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Norah Alshehri and Mohammed Guediri

Special Issue

Symmetry and Its Application in Differential Geometry and Topology III

Edited by

Prof. Dr. Tiehong Zhao and Prof. Dr. Yanlin Li



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Ricci Solitons on Spacelike Hypersurfaces of Generalized Robertson–Walker Spacetimes

Norah Alshehri ^{*,†} and Mohammed Guediri [†] 

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; mguediri@ksu.edu.sa

* Correspondence: nalshhri@ksu.edu.sa

[†] These authors contributed equally to this work.

Abstract: In this paper, we investigate Ricci solitons on spacelike hypersurfaces in a special Lorentzian warped product manifold, the so-called generalized Robertson–Walker (GRW) spacetimes. Such spacetimes admit a natural form of symmetry which is represented by the conformal vector field $f\partial t$, where f is the warping function and ∂t is the unit timelike vector field tangent to the base (which is here a one-dimensional manifold). We use this symmetry to introduce some fundamental formulas related to the Ricci soliton structures and the Ricci curvature of the fiber, the warping function, and the shape operator of the immersion. We investigate different rigidity results for Ricci solitons on the slices, in addition to the totally umbilical spacelike supersurfaces of GRW. Furthermore, our study is focused on significant GRW spacetimes such as Einstein GRW spacetimes and those which obey the well-known null convergence condition (NCC).

Keywords: warped product submanifolds; generalized Robertson–Walker spacetimes; spacelike hypersurfaces; slices; null convergence condition



Citation: Alshehri, N.; Guediri, M. Ricci Solitons on Spacelike Hypersurfaces of Generalized Robertson–Walker Spacetimes. *Symmetry* **2024**, *16*, 601. <https://doi.org/10.3390/sym16050601>

Academic Editor: Abraham A. Ungar

Received: 6 April 2024

Revised: 25 April 2024

Accepted: 9 May 2024

Published: 13 May 2024



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1. Introduction

Symmetry plays a fundamental role in physics, particularly in General Relativity, where it is often defined by a local one-parameter group of conformal transformations (resp. isometries) generated by a conformal vector field (resp. a Killing vector field). Recall that on a pseudo-Riemannian manifold (M, g) , a vector field is conformal if its local flow preserves the conformal class $[g]$ consisting of all pseudo-Riemannian metrics conformal to g . The prior assumption of such symmetries represents the key simplification in seeking exact solutions to the Einstein Equation (see [1,2]). In this article, we will focus on spacelike hypersurfaces in GRW spacetimes, where a natural form of symmetry is represented by the conformal vector field $f\partial t$, with f the warping function, and ∂t is the unit timelike vector field tangent to the base, as explained in detail below.

Let (M, g_M) be an n -dimensional Riemannian manifold, and $I \subset \mathbb{R}$ be an open interval in \mathbb{R} equipped with the metric $-dt^2$. In this work, we will use $(\overline{M}, \overline{g})$ to refer to the $(n+1)$ -dimensional product manifold $I \times M$ with the Lorentzian metric given by

$$\overline{g} = -dt^2 + f^2 g_M, \quad (1)$$

where $f > 0$ is a smooth function on I . In other words, $(\overline{M}, \overline{g})$ is a Lorentzian warped product, called generalized Robertson–Walker (GRW) spacetime with warping function f and Riemannian fiber M .

The vector field ∂t is a unit timelike vector field globally defined on \overline{M} and establishes a time orientation on \overline{M} . It is important to note that GRW spacetimes are the generalization of the co-called Robertson–Walker (RW) spacetimes, for which the fiber is precisely of constant sectional curvature. GRW spacetimes include Minkowski spacetime, de Sitter spacetime, Friedmann cosmological models, and static Einstein spacetime [3].

In [1], where the concept of GRW spacetime was first introduced, the authors raise this question: When does a complete spacelike hypersurface with constant mean curvature in a GRW spacetime become entirely umbilical and a slice? The authors showed that within spatially closed spacetimes, compact spacelike hypersurfaces are totally umbilical and, with few exceptions, are also slices. A significant portion of the studies on GRW spacetimes focus on the above query and related topics, including the curvature properties of spacelike hypersurfaces in GRW spacetimes.

In [4], a characterization of GRW spacetimes was found by B.Y. Chen. He showed that an n -dimensional Lorentzian manifold, $n \geq 3$, is isometric to a GRW spacetime specifically when it admits a timelike concircular vector field.

In [5], the authors proved that if (M, g) is an Einstein Lorentzian manifold of dimension $n \geq 4$ with zero divergence of the conformal curvature tensor, it supports a suitable concircular vector field. In particular, (M, g) is a GRW spacetime. Additionally, they establish that a spacetime consisting of a stiff matter perfect fluid or a massless scalar field with a timelike gradient and a divergence-free Weyl tensor also falls under the category of GRW spacetimes. A survey [6] is provided, mainly concentrating on Chen's description via a timelike concircular vector. Some local properties of GRW spacetimes, especially their geodesics, were studied in [7]. The study of curvature and Killing fields on GRW spacetimes was established in [8].

In [9], spacelike hypersurfaces with constant mean curvature in GRW spacetimes that satisfy the null convergence condition (NCC) have been studied. This presents various findings regarding the rigidity of these hypersurfaces in spatially parabolic GRW spacetimes.

In this paper, we examine Ricci solitons on spacelike hypersurfaces of GRW spacetimes, with a focus on investigating the conditions under which hypersurfaces in Riemannian and Lorentzian manifolds can admit Ricci soliton structures. See [10–14] for some references on Ricci solitons on Riemannian hypersurfaces in Euclidean spaces and Riemannian space forms. Additional resources concerning Ricci solitons on Riemannian manifolds and Lie groups can be found in the following works and the citations that they contain ([15–19]).

This paper is organized as follows. In Section 2, we review some concepts related to GRW spacetimes and spacelike hypersurfaces in these spacetimes.

In Section 3, we focus on Ricci solitons on spacelike hypersurfaces in a GRW spacetime. We introduce some background information, including fundamental concepts and key equations related to Ricci solitons on spacelike hypersurfaces in a GRW spacetime. We present an equation for Ricci solitons on spacelike hypersurfaces in a GRW spacetime, involving the Ricci curvature of the fiber M , the warping function f , and the shape operator of the hypersurface.

We present some Ricci soliton inequalities on compact spacelike hypersurfaces in a GRW spacetime, where the warping function f satisfies some convexity conditions. We investigate different rigidity results for Ricci solitons on compact spacelike hypersurfaces in a GRW spacetime that satisfy the so-called null convergence condition (NCC). Furthermore, we study those Ricci solitons on spacelike hypersurfaces of a GRW spacetime with the fiber M being Ricci flat. As a result, given certain natural assumptions, we present various characterizations of the Ricci solitons for which the spacelike hypersurface is a slice or totally umbilical hypersurface. Our study is primarily concerned with Einstein GRW spacetimes. We aim to fully describe Ricci solitons on compact spacelike hypersurfaces of Einstein GRW spacetimes. Among other results, we establish that there are no Ricci solitons on compact spacelike hypersurfaces of an Einstein GRW spacetime with a fiber that has a positive Ricci curvature. Additionally, we extend our research to the examination of Ricci solitons on compact spacelike hypersurfaces of Einstein GRW spacetimes, with fibers that have nonpositive Ricci curvature.

2. Preliminaries

For the following notions and formulas, we refer to [20] and [3]. Let \overline{M} be the GRW spacetime defined in the previous section. We consider the closed conformal timelike vector

field $\xi = f\partial t$ on \bar{M} . The relationship between the Levi–Civita connections of \bar{M} and M implies that

$$\bar{\nabla}_X \xi = f'X, \quad (2)$$

for any vector field X on \bar{M} , where $\bar{\nabla}$ the Levi–Civita connection \bar{M} . This means that ξ is a closed conformal vector field on \bar{M} . To calculate $\bar{Ric}(X, Y)$, we can refer to [20]. This is given as follows:

$$\bar{Ric}(X, Y) = Ric_M(X^*, Y^*) + \left(\frac{f''}{f} + (n-1) \frac{f'^2}{f^2} \right) g_M(X^*, Y^*) - n \frac{f''}{f} \bar{g}(X, \partial t) \bar{g}(Y, \partial t), \quad (3)$$

for all vector fields X and Y on \bar{M} , where \bar{Ric} and Ric_M denote the Ricci tensors of \bar{M} and M , respectively. On M , the component of the vector field X is denoted by X^* and can be expressed as $X^* = X + \bar{g}(X, \partial t)\partial t$. By using Equation (2), we see that the scalar curvature \bar{S} of \bar{M} is given by

$$\bar{S} = \frac{S_M}{f^2} + 2n \frac{f''}{f} + n(n-1) \frac{f'^2}{f^2}, \quad (4)$$

where S_M is the scalar curvature of M .

Consider a spacelike hypersurface Σ of \bar{M} , and let g be the induced metric on Σ . Let $\psi : \Sigma \rightarrow \bar{M}$ be the immersion function. In this case, it is possible to choose a unit timelike vector field N that is normal to Σ such that $\bar{g}(N, \partial t) < 0$.

Applying the Cauchy–Schwarz inequality, we obtain $\bar{g}(N, \partial t) = -\cosh \phi \leq -1$, where ϕ represents the hyperbolic angle between N and ∂t .

The closed conformal timelike vector field $\xi = f\partial t$ can be expressed as

$$\xi = \xi^T - \theta N, \quad (5)$$

where $\theta = \bar{g}(\xi, N) < 0$ is the support function on Σ and ξ^T is the tangential component of ξ , so that $\theta = -f \cosh \phi$.

Now, as ξ is a closed conformal vector field, it becomes clear when using Gauss and Weingarten formulas that

$$\nabla_X \xi^T = f'X + f \cosh \phi A(X), \quad (6)$$

and

$$A(\xi^T) = \nabla(f \cosh \phi), \quad (7)$$

where ∇ is the Levi–Civita connection of Σ , and A is the shape operator associated with N . From (6), we obtain

$$\operatorname{div}(\xi^T) = n(f' - f \cosh \phi H), \quad (8)$$

where $\operatorname{div}(\xi^T)$ is the divergence of ξ^T .

The Gauss–Codazzi equation is a widely known and used mathematical formula as follows.

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + \bar{g}(h(X, Z), h(Y, W)) - \bar{g}(h(X, W), h(Y, Z)), \quad (9)$$

for all tangent vectors X, Y, Z and W to Σ , where R and \bar{R} are the curvature tensors of M and \bar{M} , respectively.

Equation (9) results in a relationship between the Ricci curvatures Ric and \bar{Ric} of Σ and \bar{M} , respectively.

$$Ric(X, Y) = \bar{Ric}(X, Y) + \bar{g}(\bar{R}(N, X)Y, N) + g(A(X), nHY + A(Y)). \quad (10)$$

Afterward, the scalar curvature S of Σ can be expressed as

$$S = \bar{S} + 2\bar{Ric}(N, N) + |A|^2 - n^2 H^2, \quad (11)$$

where \bar{S} is the scalar curvature of \bar{M} , A is the shape operator associated with N and H is the mean curvature of Σ . Note that

$$H = -\frac{1}{n} \text{tr}(A).$$

The vector field ∂t can also be represented as

$$\partial t = (\partial t)^T + \cosh \phi N, \quad (12)$$

where $(\partial t)^T$ is the tangential part of ∂t and

$$N = N^* + \cosh \phi \partial t. \quad (13)$$

It follows from $\bar{g}(N, N) = \bar{g}(\partial t, \partial t) = -1$ that

$$\bar{g}(N^*, N^*) = \bar{g}((\partial t)^T, (\partial t)^T) = \sinh^2 \phi. \quad (14)$$

Put $\tau = \pi_I \circ \psi : \Sigma \rightarrow \mathbb{R}$, where π_I is the projection on I . A simple computation shows that $\bar{\nabla} \pi_I = -\partial t$, which yields

$$\nabla \tau = -(\partial t)^T. \quad (15)$$

In a GRW spacetime \bar{M} , there is a specific set of spacelike hypersurfaces known as its spacelike slices $\{t_0\} \times M$, where $t_0 \in I$. These spacelike slices serve as the reference frames for special observers in ∂t corresponding to each specific t_0 . A spacelike hypersurface Σ in \bar{M} is a spacelike slice if and only if the function τ remains constant on Σ . Equivalently, a spacelike hypersurface in \bar{M} is a spacelike slice if and only if the hyperbolic angle ϕ is identically zero. From Equation (6), we easily see that the shape operator A of the spacelike slice $\{t_0\} \times M$ can be expressed as $A = -\frac{f'(t_0)}{f(t_0)} Id$, where Id denotes the identity operator. As a result, the slice is totally umbilical with constant mean curvature $H = -\frac{f'(t_0)}{f(t_0)}$.

A spacetime is said to obey the null convergence condition (NCC) if its Ricci curvature satisfies $\bar{Ric}(X, X) \geq 0$, for any null vector X . In the case of a GRW spacetime $\bar{M} = I \times_f M$, it can be proved (see [21]) that \bar{M} obeys the NCC if and only if

$$Ric_M - (n-1)f^2(\log f)''g_M \geq 0. \quad (16)$$

A spacelike hypersurface Σ in the GRW spacetime $\bar{M} = I \times_f M$ is called a Ricci soliton if there exists a nonzero vector field X on Σ and a constant λ such that

$$\frac{1}{2} \mathcal{L}_X g + Ric = \lambda g, \quad (17)$$

where $\mathcal{L}_X g$ is the Lie derivative of g in the direction of X . We denote a Ricci soliton by (Σ, g, X, λ) . It is called shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. The vector field X is called the potential field of (Σ, g, X, λ) . If $\mathcal{L}_X g = 0$, the Ricci soliton is said to be trivial and from Equation (17), Σ becomes Einstein.

A noncompact and complete manifold is classified as parabolic if the only superharmonic functions that are bounded from below are constants.

3. Ricci Solitons on Spacelike Hypersurfaces of a GRW Spacetime

Assume $(\Sigma, g, \xi^T, \lambda)$ is a Ricci soliton hypersurface of the GRW spacetime $\bar{M} = I \times_f M$. That is

$$\frac{1}{2} \mathcal{L}_{\xi^T} g + Ric = \lambda g, \quad (18)$$

for some constant λ . Since ξ^T is the potential field of $(\Sigma, g, \xi^T, \lambda)$, we obtain the following equation:

$$\text{div}(\xi^T) + S = \lambda n, \quad (19)$$

where S is the scalar curvature of Σ . From (8), we obtain

$$S = n(\lambda - f' + f \cosh \phi H).$$

Then, using also (3), we obtain

$$\overline{Ric}(N, N) = Ric_M(N^*, N^*) - (n-1)(\log f)'' \sinh^2 \phi - n \frac{f''}{f}, \quad (20)$$

which in conjunction with Equations (6) and (11), enables the rewriting of Equation (19) as follows.

Lemma 1. *Let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on a spacelike hypersurface Σ of the GRW spacetime $I \times_f M$. Then,*

$$\begin{aligned} n(\lambda - f' - \theta H) &= \frac{S_M}{f^2} + 2Ric_M(N^*, N^*) - 2(n-1)(\log f)'' \sinh^2 \phi \\ &\quad + n(n-1)\left(\frac{f'^2}{f^2} - H^2\right) + |A|^2 - nH^2. \end{aligned} \quad (21)$$

If $(\log f)'' < 0$, we obtain the following theorem (compare with Theorem 3 in [21]).

Theorem 1. *Let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on the spacelike hypersurface Σ of the GRW spacetime $I \times_f M$. If $-\log f$ is strictly convex, then*

$$n(\lambda - f' - \theta H) \geq \frac{S_M}{f^2} + 2Ric_M(N^*, N^*) + n(n-1)\left(\frac{f'^2}{f^2} - H^2\right).$$

The equality holds if and only if Σ is a slice.

Let $F : I \rightarrow \mathbf{R}$ be a function such that $F' = f$. Using (15), we see that the gradient of $F \circ \tau$ on Σ is given by

$$\nabla(F \circ \tau) = F' \nabla \tau = -f(\partial t)^T = -\xi^T,$$

and so its Laplacian on Σ yields

$$\Delta(F \circ \tau) = -\operatorname{div}(\xi^T). \quad (22)$$

Since the only functions with signed Laplacian on a compact Riemannian manifold are the constants, the following theorem holds.

Theorem 2. *Let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on a compact or parabolic spacelike hypersurface Σ of a GRW spacetime $I \times_f M$. If $S \leq \lambda n$ (or $S \geq \lambda n$), then Σ is a slice.*

Proof. Assume Σ is a compact space and $S \leq \lambda n$, then it follows

$$0 = \int_{\Sigma} \operatorname{div}(\xi^T) dV = \int_{\Sigma} S - \lambda n dV \leq 0.$$

We conclude that $S = \lambda n$, and so $\Delta(F \circ \tau) = 0$. Since Σ is compact, we find that $F \circ \tau$ is constant and $\xi^T = -\nabla(F \circ \tau) = 0$. From $0 = \nabla(F \circ \tau) = f \nabla \tau$ and since f does not vanish, we conclude that τ is constant, and as a result, Σ is a slice. Assume Σ is a parabolic space and $S \leq \lambda n$ (or $S \geq \lambda n$), it follows from (22) that $\Delta(F \circ \tau) \leq 0$ (or $\Delta(F \circ \tau) \geq 0$). Therefore, $F \circ \tau$ must be a constant, implying that $\nabla(F \circ \tau) = 0$ and Σ is undoubtedly a slice. Notice that the same proof in the parabolic case works in the compact case (as any subharmonic function on a closed manifold must be constant). \square

Theorem 3. Let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on a compact spacelike hypersurface Σ of a GRW spacetime $I \times_f M$. If ξ^T is an affine vector field, then Σ is a slice.

Proof. Since ξ^T is an affine vector field, then $\operatorname{div}(\xi^T) = c$, where c is a constant. It follows that

$$0 = \int_{\Sigma} \Delta(F \circ \tau) dV = -c \operatorname{Vol}(\Sigma),$$

where $\operatorname{Vol}(\Sigma)$ is the volume of Σ . Thus, we conclude that $c = 0$. Since Σ is compact, we find that $F \circ \tau$ is constant and $\xi^T = -\nabla(F \circ \tau) = 0$. From Equation (15), we can see that τ is constant, and as a result, Σ is a slice. \square

In the following results, our attention will be on GRW spacetimes which obey NCC or are Einstein.

Theorem 4. Let $\overline{M} = I \times_f M$ be a GRW spacetime obeying the NCC. If $(\Sigma, g, \xi^T, \lambda)$ is a Ricci soliton on a compact spacelike hypersurface Σ of \overline{M} , then

$$\lambda \geq (n-1) \left(\frac{f''}{f} - H^2 \right). \quad (23)$$

The equality holds if and only if Σ is totally umbilical. Moreover, if $f'' > fH^2$, then the Ricci soliton is shrinking.

Proof. As Σ is a compact manifold, we can integrate (21) to obtain the following.

$$\begin{aligned} \int_{\Sigma} \left(\lambda n - \frac{S_M}{f^2} - n(n-1) \left(\frac{f'^2}{f^2} - H^2 \right) \right) dV &= \int_{\Sigma} (|A|^2 - nH^2) dV \\ &+ 2 \int_{\Sigma} \left(\operatorname{Ric}_M(N^*, N^*) - (n-1)(\log f)'' \sinh^2 \phi \right) dV. \end{aligned} \quad (24)$$

By applying Schwartz's inequality and taking into account (16), we deduce that

$$\lambda \geq \frac{S_M}{nf^2} + (n-1) \left(\frac{f'^2}{f^2} - H^2 \right). \quad (25)$$

Tracing the inequality (16), we derive the inequality

$$S_M \geq n(n-1)f^2(\log f)'', \quad (26)$$

which with (25) leads to (23). If the equality in (23) is satisfied, then from (24)

$$\begin{aligned} \int_{\Sigma} \frac{1}{f^2} \left(n(n-1)f^2(\log f)'' - S_M \right) dV &= \int_{\Sigma} (|A|^2 - nH^2) dV \\ &+ 2 \int_{\Sigma} \left(\operatorname{Ric}_M(N^*, N^*) - (n-1)(\log f)'' \sinh^2 \phi \right) dV. \end{aligned} \quad (27)$$

By (26), it can be seen that Σ is totally umbilical and $S_M = n(n-1)f^2(\log f)''$. If $f'' > fH^2$, then by (23) $\lambda > 0$, and the Ricci soliton is shrinking. \square

If we make the assumption that the warping function remains constant, then we have the following corollary.

Corollary 1. Let $\overline{M} = I \times_f M$ be a GRW spacetime obeying the NCC, and let the warping function be a constant. If $(\Sigma, g, \xi^T, \lambda)$ is a Ricci soliton on a compact spacelike hypersurface Σ of \overline{M} , then

$$\lambda \geq -(n-1)H^2. \quad (28)$$

The equality holds if and only if Σ is a slice.

In the context of spacelike hypersurfaces in a GRW spacetime satisfying the NCC with the condition that the manifold M is Ricci flat, we achieve the following.

Theorem 5. Let $\overline{M} = I \times_f M$ be a GRW spacetime that satisfies the NCC. Assume that M is Ricci flat, and let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on the compact spacelike hypersurface Σ of \overline{M} . Then,

$$\lambda \geq (n-1) \left(\frac{f'^2}{f^2} - H^2 \right).$$

The equality holds if and only if Σ is totally umbilical and $(\log f)'' \sinh^2 \phi = 0$.

Proof. Using Lemma 1, we have

$$n \int_{\Sigma} (\lambda - (n-1) \left(\frac{f'^2}{f^2} - H^2 \right)) dV = \int_{\Sigma} (|A|^2 - nH^2 - 2(n-1)(\log f)'' \sinh^2 \phi) dV. \quad (29)$$

Applying Schwartz's inequality and using (16), we deduce

$$\lambda \geq (n-1) \left(\frac{f'^2}{f^2} - H^2 \right).$$

It is clear that equality is achieved if and only if Σ is totally umbilical and $(\log f)'' \sinh^2 \phi = 0$. \square

If $(\log f)'' < 0$, the following holds as a direct consequence.

Corollary 2. Consider $\overline{M} = I \times_f M$ as a GRW spacetime that satisfies the NCC, and assume that M is Ricci flat, and $-\log f$ is strictly convex. If $(\Sigma, g, \xi^T, \lambda)$ is a Ricci soliton on the compact space-like hypersurface Σ of \overline{M} , then

$$\lambda \geq (n-1) \left(\frac{f'^2}{f^2} - H^2 \right).$$

The equality holds if and only if Σ is a slice.

Remark 1. In Theorem 5, if the warping function f is a constant, we obtain $\lambda \geq -(n-1)H^2$, with the equality holding if and only if Σ is totally umbilical with constant mean curvature, and $(\Sigma, g, \xi^T, \lambda)$ is steady or expanding, depending on $H = 0$ or $H \neq 0$. According to [22], the Ricci soliton is trivial. This further implies that $\operatorname{div}(\xi^T) = 0$, and by Theorem 3, Σ is a slice.

If the manifold (\overline{M}, \bar{g}) is Einstein with $\bar{Ric} = \bar{c}\bar{g}$, then (M, g_M) is also Einstein with $Ric_M = cg_M$, and the constants \bar{c} and c must satisfy the following equations (see [23,24]):

$$\bar{c} = \frac{nf''}{f},$$

and

$$c = (n-1)f^2(\log f)''.$$

Theorem 6. Let $\overline{M} = I \times_f M$ be a GRW spacetime with M being Einstein and having positive Ricci curvature, and $-\log f$ is convex. If Σ is a spacelike hypersurface in \overline{M} , then its scalar curvature satisfies

$$S \geq \frac{S_M}{f^2} + n(n-1) \left(\frac{f'^2}{f^2} - H^2 \right).$$

Moreover, the equality holds if and only if Σ is a slice.

Proof. Set $\text{Ric}_M = c g_M$, with $c > 0$. It follows from this and from the assumption on f that $S \geq \frac{S_M}{f^2} + n(n-1)\left(\frac{f'^2}{f^2} - H^2\right)$. The equality holds if and only if Σ is totally umbilical and $\text{Ric}_M(N^*, N^*) = 0$. However, $\text{Ric}_M(N^*, N^*) = \frac{c \sinh^2 \phi}{f^2}$. It follows that $\phi = 0$, that is, Σ is a slice. In addition, we have $S = \frac{S_M}{f^2}$. \square

Theorem 7. Let $\overline{M} = I \times_f M$ be a GRW spacetime, with M Einstein having nonnegative Ricci curvature. Assume that $(\Sigma, g, \xi^T, \lambda)$ is a Ricci soliton on the compact hypersurface Σ of \overline{M} . Then, we have

$$\lambda \geq (n-1)\left(\frac{f'^2}{f^2} - H^2\right) - \frac{2c}{nf^2} \sinh^2 \phi. \quad (30)$$

The equality holds if and only if M is Ricci flat and Σ is a slice.

Proof. By Equation (24) and applying Schwartz's inequality to $\overline{M} = I \times_f M$, we obtain (30). If the equality holds, then Σ is total umbilical with $\lambda = (n-1)\left(\frac{\bar{c}}{n} - H^2\right)$ and $c = 0$. According to Theorem 6.1 in [24], Σ is a slice. \square

Based on the above theorem, the following corollaries hold.

Corollary 3. There are no Ricci solitons $(\Sigma, g, \xi^T, \lambda)$ on a compact spacelike hypersurface Σ of an Einstein GRW spacetime with a fiber that has a positive Ricci curvature.

Corollary 4. Let $\overline{M} = I \times_f M$ be a GRW spacetime with M Ricci flat, and let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on the compact spacelike hypersurface Σ of \overline{M} . Then,

$$\lambda \geq (n-1)\left(\frac{\bar{c}}{n} - H^2\right).$$

The equality holds if and only if Σ is a slice. Moreover, if $\bar{c} > nH^2$, then the Ricci soliton is shrinking.

Proof. Since $c = 0$, we obtain $\frac{\bar{c}}{n} = \frac{f'^2}{f^2}$. By Equation (30), it follows $\lambda \geq (n-1)\left(\frac{\bar{c}}{n} - H^2\right)$. If the equality holds, then Σ is total umbilical with $\lambda = 0$. According to Theorem 6.1 in [24], Σ is a slice. If $\bar{c} > nH^2$, $\lambda > 0$. This indicates that the Ricci soliton is shrinking. \square

Theorem 8. Let $\overline{M} = I \times_f M$ be a GRW spacetime, with M Einstein having a nonpositive Ricci curvature. Assume that $(\Sigma, g, \xi^T, \lambda)$ is a Ricci soliton on the compact spacelike hypersurface Σ of \overline{M} . Then, we have

$$\lambda \geq 2\frac{c}{n}g_M(N^*, N^*) + \frac{c}{f^2} + (n-1)\left(\frac{f'^2}{f^2} - H^2\right). \quad (31)$$

The equality holds if and only if Σ is a slice.

Proof. By Equation (24) and by applying Schwartz's inequality to $\overline{M} = I \times_f M$, we obtain (31). If the equality holds, then Σ is total umbilical and $(\log g)'' \sinh \phi = 0$. It implies that $c = 0$ and $\lambda = (n-1)\left(\frac{\bar{c}}{n} - H^2\right)$, or $\phi = 0$ and Σ is a slice. In each case, Σ is a space-like hypersurface with constant mean curvature. According to Theorem 6.1 in [24], Σ is a slice. \square

If we assume that $c = 0$, then $\bar{c} \geq 0$ and the following consequences hold.

Corollary 5. Let $\overline{M} = I \times_f M$ be a Ricci flat GRW spacetime, with M being Ricci flat, and let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on the compact spacelike hypersurface Σ of \overline{M} . Then,

$$\lambda \geq -(n-1)H^2. \quad (32)$$

The equality holds if and only if Σ is a slice.

Proof. From the proof of Theorem 8, we obtain (32) where equality holds if and only if $\lambda \leq 0$, which implies that the Ricci soliton is trivial. By Theorem 3, Σ is a slice. \square

Corollary 6. Let $\overline{M} = I \times_f M$ be a GRW spacetime, with M Ricci flat, and let $(\Sigma, g, \xi^T, \lambda)$ be a Ricci soliton on the compact spacelike hypersurface Σ of \overline{M} . Then,

$$\lambda \geq (n-1) \left(\frac{\bar{c}}{n} - H^2 \right). \quad (33)$$

The equality holds if and only if Σ is a slice. Moreover, if $\bar{c} > nH^2$, then the Ricci soliton is shrinking.

Proof. From the proof of Theorem 8, we obtain (33). If $\bar{c} > nH^2$, then $\lambda > 0$, which implies that the Ricci soliton is shrinking. \square

4. Conclusions

In this paper, we explore Ricci solitons occurring on spacelike hypersurfaces of generalized Robertson–Walker (GRW) spacetimes. Our investigation has led to significant results that outline how a Ricci soliton behaves on such a hypersurface in the presence of a natural form of symmetry, which is represented by the conformal vector field $f\partial t$, where f is the warping function, and ∂t is the unit timelike vector field tangent to the one-dimensional base manifold. These results, subject to specific conditions, either specify particular behaviors or force the hypersurface to possess distinct characteristics, such as being a slice or a totally umbilical hypersurface. We believe that our work will serve as a valuable resource for researchers in this field, offering a deeper understanding of the mathematical and physical connections between Ricci solitons and GRW spacetimes.

Author Contributions: Conceptualization, M.G.; Investigation, N.A. and M.G.; Methodology, N.A. and M.G.; Resources, N.A.; Validation, M.G.; Writing—original draft, N.A.; Writing—review and editing, N.A. and M.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by Researchers Supporting Project number (RSPD2024R824), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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