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Special Issue
Advances in Differential Geometry and Mathematical Physics

Edited by
Dr. David D. McNutt

MDPI

https://doi.org/10.3390/axioms13090636

Article

On Spacelike Hypersurfaces in Generalized Robertson–Walker Spacetimes

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Abstract: This paper investigates generalized Robertson–Walker (GRW) spacetimes by analyzing Riemannian hypersurfaces within pseudo-Riemannian warped product manifolds of the form $(\overline{M}, \overline{g})$, where $\overline{M} = \mathbb{R} \times_f M$ and $\overline{g} = \epsilon dt^2 + f^2(t) g_M$. We focus on the scalar curvature of these hypersurfaces, establishing upper and lower bounds, particularly in the case where $(\overline{M}, \overline{g})$ is an Einstein manifold. These bounds facilitate the characterization of slices in GRW spacetimes. In addition, we use the vector field ∂_t and the so-called support function θ to derive generalized Minkowski-type integral formulas for compact Riemannian and spacelike hypersurfaces. These formulas are applied to establish, under certain conditions, results concerning the existence or non-existence of such compact hypersurfaces with scalar curvature, either bounded from above or below.

Keywords: GRW spacetimes; spacelike hypersurfaces; Minkowski-type integral formulas; scalar and mean curvatures; minimal and maximal hypersurfaces

MSC: 53A10; 53C40; 53C42; 53C65

Citation: Alessa, N.; Guediri, M. On Spacelike Hypersurfaces in Generalized Robertson–Walker Spacetimes. *Axioms* **2024**, *13*, 636. https://doi.org/10.3390/axioms13090636

Academic Editor: David D. McNutt

Received: 16 August 2024
Revised: 12 September 2024
Accepted: 16 September 2024
Published: 17 September 2024

1. Introduction

The existence of a conformal vector field on a pseudo-Riemannian manifold plays an important role in both Riemannian and Lorentzian geometry, as it signifies a symmetry in the metric tensor. In the context of general relativity, this symmetry is often employed to obtain exact solutions to the Einstein field equations.

Let (M, g_M) be an n-dimensional Riemannian manifold, $n \geq 2$, I an open interval of $\mathbb{R}$, and f a positive smooth function defined on I. By equipping I with the metric $\pm dt^2$, we obtain an $(n+1)$-dimensional pseudo-Riemannian warped product manifold $(\overline{M}, \overline{g})$, where $\overline{M} = I \times M$ and

$$\overline{g} = \pm dt^2 + f^2 g_M. \quad (1)$$

It is evident that $(\overline{M}, \overline{g})$ can be either Riemannian or Lorentzian. In the Lorentzian case, $(\overline{M}, \overline{g})$ is referred to as a generalized Robertson–Walker (GRW) spacetime, which extends the concept of Robertson–Walker (RW) spacetimes where the fiber M is three-dimensional with a constant sectional curvature. In a GRW spacetime $(\overline{M}, \overline{g})$, the vector field ∂_t is a unit vector field that is globally defined on $\overline{M}$, which is timelike in the Lorentzian case, thereby providing a time orientation for $\overline{M}$. We should note that, in general relativity theory, GRW spacetimes are also called FRWL spacetimes after Friedmann, Robertson, Walker and Lemaitre.

Many investigations have focused on the geometry of Riemannian warped product manifolds and GRW spacetimes, such as those in [1–7], and others.

A natural and interesting problem in pseudo-Riemannian warped product manifolds (particularly GRW spacetimes) is characterizing their Riemannian (or spacelike in the GRW case) hypersurfaces and determining under what conditions such a hypersurface is completely umbilical or, ideally, a slice in the ambient warped product. This area has been extensively researched by mathematicians for a long time. Some studies have focused on complete Riemannian and spacelike hypersurfaces having constant mean curvature, while others have investigated how the scalar curvature of the hypersurface relates to that of the ambient manifold (see, for example, [1,8–17]).

Research has been conducted on compact Riemannian hypersurfaces within pseudo-Riemannian warped product manifolds, with a particular focus on spacelike hypersurfaces in GRW spacetimes. This research often examines aspects such as the volume of the fiber, the warping function f, and the hyperbolic angle function, which is the inner product of the unit normal to the hypersurface and the conformal vector field $f\partial t$, where ∂t is tangent to the one-dimensional base. For example, under various geometric and physical conditions, spacelike slices are recognized as the only spacelike hypersurfaces that achieve both upper and lower volume bounds [18]. See also [19,20].

Several results characterize compact spacelike hypersurfaces of Lorentzian manifolds admitting a timelike conformal vector field (particularly a Killing vector field) using generalized Minkowski-type integral formulas, extending those first used by H. Minkowski [21]. See [17,22–26] for compact hypersurfaces in Riemannian manifolds, and [1,8–13,18,22–29] for recent references on compact spacelike hypersurfaces in Lorentzian manifolds.

This paper is organized as follows. Section 2 provides the fundamental concepts and definitions necessary for the subsequent sections, providing definitions and necessary formulas concerning hypersurfaces in pseudo-Riemannian manifolds, especially Riemannian hypersurfaces in Riemannian manifolds and spacelike hypersurfaces in Lorentzian manifolds.

In Section 3, we define Riemannian warped product manifolds and generalized Robertson–Walker (GRW) spacetimes, presenting necessary formulas for the Levi-Civita connection and the Ricci curvature at horizontal and vertical tangent vectors. We establish the necessary and sufficient conditions, in terms of the warping function, for such spaces to be Einstein manifolds, showing that a warped product is Einstein if and only if the fiber manifold is Einstein.

Section 4 estimates the relationship between the scalar curvature of the hypersurface and that of the ambient warped product manifold or GRW spacetime. We derive results concerning scalar curvature bounds based on the Ricci curvature and scalar curvature of the base manifold, considering the behavior of the warping function f and properties of the vector field ∂_t tangent to the one-dimensional base, and sometimes, the inner product of ∂_t with the unit normal to the hypersurface. For instance, the fact that $f\partial_t$ is a closed conformal vector field and the nice properties of the height both help to deduce several interesting results about the hypersurface's scalar curvature.

Section 5 serves as the main focus of the paper, concentrating on Riemannian (or spacelike) hypersurfaces within Riemannian warped product manifolds (or GRW spacetimes). We derive three Minkowski-type integral formulas for these hypersurfaces and use them to formulate several theorems regarding the characterization of both compact and non-compact spacelike hypersurfaces in GRW spacetimes. For example, we show that in a GRW spacetime, no compact spacelike hypersurface can have a mean curvature H satisfying $f'H < 0$. Additionally, we prove that if the fiber is Einstein and the hypersurface Σ is compact with constant mean curvature, then Σ is an extrinsic hypersphere, which is a totally umbilical hypersurface with a non-zero constant mean curvature. Furthermore, we show that, given regular conditions such as the convexity of the function $-\mathrm{Log} f$, there cannot be a compact spacelike hypersurface in a GRW spacetime with non-negative mean curvature and a scalar curvature exceeding that of the base manifold.

For minimal (or maximal) hypersurfaces, we show that if $\mathrm{Log} f$ (resp. $-\mathrm{Log} f$) is convex and the Ricci curvature of the base M is non-negative at the tangential part of the unit normal, then Σ is a slice, meaning it takes the form $\Sigma = \{t_0\} \times M$.

2. Preliminaries

Let (M, g) be a pseudo-Riemannian manifold of dimension $n \geq 2$ with the Levi-Civita connection ∇, and let $\mathfrak{X}(M)$ denote the collection of all vector fields on M. The curvature tensor of (M, g) is defined as the $(1, 3)$-tensor field given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

The Ricci curvature Ric is the trace of R. It is the symmetric bilinear form defined as follows. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM of M at the point p and $\epsilon_i = g(e_i, e_i)$, then

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(X, e_i)e_i, Y),$$

for all $X, Y \in T_pM$.

The scalar curvature Sc is obtained by taking the trace of the Ricci tensor Ric. It is a function on M defined by

$$Sc(p) = \sum_{i=1}^{n} \epsilon_i Ric(e_i, e_i)$$

For a function f on M, the gradient is defined as the vector field $\nabla f \in \mathfrak{X}(M)$ satisfying

$$g(\nabla f, X) = X \cdot f, \tag{2}$$

for all $X \in \mathfrak{X}(M)$.

If $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of vector fields, then the divergence of $X \in \mathfrak{X}(M)$ is defined as the function

$$\text{div}(X) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i) \tag{3}$$

The divergence of a tensor B of type $(1, 1)$ on M is defined as the vector field

$$\text{div}(B) = \text{trace}(\nabla B) = \sum_{i=1}^{n} (\nabla_{e_i} B)(e_i),$$

where the covariant derivative ∇B of B is given here by the formula

$$(\nabla_X B)(Y) = \nabla_X B(Y) - B(\nabla_X Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

The Hessian Hess$f$ of a smooth function f is the symmetric covariant $(0, 2)$-tensor given by

$$\text{Hess}f(X, Y) = g(\nabla_X(\nabla f), Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

The Laplacian Δf of f is simply

$$\Delta f = \text{div}(\nabla f) \tag{4}$$

Now consider a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ of dimension $(n + 1)$, $n \geq 2$, which may be either Riemannian or Lorentzian (that is, $\overline{g}$ has signature $(0, n + 1)$ or $(1, n)$, respectively). Additionally, (Σ, g) denotes a Riemannian manifold of dimension n, which we isometrically immerse into $(\overline{M}, \overline{g})$. Consequently, we will treat (Σ, g) as a Riemannian hypersurface within $(\overline{M}, \overline{g})$, and it will be considered spacelike if $(\overline{M}, \overline{g})$ is Lorentzian.

Let $\overline{V} \in \mathfrak{X}(\overline{M})$ be a vector field on $\overline{M}$ that we assume to be timelike when $(\overline{M}, \overline{g})$ is Lorentzian. In this case, we can select a globally defined unit timelike vector field N that is normal to Σ and aligns with the time orientation of $\overline{V}$. This implies that $\overline{g}\big(\overline{V}, N\big) < 0$ holds everywhere on Σ.

When $(\overline{M}, \overline{g})$ is Riemannian, we assume that Σ is a two-sided hypersurface, which implies the existence of a globally defined unit vector field N normal to Σ.

Let V represent the restriction of $\overline{V}$ to Σ. Then, a smooth function Θ on Σ, called the support function of V, is naturally defined by $\Theta = \overline{g}(V, N)$.

In the Lorentzian case, on Σ, we have the below inequality:

$$\Theta \leq -\sqrt{-\overline{g}(V, V)} < 0. \tag{5}$$

Let $V^{\top}$ denote the tangential component of V to Σ. Then, we have

$$V = V^{\top} + \epsilon \Theta N, \tag{6}$$

where $\epsilon = \overline{g}(N, N)$.

Let ∇ and $\overline{\nabla}$ represent the Levi-Civita connections on (Σ, g) and $(\overline{M}, \overline{g})$, respectively. Let $\mathfrak{X}(\Sigma)$ and $\mathfrak{X}(\overline{M})$ denote the sets of all tangent vector fields on Σ and $\overline{M}$, respectively, and let $\overline{\mathfrak{X}}(\Sigma)$ be the set of all vector fields on $\overline{M}$ restricted to Σ. If A represents the shape operator of Σ with respect to N, then the formulas of Gauss and Weingarten for the hypersurface Σ in $\overline{M}$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \epsilon g(A(X), Y) N$$
$$A(X) = -\overline{\nabla}_X N,$$

where $X, Y \in \mathfrak{X}(\Sigma)$.

The Gauss equation relates the curvature tensor R of (Σ, g) to the tangential component if the curvature tensor $\overline{R}$ of $(\overline{M}, \overline{g})$ and the shape operator A via the so-called Gauss equation

$$R(X, Y)Z = \big(\overline{R}(X, Y)Z\big)^{\top} + \epsilon(g(A(Y), Z)A(X) - g(A(X), Z)A(Y)), \tag{7}$$

for all $X, Y, Z \in \mathfrak{X}(\Sigma)$.

The Codazzi equation for (Σ, g) provides a formula for the normal part of $\overline{R}(X, Y)Z$, which is given by the following expression

$$\overline{R}(X, Y)N = (\nabla_Y A)X - (\nabla_X A)Y, \tag{8}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$.

The mean curvature of (Σ, g) is given by

$$H = \frac{\epsilon}{n} \mathrm{trace}(A). \tag{9}$$

The hypersurface (Σ, g) is considered totally umbilical if $A = \epsilon H I$, where I is the identity operator. It is said to be totally geodesic if $A = 0$, and it is known as an extrinsic hypersphere of $(\overline{M}, \overline{g})$ if it is totally umbilical with a non-zero constant mean curvature H. For instance, if $(\overline{M}, \overline{g})$ has a constant sectional curvature $\overline{c}$, then extrinsic hyperspheres are isometric to round spheres.

When $H = 0$, the hypersurface (Σ, g) is called minimal if $(\overline{M}, \overline{g})$ is Riemannian, and maximal if $(\overline{M}, \overline{g})$ is Lorentzian.

Equation (7) establishes a connection between the Ricci curvature Ric of Σ and the Ricci curvature $\overline{Ric}$ of $\overline{M}$. This connection is expressed by the equation

$$Ric(X, Y) = \overline{Ric}(X, Y) - \epsilon \overline{g}(\overline{R}(N, X)Y, N) + g(A(X), nHY - \epsilon A(Y)), \tag{10}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$.

Additionally, by taking the trace of Equation (10), we establish the relationship between the scalar curvature Sc of Σ and the scalar curvature $\overline{Sc}$ of $\overline{M}$, as given by the following equation:

$$Sc = \overline{Sc} - 2\epsilon\overline{Ric}(N,N) + \epsilon(n^2H^2 - \|A\|^2). \tag{11}$$

3. Pseudo-Riemannian Warped Products: The Case of GRW Spacetimes

From this point forward, we will regard $\overline{M}$ as a warped product manifold of the form $\overline{M} = \mathbb{R} \times_f M$, with the pseudo-Riemannian metric

$$\overline{g} = \epsilon dt^2 + f^2(t)g_M,$$

where (M, g_M) is an n-dimensional Riemannian manifold, f is positive smooth function on $\mathbb{R}$, and $\epsilon = \pm 1$.

If $\epsilon = 1$, then $(\overline{M}, \overline{g})$ is simply a warped product of two Riemannian manifolds. However, if $\epsilon = -1$, then $(\overline{M}, \overline{g})$ becomes a warped product of the Lorentzian one-dimensional manifold $(\mathbb{R}, -dt^2)$ and a Riemannian n-dimensional manifold (M, g_M).

In this context, $(\overline{M}, \overline{g})$ is called a generalized Robertson–Walker spacetime (GRW). This generalizes the Robertson–Walker spacetime $(\overline{M}, -dt^2 + f^2(t)g_M)$, where (M, g_M) is a 3-dimensional Riemannian manifold with constant curvature.

Let ∂_t denote the coordinate vector field on $\mathbb{R}$ and $W^\perp$ represent the normal component to M of the vector field $W \in \mathfrak{X}(\overline{M})$. For the following lemmas, we refer to [30] (see also [31]).

Lemma 1. *On $(\overline{M}, \overline{g})$, we have the following for all $X, Y \in \mathfrak{X}(M)$, where the symbol $\perp$ indicates the component orthogonal to M, which is the tangent component along the $\mathbb{R}$ factor:*

(i) $\overline{\nabla}_{\partial_t} X = \overline{\nabla}_X \partial_t = \frac{f'}{f} X$;

(ii) $(\overline{\nabla}_X Y)^\perp = -\epsilon g(X,Y)\frac{f'}{f}\partial_t$.

Lemma 2. *On $(\overline{M}, \overline{g})$, we have the following for all $X, Y \in \mathfrak{X}(M)$, where Ric_M denotes the Ricci curvature of (M, g_M):*

(i) $\overline{Ric}(\partial_t, \partial_t) = -n\frac{f''}{f}$;

(ii) $\overline{Ric}(\partial_t, X) = 0$;

(iii) $\overline{Ric}(X,Y) = Ric_M(X,Y) - \epsilon\left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right)\overline{g}(X,Y)$.

The scalar curvature $\overline{Sc}$ of $(\overline{M}, \overline{g})$ is related to the scalar curvature Sc_M of (M, g_M) and is given by

$$\overline{Sc} = \frac{Sc_M}{f^2} - 2\epsilon n\frac{f''}{f} - \epsilon n(n-1)\frac{(f')^2}{f^2}. \tag{12}$$

The following lemma describes how the Ricci curvature of $(\overline{M}, \overline{g})$ relates to the Ricci curvature of (M, g_M).

Lemma 3. *For every vector fields U and V on $\overline{M}$, with U^* and V^* being their respective components tangent to M, we have*

$$\overline{Ric}(U,V) = Ric_M(U^*,V^*) - \epsilon\Big(ff'' + (n-1)(f')^2\Big)g_M(U^*,V^*) - n\overline{g}(U,\partial_t)\overline{g}(V,\partial_t)\frac{f''}{f}. \tag{13}$$

Proof. By virtue of Lemma 2, and since $\overline{g}(U^*, V^*) = f^2 g_M(U^*, V^*)$, we have

$$\begin{aligned}\overline{Ric}(U,V) &= \overline{Ric}(U^*,V^*) + \overline{g}(U,\partial_t)\overline{g}(V,\partial_t)\overline{Ric}(\partial_t,\partial_t)\\ &= Ric_M(U^*,V^*) - \epsilon\left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right)\overline{g}(U^*,V^*) - n\overline{g}(U,\partial_t)\overline{g}(V,\partial_t)\frac{f''}{f}\\ &= Ric_M(U^*,V^*) - \epsilon\Big(ff'' + (n-1)(f')^2\Big)g_M(U^*,V^*) - n\overline{g}(U,\partial_t)\overline{g}(V,\partial_t)\frac{f''}{f}.\end{aligned}$$

□

The following proposition establishes that $(\overline{M}, \overline{g})$ is Einstein if and only if (M, g_M) is Einstein as well.

Proposition 4. *Under the notation and assumption mentioned above, $(\overline{M}, \overline{g})$ is an Einstein manifold, meaning that $\overline{Ric} = \overline{\lambda}\overline{g}$, if and only if (M, g_M) is Einstein with $Ric_M = \lambda g_M$ where $\overline{\lambda} = -\epsilon n \frac{f''}{f}$ and $\lambda = -\epsilon(n-1)\big(ff'' - (f')^2\big)$.*

Proof. If $(\overline{M}, \overline{g})$ is Einstein with $\overline{Ric} = \overline{\lambda}\overline{g}$, then

$$\overline{Ric}(U,V) = \overline{\lambda}\Big(\epsilon\overline{g}(U,\partial_t)\overline{g}(V,\partial_t) + f^2 g_M(U^*,V^*)\Big),$$

for all $U, V \in \mathfrak{X}(\overline{M})$.

Substituting this into (13), and considering that U, V are arbitrary in $\mathfrak{X}(\overline{M})$, we deduce that

$$\overline{\lambda} = -\epsilon n\frac{f''}{f} \quad \text{and} \quad Ric_M(U^*,V^*) = \Big(\overline{\lambda}f^2 + \epsilon(ff'' + (n-1)(f')^2\Big)g_M(U^*,V^*).$$

Therefore, M is Einstein with $Ric_M = \lambda g_M$, where $\lambda = -\epsilon(n-1)\big(ff'' - (f')^2\big)$. The converse is also true, as we can easily verify. □

Remark 5. *If $\epsilon\overline{\lambda} > 0$, then by setting $\omega^2 = \epsilon\overline{\lambda}$, we see that f takes the form*

$$f = A\cos\omega t + B\sin\omega t, \quad \textit{for some} \quad A, B \in \mathbb{R}.$$

If $\epsilon\overline{\lambda} < 0$, then by setting $\omega^2 = -\epsilon\overline{\lambda}$, we see that f takes the form

$$f = A\cosh\omega t + B\sinh\omega t, \quad \textit{for some} \quad A, B \in \mathbb{R}.$$

In terms of the constants A and B, it follows that

$$\lambda = \begin{cases}(n-1)\,(A^2 + B^2)\,\overline{\lambda}, & \textit{if} \quad \epsilon\overline{\lambda} > 0.\\ (n-1)\,(A^2 - B^2)\,\overline{\lambda}, & \textit{if} \quad \epsilon\overline{\lambda} < 0.\end{cases}$$

4. Estimating the Scalar Curvature of a Spacelike Hypersurface in a GRW Spacetime

We will now assume that (Σ, g) is a connected Riemannian manifold, isometrically immersed as a hypersurface in the warped product manifold $(\overline{M}, \overline{g})$.

Given that ∂_t establishes an orientation for $\overline{M}$, let N be a globally defined unit normal vector field to Σ. We will refer to θ as the support function of ∂_t, a smooth function on Σ defined by $\theta = \overline{g}(\partial_t, N)$.

Using the notation mentioned earlier, we have $\Theta = f\theta$, and it is clear from (5) that if $(\overline{M}, \overline{g})$ is Lorentzian (i.e., a GRW spacetime), then on Σ, we have the inequality

$$\theta \leq -1. \tag{14}$$

If $\partial_t^\top$ denotes the component of ∂_t that is tangent to Σ, then we can express

$$\partial_t = \partial_t^\top + \epsilon\theta N, \tag{15}$$

where $\epsilon = \overline{g}(N, N)$.

It is straightforward to observe that $\zeta = f\partial_t$ meets the following condition:

$$\overline{\nabla}_U(f\partial_t) = f'U, \tag{16}$$

for any $U \in \mathfrak{X}(\overline{M})$. In other words, ζ is a vector field on $(\overline{M}, \overline{g})$ that is closed conformal.

According to (15), ζ can be expressed as

$$\zeta = \zeta^T + \epsilon f\theta N, \tag{17}$$

where ζ^T is the tangential component of ζ.

By using (16), together with the Weingarten and Gauss formulas, we derive

$$\nabla_X\zeta^T = f'X + \epsilon f\theta A(X), \tag{18}$$

and

$$A(\zeta^T) = -f\nabla\theta - \epsilon\frac{f'}{f}\theta\zeta^T, \tag{19}$$

for all $X \in \mathfrak{X}(\Sigma)$.

From (18), we derive that

$$\mathrm{div}(\zeta^T) = n(f' + f\theta H). \tag{20}$$

We can also find the divergence of the vector field ∂t^T. This will turn out to be the Laplacian of the important function on Σ called the height function (see Lemmas 6 and 7 below).

The height function h of Σ is given by $h = \pi_{\mathbb{R}} \circ \Psi$, with $\pi_{\mathbb{R}}$ representing the projection from $\overline{M}$ onto its $\mathbb{R}$ factor, and Ψ is the isometric immersion of Σ into $\overline{M}$. The next two lemmas provide the gradient, norm, and Laplacian of h.

Lemma 6. *The gradient of h on Σ can be expressed as*

$$\nabla h = \epsilon\partial_t^\top,$$

with the norm (i.e., length) expressed as

$$\|\nabla h\|^2 = \epsilon(1 - \theta^2) \tag{21}$$

Proof. Given that $\overline{\nabla}\pi_{\mathbb{R}} = \epsilon\partial_t$, it follows that

$$\nabla h = \epsilon\partial_t^\top,$$

and from the decomposition (15), we obtain

$$\begin{aligned}\|\nabla h\|^2 &= g(\nabla h, \nabla h)\\ &= g(\epsilon\partial_t^\top, \epsilon\partial_t^\top)\\ &= \overline{g}(\partial_t - \epsilon\theta N, \partial_t - \epsilon\theta N)\\ &= \epsilon(1 - \theta^2).\end{aligned}$$

□

Lemma 7. *The Laplacian of h is*

$$\Delta h = \epsilon n(\frac{f'}{f} + H\,\theta) - \epsilon(1-\theta^2)\frac{f'}{f}. \tag{22}$$

Proof. Since the vector field $f\partial_t$ is closed conformal, applying (15), (16), and the Weingarten formula leads to

$$\begin{aligned}\nabla_X \nabla h &= \nabla_X\left(\epsilon \partial_t^\top\right)\\ &= \left(\epsilon \overline{\nabla}_X \partial_t - \theta \overline{\nabla}_X N\right)^\top\\ &= \epsilon\left(\overline{\nabla}_X \partial_t\right)^\top + \theta A(X)\\ &= \epsilon\left(\overline{\nabla}_X(\frac{1}{f} f\partial_t)\right)^\top + \theta A(X)\\ &= \epsilon \frac{f'}{f} X - \epsilon \frac{1}{f} X(f)\, \partial_t^\top + \theta A(X).\end{aligned}$$

Therefore, if $\{e_1, \ldots, e_n\}$ forms a local orthonormal frame on Σ, we obtain

$$\begin{aligned}\Delta h &= \epsilon n \frac{f'}{f} - \frac{\epsilon}{f}\sum_{i=1}^{n} g(e_i(f)\partial_t^\top, e_i) + \epsilon n H\,\theta\\ &= \epsilon n \frac{f'}{f} - \frac{\epsilon}{f}\sum_{i=1}^{n} g(\nabla_{e_i}(f\partial_t^\top), e_i) + \epsilon \sum_{i=1}^{n} g(\nabla_{e_i}\partial_t^\top, e_i) + \epsilon n H\,\theta\\ &= \epsilon n \frac{f'}{f} - \frac{\epsilon}{f}\,\mathrm{div}(f\partial_t^\top) + \epsilon\,\mathrm{div}(\partial_t^\top) + \epsilon n H\,\theta\\ &= \epsilon n \frac{f'}{f} - \frac{\epsilon}{f}\partial_t^\top(f) + \epsilon n H\,\theta\\ &= \epsilon n \frac{f'}{f} - \frac{\epsilon}{f} g\left(\nabla f, \partial_t^\top\right) + \epsilon n H\,\theta\\ &= \epsilon n \frac{f'}{f} - \frac{\epsilon}{f} g\left(\epsilon f' \partial_t^\top, \partial_t^\top\right) + \epsilon n H\,\theta.\end{aligned}$$

Since $g\left(\partial_t^\top, \partial_t^\top\right) = \|\nabla h\|^2 = \epsilon(1-\theta^2)$, then we obtain (22). □

The following result highlights the relationship between the scalar curvature of a Riemannian hypersurface (Σ, g) and that of the factor (M, g_M), while implicitly involving the scalar curvature of $\left(\overline{M} = \mathbb{R} \times_f M, \overline{g}\right)$.

Proposition 8. *Consider* (Σ, g) *as a Riemannian hypersurface within* $(\overline{M}, \overline{g})$*. Then, with the notations previously defined, the scalar curvature Sc of* (Σ, g) *is given by*

$$\begin{aligned}Sc = \frac{Sc_M}{f^2} - 2\epsilon\left[Ric_M(N^*, N^*) + (n-1)(1-\theta^2)\left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right)\right]\\ - \epsilon n(n-1)\left(\frac{(f')^2}{f^2} - H^2\right) - \epsilon\left(\|A\|^2 - nH^2\right).\end{aligned} \tag{23}$$

Proof. By Lemma 3, we have

$$\overline{Ric}(N, N) = Ric_M(N^*, N^*) - (1-\theta^2)\left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right) - n\,\theta^2 \frac{f''}{f}. \tag{24}$$

By taking the trace of (24) and applying (11) and (12), we deduce that

$$\begin{aligned}
Sc &= \frac{Sc_M}{f^2} - 2\epsilon n \frac{f''}{f} - \epsilon n(n-1)\frac{(f')^2}{f^2} + 2\epsilon n\, \theta^2 \frac{f''}{f} - 2\epsilon Ric_M(N^*, N^*) \\
&+ 2\epsilon(1-\theta^2)\left(\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right) + \epsilon\left(n^2H^2 - \|A\|^2\right) \\
&= \frac{Sc_M}{f^2} - 2\epsilon\left[Ric_M(N^*, N^*) + (n-1)(1-\theta^2)\left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right)\right] \\
&- \epsilon n(n-1)\left(\frac{(f')^2}{f^2} - H^2\right) - \epsilon\left(\|A\|^2 - nH^2\right).
\end{aligned}$$

□

If $(\overline{M}, \overline{g})$ is an Einstein manifold, then the following is a consequence of Proposition 8.

Theorem 9. *Let (Σ, g) be a Riemannian hypersurface in $(\overline{M}, \overline{g})$ with (M, g_M) being an Einstein manifold. Using the previously mentioned notations, the scalar curvature Sc of Σ is expressed as*

$$Sc = -\epsilon n(n-1)(\frac{f''}{f} - H^2) - \epsilon(\|A\|^2 - nH^2). \tag{25}$$

In particular, if we additionally assume that $f'' \geq fH^2$ everywhere, then $\epsilon S \leq 0$.

Proof. If (M, g_M) is Einstein with $Ric_M = \lambda g_M$, then, according to Proposition 4, we have $\lambda = -\epsilon(n-1)\big(ff'' - (f')^2\big)$. Consequently, we obtain

$$Sc_M = -\epsilon n(n-1)\left(ff'' - (f')^2\right),$$

$$Ric_M(N^*, N^*) = -\epsilon(n-1)\left(ff'' - (f')^2\right)g_M(N^*,\ N^*),$$

where N^* represents for the component of N tangent to M.

On the other hand, we have

$$\begin{aligned}
\epsilon = \overline{g}(N, N) &= \overline{g}(N^*,\ N^*) + \epsilon\overline{g}(N, \partial t)^2 \\
&= \overline{g}(N^*,\ N^*) + \epsilon\theta^2.
\end{aligned}$$

Since $\overline{g}(N^*, N^*) = f^2 g_M(N^*,\ N^*)$, the above equation leads to the conclusion that

$$g_M(N^*,\ N^*) = \frac{\epsilon}{f^2}(1-\theta^2).$$

Consequently, we obtain

$$\begin{aligned}
Ric_M(N^*, N^*) &= -(n-1)(1-\theta^2)\frac{1}{f^2}\left(ff'' - (f')^2\right) \\
&= -(n-1)(1-\theta^2)(\frac{f''}{f} - \frac{(f')^2}{f^2}).
\end{aligned}$$

By substituting these values into (23), we deduce that

$$Sc = -\epsilon n(n-1)(\frac{f''}{f} - H^2) + -\epsilon(\|A\|^2 - nH^2).$$

□

The following result follows from Proposition 8. It demonstrates that for a spacelike hypersurface in a GRW spacetime, if the warping function f is logarithmically concave, then the scalar curvature of that hypersurface is bounded from below.

Theorem 10. *Using the previously defined notations, let (Σ, g) be a spacelike hypersurface in a GRW spacetime $(\overline{M}, \overline{g})$ with $-\mathrm{Log} f$ being convex. Then, the scalar curvature Sc of Σ satisfies the following inequality:*

$$Sc \geq \frac{Sc_M}{f^2} + 2Ric_M(N^*, N^*) - n(n-1)H^2 \tag{26}$$

In particular, if (M, g_M) is Einstein, then, necessarily, $Sc_M \leq 0$ and

$$Sc \geq \left(1 + \frac{2\theta^2}{n}\right)\frac{Sc_M}{f^2} - n(n-1)H^2. \tag{27}$$

Proof. Given that $(\overline{M}, \overline{g})$ is now a Lorentzian manifold, Equation (23) transforms into

$$\begin{aligned} Sc = \frac{Sc_M}{f^2} &+ 2\left[Ric_M(N^*, N^*) + (n-1)(1-\theta^2)\left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right)\right] \\ &+ n(n-1)\left(\frac{(f')^2}{f^2} - H^2\right) + (\|A\|^2 - nH^2). \end{aligned}$$

Given that $-\mathrm{Log} f$ is convex, it follows that $ff'' - (f')^2 \leq 0$. Using (14) and the condition $||A||^2 - nH^2 \geq 0$, the above equation leads to inequality (26).

Assuming that (M, g_M) is Einstein, Proposition 4 implies that

$$Sc_M = n(n-1)(ff'' - (f')^2) \leq 0.$$

Additionally, we easily see that

$$Ric_M(N^*, N^*) = \left(\frac{\theta^2 - 1}{n}\right)\frac{Sc_M}{f^2},$$

and hence, inequality (27) can be directly derived from (26). □

Likewise, the following result that also follows from Proposition 8 shows that if the ambient warped product is Riemannian with a concave warping function, then the scalar curvature of the hypersurface is bounded from above.

Theorem 11. *Consider a warped product manifold $\overline{M} = \mathbb{R} \times_f M$ endowed with the Riemannian metric $\overline{g} = dt^2 + f^2(t)g_M$ with a convex $\mathrm{Log} f$. Using the previously defined notations, if (Σ, g) is a hypersurface in $(\overline{M}, \overline{g})$, then the scalar curvature Sc of Σ satisfies the following inequality:*

$$Sc \leq \frac{Sc_M}{f^2} - 2Ric_M(N^*, N^*) + n(n-1)H^2 \tag{28}$$

In particular, if (M, g_M) is Einstein, then, necessarily, $Sc_M \leq 0$ and

$$Sc \leq -\left(\frac{2}{n}\right)\frac{Sc_M}{f^2} + n(n-1)H^2 \tag{29}$$

Remark 12. *We observe that, unlike inequality (27), inequality (29) is independent of the function θ. This distinction arises because when $(\overline{M}, \overline{g})$ is Riemannian, θ is bounded (specifically, $|\theta| \leq 1$). However, when $(\overline{M}, \overline{g})$ is Lorentzian, the condition $|\theta| \geq 1$ holds.*

5. Characterizing Compact Spacelike Hypersurfaces in GRW Spacetimes

In this section, we seek to derive generalized integral formulas of Minkowski-type for compact Riemannian hypersurfaces within a pseudo-Riemannian warped product manifold of the form $(\overline{M}, \overline{g})$, where $\overline{M} = \mathbb{R} \times_f M$ and $\overline{g} = \epsilon dt^2 + f^2(t) g_M$. Our focus is particularly on compact spacelike hypersurfaces in GRW spacetimes. Using these integral formulas, we will characterize these spacelike hypersurfaces, identifying conditions under which they become extrinsic hyperspheres or slices. Additionally, some of these formulas extend existing ones related to conformal and Killing vector fields.

The first integral formula we present is general and not limited to spacelike hypersurfaces in GRW spacetimes. It is derived by integrating (20).

Theorem 13. *Under the previously defined notations, let (Σ, g) be a compact Riemannian hypersurface in the pseudo-Riemannian warped product manifold $(\overline{M}, \overline{g})$. Then, we have*

$$\int_\Sigma (f' + f H\theta)\, dV = 0, \tag{30}$$

where dV denotes the volume form of (Σ, g).

In the case where $(\overline{M}, \overline{g})$ is Riemannian, we can derive the following result from Theorem 13.

Theorem 14. *Under the previously defined notations, consider a warped product manifold $\overline{M} = \mathbb{R} \times_f M$ with a Riemannian metric $\overline{g} = dt^2 + f^2(t) g_M$. Then, there is no compact hypersurface (Σ, g) in $(\overline{M}, \overline{g})$ where θ is non-zero and does not change sign, and for which the mean curvature H satisfies $f'H < 0$ when $\theta < 0$ or $f'H > 0$ when $\theta > 0$.*

Proof. Given that θ is non-zero and does not change sign, we can assume without loss of generality that $\theta < 0$. We also assume that $f'H < 0$. The opposite case can be addressed in a similar manner. Consequently, this implies that either $f' < 0$ and $H > 0$ or $f' > 0$ and $H < 0$. Since $\theta < 0$, we deduce that either $f' + fH\theta < 0$ or $f' + fH\theta > 0$. However, this leads to a contradiction with (30), thus establishing the desired conclusion. □

Similarly, in the case of a spacelike hypersurface in a GRW spacetime, we obtain the below theorem.

Theorem 15. *In a GRW spacetime $(\overline{M}, \overline{g})$, there is no compact spacelike hypersurface (Σ, g) for which the mean curvature H satisfies $f'H < 0$.*

Proof. If $f'H < 0$, it is evident that either $f' < 0$ and $H > 0$ or $f' > 0$ and $H < 0$. Given that $\theta < 0$ and H remains constant in sign, we conclude that either $f' + fH\theta < 0$ or $f' + fH\theta > 0$. This contradicts (30), thereby proving the desired claim. □

Our second integral formula, which is of the Minkowski-type, relates to spacelike hypersurfaces in GRW spacetimes.

Proposition 16. *Let (Σ, g) be a compact Riemannian hypersurface in a pseudo-Riemannian warped product manifold of the form $(\overline{M}, \overline{g})$, where $\overline{M} = \mathbb{R} \times_f M$ and $\overline{g} = \epsilon dt^2 + f^2(t) g_M$. Using the previously defined notations, we have*

$$\int_\Sigma f\Big(\theta(\|A\|^2 - nH^2) + (n-1)\partial_t^\top(H) - \epsilon\overline{Ric}(N, \partial_t^\top)\Big) dV = 0. \tag{31}$$

Proof. If $e_1, \ldots, e_n$ is a local parallel orthonormal frame on Σ, then utilizing the Codazzi Equations (8) and (18), noting that $\nabla_X A$ is self-adjoint because A is self-adjoint, and observing that the vector $\overline{R}(e_i, \partial_t^\top)N$ lies to the tangent space of Σ, we derive

$$\begin{aligned}
\operatorname{div}\Big(A(\zeta^\top)\Big) &= \sum_{i=1}^{n} g\Big(\nabla_{e_i} A(\zeta^\top), e_i\Big) \\
&= \sum_{i=1}^{n} g\Big((\nabla_{e_i} A)\zeta^\top, e_i\Big) + \sum_{i=1}^{n} g\Big(A(\nabla_{e_i}\zeta^\top), e_i\Big) \\
&= \sum_{i=1}^{n} g\Big((\nabla_{\zeta^\top} A)e_i, e_i\Big) - \sum_{i=1}^{n} g\Big(\overline{R}(e_i, \zeta^\top)N, e_i\Big) + \sum_{i=1}^{n} g\big(f' e_i + \epsilon f\theta A(e_i), A(e_i)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{div}\Big(A(\zeta^\top)\Big) &= \sum_{i=1}^{n} g\Big(\nabla_{\zeta^\top} A(e_i), e_i\Big) - \sum_{i=1}^{n} \overline{g}\Big(\overline{R}(N, e_i)e_i, \zeta^\top\Big) + \sum_{i=1}^{n} f' g(A(e_i), e_i) \\
&\quad + \sum_{i=1}^{n} \epsilon f\theta g(A(e_i), A(e_i)) \\
&= \sum_{i=1}^{n} \zeta^\top \cdot g(A(e_i), e_i) - \overline{Ric}(N, \zeta^\top) + f' \operatorname{trace}(A) + \epsilon f\theta \|A\|^2 \\
&= \epsilon n f \partial_t^\top(H) - \overline{Ric}(N, \zeta^\top) + \epsilon n f' H + \epsilon f\theta \|A\|^2.
\end{aligned}$$

Thus, by adding and subtracting the term $\epsilon n f\theta H^2$, applying Equation (20), and using the fact that $\operatorname{div}(H\zeta^\top) = H\operatorname{div}(\zeta^\top) + \zeta^\top(H)$, we can deduce the following:

$$\operatorname{div}\Big(A(\zeta^\top)\Big) = \epsilon(n-1)f\partial_t^\top(H) - \overline{Ric}(N, \zeta^\top) + \epsilon f\theta(\|A\|^2 - nH^2) + \epsilon \operatorname{div}(H\zeta^\top). \quad (32)$$

Now, recalling that $\zeta^\top = f\partial_t^\top$, Equation (31) is obtained by integrating (32) over Σ. □

The following result, which directly follows from Proposition 16, provides a characterization of Riemannian hypersurfaces with constant mean curvature in Einstein pseudo-Riemannian warped product manifolds.

Theorem 17. *With the notations previously defined, let (Σ, g) be a compact Riemannian hypersurface in a pseudo-Riemannian warped product manifold $(\overline{M}, \overline{g})$, where $\overline{M} = \mathbb{R} \times_f M$ and $\overline{g} = \epsilon dt^2 + f^2(t) g_M$. Assume that (M, g_M) is Einstein, the mean curvature H is constant and non-zero along the integral curves of $\partial_t^\top$, and when $(\overline{M}, \overline{g})$ is Riemannian, θ is non-zero and maintains a constant sign. Then, (Σ, g) is an extrinsic hypersphere in $(\overline{M}, \overline{g})$.*

Proof. Given that H is constant and $\overline{Ric}(N, \partial_t^T) = \overline{\lambda}\, \overline{g}(N, \partial_t^T) = 0$, we deduce from (31) that

$$\int_\Sigma f\theta(\|A\|^2 - nH^2) dV = 0.$$

Given that $f > 0$, $\theta \leq -1$ if $(\overline{M}, \overline{g})$ is a GRW spacetime, and θ is non-zero, does not change sign, and is not identically zero if $(\overline{M}, \overline{g})$ is Riemannian, and since $\|A\|^2 - nH^2 \geq 0$, we can deduce from the integral above that $\|A\|^2 = nH^2$. This implies that (Σ, g) is totally umbilical, and because H is constant and non-zero, it follows that (Σ, g) is an extrinsic hypersphere. □

In particular, for compact spacelike hypersurfaces of GRW spacetimes, since $\theta \leq -1$ everywhere, we have the following theorem:

Theorem 18. *Given the previously defined notations, let (Σ, g) be a compact spacelike hypersurface with non-zero constant mean curvature H in a GRW spacetime $(\overline{M}, \overline{g})$, where $\overline{M} = \mathbb{R} \times_f M$,*

(M, g_M) is an Einstein manifold, and $\overline{g} = -dt^2 + f^2(t)g_M$. Then, (Σ, g) is an extrinsic hypersphere in $(\overline{M}, \overline{g})$.

Our third integral formula of Minkowski-type is the following:

Proposition 19. *Let (Σ, g) be a compact Riemannian hypersurface in a pseudo-Riemannian warped product manifold of the form $(\overline{M}, \overline{g})$, where $\overline{M} = \mathbb{R} \times_f M$ and $\overline{g} = \epsilon dt^2 + f^2(t)g_M$. Using the previously defined notations, we have*

$$\int_\Sigma f\theta\big(Sc - \overline{Sc} + \epsilon\overline{Ric}(N,N)\big)dV = \epsilon n \int_\Sigma f''\theta dV - \epsilon n(n-1)\int_\Sigma f' H dV. \tag{33}$$

Proof. Let $e_1, \ldots, e_n$ be a local parallel orthonormal frame on Σ. We extend this frame such that $\overline{\nabla}_N e_i = 0$, for $i = 1, \cdots, n$. Thus,

$$\begin{aligned}
\overline{R}(e_i, N)\zeta &= \overline{\nabla}_{e_i}\overline{\nabla}_N\zeta - \overline{\nabla}_N\overline{\nabla}_{e_i}\zeta - \overline{\nabla}_{\overline{\nabla}_{e_i}N}\zeta \\
&= \overline{\nabla}_{e_i}(f'N) - \overline{\nabla}_N(f'e_i) + \overline{\nabla}_{A(e_i)}\zeta \\
&= f'\overline{\nabla}_{e_i}N + e_i(f')N - N(f')e_i + f'A(e_i) \\
&= e_i(f')N - N(f')e_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
\overline{Ric}(N,\zeta) &= \sum_{i=1}^n g\big(\overline{R}(e_i,N)\zeta, e_i\big) \\
&= -nN(f') \\
&= -n\overline{g}\big(\overline{\nabla}f', N\big) \\
&= -n\overline{g}\big(\epsilon f''\partial_t, N\big) \\
&= -\epsilon n f''\theta.
\end{aligned}$$

Since $\overline{Ric}(N,\zeta) = \overline{Ric}(N,\zeta^\top) + \epsilon f\theta\overline{Ric}(N,N)$, we deduce that

$$\overline{Ric}(N,\zeta^\top) = -\epsilon n f''\theta - \epsilon f\theta\overline{Ric}(N,N). \tag{34}$$

Based on (11), and utilizing (32) and (34), we obtain

$$\begin{aligned}
f\theta\big(Sc - \overline{Sc} + \epsilon\overline{Ric}(N,N)\big) &= f\theta\Big(-\epsilon\overline{Ric}(N,N) + \epsilon(n^2H^2 - \|A\|^2)\Big) \\
&= \epsilon n f''\theta + \overline{Ric}(N,\zeta^\top) + \epsilon f\theta(n^2H^2 - \|A\|^2) \\
&= \epsilon n f''\theta - \operatorname{div}\Big(A(\zeta^\top)\Big) + \epsilon(n-1)f\partial_t^\top(H) \\
&+ \epsilon f\theta(\|A\|^2 - nH^2) + \epsilon\operatorname{div}(H\zeta^\top) + \epsilon f\theta(n^2H^2 - \|A\|^2) \\
&= -\operatorname{div}\Big(A(\zeta^\top)\Big) + \epsilon\operatorname{div}(H\zeta^\top) + \epsilon n f''\theta + \epsilon(n-1)f\partial_t^\top(H) \\
&+ \epsilon n(n-1)f\theta H^2 \\
&= -\operatorname{div}\Big(A(\zeta^\top)\Big) + \epsilon\operatorname{div}(H\zeta^\top) + \epsilon n f''\theta \\
&+ \epsilon(n-1)\Big(f\partial_t^\top(H) + nf\theta H^2\Big).
\end{aligned}$$

On the other hand, considering that

$$\operatorname{div}(H\zeta^\top) = H\operatorname{div}(\zeta^\top) + \zeta^\top(H)$$

and using (20), we have

$$\operatorname{div}(H\zeta^\top) = nH(f' + f\theta H) + f\partial_t^\top(H).$$

Thus,

$$f\partial_t^\top(H) + nf\theta H^2 = \mathrm{div}(H\zeta^\top) - nf'H.$$

By substituting this result into the previous aligned equation, we obtain

$$\begin{aligned} f\theta\big(Sc - \overline{Sc} + \epsilon\overline{Ric}(N,N)\big) &= -\mathrm{div}\Big(A(\zeta^\top)\Big) + \epsilon\mathrm{div}(H\zeta^\top) + \epsilon nf''\theta \\ &\quad + \epsilon(n-1)\Big(\mathrm{div}(H\zeta^\top) - nf'H\Big) \\ &= -\mathrm{div}\Big(A(\zeta^\top)\Big) + \epsilon n\mathrm{div}(H\zeta^\top) + \epsilon nf''\theta - \epsilon n(n-1)f'H. \end{aligned}$$

Finally, integrating both sides of this equation over Σ, we achieve (33). □

The following result concerning spacelike hypersurfaces in Einstein GRW spacetimes is one of the consequences of Proposition 19. It demonstrates that a compact spacelike hypersurface in an Einstein GRW spacetime cannot have scalar curvature $Sc \leq \frac{n}{n+1}\overline{Sc}$ while also having positive mean curvature $H > 0$.

Theorem 20. *Let $(\overline{M}, \overline{g})$ be an Einstein GRW spacetime as previously defined, where f is a non-constant function that is concave and decreasing. Given these conditions, if (Σ, g) is a compact spacelike hypersurface in $(\overline{M}, \overline{g})$ with non-negative mean curvature $H \geq 0$, then (Σ, g) is either maximal or has a scalar curvature $Sc \geq \frac{n}{n+1}\overline{Sc}$.*

Proof. Assume, for contradiction, that (Σ, g) is a compact spacelike hypersurface in $(\overline{M}, \overline{g})$ with non-negative mean curvature $H \geq 0$ and scalar curvature $Sc \leq \frac{n}{n+1}\overline{Sc}$.

Since $(\overline{M}, \overline{g})$ is an Einstein spacetime, we have by Proposition 4 the following relation:

$$\overline{Ric}(N,N) = -\frac{\overline{Sc}}{n+1} = -n\frac{f''}{f}. \tag{35}$$

Given that f is concave and decreasing, and noting that $\theta \leq -1$ and $H \geq 0$, the right-hand side of Equation (33) becomes non-positive (since $\epsilon = -1$ in this case). However, since $Sc \leq \frac{n}{n+1}\overline{Sc}$, we can conclude from (35) that

$$Sc - \overline{Sc} - \overline{Ric}(N,N) \leq 0.$$

Given that $f > 0$ and $\theta \leq -1$, it follows that the left-hand side of (33) is non-negative. Thus, we deduce that $f'' = 0$, $f'H = 0$, and $Sc = 0$ by (35). Since f is not constant, we conclude that $H = 0$, meaning that (Σ, g) is maximal. This completes the proof. □

We conclude this paper with two significant results derived from Theorem 10 and Theorem 11 for the case where the hypersurface has zero mean curvature. These results characterize a particular class of spacelike hypersurfaces known as slices of the pseudo-Riemannian warped product manifold $(\overline{M}, \overline{g})$. In the context of a GRW spacetime, spacelike slices are of particular interest in physics and general relativity as they serve as reference frames for special observers.

A Riemannian slice (or simply slice) in a pseudo-Riemannian warped product manifold $(\overline{M}, \overline{g})$ is a Riemannian hypersurface (Σ, g) where the height function h, defined in Section 4, is constant on Σ. Equivalently, according to Formula (21), (Σ, g) is a slice if and only if the function θ is identically 1 when $(\overline{M}, \overline{g})$ is Riemannian, and $\theta = -1$ when $(\overline{M}, \overline{g})$ is Lorentzian. Consequently, the shape operator A of the slice $t_0 \times M$ is given by

$$A = \epsilon \left(\frac{f'(t_0)}{f(t_0)} \right) I.$$

Thus, slices are totally umbilical with constant mean curvature $H = -\frac{f'(t_0)}{f(t_0)}$.

Theorem 21. *Let (Σ, g) be a maximal spacelike hypersurface in an Einstein GRW spacetime $(\overline{M}, \overline{g})$ with $-\mathrm{Log} f$ being convex and $Sc_M \neq 0$. Assume that $Ric_M(N^*, N^*) \geq 0$ and the scalar curvature Sc of (Σ, g) satisfies $Sc \leq \frac{Sc_M}{f^2}$. Then, (Σ, g) is a spacelike slice in $(\overline{M}, \overline{g})$.*

Proof. Under the assumptions of the theorem, it follows from (26) that $Ric_M(N^*, N^*) = 0$. However,

$$Ric_M(N^*, N^*) = \left(\frac{\theta^2 - 1}{n} \right) \frac{Sc_M}{f^2}.$$

Given that $Sc_M \neq 0$ and $\theta \leq -1$, we conclude that $\theta = -1$. Hence, (Σ, g) is a spacelike slice in $(\overline{M}, \overline{g})$. □

We have observed that if (Σ, g) is a spacelike hypersurface in a generalized Robertson–Walker (GRW) spacetime, it is always possible to choose N such that the function θ is globally defined and negative. In the case where $(\overline{M}, \overline{g})$ is Riemannian, we can assume that (Σ, g) is a two-sided hypersurface, ensuring that θ is globally defined. To prevent θ from changing sign, a more restrictive condition would be to assume that Σ is locally a graph over M. However, this condition is quite limiting, so we adopt the weaker assumption that θ does not change sign.

Theorem 22. *Consider a warped product manifold $\overline{M} = \mathbb{R} \times_f M$ with the Riemannian metric $\overline{g} = dt^2 + f^2(t) g_M$, where $\mathrm{Log} f$ is a convex function and (M, g_M) is Einstein with $Sc_M \neq 0$. Let (Σ, g) be a minimal Riemannian hypersurface in $(\overline{M}, \overline{g})$, where the function θ does not change sign. If $Ric_M(N^*, N^*) \geq 0$ and the scalar curvature Sc of (Σ, g) satisfies $Sc \geq \frac{Sc_M}{f^2}$, then (Σ, g) is a Riemannian slice in $(\overline{M}, \overline{g})$.*

Proof. The proof is similar to that of Theorem 21. Under the assumptions of the current theorem, we deduce from (28) that $Ric_M(N^*, N^*) = 0$. However, we have now

$$Ric_M(N^*, N^*) = \left(\frac{1 - \theta^2}{n} \right) \frac{Sc_M}{f^2}.$$

Given that $Sc_M \neq 0$ and θ remains constant in sign, we conclude that $\theta = \pm 1$. Hence, (Σ, g) must be a slice in $(\overline{M}, \overline{g})$.

□

6. Conclusions

This paper explores generalized Robertson–Walker (GRW) spacetimes by analyzing Riemannian hypersurfaces in pseudo-Riemannian warped product manifolds. We established scalar curvature bounds for these hypersurfaces, focusing on Einstein ambient manifolds, and used generalized Minkowski-type integral formulas to address the existence and non-existence of compact hypersurfaces with bounded scalar curvatures. For future research, we intend to extend our results to more general warped product manifolds and specific GRW spacetime models. We will investigate implications for cosmological models and universe structure and explore other functions, such as the height function, to derive new Minkowski-type integral formulas. Additionally, we will analyze how bounds on scalar curvature interact with other geometric properties of Riemannian or spacelike hypersurfaces.

Author Contributions: Conceptualization, M.G.; investigation, N.A. and M.G.; writing—original draft, N.A. and M.G.; writing—review editing, M.G.; supervision, M.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by Researchers Supporting Project number (RSPD2024R1053), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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