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Exploring perturbative constraints in higher-order curvature gravity theories

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Abstract

In the realm of general relativity (GR) and extended theories of gravity, obtaining solutions for scenarios of physical interest is a highly intricate challenge. By employing the formalism of mathematical perturbation theory within the GR framework, we demonstrate that, for a significant class of vacuum $f(R, R_{\mu\nu}R^{\mu\nu})$ theories, the corresponding solutions do not yield additional effects beyond those predicted by GR's perturbation theory. However, models characterized by terms of the form $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$ exhibit distinctive contributions not present in GR. We assert that fundamental limitations exist, explaining why solutions of certain $f(R, R_{\mu\nu}R^{\mu\nu})$ models can deviate from their GR counterparts, indicating non-connected solutions or non-analytic behavior. Conversely, in the models $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$, the solutions seamlessly connect with those of GR. This distinction highlights the nuanced interplay between higher-order curvature terms and their impact on gravitational dynamics, offering new insights into the landscape of modified gravity theories.

Keywords: higher-order curvature gravity, perturbation theory, general relativity, modified theories of gravity, non-analytic solutions, vacuum solutions, lovelock theorem

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1. Introduction

Despite the remarkable success of general relativity (GR), there are strong theoretical motivations to explore modifications to it. On the theoretical side, several significant issues stand out: (i) the presence of singularities [1, 2], where GR predicts its own breakdown, (ii) the challenges of quantization, particularly the lack of renormalizability following standard quantization methods [3, 4], (iii) the search for a consistent unification with quantum mechanics, as GR remains incompatible with the standard model of particle physics [5], and (iv) the cosmological constant problem and hierarchy problem, both unresolved in our current framework [6–8], among other theoretical arguments. Additionally, while GR can accommodate dark energy and dark matter through the inclusion of a cosmological constant and adjustments to the matter sector, the lack of direct detection of these components invites consideration of alternative frameworks where these phenomena might emerge from modifications to gravity itself. Thus, any alternative theory of gravity should ideally connect the ultraviolet and infrared regimes while preserving the core successes of GR.

Modified theories of gravity (MTG), which build upon corrections and extensions of Einstein's theory, represent the most prolific paradigms within this framework. The fundamental concept underlying this approach is to consider an effective quantum gravity action that incorporates higher-order curvature invariants and scalar fields, which may be coupled to the gravitational sector either minimally or nonminimally. These theories are constructed to recover GR at local scales and in the weak-field limit.

In this context, there has been increasing interest in MTG [9, 10]. Specifically, significant attention has been given to $f(R)$ [11, 12] and scalar-tensor theories [13] as these frameworks can effectively describe possible quantum gravity in the low-energy limit [14–16]. Additionally, these theories have been robustly incorporated into cosmological models [17, 18], where they naturally exhibit inflationary characteristics [19–21], and provide realistic models for dark energy [22–25].

The pursuit of exact solutions in MTG, such as $f(R)$ models, is of significant importance. Spherically symmetric solutions, in particular, are of great interest as they can be directly compared to the well-known Schwarzschild solution, allowing for observational constraints on any deviations. However, constructing such solutions is a challenging endeavor, and only a limited number of exact (and vacuum) solutions are currently known in $f(R)$ gravity (examples can be found in [20, 26–29]).

As in GR, the field equations in MTG have a complex and intriguing structure, often equivalent to higher-order non-linear partial differential equations for which no general solution method exists. Additionally, the nature of GR and MTG, aimed at modeling spacetime, reveals a wealth of geometric, mathematical, and physical aspects, presenting a vast array of possibilities. Our primary approach to addressing this challenge is perturbation theory, where the *background* is a spacetime in GR, and the successive perturbation terms correspond to contributions from the MTG. We should compare scenarios in the strong field limit of GR with small contributions from MTG.

The results we will show along this manuscript are perfectly applicable to the new era of multimessenger astronomy, with key areas focusing on gravitational waves from binary mergers [30] and shadows induced by superheated plasma surrounding supermassive central objects [31]. The ability to analyze perturbations in such contexts is crucial for advancing our understanding of the underlying physics and for interpreting observational data accurately. The results presented here, built on foundational work in [32] and further developed and discussed in [33] for $f(R)$ gravity, are a generalization to theories $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$. In [33], we found some mathematical properties concerning the vacuum solutions of a class of functions of the form $f(R) = R + \lambda\Psi(R)$. The solutions of these types of Lagrangians are either vacuum GR solutions, or are disconnected from GR with respect to the parameter λ, or they are not analytic in the vicinity of $\lambda = 0$.

In this work, we extend previous results to Lagrangians of the form $f(R, R_{\mu\nu}R^{\mu\nu}) = R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu})$ and we discuss their application in scalar-tensor theories and other MTG by adding the Gauss–Bonnet term. We also demonstrate how the solutions corresponding to the aforementioned theories connect with those coming from GR when $\lambda \to 0$. Finally, a conjecture about some generalizations of the results here obtained is proposed.

This work is organized as follows: In section 2, we provide a brief review of fourth-order theories of gravity, including $f(R)$, scalar-tensor, and theories constructed from general combinations of the Ricci and Riemann curvatures. Subsequently, in section 3, we present the mathematical foundations of perturbation theory and how we use it to compare different theories. In section 4, we discuss important mathematical properties and introduce the principal results and their implications for $f(R)$ and scalar-tensor theories. We explore how solutions of these extended theories connect with the corresponding background GR solutions when particular scalar invariants, included in the Lagrangian, are not zero for the background spacetime. After proposing a general conjecture, we discuss and summarize our results, highlighting current and future work, in section 5.

2. Four order theories of gravity and scalar-tensor theories

The possibilities enabled by the Lovelock theorem [34] encompass higher-order gravitational theories, which involve more than two derivatives in their field equations, as well as theories with additional fields such as scalar, vector, or tensor fields, among other types of gravitational theories. These theories have a rich history dating back to a few years following the publication of Einstein's groundbreaking GR paper and have been since then the subject of extensive research [17, 35–37]. As commented in the introduction, the motivations driving these investigations range from the quantization of matter fields within an classical spacetime to the explanation of the accelerated expansion of the Universe in its early and, potentially, late epochs [38, 39].

One approach to extending GR in this context is to incorporate a generalized Lagrangian, denoted as $f(R)$, within the geometric part of the action, as we introduce below. Even more, we can consider Lagrangians that depend on contractions of geometric quantities, such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}$ (an illustrative example of such theories is Gauss–Bonnet gravity [40]). In this section, we introduce the concept of $f(R)$ gravity, scalar tensor theories and subsequently delve into the more general combinations involving the aforementioned contractions of Ricci and Riemann curvatures.

2.1. f(R) MTG

The space-time is modeled by the pair $(M, g_{\mu\nu})$, where M is a Hausdorff and paracompact four-dimensional manifold and $g_{\mu\nu}$ is a Lorentzian metric with signature $(-,+,+,+)$ on M. The metric is determined by the $f(R)$ modified field equations, which extend the Einstein field equation.

In the $f(R)$ gravity theory, the action is written in the form:

$$S(g) = \int_M \left(\frac{1}{16\pi G} f(R) + \mathcal{L}_m \right) \sqrt{-g}\, d^4x,$$

where f is a smooth and analytic function of the Ricci scalar R and $\mathcal{L}_m$ is the Lagrangian of the matter fields. For boundary term see [41].

The variation of this action with respect to the metric $g_{\mu\nu}$ leads to the field equations:

$$\Sigma_{\mu\nu} \equiv f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) = 8\pi G T_{\mu\nu},$$

where $f'(R) \equiv \frac{\partial f}{\partial R}$.

Similar to the geometric part of the Einstein field equations, written in terms of the Einstein tensor, $G_{\mu\nu}$, the geometric part of the $f(R)$ field equations is denoted by $\Sigma_{\mu\nu}$. In addition, $T_{\mu\nu}$ is the energy-momentum tensor describing the matter distribution, $R \equiv R_{\mu\nu} g^{\mu\nu}$ is the Ricci scalar, and the Ricci tensor $R_{\mu\nu}$ is defined as $R_{\mu\nu} \equiv R_{\mu\rho\nu}{}^{\rho}$. The Riemann tensor in a local coordinate basis is given by:

$$R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^\nu} \Gamma^\sigma{}_{\mu\rho} - \frac{\partial}{\partial x^\mu} \Gamma^\sigma{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho} \Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho} \Gamma^\sigma{}_{\alpha\mu},$$

where $\Gamma^\sigma{}_{\mu\rho}$ are the Christoffel symbols, derived from the metric tensor through the expression:

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right).$$

In this context, the field equations $\Sigma_{\mu\nu} = 8\pi G T_{\mu\nu}$ describe the dynamics of the $f(R)$ gravity theory, providing a more general framework than the standard Einstein field equations.

2.2. Scalar-tensor theories and their equivalence with f(R) gravity

Scalar-tensor theories of gravity are a class of MTG where the gravitational interaction is mediated by both a metric tensor field $g_{\mu\nu}$ and one or more scalar fields. These theories generalize GR by introducing additional degrees of freedom through the scalar fields, which can vary across space-time [42].

In scalar-tensor theories, the action typically takes the form:

$$S(g,\phi) = \int_M \left[\frac{1}{16\pi G} \left(\phi R - \frac{\omega(\phi)}{\phi} (\nabla\phi)^2 - V(\phi) \right) + \mathcal{L}_m \right] \sqrt{-g}\, d^4x, \tag{1}$$

where ϕ is the scalar field, $\omega(\phi)$ is a function representing the coupling between the scalar field and the Ricci scalar R, and $V(\phi)$ is the potential of the scalar field.

The field equations obtained by varying the action with respect to the metric $g_{\mu\nu}$ and the scalar field ϕ are:

$$\begin{aligned} G_{\mu\nu} = & \frac{8\pi G}{\phi} T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2} \left(\nabla_\mu\phi \nabla_\nu\phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \right) \\ & + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box\phi) - \frac{V(\phi)}{2\phi} g_{\mu\nu}, \end{aligned} \tag{2}$$

$$\Box\phi = \frac{1}{2\omega(\phi) + 3} \left(8\pi G T - \frac{d\omega}{d\phi} (\nabla\phi)^2 + \phi \frac{dV}{d\phi} - 2V(\phi) \right), \tag{3}$$

where T is the trace of $T_{\mu\nu}$, and $\Box$ is the d'Alembertian operator [13].

One significant aspect of scalar-tensor theories is their equivalence with $f(R)$ gravity. By introducing an auxiliary field χ, in the action of $f(R)$, the action can be rewritten as [11]:

$$S(g,\chi) = \int_M \left[\frac{1}{16\pi G}\left(f'(\chi)(R-\chi)+f(\chi)\right)+\mathcal{L}_m\right]\sqrt{-g}\,d^4x. \tag{4}$$

Defining the scalar field $\phi = f'(\chi)$, the action can now be written as:

$$S(g,\phi) = \int_M \left[\frac{1}{16\pi G}\left(\phi R - V(\phi)\right)+\mathcal{L}_m\right]\sqrt{-g}\,d^4x, \tag{5}$$

where $V(\phi)$ is a potential function, $V(\phi) = \phi\,\chi(\phi) - f(\chi(\phi))$, and $\chi(\phi)$ is the inverse of $\phi = f'(\chi)$, assuming this transformation is invertible. In this form, the action (5) resembles that of a scalar-tensor theory (1) with $\omega(\phi) = 0$, under the condition that $\phi = f'(\chi)$ is indeed an invertible transformation.

This demonstrates that $f(R)$ gravity can be reformulated as a scalar-tensor theory with a specific form of the potential and coupling functions. Therefore, scalar-tensor theories provide a broad framework that encompasses $f(R)$ gravity as a special case, illustrating the deep connections between different approaches to modifying GR [10].

2.3. Including combinations of Ricci and Riemann curvatures

We now consider a family of theories of fourth order, which are more general than the $f(R)$ theory. We extend the $f(R)$ action to a new one, which is a general combination of some contractions of the Ricci and Riemann tensors. In particular, we write:

$$S(g) = \int_M \left(\frac{1}{16\pi G} f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \mathcal{L}_m\right)\sqrt{-g}\,d^4x, \tag{6}$$

where f is an arbitrary function of its arguments. The corresponding field equations are given by [17]:

$$P_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{7}$$

where

$$\begin{aligned} P^{\mu\nu} \equiv & -\frac{1}{2} f g^{\mu\nu} + f_X R^{\mu\nu} + 2 f_Y R^{\rho(\mu}R^{\nu)}{}_{\rho} + 2 f_Z R^{\delta\sigma\rho(\mu}R^{\mu)}{}_{\rho\sigma\delta} \\ & + f_{X;\rho\sigma}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right) + \Box(f_Y R^{\mu\nu}) + g^{\mu\nu}(f_Y R^{\rho\sigma})_{;\rho\sigma} \\ & - 2(f_Y R^{\rho(\mu})_{;\ \ \rho}^{\ \nu)} - 4(f_Z R^{\sigma(\mu\nu)\rho})_{;\rho\sigma}, \end{aligned} \tag{8}$$

being $X \equiv R$, $Y \equiv R_{\mu\nu}R^{\mu\nu}$ and $Z \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The derivative of f with respect to W has been denoted as f_W.

Apart from $f(R)$ gravity, a distinctive subset of these gravitational theories is referred to as Gauss–Bonnet gravity, where the action incorporates a general function $f(R,\hat{G})$, with

$$\hat{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{9}$$

These theories have been extensively investigated and are rooted in the principles of string theory [43].

3. Extended theories of gravity and perturbation theory

The intricacies of GR, particularly within the diverse facets of its field equations and covariant nature, have spurred significant advancements in the mathematical formulations of perturbation theory [44, 45]. Leveraging these formalisms, we endeavor to devise a method for comparing spacetimes in MTG with GR, even in the absence of explicit solutions, leading to interesting results.

Consider two theories: $f_0(R) = R$, representing GR, and $f_\lambda(R) = R + \lambda R^2$, corresponding to the Starobinsky model [19, 46]. Although we recover GR when $\lambda = 0$, it is an erroneous assumption that solutions of the model f_λ converge to GR solutions as λ tends to zero, as we illustrated in [33] (for a particular example in spherical symmetry, see [46]). Additionally, for any analytic function of the form $f(R) = R + \lambda\Psi(R)$ with $\Psi(0) = 0$, GR is not recovered as λ approaches zero but the solutions become either 'disconnected' or they are not analytic.

Expanding on these findings, we extend our exploration to Lagrangians of the form $f(R, R_{\mu\nu}R^{\mu\nu})$ and later we investigate how solutions become connected for Lagrangians of the type $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$. Our approach broadens our understanding of the intricate relationship between MTG and GR, emphasizing that the behavior of solutions in these theories is not merely an extrapolation of those of GR as the parameters of the theory approach zero.

Let us consider the foundational principles of perturbation theory within the realm of GR to formulate a comprehensive framework for facilitating comparisons between solutions in MTG, described by $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ and those in GR.

Assume g_0 as a solution in GR and g_λ as a solution in some MTG where its Lagrangian depends of λ and tends to the GR one when $\lambda \to 0$. We assign to each λ a spacetime $\mathcal{M}_\lambda$ and we build a 4+1-dimensional manifold, denoted as $\mathcal{N}$, foliated by spacetime manifolds $\mathcal{M}_\lambda$. Consequently, we express $\mathcal{N}$ as the Cartesian product of $\mathcal{M}$ and the real line, i.e. $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where this choice of topology implies a foliable structure without topology change among the leaves (slices) of the foliation, which is necessary to maintain analyticity. $\mathcal{M}_0$ is known as the *background* spacetime.

In the framework of GR perturbation theory [45], we employ a mapping procedure, implemented through a 'flow' $\mathcal{X}_\lambda : \mathcal{N} \to \mathcal{N}$ generated by a vector field X. This vector field is transversal to each submanifold $\mathcal{M}_\lambda$ (in a specific coordinate chart, we can select the coordinate X^5 as a non-zero constant) and is rotationless. Let T be a smooth and analytic tensor field, defined on the manifold $\mathcal{N}$. Our objective is to establish a comparison between tensor fields T_λ defined on the submanifolds $\mathcal{M}_\lambda$ and T_0 on $\mathcal{M}_0$ through $\mathcal{N}$. Within this mathematical framework, we expand the pull-back $\mathcal{X}^*_\lambda T$ of the tensor field T around $\lambda = 0$ via a Taylor series, defined in terms of the Lie derivatives along $\mathcal{X}$. This expansion is expressed as [44, 45, 47]:

$$\mathcal{X}^*_\lambda T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \pounds^k_X T, \tag{10}$$

where $\pounds^k_X T$ represents the kth Lie derivative of the tensor field T along the flow generated by the vector field X. If we denote $\overset{(k)}{T} \equiv \pounds^k_X T$ for the Lie derivatives and the pull-back $\mathcal{X}^*_\lambda T \equiv \bar{T}$, thus, the equation (10) give us

$$\bar{T} = \overset{(0)}{T} + \lambda \overset{(1)}{T} + \frac{\lambda^2}{2!} \overset{(2)}{T} + \cdots. \tag{11}$$

With this, following Wald [48], we restrict our tensor fields to the field equation on each manifold $\mathcal{M}_\lambda$. We assume that diverse gravitational models are continuously interlinked by

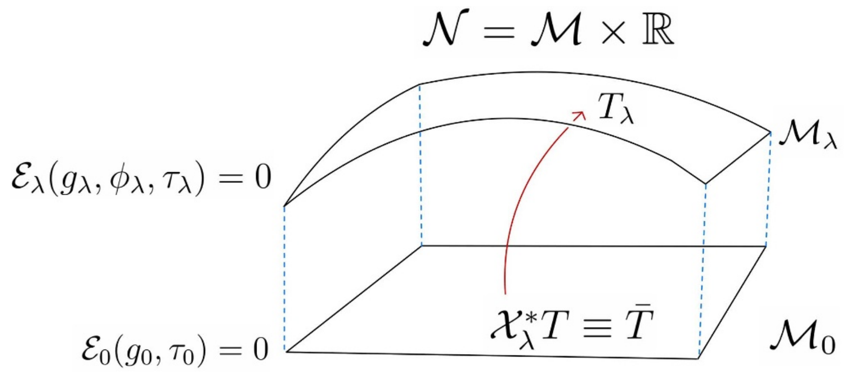

Figure 1. The (4+1)-dimensional manifold $\mathcal{N}$. Each $\mathcal{M}_\lambda$ represents a distinct gravitational model, labeled by the parameter λ, while the model within $\mathcal{M}_0$ corresponds to GR. In the domain of perturbation theory, $\mathcal{M}_0$ is the *background spacetime*. The pull-back permits the comparison of tensors in gravitational models connected by λ with GR.

the parameter λ. That is, we contemplate an Einstein field equation that is satisfied in the background, represented as follows:

$$\mathcal{E}_0(g_0, \tau_0) = 0,$$

where g_0 and τ_0 are the metric and the stress-energy tensor, respectively, in the Einstein field equation, $\mathcal{E}_0$. Now, we assume that a modified field equation is satisfied in the spacetime $\mathcal{M}_\lambda$, as

$$\mathcal{E}_\lambda(g_\lambda, \phi_\lambda, \tau_\lambda) = 0, \tag{12}$$

where $\mathcal{E}_\lambda$ denotes the field equations specific for the corresponding MTG. Also, g_λ and τ_λ represent the metric associated with the aforementioned model, and the stress-energy tensor exhibiting a dependence on λ, while ϕ_λ encompasses a set of scalar, vector or tensor fields. Figure 1 summarizes this geometric construct.

We point out that there is a mathematical freedom in this formulation regarding the choice of the flux, $\mathcal{X}$. This means that we have infinite possibilities to identify the background with the spacetime $\mathcal{M}_\lambda$. This identification is referred to as the *gauge choice* of the second kind, according to the classification of Sachs [49]. The physics of this identification should remain unchanged in each background. However, when we compare tensor-perturbed quantities using different gauge choices, they do not remain invariant in all cases. One condition to ensure such invariance is the vanishing of background quantities. This condition arises as a consequence of the generalized Stewart and Walker lemma (for details, we refer to [45, 50]). We will briefly comment on it in the following section.

4. Relation between $f(R), f(R, R_{\mu\nu}R^{\mu\nu})$ MTG and GR perturbation theories

The mathematical framework of perturbation theory introduced above is crucial as it underpins our results in this section. Essentially, we demonstrate that the interlinked metric solutions between vacuum models of $f(R, R_{\mu\nu}R^{\mu\nu})$ gravity and GR are fundamentally the same. This indicates that, within this framework, $f(R, R_{\mu\nu}R^{\mu\nu})$ does not contribute at any order when compared to GR as a background. As we will discuss, this implies that the solutions of vacuum

$f(R, R_{\mu\nu}R^{\mu\nu})$ models are either disconnected from GR or not analytic with respect to λ in a significant class of functions $f(R, R_{\mu\nu}R^{\mu\nu})$.

4.1. Preliminary Taylor expansions of important tensors

Before proving the theorem, in this section, we demonstrate how to expand each term of the modified field equations (8) in a Taylor series. Then, we state and prove the theorem.

If we consider generic tensors P, Q and O on $\mathcal{N}$ with the respective pull back $\bar{P}$, $\bar{Q}$ and $\bar{O}$ on $\mathcal{M}_0$, where we fix a gauge $\mathcal{X}$, then

$$\bar{P} = \overset{(0)}{P} + \lambda \overset{(1)}{P} + \frac{\lambda^2}{2!} \overset{(2)}{P} + \cdots , \tag{13}$$

$$\bar{Q} = \overset{(0)}{Q} + \lambda \overset{(1)}{Q} + \frac{\lambda^2}{2!} \overset{(2)}{Q} + \cdots . \tag{14}$$

$$\bar{O} = \overset{(0)}{O} + \lambda \overset{(1)}{O} + \frac{\lambda^2}{2!} \overset{(2)}{O} + \cdots . \tag{15}$$

Immediately, it can be shown that

$$\bar{P}\bar{Q}\bar{O} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!n_2!n_3!} \overset{(n_1)}{P} \overset{(n_2)}{Q} \overset{(n_3)}{O} . \tag{16}$$

Thus, the nth term of the tensor product $\bar{P}\bar{Q}\bar{O}$ is written as

$$\overbrace{PQO}^{(n)} \equiv \sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!n_2!n_3!} \overset{(n_1)}{P} \overset{(n_2)}{Q} \overset{(n_3)}{O} . \tag{17}$$

Now, we introduce the following terms

$$\bar{f_X} = \overset{(0)}{f_X} + \lambda \overset{(1)}{f_X} + \frac{\lambda^2}{2!} \overset{(2)}{f_X} + \cdots \tag{18}$$

$$\bar{f_Y} = \overset{(0)}{f_Y} + \lambda \overset{(1)}{f_Y} + \frac{\lambda^2}{2!} \overset{(2)}{f_Y} + \cdots \tag{19}$$

$$\bar{f_Z} = \overset{(0)}{f_Z} + \lambda \overset{(1)}{f_Z} + \frac{\lambda^2}{2!} \overset{(2)}{f_Z} + \cdots \tag{20}$$

With these results, we can obtain the nth term of the geometrical sector of the field equation (8). For the first term we get

$$-\frac{1}{2}\bar{f}\bar{g}_{\mu\nu} \longrightarrow -\frac{1}{2}\overset{(n)}{fg_{\mu\nu}} = -\frac{1}{2}\sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{f} \overset{(n-i)}{g_{\mu\nu}} . \tag{21}$$

The second and third terms are written as

$$\bar{f_X}\bar{R}_{\mu\nu} \longrightarrow \overset{(n)}{f_X R_{\mu\nu}} = \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{f_X} \overset{(n-i)}{R_{\mu\nu}} \tag{22}$$

$$2\bar{f_Y}\bar{R}^{\rho}{}_{(\mu}\bar{R}_{\nu)\rho} \longrightarrow 2\overbrace{f_Y R^{\rho}{}_{(\mu}R_{\nu)\rho}}^{(n)} = 2\sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!n_2!n_3!} \overset{(n_1)}{f_Y} \overset{(n_2)}{R^{\rho}}{}_{(\mu} \overset{(n_3)}{R}_{\nu)\rho} . \tag{23}$$

The following terms are covariant double derivatives of different types. To establish this, we must compare the differential operator associated with the covariant derivative $\bar{\nabla}_a$ in the perturbed manifold $\mathcal{M}_\lambda$ with the operator ∇_a in the background manifold $\mathcal{M}_0$. We denote by $\mathcal{X}_\lambda^* \bar{\nabla}_a (\mathcal{X}_\lambda^{-1})^*$ the pullback of $\bar{\nabla}_a$ to $\mathcal{M}_0$, which we represent simply as $\bar{\nabla}_a$. Consequently, we now have two distinct differential operators, ∇_a and $\bar{\nabla}_a$, defined on $\mathcal{M}_0$. We introduce the difference operator $\nabla_a - \bar{\nabla}_a$ on the dual space in $\mathcal{M}_0$, which gives rise to a new rank-$(1,2)$ tensor C^c_{ab} (see [48]). Explicitly, we have:

$$\bar{\nabla}_a \omega_b = \nabla_a \omega_b - C^c_{ab}\omega_c, \tag{24}$$

where ω_a is a dual vector field on $\mathcal{M}_0$. This property leads to the expression

$$C^c_{ab} = \frac{1}{2}\bar{g}^{cd}\left(\nabla_a \bar{g}_{bd} + \nabla_b \bar{g}_{ad} - \nabla_d \bar{g}_{ab}\right), \tag{25}$$

which characterizes the difference between the covariant derivatives $\bar{\nabla}$ and ∇. Using the expansion property of the metric perturbation (11), we can express $C_{\mu\nu}{}^{\delta}$ as a series:

$$C_{\mu\nu}{}^{\delta} = \overset{(0)}{C}_{\mu\nu}{}^{\delta} + \lambda \overset{(1)}{C}_{\mu\nu}{}^{\delta} + \frac{\lambda^2}{2!}\overset{(2)}{C}_{\mu\nu}{}^{\delta} + \cdots \tag{26}$$

In the background, we assume $\nabla_a \overset{(0)}{g}_{bc} = 0$, which implies that $\overset{(0)}{C}_{\mu\nu}{}^{\delta} = 0$. With this setup, we arrive at the identity:

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{f}_X = \nabla_\mu \nabla_\nu \bar{f}_X - C_{\mu\nu}{}^{\delta}\nabla_\delta \bar{f}_X, \tag{27}$$

which allows us to relate the action of the perturbed and background covariant derivatives on the function $\bar{f}_X$.

Thus, the fourth and fifth term read

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{f}_X \longrightarrow \overbrace{\nabla_\mu \nabla_\nu f_X}^{(n)} = \nabla_\mu \nabla_\nu \overset{(n)}{f_X} - \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{C}{}^{\delta}_{\mu\nu} \nabla_\delta \overset{(n-i)}{f_X} \tag{28}$$

and

$$\bar{g}_{\mu\nu}\bar{\Box}\bar{f}_X \longrightarrow \overbrace{g_{\mu\nu}\Box f_X}^{(n)} = \sum_{n_1+n_2+n_3=n} \frac{n!}{n_1!n_2!n_3!} \overset{(n_1)}{g}_{\mu\nu} \overset{(n_2)}{g}{}^{\sigma\rho} \overbrace{\nabla_\sigma \nabla_\rho f_X}^{(n_3)}. \tag{29}$$

For details on the Taylor expansion of the covariant derivative operator, see [44, 48].

For the sixth, seventh and eight terms we will follow the next procedure. First, we define $T_{\mu\nu} \equiv f_X R_{\mu\nu}$, $S_{\delta\mu\nu} \equiv \nabla_\delta T_{\mu\nu} = \nabla_\delta (f_X R_{\mu\nu})$ and $M_{\sigma\delta\mu\nu} \equiv \nabla_\sigma S_{\delta\mu\nu} = \nabla_\sigma \nabla_\delta (f_X R_{\mu\nu})$, thus

$$\bar{T}_{\mu\nu} = \bar{f}_X \bar{R}_{\mu\nu} \longrightarrow \overbrace{f_X R_{\mu\nu}}^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{f_X} \overset{(n-i)}{R}_{\mu\nu} \tag{30}$$

$$\begin{aligned} \bar{S}_{\delta\mu\nu} = \bar{\nabla}_\delta \bar{T}_{\mu\nu} &\longrightarrow \overbrace{\nabla_\delta T_{\mu\nu}}^{(n)} \\ &= \nabla_\delta \overset{(n)}{T}_{\mu\nu} - \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{C}{}^{\rho}_{\mu\delta} \overset{(n-i)}{T}_{\rho\nu} - \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{C}{}^{\rho}_{\delta\nu} \overset{(n-i)}{T}_{\mu\rho} \end{aligned} \tag{31}$$

$$\bar{M}_{\sigma\delta\mu\nu} = \bar{\nabla}_{\sigma}\bar{S}_{\delta\mu\nu} \longrightarrow \overbrace{\nabla_{\delta}S_{\delta\mu\nu}}^{(n)}$$
$$= \nabla_{\sigma}\overset{(n)}{S}_{\delta\mu\nu} - \sum_{i=0}^{n}\binom{n}{i}\overset{(i)}{C}{}^{\rho}_{\sigma\delta}\overset{(n-i)}{S}_{\rho\mu\nu} - \sum_{i=0}^{n}\binom{n}{i}\overset{(i)}{C}{}^{\rho}_{\sigma\mu}\overset{(n-i)}{S}_{\delta\rho\nu} - \sum_{i=0}^{n}\binom{n}{i}\overset{(i)}{C}{}^{\rho}_{\sigma\nu}\overset{(n-i)}{S}_{\delta\mu\rho} \tag{32}$$

thus, the sixth term will be

$$\bar{\Box}(\bar{f}_Y\bar{R}_{\mu\nu}) = \bar{g}^{\sigma\delta}\bar{M}_{\sigma\delta\mu\nu} \longrightarrow \overbrace{g^{\sigma\delta}M_{\sigma\delta\mu\nu}}^{(n)} = \sum_{i=0}^{n}\binom{n}{i}\overset{(i)}{g}{}^{\sigma\rho}\overset{(n-i)}{M}_{\sigma\delta\mu\nu} \tag{33}$$

and the seventh term becomes

$$\bar{g}_{\mu\nu}\bar{\nabla}_{\rho}\bar{\nabla}_{\sigma}(\bar{f}_Y\bar{R}^{\rho\sigma}) = \bar{g}_{\mu\nu}\bar{M}_{\sigma\rho}{}^{\sigma\rho} \longrightarrow \overbrace{g_{\mu\nu}M_{\sigma\rho}{}^{\sigma\rho}}^{(n)} = \sum_{i=0}^{n}\binom{n}{i}\overset{(i)}{g}_{\mu\nu}\overset{(n-i)}{M}_{\sigma\rho}{}^{\sigma\rho} \tag{34}$$

the last two terms being

$$2(f_YR^{\rho}{}_{(\mu})_{;\nu)\rho} = M_{\nu\rho}{}^{\rho}{}_{\mu} + M_{\mu\rho}{}^{\rho}{}_{\nu} \longrightarrow \overset{(n)}{M}_{\nu\rho}{}^{\rho}{}_{\mu} + \overset{(n)}{M}_{\mu\rho}{}^{\rho}{}_{\nu}\,. \tag{35}$$

Let us now generalize our previous results [33] to the case of $f(R,R_{\mu\nu}) = R + \lambda\Psi(R,R_{\mu\nu}R^{\mu\nu})$, where Ψ is an analytical function in λ, that is,

$$\bar{\Psi} = \overset{(0)}{\Psi} + \lambda\overset{(1)}{\Psi} + \frac{\lambda^2}{2!}\overset{(2)}{\Psi} + \cdots, \tag{36}$$

where $\overset{(0)}{\Psi}$ is a function of terms in the background, $\overset{(1)}{\Psi}$ is a function of first order perturbation terms,

$$\overset{(1)}{\Psi} = \pounds_X\Psi = (\frac{\partial\Psi}{\partial R})\Big|_{R=0,R_{\mu\nu}=0}\overset{(1)}{R} + (\frac{\partial\Psi}{\partial R_{\alpha\beta}R^{\alpha\beta}})\Psi_Y\Big|_{R=0,R_{\mu\nu}=0}\overbrace{R_{\alpha\beta}R^{\alpha\beta}}^{(1)} \tag{37}$$

where $\pounds_XR = \overset{(1)}{R}$, $\pounds_XR_{\alpha\beta}R^{\alpha\beta} = \overbrace{R_{\alpha\beta}R^{\alpha\beta}}^{(1)} = \overset{(1)}{R}_{\alpha\beta}\overset{(0)}{R}{}^{\alpha\beta} + \overset{(0)}{R}_{\alpha\beta}\overset{(1)}{R}{}^{\alpha\beta}$. Following this procedure, $\overset{(2)}{\Psi}$ will be a function of second order perturbation terms, and so on.

Now, for the function $\bar{f} = \bar{R} + \lambda\bar{\Psi} = \overset{(0)}{R} + \lambda\overset{(1)}{R} + \frac{\lambda^2}{2!}\overset{(2)}{R} + \cdots + \lambda(\overset{(0)}{\Psi} + \lambda\overset{(1)}{\Psi} + \frac{\lambda^2}{2!}\overset{(2)}{\Psi} + \cdots)$, we get

$$\overset{(0)}{f} = \overset{(0)}{R} \text{ for } n = 0, \tag{38}$$
$$\overset{(1)}{f} = \overset{(1)}{R} + \overset{(0)}{\Psi} \text{ for } n = 1, \tag{39}$$
$$\overset{(n)}{f} = \overset{(n)}{R} + n\overset{(n-1)}{\Psi} \text{ for } n \geqslant 1. \tag{40}$$

The same procedure applied to the function $\bar{f}_X = 1 + \lambda\bar{\Psi}_X = 1 + \lambda(\overset{(0)}{\Psi}_X + \lambda\overset{(1)}{\Psi}_X + \frac{\lambda^2}{2!}\overset{(2)}{\Psi}_X + \cdots)$ gives

$$\overset{(0)}{f}_X = 1 \text{ for } n = 0, \tag{41}$$

$$\overset{(1)}{f_X} = \overset{(0)}{\Psi_X} \text{ for } n = 1, \tag{42}$$

$$\overset{(n)}{f_X} = n \overset{(n-1)}{\Psi_X} \text{ for } n \geqslant 1. \tag{43}$$

Finally, for the function $f_Y = \lambda\Psi_Y = \lambda(\overset{(0)}{\Psi_Y} + \lambda \overset{(1)}{\Psi_Y} + \frac{\lambda^2}{2!} \overset{(2)}{\Psi_Y} + \cdots$, we get

$$\overset{(0)}{f_Y} = 0 \text{ for } n = 0, \tag{44}$$

$$\overset{(1)}{f_Y} = \overset{(0)}{\Psi_Y} \text{ for } n = 1, \tag{45}$$

$$\overset{(n)}{f_Y} = n \overset{(n-1)}{\Psi_Y} \text{ for } n \geqslant 1. \tag{46}$$

4.2. The theorem

With all these ingredients at hand, we now proceed to state and demonstrate the following

Theorem 1. *Let $\bar{P}_{ab} = 0$ be the vacuum field equations in $f(R, R_{\mu\nu}R^{\mu\nu})$ MTG for the model $f(\bar{R}, \bar{R}_{\mu\nu}\bar{R}^{\mu\nu}) = \bar{R} + \lambda\Psi(\bar{R}, \bar{R}_{\mu\nu}\bar{R}^{\mu\nu})$, where $\Psi(R, R_{\mu\nu}R^{\mu\nu})$ is analytic in a vicinity of $\lambda = 0$, and $\Psi(0,0) = 0$. Then, $\bar{P}_{ab} = \bar{G}_{ab}$ in vacuum.*

Proof. We prove it by mathematical induction. Let $n = 0$, thus

$$\begin{aligned}\overset{(0)}{P^{\mu\nu}} \equiv & -\frac{1}{2}\overbrace{fg^{\mu\nu}}^{(0)} + \overbrace{f_X R^{\mu\nu}}^{(0)} + 2\overbrace{f_Y R^{\rho(\mu} R^{\nu)}{}_{\rho}}^{(0)} \\ & + \overbrace{f_{X;\rho\sigma}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})}^{(0)} + \overbrace{\Box(f_Y R^{\mu\nu})}^{(0)} + \overbrace{g^{\mu\nu}(f_Y R^{\rho\sigma})_{;\rho\sigma}}^{(0)} \\ & - 2\overbrace{(f_Y R^{\rho(\mu})_{;}{}^{\nu)}{}_{\rho}}^{(0)}\end{aligned} \tag{47}$$

and if we have two tensor P and Q, then $\overset{(0)}{PQ} = \overset{(0)}{P}\,\overset{(0)}{Q}$. Also, $\overset{(0)}{f_X} = 1$ and $\overset{(0)}{f_Y} = 0$, thus the equation (47) give us

$$\frac{1}{2}\overset{(0)}{R}\,\overset{(0)}{g^{\mu\nu}} + \overset{(0)}{R^{\mu\nu}} = 0 \tag{48}$$

finally, we have $\overset{(0)}{P^{\mu\nu}} = \overset{(0)}{G^{\mu\nu}}$, and the trace of the last equation implies $\overset{(0)}{R^{\mu\nu}} = 0$ and $\overset{(0)}{R} = 0$. This part of the proof is just to check the consistency of our formalism, where we assume the GR field equations in the background. Now, let $n = 1$ be. First, we consider the three first terms of the perturbed field equations. Using (17), $\overset{(0)}{\Psi}=0$ and the results from $n=0$, then for $n=1$, we have,

$$\begin{aligned}
&\frac{1}{2}(\overset{(1)}{f}\,\overset{(0)}{g}_{\mu\nu}+\overset{(0)}{f}\,\overset{(1)}{g}_{\mu\nu})+\overset{(1)}{f_X}\,\overset{(0)}{R}_{\mu\nu}+\overset{(0)}{f_X}\,\overset{(1)}{R}_{\mu\nu}\\
&\quad+2(\overset{(1)}{f_Y}\,\overset{(0)}{R^\rho}{}_{(\mu}\overset{(0)}{R}_{\mu)\rho}+\overset{(0)}{f_Y}\,\overset{(1)}{R^\rho}{}_{(\mu}\overset{(0)}{R}_{\mu)\rho}+\overset{(0)}{f_Y}\,\overset{(0)}{R^\rho}{}_{(\mu}\overset{(1)}{R}_{\mu)\rho})\\
&=-\frac{1}{2}[(\overset{(1)}{R}+\overset{(0)}{\Psi})\overset{(0)}{g}_{\mu\nu}+\overset{(0)}{R}\,\overset{(1)}{g}_{\mu\nu}]+\overset{(0)}{\Psi}_X\overset{(0)}{R}_{\mu\nu}+\overset{(1)}{R}_{\mu\nu}\\
&\quad+2(\overset{(0)}{\Psi}_Y\overset{(0)}{R^\rho}{}_{(\mu}\overset{(0)}{R}_{\mu)\rho}+\overset{(0)}{\Psi}_Y\overset{(1)}{R^\rho}{}_{(\mu}\overset{(0)}{R}_{\mu)\rho}+\overset{(0)}{\Psi}_Y\overset{(0)}{R^\rho}{}_{(\mu}\overset{(1)}{R}_{\mu)\rho})\\
&=-\frac{1}{2}\overset{(1)}{R}\,\overset{(0)}{g}_{\mu\nu}+\overset{(1)}{R}_{\mu\nu}\,.
\end{aligned}\tag{49}$$

For the fourth term, we obtain

$$\begin{aligned}
\overset{(1)}{g_{\mu\nu}\Box f_X}=&\,\overset{(1)}{g}_{\mu\nu}\overset{(0)}{g}{}^{\sigma\rho}(\nabla_\sigma\nabla_\rho\overset{(0)}{f_X}-\overset{(0)}{C}{}^\delta_{\sigma\rho}\nabla_\delta\overset{(0)}{f_X})\\
&+\overset{(0)}{g}_{\mu\nu}\overset{(1)}{g}{}^{\sigma\rho}(\nabla_\sigma\nabla_\rho\overset{(0)}{f_X}-\overset{(0)}{C}{}^\delta_{\sigma\rho}\nabla_\delta\overset{(0)}{f_X})\\
&+\overset{(0)}{g}_{\mu\nu}\overset{(0)}{g}{}^{\sigma\rho}(\nabla_\sigma\nabla_\rho\overset{(1)}{f_X}-\overset{(1)}{C}{}^\delta_{\sigma\rho}\nabla_\delta\overset{(0)}{f_X}-\overset{(0)}{C}{}^\delta_{\sigma\rho}\nabla_\delta\overset{(1)}{f_X})
\end{aligned}\tag{50}$$

and since $\overset{(0)}{f_X}=1$ and $\overset{(1)}{f_X}=\overset{(0)}{\Psi}_X=\Psi_X(\overset{(0)}{R},\overset{(0)}{R}_{\mu\nu})=\Psi_X(0,0)$ is a constant, then we have that (50) vanishes. The fifth term is

$$\begin{aligned}
\overset{(1)}{\nabla_\mu\nabla_\nu f_X}&=\nabla_\mu\nabla_\nu\overset{(1)}{f_X}-\overset{(0)}{C}{}^\delta_{\mu\nu}\nabla_\delta\overset{(1)}{f_X}-\overset{(1)}{C}{}^\delta_{\mu\nu}\nabla_\delta\overset{(0)}{f_X}\\
&=\nabla_\mu\nabla_\nu\Psi_X(0,0)-\overset{(1)}{C}{}^\delta_{\mu\nu}\nabla_\delta\overset{(0)}{f_X}=0
\end{aligned}\tag{51}$$

an, for the sixth, seventh and eight terms, we use the definition (30), (31), (32). Then $\overset{(1)}{f_Y R_{\mu\nu}}=\overset{(1)}{T}_{\mu\nu}=\overset{(1)}{f_Y}\,\overset{(0)}{R}_{\mu\nu}+\overset{(0)}{f_Y}\,\overset{(1)}{R}_{\mu\nu}=\overset{(0)}{\Psi}_Y\overset{(0)}{R}_{\mu\nu}+0\cdot\overset{(1)}{R}_{\mu\nu}=0$ and $\overset{(0)}{f_Y R_{\mu\nu}}=\overset{(0)}{T}_{\mu\nu}=\overset{(0)}{f_Y}\,\overset{(0)}{R}_{\mu\nu}=0$, and this implies

$$\begin{aligned}
\overbrace{\nabla_\delta f_Y R_{\mu\nu}}^{(0)}&=\overset{(0)}{S}_{\delta\mu\nu}=\nabla_\delta\overset{(0)}{T}_{\mu\nu}=0\\
\overbrace{\nabla_\delta f_Y R_{\mu\nu}}^{(1)}&=\overset{(1)}{S}_{\delta\mu\nu}=\nabla_\delta\overset{(1)}{T}_{\mu\nu}-\overset{(0)}{C}{}^\rho_{\mu\delta}\overset{(1)}{T}_{\rho\nu}-\overset{(1)}{C}{}^\rho_{\delta\nu}\overset{(0)}{T}_{\mu\rho}=0
\end{aligned}\tag{52}$$

$$\begin{aligned}
\overbrace{\nabla_\alpha\nabla_\delta f_Y R_{\mu\nu}}^{(0)}&=\overset{(0)}{M}_{\alpha\beta\mu\nu}=\nabla_\alpha\overset{(0)}{S}_{\beta\mu\nu}=0\\
\overbrace{\nabla_\alpha\nabla_\delta f_Y R_{\mu\nu}}^{(1)}&=\overset{(1)}{M}_{\alpha\beta\mu\nu}=\overbrace{\nabla_\alpha S_{\beta\mu\nu}}^{(1)}\\
&=\nabla_\alpha\overset{(1)}{S}_{\beta\mu\nu}-\overset{(1)}{C}{}^\rho_{\sigma\delta}\overset{(0)}{S}_{\rho\mu\nu}-\overset{(1)}{C}{}^\rho_{\sigma\mu}\overset{(0)}{S}_{\delta\rho\nu}-\overset{(1)}{C}{}^\rho_{\sigma\nu}\overset{(0)}{S}_{\delta\mu\rho}=0
\end{aligned}\tag{53}$$

where we use $\overset{(0)}{C}{}^{\delta}_{\alpha\beta} = 0$, and with this notation we have

$$\overbrace{\Box f_X R_{\mu\nu}}^{(1)} = \overbrace{\Box T_{\mu\nu}}^{(1)} = \overbrace{g^{\alpha\beta}\nabla_\alpha\nabla_\beta T_{\mu\nu}}^{(1)} = \overset{(0)}{g}{}^{\alpha\beta}\overbrace{\nabla_\alpha\nabla_\beta T_{\mu\nu}}^{(1)} + \overset{(1)}{g}{}^{\alpha\beta}\overbrace{\nabla_\alpha\nabla_\beta T_{\mu\nu}}^{(0)}$$
$$= \overset{(0)}{g}{}^{\alpha\beta}\overset{(1)}{M}_{\alpha\beta\mu\nu} + \overset{(1)}{g}{}^{\alpha\beta}\overset{(0)}{M}_{\alpha\beta\mu\nu} = 0 \tag{54}$$

for the sixth term. For the seventh term we have

$$g_{\mu\nu}\overbrace{(f_Y R^{\rho\sigma})_{;\rho\sigma}}^{(1)} = \overset{(0)}{g}_{\mu\nu}\overset{(1)}{M}_{\rho\sigma}{}^{\rho\sigma} + \overset{(1)}{g}_{\mu\nu}\overset{(0)}{M}_{\rho\sigma}{}^{\rho\sigma} = 0. \tag{55}$$

Finally, and in a similar way, we get for the last term

$$\overbrace{(f_Y R^{\rho}{}_{(\mu})_{;\nu)\rho}}^{(1)} = \overset{(1)}{M}_{\nu\rho}{}^{\rho}{}_{\mu} + \overset{(1)}{M}_{\mu\rho}{}^{\rho}{}_{\nu} = 0. \tag{56}$$

Up to this point, we have proved that $\overset{(0)}{P}_{\mu\nu} = \overset{(0)}{G}_{\mu\nu}$ and $\overset{(1)}{P}_{\mu\nu} = \overset{(1)}{G}_{\mu\nu}$, this implies $\overset{(0)}{R}_{\mu\nu} = \overset{(1)}{R}_{\mu\nu} = 0$ and $\overset{(0)}{R} = \overset{(1)}{R} = 0$.

Now, we assume that the result holds for the nth term and we prove it for the $n+1$ term. Then, $\overset{(n)}{P}_{\mu\nu} = \overset{(n)}{G}_{\mu\nu}$ and $\overset{(n)}{R} = 0$, $\overset{(n)}{R}_{\mu\nu} = 0$. Also, we have $\overset{(n)}{f} = \overset{(n)}{R} + n\overset{(n-1)}{\Psi} = 0$, $\overset{(n)}{f_X} = n\overset{(n-1)}{\Psi_X} = 0$ and $\overset{(n)}{f_Y} = \overset{(n-1)}{\Psi_Y} = 0$, because $\overset{(n)}{\Psi}$, $\overset{(n-1)}{\Psi_X}$ and $\overset{(n-1)}{\Psi_Y}$ depend on a sum of factors of $\overset{(i)}{R}_{\mu\nu} = 0$ and $\overset{(i)}{R} = 0$ for $i = 0, \cdots, n-1$ (iterated derivatives of (37)). Therefore, for the first term we have

$$\begin{aligned}
-\frac{1}{2}\overset{(n+1)}{fg_{\mu\nu}} &= -\frac{1}{2}\sum_{i=0}^{n+1}\binom{n+1}{i}\overset{(i)}{f}\,\overset{(n+1-i)}{g_{\mu\nu}} \\
&= -\frac{1}{2}\left[\overset{(n+1)}{f}\,\overset{(0)}{g_{\mu\nu}} + \sum_{i=0}^{n}\binom{n+1}{i}\overset{(i)}{f}\,\overset{(n+1-i)}{g_{\mu\nu}}\right] \\
&= -\frac{1}{2}\left[(\overset{(n+1)}{R} + \overset{(n)}{\Psi})\overset{(0)}{g_{\mu\nu}} + \sum_{i=0}^{n}\binom{n+1}{i}(\overset{(i)}{R} + \overset{(i-1)}{\Psi})\overset{(n+1-i)}{g_{\mu\nu}}\right] \\
&= -\frac{1}{2}\overset{(n+1)}{R}\,\overset{(0)}{g_{\mu\nu}}\,.
\end{aligned} \tag{57}$$

For the second term we obtain

$$\begin{aligned}
\overset{(n+1)}{f_X R_{\mu\nu}} &= \sum_{i=0}^{n+1}\binom{n+1}{i}\overset{(i)}{f_X}\,\overset{(n+1-i)}{R_{\mu\nu}} \\
&= \overset{(n+1)}{f_X}\,\overset{(0)}{R_{\mu\nu}} + \sum_{i=1}^{n}\binom{n+1}{i}\overset{(i)}{f_X}\,\overset{(n+1-i)}{R_{\mu\nu}} + \overset{(0)}{f_X}\,\overset{(n+1)}{R_{\mu\nu}} \\
&= n\overset{(n)}{\Psi_X}\overset{(0)}{R_{\mu\nu}} + \sum_{i=1}^{n}\binom{n+1}{i}\overset{(i-1)}{\Psi}\,\overset{(n-1-i)}{R_{\mu\nu}} + \overset{(n+1)}{R_{\mu\nu}} \\
&= \overset{(n+1)}{R_{\mu\nu}}\,.
\end{aligned} \tag{58}$$

The third term reads

$$\overset{(n+1)}{f_Y R^{\rho}{}_{(\mu} R_{\nu)\rho}} = \sum_{n_1+n_2+n_3=n+1} \frac{(n+1)!}{n_1!n_2!n_3!} \overset{(n_1)}{f_Y} \overset{(n_2)}{R^{\rho}}{}_{(\mu} \overset{(n_3)}{R}_{\nu)\rho} = 0. \tag{59}$$

As $n_1+n_2+n_3 = n+1$ and $\overset{(i)}{R}_{\mu\nu} = 0$ for $i = 0,\ldots,n$ then the product $\overset{(n_1)}{f_Y} \overset{(n_2)}{R^{\rho}}{}_{(\mu} \overset{(n_3)}{R}_{\nu)\rho}$ is always zero.

The fourth term is calculated to be

$$\begin{aligned}
\overset{(n+1)}{g_{\mu\nu}\Box f_X} &= \sum_{n_1+n_2+n_3=n+1} \frac{(n+1)!}{n_1!n_2!n_3!} \overset{(n_1)}{g}_{\mu\nu} \overset{(n_2)}{g}_{\mu\nu} \\
&\quad \cdot \left(\nabla_\sigma \nabla_\rho \overset{(n_3)}{f_X} - \sum_{k=0}^{n_3} \binom{n_3}{k} \overset{(n_3-k)}{C^{\delta}_{\sigma\rho}} \nabla_\delta \overset{(k)}{f_X} \right) \\
&= \sum_{n_1+n_2+n_3=n+1} \frac{(n+1)!}{n_1!n_2!n_3!} \overset{(n_1)}{g}_{\mu\nu} \overset{(n_2)}{g}_{\mu\nu} \\
&\quad \cdot \left(n_3 \nabla_\sigma \nabla_\rho \overset{(n_3-1)}{\Psi_X} - k \sum_{k=0}^{n_3} \binom{n_3}{k} \overset{(n_3-k)}{C^{\delta}_{\sigma\rho}} \nabla_\delta \overset{(k-1)}{\Psi_X} \right) = 0,
\end{aligned} \tag{60}$$

because $\overset{(n_3-1)}{\Psi_X} = 0$, for $n_3 - 1 = 0,\ldots,n$ and $\overset{(k-1)}{\Psi_X} = 0$ for $k-1 = 0,\ldots,n$ by induction hypothesis.

In the same way, for the fifth term we have

$$\begin{aligned}
\overset{(n+1)}{\nabla_\mu \nabla_\nu f_X} &= \nabla_\mu \nabla_\nu \overset{(n+1)}{f_X} + \sum_{i=0}^{n+1} \binom{n+1}{i} \overset{(i)}{C^{\delta}_{\mu\nu}} \nabla_\delta \overset{(n+1-i)}{f_X} \\
&= \nabla_\mu \nabla_\nu (n+1) \overset{(n)}{\Psi_X} \\
&\quad + \sum_{i=0}^{n+1} \binom{n+1}{i} \overset{(i)}{C^{\delta}_{\mu\nu}} \nabla_\delta (n+1-i) \overset{(n-i)}{\Psi_X} = 0.
\end{aligned} \tag{61}$$

For the next terms, we use the definitions (30), (31) and (32) to obtain

$$\begin{aligned}
\overset{(n+1)}{T_{\mu\nu}} = \overset{(n+1)}{f_Y R_{\mu\nu}} &= \sum_{i=0}^{n+1} \binom{n+1}{i} \overset{(i)}{f_Y} \overset{(n+1-i)}{R_{\mu\nu}} \\
&= \overset{(n+1)}{f_Y} \overset{(0)}{R_{\mu\nu}} + \sum_{i=1}^{n} \binom{n+1}{i} \overset{(i)}{f_Y} \overset{(n+1-i)}{R_{\mu\nu}} + \overset{(0)}{f_Y} \overset{(n+1)}{R_{\mu\nu}} \\
&= (n+1) \overset{(n)}{\Psi_Y} \overset{(0)}{R_{\mu\nu}} + \sum_{i=1}^{n} \binom{n+1}{i} \overset{(i)}{\Psi_Y} \overset{(n+1-i)}{R_{\mu\nu}} = 0
\end{aligned} \tag{62}$$

because $\overset{(i)}{\Psi_Y} = 0$ for $i = 0,\ldots,n$, and $\overset{(n)}{\Psi_Y} = 0$, by induction hypothesis, and then the same for $\overset{(i)}{T}_{\mu\nu}$ for $i = 0,\ldots,n+1$. As $\overset{(l)}{S}_{\delta\mu\nu}$ depends of $\overset{(k)}{T}_{\mu\nu}$ for all i and $k = 0,\ldots,l$, thus $\overset{(i)}{S}_{\delta\mu\nu} = 0$ for $i =$

$0,\cdots,n+1$. Then, $\overset{(l)}{M}_{\sigma\delta\mu\nu}$ depends on $\overset{(i)}{S}_{\delta\mu\nu}$ for all i and finally $\overset{(i)}{M}_{\sigma\delta\mu\nu}=0$ for $i=0,\cdots,n+1$. Thus, for the sixth term we arrive to

$$\overbrace{\Box f_Y R_{\mu\nu}}^{(n+1)} = \overbrace{g^{\rho\delta}\nabla_\rho\nabla_\delta T_{\mu\nu}}^{(n+1)} = \overbrace{g^{\rho\delta}M_{\rho\delta\mu\nu}}^{(n+1)} = \sum_{i=0}^{n+1}\binom{n}{i}\overset{(i)}{g}{}^{\sigma\rho}\overset{(n-i)}{M}_{\sigma\delta\mu\nu}=0. \tag{63}$$

The seventh term is

$$\overbrace{g_{\mu\nu}\nabla_\rho\nabla_\sigma(f_Y R^{\rho\sigma})}^{(n+1)} = \overbrace{g_{\mu\nu}M_{\sigma\rho}{}^{\sigma\rho}}^{(n+1)} = \sum_{i=0}^{n}\binom{n}{i}\overset{(i)}{g}_{\mu\nu}\overset{(n-i)}{M}_{\sigma\rho}{}^{\sigma\rho}=0. \tag{64}$$

Finally, the last terms are written as

$$2(f_Y \overset{(n+1)}{R}{}^{\rho}{}_{(\mu})_{;\nu)\rho} = \overset{(n+1)}{M}_{\nu\rho}{}^{\rho}{}_{\mu} + \overset{(n+1)}{M}_{\mu\rho}{}^{\rho}{}_{\nu}=0. \tag{65}$$

The sum of all these results gives us

$$\overset{(n+1)}{P}_{\mu\nu} = -\frac{1}{2}\overset{(n+1)}{R}\,\overset{(0)}{g}_{\mu\nu}+\overset{(n+1)}{R}_{\mu\nu}=0 \tag{66}$$

and, finally, this implies $\overset{(i)}{P}_{\mu\nu}=\overset{(i)}{G}_{\mu\nu}$ for all i and, therefore, $\bar{P}_{\mu\nu}=\bar{G}_{\mu\nu}$.

□

Hence, we have once again determined that the asymptotic expansions for $\bar{P}_{\mu\nu}$ and $\bar{G}_{\mu\nu}$ are identical, suggesting that these two tensors differ at most in singular terms with respect to λ. Additionally, in the context of perturbative constraints, these results imply that the approximations for the vacuum field equations $f(R,R_{\mu\nu}R^{\mu\nu})$ are equivalent to those achievable in GR. Consequently, within this perturbative framework, the MTG described by $f(R,R_{\mu\nu}R^{\mu\nu})=R+\lambda\Psi(R,R_{\mu\nu}R^{\mu\nu})$ will not introduce any modifications to the findings within GR.

The reader should note that we have proved $\overset{(i)}{R}_{\mu\nu}=0$ for all i. Consequently, $\bar{R}_{\mu\nu}=\overset{(0)}{R}_{\mu\nu}+\lambda\overset{(1)}{R}_{\mu\nu}+\frac{\lambda^2}{2}\overset{(2)}{R}_{\mu\nu}+\cdots=0$, and the same applies to the Ricci scalar $\bar{R}=\overset{(0)}{R}+\lambda\overset{(1)}{R}+\frac{\lambda^2}{2}\overset{(2)}{R}+\cdots=0$. It is important to emphasize that, in our proof, we do not impose any symmetry considerations. Therefore, this result holds for any vacuum spacetime, whether it exhibits spherical, axial symmetry, or it represent a gravitational wave or any other configuration. Another significant point is that there could exist solutions different from those of GR within this family of theories. However, such solutions are either disconnected by the parameter λ or do not admit a Taylor expansion.

Regarding the gauge freedom of perturbation theory, mentioned above in the end of section 3, it is important to be aware that results such as the previous theorem just proved could be, in general, gauge-dependent. However, in this case, we are comparing vacuum theories. That is, $\Sigma(\lambda)_{\mu\nu}=0$ in each slide labeled by λ. This assumption is very important for our results since it allows us to disregard the gauge freedom problem by virtue of the generalized Stewart-Walker lemma. This lemma implies, for a $\Sigma_{\mu\nu}$ which vanishes in the background, that for any two different gauges X and Y, $\overset{(n)}{\Sigma}_X=\overset{(n)}{\Sigma}_Y$ for any order n. The case of a non-vanishing energy momentum tensor $T_{\mu\nu}\neq 0$ involves the construction of gauge-invariant quantities which are non-trivial and, therefore, it must be studied separately [33]. In a future work, we aim to provide a detailed description of the behavior of the solutions under gauge choices within this formulation.

4.3. Some implications

We enumerate some important consequences of theorem 1.

Corollary 1.1. *Let $\bar{\Sigma}_{ab} = 0$ be the f(R) vacuum field equations for the model $f(\bar{R}) = \bar{R} + \lambda\Psi(\bar{R})$. Then, $\bar{\Sigma}_{ab} = \bar{G}_{ab}$ in vacuum.*

This corollary, previously reported in [33], appears here as a particular case of theorem 1.

Even more, with the established equivalence between $f(R)$ and scalar-tensor theories, we possess a robust foundation for exploring the connections and implications inherent in these two formulations. By assuming the same functional form $f(x) = x + \lambda\Psi(x)$ and letting χ be the inverse function of Ψ', we can state the following:

Lemma 2. *The Lagrangian $\mathcal{L} = f(R) = R + \lambda\Psi(R)$ is equivalent to the scalar-tensor Lagrangian $\mathcal{L} = R + \lambda\phi R - \lambda\tilde{V}(\phi)$, where $\tilde{V}(\phi) = \phi\chi(\phi) - \Psi(\chi(\phi))$, assuming Ψ' is invertible and $\phi = \Psi'(\chi)$.*

Proof. As shown in the scalar-tensor section, we introduce an auxiliary field χ, then

$$\begin{aligned} \mathcal{L} &= f'(\chi)(R-\chi) + f(\chi) \\ &= (1+\lambda\Psi'(\chi))(R-\chi) + (\chi + \lambda\Psi(\chi)), \end{aligned} \tag{67}$$

but since $\phi = \Psi'(\chi)$ and $\chi(\phi) = \Psi'^{-1}(\phi)$, we have

$$\begin{aligned} &= (1+\lambda\phi)(R-\chi(\phi)) + (\chi(\phi) + \lambda\Psi(\chi(\phi))) \\ &= R + \lambda\phi R - \lambda(\phi\chi(\phi) - \Psi(\chi(\phi))). \end{aligned} \tag{68}$$

□

Corollary 2.1. *Let $\Phi_{\mu\nu}$ be the vacuum field equation in scalar-tensor theories of gravity with a Lagrangian of the form $\mathcal{L} = R + \lambda\phi R - \lambda\tilde{V}(\phi)$, where $\tilde{V}(\phi) = \phi\chi(\phi) - \Psi(\chi(\phi))$, assuming Ψ' is invertible and $\Psi'(\chi) = \phi$. Then, $\bar{\Phi}\mu\nu = \bar{G}\mu\nu$.*

Proof. By applying corollary 1.1 and lemma 2, the above corollary follows. □

In this case, the parameter for the kinetic term is $\omega = 0$, and the potential $\tilde{V}(\phi)$ takes a specific form. Additionally, we observe the absence of minimal coupling with the scalar field in the first term of the Lagrangian, while the second term involves coupling and the third term includes a potential.

Let us now try to elucidate similar behaviors in other theories with specific constraints.

4.4. Case $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$

In the general context, where we consider theories of gravity in the form $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$, we introduce a Kretschmann scalar that is non-zero for GR vacuum solutions. Initially, we analyze the weak-field limit of these theories, specifically for those with a Lagrangian density of the form,

$$\mathcal{L} = R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{69}$$

where α, β, and γ are constants. Utilizing the well-known result that the Gauss–Bonnet term, $4R_{\mu\nu}R^{\mu\nu} - R^2 - R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, is a total divergence [51], we can rewrite (69) as

$$\mathcal{L} = R + \alpha_1 R^2 + \beta_1 R_{\mu\nu}R^{\mu\nu} \tag{70}$$

with a redefinition of constants α_1, β_1. Consequently, applying our theorem yields the following

Corollary 2.2. *Let $\tilde{P}_{\mu\nu}$ be the field equations for the Lagrangian density given in (69). Then, $\bar{P}_{\mu\nu} \equiv \bar{G}_{\mu\nu}$*

For the case of a nonlinear form of $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$, a new system of differential equations arises for the perturbed quantities. For instance, at first order in vacuum, we have

$$\begin{aligned}\overset{(1)}{P}{}^{\mu\nu} = \overset{(1)}{R}{}^{\mu\nu} - \frac{1}{2}\overset{(1)}{R}\,\overset{(0)}{g}{}^{\mu\nu} - \overset{(0)}{\Psi}\,\overset{(0)}{g}{}^{\mu\nu} + 4\,\overset{(0)}{\Psi}_Z \overbrace{R^{\delta\sigma\rho(\mu}}^{(0)}\overbrace{R^{\nu)}}^{(0)}{}_{\rho\sigma\delta} \\ - 8\nabla_\rho\nabla_\sigma\,\overset{(0)}{\Psi}_Z \overbrace{R^{\sigma(\mu\nu)\rho}}^{(0)} = 0\end{aligned} \tag{71}$$

for the case $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}) = \frac{1}{2}R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$. Notably, the last three terms in (71) exhibit characteristics resembling a non vanishing energy-momentum tensor. If those terms are non-zero, they introduce a new contribution to the first-order metric perturbation. Evidently, the Riemann tensor has non-zero components in the background, such as in spherical symmetry, and the Kretschmann scalar does as well. In the case of $\overset{(0)}{\Psi}_Z = \Psi_Z(0, 0, \overset{(0)}{K})$ with $\overset{(0)}{K}$ being the Kretschmann scalar in the background, as argued earlier, the function Ψ must be nonlinear in its arguments. Consequently, the last three terms in (71) are not necessary zero. In fact, there exists a significant class of models of interest to the community where $\overset{(0)}{\Psi}_Z \neq 0$ [17, 52].

Up to now, we have shown that, for a significant class of Lagrangians in the metric formalism, the solutions to their corresponding field equations do not either exhibit analytic behavior or become disconnected with respect to the parameter λ. An important feature of these (vacuum) Lagrangians is that the relevant quantities vanish in the background. For example, in the Starobinsky model, where $f(R) = R + \lambda R^2$, the Ricci scalar R is zero in the background and, consequently, $f(R)$ is also zero in the background. Note that, in our central result, where $f(R, R_{\mu\nu}R^{\mu\nu}) = R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu})$, we assume that $\Psi(0, 0) = 0$. Therefore, $f(R, R_{\mu\nu}R^{\mu\nu})$ is zero in the background. However, when non-zero terms such as the Kretschmann scalar appear in the background Lagrangian, the solutions exhibit analytic behavior. Based on these observations, we propose the following conjecture:

Conjecture 1. If the geometric part of the family of the Lagrangians $\mathcal{L}_\lambda(g_{\mu\nu}, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\delta}, \ldots) = 0$ in the background, where $\mathcal{L}_0 = R$, then the class of solutions is either GR or another solution that is disconnected from GR or not analytic with respect to λ (in vacuum).

Another possibility is that the geometric Lagrangian is constant in the background. In that case, we would have the same situation but with a cosmological constant.

4.5. Stability considerations

Higher-order curvature gravity theories, which include terms like $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}$, or more general forms such as $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$ in their Lagrangians, are well-known to potentially suffer from instabilities. For instance, the Ostrogradsky instability arises in higher-derivative theories when the action contains terms with higher-order time derivatives, leading to unbounded Hamiltonians and, consequently, an unstable vacuum [53, 54]. Interestingly, $f(R)$ gravity appears to be an exception that avoids this instability, as it does not introduce derivatives of the metric beyond second order in the scalar-tensor framework [11, 55].

Additionally, certain choices of functional forms can lead to the Dolgov–Kawasaki instability, which arises when the effective mass squared of a scalar degree of freedom becomes negative. In particular, some $f(R)$ gravity theories may exhibit this instability if $f''(R) < 0$ under specific conditions [56].

Our analysis focuses on a class of functions $f(R, R_{\mu\nu}R^{\mu\nu})$, where $f(R)$ is a particular case that inherently avoids the Ostrogradsky instability. We also consider functions that satisfy $f''(R) > 0$, which in certain cases ensures that neither the Ostrogradsky nor the Dolgov–Kawasaki instabilities arise in the scenarios we study. However, when extending to a broader class of functions $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta})$, we find that, although conditions for analyticity or connected solutions may be met, these functions could still allow for the presence of instabilities.

While our analysis is centered on demonstrating analyticity properties of specific functional forms and does not directly address stability analysis, we acknowledge that such instabilities are critical considerations in the physical viability of higher-order gravity theories.

5. Final remarks and future work

The exploration of MTG, spurred by the Lovelock theorem and various theoretical and observational challenges to GR, has resulted in a plethora of extended gravitational theories. The mathematical richness, inherent issues, and advantages of each theory present a vast and complex landscape in the search for the 'best' theory of gravitation. The direction forward remains uncertain, requiring robust tools to compare GR with other modified theories.

To navigate this complex landscape of possibilities, we have developed an initial tool that facilitates a systematic comparison between GR and other modified gravity theories. While the weak-field limit typically implies a post-Minkowskian expansion, leading to solutions that start from a flat (Minkowski) background and incorporate relativistic corrections, the post-Newtonian expansion begins with a Newtonian background and is commonly applied when examining non-relativistic sources [57]. Our approach, which we refer to as a *post-Einstein theory*, adapts and extends these methodologies to provide a novel framework for analyzing how new theoretical contributions manifest in gravitational perturbations.

Using the formal mathematical tools of perturbation theory developed in GR [33], we extend our investigation to systematically compare solutions in modified gravity theories with those in GR. This approach allows us to explore subtle effects introduced by perturbing gravitational fields within these theories, offering a post-Einstein perspective that highlights the interplay between established gravitational models and their modifications. In figure 2, we introduce a conceptual sketch of this framework.

On one hand, we found that for Lagrangians of the form $f(R, R_{\mu\nu}R^{\mu\nu}) = R + \lambda\Psi(R, R_{\mu\nu}R^{\mu\nu})$, there are no perturbed contributions with respect to GR. Theorem 1 states that the field equations $\bar{G}_{\mu\nu} = \bar{P}_{\mu\nu}$ mean that the Taylor expansion of the pullback of the diffeomorphism $\mathcal{X}$ is the same for the tensor fields $G_{\mu\nu}$ and $P_{\mu\nu}$ around $\lambda = 0$ in the manifold $\mathcal{N}$. We prove the theorem by induction, order by order, in the Taylor expansion. Due to the mathematical construction of perturbation theory, this implies that the metric vacuum solutions in the modified spacetime, which is connected with vacuum GR solutions through the parameter

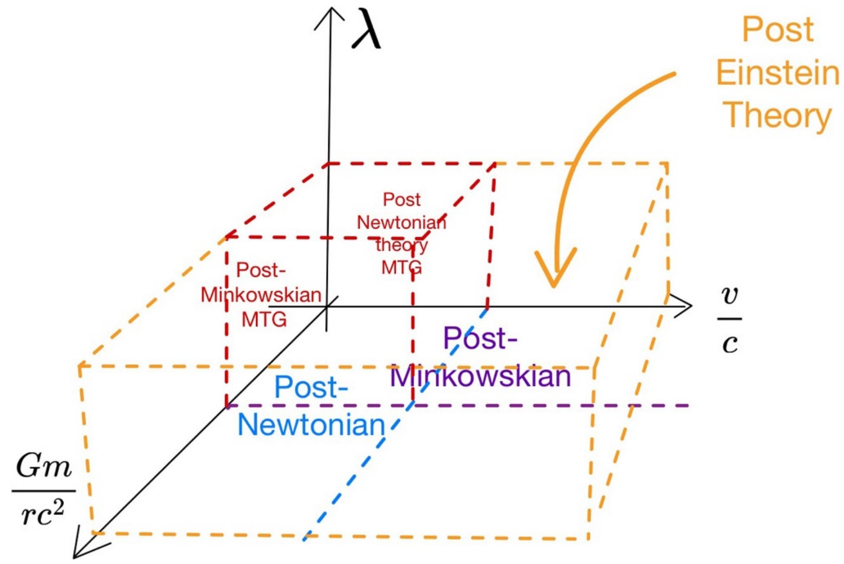

Figure 2. This diagram illustrates the relevance of different *weak field scenarios*. In the plane $\lambda = 0$, we have GR: the blue region represents the post-Newtonian expansion, and the purple region represents the post-Minkowskian expansion. The red region depicts the weak-field limit in MTG for $\lambda \gtrsim 0$, typically used for Solar System constraints on MTG or compact binary systems. In the strong-field limit of GR, small contributions from MTG are shown in the orange region.

λ, are only solutions of the vacuum field equations of GR. For example, the Schwarzschild GR vacuum solution is a solution for a substantial class of $f(R, R_{\mu\nu}R^{\mu\nu})$ models. Broadly speaking, if a solution is valid within the framework of GR, it automatically qualifies as a solution in the extended class of $f(R, R_{\mu\nu}R^{\mu\nu})$ theories, as shown in (8). However, the implications of these results become more intricate when boundary conditions differ, leading to distinct solutions. This nuanced behavior is particularly evident in the $f(R)$ scenario, highlighting the potential disconnection or analytic issues of the solutions. Thus, corollary 1.1 is presented here as a particular case of theorem 1 and states the same behavior in $f(R)$ theories of gravity.

In the context of $f(R)$ and $f(R, R_{\mu\nu}R^{\mu\nu})$ MTG, it is important to recall that the class of field equations described by $\mathcal{E}_\lambda$ must be supplemented by a set of boundary conditions, even for $\lambda \to 0$. The unperturbed solution might not satisfy such boundary conditions since the corresponding equations do not involve higher-order derivatives. This issue, known as the presence of boundary layers, conditions the possible solutions for $\lambda \to 0$. As illustrated by [46], the resulting g_0 is not a solution of Einstein's equations (which are the $\lambda \to 0$ limit of Starobinsky model's field equations) but is a metric with pathological features such as non-differentiability, discontinuity, or non-analyticity, all induced by the boundary layer present in the problem. This issue reflects the problems involved in obtaining limits of spacetimes [58] and must be considered when constructing a perturbation theory around the background solution g_0.

The work of Clifton and Barrow [59] provides a spherically symmetric solution in the $f(R) = R^{1+\lambda}$ model, Çıkıntoğlu presents a solution in the Starobinsky model [46], and Sunny *et al* [60] explore solutions in models $f(R) = (\alpha_0 + \alpha_1 R)^p$, among others. These solutions exemplify how the theorem applies in various contexts. To clarify, we state that the counter-reciprocal of the theorem reduces to one of the following possibilities:

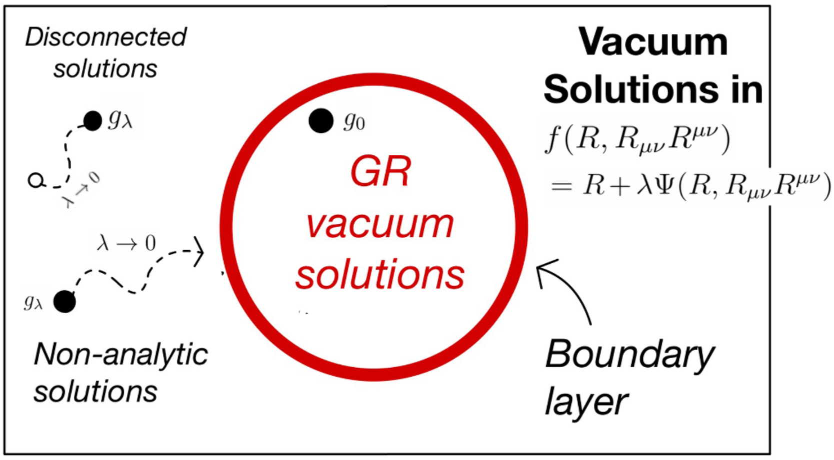

Figure 3. This diagram represents a sketch of a set of solutions in theories with Lagrangians of the form $R+\lambda\Psi(R,R_{\mu\nu}R^{\mu\nu})$. GR solutions are included in this kind of MTG, and if we assume a solution different from that of GR, we could have solutions that are either disconnected, through the λ parameter, or not analytic around $\lambda=0$.

(i) The presence of matter characterizes the case under consideration.
(ii) The solution is not analytic with respect to λ around $\lambda=0$ (possibly disconnected).
(iii) The function $f(R,R_{\mu\nu}R^{\mu\nu})$ deviates from the prescribed form $R+\lambda\Psi(R,R_{\mu\nu}R^{\mu\nu})$, with $\Psi(0,0)=0$.

Another illustrative example is the detection of the gravitational wave event GW170817, along with its electromagnetic counterpart GRB170817A, which has provided stringent constraints on modified gravity theories in cosmological contexts. For instance, in $f(T)$ gravity, deviations from GR arise in the dispersion relation and frequency of cosmological gravitational waves, introducing a new parameter that quantifies these effects [61]. However, it is important to clarify that our results focus on vacuum solutions. Consequently, the constraints on the dispersion relation derived from GW170817 are not directly applicable to our study.

Figure 3 represents the set of possible solutions in this context.

The implications of the theorem, derived within the vacuum context, extend beyond scenarios characterized solely by spherical symmetry. The outcomes presented here transcend specific spatial symmetries, offering broader applicability across diverse gravitational scenarios, including gravitational waves. This underscores the robustness and wide relevance of our findings, to encompass a wide range of gravitational contexts.

To explore additional effects of theories related to theorem 1, one must delve into the realm of full and exact solutions, where the intricacies of MTG can fully unfold. In the perturbed context, the absence of additional effects underscores a form of rigidity inherent in the GR solution. This rigidity, observed in the face of perturbations, highlights the robustness and stability of the standard GR solutions under certain conditions. Nevertheless, the study of a one-parameter family of spacetimes denoted as $(\mathcal{M}_\lambda, g_\lambda)$ within the framework of GR in a vacuum reveals intriguing aspects [58].

These results can be extended to scalar-tensor MTG, using the equivalence with $f(R)$, in a particular class of scalar-tensor theories. The general case in scalar-tensor theories is currently being explored, with special care in the case $\omega \neq 0$, and we hope to compare it with the class of solutions in the literature (see [62] and references therein).

On the other hand, we found that the inclusion of terms like $R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}$ presents two possibilities. First, if the function f of the Lagrangian (6) is linear in its arguments, the Lagrangian converts to a form covered by the theorem, and then we have corollary 2.2 and all the consequences discussed above apply. Second, if the function f is not linear in its third argument, it can break the rigidity observed in this formalism, allowing for a richer variety of perturbative solutions that may reveal deeper insights into gravitational interactions, as seen in equation (71). The particularity lies in the inclusion of non-zero terms in the background in the Lagrangian and non-linearity in its second derivatives, leading us to propose conjecture 1. This opens the possibility of studying perturbed solutions without the complications of finding exact solutions, allowing for a clear comparison with GR. We are currently working within this framework to obtain first-order solutions, with explicit results to be presented in future works.

Data availability statement

No new data were created or analysed in this study.

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References

[1] Penrose R 1965 *Phys. Rev. Lett.* **14** 57–59
[2] Ellis G F R and Hawking S 1973 *The Large Scale Structure of Space-Time* (Cambridge University Press)
[3] Percacci R 2017 *An Introduction to Covariant Quantum Gravity and Asymptotic Safety* (World Scientific)
[4] 't Hooft G and Veltman M 1974 *Ann. Inst. Phys. Theor.* **20** 69–94
[5] Rovelli C 2004 *Quantum Gravity* (Cambridge University Press)
[6] Weinberg S 1989 *Rev. Mod. Phys.* **61** 1
[7] Adler R J, Bazin M and Schiffer M 1995 *Introduction to General Relativity* (McGraw-Hill)
[8] Martin J 2012 *C. R. Physique* **13** 566–665
[9] Capozziello S and De Laurentis M 2011 *Phys. Rep.* **509** 167–321

[10] Faraoni V and Capozziello S 2011 *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics* (Springer)
[11] Sotiriou T P and Faraoni V 2010 *Rev. Mod. Phys.* **82** 451–97
[12] De Felice A and Tsujikawa S 2010 *Living Rev. Relativ.* **13** 3
[13] Fujii Y and Maeda K i 2003 *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press)
[14] Buchbinder I 1992 *Effective Action in Quantum Gravity* (Institute of Physics)
[15] Gasperini M and Veneziano G 1992 *Phys. Lett.* B **277** 256–64
[16] Green M B, Schwarz J H and Witten E 2012 *Superstring Theory Vol. 2: 25th Anniversary edn Cambridge Monographs on Mathematical Physics* (Cambridge University Press)
[17] Clifton T, Ferreira P G, Padilla A and Skordis C 2012 *Phys. Rep.* **513** 1–189
[18] Faraoni V 2004 *Cosmology in Scalar Tensor Gravity* (Springer)
[19] Starobinsky A 1980 *Phys. Lett.* B **91** 99–102
[20] Nojiri S, Odintsov S and Oikonomou V 2017 *Phys. Rep.* **692** 1–104
[21] La D and Steinhardt P J 1989 *Phys. Rev. Lett.* **62** 376–8
[22] Artymowski M and Lalak Z 2014 *J. Cosmol. Astropart. Phys.* JCAP09(2014)036
[23] Capozziello S, Martin-Moruno P and Rubano C 2008 *Phys. Lett.* B **664** 12–15
[24] Oikonomou V K and Giannakoudi I 2022 *Int. J. Mod. Phys.* D **31** 2250075
[25] Langlois D 2019 *Int. J. Mod. Phys.* D **28** 1942006
[26] Multamäki T and Vilja I 2006 *Phys. Rev.* D **74** 064022
[27] Capozziello S, Stabile A and Troisi A 2008 *Class. Quantum Grav.* **25** 085004
[28] Sebastiani L and Zerbini S 2011 *Eur. Phys. J.* C **71** 1–5
[29] Guilabert T, Calzada P V, Bargueño P and Miret-Artés S 2024 *Eur. Phys. J.* C **84** 1
[30] Abbott B P (LIGO Scientific Collaboration and Virgo Collaboration) 2016 *Phys. Rev. Lett.* **116** 061102
[31] The Event Horizon Telescope Collaboration 2019 *Astrophys. J. Lett.* **875** L1
[32] Molano D 2015 Teoría de perturbaciones cosmológicas en teorías de gravedad modificada f(r) (available at: https://repositorio.unal.edu.co/handle/unal/553 58)
[33] Molano D, Villalba F D, Castañeda L and Bargueño P 2020 *Class. Quantum Grav.* **37** 145006
[34] Lovelock D 1971 *J. Math. Phys.* **12** 498–501
[35] Charmousis C 2015 *Lect. Notes Phys.* **892** 25–56
[36] Fujii Y and Maeda K I 2004 *The Scalar–Tensor Theory of Gravitation* (Cambridge University Press)
[37] Pauli W 1958 *Theory of Relativity* (Pergamon)
[38] Capozziello S and Faraoni V 2011 *Beyond Einstein Gravity. A Survey of Gravitational Theories for Cosmology and Astrophysics* (Springer)
[39] Shankaranarayanan S and Johnson J P 2022 *Gen. Relativ. Gravit.* **54** 44
[40] Fernandes P G S, Carrilho P, Clifton T and Mulryne D J 2022 *Class. Quantum Grav.* **39** 063001
[41] Guarnizo A, Castañeda L and Tejeiro J M 2010 *Gen. Relativ. Gravit.* **42** 2713–28
[42] Brans C and Dicke R H 1961 *Phys. Rev.* **124** 925
[43] Zwiebach B 1985 *Phys. Lett.* B **156** 315–7
[44] Nakamura K 2019 (arXiv:1912.12805)
[45] Bruni M, Matarrese S, Mollerach S and Sonego S 1997 *Class. Quantum Grav.* **14** 2585
[46] Çıkıntoğlu S 2018 *Phys. Rev.* D **97** 044040
[47] Schouten J A 1954 *Ricci-Calculus* (Springer)
[48] Wald R 1984 *General Relativity* (The University of Chicago Press)
[49] Sachs R K 1964 Gravitational radiation *Relativity Groups and Topology* ed C M DeWitt and B S DeWitt (Gordon and Breach) pp 523–63
[50] Stewart J M and Walker M 1974 *Proc. R. Soc.* A **341** 49–74
[51] Lanczos C 1938 *Ann. Math.* **39** 842–50
[52] De Felice A and Tsujikawa S 2009 *Phys. Rev.* D **80** 063516
[53] Ostrogradsky M 1850 *Mem. Acad. St. Petersbourg* **6** 385–517
[54] Woodard R P 2015 *Scholarpedia* **10** 32243
[55] Woodard R P 2007 *Lectures on Astrophysics and Cosmology, Proc. 43rd Karpacz Winter School of Theoretical Physics* (arXiv:astro-ph/0601672)
[56] Dolgov A D and Kawasaki M 2003 *Phys. Lett.* B **573** 1–4

[57] Blanchet L 2019 *Int. J. Mod. Phys.* D **28** 1930011
[58] Geroch R 1969 *Commun. Math. Phys.* **13** 180–93
[59] Clifton T and Barrow J D 2005 *Phys. Rev.* D **72** 103005
[60] Sunny A, Xavier S and Shankaranarayanan S 2024 *Class. Quantum Grav.* **41** 135002
[61] Cai Y F, Li C, Saridakis E N and Xue L 2018 *Phys. Rev.* D **97** 103513
[62] Faraoni V, Giusti A and Fahim B H 2021 *Phys. Rep.* **925** 1–58