be a Finsler manifold and let p and q be two points of $\mathcal{M}$, then geodesics joining p and q for the Finsler metric F are the critical points of the action integral $E(x)$ restricted to the submanifold $\Omega^{1,2}(p, q; \mathcal{M})$ of $H^{1,2}(I, \mathcal{M})$. Moreover the closed geodesics for F are the critical points of the restriction of $E(x)$ to the sumbmanifold $\Lambda^{1,2}(\mathcal{M})$ of $H^{1,2}(I, \mathcal{M})$. In both cases the geodesics satisfy a variational principle, indeed if $x(s) \colon [0, 1] \to \mathcal{M}$ is a geodesic joining two fixed* *points p and q, or it is a closed geodesic, then there exists a non-negative constant E_x, such that for any $s \in [0,1]$,*

$$E_x = F^2(x(s), \dot{x}(s)).$$

The previous variational principle holds for more general boundary conditions, see [17]. We point out that the constant E_x is null if, and only if, x is a trivial geodesic, that is, it is reduced to a point.

The previous theorem and the abstract critical theorems stated in Section 6 allow to produce some results on the existence and multiplicity for geodesics joining two points on a Finsler manifold, satisfying some completeness assumptions.

We first state an existence result for geodesics joining two points.

Theorem 10. *Let $(\mathcal{M}, F)$ be a forward or backward complete Finsler manifold, then for any couple of points p and q, there exists a geodesic joining p and q for the Finsler metric F.*

A proof of the previous can be found in [17], where it is shown that the action integral $E(x)$, which is obviously bounded from below, satisfies the (PS) condition, under the assumption that F is forward or backward complete.

In order to state a multiplicity result for the geodesics joining two points, the topological richness of the infinite dimensional manifold $\Omega^{1,2}(p,q;\mathcal{M})$ has to be studied. This was performed by Fadell and Husseini in the two papers [50,51], which study the topological properties of the based loop space of a manifold. The main results of Fadell and Husseini are stated in the following theorem:

Theorem 11 (Fadell-Husseini)**.** *Let $\mathcal{M}$ be a smooth and non-contractible manifold and let p and q two points of $\mathcal{M}$, then the Ljusternik–Scnirelmann category of the Sobolev manifold $\Omega^{1,2}(p,q;\mathcal{M})$ is equal to $+\infty$. Moreover, there exists a sequence $(K_m)_{m\in\mathbb{N}}$ of compact subsets of $\Omega^{1,2}(p,q;\mathcal{M})$, such that*

$$\lim_{m\to\infty} cat_{\Omega^{1,2}(p,q;\mathcal{M})}(K_M) = +\infty.$$

By applying the previous theorem to the restriction of the action integral on $\Omega^{1,2}(p,q;\mathcal{M})$ we attain the following multiplicity result on the geodesics joining two points on a Finsler manifold, see [17].

Theorem 12. *Let $(\mathcal{M}, F)$ be a forward or backward complete Finsler manifolds, and assume that the manifold $\mathcal{M}$ is non-contractible.*

Then, for any couple points p and q of $\mathcal{M}$, there exists a sequence $(x_m)_{m\in\mathbb{N}}$ of geodesics for the Finsler metric F, such that

$$\lim_{m\to\infty} E_{x_m} = \lim_{m\to\infty} \frac{1}{2}\int_0^1 F^2(x_m, \dot{x}_m)ds = +\infty$$

We examine now the Morse theory for geodesics on a Finsler manifold. Now there are remarkable differences between the Riemannian geometry and Finsler geometry. Indeed, in Riemannian geometry the action functional (10) is of class C^2 and this fact permits the finding of the classical Morse Relations developed by Palais in [9], see also [34] for a proof using the classical curve shortening procedure of Marston Morse.

Instead, for a generic Finsler metric the action integral is only of class $C^{1,1}$ and so we cannot apply Theorem 8 to obtain the Morse relations for geodesics. In spite of this, the Morse relations can be obtained also in the case of Finsler manifolds.

We now start the Riemannian case. Let $(\mathcal{M}, g)$ be a Riemannian manifold and let p and q two points of $\mathcal{M}$. The action integral $E(x)\colon \Omega^{1,2}(p,q;\mathcal{M}) \to \mathbb{R}$ is of class C^2 (see for instance [48]). Let x be a critical point of E on $\Omega^{1,2}(p,q;\mathcal{M})$ (that is x is a geodesic joining

p and q), then the Hessian $E''(x)\colon T_x\Omega^{1,2}(p,q;\mathcal{M}) \times T_x\Omega^{1,2}(p,q;\mathcal{M}) \to \mathbb{R}$ ig given for any $\xi, \xi' \in T_x\Omega^{1,2}(p,q;\mathcal{M})$, by

$$E''(x)[\xi,\xi'] = \int_0^1 \left[g(x)[D_s\xi, D_s\xi'] - g(x)[R(\dot{x},\xi)\dot{x},\xi'] \right] ds,$$

where D_s is the Levi-Civita connection associated to the Riemannian metric g, $R(\cdot,\cdot)\cdot$ denotes the Riemann tensor for the metric g, see also (15). Now assume that p and q are *non-conjugate*, that is there are no non-trivial $\xi(s) \in T_x\Omega^{1,2}(p,q;\mathcal{M})$ solutions of the Jacobi equations

$$D_s^2\xi + R(\dot{x},\xi)\dot{x} = 0,$$

then the action integral is a *Morse function*, that is all its critical points are non-degenerate. Moreover, it can be proved that any geodesic joining p and q has finite Morse index $m(x,E)$ and co-index $m^*(X,E)$, because the Hessian $E''(x)$ is a perturbation of the positive definite bilinear form $Q(x)[\xi,\xi'] = \int_0^1 g(x)[D_s\xi, D_s\xi']ds$ by the bilinear form defining a compact operator $K(x)[\xi,\xi'] = \int_0^1 g(x)[R(\dot{x},\xi)\dot{x},\xi']ds$. Moreover, the fundamental *Morse index theorem* holds:

Theorem 13. *The Morse index $m(x,E)$ of a geodesic x joining the two non-conjugate points p and q is finite and it is equal to the geometric index $\mu(x)$ of the geodesic, that is number of conjugate points $x(s)$, $s \in]0,1[$ along x, counted with their multiplicities.*

Putting together these results and applying Theorem 8, we obtain the Morse relations for the geodesics joining two non-conjugate points of a Riemannian manifold.

Theorem 14. *Let $(\mathcal{M},g)$ be a complete Riemannian manifold and let p, $q \in \mathcal{M}$ be two non-conjugate points for the metric g.*

Then, for any field $\mathcal{K}$ there exists a formal series $Q(r)$ whose coefficient belong to $\mathbb{N} \cup \{+\infty\}$ such that

$$\sum_{x \in K(E)} r^{m(x,E)} = \mathcal{P}(\Omega^{1,2}(p,q;\mathcal{M}),\mathcal{K}) + (1+r)Q(r),$$

where we have denoted by $K(p,q)$ the set of the geodesics joining p and q.

We consider the case now of a Finsler manifold $(\mathcal{M},F)$, the action integral

$$E(x) = \frac{1}{2}\int_0^1 F^2(x,\dot{x})ds$$

is only of class $C^{1,1}$ on the Sobolev manifold $H^{1,2}(I,\mathcal{M})$, see, for instance, [17,44,49], so we cannot obtain the Morse relations for geodesics directly from the abstract Theorem 8. However the Morse relations for the geodesics joining two non-conjuagte points still hold.

Fix two points p and q in the manifold $\mathcal{M}$ and consider the restriction of the action integral E to the manifold $\Omega^{1,2}(p,q;\mathcal{M})$, so that the critical points of E on $\Omega^{1,2}(p,q;\mathcal{M})$ are the geodesics joining p and q. Fix a geodesic $x \in \Omega^{1,2}(p,q;\mathcal{M})$, we can define, at least in a formal way, the Hessian $E''(x)\colon T_x\Omega^{1,2}(p,q;\mathcal{M}) \times T_x\Omega^{1,2}(p,q;\mathcal{M}) \to \mathbb{R}$, setting, for any ξ, $\xi' \in T_x\Omega^{1,2}(p,q;\mathcal{M})$, see [28]

$$E''(x)[\xi,\xi'] = \int_0^1 \left[g(x,\dot{x})[D_s\xi, D_s\xi'] - g(x,\dot{x})[R(x,\dot{x})(\dot{x},\xi)\dot{x},\xi'] \right], \tag{39}$$

where $g(\dot{x},\dot{x})$ is the fundamental tensor associated to the Finsler metric F, defined at (4), D_s denotes the Chern connection with reference $\dot{x}$ of ther Finsler metric F and $R(x,\dot{x})$ denotes the *flag curvature* associated to the Finsler metric F. The Hessian can be defined first for smooth vector fields ξ along the geodesic x and then can be extended by density on $T_x\Omega^{1,2}(p,q;\mathcal{M})$.

Now, as for Riemannian metrics, the Hessian $E''(x)$ is a bilinear form on the tangent space $T_x\Omega^{1,2}(p,q;\mathcal{M})$, and the notions of non-degenerate geodesics, Morse index and co-index, Jacobi equations and conjugate points can be extended to Finsler metrics. In particular any geodesic has finite index $m(x;E)$ and coindex $m^*(x,E)$, because $E''(x)$ is a bilinear form which is a compact perturbation of a positive definite bilinear form. Moroever, as for the Riemannain case, the Morse index $m(x,E)$ of a geodesic is equal to the geometric index $\mu(x)$, the number of conjugate points counted with their multiplicity.

Using a fundamental result by Richard Palais, see [37], on the homotopical equivalence between the Sobolev manifold $H^{1,2}(I,\mathcal{M})$ and the Banach manifold $C^0(I,\mathcal{M})$ of the continuous curves on $\mathcal{M}$, see also [40], the following result has been proven in [42,43].

Theorem 15. *Let $(\mathcal{M},F)$ be a Finsler manifold, and assume that F is forward (or backward) complete. Let p and q on $\mathcal{M}$ and assume that they are non-conjugate, that is any geodesic x joining p and q is a non-degenerate critical point of the action integral E.*

Then, for any field $\mathcal{K}$, there exists a formal series $Q(r)$ whose coefficient belong to $\mathbb{N}\cup\{+\infty\}$, such that

$$\sum_{x\in K(E)} r^{m(x,E)} = \mathcal{P}(\Omega^{1,2}(p,q;\mathcal{M});\mathcal{K}) + (1+r)Q(r). \tag{40}$$

The Morse relations for the geodesics joining two non-conjugate points of a forward (or backward) Finsler manifolds hold as in Riemannian geometry. A consequence of all the previous results and which has an interesting applications in general relativity is the following theorem on the number of geodesics joining two non-conjugate points on a Finsler manifold (obviously it holds also for Rimennian manifolds).

Theorem 16. *Let $(\mathcal{M},F)$ be a forward (or backward) complete Finsler manifold and let p and q be two non-conjugate points for F. Then the number of geodesics joining p and q is infinite or it is an odd number.*

Indeed, if the manifold $\mathcal{M}$ is non-contractible, the Ljusternik–Schnirelmann category of $\Omega^{1,2}(p,q;\mathcal{M})$ is equal to $+\infty$ and so that there exists infinitely many geodesics joining p and q. Assume that $\mathcal{M}$ is contractible into itself, then also the Hilbert manifold $\Omega^{1,2}(p,q;\mathcal{M})$ is contractible into itself, so its Poincaré polynomial is equal to 1. Setting $r=1$ in (40), we obtain:

$$\#\{x\in(\Omega^{1,2}(p,q;\mathcal{M}) : E'(x)=0\} = 1+2Q(1).$$

If $Q(1)\in\mathbb{N}\cup\{+\infty\}$ is equal to $+\infty$, then the set of the geodesics joining p and q is still equal to $+\infty$. However, if $Q(1)$ is finite, then the number of geodesics joining p and q is equal to $1+2Q(1)$, so it is an odd number. A condition which guarantees that the number of geodesics is finite is the existence of a function $L\colon\mathcal{M}\to\mathbb{R}$, which is strictly convex with respect to the Finsler structure F, and with only one non-degenerate at the absolute minimum, see [52,53].

We conclude this section recalling some results about the existence of closed geodesics. Additionally, for this problem, there are deep differences between the Riemannian and the Finslerian case. A long standing conjecture about Riemannian geometry was the existence of infinitely many, geometrically closed geodesics on a compact Riemannian manifold, see for instance [48]. Two closed geodesics are geometrically distinct if they have different image on the manifold. This conjecture is still open for a generic Riemannian manifold, but it was solved, for instance, on the two dimensional sphere S^2, where Bangert and Franks, see [54,55], proved that *any* Riemannian metric admits infinitely geometrically distinct closed geodesics. However, this result is false for Finsler manifolds, by the very famous Katok counterexamples [20,56], which are perturbations of the standard metric on S^2 admitting only 2 geometrically closed geodesics. The problem on the multiplicity of closed geodesics on Riemannian and Finsler manifolds is still a very active research field, see for instance [57], which gives also a survey on the closed geodesic problem on Finsler manifolds.

8. An Alternative Variational Principle for Geodesics for a Randers Metric

We have seen in the last section that one of the main differences between Riemannian geometry and Finsler geometry is that the regularity of the action integral on the Hilbert–Sobolev manifold $H^{1,2}(I,\mathcal{M})$ and its submanifolds. It is of class at least C^2 in Riemannian geometry and it is of class $C^{1,1}$ in Finsler geometry. However, for the geodesics of *Randers metrics*, it can be proved an alternative variational involving a functional of class C^2. Let $(\mathcal{M}_0, F)$, be a Randers metric, with

$$F(x,v) = \sqrt{h(x)[v,v]} + h(x)[W(x),v],$$

see (7). The action integral for a Randers metric has the form

$$E(x) = \int_0^1 F^2(x)ds$$

$$= \int_0^1 \left(\sqrt{h(x)[\dot{x},\dot{x}] + h(x)[W(x),\dot{x}]^2} + h(x)[W(x),\dot{x}]\right)^2 ds,$$

and it clearly of class $C^{1,1}$ on the manifold $H^{1,2}(I,\mathcal{M}_0)$. Now consider the functional

$$I(x) = \sqrt{\int_0^1 (h(x)[\dot{x},\dot{x}] + h(x)[W(x),\dot{x}]^2)ds} + \int_0^1 h(x)[W(x),\dot{x}]ds. \tag{41}$$

The functional $I(x)$ is of class $C^{1,1}$ on $H^{1,2}(I,\mathcal{M}_0)$ and it is of class C^2 on the open submanifold of non-constant curves. In particular $I(x)$ is of class C^2 on the submanifold $\Omega^{1,2}(p,q;\mathcal{M}_0)$, if $p \neq q$, while it is of class $C^{1,1}$ on the submanifold $\Lambda^{1,2}(I,\mathcal{M}_0)$. The following alternative variational principle has been proven in [58]. We restrict to the case of two points p and q.

Theorem 17. *$(\mathcal{M}_0, F)$ be a Finsler manifold, let F be a Randers metric and let p and q two points of $\mathcal{M}_0$. Then a curve $x(s) \in \Omega^{1,2}(p,q;\mathcal{M}_0)$ is a critical point of the functional I on $\Omega^{1,2}(p,q;\mathcal{M}_0)$ if, and only if, x is a pregeodesic for the Randers metric F and the parameterisation of $x(s)$ satisfies the property that $h(x)[\dot{x},\dot{x}]$ is constant.*

An analogous result holds for closed geodesics for the Randers metric F. By the previous theorem, the results on the existence and multiplicity for geodesics on a Randers can be obtained using the C^2 functional $I(x)$, rather than the action integral E.

9. Multiplicity of Light Rays and Applications to the Gravitational Lensing

In this section, we shall apply the results of the previous sections to obtain some results on the existence and multiplicity of light rays on a standard stationary Lorentzian manifolds. In particular the multiplicity results allow to prove under which geometric conditions the physical phenomena of multiple images to the gravitational lens effect can occur.

We fix a standard stationary Lorentzian manifold $(\mathcal{M}, g)$, with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and g is given by (18). Let $p = (x_0, 0)$ be a point of $\mathcal{M}$ (the choice of the time coordinate equal to 0 is not restrictive) and a *vertical line* $\gamma(s) = (x_1, s)\colon \mathbb{R} \to \mathcal{M}$ on the manifold $\mathcal{M}$, with $x_0 \neq x_1$. The following results on the existence and multiplicity of light rays joining p and γ hold, see [17].

Theorem 18. *Assume that the Fermat metric F_+, see (23) associated to $(\mathcal{M}, g)$ is forward complete, then there exists a future pointing light ray $z(s) = (x(s), t(s))$ joining p and γ, such that the travel time $T(z) = t(1) = F_+(x)$.*

Moreover assume that the manifold $\mathcal{M}$ is non-contractible into itself, then there exists a sequence $z_m = (x_m, t_m)$ of future pointing light rays joining p and γ with arrival times $T(z_m)$, such that

$$\lim_{m\to\infty} T(z_m) = F_+(x_m) = +\infty.$$

An analogous result holds for past pointing light rays.

Theorem 19. *Assume that the Fermat metric F_-, see (24) associated to $(\mathcal{M}, g)$ is forward complete, then there exists a past pointing light ray $z(s) = (x(s), t(s))$ joining p and γ, such that the travel time $T(z) = t(1) = -F_-(x)$.*

Moreover assume that the manifold $\mathcal{M}$ is non-contractible into itself, then there exists a sequence $z_m = (x_m, t_m)$ of past pointing light rays joining p and γ with arrival times $T(z_m)$, such that

$$\lim_{m\to\infty} T(z_m) = -F_-(x_m) = -\infty.$$

Theorems 18 and 19 can be proven by applying the critical point theory on geodesics on Finsler metrics to the Fermat metrics F_+ and F_-. Indeed, by the Fermat principle, the existence and multiplicity of the future pointing (respectively, past pointing) light rays joining p and γ is equivalent to the same problem for the geodesics of the Fermat metrics F_+ (respectively, F_-), so we can apply the critical point theory to the action functionals

$$E_+(x) = \int_0^1 \left(\sqrt{g_0(x)[\dot{x}, \dot{x}] + g_0(x)[\delta(x), \dot{x}]^2} + g_0(x)[\delta(x), \dot{x}] \right)^2 ds,$$

$$E_-(x) = \int_0^1 \left(\sqrt{g_0(x)[\dot{x}, \dot{x}] + g_0(x)[\delta(x), \dot{x}]^2} - g_0(x)[\delta(x), \dot{x}] \right)^2 ds,$$

defined on the Sobolev manifold $\Omega^{1,2}(\mathcal{M}_0, x_0, x_1)$. The functionals are of class $C^{1,1}$ on $\Omega^{1,2}(\mathcal{M}_0, x_0, x_1)$, are bounded from below and satisfies the Palais–Smale condition, assuming the forward completeness of the Fermat metrics F_+ (respectively, F_-).

The multiplicity result follows by the fact that if the manifold $\mathcal{M}$, or equivalently the manifold $\mathcal{M}_0$ is non-contractible, then the Ljusternik–Schnirelmann category oh the Sobolev manifold $\Omega^{1,2}(\mathcal{M}_0, x_0, x_1)$ is equal to $+\infty$, and so we can apply Theorem 7. Notice that these results can be obtained also using the alternative variational principle for geodesics for a Randers metric, see [58,59].

This multiplicity result allows to obtain a mathematical description of the so called gravitational lensing effect in a standard stationary spacetime, that is multiple images of a light source in the spacetime can be detected by an observer, if the spacetime is non-contractible.

We state now the Morse relations for the light rays joining p and γ, assuming that p and γ are not *light-conjugate*. We still consider a standard stationary Lorentzian manifold $(\mathcal{M}, g)$, with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and g is given by (18). Let $p = (x_0, 0)$ be a point of $\mathcal{M}$ and let $\gamma(s) = (x_1, s) \colon \mathbb{R} \to \mathcal{M}$ be a vertical line on the manifold $\mathcal{M}$, with $x_0 \neq x_1$.

Definition 18. *The point p and the vertical line γ are said to be light-non-conjugate if any light-like geodesic $z(s) = (x(s), t(s)) \colon [0,1] \to \mathcal{M}$ joining p and γ is a non-degenerate geodesic according to Definition 11.*

There is a strict relation between the light-non-degeneracy of the point $p = (x_0, 0)$ and the vertical line $\gamma(s) = (x_1, s)$ on the manifold $\mathcal{M}$, with the non-degeneracy of the points x_0 and x_1 with respect to the Fermat metrics F_+ and F_- on F_-. Indeed, the following theorem has been proved in [42], see also [59] for a proof involving the alternative variational principle.

Theorem 20. *The point $p = (x_0, 0)$ and the vertical line $\gamma(s) = (x_1, s)$ are light-non-conjugate if, and only if, x_0 and x_1 are not conjugate with respect to the Fermat metric F_+. Moreover, let $z(s) = (x(s), t(s))$ be a light-like geodesic joining p and γ, then the Morse index of the reparameterisation of $x(s)$ which is a geodesic for the Fermat metric F_+, is equal to the geometric index $\mu(z)$ of the light-like geodesic $z(s)$, that is the number of conjugate points of z, counted with their multiplicity.*

Thanks to the previous theorem, we can obtain the Morse relations for the light rays joining p and γ, assuming that p and γ are light-conjugate, by applying the Morse relations to the action integral of the Fermat metric F_+ for the non-conjugate points x_0 and x_1. We shall denote by $\mathcal{L}^{1,2}_+(p, \gamma; \mathcal{M})$ the Hilbert manifold of the future pointing light-like curves joining p and γ. Clearly $\mathcal{L}^{1,2}_+(p, \gamma; \mathcal{M})$ is homotopically equivalent to $\Omega^{1,2}(x_0, x_1; \mathcal{M}_0)$.

Theorem 21. *Assume that the Fermat metric F_+ is forward complete and assume that the point p and the vertical γ are light-non-conjugate and let Z be the set of future pointing light-like geodesics joining p and γ.*

Then, for any field $\mathcal{K}$, there exists a formal series $Q(r)$ with coefficients in $\mathbb{N} \cup \{+\infty\}$, such that

$$\sum_{z \in Z} r^{\mu(z)} = \mathcal{P}(\mathcal{L}^{1,2}_+(p, \gamma; \mathcal{M}), \mathcal{K}) + (1 + r)Q(r). \tag{42}$$

An analogous result hold for past pointing light-like geodesics joining p and γ, assuming that F_- is forward complete.

The proof follows from Theorems 15 and 20. Indeed we have the Morse relations for the action integral for the Fermat metric F_+ and the points x_0 and x_1 and then we can replace the set of the geodesics joining x_0 and x_1 with the set Z of the future pointing light-like geodesics joining p and γ. Moreover, the Hilbert manifolds $\mathcal{L}^{1,2}_+(p, \gamma; \mathcal{M})$ and $\Omega^{1,2}(x_0, x_1; \mathcal{M}_0)$ have the same Poincaré series, because they are homotopically equivalent.

In particular, applying Theorem 16 to the Fermat metric, we get the following result on the number of light rays joining p and γ, assuming that they are light-non-conjugate

Theorem 22. *Assume that p and γ are light-non-conjugate, then the number of future pointing light rays joining p and γ are infinite or it is an odd number.*

The previous theorem gives a theoretical proof of the *odd image theorem*, which claims that in the gravitational lens effect the number of images is odd. We see that this is true if the number of images is finite. We point out that the assumption that p and γ are light-non-conjugate is true except for a residual set. This follows by the Sard theorem.

Geometric conditions which gurantee that the number of images is finite has been studied in [52,53].

The results obtained on the existence and multiplicity of light rays joining the point p and the curve γ can be extended to the case the manifold $\mathcal{M}$ has a boundary. This case is physically very relevant, since many spacetimes in general relativity, as the Scwarzschild, the Reissner–Nordstrom and Kerr spacetimes have a boundary, or are a stationary structure only on some open subset of the spacetime. The boundary has to satisfy the following condition of *light-convexity* introduced in [59].

Definition 19. *Let $(\mathcal{M}, g)$ be a Lorentzian manifold and let $\mathcal{N}$ be an open subset of $\mathcal{M}$, with boundary $\partial\mathcal{N}$. We say that the boundary $\partial\mathcal{N}$ of $\mathcal{N}$ is light convex if for any light-like geodesic $z\colon [0, 1] \to \mathcal{N} \cup \partial\mathcal{N}$, such that $z(0) \in \mathcal{N}$ and $z(1) \in \mathcal{N}$, then $z([0, 1]) \subset \mathcal{N}$.*

If the boundary $\partial\mathcal{N}$ of $\mathcal{N}$ is a smooth submanifold of $\mathcal{M}$, then the light-convexity can be expressed by the Hessian, evaluated along light-like vectors, of a smooth function $\Phi\colon \mathcal{M} \to \mathbb{R}$, such that $\mathcal{N} = \Phi^{-1}(]0,+\infty[)$ and $\partial\mathcal{N} = \Phi^{-1}(0)$. Light-convexity can be described also using the second fundamental form of the submanifold $\partial\mathcal{N}$, see [60] for the details and other applications to general relativity. In [59] it has been proved that some open subsets of the Kerr spacetimes outside the stationary limit surface have light-convex boundary. The light-convexity of open stationary subsets in a stationary spacetime can be related to the geometric properties of the Fermat metrics, see [61].

Theorem 23. *Let $(\mathcal{M}, g)$ be a stationary Lorentzian manifold, with with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and g is given by* (18)*. Let $\mathcal{N}_0$ be an open subset of $\mathcal{M}_0$ with smooth boundary of class C^2 and set $\mathcal{N} = \mathcal{N}_0 \times \mathbb{R}$, and assume that the Fermat metric F_+ is forward complete on $\mathcal{N}_0 \cup \partial\mathcal{N}_0$. Moreover assume that the boundary $\partial\mathcal{N} = \partial\mathcal{N}_0 \times \mathbb{R}$ of $\mathcal{N}$ is light-convex.*

Let $p = (x_0, 0)$ be a point of $\mathcal{N}$ and let $\gamma(s) = (x_1, s)\colon \mathbb{R} \to \mathcal{N}$ be a vertical line on the manifold $\mathcal{N}$, with $x_0 \neq x_1$. then there exists a future pointing light ray $z(s) = (x(s), t(s))$ joining p and γ, such that the travel time $T(z) = t(1) = F_+(x)$.

Moreover, if the the manifold $\mathcal{N}$ is non-contractible into itself, then there exists a sequence $z_m = (x_m, t_m)$ of future pointing light rays joining p and γ with arrival times $T(z_m)$, such that

$$\lim_{m\to\infty} T(z_m) = F_+(x_m) = +\infty.$$

Finally, if the p and the curve are not light-conjugate, the Morse relations hold.

The same results hold for past pointing light rays, assuming that the Fermat metric F_- is forward complete on $\mathcal{N}_0 \cup \partial\mathcal{N}_0$.

Applications of this theorem are presented in the papers [28,59,62,63], where it is shown that some open subsets of physically relevant spacetimes of general relativity as the Schwarzschild, the Reissner–Nordstrom and the Kerr spacetimes contain non-contractible and open subsets with light-convex boundary, so that multiplicity of light rays hold and the gravitational lensing effect occur. In all these cases, in particular the Schwarzschild spactime, the manifold are non-contractible, and infinitely many images occur. Conditions which guarantee the finiteness of images can be found in the papers [53,64].

We conclude this section presenting another interesting class of light rays in a standard stationary Lorentzian manifold, the *spatially closed light rays*, which are light-like geodesics $z(s) = (x(s), t(s))\colon [0,1] \to \mathcal{M}$, such that the spatial component $x(s)$ is a closed curve, that is $x(0) = x(1)$. If we assume without any restriction that $t(0) = 0$, then z is called a T-spatially periodic light rays if $t(1) = T$. These classes of light rays are very interesting, because they are a generalisation to an arbitrary stationary spacetime of *circular geodesics* in static or stationary axis symmetric spacetimes, where circular geodesics define the *light sphere* and the *shadow of a black hole*, see for instance [32]. Additionally, spatially closed light rays satisfy a Fermat principle, because their spatial parts have to be closed pregeodesics for one of the Fermat metrics F_+ or F_-, so the existence of spatially closed light rays can be studied searching for non-constant critical points of the action integral for the Fermat metrics on the manifold $\Lambda^{1,2}(\mathcal{M}_0)$ of the closed curves. Some results on the existence of spatially closed light rays have been obtained in [60,65,66].

10. Causal Properties of a Stationary Spacetime and Fermat Metrics

In this section we shall see how the Fermat metrics influence the causal structure of standard stationary spacetime. We first recall some notions from causality theory of Lorentzian manifolds, we refer to [29–31] for details.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold, two points p and q on $\mathcal{M}$ are said *causally related*, there exists a curve $\gamma\colon [0,1] \to \mathcal{M}$, such that $\gamma(0) = p$, $\gamma(1) = q$ and γ is causal, that is $\dot\gamma(s)$ is a causal vector for any $s \in [0,1]$, Section 3 for the definition of causal curve. Let $p \in \mathcal{M}$, the causal future $I^+(p)$ is the subset of $\mathcal{M}$ given by the points q such that there exists a causal curve joining p and q. The causal past $I^-(p)$ is the subset of $\mathcal{M}$ given by the points q, such that there exists a causal curve joining q and p. We introduce now the notion of *globally hyperbolicity* of a Lorentzian manifolds. This notion was introduced by Jean Leray in the 1950s in order to study the well posedness of strictly hyperbolic systems of partial differential equations. The notion of global hyperbolicity is a fundamental notion in general relativity in order to study evolution equations.

Definition 20. *A Lorentzian manifold $(\mathcal{M}, g)$ is said globally hyperbolic if $(\mathcal{M}, g)$ is causal, that is, it does not contain causal closed curves, and for any couple of causally related points ts p and q, the intersection of the future $I^+(p)$ of p with the past $I^-(q)$ of q is a compact subset of $\mathcal{M}$.*

It is possible to prove that a Lorentzian manifold $(\mathcal{M}, g)$ is globally hyperbolic if, and only if, it contains a Cauchy hypersurface $\mathcal{M}_0$, that is a subset of $\mathcal{M}$ such that any inextendible causal curve z of the manifold intersects $\mathcal{M}_0$ exactly one, and only one, time.

Global hyperbolicity is an important assumption in the study of evolution equations in general relativity and, in particular, for Einstein equations.

The global hyperbolicity of a standard stationary are deeply related to the completeness of the Fermat metrics. Indeed the following result was proved in [17].

Theorem 24. *Let $(\mathcal{M}, g)$ be a standard stationary spacetime, with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and let F_+, F_- be the Fermat metrics associated on $\mathcal{M}_0$. If $(\mathcal{M}, g)$ is globally hyperbolic and $\mathcal{M}_0$ is a Cauchy surface, then F_+ and F_- are backward and forward complete.*

Conversely, if one of F_+ or F_- is complete in a direction, then $(\mathcal{M}, g)$ is globally hyperbolic.

A necessary and sufficient condition for the global hyperbolicity of $(\mathcal{M}, g)$ has been successively proved in [67].

Theorem 25. *A standard stationary spacetime $(\mathcal{M}, g)$ is globally hyperbolic if, and only if, for any $p \in \mathcal{M}$ and for any $r > 0$, the intersection $B_+(p,r) \cap B_-(p,r)$ is a compact subset of $\mathcal{M}$, where $B_+(p,r)$ and $B_-(p,r)$ are, respectively, the forward balls centred at p of radius r with respect to the Fermat metrics F_+ and F_-.*

In particular it follows that if the standard spacetime is globally hyperbolic, all the results on the existence, multiplicity and Morse theory joining a point p and a vertical line $\gamma(s) = (x_1, s)$ hold.

11. Geodesic Connectedness of a Lorentzian Manifold

A Lorentzian manifold $(\mathcal{M}, g)$ is said *geodesically connected* if for any couple of points p and q there is a geodesic for g joining p and q. Geodesics joining p and q are the critical points of the action functional

$$E(z) = \int_0^1 g(z(s))[\dot{z}, \dot{z}]ds$$

defined on the Hilbert manifold $\Omega^{1,2}(\mathcal{M}; p, q)$. Since the metric g is not positive definite, the action integral E is not bounded not from below nor from above, so a geodesic joining p and q cannot be found by minimisation of the functional E as for complete Riemannian manifolds. Moreover, it can be proven that all the stationary of the action functional E are saddle points of infinite Morse index and co-index, and this fact creates many difficulties to apply the critical point techniques developed in the last years to find a critical point of E.

The problem has been widely studied in a standard stationary spacetime, using critical point theorems for strongly indefinite functionals, or reducing the problem to the search of the critical points of E on a natural constraint, on which the action functional is bounded from below, which exists thanks to the stationarity of the metric.

Let $(\mathcal{M}, g)$ be a standard stationary Lorentzian manifold, with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and g given by (18). The action integral has the form

$$E(z) = \frac{1}{2}\int_0^1 \left[\langle \dot{x}, \dot{x}\rangle + 2\langle \delta(x), \dot{x}\rangle \dot{t} - \beta(x)\dot{t}^2\right] ds. \tag{43}$$

defined on the Sobolev manifold $H^{1,2}(I, \mathcal{M}) = H^{1,2}(I, \mathcal{M}_0) \times H^{1,2}(I, \mathbb{R})$. Let $z_0 = (x_0, t_0)$ and $z_1 = (x_1, t_1)$ be two points of $\mathcal{M}$, then the geodesics joining z_0 and z_1 for the metric g are the critical points of the action integral on the Sobolev submanifold $\Omega^{1,2}(z_0, z_1; \mathcal{M}) = \Omega^{1,2}(x_0, x_1; \mathcal{M}_0) \times \Omega^{1,2}(t_0, t_1; \mathbb{R})$.

From (43), it immediately appears that the action integral E is unbounded both from below and from above, so a critical point cannot be found by a minimum argument. In spite of this the problems of the existence and multiplicity of the geodesics joining z_0 and z_1 have been intensively studied, and a quite definitive answer can be given to this problem. Here we present the main results obtained about these problems

- The first result about the geodesic connectedness of a standard stationary Lorentzian manifolds was obtained by Vieri Benci and Dino Fortunato in [68], assuming that the spatial hypersurface $\mathcal{M}_0$ is compact and using techniques of strongly indefinite functionals;
- Successively the geodesic connectedness was proved in [69], assuming that the Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ is complete and some assumptions on the growth of the shift vector field $\delta(x)$ and the lapse function $\beta(x)$ are satisfied. In [69] a natural constraint for the action integral E is proven and it allows to prove that the existence of the geodesics joining $z_0 = (x_0, t_0)$ and $z_1 = (x_1, t_1)$ is equivalent to the search of the critical points of a new functional J defined only on the submanifold $\Omega^{1,2}(x_0, x_1; \mathcal{M}_0)$. The functional $J\colon \Omega^{1,2}(x_0, x_1; \mathcal{M}_0) \to \mathbb{R}$ is defined in the following way:

  $$J(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle ds + \frac{1}{2}\int_0^1 \frac{\langle \dot{x}, \dot{x}\rangle^2}{\beta(x)} ds - \frac{1}{2}K^2(x)\int_0^1 \frac{1}{\beta(x)} ds,$$

  where

  $$K(x) = \frac{(t_1 - t_0) - \int_0^1 \frac{\langle \delta(x), \dot{x}\rangle}{\beta(x)} ds}{\int_0^1 \frac{1}{\beta(x)} ds}.$$

  Under the growth assumptions on $\delta(x)$ and $\beta(x)$, it can be proved that J is bounded from below. Moreover, the completeness of the Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ guarantees that J satisfies the Palais–Smale condition, so a minimum of the functional J exists and it corresponds, by the variational principle, to a geodesic joining z_0 and z_1. Finally, if the manifold is non-contractible, Theorem 11 guarantees the existence of infinitely many critical points of the functional J and also the existence of infinitely many geodesics joining z_0 and z_1. Moreover in [69], the case of manifold with convex boundary is also considered;
- In [70], it is studied the geodesic connectedness of a non-necessarily standard stationary spacetime;
- Finally in [71] it is proved the geodesic connectedness of globally hyperbolic standard stationary Lorentzian manifold, assuming that the Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ is complete, see (18).

Now, by the characterisation of the global hyperbolicity of a standard stationary Lorentzian manifold in terms of the Fermat metrics, we state the following proposition, where the assumptions are given only on the initial conditions, the spatial component $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$, the shift $\delta(x)$ and the lapse $\beta(x)$ and not on the whole stationary metric, i.e., the global hyperbolicity.

Proposition 1. *Let $(\mathcal{M}, g)$ be a standard stationary Lorentzian manifold as in (18) and assume that $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a complete Riemannian manifold and one of the Fermat metrics F_+ or F_- is forward complete. Then $(\mathcal{M}, g)$ is geodesically connected.*

12. Finsler Geometry and the Sagnac Effect

In this section we present another application of Fermat metrics, a geometric description of the celebrated Sagnac effect using the Fermat metrics. We shall see that the Sagnac effect is a measure of the asymmetry of the Fermat metrics F_+ and F_- and, indeed, if the metric is static, the Sagnac effect does not appear. This observation is not new and has already been exploited in many papers. Here we emphasise the role of the Fermat metrics in the Sagnac and we think that it strongly put in evidence the value of Finsler geometries in the study of physical phenomena. In this case, the application of Finsler geomtery to the Sagnac effect is not a mere generalisation of a phenomena geometrically described using Riemannian geometry, because in that case the Sagnac effect does not appear.

We present, now, the geometric description of the Sagnac effect. We refer to [72] for the original paper by George Sagnac, to [73] for the Paul Langevin interpretation of the Sagnac effect in special relativity, to [74] for a geometric approach to the Sagnac effect, where the role of the Fermat metrics is not exploited, to [75] for a more intrinsic approach to the Sagnac effect, and to [76] for technological applications to the the Sagnac effect as in the global positioning system (GPS). Let $(\mathcal{M}, g)$ be a standard stationary spacetime with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and the metric is given by (18).

Let $x(s)\colon [0,1] \to \mathcal{M}_0$ be a spatial closed curve, so $x(0) = x(1)$ and take the lift $z(s) = (x(s), t(s))$ of x to a future pointing light-like curve, so $g(z)[\dot z, \dot z] = 0$ and $t(s)$ solves (21), that is

$$\dot t(s) = \sqrt{g_0(x)[\dot x, \dot x] + g_0(x)[\delta(x), \dot x]^2} + g_0(x)[\delta(x), \dot x] > 0.$$

Now let $y(s) = x(1-s)\colon [0,1] \to \mathcal{M}_0$ the closed curve obtained reversing the orientation of $x(s)$ and let $w(s) = (y(s), k(s))$ the lift of $y(s)$ to a future pointing light-like curve, so that

$$\dot k(s) = \sqrt{g_0(y)[\dot y, \dot y] + g_0(y)[\delta(y), \dot y]^2} + g_0(y)[\delta(y), \dot y] > 0.$$

Now we evaluate the travel times of the two future pointing light-like curves $z(s)$ and $w(s)$ and we obtain

$$T(z) = F_+(x) = \int_0^1 \left(\sqrt{g_0(x)[\dot x, \dot x] + g_0(x)[\delta(x), \dot x]^2} + g_0(x)[\delta(x), \dot x]\right) ds,$$

$$T(w) = F_+(y) = \int_0^1 \left(\sqrt{g_0(y)[\dot y, \dot y] + g_0(y)[\delta(y), \dot y]^2} + g_0(y)[\delta(y), \dot y]\right) ds.$$

On the other hand, since $y(s)$ has been defined reversing the orientation of $x(s)$, we have

$$T(w) = F_+(y) = \int_0^1 \left(\sqrt{g_0(y)[\dot y, \dot y] + g_0(y)[\delta(y), \dot y]^2} + g_0(y)[\delta(y), \dot y]\right) ds$$
$$= \int_0^1 \left(\sqrt{g_0(x)[\dot x, \dot x] + g_0(x)[\delta(x), \dot x]^2} - g_0(x)[\delta(x), \dot x]\right) ds = F_-(x)$$

so we find there the travel times of z and w can have a different value and the difference ΔT is given by

$$\Delta T = T(z) - T(w) = F_+(x) - F_-(x) = 2\int_0^1 g_0(x)[\delta(x), \dot x] ds. \quad (44)$$

The difference of travel times causes an optical diffraction, which is called the Sagnac effect along the closed spatial curve $x(s)$.

Remark 7. *We point out that both the Fermat metrics, F_+ and F_-, contribute to the evaluation of the Sagnac effect. Moreover if the metric is static, and there is only one Fermat metric g_0 which is Riemannian, then $\Delta T = 0$ and the Sagnac effect does not appear.*

In particular, in the Boyer–Lindquist coordinates the Sagnac effect does not appear on the Schwarzschild and Reissner–Nordstrom. On the Kerr spacetime outside the stationary limit surface, the Sagnac effect appear.

Let $x(s) = (r(s), \theta(s), \phi(s)) \colon [0,1] \to \mathcal{M}_a$ be a closed curve whose support is contained on the spatial part of Kerr spacetime outside the stationary limit surface, then by (44) and (30)–(31), the Sagnac effect along $x(s)$ is equal to:

$$\Delta T = 4 \int_0^1 \frac{marsen^2\theta}{\lambda(r,\theta)\beta(r)} \dot{\phi} ds.$$

13. Conclusions and New Directions

In this paper we have introduced the Fermat metrics associated to a standard stationary spacetime. They are two Finsler metrics of Randers type and their definition is obtained evaluating the travel time of light-like curve of the spacetime. We have recalled the Fermat principle which relates the light rays of a standard stationary spacetimes to the geodesics of the Fermat metrics and we have presented some applications to the study of the geometrical and physical properties of this class of spacetimes. In particular we have studied the multiplicity of light rays joining an event and a time-like curve, giving a mathematical description of the multiple image effect. We have studied some causal property of a standard stationary spacetime, in particular, we related the global hyperbolicity to the completeness of the Fermat metrics. A result on the geodesic connectedness has been obtained involving again the completeness of the Fermat metrics. Finally, a geometric formulation of the Sagnac effect with the Fermat metrics has been presented.

Those are only some of the possible applications of the Fermat metrics. Some other application of the Fermat metrics concerns the strict relations between the Randers–Finsler geometry and the Zermelo navigation problem, see [77–79] for the formulation of the Zermelo navigation problem and its link with Randers geometry.

Another very interesting application of Fermat metrics is the description of the gravitational lensing in Kerr–Randers geometry using the Gauss–Bonnet theorem, see [80] and also the recent paper [81]. Finally, in the paper [82], the case in which the Fermat metrics have no sign restriction is considered.

Now we present other research fields in which some optical metrics, that we always define as Fermat metrics, appear. These research fields are actually strongly active and have some interesting physical applications.

13.1. Spacetimes with a Light-like Killing Field

We have introduced the Fermat metrics in the class of standard stationary Lorentzian manifolds, where the vector field $Y(z) = (0,1)$ is killing and time-like, see Remark 1. Recently it has been considered the case in which $Y(z) = (0,1)$ is a *light-like killing vector field*, see [83]. As in Section 4, let $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold and let $\delta(x)$ be a smooth surface, such that $\delta(x) \neq 0$, for all $x \in \mathcal{M}_0$. It can be proved that the following covariant symmetric tensor field

$$g(z)[\zeta, \zeta'] = g(x,t)[(\xi,\tau),(\xi',\tau')] = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau \qquad (45)$$

is non-degenerate and it defines a Lorentzian metric on the product $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$. These class of metrics are often used to model *gravitational waves*. The differences between the standard stationary Lorentzian manifolds are that the lapse function $\beta(x)$ is null and the shift $\delta(x)$ cannot be equal to 0 to any point. This fact imposes some topological restrictions on the manifold $\mathcal{M}_0$. Indeed, if $\mathcal{M}_0$ is a compact manifold, then its Euler–Poincaré characteristic $\chi(\mathcal{M}_0)$ must be different from 0. The vector field $Y(z) = (0,1)$ is still a Killing vector field, because the coefficients of the metric g do not depend on the variable t, but now it is light-like, as it can be easily proved showing that $g(x,t)[(0,1),(0,1)] = 0$. The Lorentzian manifold $(\mathcal{M}, g)$ is still time-oriented and a global time-like vector field is the gradient ∇T of the temporal function $T(x,t) = t \colon \mathcal{M} \to \mathbb{R}$. The fact that the shift $\delta(x)$ is never null needs to prove that ∇T is time-like everywhere.

Now, if we consider a future light-like curve $z(s) = (x(s), t(s)) \colon [0,1] \to \mathcal{M}$, for the metric g, arguing as in Section 4, we obtain:

$$g(z)[\dot{z}, \dot{z}] = \langle \dot{x}, \dot{x} \rangle + 2\langle \delta(x), \dot{x} \rangle \dot{t} = 0$$

and solving with respect to $\dot{t}$, we obtain:

$$\dot{t}(s) = \frac{\langle \dot{x}, \dot{x} \rangle}{2\langle \delta(x), \dot{x} \rangle}.$$

The travel time of the light-like curve $z(s)$ is given by

$$T(z) = t(1) - t(0) = \frac{1}{2} \int_0^1 \frac{\langle \dot{x}, \dot{x} \rangle}{2\langle \delta(x), \dot{x} \rangle} ds.$$

So in this case the Fermat metric is given by the function

$$F(x,v) = \frac{\langle v, v \rangle}{2\langle \delta(x), v \rangle}, \tag{46}$$

which a positively homogeneous function of degree 1 as a Finsler metric, but it is not defined on the whole tangent bundle, because it is not defined on the set

$$Ker(\delta) = \{(x,v) \in T\mathcal{M}_0 : \langle \delta(x), v \rangle = 0\}.$$

The metrics (46) are known in literature as Kropina metrics and are a very important class of the so called *conical Finsler metrics*, see [84,85], where Kropina metrics were introduced first.

A Fermat principle relating the light-like geodesics for the metric g defined at (45) and the geodesics for the Kropina metric (46) has been proved in [83]. Some results on the existence of geodesics connecting two points or closed geodesics for a Kropina metric have been recently proved in [86]. We also mention the paper [87], where the existence of geodesics joining two points for the Lorentzian metric (45) is studied. We point out that, since a Kropina metric is not defined on the whole tangent bundle, there always exists couple of point which are not joined by any smooth curve and by no geodesic, see [86]. Instead, the problem on the existence of a closed geodesic for an arbitrary Kropina metric on a compact manifold is still open.

13.2. Finsler Spacetimes

In the recent years, some alternative models of the gravitational field have been introduced, in order to justify some observational results in astronomy. The first alternative model consists in the modification of the Einstein–Hilbert functional, whose Euler–Lagrange equations are the Einstein equations, with functionals whose Lagrangian density is a function of the scalar curvature. Another model, involved especially in observations of an asymmetric behaviour of matter, requires the substitution of the classical metric tensor given by a Lorentzian metric, with a *Lorentz–Finsler metric*, that is an homogeneous function defined on the tangent bundle of a manifold, such that the fundamental tensor defined as in (4) is a non-degenerate bilinear form of index 1, that is a Minkowski scalar product. This research field is actually very hot and it has many open problems, as the same definition of a Lorentz–Finsler metric F. Indeed some authors consider 1-homogeneous functions F, as in classical Finsler geometry, while authors prefer to define 2-homogeneous functions F, viewing F a sort of Lagrangian density and avoiding deep regularity problems given by the choice of 1-homogeneous functions. We refer to the review papers [88–90] and to the recent papers [91–95] for the main definitions and recent developments of Lorentz–Finsler geometry and its applications to gravitational physics.

A Fermat principle for light rays on Lorentz–Finsler spacetime was stated in [96], while in the papers [97,98], the notions of static and stationary spacetimes are extended to Lorentz–Finsler geometry, see also [99,100] for further results in this direction. For these classes of Lorentz–Finsler a detailed discussion of optical geometry and an analysis on the geometric and physical assumptions which guarantee the multiple image effect is still an open problem.

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