Spectral Analysis of Volume Operators in Loop Quantum Gravity for Kinematical Case

Ignatius Sebastian[1], Idrus Husin[1], Seramika Ariwahjoedi[1], Freddy P. Zen[1,2]

[1] Theoretical Physics Laboratory, THEPI Division, Department of Physics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia.
[2] Indonesia Center for Theoretical and Mathematical Physics (ICTMP), Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia.

E-mail: `ignsebastian96@gmail.com`

Abstract. Loop Quantum Gravity has become one of the alternative solutions to quantum gravity. This formulation introduced geometrical operators which successfully used to model that in the quantum scale, the space is actually discretized in the order of Planck length. These operators are area and volume operator. The regularization process of these operators came from the classical definition of area and volume, thus, the eigenvalues of area operator and volume operator are respectively the area and volume of the space. However, there exists two types of volume operator, the Ashtekar-Lewandowski operator and the Rovelli-Smolin operator. The significant difference between these two operators is the fact that Ashtekar-Lewandowski operator is sensitive to the direction of the spin networks link, while Rovelli-Smolin operator is not. This difference will produce different spectral. In this article, we compare the resulting spectral of the two volume operators, where both of them is used to calculate the volume of the monochromatic 4-valent and 6-valent spin network for the kinematical case.

1. Introduction

One of the most challenging problem in physics is the formulation of quantum gravity. There is a lot of attempts to solve this problem and Loop Quantum Gravity (LQG) acts as one of those alternative solutions. This approach combines quantum mechanics with gravity using two different formulations, the canonical and the covariant formulations. In this article, we consider the canonical formulation. It uses the 3+1 decomposition and tetrad formulation to rewrite the Einstein-Hilbert action [1]. From this formulation, one will reach one of the biggest achievement in LQG, which is the ability to model that space is actually discrete in the quantum scale [2, 3, 4]. This is a consequence of the introduction of the geometric operators, area and volume, which have discrete eigenvalues. The first attempt to build this operator had been done by Rovelli and Smolin [2]. They showed that by using the classical definition of area and volume, one could create quantum geometric operators. Ashtekar and Lewandowski also constructed these operators with different regularization and produced the same area operator but a different volume operator [[3, 4]]. Nevertheless, these operators are built through the loop representation in Hilbert space $L^2[SU(2)]$. Since the regularization of these operators starts from the classical

definition of area and volume, the eigenvalues will represent the size of area and volume of a space. Therefore, a space in a manifold Σ is discretized and formed by quanta of space [2, 3, 4].

In this article, we will calculate the volume of a monochromatic 4-vertex spin network and a monochromatic 6-vertex spin network using the Rovelli-Smolin volume operator $\hat{V}_{RS}$ and Ashtekar-Lewandowski volume operator $\hat{V}_{AL}$. We use $j = \frac{1}{2}$ and $j = 1$ for the 4-vertex spin network and $j = \frac{1}{2}$ for 6-vertex spin network. We will compare the spectral of $\hat{V}_{RS}$ and $\hat{V}_{AL}$ to see the difference between them. The scope of this article is restricted to the kinematical case, where we only consider the Gauss constraint and neglecting the two remaining constraints [1, 5].

2. Volume Operator

There are two types of volume operator, the Rovelli-Smolin volume operator $(\hat{V}_{RS})$ [2, 6, 7, 8] and the Ashtekar-Lewandowski volume operator $\hat{V}_{AL}$ [4, 9]. These operators came from the same classical definition of volume. However, the differences in the regularization process yield different forms of the operator. The differences between these two operators can be seen in the appearance of the sign factor in $\hat{V}_{AL}$, the coefficient in front of the operator, and the way the operator sums up the value for each link.

The classical definition of a volume of a region R in 3 dimensional Σ is defined as

$$V_R(x) = \int_R d^3x \sqrt{q(x)} \tag{1}$$

with q is the determinant of the spatial metric q^{ab}. Using the fact that q^{ab} is the inverse of q_{ab}, we can write q in terms of densitized triad's determinant E as follows:

$$q = E \tag{2}$$

Thus, equation (1) can be rewritten into:

$$\begin{aligned} V_R(x) &= \int_R d^3x \sqrt{E(x)} \\ &= \int_R d^3x \sqrt{\left| \frac{1}{3!} \varepsilon^{ijk} \varepsilon_{abc} E_i^a(x) E_j^b(x) E_k^c(x) \right|} \end{aligned} \tag{3}$$

In LQG, we can promote E to be an operator, so that we can rewrite equation (3) as follows:

$$\hat{V}_R(x) = \int_R d^3x \sqrt{\left| \frac{1}{3!} \varepsilon^{ijk} \varepsilon_{abc} \hat{E}_i^a(x) \hat{E}_j^b(x) \hat{E}_k^c(x) \right|} \tag{4}$$

This operator acts on an n-vertex spin network. However, the operator can not be diagonalized by the spin network state on the standard basis i.e., the direct-sum basis of SU(2) representation. Each operators for different spin network is diagonalized by different state vector. Therefore, the eigenvalue of this operator can not be expressed in a general form [2, 4].

2.1. Rovelli-Smolin Volume Operator

Rovelli and Smolin introduced the first version of this operator [2]. The development of this operator leads us to two types of operator [6, 7]. In this article, we will use the one introduced in [7]. After applying some regularization processes, one obtains the volume operator as:

$$\hat{V}_{RS} = \sum_{I<J<K} 3! \sqrt{\left| Z \hat{q}_{IJK} \right|} \tag{5}$$

$$\hat{q}_{IJK} = \varepsilon_{ijk} J_I^i J_J^j J_K^k = \frac{i}{4}\left[(J_{IJ})^2, (J_{JK})^2\right] \tag{6}$$

with $J_{IJ} = J_I + J_J$, $\left[(J_{IJ})^2, (J_{JK})^2\right] = \sum_{i,j=1}^3 \left[(J_I^i + J_J^i)^2, (J_J^j + J_K^j)^2\right]$ and $Z = \frac{i}{192}$.

2.2. Ashtekar-Lewandowski Volume Operator

On the other hand, the $\hat{V}_{AL}$ goes through different regularization. In this article, we used the derivation presented in [9]. In his article, Thiemman introduced a cube in the beginning of the regularization. However, each cube acts on different densitized triad, hence, one has three different cubes, each accompanied by a characteristic function. This approach will give different result although the angular momentum operator part inside the operator is equal [9]. Moreover, the operator will be slightly different than the $\hat{V}_{RS}$. It contains a new factor called the sign factor $\epsilon(e_I, e_J, e_K)$, where the value is determined by the cross product of three tangent vectors of the link at a vertex. Again, using the identity from (6), the $\hat{V}_{AL}$ can be written as

$$\hat{V}_{AL} = \sqrt{\left|3!Z \sum_{I,J,K} \epsilon(I,J,K)\hat{q}_{IJK}\right|} \tag{7}$$

Notice that there is a regularization coefficient Z in each operators ($\hat{V}_{RS}$ and $\hat{V}_{AL}$). The value of this coefficient varies depending on the regularization process. Nevertheless, Giesel and Thiemann [7, 8] found that there is a fixed value for Z, considering the consistency of the volume operator, triad, and flux quantizations. The value is $Z = \frac{i\beta^3}{4}C_{reg}$ with β is the Immirzi parameter and $C_{reg} = \frac{1}{3!8}$. However, it has been proved that $\hat{V}_{RS}$ is not consistent if one consider the dynamical part of the theory. Nevertheless, it is still very useful for the kinematical case. In the following calculation, we will consider the value of $Z = 1$ to simplify the calculation.

3. Spectral of Volume Operator

In this section, we will calculate the spectral of volume operator in monochromatic 4-vertex and 6-vertex spin network using the Rovelli-Smolin volume operator ($\hat{V}_{RS}$) and the Ashtekar-Lewandowski volume operator ($\hat{V}_{AL}$). Since the form of these operators is different, the eigenvalue will also be different. Our calculation follows the steps introduced in [10, 11].
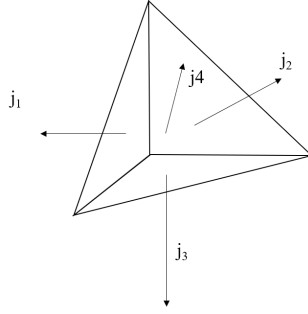
3.1. Spectral of 4-vertex

The direction of links in this spin network are chosen outwards such that it represents a real tetrahedron. We label each link with spin j_1, j_2, j_3, j_4. However, since we deal with monochromatic spin network, all links will have the same spin, j. Moreover, since we consider the kinematical case, the gauge invariance condition is imposed:

$$\hat{J}_1 + \hat{J}_2 + \hat{J}_3 + \hat{J}_4 = 0 \tag{8}$$

With this condition, the total value of spin j_{1234} will become 0 and the value of spin j_{123} will equal to j_4. This condition also create a new identity from eq. (6), that is:

$$\hat{q}_{IJJ} + \hat{q}_{IJI} = \hat{q}_{IJJ} - \hat{q}_{JII}$$
$$= 0 \tag{9}$$

Using eq. (8) and eq. (9), the volume operator will become:

Figure 1 The Tetrahedron

$$\hat{q}_{124} = -(\hat{q}_{121} + \hat{q}_{122} + \hat{q}_{123}) = -\hat{q}_{123}$$
$$\hat{q}_{134} = -(\hat{q}_{131} + \hat{q}_{132} + \hat{q}_{133}) = -\hat{q}_{132}$$
$$\hat{q}_{234} = -(\hat{q}_{231} + \hat{q}_{232} + \hat{q}_{233}) = -\hat{q}_{231} \tag{10}$$

Therefore, the only combination of links which needs to be calculated is $I, J, K = 1, 2, 3$.

The calculation of the eigenvalues of volume operator starts from finding the matrix form of the operator for the 4-vertex. This calculation uses the standard bases and the completeness properties. The matrix form can be calculated as follows:

$$\langle \vec{g}(12)|\hat{q}_{123}|\vec{g}'(12)\rangle = \langle \vec{g}(12)|\big[J_{12}^2, J_{23}^2\big]|\vec{g}'(12)\rangle$$
$$= \big[g_2(12)(g_2(12) + 1) - g_2'(12)(g_2'(12) + 1)\big]$$
$$\sum_{\vec{g}(23)} g_2(23)(g_2(23) + 1)$$
$$\langle g_2(j_2, j_3), g_3(g_2, j_1) = j_4|g_2'(j_1, j_2), g_3'(g_2', j_3) = j_4\rangle$$
$$\langle g_2(j_2, j_3), g_3(g_2, j_1) = j_4|g_2'(j_1, j_2), g_3'(g_2', j_3) = j_4\rangle \tag{11}$$

Then, using the properties of the 6j symbols, eq. (11) will become:

$$\langle \vec{g}(12)|\hat{q}_{123}|\vec{g}'(12)\rangle = \big[g_2(12)(g_2(12) + 1) - g_2'(12)(g_2'(12) + 1)\big]$$
$$\sum_{\vec{g}(23)} g_2(23)(g_2(23) + 1)\big[(2g_2(12) + 1)(2g_2(23) + 1)\big]^{\frac{1}{2}}\big[(2g_2'(12) + 1)(2g_2(23) + 1)\big]^{\frac{1}{2}}$$
$$(-1)^{j_1+j_2+j_3+j}(-1)^{j_1+j_2+j_3+j_4}\begin{Bmatrix} j_1 & j_2 & g_2(12) \\ j_3 & j_4 & g_2(23) \end{Bmatrix}\begin{Bmatrix} j_2 & j_3 & g_2(23) \\ j_1 & j_4 & g_2'(12) \end{Bmatrix} \tag{12}$$

with $g_2(12) = j_{12}, g_2'(12) = j_{12}'$ and $g_2(23) = j_{23}$.

3.1.1. The $j = \frac{1}{2}$ Case. In this case, we will evaluate the eigenvalue of volume operator for 4-vertex using $j_1 = j_2 = j_3 = j_4 = \frac{1}{2}$. This condition gives us the possible value of j_{12} and j_{23} which will be used in the calculation.

$$0 \le j_{12} \le 1 \qquad , 0 \le j_{23} \le 1 \tag{13}$$

Using this possible value and eq. (12), therefore, the matrix of the volume operator of 4-vertex is:

$$Q_{123}^{j_{12}j'_{12}} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix} \tag{14}$$

The matrix's eigenvalue is $\lambda = \pm i\sqrt{3}$. Then, the matrix will be used to calculate the operator $\hat{V}_{RS}$ and $\hat{V}_{AL}$. First, using eq. (5), we will calculate the operator $\hat{V}_{RS}$ as follows:

$$\begin{aligned} \hat{V}_{RS} &= 3!(\sqrt{|q_{123}|} + \sqrt{|q_{124}|} + \sqrt{|q_{134}|} + \sqrt{|q_{234}|}) \\ &= 4 \times 3!\sqrt{|q_{123}|} \end{aligned} \tag{15}$$

For the next step, the eigenvalue of $\hat{V}_{AL}$ will be calculated. Using eq. (7), we have:

$$\begin{aligned} \hat{V}_{AL} &= \sqrt{3!(|\epsilon_{123}\hat{q}_{123} + \epsilon_{124}\hat{q}_{124} + \epsilon_{134}\hat{q}_{134} + \epsilon_{234}\hat{q}_{234}|)} \\ &= \sqrt{3!|(\epsilon_{123} - \epsilon_{124} + \epsilon_{134} - \epsilon_{234})\hat{q}_{123}|} \end{aligned} \tag{16}$$

By using the direction of the link which has been determined in the beginning of the calculation, we have:

$$\begin{aligned} \hat{V}_{AL} &= \sqrt{3!|(1 - (-1) + 1 - (-1))\hat{q}_{123}|} \\ &= 2\sqrt{3!|\hat{q}_{123}|} \end{aligned} \tag{17}$$

Hence, the eigenvalue of $\hat{V}_{RS}$ and $\hat{V}_{AL}$ for this case are:

$$v_{RS} = 24\sqrt{\sqrt{3}} = 31.586 \tag{18}$$

$$v_{AL} = 2\sqrt{3!\sqrt{3}} = 6.447 \tag{19}$$

3.1.2. The $j = 1$ Case. In this case, we use $j_1 = j_2 = j_3 = j_4 = 1$, so that

$$0 \leq j_{12} \leq 2 \qquad , 0 \leq j_{23} \leq 2 \tag{20}$$

With this condition, first, we will calculate the condition for $j_{12=1,j'_{12}=0}$:

$$\begin{aligned} Q_{123}^{10} &= 12\sqrt{3} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{Bmatrix} + 60\sqrt{3} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} \\ &= \frac{8\sqrt{3}}{3} \end{aligned} \tag{21}$$

After that, for $j_{12} = 2, j'_{12} = 0$

$$\begin{aligned} Q_{123}^{20} &= 36\sqrt{5} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{Bmatrix} + 180\sqrt{5} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix} \\ &= 0 \end{aligned} \tag{22}$$

Then, for $j_{12} = 2, j'_{12} = 1$

$$\begin{aligned} Q_{123}^{21} &= 24\sqrt{15} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} + 120\sqrt{15} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{Bmatrix} \\ &= \frac{4}{3}\sqrt{15} \end{aligned} \tag{23}$$

For $j_{12} = j'_{12}$, the matrix's element will become 0 and if we switch the value of j_{12}, j'_{12}, the matrix's elements will have the same value but with different sign. Therefore, we have:

$$Q_{123}^{j_{12}j'_{12}} = \begin{pmatrix} 0 & \frac{8}{3}\sqrt{3} & 0 \\ -\frac{8}{3}\sqrt{3} & 0 & \frac{4}{3}\sqrt{15} \\ 0 & -\frac{4}{3}\sqrt{15} & 0 \end{pmatrix} \tag{24}$$

The eigenvalues of the matrix are $\lambda = 0, \pm i 6.928$. Hence, the eigenvalues of $\hat{V}_{RS}$ and $\hat{V}_{AL}$ are:

$$v_{RS} = 24\sqrt{6.928} = 63.171 \tag{25}$$

$$v_{AL} = 2\sqrt{3! \times 6.928} = 12.895 \tag{26}$$

3.2. Spectral of 6-vertex

In this case, we define a monochromatic 6-vertex spin network with all of its links' direction are chosen outward. We label each link with $j_1, j_2, j_3, j_4, j_5, j_6$. For this case, we only calculate $j = \frac{1}{2}$. Using the gauge invariance condition, we have:

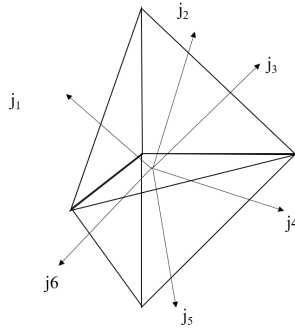$$\hat{J}_1 + \hat{J}_2 + \hat{J}_3 + \hat{J}_4 + \hat{J}_5 = -\hat{J}_6 \tag{27}$$

Figure 2 Two Tetrahedrons Glued Together

so the total spin $j = 0$ and $g_5(12) = j_{12345} = j_6$. Then, with the properties of $\hat{q}_{IJK}$ from eq. (9), the volume operator which works to the 6-vertex will become:

$$\hat{q}_{126} = -(\hat{q}_{121} + \hat{q}_{122} + \hat{q}_{123} + \hat{q}_{124} + \hat{q}_{125}) = -\hat{q}_{123} - \hat{q}_{124} - \hat{q}_{125}$$
$$\hat{q}_{136} = -(\hat{q}_{131} + \hat{q}_{132} + \hat{q}_{133} + \hat{q}_{134} + \hat{q}_{135}) = \hat{q}_{123} - \hat{q}_{134} - \hat{q}_{135}$$
$$\hat{q}_{146} = -(\hat{q}_{141} + \hat{q}_{142} + \hat{q}_{143} + \hat{q}_{144} + \hat{q}_{145}) = \hat{q}_{124} + \hat{q}_{134} - \hat{q}_{145}$$
$$\hat{q}_{156} = -(\hat{q}_{151} + \hat{q}_{152} + \hat{q}_{153} + \hat{q}_{154} + \hat{q}_{155}) = \hat{q}_{125} + \hat{q}_{135} + \hat{q}_{145}$$
$$\hat{q}_{236} = -(\hat{q}_{231} + \hat{q}_{232} + \hat{q}_{233} + \hat{q}_{234} + \hat{q}_{235}) = -\hat{q}_{123} - \hat{q}_{234} - \hat{q}_{235}$$
$$\hat{q}_{246} = -(\hat{q}_{241} + \hat{q}_{242} + \hat{q}_{243} + \hat{q}_{244} + \hat{q}_{245}) = -\hat{q}_{124} + \hat{q}_{234} - \hat{q}_{245}$$
$$\hat{q}_{256} = -(\hat{q}_{251} + \hat{q}_{252} + \hat{q}_{253} + \hat{q}_{254} + \hat{q}_{255}) = -\hat{q}_{125} + \hat{q}_{235} + \hat{q}_{245}$$
$$\hat{q}_{346} = -(\hat{q}_{341} + \hat{q}_{342} + \hat{q}_{343} + \hat{q}_{344} + \hat{q}_{345}) = -\hat{q}_{134} - \hat{q}_{234} - \hat{q}_{345}$$
$$\hat{q}_{356} = -(\hat{q}_{351} + \hat{q}_{352} + \hat{q}_{353} + \hat{q}_{354} + \hat{q}_{355}) = -\hat{q}_{135} - \hat{q}_{235} + \hat{q}_{345}$$
$$\hat{q}_{456} = -(\hat{q}_{451} + \hat{q}_{452} + \hat{q}_{453} + \hat{q}_{454} + \hat{q}_{455}) = -\hat{q}_{145} - \hat{q}_{245} - \hat{q}_{345} \tag{28}$$

In the same way as before, the calculation of eigenvalues starts by calculating the element of the operator. Since the operator depends on 6 different indices, it will become a tensor with 6 indices $Q^{(g_2 g_3 g_4 g_2' g_3' g_4')}$. To find the eigenvalues, we can use the tensor unfolding process, thus the 6-rank tensor will become a matrix [12]. The calculation of this case will follow this matrix:

$$
\begin{bmatrix}
111111 & 111112 & 111121 & 111122 & 111211 & 111212 & 111221 & 111222 \\
112111 & 112112 & 112121 & 112122 & 112211 & 112212 & 112221 & 112222 \\
121111 & 121112 & 121121 & 121122 & 121211 & 121212 & 121221 & 121222 \\
122111 & 122112 & 122121 & 122122 & 122211 & 122212 & 122221 & 122222 \\
211111 & 211112 & 211121 & 211122 & 211211 & 211212 & 211221 & 211222 \\
212111 & 212112 & 212121 & 212122 & 212211 & 212212 & 212221 & 212222 \\
221111 & 221112 & 221121 & 221122 & 221211 & 221212 & 221221 & 221222 \\
222111 & 222112 & 222121 & 222122 & 222211 & 222212 & 222221 & 222222
\end{bmatrix}
\tag{29}
$$

The index 1 and 2 show the possible components of the $g_2 = j_{12}, g_3 = j_{123}$ and $g_4 = j_{1234}$ as well as $g_2' = j_{12}', g_3' = j_{123}'$ and $g_4' = j_{1234}'$. Then, we use Brunnemann and Thiemann's general formula in [10] to calculate the matrix element of the volume operator. Therefore, using this general formula and the method from [12], we will have

$$
v_{RS} = 46.723, 146.302, 149.542, 161.200, 181.366
\tag{30}
$$

$$
v_{AL} = 7.406, 9.460;
\tag{31}
$$

4. Conclusion

Calculations in the previous section give a result that in 4-vertex case, the eigenvalues of $\hat{V}_{RS}$ are always larger than the eigenvalues of $\hat{V}_{AL}$, as can be seen from eq. (15) and (17). The eigenvalues of 6-vertex spin network which is calculated in the previous section also gives the same result, that $\hat{V}_{RS}$'s eigenvalues are bigger than $\hat{V}_{AL}$. However, this result may vary depending on the link's orientation, and moreover, this conclusion needs to be validated by other choices of link orientation.

Another fact that can be seen from the calculation is that for $j = \frac{1}{2}$, the smallest eigenvalue of 6-vertex and all of 4-vertex's eigenvalues which are calculated by using $\hat{V}_{RS}$ satisfies $v_{4-vertex} + v_{4-vertex} > v_{6-vertex}$ inequality. However, others eigenvalues of 6-vertex satisfies $v_{4-vertex} + v_{4-vertex} < v_{6-vertex}$ inequality. Thus, although the classical geometric shape of 6-vertex is actually the same as two tetrahedron added together, the volume of those two shapes is not equivalent. The calculation of $\hat{V}_{AL}$'s eigenvalues shows similar result with a difference. The eigenvalue of 4-vertex and 6-vertex satisfies $v_{4-vertex} + v_{4-vertex} > v_{6-vertex}$ inequality. These results show that 4-vertex and 6-vertex are unique in the sense that we can not form 6-vertex's volume from 4-vertex's. Thus, the choice of the discretization of space affects the volume of the actual space. Moreover, the results are not in accordance with classical conditions. In the classical geometry, two tetrahedron which are glued together on one side will have the same volume as two tetrahedron with the same size. However, in this case, we get the inequality $v_{4-vertex} + v_{4-vertex} > v_{6-vertex}$ for $\hat{V}_{AL}$, also the inequality $v_{4-vertex} + v_{4-vertex} > v_{6-vertex}$ for the smallest eigenvalue of 6-vertex and $v_{4-vertex} + v_{4-vertex} < v_{6-vertex}$ for the rest of 6-vertex's eigenvalues from $\hat{V}_{RS}$. Nonetheless, this condition only applies for $j = \frac{1}{2}$. This might be an effect of the quantum fluctuation of geometries arising from the uncertainty principle. It might be interesting to carry out the research on this subject further.

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