Amplitudes in QFT and CFT

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To my friend Sourav

»Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018 angegebenen Hilfsmittel angefertigt habe.«

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Zusammenfassung

In dieser Dissertationsschrift werden Amplituden in Quantenfeldtheorien und konformen Feldtheorien (CFTs) studiert. Zunächst wird mittels der **»double copy«**-Methode gezeigt, inwiefern Integranden von Gravitationsamplituden aus Integranden von Eichtheorien gewonnen werden können. Um diese Methode anzuwenden, bedarf es einer konkreten Darstellung der Eichtheorieintegranden, sodass die kinematischen Faktoren des Integranden die gleichen algebraischen Relationen erfüllen wie die Strukturkonstanten der Eichtheorie; insbesondere muss die Jacobiidentität erfüllt sein. Mithilfe dieser Methode werden Vierpunktsamplituden in $\mathcal{N} = 0$ Supergravitation gekoppelt mit Yang-Mills in erster Ordnung der Störungsreihe berechnet, welche als asymptotische Zustände Gravitonen oder Gluonen positiver Helizität enthalten. Weiterhin ist es möglich die virtuellen Axionen und Dilatonen in Spezialfällen zu entfernen, sodass einige Resultate in reiner Einstein-Yang-Mills präsentiert werden können.

Das Analogon der Amplituden in konform invarianten Theorien, genannt **Mellinamplituden**, wird anschließend im zweiten Teil diskutiert. Nicht nur durch ihre Beschreibung als Funktionen »lorentzinvarianter« Variablen, welche durch eine Art »LSZ-Reduktion« gewonnen werden, weisen Mellinamplituden eine formal heuristische Ähnlichkeit zu Amplituden auf, sondern auch können alle physikalischen Größen einer CFT aus ihnen berechnet werden. D.h., ebenso wie Amplituden einen Streuprozess vollständig charakterisieren, ist eine CFT eindeutig über ihre Mellinamplituden festgelegt. Fermionische Mellinamplituden wurden zum ersten Mal, in der Veröffentlichung worauf diese Dissertationsschrift basiert, studiert. Jede Komponente der fermionischen Mellinamplituden ist einer bestimmen Tensorstruktur zugeordnet, deren Polstruktur im einzelnen diskutiert wird. Es werden die analytische Eigenschaften der fermionischen Mellinamplituden der gemischten Vierpunktskorrelationsfunktion von zwei Fermionen und Skalaren, sowie von vier Fermionen studiert und darauffolgend werden diese Resultate durch störungstheoretische Rechnungen bei schwacher und starker Kopplung bestätigt.

Abstract

In this thesis, amplitudes in quantum field theory (QFT) and conformal field theory (CFT) are studied. In the first chapter a modern technique to obtain integrands for gravity theories from gauge theory integrands is discussed. This formalism is called the **doubly copy** method and it can be applied if the gauge theory integrand is given in a specific representation where the kinematic numerator factors obey the same algebraic relations as the colour factors, e.g. the Jacobi identity. This method is applied to obtain the positive helicity sector of amplitudes in $\mathcal{N} = 0$ supergravity coupled to Yang-Mills with external gravitons and gluons at one loop. Only the special case of four external particles is studied. Partial results are also obtained for pure Einstein-Yang-Mills amplitudes, where the axion and dilaton as virtual particles have been removed.

In the second chapter, the natural analogue of amplitudes in CFTs is studied. These mathematical objects are called **Mellin amplitudes**. Mellin amplitudes can be understood as the CFT analogue of QFT amplitudes, because they are functions of "Lorentz invariant" quantities of their "momenta". In addition all the CFT data is encoded in the Mellin amplitudes as all the data of a scattering process is included in usual amplitudes. The study of fermionic Mellin amplitudes has been carried out for the first time in the associated publication. These Mellin amplitudes have several components each associated to a certain tensor structure. The analytic properties of fermionic Mellin amplitudes corresponding to mixed four fermion-scalar conformal correlators and four fermion conformal correlators are deduced and finally these general results are confirmed by explicit perturbative calculations at weak and strong coupling.

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Publications by the author

This thesis is based on the following publications by the author:

[1] J. Faller, S. Sarkar, and M. Verma, “Mellin Amplitudes for Fermionic Conformal Correlators,” JHEP **03** (2018) 106, `arXiv:1711.07929 [hep-th]`.

[2] J. Faller, J. Plefka “Positive helicity Einstein-Yang-Mills amplitudes from the double copy method,” Phys. Rev. D **99**, 046008 `arXiv:1812.04053 [hep-th]`.

Software

- This thesis has been typeset with LaTeX.
- Several calculations have been carried out with Mathematica by Wolfram Research including the packages FeynCalc [3, 4] and S@M [5].
- Graphics have been created with FeynMF, TikZ and Inkscape.

Contents

Chapter 1

Introduction

Physics describes the phenomena of nature in mathematical terms and verifies its claims by experiments. The mathematical formulation of these phenomena is realized by abstract models. The modelling process is reductionistic and it assigns a quantity to the phenomena which also appears in the mathematical equations. E.g. in the solar system the force of gravity is assigned to the planet which pulls it towards the sun, but exactly this force also appears in Newton's second law. In this sense the model can also be seen as an interpretation. Ideally the construction is based on a theory which includes fundamental principles and instructions on how to use these principles to apply them to the observed empirical data. However, the empirical data is not sufficient to uniquely specify a theory. Therefore different theories which model the same class of phenomena can predict the same observables [6, p. 76] [7, p. 168f]. There are plenty of examples like string dualities, particle-wave duality, Bohmian and Kopenhagener quantum mechanics etc. which illustrate this. This is well studied in [6]. It is important to note that the ontology of these theories might be very different [6, p. 101f, 206f]. Therefore a physicist has the freedom to choose the kind of theory to model a certain natural phenomenon. The chosen model has to satisfy at least two requirements to be useful in physics: Firstly the dynamics of the system should be formulated in such a way that it can be solved (analytically or numerically, completely or perturbatively etc.). Secondly all the essential features of the phenomenon have to be describable within this model.

Nowadays local quantum field theory (QFT) is the most efficient framework to incorporate these ideas in high energy physics and it is also frequently used for condensed matter and statistical systems. In this thesis the focus lies in the realm of high energy physics. The importance of QFT stems from two observations: Firstly, the fundamental principles of quantum mechanics, Lorentz invariance and cluster decomposition necessarily imply the possible description of relativistic phenomena by a QFT [8, 9]. Although this description might break down at high energies, it is still useful to model low energy physics according to the folk theorem given by S. Weinberg: "It is very likely that any quantum theory that at sufficiently low energy and large distances looks Lorentz invariant and satisfies the cluster decomposition principle will also at sufficiently low energy *look* like a quantum field theory [9, p. 8]." Secondly, it combines the two aforementioned necessary requirements to construct models. The basic principles and the mathematical methods of QFT provide a description of processes at a characteristic energy scale such that the essential features of it are described. This is established by constructing the most general Lagrangian consistent with the symmetries of the process which allows to calculate expectation values at least perturbatively. In addition, this Lagrangian defines the fundamental degrees of freedom at each scale, because the fields which appear in the Lagrangian are elementary [9]. The success of this approach is evident in particle physics, e.g. collision experiments are well described by perturbative methods of QFT. The mathematical object to study these collision experiments is the unitary S-matrix $S = \mathbb{1} + iT$ which maps in-states to out-states [10–13]. The non-trivial part of a scattering process is included in the transfer matrix T and its matrix elements are amplitudes. Hence, the art of calculating these amplitudes is a very crucial part in theoretical high energy physics to test the validity of specific models constructed using QFT.

The need for a detailed and accurate theoretical comprehension of collision experiments is based on the facts that these experiments are among the most conducted ones in the current field of high energy physics and that in general the success of experimental physics is intertwined with the theoretical development of the field itself. These experiments paved the way to the most significant theoretical model constructed by quantum field theorists: the Standard Model of particle physics. Its development started by successfully combining the electromagnetic and weak interaction by Glashow [14] and its inclusion of the Higgs-Englert-Brout-mechanism [15, 16] by Weinberg and Salam [17, 18]. Among other things the importance of scattering experiments results from the fact that the last elementary constituents of the Standard Model were experimentally verified in colliders, like the top quark [19], the tau neutrino [20] and the Higgs boson [21, 22].

Weinberg's definition of QFT as the description of low energy physics combines well with Wilson's notion of QFT as a way to parametrize the effective degrees of freedom of a system only in form of an effective field theory (EFT). This concept along with the renormalization group method has shaped the modern understanding of QFT and it paves the way to understand QFT beyond perturbation theory [23–25]. This idea is implemented by constructing the most general Lagrangian consistent with the symmetries of the system and integrating out high momentum fluctuations. The different coupling constants of the Lagrangian may start to grow and become relevant at a certain energy scale. At each scale the system is described by an EFT. A change in scale is obtained by a so-called renormalization group transformation (RGT). It is interesting to note that many high energy EFTs may lead to the same low energy EFT. Hence, in the process of integrating out high momentum fluctuations information about the system is lost. Contrary to a Feynman diagrammatic expansion, Wilson's approach does not treat all wave lengths the same way, because it integrates out high energy modes [26].

This approach is well suited to describe physical systems at criticality (for example phase transitions) near the fixed point of the RGT. At the critical surface the correlation length diverges such that there is no natural scale associated to the model any more; hence the system is scale invariant and does only depend on universal properties, but not on the details of the interaction. Critical exponents which describe physical quantities near phase transitions are universal. Close to the fixed point the RGTs can be used to determine the critical exponents of the system [26, 27]. If scale invariance is enhanced to conformal invariance, systems at criticality can be described by conformal field theory (CFT) [28]. For example this is the case if the energy momentum tensor of the model can be made traceless [29]. Thus CFTs can be used to study non-perturbative phenomena like phase transitions.

Not only is the goal of physics to simply describe physical systems, but also to understand the fundamental principles and laws which govern them. Hence, the construction of a coherent and all-encompassing theory which unifies all four known interactions is one of the major aims in theoretical physics. The next step to achieve this goal is the unification of gravity with the strong, weak and electromagnetic interaction in a theory of quantum gravity. So far there is no satisfying theory of quantum gravity, the best current candidate can be found in the framework of string theory. Especially, in the early 90^{th} Maldacena conjectured that superconformal gauge theories in the large N limit, N being the number of colours, can be used to describe supergravity in anti de Sitter (AdS) space which is the low energy limit of string theory [30]. This idea, developed further by [31, 32], is one of most active research areas in high energy physics nowadays. It is intriguing that the conjectured AdS/CFT correspondence can be used to define quantum gravity with asymptotically AdS boundary conditions by a CFT. This highlights the importance of CFTs again, because it implies that quantum gravity in AdS can be seen as emerging from a boundary CFT. It has to be emphasized that both theories describe the same observables. The AdS/CFT correspondence is a strong/weak duality which implies that strongly coupled systems in the CFT are weakly coupled in the dual gravity theory and vice versa. Especially, in the classical limit where the gravity theory can be solved, this duality enables one to study the dual CFT at strong coupling in the planar limit. Perturbatively the strong coupling regime of a CFT can be studied by Witten diagrams due to this duality [32–34]. Even though this correspondence is not proven yet, it has led to a lot of insight into current physical research problems, like the black hole information paradox [35, 36], holography [37, 38], the emergence of (approximate) symmetries [39] etc.

Therefore, to understand physics on a fundamental level as well as for phenomenology the study of amplitudes in QFT and to find an appropriate description for CFTs is of great importance and demands further investigation. These two aspects are diverse fields of research in their own right and have developed massively in the recent years as it is discussed in the following.

The study of **amplitudes** has a long history. Nowadays, the canonical algorithm to compute these amplitudes perturbatively is based on Feynman diagrams which are the graph-theoretical building blocks in this perturbative expansion. Although, in principle, this algorithm works to all orders in perturbation theory, it is, in general, not very efficient. E.g. in quantum electrodynamics the number of diagrams grows rapidly in terms of its loop order L. Asymptotically it is of the order $\mathcal{O}(L!)$ [40]. Therefore, physicists have developed various tools to obtain the amplitude by different, more efficient methods. To name a few, the methods are ranging from standard techniques like colour decomposition [41, 42], spinor-helicity formalism [43, 44], supersymmetric identities [45, 46], Berends-Giele recurrence relations [47] and generalized unitarity [48–50] to more modern approaches, e.g. twistor space methods [51, 52], BCFW recurrence relations [53, 54], the amplituhedron [55, 56], CHY formalism [57–59] and the double copy method [60, 61]. However, all of this methods lack some sort of completeness. Either they can only be applied to special kind of particle types, to particular dimensions, to a limited order in perturbation theory or they still need information which is obtained from Feynman diagrams. Therefore it can be said that, although a lot of technical progress has been made in the calculation of amplitudes, Feynman diagrams still form the basis of this field.

In this thesis the focus is on the calculation of gravity amplitudes. The computation of gravity amplitudes using conventional Feynman diagrams is even harder than it is for the other three fundamental interactions, because for gravity it is already difficult to construct the Feynman integrand [62]. An efficient method to determine the gravity integrand has been proposed by Bern, Carrasco and Johansson in [60, 61]. The general idea is to construct gravity integrands from gauge theory integrands where the latter are much easier to build. This method has triggered a broad field of interest: It has made the direct calculation of many gravity amplitudes possible or easier [63–70] and it has been extended to classical physics too [71–79] like to the study of perturbative black hole solutions.

Concretely, in this thesis all amplitudes in $\mathcal{N} = 0$ supergravity coupled to Yang-Mills with four external positive-helicity gluons and gravitons are calculated at one-loop order using the aforementioned double copy method [2]. This extends the work done in [80] where pure Einstein-Yang-Mills amplitudes with four external particles have been calculated at one-loop in leading order of the Einstein constant.

Even though there are doubts about a graviton being detectable [81, 82], there are still experimental reasons to study gravity scattering amplitudes. Recently, there has been a growing interest in relating amplitudes to classical, perturbative solutions of gravity theories [83–89] which extends earlier approaches [90–92]. The interest has been enhanced due to the recent detection of gravitational waves by LIGO and VIRGO [93, 94].

In order to solve a concrete physical problem, a formulation of it in appropriate mathematical terms which concisely incorporate the dynamics of the system, is even more important than the different calculation techniques. An adequate choice of language can simplify or even trivialize physical statements like the covariant formulation of Maxwell's equations shows. Here it is argued that the Mellin-Barnes representation, notably the **Mellin amplitude**, provides an efficient formalism to study CFTs in dimensions higher than two.

Recall that the complete information to construct any n-point correlation function in a CFT is given by the CFT data. The CFT data is the combined knowledge of all the operator product expansion (OPE) coefficients and the spectrum of local primary operators, i.e. their scaling dimensions and their Lorentz representations [95–98]. A mathematical formalism which makes this data manifest is a proper way to describe CFTs; and the Mellin-Barnes representation provides such a formalism. This representation has been studied by Mack in [99, 100] systematizing earlier ideas presented by Symanzik [101]. In the Mellin-Barnes representation of conformal correlation functions the complete spectrum and the OPE coefficients are encoded in the analytic structure of the Mellin amplitude. The location of its poles gives the twist of the exchanged operators and its residues at these poles give the OPE coefficients as well as their Lorentz representations.

Moreover the formal structure of Mellin amplitudes is very similar to massive momentum space scattering amplitudes. According to this analogy the Mellin variables become products of fictitious momenta and the scaling dimensions of the exchanged single trace operators play the role of the masses in massive QFTs. This observation has stimulated many research ideas: It has been argued that Mellin amplitudes are the proper object to define scattering amplitudes in AdS. These ideas were further developed by constructing perturbative "Feynman rules" in Mellin space in the weakly interacting gravity theory which is dual to the strongly coupled CFT [102, 103]. "Feynman rules" in the weak coupling regime of the CFT have been determined in [104]. This shows that the Mellin-Barnes representation is in particular fruitful in studying the AdS/CFT correspondence. Further progress in this direction has been achieved by investigating Regge theory in AdS [105] and the flat-space limit in AdS which relates Mellin amplitudes to QFT amplitudes in flat space. The flat-space limit refers to a scattering process where all length scales are much smaller than the radius of AdS [106–108]. Further, the flat-space limit has shed some light on a broader field of applications of Mellin amplitudes. It has been used to derive bounds on the cubic coupling constants of the flat-space scattering matrix of massive QFTs in two dimensions by combining it with the program of numerical conformal bootstrap [109]. In general the Mellin amplitude is not a meromorphic function, because the spectrum of a generic CFT contains "double twist" operators which accumulate for increasing spin $l \to \infty$ [110, 111] and the location of the poles of a Mellin amplitude is dictated by the twist of the exchanged operators which implies that the location of the poles of the Mellin amplitude accumulate. Hence it cannot be meromorphic [112]. However, in the large N limit of conformal gauge theories the analytic structure of a Mellin amplitude simplifies drastically, because it describes the spectrum of single trace operators only whose values of twist do not accumulate. Thus the Mellin amplitude is meromorphic in the large N limit. The nice analytic structure of Mellin amplitudes in this limit is reminiscent of the structure of tree-level amplitudes as manifest in the BCFW recursion relations [53, 54]. In BCFW recursion relations the amplitude (as a complex function) is described by a sum over simple poles. The residues at these poles are given by lower-point amplitudes. Thus it is a tempting challenge to prove similar recursion relations for Mellin amplitudes. Even though this similarity is striking, there is still no proof. A first step to analyze the factorization properties of Mellin amplitudes more closely has been taken in [113]. Another field of current interest is to rephrase the conformal bootstrap ideas stated by Polyakov [96] in Mellin-Barnes representation [114, 115]. This is one of the most promising applications of Mellin amplitudes [116–122].

However, most of these research ideas have been applied to scalar Mellin amplitudes only. There has been less progress in studying spinning Mellin amplitudes [113, 123]. Especially tackling the problem of fermionic Mellin amplitudes had not been considered thus far. But to obtain information about fermionic CFT data using Mellin amplitudes, it is necessary to define Mellin amplitudes for fermionic operators and to examine their analytic structure. This is discussed in chapter 3 based on the paper [1].

This thesis has two main chapters. In chapter 2, it is explained how to obtain amplitudes in $\mathcal{N} = 0$ supergravity coupled to Yang-Mills at one-loop in the all-plus helicity sector using the double copy method. This chapter starts with the definition and different properties of amplitudes in QFT. Afterwards, in section 2.2, the regularization procedure of these quantities is discussed. In section 2.3 it is shown how tensorial Feynman diagrams can be decomposed into a basis of scalar Feynman diagrams. In the present case this procedure reduces all Feynman integrals of section 2.5 to known scalar integrals. In section 2.4 the double copy method is introduced. The concrete realization of this double copy method for $\mathcal{N} = 0$ supergravity coupled to Yang-Mills is stated in 2.5. In the last section the corresponding amplitudes at one-loop are calculated. This chapter is also supplemented by two appendices: In appendix A.1 more properties of the spinor-helicity formalism are studied and appendix A.2 contains further details of the calculations done in section 2.5.

In chapter 3, Mellin amplitudes for fermionic conformal correlation functions are studied. Section 3.1 is a short summary of general properties of CFTs. This discussion is followed by presenting a method to determine a basis of tensor structures of conformal correlators in section 3.2. Furthermore, a concrete basis for three- and four-point correlators in three dimensions is given. In subsection 3.3.1 scalar Mellin amplitudes are introduced. A general definition for spinning Mellin amplitudes is stated in subsection 3.3.2. In the following subsections the pole structure of fermionic Mellin amplitudes is analyzed in three dimensions. The results of this analysis are confirmed by perturbative calculations in terms of Witten diagrams and conformal Feynman diagrams which are computed in section 3.4. More details about the conformal algebra are spelled out in appendix B.1. To construct a basis of tensor structures a concrete representation of the Lorentz algebra has to be given. This is done in appendix B.2. In appendix B.3 it is illustrated how to calculate the Mellin amplitudes of scalar contact diagrams at weak and strong coupling. In the last appendix B.5 the spinor exchange Witten diagram is calculated in detail.

Remark: The signature of the metric is chosen like it is common in the respective research fields. I.e. in chapter 2 the signature reads $(+,-,-,\dots)$ except the general analysis of dimensional regularization and in chapter 3 general properties of CFTs are explained in Euclidean signature $(+,+,+,\dots)$ whereas the concrete analysis of Mellin amplitudes is performed in Lorentzian signature $(-,+,+)$.

Chapter 2

Amplitudes

In this chapter the signature of Minkowski space $\mathbb{R}^{1,3}$ is $(+,-,-,-)$. Thus a scalar product of four-vectors $q, p \in \mathbb{R}^{1,3}$ can be decomposed into $p \cdot q = p^0 q^0 - \mathbf{p} \cdot \mathbf{q}$. Further, Euclidean vectors $\mathbf{q} \in \mathbb{R}^d$ are denoted in bold whereas Minkowskian vectors are not typographically emphasized and therefore they are written like scalar quantities.

In section 2.1 the notion of amplitudes is defined and the relevant properties of them are explained. The next section 2.2 explains dimensional regularization, which is needed to obtain finite, physical results from divergent expressions. The general concept of dimensional regularization for scalar amplitudes is discussed in Euclidean signature in section 2.2.1, whereas the four-dimensional-helicity scheme is described in Minkowskian signature in section 2.2.2. An important concept to calculate amplitudes of non-scalar fields at one-loop is the technique of Veltman-Passarino which is explained in section 2.3. The remaining sections deal with the calculation of amplitudes in $\mathcal{N} = 0$ supergravity coupled to Yang-Mills using the double copy method.

2.1 Basics of Amplitudes

A classical Klein-Gordon scalar field theory described by the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ shall be quantized by foliating the four-dimensional spacetime $\mathbb{R}^{1,3}$ along the isometry ∂_t in three-dimensional hypersurfaces $\Sigma_t \cong \mathbb{R}^3$. Each Σ_t is endowed with a Hilbert space where canonical commutation relations on the fields ϕ and $\Pi = \frac{\partial \mathcal{L}}{\partial \partial_t \phi}$ are imposed: $[\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$. Π is called the canonical momentum of the field ϕ. The unitary time evolution operator

$$U(t_2 - t_1) = e^{-iH(t_2 - t_1)} : \mathcal{V}_1 \to \mathcal{V}_2 \tag{2.1}$$

evolves states $\{|\phi_{1;i}\rangle \in \mathcal{V}_1\}$ from one Hilbert space to another $\{U(t)\,|\phi_{1;i}\rangle := |\phi_{2;i}\rangle \in \mathcal{V}_2\}$, where $H(\phi, \Pi) = \int d^3x \mathcal{H}$ is the Hamiltonian of the theory which is the conserved quantity with respect to time translation symmetry. Since all Hilbert spaces $\mathcal{V}_i$ are related by a symmetry transformation U, they are isomorphic [98, 124].

The probability that an initial vacuum state $|0\rangle \in \mathcal{V}_{\text{in}}$ evolves to a final vacuum state $|0\rangle \in \mathcal{V}_{\text{out}}$ is given by the transition amplitude

$$\langle 0|\, U(t_{\text{out}} - t_{\text{in}})\, |0\rangle = \int \mathcal{D}[\phi]\, \mathcal{D}[\Pi]\, e^{i \int d^4x (\partial_t \phi\, \Pi - \mathcal{H}(\phi, \Pi))} = \int \mathcal{D}[\phi]\, \mathcal{D}[\Pi]\, e^{iS_{\text{H}}}, \tag{2.2}$$

where (2.2) integrates over all classical field configurations with the boundary conditions that $\phi(t_{\text{in}}, \mathbf{x})$, $\phi(t_{\text{out}}, \mathbf{y})$ match onto free fields. These boundary values define the correct $i\epsilon$-prescription of the Feynman propagator [125, sec. 14.4.1]. Equation (2.2) can be derived by calculating the transition amplitude for infinitesimal time steps $\tilde{\epsilon}$ successively and inserting a complete set of eigenstates for Π at each hypersurface Σ_{t_i}. The path integral measure $\mathcal{D}[\phi]$ is defined by the formal limit $\tilde{\epsilon} \to 0$, i.e. in the continuous limit where the time intervals tend to zero. This is the Hamiltonian path integral, i.e. the sum over phase space paths weighted by $e^{iS_{\text{H}}}$. If the Hamiltonian does only depend quadratically on the conjugate momentum Π the integral over $\mathcal{D}[\Pi]$ is Gaussian and can be performed which leads to the **Lagrangian path integral**

$$\int \mathcal{D}[\phi]\, e^{iS_{\text{L}}} = \int \mathcal{D}[\phi]\, e^{i\int d^4x \mathcal{L}(\phi, \partial_\mu \phi)}, \tag{2.3}$$

i.e. the path integral over all paths in configuration space weighted by $e^{iS_{\text{L}}}$. The Lagrangian is related to the Hamiltonian density by $\mathcal{L} = \partial_t \phi\, \Pi - \mathcal{H}$ [8, 126].

The path integral (2.3) shall be taken as the definition of any interacting theory and the theory is completely specified by its action S_L. All physical information can be deduced from the path integral. The time-ordered correlation functions can be obtained by

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \langle 0 \left| T\phi(x_1) \dots \phi(x_n) \right| 0 \rangle = \frac{\int \mathcal{D}[\phi]\, e^{iS_{\text{L}}} \phi(x_1) \dots \phi(x_n)}{\int \mathcal{D}[\phi]\, e^{iS_{\text{L}}}}, \tag{2.4}$$

where the state $|0\rangle$ is the vacuum of the Hilbert space on the given hypersurfaces $\mathcal{V}_{\text{in}}$ and $\mathcal{V}_{\text{out}}$.[1]

By introducing a source term $J(x)$ (2.3) is generalized to the **generating functional**

$$Z[J] = \int \mathcal{D}[\phi]\, e^{iS_{\text{L}} + i\int d^4x J\, \phi} \quad \text{with} \quad Z[0] = \int \mathcal{D}[\phi]\, e^{iS_{\text{L}}} \tag{2.5}$$

such that (2.4) can be rewritten as

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = (-i)^n Z[0]^{-1} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0}. \tag{2.6}$$

$Z[J]$ can be solved exactly for free field theory, however, for an interacting theory this is not possible in general. But (2.5) can be computed perturbatively using the free field theory result. This expansion is usually translated into a graph-theoretical language and each algebraic expression can be represented by a **Feynman graph** or **Feynman diagram** which, among other things, is an mnemonic device to construct this perturbative series of the generating functional [125]. Feynman graphs can be constructed by Feynman rules which are completely fixed by the Lagrangian. Calculating perturbatively the generating functional $Z[J]$ amounts to the sum of all non-vanishing Feynman graphs where the external lines are attached to the sources J. But $Z[0]$ generates all possible vacuum diagrams (diagrams without external lines) and subtracts them from the numerator of (2.4) which implies that in (2.6) all, possible disconnected, graphs which include vacuum diagrams drop out of the sum. In particular (2.6) contains all non-vacuum Feynman graphs which can be constructed

[1] Note that on the left hand side the field ϕ are quantum operator insertions whereas on the right hand side the functional integral is over classical fields ϕ, i.e. the eigenvalues of the quantum operators. However, notationally it shall not be distinguished between these two cases.

from the Feynman rules. The disconnected parts, however, factorize in a neat way such that the generating function can be written as

$$Z[J] = e^{iW[J]}$$

where $iW[J]$ is the sum over all connected graphs omitting vacuum diagrams [127, 128].

The correlation function generates all kinds of single- and multi-particle states. But to calculate the probability that a certain set of one-particle states evolves into another set of one-particle states it is necessary to project onto the latter one.[2] Let the interaction be short-ranged such that the initial set of one-particle states $|i\rangle = |\mathbf{p}_1 \dots \mathbf{p}_k\rangle_\infty \in \mathcal{V}_\infty$ is an asymptotic state at time $t = \infty$. In the same way the final set of one-particle states $|f\rangle = |\mathbf{p}_{k+1} \dots \mathbf{p}_n\rangle_{-\infty} \in \mathcal{V}_{-\infty}$ is defined at $t = -\infty$. Since it is assumed that the interaction is short ranged the Hilbert spaces $\mathcal{V}_\infty$, $\mathcal{V}_{-\infty}$ are isomorphic to free Hilbert spaces. Therefore the asymptotic states satisfy the on-shell condition $p^2 = m^2$. Next the correlation function (2.6) shall be projected onto these states which can be carried out by the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [130]. This procedure defines the important concept of matrix elements

$$\langle f| S |i\rangle = \left[\prod_{j=1}^{k} i \int d^4x_j e^{-ip_j \cdot x_j} \frac{\Box_j + m^2}{\sqrt{Z}}\right]\left[\prod_{j=k+1}^{n} i \int d^4x_j e^{ip_j \cdot x_j} \frac{\Box_j + m^2}{\sqrt{Z}}\right] \langle \phi(x_1) \dots \phi(x_n)\rangle \quad (2.7)$$

of the scattering matrix S. Thus the S-matrix maps the initial state $|i\rangle \in \mathcal{V}_\infty$ into the Hilbert space $\mathcal{V}_{-\infty} \ni S|i\rangle$ such that the overlap with the state $|f\rangle \in \mathcal{V}_{-\infty}$ can be computed. Here $\Box_j = \partial_{j,\mu}\partial_j^\mu$ is the D'Alembertian and Z is the wave function renormalization constant.[3] Note that the k incoming asymptotic states have the exponential factor $-ipx$ whereas the $n-k$ outgoing states have the factor ipx [125]. Therefore, an incoming particle can be transformed into an outgoing particle simply by $p \to -p$ which also changes the helicity $h \to -h$, i.e. the spin projection onto the three-momentum $\mathbf{p}$. This implies that for the S-matrix an incoming particle with quantum numbers p and h is equivalent to an outgoing antiparticle with quantum numbers $-p$ and $-h$. This property is called crossing symmetry and it can be used to describe a scattering process with outgoing particles only, which is the convention used in this thesis [131]. However, the helicity h is only a quantum number for massless particles.

The S-matrix is the central object in scattering theory because the entire information of a scattering process is encoded in it. To see that the LSZ reduction formula projects onto one-particle states it is useful to analyze its pole structure and the on-shell condition

[2] Mathematically a particle is defined to transform in an irreducible unitary representation of the Poincaré group. The representation has to be irreducible, because any observer (in any Poincaré frame) should be able to measure one (and only one) particle and unitarity is required to leave the matrix elements (2.7) invariant. This classification has been done by Wigner [129] and is well explained in [8, 125] .
The Poincaré group is non-compact which makes all unitary representations infinite dimensional. These representations can be found by studying the little (stabilizer) group of the Poincaré group. The little group for the massive case is SO(3) and has therefore finite dimensional unitary representations. The irreducible representations are labelled by the spin. In the massless case the little group can be effectively described by the two helicity states.

[3] The wave function renormalization arises from the on-shell pole of the two point function $\langle\phi(x_1)\phi(x_2)\rangle$ in momentum space, i.e. in the limit $p_j^2 \to m^2$ the Fourier transform of $\langle\phi(x_1)\phi(x_2)\rangle$ is given by $\sim \frac{iZ}{p_j^2 - m^2 + i\epsilon}$.

of the particles: On the one hand propagators of asymptotic particles which satisfy the on-shell condition $p_j^2 = m^2$ have poles of the form $\sim \frac{1}{p_j^2-m^2}$ and on the other hand the Fourier transform of $\Box_j + m^2$ is $-p_j^2 + m^2$ which vanishes on-shell. Hence, only terms with poles of the form $\sim \frac{1}{p_j^2-m^2}$ shall be non-zero and the S-matrix elements (2.7) are given by the residues of these poles. For spinning asymptotic particles the LSZ formula also multiplies the corresponding polarization vectors to project out the desired spin state [125].

Diagrammatically the LSZ reduction can be carried out by amputating all external external lines which means that they are cut off until they begin to interact with other fields. Thus only amputated diagrams contribute to (2.7). In general there is a certain probability that the in- and out-state are the same $\langle f| S |i\rangle = \langle i| S |i\rangle \neq 0$ which means that no scattering at all occurred. To describe scattering processes only it is common to define the interacting part, the transfer matrix T, of the scattering matrix by the relation $S = \mathbb{1} + iT$ [125].

An important axiom of QFT is the so-called cluster decomposition principle which states that distant uncorrelated experiments do not affect each other. This principle is automatically satisfied if the Lagrangian is constructed out of fields or rather the Hamiltonian is made of creation and annihilation operators. To make this property manifest on the level of the scattering matrix it is useful to partition it into connected (sub-)parts, i.e. to group the particles into clusters. For the four-point S-matrix element ${}_{\infty}\langle \mathbf{k}_1\mathbf{k}_2|S\,|\mathbf{p}_1\mathbf{p}_2\rangle_{-\infty} = S_{\mathbf{k}_1\mathbf{k}_2;\mathbf{p}_1\mathbf{p}_2}$ this principle states

$$S_{\mathbf{k}_1\mathbf{k}_2;\mathbf{p}_1\mathbf{p}_2} = S^c_{\mathbf{k}_1\mathbf{k}_2;\mathbf{p}_1\mathbf{p}_2} + S^c_{\mathbf{k}_1;\mathbf{p}_1}S^c_{\mathbf{k}_2;\mathbf{p}_2} + S^c_{\mathbf{k}_1;\mathbf{p}_2}S^c_{\mathbf{k}_2;\mathbf{p}_1}$$

which clusters the particles into a proper $2 \to 2$ scattering process $S^c_{\mathbf{k}_1\mathbf{k}_2;\mathbf{p}_1\mathbf{p}_2}$ and two different $1 \to 1$ processes which correspond to a free propagation of the particles. However, if the particles $\mathbf{k}_2$ and $\mathbf{p}_2$ are separated from the other two the only contributing element is given by

$$S^c_{\mathbf{k}_1;\mathbf{p}_1}S^c_{\mathbf{k}_2;\mathbf{p}_2}$$

because these two particles cannot interact with $\mathbf{k}_1$ and $\mathbf{p}_1$ any more, respectively. For a scattering process involving more external particles the S-matrix can be partitioned in the same way. This shows that the general S-matrix factorizes into connected S-matrix elements. In the language of Feynman graphs the connected S-matrix element is constructed by connected Feynman graphs only. Another important property is that each connected part of the S-matrix contains a single four-momentum conserving δ-distribution. This property is also manifest using Feynman graphs [8].

The partitioning of the S-matrix elements shows that it is enough to consider their connected part only, because the full matrix element is a sum of products of connected S-matrix elements. Expanding the S-matrix in a series $S = \mathbb{1} + g\, iT^0 + g^2\, iT^1 + \dots$ in terms of its coupling constant g and applying this expansion to (2.7) leads to the important concept of (n-point) amplitudes

$$\begin{aligned}&\langle f|\, \mathbb{1}\, |i\rangle + g\, i\, \langle f|\, T^0\, |i\rangle + g^2\, i\, \langle f|\, T^1\, |i\rangle + \mathcal{O}\left(g^3\right)\\ =&\text{free part} + i(2\pi)^4\delta^4\Big(\sum_{j=1}^{n} p_j\Big)\left(g\mathcal{A}_n^0 + g^2\mathcal{A}_n^1 + \mathcal{O}\left(g^3\right)\right).\end{aligned} \tag{2.8}$$

The overall momentum-conserving δ-distribution, which follows from the cluster decomposition theorem, has been explicitly written out on the r.h.s. The amplitude $i\mathcal{A}_n^m$ can be constructed from momentum-space Feynman rules and the superscript m denotes the order in perturbation theory. To evaluate $i\mathcal{A}_n^m$ at order g^{m+1} all connected Feynman graphs with m-loops have to be evaluated and summed up. Hence, amplitudes contain the non-trivial contribution to the scattering process.

There are two kind of particle experiments to carry out: Either the decay of one particle into $n > 1$ particles or a collision experiment in which two particles scatter into $n > 1$ particles, since it is very unlikely that more than two particles collide at an instant of time.

To show how the theoretical framework of QFT is used to make predictions the second process, $2 \to n$ scattering, is outlined. The mathematical quantity which describes this process is the differential cross section $d\sigma/d\Omega$. It yields the number of particles which scatter into an area described by the solid angle $d\Omega$. It is defined by the quantum mechanical normalized probability

$$P = \frac{|\langle f|\, S\, |i\rangle|^2}{\langle i \mid i\rangle \langle f \mid f\rangle}$$

that a scattering process occurs normalized by the elapsed time t and volume V where the event happens, divided by the particle densities ϱ_i and the relative velocity $\mathbf{v}_r$ of the beams. In the center-of-mass frame $\mathbf{p}_1 = -\mathbf{p}_2$ it is given by

$$d\sigma = \frac{1}{tV}\frac{1}{\varrho_1\varrho_2\mathbf{v}_r}dP = \frac{1}{tV}\frac{V^2}{|\mathbf{v}_1 - \mathbf{v}_2|}dP = \frac{V}{t}\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}dP \tag{2.9}$$

because the velocities are collinear.[4] The particle densities are normalized such that one particle per unit volume occurs, i.e. $\varrho_i = V^{-1}$.

$$dP = \frac{|\langle f|\, S\, |i\rangle|^2}{\langle i \mid i\rangle \langle f \mid f\rangle}\prod_{j=1}^{n}\frac{V}{(2\pi)^3}d^3p_j$$

is the probability to detect the outgoing particle j in the momentum range d^3p_j.[5] Free particle states $|\mathbf{p}\rangle$ (in a box) are normalized such that $\langle \mathbf{p} \mid \mathbf{p}\rangle = E_pV$ which fixes the norm of $|i\rangle = |\mathbf{k}_1\mathbf{k}_2\rangle_\infty$ and $|f\rangle = |\mathbf{p}_1 \dots \mathbf{p}_n\rangle_{-\infty}$. If it is assumed that a pure scattering is happening, the overlap is given by $\langle f|\, S\, |i\rangle = i\, \langle f|\, T\, |i\rangle = (2\pi)^4\delta^4(\sum_i p_i)i\mathcal{A}$ where in the last step the cluster decomposition principle has been used to extract the δ-distribution.[6] Hence, the differential cross section for a $2 \to n$ scattering reads

$$d\sigma = \frac{V}{t}\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\frac{|\langle f|\, S\, |i\rangle|^2}{\langle i \mid i\rangle \langle f \mid f\rangle}\prod_{j=1}^{n}\frac{V}{(2\pi)^3}d^3p_j = \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\frac{|\mathcal{A}|^2}{4E_{k_1}E_{k_2}}d\Pi_{\mathrm{LIPS}} \tag{2.10}$$

[4] Actually, it is coincidence that the non-relativistic formula for the relative velocity appears in (2.9). The general formula for the relativistic velocity is $\mathbf{v}_{\mathrm{rel}} = \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v_2})^2}\sqrt{1 - \mathbf{v}_1 \cdot \mathbf{v_2}}^{-1}$ [132]. Thus the flux of the particle beams reads $\Phi = \varrho_1\varrho_2\sqrt{1 - \mathbf{v}_1 \cdot \mathbf{v_2}}\mathbf{v}_{\mathrm{rel}} = \varrho_1\varrho_2\mathbf{v}_r$ [133].

[5] The prefactor follows from the normalization that one particle is detected after integrating over a certain range: $1 = \int d^3n_j = \frac{V}{(2\pi)^3}\int d^3p_j$. The last equality states the simple fact that a particle in the box $V = L^3$ with momentum $\mathbf{p}_j = \left(\frac{2\pi}{L}\right)^3\mathbf{n}_j$ is quantized with the momentum quantum numbers $\mathbf{n}_j \in \mathbb{Z}^3$.

[6] $\mathcal{A}$ includes the connected as well as the disconnected S-matrix elements. Even though the disconnected S-matrix elements do not come with an overall momentum conserving δ-distribution, the identity $\delta(x)\delta(y) = \delta(x+y)\delta(y)$ can be used to obtain one.

with the Lorentz-invariant phase space (LIPS) measure

$$
\begin{aligned}
d\Pi_{\text{LIPS}} &= (2\pi)^4 \delta^4 \left(\sum_i p_i\right) \prod_{j=1}^{n} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_{p_j}} \\
&= (2\pi)^4 \, \delta^4 \left(\sum_i p_i\right) \prod_{j=1}^{n} \frac{d^4 p_j}{(2\pi)^4} \, (2\pi) \, \delta\left(p_j^2 + m^2\right) \Theta\left(p^0\right) .
\end{aligned}
\tag{2.11}
$$

In the last step the identity $\delta(x+y)\delta(x) = \delta(y)\delta(x)$ has been used as well as the fact that in the box the δ-distribution obeys $(2\pi)^4\delta^4(0) = \int d^4x = tV$ which follows from Fourier analysis. The total cross section is obtained by integrating this expression in the region where the particles $\mathbf{p}_j$ are measured. Note that the final expression is independent of t and V. [125, 127].

Due to probability conservation $\langle i \mid i\rangle = \langle f \mid f\rangle = \langle i| \, S^\dagger S \, |i\rangle$ the S-matrix has to be unitary $SS^\dagger = S^\dagger S = \mathbb{1}$ which implies that the transfer matrix T obeys the **generalized optical theorem**

$$
TT^\dagger = -i\left(T - T^\dagger\right) = 2\,\Im(T) \, . \tag{2.12}
$$

The generalized optical theorem is a very powerful statement, because it allows to determine the imaginary part of the T-matrix at a given order from results of lower order in perturbation theory. This is easy to see by expanding the (connected part of the) T-matrix in a power series $T = gT^0 + g^2T^1 + \mathcal{O}(g^3)$ of the coupling constant g.[7] Applying this expansion to (2.12) and inserting a complete set of state[8] $\mathbb{1} = \sum_a \int d\Pi_a \, |a\rangle \, \langle a|$ gives

$$
2\,\Im\Big(\langle f| \, T^1 \, |i\rangle\Big) = \sum_a \int d\Pi_a \, \langle f| \, T^0 \, |a\rangle \, \langle a| \left(T^0\right)^\dagger |i\rangle \, . \tag{2.13}
$$

The sum runs over all (on-shell) single- and multi-particle states $|a\rangle$, i.e. all non-kinematic quantum numbers which label distinguished particles [125]. For example in SU(N) Yang-Mills theory there is only one particles species (the gluon) which can have the two helicity configurations ±1 and $N^2 - 1$ different colour labels. Thus one has to sum over all of these quantum numbers. The integration measure is proportional to the Lorentz-invariant phase space measure (2.11)

$$
d\Pi_{\text{LIPS}} = (2\pi)^4 \, \delta^4 \left(\sum_i p_i\right) d\Pi_a.
$$

The unitarity requirement of the S-matrix manifests itself in terms of a Feynman diagrammatic expansion in terms of the cutting equations [48, 135] which state that the imaginary part of a loop amplitude comes from setting propagators on-shell, i.e. a cut is given by the prescription

$$
\frac{i}{p^2 - m^2 + i\epsilon} \to 2\pi\delta(p^2 - m^2)\Theta(p^0),
$$

[7] The analytic S-matrix program tried to bootstrap the S-matrix by obtaining the real part of the T-matrix from the imaginary part by so-called dispersion relations. However, the program did not succeed [134].

[8] The completeness relation is defined in analogy to the non-relativistic quantum mechanical completeness relation $\mathbb{1} = (2\pi)^{-3} \int d^3p \, |p\rangle \, \langle p|$ where the states are simply labelled by their momenta.

which can be seen by comparing (2.11) and (2.13) on a diagrammatic level.

Nowadays this method has been refined and allows to decompose a loop-level amplitude into a finite basis of master integrals $\{I_k\}$. This method of **generalized unitarity cuts** is one of the most efficient ways to determine loop-level integrands by studying the analytical structure of loop-integrands which are rational functions of Lorentz invariant quantities. The idea is to find a basis $\{I_k\}$ (i.e. a spanning set of cuts) in which the n-point m-loop amplitude $\mathcal{A}_n^m$ can be expressed

$$\mathcal{A}_n^m = \sum_k c_k I_k \tag{2.14}$$

where the coefficients c_k are given by manifest gauge invariant functions of Lorentz invariants. It can be rather difficult to construct a good basis, however, in general a basis can be found by making use of different constraints on the amplitude like power counting. Due to the fact that the final form of the amplitude (2.14) depends on the chosen basis, it is important to choose a suitable basis. The method of prescriptive unitarity [49] suggest that a good basis is given if every coefficient c_k is determined by a single field-theory cut. The next step is to make an ansatz for the integrand and to determine all the unknown coefficients c_k using linear algebra by the criterion that the residues match the field theory [49, 50, 136].[9] Whenever a decomposition of the form (2.14) is possible using generalized unitarity the integrand is solely determined by its residues.[10] However, not all amplitudes are cut-constructable. In particular the rational terms as functions of Lorentz invariant quantities in the amplitude cannot be obtained by cuts, because these do not have any discontinuities (and therefore no imaginary part). There are different methods to overcome this obstacle like using d-dimensional unitarity cuts [137], because functions which are rational in four dimensions are not rational in generic dimensions d and therefore former rational functions develop branch cuts.

A further advantage of this method is that properties of tree-level amplitudes can be carried over to loop-level if the property is basically unchanged if a propagator is cut [136]. In section 2.3 the method of Veltman-Passarino reduction is explained which also uses the idea that a certain one-loop amplitude can be decomposed in a set of basis integrals.

2.1.1 Supersymmetric Ward-Identities

The generating functional (2.5) is independent of the integrated fields. Hence, applying an infinitesimal shift $\phi(x) \to \phi(x) + \delta\phi(x)$ does not change the generating functional:

$$\begin{aligned}
Z[J] &= \int \mathcal{D}[\phi]\, e^{iS_{\mathrm{L}}[\phi+\delta\phi, \partial_\mu\phi+\partial_\mu\delta\phi]+i\int d^4x J\,(\phi+\delta\phi)} \\
&= \int \mathcal{D}[\phi]\, e^{iS_{\mathrm{L}}[\phi,\partial_\mu\phi]+i\int d^4x J\,\phi} \left(1 + i\int d^4x \left(\frac{\delta S_L}{\delta\phi(x)} + J(x)\right)\delta\phi(x) + \dots\right) \\
&= Z[J] + \delta Z[J] + \cdots
\end{aligned}$$

[9] In the light of generalized unitarity this is done by cutting a certain number of internal propagators and calculating their contributions using the generalized optical theorem (2.13) (or Cutkosky rules). This expression has to be matched with the basis $\{I_k\}$ to find the coefficients c_k.

[10] Residues of the integrand manifest itself in discontinuities of the integrated amplitude, e.g. the rational function $\frac{1}{z}$ has a pole at $z = 0$ but its integrated expression $\ln z$ has a branch cut along the negative real axis. Thus one can either talk about the discontinuities of the amplitude or the residues of the integrand.

where it has been assumed that the measure $\mathcal{D}[\phi]$ is invariant under the shift.[11] In particular this implies that $\delta Z[J] = 0$, because this relation has to hold order by order in $\delta\phi$. Taking functional derivatives of $\delta Z[J]$ yields

$$\begin{aligned} 0 &= (-i)^n \left.\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}\delta Z[J]\right|_{J=0} \\ &= \int \mathcal{D}[\phi]\, e^{iS_{\mathrm{L}}} \int d^4x \left[i\frac{\delta S_L}{\delta\phi(x)}\phi(x_1)\dots\phi(x_n) + \sum_{k=1}^{n}\phi(x_1)\dots\hat{\phi}(x_k)\dots\phi(x_n)\delta^4(x-x_k)\right]\delta\phi(x). \end{aligned}$$

The hat denotes that this quantity is omitted in the sum. Since the variation $\delta\phi$ is arbitrary this equation has to hold independently of it and also prior of the integration. This leads to the famous Schwinger-Dyson equations

$$\left\langle \frac{\delta S_L}{\delta\phi(x)}\phi(x_1)\dots\phi(x_n)\right\rangle = i\sum_{k=1}^{n}\delta^4(x-x_k)\left\langle\phi(x_1)\dots\hat{\phi}(x_k)\dots\phi(x_n)\right\rangle \tag{2.15}$$

which indicate how a quantum theory deviates from the corresponding classical theory by the additional contact terms that are proportional to the δ-distribution.[12]

The **Schwinger-Dyson** equations can be seen as the QFT-analogue to the classical equations of motions. However, in a classical theory not only the equations of motions are important, but also the conserved charges and currents which can be derived from a continuous symmetry of the action according to Noether's theorem. The QFT-analogue to Noether's theorem is given by the famous **Ward-Takahashi identities**. They can be derived by the following considerations. The assumption that $\delta\phi$ is the infinitesimal change of a symmetry transformation implies that the Lagrangian changes by a total derivative only, i.e.

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}(x)}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}\partial_\mu\delta\phi(x) = \partial_\mu K^\mu(x).$$

Using the equation of motions

$$\frac{\delta S_L}{\delta\phi(x)} = \int d^4y\frac{\delta\mathcal{L}(y)}{\delta\phi(x)} = \frac{\partial\mathcal{L}(x)}{\partial\phi(x)} - \partial_\mu\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}$$

this can be rewritten as

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}\delta\phi(x)\right) - \delta\mathcal{L}(x) + \frac{\delta S_L}{\delta\phi(x)}\delta\phi(x) := \partial_\mu j^\mu(x) + \frac{\delta S_L}{\delta\phi(x)}\delta\phi(x) \tag{2.16}$$

with the Noether current

$$j^\mu(x) = \left(\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))}\delta\phi(x) - K^\mu(x)\right) \tag{2.17}$$

[11] Hamilton's principle $\frac{\delta S_L}{\delta\phi_{\mathrm{cl}}(x)} = 0$ holds for classical field configurations ϕ_{cl} only, but the path integral integrates over all field configurations.

[12] For a massless scalar free-field theory $\mathcal{L} = \frac{1}{2}\phi\Box\phi$, the equation of motions are $\frac{\delta S_L}{\delta\phi(x)} = -\Box_x\phi(x)$ such that (2.15) reproduces the fact that the two-point correlator is the Green's function of the free-field theory: $\Box_x\langle\phi_x\phi_{x_1}\rangle = -i\delta^4(x-x_1)$.
However, one has to take care that the operator $\Box$ is outside of the path integral such that it also appears outside the correlation function. This subtlety of the relation between canonical and path integral quantization is explained in [125] in chapter 14.7.

which is conserved for a classical field configuration $\frac{\delta S_L}{\delta\phi_{\mathrm{cl}}(x)} = 0$ according to (2.16). In addition if the coordinates change too, which corresponds to a global space-time symmetry $x^\nu \to x'^\nu = x^\nu - \xi^\nu$, the energy-momentum tensor $T^{\mu\nu}(x)\xi_\nu$ has to be added to the r.h.s. of (2.17) as well. Plugging (2.16) into the Schwinger-Dyson equation (2.15) yields the Ward-Takahashi identities

$$\partial_\mu \langle j^\mu \phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{k=1}^{n}\delta^4(x-x_k)\left\langle \phi(x_1)\dots\hat{\phi}(x_k)\dots\phi(x_n)\right\rangle . \tag{2.18}$$

Thus a Noether current is conserved in a QFT upto contact terms. Note that the partial derivative has to be written outside the correlation function (see footnote 12) [127].

The identities (2.18) are very powerful. They imply that in a theory invariant under global supersymmetry transformations certain amplitudes vanish identically to all loop-orders. If for a purely bosonic amplitude the helicity configuration of all particles is identical or identical except one particle the amplitudes vanish. In particular this implies that in a theory where Yang-Mills is coupled to Einstein gravity (see (2.65)) the amplitudes satisfy

$$\begin{aligned} \mathcal{M}(1^+,\dots,n^+,1^{++},\dots,m^{++}) &= 0 \\ \mathcal{M}(1^-,2^+,\dots,n^+,1^{++},\dots,m^{++}) = \mathcal{M}(1^+,\dots,n^+,1^{--},2^{++},\dots,m^{++}) &= 0 \end{aligned} \tag{2.19}$$

where n and m are the numbers of gluons and gravitons, respectively. This result also holds for non-supersymmetric theories at tree-level, because the scalars and the fermions couple at least quadratically to the gluon and graviton in the supersymmetric Lagrangian. This statement can also be visualized diagrammatically: If at tree-level a graph contains internal fermionic or scalar lines, the graph has to have external fermionic or scalar lines, too [45].

2.1.2 Spinor-Helicity Formalism

In physics the descriptive power of a given formalism is always based on the effectiveness and conciseness of the language in use. Well-known examples are the covariant formulation of Maxwell's equations and the bra-ket formalism of Dirac for non-relativistic quantum mechanics.

A very efficient language for massless scattering processes in four dimensions is provided by the **spinor-helicity formalism** [43, 44]. The key point is that this formalism unifies the description of the on-shell degrees of freedom momentum and helicity into a single object. Since in spinor-helicity formalism all the operators transform under the double cover of the Lorentz group $\mathrm{SO}_+(1,3)$ which is given by $\mathrm{SL}(2,\mathbb{C})$, it is best to start with the massless Dirac equation to introduce this formalism:

$$0 = i\partial\!\!\!/\psi(x) := i\gamma^\mu\partial_\mu\psi(x) = \begin{pmatrix} 0 & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & 0\end{pmatrix}\begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix} = \begin{pmatrix} i\sigma^\mu\partial_\mu\psi_R(x) \\ i\bar{\sigma}^\mu\partial_\mu\psi_L(x)\end{pmatrix} \tag{2.20}$$

where $\sigma^\mu = (\mathbb{1}_2,\sigma)$, $\bar{\sigma}^\mu = (\mathbb{1}_2,-\sigma)$. In (2.20) the Weyl representation (A.2) of the γ-matrices has been chosen. The defining property of the γ-matrices is that they obey the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}_4$. The Dirac spinor $\psi(x)$ can be decomposed into a pair of two-component Weyl spinors ψ_L and ψ_R, because the spinor representation of the Lorentz group is reducible which is shown in the appendix A.1. The vector σ contains the three Pauli matrices, which are given in (A.1). The corresponding conjugate spinor $\bar{\psi} = \psi^\dagger\gamma^0$ is defined such that $\bar{\psi}\psi$ transforms as a Lorentz scalar. Furthermore, for the massless Dirac equation the two Weyl spinors decouple, hence there are only two instead of four independent solutions to (2.20).

Since ψ obeys the massless Klein-Gordon equation $\partial\!\!\!/\partial\!\!\!/\psi = \partial^2\psi = 0$, it can be constructed by a superposition of plane waves $\psi_p(x) = u(\mathbf{p})e^{-ipx}+v(\mathbf{p})e^{ipx}$ of positive, $u(\mathbf{p})$, and negative, $v(\mathbf{p})$, energy solutions which obey the following set of equations:[13]

$$p\!\!\!/u(\mathbf{p}) = \begin{pmatrix} (E_p - \sigma\cdot\mathbf{p})\, u_R(\mathbf{p}) \\ (E_p + \sigma\cdot\mathbf{p})\, u_L(\mathbf{p}) \end{pmatrix} = 0 = -p\!\!\!/v(\mathbf{p}) = \begin{pmatrix} (-E_p + \sigma\cdot\mathbf{p})\, v_R(\mathbf{p}) \\ (-E_p - \sigma\cdot\mathbf{p})\, v_L(\mathbf{p}) \end{pmatrix}.$$

This confirms that there are only two independent solutions, since the two positive energy solutions coincide with the two negative energy solutions. Thus one can identify $u_L(\mathbf{p}) = v_L(\mathbf{p}) := \lambda(\mathbf{p})$ and $u_R(\mathbf{p}) = v_R(\mathbf{p}) = \bar{\lambda}(\mathbf{p})$. Hence the massless fermion shall be described by a two-component left- or right-chiral Weyl spinor:

$$\chi_w = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix} \quad \text{(left-chiral)}, \qquad \text{or} \quad \xi_w = \begin{pmatrix} 0 \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix} \quad \text{(right-chiral)}. \tag{2.21}$$

The indices $\alpha, \dot{\alpha} = 1,2$ distinguish between the fundamental $\underline{2}$ and anti-fundamental $\overline{\underline{2}}$ representation of $\mathrm{SL}(2,\mathbb{C})$. Writing the indices explicitly (2.20) reads

$$0 = \begin{pmatrix} i\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\partial_\mu\psi_R^{\dot{\alpha}}(x) \\ i\,(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\,\partial_\mu\psi_{L,\alpha}(x) \end{pmatrix} = \begin{pmatrix} i\,(\sigma^\mu\partial_\mu)_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha}}(x) \\ i\,(\bar{\sigma}^\mu\partial_\mu)^{\dot{\alpha}\alpha}\,\chi_\alpha(x) \end{pmatrix}. \tag{2.22}$$

Indices are raised and lowered with the $\mathrm{SL}(2,\mathbb{C})$-invariant tensor $\epsilon = i\sigma_2$, i.e. $\chi^\alpha := (\epsilon\chi)^\alpha = \epsilon^{\alpha\beta}\chi_\beta$ and $\bar{\xi}_{\dot{\alpha}} := (\epsilon\bar{\xi})_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}}$. Thus in momentum space the lower component of (2.22) reads

$$p^{\dot{\alpha}\alpha}\lambda_\alpha(\mathbf{p}) = 0 \qquad \text{with} \quad p^{\dot{\alpha}\alpha} = p^\mu\bar{\sigma}_\mu^{\dot{\alpha}\alpha} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \tag{2.23}$$

where the plane wave decomposition of the free Weyl fermion is given by

$$\chi_\alpha(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\lambda_\alpha(\mathbf{p})\left(a_\mathbf{p}\, e^{-ipx} + b_\mathbf{p}^\dagger\, e^{ipx}\right). \tag{2.24}$$

Next it shall be shown that the momentum can be represented in terms of these commuting Weyl spinors. Solving (2.23) and the complex conjugated equation $p_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} = 0$ in the Lorentz frame $p_E = (E_0, 0, 0, E_0)$ gives

$$\lambda_\alpha(\mathbf{p_E}) = \begin{pmatrix} 0 \\ \sqrt{2E_0} \end{pmatrix} \quad \text{and} \quad \bar{\lambda}^{\dot{\alpha}}(\mathbf{p_E}) = \begin{pmatrix} \sqrt{2E_0} \\ 0 \end{pmatrix}.$$

Comparing their product with (2.23) implies

$$\bar{\lambda}^{\dot{\alpha}}(\mathbf{p_E})\lambda^\alpha(\mathbf{p_E}) = \begin{pmatrix} 2E_0 & 0 \\ 0 & 0 \end{pmatrix} = p_E^{\dot{\alpha}\alpha}$$

[13]For the negative energy solutions instead of considering $p^0 = E_p < 0$ the sign of the exponential has been changed.
In this section the vectors u and v are written in the chiral eigenbasis. However, it is common to write them in the helicity eigenbasis, which is equivalent for massless particles and opposite for massless antiparticles. In particular, it holds $u_R = u_+$, $u_L = u_-$, $v_R = v_-$ and $v_L = v_+$ (see [127, 138, 139] for more information).

for the frame $(E_0, 0, 0, E_0)$. This expression can be boosted into an arbitrary Lorentz frame which yields $\bar{\lambda}^{\dot{\alpha}}\lambda^{\alpha} = p^{\dot{\alpha}\alpha}$. To obtain a Lorentz invariant quantity constructed from λ_α and $\bar{\lambda}^{\dot{\alpha}}$ one has to contract pairs of these or concatenate them with momenta:

$$\begin{aligned} \langle ij\rangle = \langle\lambda_i\lambda_j\rangle &:= \lambda_i^{\alpha}\lambda_{j\,\alpha} = -\langle ji\rangle, & [ij] = [\lambda_i\lambda_j] &:= \bar{\lambda}_{i\,\dot{\alpha}}\bar{\lambda}_j^{\dot{\alpha}} = -\,[ji]\,, \\ \langle 1\,|2|\,3] &:= \lambda_1^{\alpha_2} p_{2\,\alpha_2\dot{\alpha}_2}\bar{\lambda}_3^{\dot{\alpha}_2} = \langle 12\rangle\,[23]\,, & [1\,|2|\,3\rangle &:= \bar{\lambda}_{1\,\dot{\alpha}_2} p_2^{\dot{\alpha}_2\alpha_2}\lambda_3^{\alpha_2} = [12]\,\langle 23\rangle. \end{aligned} \tag{2.25}$$

These elements are the arguments of any gluon amplitude viewed as a function of Lorentz invariant quantities in four dimensions.

In order to obtain an explicit representation of the polarization vectors $\varepsilon_{\pm}^{\dot{\alpha}\alpha} = (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}\varepsilon_{\pm}^{\mu}$ in terms of momentum spinors, it is also instructive to choose the Lorentz frame $(E_0, 0, 0, E_0)$. One possible choice of the polarization vectors such that they are orthogonal to the momentum is given by the circular polarized vectors (helicity eigenstates) which read

$$\varepsilon_{\pm}^{\mu}(\mathbf{p_E}) := \varepsilon_{\pm,p_E}^{\mu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}.$$

They satisfy $\varepsilon_{\pm,p_E}\cdot\varepsilon_{\pm,p_E} = 0$, $\varepsilon_{+,p_E} = (\varepsilon_{-,p_E})^*$, $p_E\cdot\varepsilon_{\pm,p_E} = q_0\cdot\varepsilon_{\pm,p_E} = 0$ with the reference vector $q_0 = (1, 0, 0, -1)$. The corresponding spinors of the reference momentum are defined by $q^{\dot{\alpha}\alpha} := \bar{r}^{\dot{\alpha}}r^{\alpha}$. Due to the antisymmetric metric tensor ϵ, identical pairs of spinor products vanish: $\lambda^\alpha\lambda_\alpha = \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} = r^\alpha r_\alpha = \bar{r}_{\dot{\alpha}}\bar{r}^{\dot{\alpha}} = 0$. Hence if the numerator of $\varepsilon_{\pm,p_E}^{\dot{\alpha}\alpha}$ is of the form $\bar{r}^{\dot{\alpha}}\lambda^{\alpha}$ or $\bar{\lambda}^{\dot{\alpha}}r^{\alpha}$ the orthogonality relations are trivially obeyed. Further λ^α ($\bar{\lambda}^{\dot{\alpha}}$) has -1/2 (+1/2) helicity because it is the momentum spinor of the left (right) handed Weyl spinor from (2.24). On the other hand the polarization vectors $\varepsilon_{+}^{\dot{\alpha}\alpha}$ ($\varepsilon_{-}^{\dot{\alpha}\alpha}$) carry helicity +1 (-1) which implies, after a suitable normalization, that the polarization vectors are given by

$$\varepsilon_{+,p_E}^{\dot{\alpha}\alpha} = -\sqrt{2}\frac{\bar{\lambda}^{\dot{\alpha}}(p_E)r^{\alpha}(q_0)}{\langle\lambda(p_E)r(q_0)\rangle}, \quad \varepsilon_{-,p_E}^{\dot{\alpha}\alpha} = \sqrt{2}\frac{\bar{r}^{\dot{\alpha}}(q_0)\lambda^{\alpha}(p_E)}{[\lambda(p_E)r(q_0)]}.$$

These expression can be boosted into an arbitrary Lorentz frame, too. Therefore it can be concluded that

$$\bar{\lambda}^{\dot{\alpha}}\lambda^{\alpha} = p^{\dot{\alpha}\alpha}, \quad \varepsilon_{+}^{\dot{\alpha}\alpha} = -\sqrt{2}\frac{\bar{\lambda}^{\dot{\alpha}}r^{\alpha}}{\langle\lambda r\rangle}, \quad \varepsilon_{-}^{\dot{\alpha}\alpha} = \sqrt{2}\frac{\bar{r}^{\dot{\alpha}}\lambda^{\alpha}}{[\lambda r]}, \tag{2.26}$$

which proves that the momentum Weyl spinors are the basic building blocks of the scattering amplitudes in four dimensions. It can be checked that

$$\varepsilon_{\pm}^{\dot{\alpha}\alpha}(p, q') = \varepsilon_{\pm}^{\dot{\alpha}\alpha}(p, q) + s_{\pm}p^{\dot{\alpha}\alpha} \quad \text{with} \quad s_{+} = s_{-}^{*} = \sqrt{2}\frac{\langle rr'\rangle}{\langle\lambda r\rangle\langle\lambda r'\rangle}.$$

Thus the freedom of choosing the reference vector q encodes the gauge freedom of massless scattering processes and it has to be chosen such that $q \neq p$ [131].

The power of this formalism relies on the fact that it is easy to manipulate these spinor products. Consider for example

$$-\langle 12\rangle\,[41]\,[23]\,\langle 34\rangle = \langle 14\rangle\langle 23\rangle\,[12]\,[34]$$

which implies that

$$\frac{\langle 12\rangle\langle 34\rangle}{[12]\,[34]} = \frac{\langle 14\rangle\langle 23\rangle}{[14]\,[23]}. \tag{2.27}$$

This identity shall be useful later on to calculate amplitudes in $\mathcal{N} = 0$ supergravity coupled with Yang Mills.

2.1.3 Colour-Ordering

One of the advantages of Feynman diagrams is that they represent complicated algebraic expressions in a mnemonic, diagrammatic manner. However, they do not describe physical quantities or processes because amongst other things they are not gauge invariant. Generally, it is desirable to have gauge invariant building blocks for an amplitude, since they are easier to control.

Hence, one would like to write an amplitude as a linear combination of gauge invariant objects. The authors of [41] proposed such a form for an n-gluon tree-level amplitude which is called its **colour decomposition**:

$$\mathcal{A}_n^0\left(1_{a_1}^{h_1}, \dots, n_{a_n}^{h_n}\right) = g^{n-2} \sum_{\sigma\in\mathcal{S}_{n-1}} \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right) A_{n;1}^0\left(\sigma(1^{h_1}), \dots, \sigma(n^{h_n})\right). \tag{2.28}$$

Here g is the coupling constant and $k_{a_k}^{h_k}$ represents the k^{th} particle with momentum p_k, helicity h_k and colour factor a_k. The kinematic information of the amplitude $\mathcal{A}_n^0$ is contained in the gauge invariant partial amplitudes $A_{n;1}^0$, whereas the colour information is separated into the traces $\mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right)$ of fundamental generators $(T^a)_i^{\bar{j}}$ of the gauge group $\mathrm{SU}(N) = \{U \in \mathrm{GL}(N,\mathbb{R}) | U^\dagger U = \mathbb{1} \text{ and } \det U = 1\}$, which are normalized such that

$$\mathrm{Tr}(T^a T^b) = \delta^{ab} \tag{2.29}$$

holds. The partial amplitudes are gauge invariant because the traces form a linear independent basis in colour space. Due to the cyclic invariance of the trace the sum runs only over all non-cyclic permutations $\mathcal{S}_{n-1}$. Hence, the colour degrees of freedom have been completely decoupled from the kinematic degrees of freedom. Any $\mathrm{SU}(N)$ tree-level amplitude whose particles transform in the adjoint representation can be written in the form (2.28)[14] which can be easily confirmed by decomposing the adjoint generators

$$f^{abc} = -\frac{i}{\sqrt{2}}\mathrm{Tr}\left(T^a\left[T^b, T^c\right]\right) \tag{2.30}$$

which appear in the Feynman diagrams of gluon scattering into fundamental generators T^a. Therefore a tree-level Feynman diagram for an n-particle scattering amplitude is given by products of traces which contain three fundamental generators. Schematically the colour structure is of the form:

$$f^{a_1a_2b_1}f^{b_1a_3b_2}\dots f^{b_{n-2}a_{n-1}a_n} \sim \mathrm{Tr}(T^{a_1}T^{a_2}T^{b_1})\mathrm{Tr}(T^{b_1}T^{a_3}T^{b_2})\dots\mathrm{Tr}(T^{b_{n-2}}T^{a_{n-1}}T^{a_n})$$
$$\pm \text{ other traces.}$$

[14] This decomposition works for other gauge groups, too.

Note that in each trace some of the adjoint indices $a_1, a_2, \ldots, b_1, b_2, \ldots$ are contracted with the adjoint indices of other traces. Therefore one can combine these traces to form one trace by using the completeness relations

$$\sum_{a=1}^{N^2-1} (T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} - \frac{1}{N}\delta_{i_1}^{\bar{j}_1}\delta_{i_2}^{\bar{j}_2} \quad \text{for SU}(N)$$

which can be proven from fact that every Hermitian matrix B can be written as a linear combination of the traceless Hermitian matrices T^a and the unit matrix: $B = c_0\mathbb{1} + c_a T^a$. It turns out that $c_0 = N^{-1}\text{Tr}B$ and $c_a = \text{Tr}BT^a$ which implies that $0 = -B_{j_1}^{\bar{i}_1}((T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{i}_2} + \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} - \frac{1}{N}\delta_{i_1}^{\bar{j}_1}\delta_{i_2}^{\bar{j}_2})$. However, if all particles charged under the gauge group transform in the adjoint representation the simpler completeness relation of $\text{U}(N)$ can be used:

$$\sum_{a=0}^{N^2-1} (T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} \quad \text{for U}(N). \tag{2.31}$$

Due to an isomorphism $\text{U}(N) \cong \text{SU}(N) \times \text{U}(1)$ one can add the additional $\text{U}(1)$-generator $(T^0)_i^{\bar{j}} := \frac{1}{\sqrt{N}}\delta_i^{\bar{j}}$ to the $\text{SU}(N)$-generators. However, this generator commutes with all other generators which implies that $f^{0ab} = 0$.

Since the partial amplitudes $A_{n;1}^0\left(1^{h_1}, \ldots, n^{h_n}\right)$ are the coefficients of certain colour traces $\text{Tr}\left(T^{a_1} \cdots T^{a_n}\right)$ they just receive contributions from a fixed ordering of the external particles. Hence their analytic structure is much simpler than the full amplitude. For example they can develop poles in channels of adjacent momenta $(p_i + p_{i+1} + \cdots + p_{i+j})^2$ only. By defining colour-ordered Feynman rules which are obtained from the usual Feynman rules by removing the colour degrees of freedom , this property is manifest, because with these rules only planar diagrams w.r.t. a fixed ordering of the external momenta have to be drawn to calculate a certain partial amplitude [138].

Colour decomposition of loop-amplitudes can be determined in a similar fashion. The authors of [42] explained that for one-loop amplitudes, in addition, new kind of trace products can appear where two pairs of generators are contracted if one closes a loop. Thus one shall eventually obtain trace products of the form:

$$\text{Tr}(T^{a_1} \cdots T^{a_m} T^{a_l} T^{a_l} T^{a_{m+1}} \cdots T^{a_n}), \quad \text{or} \quad \text{Tr}(T^{a_1} \cdots T^{a_m} T^{a_l} T^{a_i} \cdots T^{a_l} T^{a_{m+1}} \cdots T^{a_n}).$$

If two generators are positioned next to each other they produce $(T^{a_l}T^{a_l})_i^{\bar{j}} = N\delta_i^{\bar{j}}$ due to (2.31), which is the quadratic Casimir operator of $\text{U}(N)$ in the fundamental representation. Similarly, one obtains two traces if the operators are not adjacent. Hence, the general one-loop colour decomposition can be written as:

$$\begin{aligned}\mathcal{A}_n^1\left(1_{a_1}^{h_1}, \ldots, n_{a_n}^{h_n}\right) = g^n \Big[&\sum_{\sigma \in \mathcal{S}_{n-1}} N\ \text{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right) A_{n;1}^1\left(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n})\right) \\ &\sum_{c=2}^{\lfloor \frac{n}{2} \rfloor + 1} \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n;c}} \text{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}\right) \text{Tr}\left(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}\right) \\ &\times\ A_{n;c}^1\left(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n})\right) \Big],\end{aligned} \tag{2.32}$$

where $A_{n;c}^1$ are the partial amplitudes and $\mathcal{S}_{n;c}$ are the subsets of permutations which leave the double traces invariant. $\lfloor n \rfloor$ is the greatest integer less than or equal to n. The primitive amplitudes $A^1_{n;1}$, which are also colour-ordered because they are the coefficients of the single trace structures, can be calculated by drawing all planar diagrams. On the other hand the remaining partial amplitudes $A^1_{n;c}$ for $c > 1$ can be determined from the primitive amplitudes $A^1_{n;1}$. For example four $n = 4$ they satisfy the simple relation:

$$A^1_{4,3}\left(1^{h_1},\ldots,4^{h_4}\right) = \sum_{\sigma\in\mathcal{S}_3} A^1_{4;1}\left(\sigma(1^{h_1}),\ldots,\sigma(4^{h_4})\right), \tag{2.33}$$

where $\mathcal{S}_3$ is the sum over all non-cyclic permutations.

2.2 Regularization

In this thesis a perturbative approach to QFT is described by the expansion in the coupling constants of the amputated correlators as described in section 2.1. The common tool to calculate this series are the former mentioned Feynman graphs. However, a straight forward calculation of many diagrams is not possible, because they may diverge when the loop momenta go to infinity for fixed external momenta. These ultraviolet divergences are caused by large fluctuations of the fields at short distances. To obtain sensible physical results one has to remove these divergences systematically: This program goes under the name **regularization** and **renormalization**. The general idea of regularization is to introduce a regulator and to evaluate the regularized expression in the region where it converges. After the calculation the expression can be continued back to the region of interest. Technically, this procedure is very delicate, because on the one hand divergent expressions cannot be manipulated in a meaningful way and on the other hand a regulator may break some of the theory's symmetries like Poincaré or gauge symmetries.

2.2.1 Dimensional Regularization

In this thesis both of these problems are handled by **dimensional regularization** (DR). The basic concept is to consider the integrals in generic dimensions d and do all the manipulations in the region of convergence, i.e. for the dimension for which the integral converges. After the expression is computed it is continued back to the dimension of interest: $d = 4$. Since the expression is regularized, the integral is finite. A further advantage of this regularization procedure is that neither Poincaré nor gauge symmetries are broken. The following discussion is in the line of [140].

A sensible definition for integrals in non-integer $d \in \mathbb{C}\backslash\mathbb{N}^*$ dimensions has been given by Wilson [23]. It turns out that the dimension of this vector space $\mathcal{V}_d$ has to be infinite dimensional for generic d.[15] To obtain a vector space in $n \in \mathbb{N}^*$ dimensions a δ-distribution is introduced effectively, which localizes the integral on an n-dimensional hyperplane. The integration is defined such that it obeys general properties of radial symmetric Euclidean

[15] $\mathcal{V}_d$ has to be infinite dimensional, because one can regulate any integer dimensional vector space $\mathcal{V}_n$ with $n \in \mathbb{N}^*$ with it, which implies that $\forall n \in \mathbb{N}^* : \mathcal{V}_n \subset \mathcal{V}_d$ has to be a vector subspace.

integrals, like linearity and scaling as well as translation and rotation invariance

$$\int d^dL\,[af(\mathbf{L}) + b\,g(\mathbf{L})] = a\int d^dL\ f(\mathbf{L}) + b\int d^dL\ g(\mathbf{L})$$
$$\int d^dL\ f(a\mathbf{L}) = a^{-d}\int d^dL\ f(\mathbf{L})$$
$$\int d^dL\ f(\mathbf{L}+\mathbf{q}) = \int d^dL\ f(\mathbf{L})$$

for $a, b \in \mathbb{C}$ and any vector $\mathbf{q}$. Integrals defined over Minkowski space are obtained by a Wick rotation. The uniqueness of d-dimensional integrals can be proven by showing that the integral is unique for a set of basis functions $f_{a,\mathbf{q}}(\mathbf{L}) = \exp\left(-a^2(\mathbf{L}+\mathbf{q})^2\right)$ due to linearity. The radial basis functions are chosen to be Gaussian. Requiring that the integral of the basis function matches the result of integer-dimensional integrals

$$\int d^dL\ f_{a,\mathbf{q}}(\mathbf{L}) \quad = a^{-d}\int d^dL\ e^{-\mathbf{L}^2} = a^{-d}\pi^{d/2} \tag{2.34}$$

fixes the integral uniquely. Further it is demanded that

$$\int d^{d_1}L_1\, d^{d_2}L_2\ e^{-\mathbf{L}_1^2-\mathbf{L}_2^2} = \int d^{d_1+d_2}L\ e^{-\mathbf{L}^2}.$$

The next step is to construct a concrete formula to evaluate d-dimensional integrals. To do so, it is useful to separate the finite n-dimensional subspace containing all the external momenta $\mathbf{q}_j \in \mathbb{R}^n$ from the remaining (infinitely many orthogonal) components.

$$\begin{aligned} \mathbf{L} &= (L_1, L_2, \ldots, L_n, L_{n+1}, \ldots) = \mathbf{L}_{\|} + \mathbf{L}_{\perp} \quad \text{with} \\ \mathbf{L}_{\|} &= (L_1, L_2, \ldots, L_n, 0, \ldots) \quad \text{and} \quad \mathbf{L}_{\perp} = (0, \ldots, L_{n+1}, L_{n+2}, \ldots)\,. \end{aligned} \tag{2.35}$$

Finally the d-dimensional integral is defined to be

$$\int d^dL\ f(\mathbf{L}) := \int dL_1 dL_2 \cdots dL_n \int d^{d-n}L_{\perp}\ f(\mathbf{L}), \tag{2.36}$$

where the ordinary n-dimensional integral is performed *after* the integral over $d^{d-n}L_{\perp}$. Since only rotational invariant functions are considered, the integral over $d^{d-n}L_{\perp}$ shall be defined by general spherical coordinates

$$\int d^{d-n}L_{\perp}\ f(\mathbf{L}) := \mathrm{Vol}(S^{d-n-1})\int_0^{\infty} dL_{\perp}\ L_{\perp}^{d-n-1} f(\mathbf{L}) \quad \text{with} \quad \mathrm{Vol}(S^{m-1}) = 2\frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}. \tag{2.37}$$

$\mathrm{Vol}(S^{d-n})$ is the volume of the hypersphere S^{d-n}.

Hence the d-dimensional integral reads

$$\int d^dL\ f(\mathbf{L}) = \frac{2\pi^{(d-n)/2}}{\Gamma\left(\frac{d-n}{2}\right)}\int dL_1 dL_2 \cdots dL_n \int_0^{\infty} dL_{\perp}\ L_{\perp}^{d-n-1} f(\mathbf{L}). \tag{2.38}$$

The convergence of (2.38) depends on the dimension d. At $L_{\perp} = \infty$ the convergence improves for smaller d, but at $L_{\perp} = 0$ it improves for greater d. Next the formula (2.38) has to be extended to regularize potentially divergent integrals, i.e. a generalized formula should hold for all values of d. To do so (2.38) is manipulated in a dimension where it converges and then it is analytically continued to smaller values of d by explicitly subtracting and adding the singularities. In the following it shall be assumed that the function $f(\mathbf{L})$ does not depend on external momenta such that the parallel space can be set to zero. Since the function has to be invariant under rotations it has to be a function of L^2 effectively. It is assumed that $f(L^2)$ is analytic in L and that the integral

$$\int d^d p\, f(\mathbf{L}^2) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty dL\, L^{d-1} f(L^2). \tag{2.39}$$

converges for $0 < \Re(d) < d_{\max}$. Note that (2.39) is analytic in d, because $\Gamma(x)^{-1}$ is analytic in x. Now one can analytically continue the range of convergence of this integral to the range $-2 < \Re(d) < d_{\max}$ by rewriting (2.39) in the following way

$$\int d^d L\, f(\mathbf{L}^2) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \left[\int_r^\infty dL\, L^{d-1} f(L^2) + \int_0^r dL\, L^{d-1} \left[f(L^2) - f(0)\right] + f(0)\frac{r^d}{d}\right] \tag{2.40}$$

for any $r > 0$ which follows from power counting. Due to the analyticity of $f(L^2)$ the function can be Taylor expanded $f(L^2) = f(0) + L^2 f'(0) + \cdots$ and the most divergent part $f(0)$ for small d is simply subtracted. Further it immediately follows from (2.40) that

$$\begin{aligned}
\int d^0 L\, f(\mathbf{L}^2) =& \lim_{d\to 0} \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \left[\int_r^\infty dL\, L^{d-1} f(L^2) + \int_0^r dL\, L^{d-1} \left[f(L^2) - f(0)\right] + f(0)\frac{r^d}{d}\right] \\
=& 2f(0) \lim_{d\to 0} \frac{1}{d\,\Gamma\left(\frac{d}{2}\right)} = f(0).
\end{aligned}$$

If one is only interested in the range $-2 < \Re(d) < 0$ the formula can be further simplified by taking the limit $r \to \infty$ to obtain the simple formula

$$\int d^d L\, f(\mathbf{L}^2) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty dL\, L^{d-1} \left[f(L^2) - f(0)\right].$$

Hence continuing this procedure for simple poles which are polynomials in L^2 the divergence can be systematically subtracted (and added) to obtain a finite integral for any dimension d with

$$\begin{aligned}
\int d^d L\, f(\mathbf{L}^2) &= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty dL\, L^{d-1} \left[f(L^2) - f(0) - L^2 f'(0) - \cdots - \left(L^2\right)^l \frac{f^{(l)}(0)}{l!}\right], \\
\int d^{-2l} L\, f(\mathbf{L^2}) &= (-\pi)^{-l} f^{(l)}(0).
\end{aligned} \tag{2.41}$$

The first formula is valid in the range $-2l - 2 < \Re(d) < -2l$ for any integer $l > 0$ and the second one for any integer $l \geq 0$. So far (2.41) has been derived for a function which converges at least for $\Re(d) > 0$, but from now on (2.41) shall serve as a *definition* for general d-dimensional integrals even though if the integral does not converge for *any* d. This definition is sensible as long as the divergence is polynomial in L^2 at $L = \infty$. This is important. DR has been introduced to regulate Feynman graphs, however, these graphs may posse ultraviolet divergences in $d = 4$ which implies they diverge at $L = \infty$ for any

$\Re(d) > 0$. Thus the continuation process is not valid, but if (2.41) is taken as a definition of the integral, one can always find a region of convergence for some d such that the integral is finite, i.e. (2.41) gives a regulated well-defined definition of Feynman graphs.

The explicit continuation forces

$$\int d^dL\ (\mathbf{L}^2)^a = 0 \quad \text{for} \quad a \in \mathbb{C}, \tag{2.42}$$

because the integrals are regulated by subtracting powers of L^2 to render them finite.

To show the power of the formalism this discussion shall be finished by an example. Consider the function

$$g(\mathbf{L}^2) = \frac{(\mathbf{L}^2)^\alpha}{(\mathbf{L}^2 + M^2)^\beta} = \left(\mathbf{L}^2\right)^\alpha \frac{1}{\Gamma(\beta)} \int_0^\infty \frac{du}{u} u^\beta e^{-u\left(\mathbf{L}^2 + M^2\right)}.$$

In the last expression the generalized propagator has been Schwinger parametrized which is valid for $\Re(\beta) > 0$. This function can be integrated in general dimension using (2.41)

$$\begin{aligned}\int d^dL\ g(\mathbf{L^2}) &= \frac{\pi^{d/2}}{\Gamma(\beta)\Gamma(d/2)} \int_0^\infty \frac{du}{u} u^\beta e^{-uM^2} \int_0^\infty \frac{dL^2}{L^2} \left(L^2\right)^{\alpha + d/2} e^{-uL^2} \\ &= \pi^{d/2} \frac{\Gamma(d/2+\alpha)\Gamma(\beta-\alpha-d/2)}{\Gamma(\beta)\Gamma(d/2)} M^{2\alpha+d-2\beta}.\end{aligned} \tag{2.43}$$

The integral of $g(\mathbf{L}^2)$ converges (absolutely) for $\Re(\beta-\alpha-d/2) > 0$ and $\Re(\alpha+d/2) > 0$. However, the final result of (2.43) is valid for any value as long as the Γ-functions in the numerator are regular. Observe that $\Gamma(m)$ only has simple poles at $-m \in \mathbb{N}$. Thus the (possible) divergence of the integral (2.43) now resides in the simple poles of the Γ-function. Hence, even if the integral of the function $\mathbf{L}^2/\left(\mathbf{L}^2 + M^2\right)$ had been considered which does not converge for any positive d, (2.43) gives a regularized version of this integrated expression.

Therefore the actual quantity which has to be evaluated for Feynman graphs is given by (2.41). Generally this integral is convergent for some region such that the integral can be evaluated.

2.2.2 Four-Dimensional-Helicity Regularization

So far it has been discussed how to regularize scalar integrals. Tensor integrals shall be regularized in the same way, whereas the regularization is done for each component separately. But for particles transforming under a non-trivial representation of the Lorentz group like gluons which transform in the vector representation an additional subtlety arises. What happens to the helicity degrees of freedom? Are they continued to $d = 4 - 2\epsilon$ dimensions as well or can one keep them in four dimensions?

There are at least three DR schemes which enables one to regularize integrals for vector particles [141, 142].

1. It is possible to continue the polarization vectors of the observed and unobserved particles to $4-2\epsilon$ dimensions such that a gluon has $2-2\epsilon$ helicity states. This scheme is called **conventional dimensional regularization** [140].

		Conventional	't Hooft and Veltman	FDH
momentum	observed particles	$4-2\epsilon$	4	4
	unobserved particles	$4-2\epsilon$	$4-2\epsilon$	$4-2\epsilon$
helicity states	observed particles	$2-2\epsilon$	2	2
	unobserved particles	$2-2\epsilon$	$2-2\epsilon$	2

Table 2.1: Summarizing the three different regularization schemes.

2. Another scheme is that only the unobserved polarization vectors are continued to $4-2\epsilon$ dimensions but the observed polarization vectors are kept in four dimension. This scheme is known as the **'t Hooft and Veltman scheme** [143].
3. The **four-dimensional-helicity scheme** (FDH) keeps the polarization vectors of the observed and unobserved particles in four dimensions.

The defining properties of the three different regularization schemes are listed in table 2.1. Each scheme has its advantages and disadvantages: For example the conventional regularization scheme is conceptually most appealing since it treats all quantities in a uniform way. On the other hand it is incompatible with the spinor-helicity formalism which cannot be defined in $4-2\epsilon$ dimensions. However, the FDH scheme keeps the external helicity states and momenta fixed; hence it is compatible with spinor-helicity variables. It can be shown that at one-loop all regularization schemes are equivalent [144–146].

Later the calculations shall be done with spinor-helicity variables, hence to regularize Feynman integrals the FDH scheme is used in $4-2\epsilon$ dimensions with $\epsilon<0$. Thus one separates the d-dimensional Minkowskian vector

$$L := (l,\mu) \in \mathbb{R}^{1,3} \times \mathbb{R}^{-2\epsilon}, \tag{2.44}$$

where the two vector spaces are orthogonal $l\cdot\mu=0$. Therefore the momentum squared is given by $L^2=l^2-\mu^2$ using the mostly minus convention $(+,-,-,\ldots)$. Thus a higher dimensional vector can be viewed as a lower dimensional vector whose mass-squared is shifted by μ^2.[16] In particular a scalar product with a four-dimensional vector projects always onto the four-dimensional vector space e.g. $\varepsilon_4\cdot L=\varepsilon_4\cdot l$. This implies one can treat the loop integration with "massless" loop momentum L in d dimensions as a massive loop momentum l in four dimensions.

Following the conventions of [80] the d-dimensional scalar Feynman integrals at one loop are defined by the expression

$$\frac{i}{(4\pi)^{2-\epsilon}} I_n\left[\mu^{2r}\right] := \int \frac{d^4 l}{(2\pi)^4} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\mu^{2r}}{D_0\cdots D_{n-1}}, \tag{2.45}$$

where

$$D_i = Q_i^2 + i\epsilon = (q_i+L)^2 + i\epsilon = (q_i+l)^2 - \mu^2 + i\epsilon, \qquad q_j = \sum_{i=1}^{j} p_i \tag{2.46}$$

[16] Note that $L^2=l^2-\mu^2$ due to the mostly minus signature. This implies that if $L^2=M^2$ then $l^2=L^2+\mu^2=M^2+\mu^2$. Furthermore the elements of the vector space $\mathbb{R}^{-2\epsilon}$ are defined in (2.35).

and the p_i are the massless external momenta. Thus the dependence on the non-integer integration dimension (-2ϵ) is fully encoded in μ^2. It serves as mass term. Hence the FDH scheme can be viewed as integrating a fictitious mass term.

The expressions for the scalar bubble $I_2\,[1;S]$, the one-mass triangle $I_3\,[1;S]$ and the zero-mass box $I_4\,[1;S,T]$ are known.[17] They are simple functions of the Mandelstam invariants $S = \langle 12\rangle\,[21]$, $T = \langle 14\rangle\,[41]$, and $U = \langle 13\rangle\,[31]$.

$$\begin{aligned} I_2\,[1;S] &= r_\Gamma \frac{(-S)^{-\epsilon}}{\epsilon\,(1-2\epsilon)}, \qquad \text{with} \qquad r_\Gamma := \frac{\Gamma\,(1+\epsilon)\,\Gamma^2\,(1-\epsilon)}{\Gamma\,(1-2\epsilon)}, \\ I_3\,[1;S] &= -\frac{r_\Gamma}{\epsilon^2}\,(-S)^{-1-\epsilon}, \\ I_4\,[1;S,T] &= r_\Gamma \frac{2}{ST}\left[\frac{(-S)^{-\epsilon}}{\epsilon^2}\,{}_2F_1\left(1,-\epsilon,1-\epsilon;1+\frac{S}{T}\right) + \frac{(-T)^{-\epsilon}}{\epsilon^2}\,{}_2F_1\left(1,-\epsilon,1-\epsilon;1+\frac{T}{S}\right)\right]. \end{aligned} \tag{2.47}$$

These expressions are exact for any ϵ. Thus they are valid in any dimension. This can be made manifest by a shift of ϵ, e.g.

$$d = 6-2\epsilon \text{ is obtained by } \epsilon \to \epsilon - 1, \qquad d = 8-2\epsilon \text{ is obtained by } \epsilon \to \epsilon - 2.$$

One the other hand the μ^2 dependence can be removed by shifting the dimension of the integral under consideration using the formula

$$I_n^{d=4-2\epsilon}\left[\mu^{2r}\right] = -\epsilon\,(1-\epsilon)\,(2-\epsilon)\cdots(r-1-\epsilon)\,I_n^{d=4+2r-2\epsilon}\,[1]\,. \tag{2.48}$$

Hence, this formula solves all the integrals of the form (2.47) for any μ^2 factor, because (2.48) removes the dependence on the fictitious mass μ^{2r} parameter by shifting the dimension of the integration variable to $4+2r-2\epsilon$. The formula (2.48) can be derived by using (2.41)

$$\begin{aligned} \int \frac{d^{4-2\epsilon}L}{(2\pi)^{4-2\epsilon}}\left(\mu^2\right)^r f\left(p,\mu^2\right) &= \int \frac{d^4l}{(2\pi)^4}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\left(\mu^2\right)^r f\left(p,\mu^2\right) \\ &= \mathrm{Vol}(S^{-2\epsilon-1})\int\frac{d^4l}{(2\pi)^4}\int_0^\infty \frac{d\mu^2}{2(2\pi)^{-2\epsilon}}\left(\mu^2\right)^{-1-\epsilon+r} f\left(p,\mu^2\right) \\ &= (2\pi)^{2r}\,\frac{\mathrm{Vol}(S^{-2\epsilon-1})}{\mathrm{Vol}(S^{2r-2\epsilon-1})}\int\frac{d^4l}{(2\pi)^4}\int\frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f\left(p,\mu^2\right) \\ &= (4\pi)^r\,\frac{\Gamma(r-\epsilon)}{\Gamma(-\epsilon)}\int\frac{d^4l}{(2\pi)^4}\int\frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f\left(p,\mu^2\right) \\ &= -\epsilon(1-\epsilon)\cdots(r-1-\epsilon)(4\pi)^r\int\frac{d^{4+2r-2\epsilon}L}{(2\pi)^{4+2r-2\epsilon}} f\left(p,\mu^2\right). \end{aligned}$$

For the particular case $f\left(p,\mu^2\right) = 1$ the desired relation is obtained [137]. Later the following

[17] Even if the theory under consideration is massless ($p_i^2 = 0$) the triangle diagram depends on an additional (mass-like) parameter since on one of the three legs two momenta are attached which do not square to zero, e.g. $(p_1+p_2)^2 = S \neq 0$.

explicit expressions are used in four dimensions:

$$
\begin{aligned}
I_2\left[\mu^2; S\right] &= -\frac{S}{6} + \mathcal{O}(\epsilon), & I_2\left[\mu^4; S\right] &= -\frac{S^2}{60} + \mathcal{O}(\epsilon), \\
I_2\left[\mu^6; S\right] &= -\frac{S^3}{420} + \mathcal{O}(\epsilon), \\
I_3\left[\mu^2; S\right] &= \frac{1}{2} + \mathcal{O}(\epsilon), & I_3\left[\mu^4; S\right] &= \frac{S}{24} + \mathcal{O}(\epsilon), \\
I_3\left[\mu^6; S\right] &= \frac{S^2}{180} + \mathcal{O}(\epsilon), \\
I_4\left[\mu^2; S, T\right] &= \mathcal{O}(\epsilon), & I_4\left[\mu^4; S, T\right] &= -\frac{1}{6} + \mathcal{O}(\epsilon), \\
I_4\left[\mu^6; S, T\right] &= -\frac{S+T}{60} + \mathcal{O}(\epsilon), & I_4\left[\mu^8; S, T\right] &= -\frac{1}{840}\left(2S^2 + ST + 2T^2\right) + \mathcal{O}(\epsilon).
\end{aligned}
\tag{2.49}
$$

2.3 Veltman-Passarino Reduction

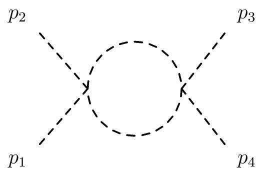
Evaluating Feynman integrals is a daunting task. Thus different methods have been developed to rewrite unknown integrals in terms of known ones. A widely used method for computing Feynman integrals at one-loop order is the integral reduction invented by Veltman and Passarino called **Veltman-Passarino reduction** [147, 148]. The idea is to reduce a generic one-loop integral $T_{N,\mathcal{N}(L)}$ with N external points and numerator factor $\mathcal{N}(L)$ to a linear combination of one-loop scalar integrals which provide a basis for the tensor integrand. In this thesis only the integral reduction of four-point massless amplitudes is discussed, since these are the type of integrals which shall appear in section 2.6. For further information the reader is referred to [149] and [138]. The following discussion is based on the second reference.

It shall be shown that for a massless theory any four-point tensor integral $T_{4,\mathcal{N}(L)}$ with a numerator $\mathcal{N}(L)$ which is polynomially bounded in the loop momentum L^a, $a \leq 4$ can be written as a linear combination of scalar integrals $I_N^{(j_N)}$ with $N \leq 4$ external particles:

$$
T_{4,\mathcal{N}(l)} = \frac{i}{(4\pi)^{2-\epsilon}} \left(\sum_{j_4} c_{4,j_4} I_4^{(j_4)} + \sum_{j_3} c_{3,j_3} I_3^{(j_3)} + \sum_{j_2} c_{2,j_2} I_2^{(j_2)} \right). \tag{2.50}
$$

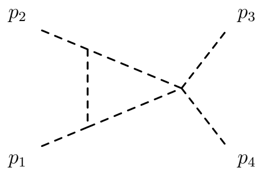
The summations j_N are taken over the different possible distributions of the external momenta on the $N \leq 4$ legs of $I_N^{(j_N)}$ and the coefficients c_{N,j_N} are algebraic four-dimensional quantities which depend on external data i.e. momenta and polarizations. The definition of the scalar integrals is given in (2.45) and they are diagrammatically listed in table 2.2. Note that no tadpole scalar integral appears in the decomposition of (2.50). The reason is that for a massless theory the tadpole integral is of the form (2.42); hence it has to vanish in DR. It is instructive to separate the integration measure into the region of physical dimensions d_p and the transverse dimensions d_t such that $d = d_p + d_t$. Due to momentum conservation there are only $N-1$ independent momenta for an amplitude with $N \leq 4$ external points. Thus it can be concluded that $d_p = N - 1$. To perform the reduction it is appropriate to use a dual basis $\{v_i \in \mathbb{R}^{1,3} | 1 \leq i \leq d_p\}$ to the region momenta $q_i = \sum_{j=1}^{i} p_j$ which satisfies $v_i \cdot q_j = \delta_{ij}$. This basis is called the Neerven-Vermaseren basis. The explicit construction of $v_i(q_1, \ldots, q_{d_p})$ can be found in [138]. For the following discussion it shall be remarked that

Bubbles:



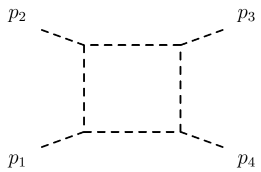
$$\frac{i}{(4\pi)^{2-\epsilon}} I_2\left[1; S\right] = \int \frac{d^d L}{(2\pi)^d} \frac{1}{D_0 D_2}$$

Triangles:



$$\frac{i}{(4\pi)^{2-\epsilon}} I_3\left[1; S\right] = \int \frac{d^d L}{(2\pi)^d} \frac{1}{D_0 D_1 D_2}$$

Boxes:



$$\frac{i}{(4\pi)^{2-\epsilon}} I_4\left[1; S, T\right] = \int \frac{d^d L}{(2\pi)^d} \frac{1}{D_0 D_1 D_2 D_3}$$

Table 2.2: Scalar integral basis for tensor integrals with four external legs. The inverse propagators are defined by $D_i = (L + q_i)^2 + i\epsilon$ and $q_i = \sum_{j=1}^{i} p_j$. S and T are Mandelstam variables. In this basis it has already been anticipated that only cubic and quartic vertices shall appear in the theory which is considered later on.

the basis exists and posses the following properties:

$$n_r \cdot n_s = \delta_{rs}, \quad v_i \cdot n_r = 0, \quad q_i \cdot n_r = 0 \quad \text{and} \quad v_i \cdot q_j = \delta_{ij}, \tag{2.51}$$

where $\{n_r \in \mathbb{R}^{1,d-1} | 1 \leq i \leq d_t\}$ is the dual basis of the transverse dimension.[18] The first condition states simply orthonormality of this basis whereas the second and third condition imply that this vector space is orthogonal to the physical (dual) vector space. Therefore the loop momentum decomposes into

$$L^\mu = \sum_{i=1}^{d_p} (L \cdot q_i)\, v_i^\mu + \sum_{r=1}^{d_t} (L \cdot n_i)\, n_i^\mu .$$

In general, the loop momentum L is contracted with vectors of external data u_i in the numerator $\mathcal{N}(L)$. In particular if L is contracted with the region momenta q_i one can rewrite this scalar product in terms of inverse propagators:

$$(L \cdot q_i) = \frac{1}{2}\left(\left[(L + q_i)^2 + i\epsilon\right] - \left[L^2 + i\epsilon\right] - q_i^2\right) = \frac{1}{2}\left(D_i - D_0 - q_i^2\right). \tag{2.52}$$

This reduces the degree of L in $\mathcal{N}(L)$ because this substitution re-expresses any tensor integral $T_{N,\mathcal{N}(L)}$ as a linear combination of integrals with $N' \leq N$ legs.

[18]Regarding the discussion of chapter 2.2.1 the dual transverse space has to be infinite dimensional since $d = 4 - 2\epsilon$; hence the vector space $V_{-2\epsilon}$ has infinitely many basis vectors. However, only the dual vector n_ϵ to $l_{-2\epsilon}$ is needed which satisfies $L \cdot n_\epsilon = l_{-2\epsilon} \cdot n_\epsilon = 1$. The notation (2.44) has been used.

After these preliminary considerations the reduction procedure of a four-point tensor integral can be applied straight forwardly. The tensor integral is of the form

$$T_{4,\mathcal{N}(L)} = \int \frac{d^d L}{(2\pi)^d} \frac{\mathcal{N}_r(L)}{D_0 D_1 D_2 D_3} \quad \text{with} \quad \mathcal{N}_r(L) = \prod_{i=1}^{r} u_i \cdot L, \tag{2.53}$$

where u_i is a vector which depends on physical momenta and polarization vectors. Since a four-point amplitude is considered, the physical space has dimension $d_p = 3$ spanned by q_1, q_2 and q_3 and the transverse space has dimension $d_t = 1 - 2\epsilon$. Thus

$$\begin{aligned} L^\mu &= \sum_{i=1}^{3} (L \cdot q_i) v_i^\mu + (L \cdot n_4) n_4^\mu + (L \cdot n_\epsilon) n_\epsilon^\mu, \\ u_j \cdot L &= \sum_{i=1}^{3} (L \cdot q_i)(u_j \cdot v_i) + (L \cdot n_4)(u_j \cdot n_4) \end{aligned} \tag{2.54}$$

by using $u_j \cdot n_\epsilon = 0$. This follows from the fact that n_ϵ^μ is orthogonal to all vectors of the physical space. Inserting (2.52) into (2.54) gives

$$u_j \cdot L = (L \cdot n_4)(u_j \cdot n_4) + \mathcal{O}(D_i) + const,$$

where $\mathcal{O}(D_i)$ and *const* refer to terms proportional to inverse propagators and independent of L, respectively. These steps already decrease the L dependence of the numerator:

$$\frac{\mathcal{N}_r(L)}{D_0 D_1 D_2 D_3} = \frac{\prod_{i=1}^{r} u_i \cdot L}{D_0 D_1 D_2 D_3} = \sum_{i=1}^{r} c_i \frac{(L \cdot n_4)^i}{D_0 D_1 D_2 D_3} + \frac{const}{D_0 D_1 D_2 D_3} + \text{lower point integrands.}$$

It follows from (2.54) that the coefficients c_i depend on external data.

It can be seen that the second and third term have been successfully reduced to scalar integrals or tensor integrals with less external points (one, two or three), respectively.[19] Further simplifications can be achieved by squaring the first equation in (2.54) and using the relations (2.51) and $L^2 + i\epsilon = D_0$ as well as (2.52) to rewrite the factor

$$(L \cdot n_4)^2 = -(L \cdot n_\epsilon)^2 + const + \mathcal{O}(D_i) = -\mu^2 + const + \mathcal{O}(D_i),$$

where the decomposition $L = (l, \mu)$ has been used. Hence the numerator is of the form

$$\mathcal{N}_r(L) = b_0 + b_1 (L \cdot n_4) + b_2 \mu^2 + b_3 (L \cdot n_4) \mu^2 + b_4 \mu^4 + \text{lower point integrands} \tag{2.55}$$

with the coefficients b_j depending on external polarization vectors and momenta.

The same reduction procedure works for tensor integrals with two or three external points. However, for tensor integrals with two (three) external points the physical space has dimension $d_p = 1$ ($d_p = 2$) and therefore the decomposition of L^μ is different.

Now it shall be shown that $T_{4,\mathcal{N}(L)}$ can be written as a linear combination of scalar integrals and lower point tensor integrals. Plugging into (2.53) the numerator representation (2.55) gives

$$T_{4,\mathcal{N}(L)} = \int \frac{d^{d_p + d_t} L}{(2\pi)^d} \frac{1}{D_0 D_1 D_2 D_3} \left(b_0 + b_1 (L \cdot n_4) + b_2 \mu^2 + b_3 (L \cdot n_4) \mu^2 + b_4 \mu^4 \right), \tag{2.56}$$

[19] Removing a propagator implies that two external legs are fused together. This product only counts as one external point after the fusion process.

where the lower point integrals have been neglected. Due to rotational invariance of the transverse space all terms proportional to $(L \cdot n_4)$ shall vanish. This can be seen by decomposing the loop momentum

$$L_{\|}^{\mu} = \sum_{i=1}^{3} (L \cdot q_i) v_i^{\mu} \in \mathbb{R}^{1,3}, \quad \text{and} \quad L_{\perp}^{\mu} = (L \cdot n_4) n_4^{\mu} + (L \cdot n_\epsilon) n_\epsilon^{\mu} \in \mathbb{R}^{1,d-1}$$

which implies that the denominators of the propagators are of the form

$$D_i = (L + q_i)^2 + i\epsilon = L_{\perp}^2 + (L_{\|} + q_i)^2 + i\epsilon.$$

Hence they are rotational invariant in the transverse dimension. The numerator $\mathcal{N}_r(L)$ is of the order $L_{\perp}^4$ such that a general integral can be of the form of any of the following components

$$\begin{aligned} &\int \frac{d^{d_t} L_{\perp}}{(2\pi)^{d_t}} \frac{1}{D_0 D_1 D_2 D_3} \begin{pmatrix} L_{\perp}^{\mu_1} \\ L_{\perp}^{\mu_1} L_{\perp}^{\mu_2} \\ L_{\perp}^{\mu_1} L_{\perp}^{\mu_2} L_{\perp}^{\mu_3} \\ L_{\perp}^{\mu_1} L_{\perp}^{\mu_2} L_{\perp}^{\mu_3} L_{\perp}^{\mu_4} \end{pmatrix} \\ &= \int \frac{d^{d_t} L_{\perp}}{(2\pi)^{d_t}} \frac{1}{D_0 D_1 D_2 D_3} \begin{pmatrix} 0 \\ d_t^{-1} \eta^{\mu_1\mu_2} L_{\perp}^2 \\ 0 \\ (d_t^2 + 2d_t)^{-1} \left(L_{\perp}^4 \eta^{\mu_1\mu_2} \eta^{\mu_3\mu_4} + \text{perm} \right) \end{pmatrix} \end{aligned}$$

because the first and third component are odd functions. Applying this argument to (2.56) simplifies the integral schematically to

$$T_{4,\mathcal{N}(L)} = \frac{i}{(4\pi)^{2-\epsilon}} \left(b_0 I_4[1] + b_2 I_4[\mu^2] + b_4 I_4[\mu^4] \right) + \text{lower point integrals},$$

where the notation of (2.45) has been used. Similarly this technique can be applied to tensor integrals with two or three external points. This shows that any tensor integral can be written as a linear combination of scalar integrals. Since the scalar integrals are analytically known the computation of tensor integrals is reduced to a problem in linear algebra, i.e. finding the coefficients c_{k,j_k}.

2.4 Double Copy Method

In the last section it has been shown that the evaluation of certain integrals can be trivialized by rewriting them as a linear combination of known scalar integrals. However, not only is it difficult to compute the integrals which appear in perturbation theory but also to construct the integrand of a certain amplitude can be quite challenging. In particular this holds true for integrands which are derived from a gravity action. For example DeWitt has shown that the cubic and quartic vertex of Einstein gravity contain 171 and 2850 separate terms, respectively. The reason is that the vertices have to be invariant under arbitrary permutations of the external momentum indices [62]. Furthermore linearizing $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{2\kappa} \int d^4x \sqrt{-G}\, R, \tag{2.57}$$

around the Minkowskian background metric $\eta_{\mu\nu}$ yields infinitely many interaction vertices due to the Taylor expansion of the square root of the determinant of the metric $\sqrt{-G}$. The kinetic term of the fluctuation $h_{\mu\nu}$ is contained in the Ricci scalar R. In a quantized theory $h_{\mu\nu}$ is interpreted as the graviton. κ is the Einstein constant.

Hence, it is not advisable to calculate gravity amplitudes using standard Feynman diagrammatic methods. In comparison building integrands in a gauge theory is much simpler since their Lagrangians have a better structure for the purpose of quantum field theoretic calculations. The famous Yang-Mills (YM) Lagrangian

$$\mathcal{L}^{\mathrm{YM}} = -\frac{1}{4} F^{\mu\nu,a} F^{a}_{\mu\nu} - \frac{1}{2\xi} \left(\partial^{\mu} A^{a}_{\mu} \right) , \tag{2.58}$$

with the field strength tensor $F^{c}_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu$ is an example of this, because it posses a cubic and quartic vertex only. The last term is the gauge fixing term which is necessary to invert the free equations of motions, i.e. to obtain the propagator. g is the coupling constant of the vertices and f^{abc} is the gauge group structure constant under which the gauge field A^a_μ is charged. In the following the gauge group is given by $\mathrm{SU}(N)$. The Feynman rules for this theory are given in appendix A.2.1.

Due to this observation it might be tempting to determine gravity integrands from gauge theory integrands. The first realization of this idea has been developed by Kawai, Lewellen and Tye (KLT) who derived a formula which relates closed string amplitudes to a sum of products of open string amplitudes at tree-level [150]. Taking the point-particle limit where the string becomes point-like their formula expresses tree-level gravity amplitudes as a sum of products of tree-level gauge theory amplitudes. Combining this method with generalized unitarity even loop-level gravity integrands can be derived from gauge theory tree-level amplitudes [151]. Furthermore, the KLT formula has been generalized to one-loop amplitudes as well [152, 153].[20]

A similar but different method to generate the gravity integrand is simply given by fusing two gauge theory integrands together. A decade ago Bern, Carrasco and Johansson [60] have argued that this is possible if at least one set of gauge theory numerators obeys certain properties. If one represents the gauge theory integrands with trivalent vertices only such that all kinematic numerators are arranged to obey a Jacobi-like relation mirroring the property of the colour degrees of freedom, then the numerators of a gravity amplitude follow by simply multiplying pairs of kinematic numerators of two gauge theories.

To apply this technique it is necessary to express an n-loop amplitude in a trivalent fashion, which shall be illustrated for a $\mathrm{SU}(N)$ gauge theory amplitude with all particles transforming in the adjoint representation. Such an amplitude can be written in the following way

$$\mathcal{A}^n_m = i^{n-1} g^{m-2+2n} \sum_{\mathcal{S}_m} \sum_{j \in \Gamma} \int \frac{d^{dn} L}{(2\pi)^{dn}} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} D_{\alpha_j}} , \tag{2.59}$$

where the amplitude consists of three main ingredients:

- The colour dependence is encoded in the colour factors c_j which are a chain of the

[20] Note that the generalization at one loop needs the inverse of all but one propagator to be linear in the loop momentum.

adjoint generators if^{abc}.[21] They obey the Jacobi identity, which for the four particle case may be sketched as $c_s = c_t + c_u$ with $c_s := if^{abe'}if^{e'cd}$, $c_t := -if^{ade'}if^{e'bc}$ and $c_u := -if^{ace'}if^{e'db}$. Furthermore, the adjoint generators if^{abc} are antisymmetric in their indices which implies that the colour factors are antisymmetric under transposition. $c_i = -c_j$.

- The set of all reduced Feynman propagators $1/(L^2 - m^2)$ associated to the j^{th} graph are denoted by the inverse of the product $\prod_{\alpha_j} D_{\alpha_j}$.
- The numerators n_j account for the remaining kinematic dependence of the amplitude. Note that the factors $\pm i$ of the Feynman propagators are also absorbed in n_j.

The second sum runs over all distinct, non-isomorphic, trivalent graphs Γ and the first one over all $|\mathcal{S}_m| = m!$ permutations of the external legs. Any over-counting of the j^{th} diagram is removed by the symmetry factor S_j (including these of internal automorphism symmetries with external legs fixed). Note that by using the identity $1 = D_{\alpha_j}/D_{\alpha_j}$ any graph in a diagrammatic expansion can be made trivalent formally.

Representing the amplitude in the form given in (2.59) reveals the parallel treatment of colour degrees of freedom c_j and kinematic degrees of freedom n_j and is especially powerful if one arranges the kinematic numerators in such a way that they obey the same algebraic relations as the corresponding colour factors

$$\left.\begin{aligned} c_s &= c_t + c_u \\ c_i &= -c_j \end{aligned}\right\} \implies \begin{cases} n_s = n_t + n_u \\ n_i = -n_j \end{cases}. \tag{2.60}$$

This relation is called **colour-kinematics duality** (CKD).

It has been conjectured by Bern, Carrasco and Johanson [60, 61] and shown at tree-level ($n = 0$) in refs. [154–159] that it is always possible to arrange all the numerators n_i of a diagram in such a way that they obey (2.60).

To illustrate this property the simple example of a four-point tree-level amplitude of YM theory in four dimensions shall be calculated with this method [60]. According to (2.59) the $(+,+,-,-)$ helicity amplitude can be written in the following way:

$$\mathcal{A}_4^0 = -ig^2\left(\frac{c_s n_s}{S} + \frac{c_t n_t}{T} + \frac{c_u n_u}{U}\right)$$

To obtain a simple result the numerators are expressed in spinor-helicity variables. A suitable gauge choice for the four polarization vectors $\epsilon_i(p_i)$ is given by $r_1 = r_2 = p_4$ and $r_3 = r_4 = p_1$ such that the four-point vertex gives no contribution, because at least two pairs of polarization vectors are contracted. A straightforward calculation of this amplitude with the Feynman rules (A.1) leads to

$$n_s = 2\frac{\langle 12\rangle^2\,[34]^2}{T}, \quad n_u = 2\frac{\langle 12\rangle^2\,[34]^2}{T} \quad \text{and} \quad n_t = 0, \tag{2.61}$$

which shows that (2.60) is satisfied. However, this is only one particular gauge choice and there is an entire family of CKD representations. For the numerators (2.61) CKD holds

[21] This coincides with the conventions given in [64] which differ from the conventions in [63].

for any reparametrization of the form $n_S \to n_S + S\ \Delta(p_i, \epsilon_i)$ and $n_j \to n_j - j\ \Delta(p_i, \epsilon_i)$ for $j \in \{, T, U\}$, because the colour factors c_j obey the Jacobi relation $c_s = c_t + c_u$. These kind of transformations are called generalized gauge transformations and they correspond to shifting the contribution of the quartic vertex into different channels.

A map of the form $n_j \to n_j + \Delta_j(p_i, \epsilon_i)$ is a generalized gauge transformation if it leaves the integrand of (2.59) invariant. I.e. any choice of Δ_i such that the constraint

$$\sum_{j\in\Gamma} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} D_{\alpha_j}} = 0 \tag{2.62}$$

is satisfied is a valid generalized gauge transformation [154].

It is a striking feature of CKD that integrands for gravity amplitudes can be easily constructed from integrands of gauge theories if at least one set of the gauge theory numerators n_j or $\tilde{n}_j$ satisfies (2.60):

$$\mathcal{M}_m^n = i^{n-1} \left(\frac{\kappa}{4}\right)^{m-2+2n} \sum_{\mathcal{S}_m} \sum_{j\in\Gamma} \int \frac{d^{dn}L}{(2\pi)^{dn}} \frac{1}{S_j} \frac{\tilde{n}_j n_j}{\prod_{\alpha_j} D_{\alpha_j}}. \tag{2.63}$$

The reason that only one of the numerators has to satisfy the duality stems from the relation (2.62). Assuming that the sets $\{n_i\}$ and $\{\tilde{n}_i\}$ satisfy CKD while the numerators $n_i^{\cancel{\text{CKD}}} = n_i + \Delta_i$ do not obey CKD the double copied amplitude (2.63) is still the same. This feature is due to the generalized gauge transformations (2.62) because the part Δ_i which does not satisfy the duality is simply projected out:

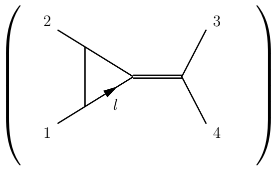
$$\sum_{j\in\Gamma} \frac{1}{S_j} \frac{\tilde{n}_j n_j^{\cancel{\text{CKD}}}}{\prod_{\alpha_j} D_{\alpha_j}} = \sum_{j\in\Gamma} \frac{1}{S_j} \frac{\tilde{n}_j n_j}{\prod_{\alpha_j} D_{\alpha_j}} + \sum_{j\in\Gamma} \frac{1}{S_j} \frac{\tilde{n}_j \Delta_j}{\prod_{\alpha_j} D_{\alpha_j}} = \sum_{j\in\Gamma} \frac{1}{S_j} \frac{\tilde{n}_j n_j}{\prod_{\alpha_j} D_{\alpha_j}},$$

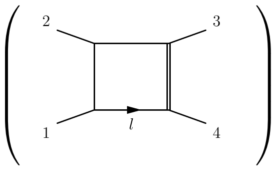
since the numerators $\tilde{n}_i$ inherit the same algebraic properties as the colour factors and therefore satisfy (2.62) with c_j replaced by $\tilde{n}_j$. Furthermore, gauge invariance of both gauge theory amplitudes implies invariance of the corresponding double copied amplitude under linearized diffeomorphisms, i.e. (2.63) is an amplitude of *some* gravity theory [64]. The unitarity method motivates an extension of this feature to loop-level by reducing amplitudes containing loops to tree-level amplitudes and demanding that CKD holds for all cuts. This feature of obtaining gravity amplitude integrands has been proven at tree-level for pure gravity using BCFW recursion relations [54] in four dimensions [154]. However, a general proof is still missing. The **double copy** (DC) construction can be viewed as a generalization of the Kawai-Lewellen-Tye relations [150] and a loop generalization thereof.

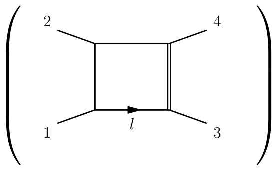
If a gravity integrand of theory $\mathcal{G}$ is constructed according to (2.63) from the integrands of the gauge theories $\mathcal{Q}_i$ the notation $\mathcal{G} = \mathcal{Q}_1 \otimes_{\text{DC}} \mathcal{Q}_2$ is used.

However, in general it might be quite challenging to find a CKD representation of a certain gauge theory. Therefore a modified DC construction has been developed where the numerators do not have to satisfy CKD. To compensate for the failure of the Jacobi identity of the numerators, contact terms have to be added to the gravity integrand. This method has been successfully used to obtain the five-loop $\mathcal{N} = 8$ supergravity integrand for four external points [67, 68].

Since gauge theory integrands are much easier to calculate than gravity integrands, the DC prescription gives a powerful tool to build gravity integrands. In particular, it is enough to construct a certain number of master integrands, because all other integrands are determined by the relation (2.60). Diagrammatically this can be sketched for four points as

n 

$= n$ 

$- n$ 

, (2.64)

where $n(\ldots)$ represents the numerator of the corresponding graph in brackets.

Applying the same reasoning the bubble graphs may be re-expressed as the difference of triangle graphs. Therefore the master numerators for four points at one-loop are given by the numerators of the box diagrams since all other numerators can be obtained by applying the kinematic Jacobi identity (2.60).

2.5 Computing Amplitudes in $\mathcal{N} = 0$ Supergravity Coupled to Yang-Mills

2.5.1 Equivalence of the Antisymmetric $B_{\mu\nu}$-Field and the Axion χ

It has been shown that $\mathcal{N} = 0$ supergravity (SUGRA) can be obtained by double copying YM: SUGRA = YM $\otimes_{\mathrm{DC}}$ YM [154]. This theory contains the graviton, the Kalb-Ramond field B and the dilaton φ. But classically B is dual to a scalar, called axion χ, in four dimension. This is explained in [160]. A short review about this relation between both theories has been given in the appendix of [79].

However, this duality only proves classical but not quantum equivalence. On the one hand in the literature there have been arguments that both theories are different at the quantum level [161]. It is for example known that they differ by topological terms. On the other hand it has been argued that this duality transformation even holds quantum mechanically, at least for expectation values [162–165]. In addition, perturbative calculations at two loops have shown that even though the ultraviolet behaviour of both theories differ their expectation values of the observables are identical [166]. Although not conclusive there are many reasons which support the claim that $\mathcal{N} = 0$ supergravity is quantum equivalent to axion-dilaton gravity. In this work one-loop amplitudes are calculated only. It has been shown that at one-loop order the effective actions of both theories differ only by a topological term which does not affect perturbation theory. This implies that both theories yield the same amplitudes at one-loop [167].

An analysis of the spectrum of YM $\otimes_{\mathrm{DC}}$ YM+ϕ^3 shows that this theory equals $\mathcal{N} = 0$ supergravity coupled to YM, but in accordance with the evidence of the aforementioned references this theory is equivalent to Einstein-Yang-Mills theory coupled to a dilaton and axion in the regime of interest. Hence, in this thesis YM $\otimes_{\mathrm{DC}}$ YM+ϕ^3 shall be referred as Einstein-Yang-Mills theory coupled to a dilaton and axion (EYM). For pure gravity coupled to YM the name "pure EYM" is used. Therefore removing the dilaton and axion in EYM leads to pure EYM.

2.5.2 Double copy of YM and YM+ϕ^3

Four-point positive helicity amplitudes at one-loop in EYM are of relative simple form because they are given by rational functions of the external data. This follows from the supersymmetric Ward-Takahashi identities (2.19) which imply that all positive and all-but-one positive helicity amplitudes in EYM vanish at tree-level. Thus the one-loop amplitude has vanishing unitarity cuts in four dimensions so that the amplitude cannot have any discontinuities in the Mandelstam invariants. Thus it can be concluded that they have to be rational functions of these.

In this section all relevant contributions at one-loop order of EYM for particles with identical helicities are established in four dimensions. The theory is defined by the Lagrangian

$$\mathcal{L}^{\mathrm{EYM}}=\frac{\sqrt{-G}}{\kappa^2}\left(-2\,R+\partial_\mu\varphi\partial^\mu\varphi+e^{2\varphi}\partial_\mu\chi\partial^\mu\chi\right)-\frac{\sqrt{-G}}{4}\left(e^{-\varphi}F^a_{\mu\nu}F^{\mu\nu,a}+i\chi F^a_{\mu\nu}\tilde{F}^{\mu\nu,a}\right) \tag{2.65}$$

where G is the determinant of the metric $G_{\mu\nu}$, the scalar curvature is encoded in the Ricci scalar R and $\tilde{F}^a_{\mu\nu}=\frac{i}{2}\sqrt{-G}\,\epsilon_{\mu\nu\varrho\sigma}F^{\varrho\sigma,a}$ represents the dual field strength tensor. The scalars φ and χ are the dilaton and axion, respectively. Comparing (2.65) with pure Einstein gravity (2.57) and YM (2.58) reveals that EYM is basically given by gravity coupled to YM and the two scalars, the axion and the dilaton.

Obviously the structure of the Lagrangian (2.65) reveals that it is rather tedious to calculate amplitudes by the standard method of Feynman diagrams. However, an efficient method to compute amplitudes in this theory is established by the following algorithm:

1. Construct the gravity integrand using the DC method.
2. Veltman-Passarino reduce the gravity integrand to a linear combination of scalar integrands.
3. Evaluate the known scalar integrals.

The 2. and 3. step shall be automatized which makes the algorithm very efficient. The Veltman-Passarino reduction is implemented by the `Mathematica` package `FeynCalc` [3, 4] and re-expresses the amplitude in terms of scalar integrals of the form (2.45), which can immediately be written as rational functions of Mandelstam invariants (2.49). Hence, the only non-trivial step is to build the integrand of an amplitude. It shall be shown that this can be easily done using the double copy method described in section 2.4.

The authors of ref. [63, 64, 66] confirmed at tree level that the DC procedure of YM defined in (2.58), and Yang-Mills coupled to a biadjoint scalar (YM+ϕ^3)[22]

$$\begin{aligned}\mathcal{L}^{\mathrm{YM}+\phi^3} &= \mathcal{L}^{\mathrm{YM}} + \frac{1}{2}\left(D_\mu\phi^A\right)^a\left(D^\mu\phi^A\right)^a - \frac{g^2}{4}f^{abe}f^{ecd}\phi^{Aa}\phi^{Bb}\phi^{Ac}\phi^{Bd} \\ &+ \frac{1}{3!}\lambda g F^{ABC}f^{abc}\phi^{Aa}\phi^{Bb}\phi^{Cc}\end{aligned} \tag{2.66}$$

with the definitions

$$\begin{aligned}F^c_{\mu\nu} &= \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc}A^a_\mu A^b_\nu, \\ \left(D_\mu\phi^A\right)^a &= \partial_\mu\phi^{Aa} + g f^{abc}A^b_\mu\phi^{Ac}.\end{aligned}$$

give the same amplitudes as the one derived from (2.65). Here F^{ABC} are the structure constants of the global group and it is simply demanded that they shall obey the Jacobi identity and that they are antisymmetric in all of their indices $A, B, C, \ldots$. After double copying the integrands the global group is promoted to the local gauge group SU(N) of EYM. Dimensional analysis reveals that the coupling constant λ is of mass dimensions one.

After discussing the relevant Lagrangians of the theories, the first step is to determine both numerators n^{YM} and $n^{\mathrm{YM}+\phi^3}$ which can be extracted from the corresponding one-loop integrands of pure YM and YM+ϕ^3. It turns out that the four-point YM numerator $n^{\mathrm{YM}}_{1^+2^+3^+4^+}$ at one-loop for an identical helicity amplitude is very simple because it is exclusively given by graphs with box topology.[23]

This is easy to verify using the trace-base representation of one-loop amplitudes (2.32). The primitive amplitude $A^1_{4;1}$ has been calculated in several papers to all orders in ϵ [137, 141, 168, 169] and is given by

$$A^1_{4;1}(1^+, 2^+, 3^+, 4^+) = \frac{2i}{(4\pi)^{2-\epsilon}}\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle} I_4\left[\mu^4; S, T\right], \tag{2.67}$$

where $I_4\left[\mu^4; S, T\right]$ is the scalar box integral and μ is the fictitious mass of the propagating complex scalar field in the loop which needs to be integrated over in order to emulate $4-2\epsilon$ dimensions. This implies that the integrand of $A^1_{4;1}$ reads

$$a^{\mathrm{YM}}_{1^+2^+3^+4^+} = 2i\mu^4\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}. \tag{2.68}$$

The remaining partial amplitudes $A^1_{4;n}$ with $n > 1$ can be obtained from the primitive amplitude $A^1_{4;1}$. Note that the partial amplitude $A^1_{4;2}$ cannot contribute because SU(N) generators are traceless. So there is only the partial amplitude $A^1_{4;3}$ left and this one can be determined by the relation (2.33). This relation gives $A^1_{4;3} = 6A^1_{4;1}$ by assuming that for all orderings of the external momenta, the primitive amplitude (2.67) is the same. It is easy to

[22] In particular this has been worked out in [63, 64]. To show this they have studied the Abelian and non-Abelian case. For example the Abelian Maxwell-Einstein theory is uniquely defined by its spectrum and its cubic interaction if the theory has $\mathcal{N}=4$ and $\mathcal{N}=2$ supersymmetry and can be lifted to five dimensions. A consistent truncation of this theory to $\mathcal{N}=0$ transfers this uniqueness property to it. Since $\mathcal{N}=4$ super YM (SYM) can be consistently truncated to YM it follows from the DC procedure that $\mathcal{N}=4$ Maxwell-Einstein (which is $(\mathcal{N}=4\,\mathrm{SYM}) \otimes_{\mathrm{DC}} \left(\mathrm{YM}+\phi^3\right)$) can be truncated to $\mathcal{N}=0$ Maxwell-Einstein. Hence, the Abelian theory is uniquely determined by its spectrum and interaction. By promoting the Abelian field strength to the non-Abelian generalization this claim can be extended to EYM.

[23] This property follows from maximal cuts [53], because only box type diagrams of YM theory are consistent with its analytic structure and all other diagrams shall cancel.

show that all primitive amplitudes (2.67) for any particular ordering of the external particles are the same. This follows from the fact that the integral gives a constant and that the numerator factor $a^{\mathrm{YM}}_{1^+2^+3^+4^+}$ is invariant under permutations, which shall be shown below.

The next step is to transform the integrand (2.68) into the structure constant basis. A general box colour factor reads

$$\begin{aligned} c^{a_1a_2a_3a_4} &:= f^{a'a_1b'} f^{b'a_2c'} f^{c'a_3d'} f^{d'a_4a'} \\ &= \frac{1}{4}\big[N\,\mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + N\,\mathrm{Tr}\,(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) \\ &+ 2\,\delta^{a_1a_2}\delta^{a_3a_4} + 2\,\delta^{a_1a_3}\delta^{a_2a_4} + 2\,\delta^{a_1a_4}\delta^{a_2a_3}\big], \end{aligned} \tag{2.69}$$

where (2.30) and the normalization condition (2.29) as well as the completeness relation (2.31) have been used. The basis elements of the box colour structure can be chosen as

$$c^{a_1a_2a_3a_4}, \quad c^{a_1a_2a_4a_3} \quad \text{and} \quad c^{a_1a_4a_2a_3}. \tag{2.70}$$

Using (2.32) and the fact that all primitive amplitudes are the same the full amplitude is given by

$$\begin{aligned} \mathcal{A}^1_4 &= g^4\Big[\sum_{\sigma\in\mathcal{S}_3} N\,\mathrm{Tr}\,(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(4)}}) + 6\sum_{\sigma\in\mathcal{S}_4/\mathcal{S}_{4;3}} \delta^{a_{\sigma(1)}a_{\sigma(2)}}\,\delta^{a_{\sigma(3)}a_{\sigma(4)}}\Big] A^1_{4;1}(1^+,2^+,3^+,4^+) \\ &= 4\,g^4\left[c^{a_1a_2a_3a_4} + c^{a_1a_2a_4a_3} + c^{a_1a_4a_2a_3}\right] A^1_{4;1}(1^+,2^+,3^+,4^+) \\ &= g^4\sum_{\mathcal{S}_4}\sum_{j\in\Gamma}\int\frac{d^dL}{(2\pi)^d}\frac{1}{S_j}\frac{c_j n^{\mathrm{YM}}_j}{\prod_{\alpha_j} D_{\alpha_j}} \end{aligned}$$

In the second line the full amplitude has been rewritten in the basis (2.70) which can be done using the decomposition (2.69). In the last line it is simply demanded that this amplitude can be written in the form (2.59). It can be inferred from these arguments that the numerator of the box colour structure basis is given by

$$n^{\mathrm{YM}}_{1^+2^+3^+4^+} = 4a^{\mathrm{YM}}_{1^+2^+3^+4^+}. \tag{2.71}$$

Furthermore the numerators obey the following symmetry properties

$$\begin{aligned} n^{\mathrm{YM}}_{i^+j^+k^+l^+} &= n^{\mathrm{YM}}_{i^+j^+l^+k^+}, \\ n^{\mathrm{YM}}_{i^+j^+k^+l^+} &= n^{\mathrm{YM}}_{k^+l^+i^+j^+}, \\ n^{\mathrm{YM}}_{i^+j^+k^+l^+} &= n^{\mathrm{YM}}_{j^+k^+i^+l^+} = n^{\mathrm{YM}}_{k^+i^+j^+l^+}. \end{aligned} \tag{2.72}$$

which can be proven in the same lines as (2.27). These properties imply that for all 24 positions of the external legs the numerator reads as (2.71). Once the symmetries of the numerator are known it is trivial to prove that they obey CKD. CKD demands the relation (2.60) to hold which in this case translates into

$$\begin{aligned} & c^{a_1a_2a_3a_4} - c^{a_1a_2a_4a_3} = c^{\mathrm{triangle}} := f^{a'a_1b'} f^{b'a_2c'} f^{c'a'd'} f^{d'a_4a_3} \\ \Longrightarrow\ & n^{\mathrm{YM}}_{1^+2^+3^+4^+} - n^{\mathrm{YM}}_{1^+2^+4^+3^+} = n^{\mathrm{YM}}_{\mathrm{triangle}}. \end{aligned} \tag{2.73}$$

It has been discussed that $n^{\mathrm{YM}}_{\mathrm{triangle}} = 0$. Besides, it has been shown that all numerators with box topology are the same $n^{\mathrm{YM}}_{i^+j^+k^+l^+} = n^{\mathrm{YM}}_{1^+2^+3^+4^+}$. These facts imply that (2.73) is trivially obeyed.

Thus, in restricting to the all-plus sector, only the colour structures (2.70) appear for the YM amplitudes and their numerators satisfy CKD automatically. Therefore to obtain integrands of amplitudes in EYM all numerators of YM+ϕ^3 theory have to be collected which have the same colour structures as (2.70) and multiply the two numerators divided by the stripped propagators.

A general one-loop amplitude of YM+ϕ^3 is of the form

$$\mathcal{A}_4^1\Big|_{g^4\lambda^n} = g^4\lambda^n \sum_{\mathcal{S}_m}\sum_{j\in\Gamma}\int \frac{d^dL}{(2\pi)^d}\frac{1}{S_j}\frac{c_j n_j^{\mathrm{YM}+\phi^3}}{\prod_{\alpha_j} D_{\alpha_j}}. \tag{2.74}$$

It follows from the Lagrangian (2.66) that the ϕ^3 interaction in YM+ϕ^3 is proportional to the coupling constants $g\lambda$. Thus the exponent $n \leq 4$ indicates how often the ϕ^3-interaction appears in the Feynman diagrammatic decomposition of the amplitude. $n_j^{\mathrm{YM}+\phi^3}$ shall be computed by Feynman diagrams generated from the Feynman rules of the theory given in appendix A.2.1.

Once both gauge theory numerators are calculated the gravity amplitude reads

$$\mathcal{M}_4^1\Big|_{\kappa^{4-n}g_{\mathrm{YM}}^n} = \left(\frac{\kappa}{4}\right)^{4-n} g_{\mathrm{YM}}^n \sum_{\mathcal{S}_m}\int \frac{d^dL}{(2\pi)^d} n^{\mathrm{YM}} \sum_{j\in\Gamma_g}\frac{1}{S_j}\frac{n_j^{\mathrm{YM}+\phi^3}}{\prod_{\alpha_j} D_{\alpha_j}}. \tag{2.75}$$

according to (2.63). The set Γ_g only includes all the numerators $n_j^{\mathrm{YM}+\phi^3}$ which have a colour structure of the form (2.70) and g_{YM} is the YM coupling constant in EYM. In [63, 64] it has been deduced how the global colour structure constant F^{ABC} of YM+ϕ^3 maps into the local one f^{abc} of EYM. Furthermore, it follows from (2.74) and (2.75) that the mapping of the coupling constants from the gauge theories to EYM is given by

$$F^{ABC} \to f^{abc}, \qquad (g^2, g\lambda) \to \left(\frac{\kappa}{4}, 4\frac{g_{\mathrm{YM}}}{\kappa}\right).$$

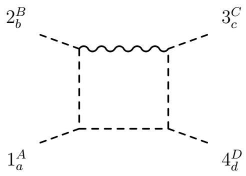
2.6 Amplitudes

In this section the missing step to obtain the amplitudes in EYM is accomplished: All the integrands of YM+ϕ^3 are evaluated and in the end of this section the full result for the EYM amplitudes is presented. New results are obtained for $\langle 1^+2^+3^+4^{++}\rangle$ and $\langle 1^+2^+3^{++}4^{++}\rangle$ at order κ^3 and κ^4, respectively. Furthermore, the κ-corrections at order κ^2 and κ^4 for the four-gluon amplitude $\langle 1^+2^+3^+4^+\rangle$ are also shown. Note that a DC expression of these integrands for arbitrary helicity configurations has been given in [63] by using the results of [170], however, the integrated amplitude has not been published yet. In the last part of this section the result for $\langle 1^{++}2^{++}3^{++}4^{++}\rangle$ is evaluated [2]. The relevant integrands of YM+ϕ^3 which shall be used in the DC prescription are collected in appendix A.2.2. In the following Feynman diagrams curly lines represent the gluon and the dashed lines sketch the real scalar.

2.6.1 Amplitudes: $\langle 1^+2^+3^+4^+\rangle$

First the κ^2 correction to $\langle 1^+2^+3^+4^+\rangle\big|_{\kappa^2}$ is computed with the DC method introduced in the previous section. Therefore in YM+ϕ^3 all Feynman graphs which are proportional to λ^2

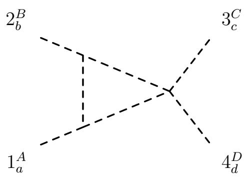
have to be calculated according to equ. (2.75). It has also been pointed out that only the box diagrams are non-vanishing for the all-plus YM amplitude. Hence, only the diagrams which carry the same colour structure as the box diagrams have to be determined in YM+ϕ^3. A careful analysis shows that only the graph topologies shown in figure 2.1 contribute. The integrands in figure 2.1 have a very simple form and can be obtained by the Feynman rules given in appendix A.2.1. For example the integrand of the first graph in figure 2.1 reads:



$$= \frac{g^4 \lambda^2 c^{abcd}}{D_0 D_1 D_2 D_3} F^{ABE'} F^{E'CD} \left[(p_3 + q_3 + l) \cdot (q_1 + l - p_2) - \mu^2 \right].$$

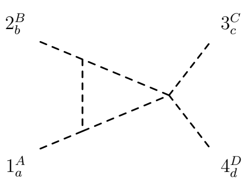
The factors in the denominator $D_i := Q_i^2 + i\epsilon = (l + q_i)^2 - \mu^2 + i\epsilon$ are the denominators of the Feynman propagators. The colour structure reads $c^{abcd} = f^{a'ab'} f^{b'bc'} f^{c'cd'} f^{d'da'}$ and the global SU(N) group information is encoded in $F^{ABE'} F^{E'CD}$, which shall be mapped into the adjoint gauge group generators of EYM after double coping.

The amplitude representation (2.74) only contains cubic graphs, however, it can be seen that in figure 2.1 Feynman graphs with quartic vertices also appear, e.g. the first graph in the second line of figure 2.1 reads



$$= \frac{g^4 \lambda^2}{D_0 D_1 D_2} \Big[c^{abcd} \left(F^{ACE'} F^{E'DB} - \delta^{CD} F^{A'AB'} F^{B'BA'} \right)$$
$$+ c^{abdc} \left(F^{ADE'} F^{E'CB} - \delta^{CD} F^{A'AB'} F^{B'BA'} \right) + c^{\text{triangle}} \ldots \Big].$$

This expression can be simplified using the Jacobi identity $c^{abdc} \equiv c^{abcd} + c^{\text{triangle}}$ of (2.73) which implies that the first two terms add up. However, it has also been extensively discussed that all numerators in YM theory associated to colour structures different from (2.70) vanish. This implies that the parts of the integrand which effectively contribute to EYM are



$$\overset{\text{eff}}{=} \frac{g^4 \lambda^2}{D_0 D_1 D_2} c^{abcd} \Big(F^{ACE'} F^{E'DB} + F^{ADE'} F^{E'CB}$$
$$- 2\delta^{CD} F^{A'AB'} F^{B'BA'} \Big).$$

The next step is to insert $\frac{D_3}{D_3}$ which makes the graph trivalent formally such that the gravity integrand can be obtained by (2.75).

After collecting all the non-vanishing contributions and building up the complete integrand, the amplitude is Veltman-Passarino-reduced with the `Mathematica` package `FeynCalc`

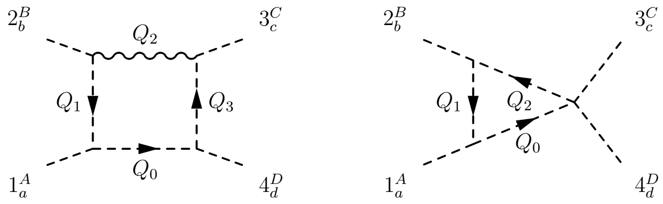


Figure 2.1: Graphs of these topologies are the only ones that have to be considered in YM+ϕ^3 at order $\lambda^2 g^4$. The other graphs do not contain the colour structures (2.70). Curly lines represent propagating gluons and dashed lines represent scalar fields. The internal momenta Q_i are d-dimensional.

[3, 4].Thus the full amplitude at order κ^2 reads

$$\begin{aligned}\mathcal{M}_4^1\left(1^+2^+3^+4^+\right)\Big|_{\kappa^2 g_{\mathrm{YM}}^2} &= -\frac{i}{(4\pi)^2}\left(\frac{\kappa g_{\mathrm{YM}}}{4}\right)^2 \frac{4}{3}\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle} \\ &\times\Big(4f^{abe'}f^{e'cd}\,(U-T)+4f^{ade'}f^{e'bc}\,(S-U)+\,4f^{ace'}f^{e'db}\,(T-S) \\ &+NS\;\delta^{ab}\delta^{cd}+NT\;\delta^{ad}\delta^{cb}+NU\;\delta^{ac}\delta^{bd}\Big) \\ &= -i\frac{\kappa^2 g_{\mathrm{YM}}^2}{192\pi^2}\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\Big(4f^{abe'}f^{e'cd}\,(U-T)+NS\;\delta^{ab}\delta^{cd}+\mathrm{perm}\Big)\,,\end{aligned}$$

where perm indicates the permutations of the legs two and three as well as two and four. The kinematic dependence is encoded in the spinor brackets and the Mandelstam variables $S=\langle 12\rangle\,[21]$, $T=\langle 23\rangle\,[32]$ and $U=\langle 13\rangle\,[31]$.

The next correction term $\langle 1^+2^+3^+4^+\rangle|_{\kappa^4}$ can be obtained by the same technique as for $\langle 1^+2^+3^+4^+\rangle|_{\kappa^2 g_{\mathrm{YM}}^2}$. All the graphs which contribute are depicted in figure 2.2. The numerators are fairly simple and are listed in the appendix A.2.2. A straight forward calculation gives the following integrated amplitude

$$\begin{aligned}\mathcal{M}_4^1\left(1_a^+,2_b^+,3_c^+,4_d^+\right)\Big|_{\kappa^4} &= \frac{i}{(4\pi)^2}\frac{\kappa^4}{4^4}\frac{4}{15}\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\Big(\delta^{ab}\delta^{cd}\Big(40\,TU-(2+N_g)\,S^2\Big) \\ &+\delta^{ac}\delta^{bd}\Big(40\,ST-(2+N_g)\,U^2\Big)+\delta^{ad}\delta^{bc}\Big(40\,SU-(2+N_g)\,T^2\Big)\Big) \\ &= \frac{i}{(16\pi)^2}\frac{\kappa^4}{60}\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\Big[\delta^{ab}\delta^{cd}\Big(40\,TU-(2+N_g)\,S^2\Big)+\mathrm{perm}\Big]\,.\end{aligned}$$

where again perm indicates the permutations of legs two and three as well as two and four and $N_g=\delta^{a'a'}=N^2-1$ is the number of adjoint generators of the Lie algebra.

2.6.2 Amplitudes: $\langle 1^+2^+3^+4^{++}\rangle$

It has been explicitly shown that $\langle 1^+2^+3^+4^{++}\rangle|_{\kappa g_{\mathrm{YM}}^3}$ vanishes in four dimensions [80] using both generalized unitarity and the DC method. Therefore the DC calculation is not reproduced in this thesis.

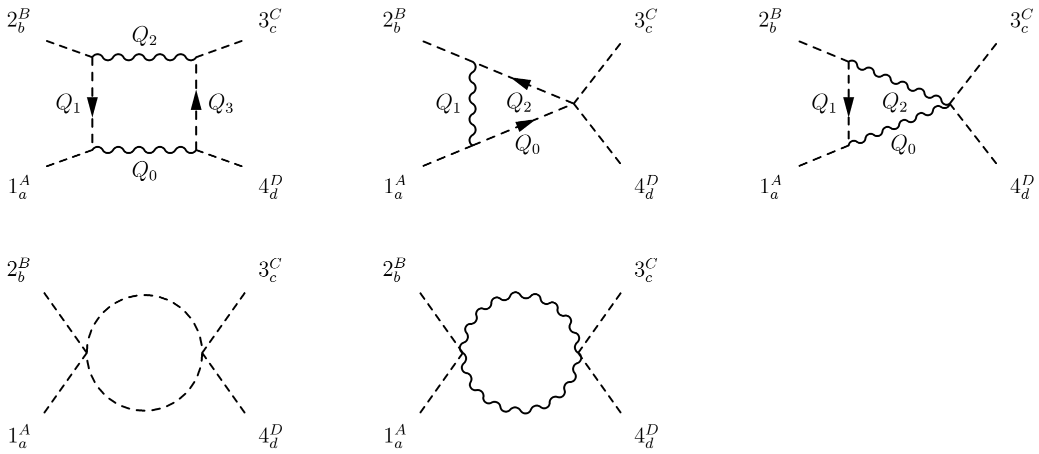
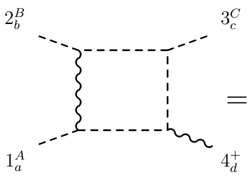


Figure 2.2: At order g^4 these box, triangle and bubble topologies are non-vanishing after double copying.

The next contribution is at order κ^3. The corresponding graphs are drawn in figure 2.3 and the first integrand is of the form



$$= \frac{-ig^4\lambda}{D_0 D_1 D_2 D_3} c^{abcd} F^{ABC} \, (p_2 + Q_2) \cdot (Q_0 - p_1) \, \frac{\langle r_4 \, | q_0 + q_3 + 2l | \, 4]}{\sqrt{2} \langle r_4 4 \rangle},$$

where four-dimensional spinor-helicity variables are used to represent the polarization vector, which is defined in (2.26). Compared to the previous two amplitudes there is an additional gauge freedom that is encoded in the reference vector r_4 which can be chosen arbitrarily but not such that it is proportional to p_4. The full calculation has been done for the gauge choices $r_4 \in \{p_1, p_2, p_3\}$. Since the amplitude has to be invariant under different gauges, this is a powerful crosscheck for the final result.

After all integrands of YM+ϕ^3 have been determined one can construct the gravity integrand using (2.75). Evaluating and reducing this expression yields the simple result

$$\mathcal{M}_4^1(1_a^+, 2_b^+, 3_c^+, 4^{++})\Big|_{g_{\rm YM}\kappa^3} = -\frac{\kappa^3 g_{\rm YM}}{(8\pi)^2} \frac{f^{abc}}{\sqrt{2}} \frac{[41]\,[42]\,[43]\,[12]}{\langle 34 \rangle}. \tag{2.76}$$

2.6.3 Amplitudes: $\langle 1^+ 2^+ 3^{++} 4^{++} \rangle$

The leading order in κ^2 for the amplitude $\langle 1^+ 2^+ 3^{++} 4^{++} \rangle|_{\kappa^2 g_{\rm YM}^2}$ has been determined in [80] using the unitarity based two-cut method. Here it shall be shown that with the DC prescription (2.75) this result is much easier obtained. Only the graph topologies drawn in figure 2.4 have to be evaluated on the YM+ϕ^3 side. Besides, the calculation can even further be simplified by choosing the reference momenta r_i for the gluon polarization vectors at the

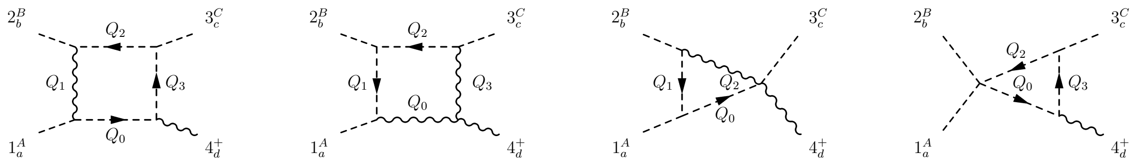


Figure 2.3: To obtain $\langle 1^+ 2^+ 3^+ 4^{++}\rangle|_{\kappa^3 g_{\mathrm{YM}}}$ these type of graphs in YM+ϕ^3 have to be analyzed.

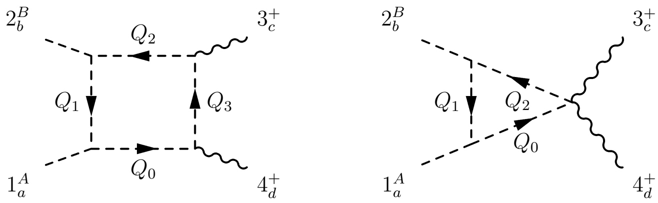


Figure 2.4: This pair of graph topologies represent graphs which appear at leading order in $\lambda^2 g^2$ which contribute to $\langle 1^+ 2^+ 3^{++} 4^{++}\rangle$.

legs three and four to be the same such that the integrands containing the quartic vertex give zero identically.[24]

Thus only the first type of graph from figure 2.4 has to be determined. The resulting amplitude is given by

$$\mathcal{M}_4^1(1_a^+, 2_b^+, 3^{++}, 4^{++})\Big|_{g_{\mathrm{YM}}^2 \kappa^2} = \frac{i}{(4\pi)^2}\left(\frac{\kappa\, g_{\mathrm{YM}}}{2}\right)^2 f^{a'ab'} f^{b'ba'} \frac{S}{6} \frac{[12]\,[34]^2}{\langle 12\rangle \langle 34\rangle^2}. \tag{2.77}$$

This result agrees with the expression given in [80] if one inserts into their results the pair of coupling constants $(g_{\mathrm{YM}}\kappa/2)^2$ and the colour structure $N\,\mathrm{Tr}\left(T^a T^b\right) = N\,\delta^{ab} = f^{a'ab'} f^{b'ba'}$ following from the decomposition of a one-loop amplitude into partial amplitudes (2.32). Since the result stated in [80] is the only partial amplitude which contributes, both expression coincide.

The κ^4 contribution can be determined by the graphs given in figure 2.5. Applying the same steps as before the result reads

$$\mathcal{M}_4^1(1_a^+, 2_b^+, 3^{++}, 4^{++})\Big|_{\kappa^4} = i\frac{\kappa^4}{(16\pi)^2} \frac{[21]^2\,[43]^3}{\langle 34\rangle} \frac{2+N_g}{90}\delta^{ab}. \tag{2.78}$$

Here N_g represents again the dimension of the adjoint representation of the gauge group. This result has been calculated for the following choices of reference momenta $r_3 \in \{p_1, p_2, p_4\}$, $r_4 \in \{p_1, p_2, p_3\}$ yielding identical results.

[24]This follows from $\epsilon_3^+ \cdot \epsilon_4^+ \sim \langle r_3 r_4\rangle$.

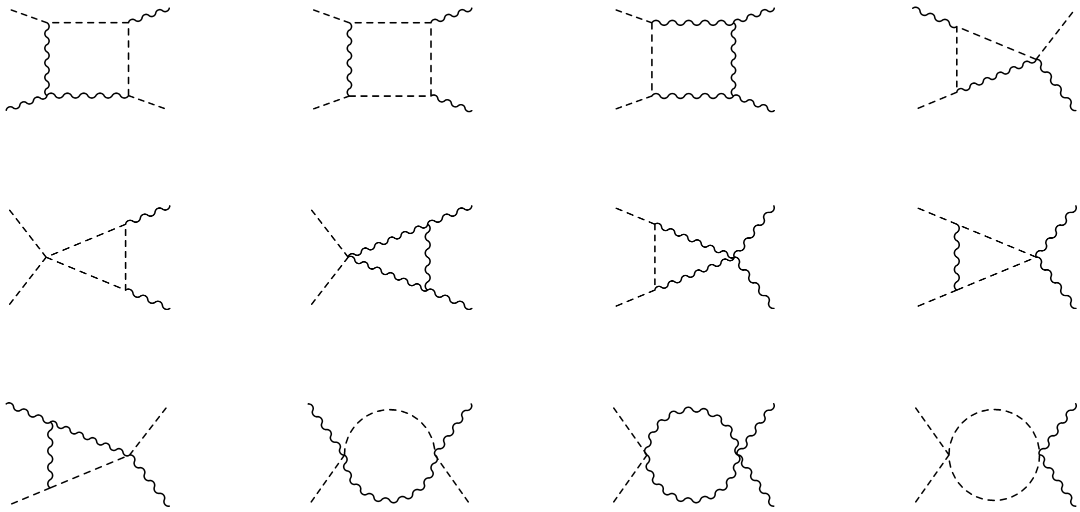


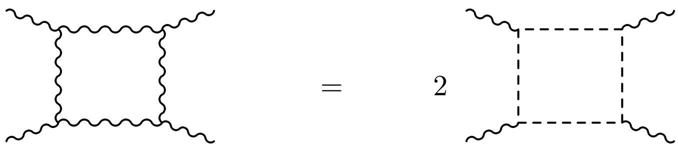
Figure 2.5: At g^4 these graph topologies shall contribute to the gravity amplitude $\langle 1^{+}2^{+}3^{++}4^{++}\rangle|_{\kappa^4}$.

2.6.4 Amplitude: $\langle 1^{++}2^{++}3^{++}4^{++}\rangle$

The remaining all-plus amplitude of EYM only contains gravitons as asymptotic states. In principle all the diagrams given in figure 2.6 contribute to the integrand. It can be seen that the graphs in the first row arise from pure YM theory. It has been discussed in section 2.5 that the calculation of these diagrams simplifies by choosing an appropriate gauge such that the graphs with box topology are the only non-vanishing ones, i.e. the only integrand which survives is exactly given by (2.71). For example this has been demonstrated in [169] using unitarity cuts. To simplify their analysis, the propagating gluon has been replaced by a complex scalar which is possible due to the supersymmetric Ward–Takahashi identities. They relate the amplitudes which inherit a circulating gluon to amplitudes with a circulating complex scalar in the following way

$$\mathcal{M}_4^{1,\mathrm{gluon}}(1^{++},2^{++},3^{++},4^{++}) = \mathcal{M}_4^{1,\mathrm{scalar}}(1^{++},2^{++},3^{++},4^{++}).$$

The superscripts “scalar” and “gluon” refer to the particle types which are circulating in the loop. Since a complex scalar has two degrees of freedom this relation can be represented diagrammatically by



This immediately shows that the graphs in the first and second row are intimately related. Hence one can write the amplitude as the sum of three scalar boxes in $4-2\epsilon$ dimensions,

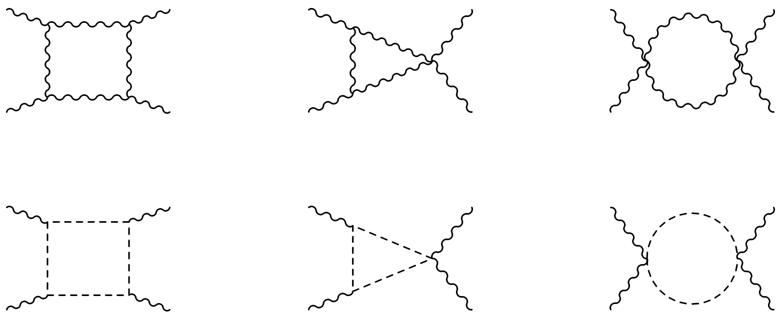


Figure 2.6: These topologies in YM+ϕ^3 have to be evaluated to obtain $\langle 1^{++}2^{++}3^{++}4^{++}\rangle|_{\kappa^4}$.

which are defined in equation (2.45):

$$\begin{aligned}\mathcal{M}_4^1(1^{++},2^{++},3^{++},4^{++}) &= \frac{i}{(4\pi)^{2-\epsilon}}\frac{\kappa^4}{4}\left(\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\right)^2 \\ &\times\Big(I_4\left[\mu^8;S,T\right]+I_4\left[\mu^8;T,U\right]+I_4\left[\mu^8;U,S\right]\Big)\left(1+\frac{N_g}{2}\right).\end{aligned}$$

In four dimensions the result simply reduces to

$$\mathcal{M}_4^1(1^{++},2^{++},3^{++},4^{++}) = -\frac{i}{(4\pi)^2}\kappa^4\left(\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\right)^2\frac{S^2+T^2+U^2}{1920}\left(2+N_g\right). \tag{2.79}$$

According to [65] the dilaton and axion can be removed by subtracting twice the contribution generated by the adjoint scalar circulating in the loop. For the all-plus amplitude the scalar part is given by

$$\mathcal{M}_4^{1,\text{scalar}}(1^{++},2^{++},3^{++},4^{++}) = -\frac{i}{(4\pi)^2}\kappa^4\left(\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\right)^2\frac{S^2+T^2+U^2}{3840}. \tag{2.80}$$

This result implies that the pure EYM (i.e. without the axion and dilaton) amplitude reads

$$\mathcal{M}_{4,\text{pEYM}}^1(1^{++},2^{++},3^{++},4^{++}) = -\frac{i}{(4\pi)^2}\kappa^4\left(\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\right)^2\frac{S^2+T^2+U^2}{1920}\left(1+N_g\right). \tag{2.81}$$

The result for the pure gravity part of (2.81) agrees with [151, 171] which can be seen using the identity

$$\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle} = -\frac{ST}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.$$

2.6.5 Final Results

In this chapter all four-point amplitudes with positive helicity configuration have been calculated at one-loop order for EYM. As expected the amplitudes are rational functions and read:

$$
\begin{aligned}
\mathcal{M}_4^1(1_a^+,2_b^+,3_c^+,4_d^+) &= \frac{i}{(4\pi)^2}\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\Big(\Big[-\frac{4}{3}\,g_{\mathrm{YM}}^4\,f^{a'ab'}f^{b'bc'}f^{c'cd'}f^{d'da'}\\
&\quad-\frac{\kappa^2 g_{\mathrm{YM}}^2}{12}\Big(4f^{abe'}f^{e'cd}\,(U-T)+NS\,\delta^{ab}\delta^{cd}\Big)\\
&\quad+\frac{\kappa^4}{960}\delta^{ab}\delta^{cd}\Big(40\,TU-(2+N_g)\,S^2\Big)\Big]+\mathrm{perm}\Big),\\
\mathcal{M}_4^1(1_a^+,2_b^+,3_c^+,4^{++}) &= -\frac{\kappa^3 g_{\mathrm{YM}}}{(8\pi)^2}\frac{f^{abc}}{\sqrt{2}}\frac{[41]\,[42]\,[43]\,[12]}{\langle 34\rangle},\\
\mathcal{M}_4^1(1_a^+,2_b^+,3^{++},4^{++}) &= \frac{i}{(4\pi)^2}\delta^{ab}\frac{[12]^2\,[34]^2}{\langle 34\rangle^2}\Big(-\frac{\kappa^2 g_{\mathrm{YM}}^2}{24}N+\frac{\kappa^4}{1440}S\,(2+N_g)\Big),\\
\mathcal{M}_4^1(1^{++},2^{++},3^{++},4^{++}) &= -\frac{i}{(4\pi)^2}\kappa^4\left(\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\right)^2\frac{S^2+T^2+U^2}{1920}\,(2+N_g)\,.
\end{aligned}
$$

Moreover, this result can be extended to arbitrary helicity configurations at four points. These amplitudes can be obtained by using the DC method in the same way as it has been discussed in this thesis because the authors of [170] constructed a CKD satisfying representation for four-point YM amplitudes at one-loop for arbitrary helicity configurations. However, using their numerators the calculation is much more tedious since these are non-zero for all possible graph topologies which increases the number of diagrams to be calculated in YM+ϕ^3 enormously. Hence, it might be more efficient to use a different technique to compute these amplitudes.

2.6.6 Pure Einstein-Yang-Mills Amplitudes

Generally, it would be very interesting to be able to obtain the pure EYM results from the amplitudes calculated in this paper. An analysis of possible interaction terms generated by the Lagrangian (2.65) shows that the axion and dilaton cannot contribute at one loop for a four-point amplitude if the number of asymptotic gluon states matches the power of the coupling constant g_{YM}. The reason is that in this case only the gauge fields can propagate in the loop, because all fields couple at least quadratically to it.

Further for the four-point graviton amplitude $\mathcal{M}_4^1(1^{++},2^{++},3^{++},4^{++})$ it has been possible to separate the axion and dilaton contributions from the gluon and graviton contributions to obtain the pure EYM result. This has been achieved by subtracting the unwanted particle types circulating in the loop according to [65].

The authors of [65] have motivated and illustrated on several examples that pure gravity theories can be directly obtained by considering a generalized double copy procedure in which particles transforming in the (anti-)fundamental and adjoint representation of the gauge group are studied. The reason is that the dilaton $\underline{\varphi}$ and the axion $\underline{\chi}$ on-shell states do not only appear in the spectrum of the tensor product of two on-shell gluon states $\underline{A}^\mu$, but also in the tensor product of fundamental and anti-fundamental Weyl fermions $\underline{\psi}^+$ and $\underline{\psi}^-$, respectively. Thus subtracting the latter contribution from the former is supposed to yield a pure gravity theory.[25] Consider the following decomposition into irreducible representations

[25]This implies that the fundamental Weyl fermions have to have opposite statics. Otherwise both contri-

of the Poincaré group

$$\underline{A}^{\mu} \otimes \underline{A}^{\nu} = \underline{h}^{\mu\nu} \oplus \underline{\phi}^{+-} \oplus \underline{\phi}^{-+},$$

where the graviton on-shell state $\underline{h}^{\mu\nu}$ has two degrees of freedom and the states $\underline{\phi}^{+-} \sim \underline{\varphi}+i\underline{\chi}$ and $\underline{\phi}^{-+}$ have one each. In comparison the on-shell Weyl fermions tensor product can be written as

$$\underline{\psi}^{+} \otimes \underline{\psi}^{-} = \underline{\phi}^{+-} \quad \text{and} \quad \underline{\psi}^{-} \otimes \underline{\psi}^{+} = \underline{\phi}^{-+}.$$

Using the generalized DC method of two YM theories coupled to (opposite statistic) fundamental Weyl fermions yields an amplitude in a pure gravity theory. The authors of [65] could conclude that by studying the internal degrees of freedom of the loop amplitudes and by applying generic unitarity cuts of the gravity amplitude in four dimensions. This analysis has revealed that the only internal propagating states are gravitons. Hence the resulting amplitude has to be a pure gravity amplitude.

However, the four-point graviton amplitude $\mathcal{M}_4^1(1^{++}, 2^{++}, 3^{++}, 4^{++})$ and the analysis of [65] have in common that only one particle type is circulating in the loop. But all the remaining amplitudes computed in this thesis contain a mixed type of particles circulating in the loop, which makes a subtraction of the axion and dilaton rather difficult.

butions would add up.

In general the subtraction has to be performed by the fundamental Weyl fermions, however, in the case of a four-point pure gravity amplitude at one-loop the authors of [65] have shown that this is equivalent to subtracting twice the contribution (2.80).

Chapter 3

Mellin Amplitudes

Comments about the Signature

It is common to study several properties of conformal field theories (CFTs) in Euclidean spacetime. However, Euclidean and Lorentzian CFTs contain the same information, i.e. the CFT data is identical. They are related by analytic continuation which is established by a Wick rotation from Euclidean time τ_j to Lorentzian time t_i. The choice of contour of this Wick rotation is important, because it dictates the time ordering of the correlator, e.g. if it is time-ordered, anti-time-ordered or of mixed ordering. A neat way to implement this choice of contour is by fixing the $i\epsilon$-prescription. This result is known as the Osterwalder-Schrader reconstruction theorem which states that well-behaved Euclidean correlation functions can be analytically continued to Wightman functions [172–174]. This is nicely reviewed in [175, 176] and more mathematically in the standard textbook [177, Theorem 3-5].

To illustrate this take the simple example of an Euclidean two-point correlator

$$\langle \phi_E(\tau_1, \mathbf{x}_1)\phi_E(\tau_2, \mathbf{x}_2)\rangle = \frac{1}{(\tau_{12}^2 + \mathbf{x}_{12}^2)^\Delta}. \tag{3.1}$$

This function is analytic for non-coincident points $(\tau_1, \mathbf{x}_1) \neq (\tau_2, \mathbf{x}_2)$. (3.1) can be analytically continued to a Wightman function by setting $\tau_i = \epsilon_i + it_i$ with $\epsilon_1 > \epsilon_2$. The inequality $\epsilon_1 > \epsilon_2$ fixes the ordering of the operators. This leads to

$$\langle \phi_E(\epsilon_1 + it_1, \mathbf{x}_1)\phi_E(\epsilon_2 + it_2, \mathbf{x}_2)\rangle = \frac{1}{(-t_{12}^2 + \mathbf{x}_{12}^2 + 2i\epsilon_{12}t_{12} + \epsilon_{12}^2)^\Delta}.$$

Finally one sends $\epsilon_{12} := \epsilon > 0$ to zero which gives the Lorentzian correlator

$$\langle \phi_L(t_1, \mathbf{x}_1)\phi_L(t_2, \mathbf{x}_2)\rangle = \lim_{\epsilon\to 0} \frac{1}{(x_{12}^2 + 2i\epsilon t_{12})^\Delta} = \lim_{\epsilon\to 0} \frac{1}{(x_{12}^2 + i\epsilon)^\Delta} = \frac{1}{|x_{12}^2|^\Delta} e^{-i\pi\Delta} \tag{3.2}$$

with $x_{12}^2 = -t_{12}^2 + \mathbf{x}_{12}^2$. This is the result for a time-ordered correlator, e.g. $(t_1 > t_2)$. For an anti-time-ordered correlator $(t_2 > t_1)$ the phase would have opposite sign. The difference in both time orderings corresponds to crossing the branch cut along the real negative axis. Hence the $i\epsilon$-prescription defines the choice of contour uniquely. This analysis can be generalized to n-point correlators.

In both correlators (3.1) and (3.2) the same conformal dimension is encoded which implies they carry the same information. Therefore the signature can be chosen either way. In this thesis the choice of signature is mostly the same as it is in the literature, e.g. parts of the introduction to CFTs (section 3.1), the definition of the Mellin amplitude (section 3.3) and the perturbative calculations (section 3.4) shall be presented with the metric signature $(+,\dots,+)$. To obtain the definition for the Mellin amplitude in Lorentzian spacetime the correct $i\epsilon$-prescription is included. Since the final results in this thesis shall be presented in Lorentzian spacetime all the three-dimensional tensor structures in section 3.2 are constructed for the metric signature $(-,+,+)$. The conformal group/algebra in section 3.1 is studied in Lorentzian spacetime, too.

3.1 Conformal Symmetry

Generally symmetries are generated by Killing vector fields, because these are the (infinitesimal) generators of isometries

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \overset{\text{isometry}}{=} g_{\mu\nu}(x) \tag{3.3}$$

of the metric g, i.e. they generate distance preserving transformations of the metric. The indices can take the values $0 \le \mu,\nu,\dots, \le d-1$. Since the metric is invariant under the flow of Killing vector fields, the action posses a symmetry according to Noether's theorem. Mathematically, the flow of a Killing vector field X has to vanish along the metric g

$$\mathcal{L}_X g = 0, \tag{3.4}$$

where the Lie derivative $\mathcal{L}_X$ evaluates the change of the metric g along the flow of the vector field X. In local coordinates (3.4) translates into the Killing equations[1]

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \quad \text{with the covariant derivatives } \nabla_\mu. \tag{3.5}$$

However, if one is interested in the class of phenomena invariant under conformal transformations

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x), \tag{3.6}$$

symmetries are given by **conformal Killing vectors fields** $\epsilon = \epsilon^\mu\partial_\mu$ which change the metric under the flow of ϵ only by a local rescaling $\Omega(x)$ and satisfy locally the conformal Killing equations

$$\mathcal{L}_\epsilon g = \Omega^2(x) g \quad \text{or locally} \quad \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu = \frac{2}{d}\nabla_\rho\epsilon^\rho g_{\mu\nu}.$$

Assuming that the metric is the constant Minkowskian metric $g=\eta$ with the signature $(-,+,\dots,+)$ and considering the subclass of transformations that rescale the Minkowskian metric η only, the conformal Killing equations take the simple form

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}\partial_\rho\epsilon^\rho\eta_{\mu\nu}. \tag{3.7}$$

[1]Locally an isometry (3.3) yields the Killing equation for the infinitesimal displacement $x'^\mu = x^\mu + X^\mu$ and X^μ has to obey the Killing vector equation (3.5) which can be checked by $\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = g'_{\mu\nu}(x' - X) - g_{\mu\nu}(x)$.

The most general vector in $d > 2$ dimensions which satisfies (3.7) is given by

$$\epsilon = \epsilon^\mu \partial_\mu = i a^\mu P_\mu + i\lambda D + \frac{i}{2} w^{\mu\nu} M_{\mu\nu} + i b^\mu K_\mu \tag{3.8}$$

where the corresponding generators are defined by

$$\begin{aligned}
P_\mu &:= -i\partial_\mu && \text{generates translations,} \\
M_{\mu\nu} &:= i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) && \text{generates rotations,} \\
D &:= -i x^\mu \partial_\mu && \text{generates dilatations,} \\
K_\mu &:= -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) && \text{generates special conformal transformations.}
\end{aligned} \tag{3.9}$$

These generators characterize the **conformal algebra** and obey the following (non-vanishing) commutation relations

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i\left(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu\right), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma}\right), \\
[D, P_\mu] &= i P_\mu, \\
[D, K_\mu] &= -i K_\mu, \\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu} D - M_{\mu\nu}\right), \\
[M_{\mu\nu}, K_\rho] &= i\left(\eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu\right).
\end{aligned} \tag{3.10}$$

The conformal algebra includes the Poincaré algebra as a subalgebra, since the first two commutation relations of (3.10) form a closed subalgebra. In fact the conformal algebra acting on $\mathbb{R}^{1,d-1}$ is isomorphic to $\mathrm{SO}(2,d)$. The isomorphism is given by mapping the generators (3.9) to the generators

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu,d} = -\frac{1}{2}\left(P_\mu + K_\mu\right), \quad J_{\mu,d+1} = \frac{1}{2}\left(K_\mu - P_\mu\right), \quad J_{d+1,d} = D. \tag{3.11}$$

It can be verified that in this representation the conformal algebra satisfies

$$[J_{MN}, J_{RS}] \;\; = i\left(\eta_{NR} J_{MS} + \eta_{MS} J_{NR} - \eta_{NS} J_{MR} - \eta_{MR} J_{NS}\right), \tag{3.12}$$

where the diagonal metric η_{MN} has the signature $(-,+,\dots,+,-)$. The coordinates can take the following values $0 \le M, N, R, S \le d+1$ [29, 98, 124].

In a quantum theory one is interested in calculating observables which, in a conformal theory, can be obtained from local operators. Hence, it is important to analyze how the conformal symmetry acts on operators. In CFTs it is useful to distinguish two types of local operators: **primaries** and **descendants**. Primary operators $\mathcal{O}_\Delta(x)$ are of special interest because all descendant operators can be obtained from them, and they satisfy very simple transformation rules at the origin $x = 0$, i.e. they are eigenvectors of the dilatation operator and transform in irreducible representations of the Lorentz group[2]

$$[M_{\mu\nu}, \mathcal{O}_\Delta(0)] = S_{\mu\nu} \mathcal{O}_\Delta(0), \quad [D, \mathcal{O}_\Delta(0)] = -i\Delta \mathcal{O}_\Delta(0), \quad [K_\mu, \mathcal{O}_\Delta(0)] = 0. \tag{3.13}$$

[2]If the set $\{\mathcal{O}_{\Delta_i}(0) | i \in I\}$ forms a basis of an invariant subspace of the representation space of the Lorentz group (i.e. the representation of the Lorentz group is irreducible), then every generator that commutes with generators of the Lorentz group have to be proportional to the identity by Schur's lemma. This implies that $\mathcal{O}_\Delta(0)$ is an eigenvector of the (matrix-) representation of the dilatation operator D, since $[D, M_{\mu\nu}] = 0$.

Δ is called the scaling dimension of the operator and it gives information about physical data like critical exponents. $S_{\mu\nu}$ is a matrix-valued representation of (Lorentz-)rotations $M_{\mu\nu}$. Further details can be found in the appendix B.1. The last commutator $[K_\mu, \mathcal{O}_\Delta(0)] = 0$ follows from the fact that the matrix-representation of D is proportional to the unit matrix combined with the commutation relations (3.10). In addition this commutation relation suggest that there is a lowest weight state of D from which all other states of a conformal multiplet can be constructed by applying conformal transformations, i.e. the irreducible representation space can be constructed in similar way as the algebra of SU(2). Let $\mathcal{O}_\Delta(0)$ denote the primary operator which satisfies $[D, \mathcal{O}_\Delta(0)] = -i\Delta\mathcal{O}_\Delta(0)$. Using $[D, P_\mu] = iP_\mu$ one can deduce

$$\mathcal{O}_\Delta(0) \overset{P^{\mu_1}}{\to} \mathcal{O}^{\mu_1}_{\Delta+1}(0) \overset{P^{\mu_2}}{\to} \mathcal{O}^{\mu_1\mu_2}_{\Delta+2}(0) \overset{P^{\mu_3}}{\to} \dots \tag{3.14}$$

by applying the Jacobi identity. Thus the entire ladder of dilatation eigenvalues can be walked up by applying the operator P_μ. Note that these operators $\mathcal{O}^{\mu_1}_{\Delta+1}(0)$ do not commute with K_μ; hence they are not primaries but descendants. In general a descendant can be constructed from a primary by applying the operator P_μ as can be checked with the commutation relations (3.10). It can be proven that all local operators in a CFT are a linear combination of primaries and descendants [124].

3.1.1 Operator-State-Correspondence and Operator-Product-Expansion

In section 2.1 it has been pointed out that spacetime should be quantized along one of its isometries. As shall be shown, for CFTs it is useful to choose the direction of quantization along scale transformations. This implies that the d-dimensional spacetime is foliated into spheres S^{d-1} where each of these is equipped with its own Hilbert space $\mathcal{V}$. To act on $\mathcal{V}$, operators have to be inserted on the spheres. To map an operator from one Hilbert space to another the Euclidean evolution operator[3]

$$U(r_2 - r_1) = e^{-D(\ln r_2 - \ln r_2)} : \mathcal{V}_1 \to \mathcal{V}_2 \tag{3.15}$$

can be used.[4] It follows from the comparison of (3.15) and (2.1) that the dilatation operator D plays the role of the Hamiltonian in a radially quantized theory. The origin shall be placed at $x = 0$, however, any other point is equally fine. This quantization scheme is sketched in 3.1.

Radial quantization is a very convenient quantization-scheme for CFTs. In particular, this scheme is useful to prove the **operator-state-correspondence** which states that every state in the CFT can be created by operators locally acting at a small neighbourhood close to the origin. This means that the entire Hilbert space can be constructed from one, single point. As usual in quantum mechanics, states are classified by its quantum numbers, i.e. in CFTs they are given by its scaling dimension Δ and its SO(d) representation denoted by

[3]This and the next subsection are explained in Euclidean signature.

[4]Scale transformations become an isometry (i.e. they obey (3.3)) if a Weyl rescaling is performed which maps $\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$. On the cylinder $\mathbb{R} \times S^{d-1}$, scale transformations are an isometry and the "time evolution operator" is given as in (3.15). Both metrics are related by

$$ds^2_{\mathbb{R}^d} = dr^2 + r^2 ds^2_{S^{d-1}} = e^{2\tau}\left(d\tau^2 + ds^2_{S^{d-1}}\right) = e^{2\tau} ds^2_{\mathbb{R}\times S^{d-1}} \quad \text{and} \quad r = e^\tau.$$

In particular, radial quantization on the plane is equivalent to usual quantization on the cylinder [124].

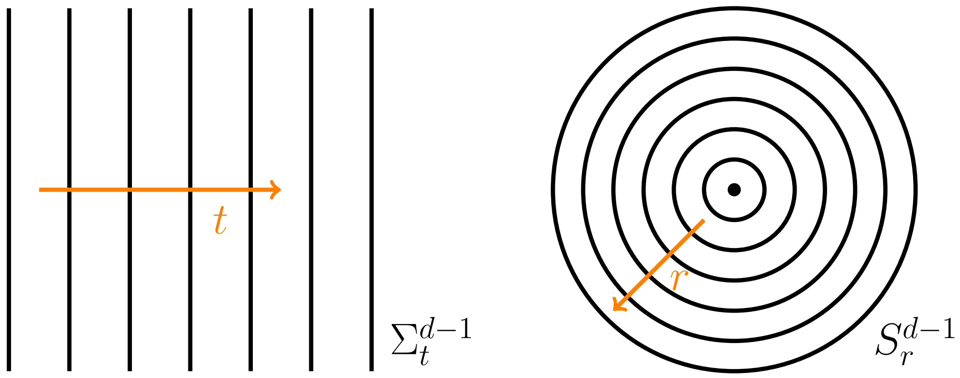


Figure 3.1: On the l.h.s. spacetime is quantized along the time direction t. Each hypersurface Σ_t^{d-1} is equipped with a Hilbert space. On the r.h.s. spacetime is quantized radially, each sphere S_r^{d-1} is endowed with its own Hilbert space.

its spin l. Hence, a state is uniquely characterized by $|\Delta, l\rangle_i$, where the index i is given in the appropriate SO(d) representation under which the state transforms, e.g. scalar, tensor, spinor etc. representation. However, in the following discussion the SO(d) spin value l and index i shall not be written explicitly, i.e. the short-hand notation $|\Delta, l\rangle_i := |\Delta\rangle$ is used. This notional simplification is not relevant for the proof.

Each local operators constructs a state

It has been pointed out, in a CFT the complete basis of operators is given by primaries and descendants. Further, it has been shown that each descendant can be created from primaries by acting with the momentum operator. Thus, to construct a state from an operator, it is best to start to map primary operators inserted at the origin $x = 0$ to primary states. This is established by

$$\mathcal{O}_\Delta(0) \longrightarrow \mathcal{O}_\Delta(0)\,|0\rangle := |\Delta\rangle \tag{3.16}$$

with the conformal invariant vacuum state $|0\rangle$. This implies that the state $|\Delta\rangle$ and the operator $\mathcal{O}_\Delta(0)$ obey the same commutation relations (3.13). A general primary operator $\mathcal{O}_\Delta(x)$ inserted at any point but the origin can be written as a linear combination of local operators $\mathcal{O}_\Delta$ inserted at $x = 0$ (or rather the identified state $|\Delta\rangle$). Hence,

$$\mathcal{O}_\Delta(x)\,|0\rangle = e^{P\cdot x}\mathcal{O}_\Delta(0)e^{-P\cdot x}\,|0\rangle = e^{P\cdot x}\,|\Delta\rangle = \sum_{n=0}^{\infty} \frac{(\,P\cdot x)^n}{n!}\,|\Delta\rangle\,, \tag{3.17}$$

which is a linear combination of the basis vectors (3.16) [98, 124].

In comparison descendant operators can be generated by (3.14), but due to the identification (3.16) their construction works the same way, e.g.

$$\begin{aligned} DP^\mu\,|\Delta\rangle &= [D,[P_\mu,\mathcal{O}_\Delta]]\,|0\rangle = ([[D,P_\mu]\,,\mathcal{O}_\Delta] - [P_\mu,[D,\mathcal{O}_\Delta]])\,|0\rangle \\ &= ([D,P^\mu] - P^\mu D)\,|\Delta\rangle \sim (\Delta+1)P^\mu\,|\Delta\rangle \\ \Longrightarrow\;\; P^\mu\,|\Delta\rangle &\sim |\Delta+1\rangle\,. \end{aligned} \tag{3.18}$$

Therefore acting with all possible operators on the vacuum constructs a state in the Hilbert space which is uniquely characterized by both of its quantum numbers Δ and l.

Each state constructs a local operator

The converse direction holds also in a CFT. I.e. to each state a local operator can be constructed. E.g. to the primary state $|\Delta\rangle$ the corresponding local primary operator $\mathcal{O}_\Delta(0)$ can be created by its action onto conformal correlation functions.

$$\langle \mathcal{O}_{\Delta_1}(x_1)\ldots\mathcal{O}_{\Delta_n}(x_n)\mathcal{O}_\Delta(0)\rangle := \langle 0|\,\mathcal{O}_{\Delta_1}(x_1)\ldots\mathcal{O}_{\Delta_n}(x_n)\,|\Delta\rangle \tag{3.19}$$

Due to the identification (3.16) both sides of the equation obey the same transformation properties under conformal transformations. Hence, (3.19) defines to each state an appropriate local operator such that both conformal correlation functions yield the same result. Therefore the state and the so-defined operator can be identified [98]. This generalizes (3.16)

$$\mathcal{O}_\Delta(0) \longleftrightarrow |\Delta\rangle$$

to a one-to-one map.

This shows that in radially quantized CFT, the operator-state correspondence is easy to prove since the vacuum is located at a single point. Due to this isomorphism between local operators and states, an arbitrary state $|\psi\rangle$ state can be decomposed into the basis $|\Delta_k\rangle$. In particular,

$$|\psi\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,|0\rangle = \sum_{k=0}\tilde{\lambda}_{12k}C_{12k}\left(x_{12},x_{23},P\right)|\Delta_k\rangle\,, \tag{3.20}$$

where the sum runs over all primary states and the function $C_{12k}\left(x_{12},x_{23},P\right)$ generates all possible descendants by acting with the momentum operator P^μ on primaries. The OPE coefficient $\tilde{\lambda}_{12k}$ is a simple constant.[5] It is an important fact that the form of the function C_{12k} is completely fixed by conformal symmetry. But implicitly C_{12k} depends on the scaling dimension Δ_i of the involved operators. Using the operator-state-correspondence, equ. (3.20) can be written as an operator equation

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{k=0}\tilde{\lambda}_{12k}C_{12k}\left(x_{12},x_{23},\partial_3\right)\mathcal{O}_{\Delta_k}(x_3) \tag{3.21}$$

which holds inside any correlation function. The **operator-product-expansion** (OPE) (3.21) can be used to reduce the calculation of n-point functions to the computation of n-1-point functions, because it fuses a pair of operators into a sum of local operators. The initial value to solve this recurrence relation is fixed by dimensional analysis to be

$$\langle\mathcal{O}(x)\rangle = \begin{cases} 1, & \text{if } \mathcal{O} \text{ is the unit operator } \mathbb{1} \\ 0, & \text{else.} \end{cases}$$

Due to the operator-state-correspondence the set of operators which are included in a ball create a state on its boundary. Hence, (3.21) does not just hold asymptotically, it has a finite radius of convergence. It converges if it is possible to encircle the two operators $\mathcal{O}_1(x_1)$, $\mathcal{O}_2(x_2)$ without any other operator as it is shown in figure 3.2 [98, 124, 178, 179].

[5]For notational simplicity spin indices have been omitted.

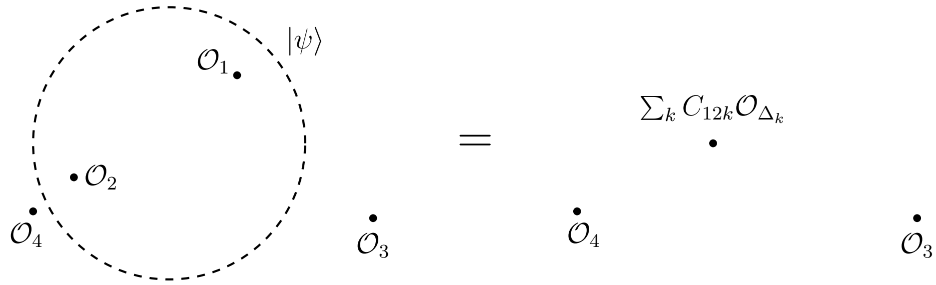


Figure 3.2: The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ can be enclosed by a sphere separating it from other operator insertions. Hence, they produce a state $|\psi\rangle$ on the dashed sphere. The state $|\psi\rangle$ is of the form (3.20).

3.1.2 Conformal Correlators

Conformal symmetry imposes very strong constraints on correlation functions. For example the form of the two- and three-point functions cannot contain any dynamical information, i.e. their form is completely fixed by its symmetry. For symmetric traceless tensor operators with integer spin $l \geq 0$ the two-point function is given by

$$\begin{aligned} \langle \mathcal{O}^{\mu_1\dots\mu_l}(x_1)\mathcal{O}'^{\nu_1\dots\nu_l}(x_2)\rangle &= \delta_{\mathcal{O}\mathcal{O}'}\frac{I^{\mu_1\dots\mu_l\nu_1\dots\nu_l}(x_{12})}{|x_{12}|^{2\Delta_\mathcal{O}}} \\ \text{with}\quad I^{\mu_1\dots\mu_l\nu_1\dots\nu_l}(x) &= \sum_{\text{sym}\,\mu_i,\,\nu_j}\frac{I^{\mu_1\nu_1}(x)\dots I^{\mu_l\nu_l}(x)}{|x|^{2\Delta}} - \text{ traces}, \\ I^{\mu\nu}(x) &= \delta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2} \quad \text{and} \quad x_{12} = x_1 - x_2. \end{aligned} \tag{3.22}$$

The summation symmetrizes the expression in the indices μ_i and ν_j, respectively and the subtraction of traces corresponds to removing terms of the form $\delta^{\mu_i\mu_j}$ and $\delta^{\nu_i\nu_j}$ such that the two-point function is traceless in its μ_i and ν_j indices separately, however, it does not have to be traceless in μ_i-ν_j-contractions. In (3.22) the operators have been normalized according to $\delta_{\mathcal{O}\mathcal{O}'}$ [180].

Equivalently, the form of the three-point function is fully dictated by conformal symmetry. In case of three scalar operators ϕ_i the three-point function reads

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{3.23}$$

where the constant λ_{123} is not determined by symmetry. Alternatively, the three-point correlator can be reduced to a two-point correlator using the OPE (3.21):

$$\begin{aligned} \langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle &= \sum_k \tilde{\lambda}_{12k}C_{12k}\left(x_{12},\partial_{x_2}\right)\langle\mathcal{O}_{\Delta_k}(x_2)\phi_3(x_3)\rangle\,. \\ &=\tilde{\lambda}_{123}C_{123}\left(x_{12},\partial_{x_2}\right)\frac{1}{|x_{23}|^{2\Delta_3}} \\ &\stackrel{!}{=}\frac{\lambda_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \\ &=\lambda_{123}\frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}}\left(1+\dots\right)\frac{1}{|x_{23}|^{2\Delta_3}} \end{aligned} \tag{3.24}$$

Due to the diagonal nature of the two-point correlator (3.22) the sum in (3.24) has one non-vanishing term only such that this expression reduces to the second line. In the second line is the two-point function (3.22) of a pair of scalar primaries and all the descendants of the conformal multiplet are generated by the function C_{123}. In the third line the explicit form (3.23) of the three-point correlator has been plugged in. And in the last line a Taylor expansion around $x_{12} = 0$ has been performed. The derivation (3.24) is quite important. Firstly, it determines the function C_{ijk} by Taylor expanding (3.23) around $x_{12} = 0$ and secondly, it shows that according to the normalization used in this thesis the relation $\lambda_{ijk} = \tilde{\lambda}_{ijk}$ holds [98].

A simple analysis of the number of conformal invariant terms made of the arguments x_i with $1 \leq i \leq 4$ demonstrates that the four-point correlator has to be a function of two independent variables, e.g. u and v. The scalar four-point correlator can be written as

$$\begin{aligned} \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle &= \left|\frac{x_{24}}{x_{14}}\right|^{\Delta_1-\Delta_2} \left|\frac{x_{14}}{x_{13}}\right|^{\Delta_3-\Delta_4} \frac{\mathcal{A}(u,v)}{|x_{12}|^{\Delta_1+\Delta_2}\,|x_{34}|^{\Delta_3+\Delta_4}} \\ \text{with the conformal cross ratios}\quad u &= \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad v = \frac{x_{14}x_{23}}{x_{13}x_{24}}. \end{aligned} \tag{3.25}$$

The conformal invariant reduced correlator $\mathcal{A}$ is only a function of the two cross ratios u and v. Applying the OPE (3.21) and using the diagonal nature of the two-point correlator, the four-point correlator can be written as

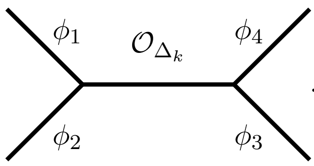
$$\begin{aligned} \langle \phi_1(x_1)\dots\phi_4(x_4)\rangle &= \sum_{k,k'} \lambda_{12k}\lambda_{34k'} C_{12k}^{a_l}(x_{12},\partial_{x_2})\, C_{34k'}^{b_{l'}}(x_{34},\partial_{x_4}) \left\langle \mathcal{O}^{a_l}(x_2)\mathcal{O}'^{b_{l'}}(x_4)\right\rangle \\ &= \sum_k \lambda_{12k}\lambda_{34k} C_{12k}^{a_l}(x_{12},\partial_{x_2})\, C_{34k}^{b_l}(x_{34},\partial_{x_4}) \frac{I^{a_l b_l}(x_{24})}{|x_{24}^2|^{\Delta_k}} \\ &:= \left|\frac{x_{24}}{x_{14}}\right|^{\Delta_1-\Delta_2} \left|\frac{x_{14}}{x_{13}}\right|^{\Delta_3-\Delta_4} \frac{\sum_k \lambda_{12k}\lambda_{34k} g_{\Delta_k,l_k}(u,v)}{|x_{12}|^{\Delta_1+\Delta_2}\,|x_{34}|^{\Delta_3+\Delta_4}} \end{aligned} \tag{3.26}$$

with the multi-indices $a_l = \mu_1 \dots \mu_l$ and $b_l = \nu_1 \dots \nu_l$.[6] g_{Δ_k,l_k} is the conformal block. Its definition shows that its form is completely fixed by the conformal symmetry, since C_{ijk} and I^{ab} are determined by it. Hence, the reduced correlator is given by

$$\mathcal{A}(u,v) = \sum_k \lambda_{12k}\lambda_{34k} g_{\Delta_k,l_k}(u,v). \tag{3.27}$$

In (3.26) the pairs of operators (12) and (34) have been fused together to build the conformal block which can be diagrammatically depicted by

$$\langle \phi_1(x_1)\dots\phi_4(x_4)\rangle = \sum_k \lambda_{12k}\lambda_{34k}$$



This fusion process is called the s-channel. But it is also possible to fuse the operators (13) and (24), which is called the t-channel. However, independent of the fusing process the result has to be the same. This requirement, called OPE associativity, is an non-trivial constraint

[6] Note that in the summation over all primary operators denoted by k, l is the spin of the k^{th} operator.

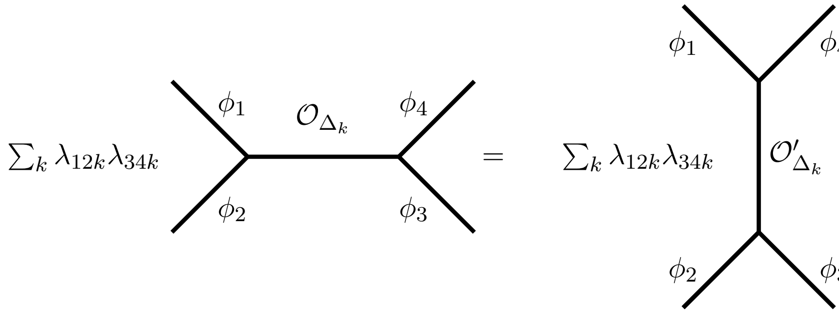


Figure 3.3: OPE associativity requires that in either case the summation over all exchanged operators represents the same four-point correlator. This condition imposes non-trivial constraints on the scaling dimensions and the OPE coefficients of the CFT data.

on the CFT which is actively used in the conformal bootstrap program to obtain information about the operator content and the possible interactions of the CFT [181–190]. Even though this line of research is quite modern, the main idea has already been stated in [95–97]. The OPE associativity is sketched in figure 3.3. To impose the constraints derived from OPE associativity, it is important to determine the exact form of the conformal block [191–202].

This example shows an interesting feature of CFTs. The four-point correlator is fixed by the knowledge of its lower-point correlators. Whereas the conformal block is theory independent and its form depends merely on the conformal symmetry, the OPE coefficients and the scaling dimensions depend on the operators which are included in the theory. Therefore it seems tempting to *define* CFTs by its **CFT data** and the requirement that its data is consistent with OPE associativity [95–98]. The CFT data is the combined physical input of all scaling dimension of the operators including their $\mathrm{SO}(d)$ representation and the knowledge of all kinds of possible interactions which is encoded in the OPE coefficients of the non-vanishing three-point functions.

3.1.3 Embedding Space Formalism

The local isomorphism (3.12) is quite intriguing because it linearizes the action of the conformal algebra acting on an auxiliary space $\mathbb{R}^{2,d}$ which is called **embedding space** [203]. The coordinates on the embedding space shall be written as X^M and the action of the conformal algebra on it is given by

$$X^M \to \Lambda^M_N X^N, \quad \eta_{MN}\Lambda^M_K\Lambda^N_L = \eta_{KL}, \quad \det\Lambda = 1$$

where Λ^M_N is a finite Lorentz transformation generated by (3.11) given in the vector representation. However, to connect the embedding space $\mathbb{R}^{2,d}$ to the physical space $\mathbb{R}^{1,d-1}$ one has to choose a d-dimensional submanifold of $\mathbb{R}^{2,d}$ which respects the symmetry. Since the action of $\mathrm{SO}(2,d)$ transformations leaves the lightcone

$$X^2 = X^M X^N \eta_{MN} = 0$$

invariant one degree of freedom is removed by restricting to it. This can be seen by describing the hypersurface in appropriate lightcone coordinates $X^0, \dots, X^{d-1}, X^+, X^-$ with $X^+ = X^d + X^{d+1}$ and $X^- = X^{d+1} - X^d$. Thus on the lightcone one coordinate can be expressed

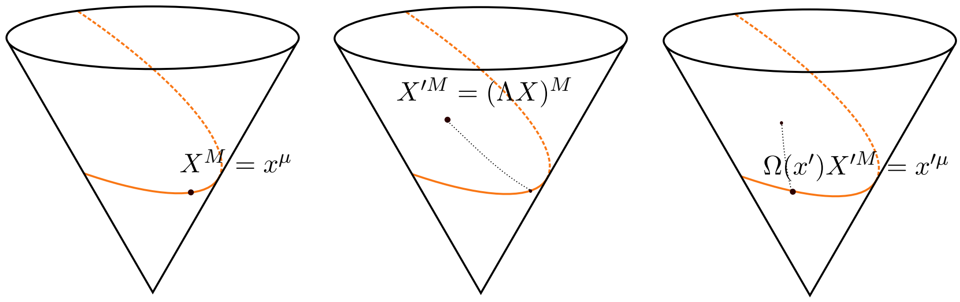


Figure 3.4: The physical space is represented by the section (orange line). After an SO$(2,d)$ transformation X^M is mapped to $X'^M = \Lambda^M_N X^N$ which is generally not on the section. However, a scale transformation $X'^M \to \Omega X'^M$ moves the point again onto the section.

in terms of the others

$$X^- = \frac{-(X^0)^2 + (X^1)^2 + \ldots + (X^{d-1})^2}{X^+}.$$

The second coordinate X^+ is removed by restricting to the section of the lightcone defined by

$$x^\mu = \frac{X^\mu}{X^d + X^{d+1}} = \frac{X^\mu}{X^+}, \quad \text{and} \quad X^+ = \text{const}.$$

such that the d-dimensional section of $\mathbb{R}^{2,d}$ is parametrized by

$$X^M = (X^+, X^-, X^\mu) \overset{\text{section}}{=} X^+(1, x^2, x^\mu). \tag{3.28}$$

The action of (3.11) on $\mathbb{R}^{1,d-1}$ can be analyzed in two steps:

1. The embedding coordinate is mapped from the lightcone onto the lightcone by a Lorentz transformation $X^M \to \Lambda^M_N X^N$.
2. If X^M has been on the section (3.28), the point $X'^M = \Lambda^M_N X^N$ is generally not on the section, but just on the lightcone. However, the point X'^M can be rescaled to the section by

   $$X'^M = \big(c(x'), d(x'), \tilde{x}'^\mu(x')\big) \to \Omega(x')X'^M = X^+(1, x'^2, x'^\mu)$$

   where $\Omega(x') = c(x')^{-1}X^+$, $d(x')\Omega(x') = X^+x'^2$ and $\tilde{x}'^\mu\Omega(x') = X^+x'^\mu$. This corresponds exactly to a Weyl transformation of the induced metric on $\mathbb{R}^{1,d-1}$, which is drawn in figure 3.4.

To prove the second statement, it is best to analyze the infinitesimal line element

$$\begin{aligned} ds^2 = dX^M dX_M &= -(dX^0)^2 + \sum_{i=1}^{d-1}(dX^i)^2 - dX^+dX^- \\ &\overset{\text{section}}{=} -(dX^0)^2 + \sum_{i=1}^{d-1}(dX^i)^2 = \left(X^+\right)^2 \eta_{\mu\nu}dx^\mu dx^\nu \end{aligned}$$

since X^+ is constant. Plugging into ds^2 the rescaled variable $\Omega(x)X^M$ the line element transforms as

$$\begin{aligned} ds^2 = dX'^M dX'_M = \eta_{MN} d(\Omega(x)X)^M d(\Omega(x)X)^N = \Omega^2(x)\eta_{MN} dX^M dX^N \\ = \left(X^+\right)^2 \Omega^2(x)\eta_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{3.29}$$

which is exactly a Weyl transformation. In the second step of the computation it has been used that $X^2 = \eta_{MN}X^M dX^N = 0$ and in the last step the parametrization of the section (3.28) has been applied.[7]

Thus one has traded the complicated action of the conformal generators (3.9) on the simple manifold $\mathbb{R}^{1,d-1}$ with the linearized action of the generators of $\mathrm{SO}(2,d)$ on the manifold parametrized by (3.28) which is the projective lightcone [33, 98, 204].

To describe conformal field theories in embedding space it is also necessary to define operators in it. The easiest type of operators to analyze are scalar primary operators $\phi(x)$. Spinning operators shall be treated in a similar fashion afterwards. The corresponding embedding space operator to $\phi(x)$ is $\Phi(X)$. $\Phi(X)$ has to be homogeneous to be well-defined on the lightcone, i.e. for $X^M \sim \lambda X^M$ it follows that $\Phi(X) \sim \Phi(\lambda X) = \lambda^c \Phi(X)$ for some $c \in \mathbb{R}\backslash\{0\}$.

This implies the operator in embedding space has to be related to the operator in physical space by[8]

$$\Phi(X)\Big|_{\text{section}} = \Phi(X^+(1, x^2, x)) = (X^+)^{-c}\Phi(1, x^2, x) = (X^+)^{-c}\phi(x)$$

Comparing the action of a conformal transformation on $\phi(x)$, which is given in the appendix B.1, and the corresponding rescaling on $\Phi(X)$ leads to $c = \Delta$ because

$$\phi(x) \to \phi'(x') = \Omega^{-\Delta}(x)\phi(x), \quad \Phi(X) \to \Phi'(X') = \Phi(\lambda X) = \lambda^{-c}\Phi(X).$$

To scale the operator Φ back to the section, $\lambda = \Omega(x)$ has to hold according to the previous analyses (3.29). This implies the aforementioned made statement that $c = \Delta$. Thus scalar operators in embedding space have to satisfy:

$$\begin{aligned} \Phi(X)\Big|_{\text{section}} = \frac{1}{(X^+)^\Delta}\phi(x) \quad \text{is the section condition,} \\ \Phi(\lambda X) = \lambda^{-\Delta}\Phi(X) \quad \text{is the homogeneity condition.} \end{aligned} \tag{3.30}$$

(Fermionic) Tensor operators contain tensor indices in addition. Thus to construct fermionic operators in embedding space one has to project these spinor indices onto the physical coordinates as well. An elegant way to deal with these spinor indices is to introduce

[7] Actually, this proves that the lightcone is a projective space. Two points on the lighcone $X^M(x)$ and $\Omega(x)X^M(x)$ with $\Omega(x) \in \mathbb{R}\backslash\{0\}$ which are related by a scale transformation differ in physical space by a conformal transformation. Since points related by a conformal transformation are equivalent, points on the lightcone related by a scale transformation are equivalent. Hence, the points $X^M(x)$ and $\Omega(x)X^M(x)$ belong to the same equivalence class.

[8] This follows from (3.28) and $\Phi(\lambda X) = \lambda^c\Phi(x)$, because $\Phi(1, x^2, x)$ depends neither on X^+ nor on X^- it is demanded that $\Phi(1, x^2, x) = \phi(x)$.

an auxiliary spinor s_α which transforms in the anti-fundamental spinor representation and is a primary of vanishing dimension. Thus one can write

$$\psi(x,s) = s_\alpha \psi^\alpha(x) \quad \text{with} \quad \psi^\alpha(x) = \frac{\partial}{\partial s_\alpha}\psi(x,s). \tag{3.31}$$

Since $\psi(x,s)$ and $\psi^\alpha(x)$ are in one-to-one correspondence $\psi(x,s)$ contains the entire information of the spinor operator $\psi^\alpha(x)$. Similarly an auxiliary spinor S_I can be introduced for the embedding spinor $\Psi^I(X)$ which shall be defined by

$$\Psi(X,S) = S_I \Psi^I(X). \tag{3.32}$$

In (3.31) and (3.32) all spinor indices are contracted; hence these objects transform like scalar operators (3.30):

$$\Psi(X,S)\Big|_{\text{section}} \; = \tfrac{1}{(X^+)^\Delta}\psi(x,s) \tag{3.33}$$

and obey the homogeneity property

$$\Psi(\lambda_1 X, \lambda_2 S) \; = \lambda_1^{-\Delta-\frac{1}{2}}\lambda_2 \Psi(X,S), \tag{3.34}$$

which follows from the transformation behaviour of fermions under conformal transformations and the fact that $\Psi(X,S)$ depends linearly on S_I. This analysis can be generalized to arbitrary fermionic tensors.

To be concrete, consider physical primary spinor operators $\psi^\alpha(x)$ in three dimensions with signature $(-,+,+)$ which transform in a representation of the double cover of SO(2,1), i.e. $\mathrm{SP}(2,\mathbb{R}) = \{M \in \mathrm{GL}(2,\mathbb{R}) | M^T\omega M = \omega\}$. This implies that $\psi^\alpha(x)$ is a Majorana spinor because the representation is real. The operator $\Psi^I(X,S)$ transforms in the fundamental representation of $\mathrm{SP}(4,\mathbb{R}) = \{M \in \mathrm{GL}(4,\mathbb{R}) | M^T\Omega M = \Omega\}$ which is the double cover of SO(3,2). The corresponding symplectic forms (to raise and lower indices) are defined by

$$\omega_{\alpha\beta} = \omega^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega_{IJ} = \Omega^{IJ} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}. \tag{3.35}$$

However, the embedding spinor $\Psi(X,S)$ does still have the double amount of components compared to the physical spinor $\psi(x,s)$. The correct identification of $\Psi(X,S)$ and $\psi(x,s)$ is obtained by the transversality condition

$$S_I X^I{}_J = 0 \quad \text{with} \quad X^I{}_J := X^A \left(\Gamma_A\right)^I{}_J\,. \tag{3.36}$$

It can be checked that this requirement yields the correct conformal transformation for the physical operators if and only if the embedding space operators transform covariantly under Lorentz transformations. The transversality condition (3.36) identifies the auxiliary spinors in the following way

$$S_I = \sqrt{X^+}\begin{pmatrix} s_\alpha \\ -x^\alpha_{\;\beta}s^\beta \end{pmatrix} \quad \text{with} \quad x^\alpha_{\;\beta} := x^\mu \left(\gamma_\mu\right)^\alpha_{\;\beta} \tag{3.37}$$

where the matrices $\{\gamma^\mu | 0 \le \mu \le 2\}$ satisfy the Clifford algebra.

This analysis completes the study of fermionic operators in embedding space. More information about the concrete algebra and the matrix-representation of the γ-matrices can be found in the appendix B.2.

The advantage of the embedding space formalism is that it is more suitable to construct correlation functions in it than in physical space. For example the two point function is given by

$$\langle \Psi(X_1, S_1)\Psi(X_2, S_2)\rangle = i\frac{\langle S_1 S_2\rangle}{X_{12}^{\Delta+\frac{1}{2}}},$$

which follows from the homogeneity property (3.34) and Lorentz invariance. To simplify the notation $\langle S_1 X_2 X_3 \dots S_n\rangle = (S_1)_I (X_2)^I{}_J (X_3)^I{}_J \dots \Omega^{LM}(S_n)_M$ and $X_{ij} = -2X_i \cdot X_j = (x_i - x_j)^2$ is used at the Poincaré section $X^+ = 1$.[9] The correlation function in physical space reads

$$\langle \psi^{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\rangle = i\frac{(x_{12})^{\alpha_1}{}_{\alpha_2}}{(x_{12}^2)^{\Delta+1}}$$

with $x_{12} = x_1 - x_2$. This follows from (3.37) and (3.33).

Another advantage of the embedding space formalism is that constructing tensor structures is easier to handle in it than in physical space because one has to build Lorentz invariant quantities only. This shall be important in section 3.2 [199, 204]. But before doing that it is useful to study the appearance of tensor structure of correlation functions in embedding space.

An operator $\mathcal{O}$ of spin l has $2l$ auxiliary vectors S_i [199, 205, 206] which leads to a generalization of the homogeneity property (3.34) in a straightforward manner

$$\mathcal{O}(aX, bS) = a^{-\Delta-l} b^{2l}\, \mathcal{O}(X, S). \tag{3.38}$$

Therefore, an n-point conformal correlator of operators $\mathcal{O}_i$ with dimension Δ_i and spin l_i obeys the following homogeneity property,

$$\langle \mathcal{O}_1 (a_1 X_1, b_1 S_1) \cdots \mathcal{O}_n (a_n X_n, b_n S_n)\rangle = \prod_{i=1}^{n} a_i^{-\Delta_i} \left(\frac{b_i^{2l_i}}{a_i^{l_i}}\right) \langle \mathcal{O}_1 (X_1, S_1) \cdots \mathcal{O}_n (X_n, S_n)\rangle \tag{3.39}$$

For example, an n-point correlator of $2K$ spin one-half fermionic and M scalar operators, which shall be of interest in section 3.3.2, has to satisfy

$$\begin{aligned}&\langle \Psi_1 (a_1 X_1, b_1 S_1) \cdots \Psi_{2K} (a_{2K} X_{2K}, b_{2K} S_{2K})\, \Phi_{2K+1} (a_{2K+1} X_{2K+1}) \cdots \Phi_n (a_n X_n)\rangle \\ =&\prod_{i=1}^{n} a_i^{-\Delta_i} \prod_{j=1}^{2K} \frac{b_j}{\sqrt{a_j}} \langle \Psi_1 (X_1, S_1) \cdots \Psi_{2K} (X_{2K}, S_{2K})\, \Phi_{2K+1} (X_{2K+1}) \cdots \Phi_n (X_n)\rangle\end{aligned}$$

with $2K + M = n$.

Tensor structures $\mathbb{T}_k$ for a generic spinning correlator shall be defined such that they entirely account for the factor $\prod_{i=1}^{n} \frac{b_i^{2l_i}}{a_i^{l_i}}$ in the homogeneity relation (3.39). Concretely, the

[9] Alternatively one can also rescale the factor X^+ since the lightcone is a projective space, but in this thesis the representative of the equivalence class shall be at $X^+ = 1$ for all coordinates.

tensor structures $\mathbb{T}_k$ are chosen such that they obey

$$\mathbb{T}_k\left(b_1 S_1, \cdots, b_n S_n;\, a_1 X_1, \cdots, a_n X_n\right) = \prod_{i=1}^{n} \frac{b_i^{2l_i}}{a_i^{l_i}} \mathbb{T}_k\left(S_1, \cdots, S_n;\, X_1, \cdots, X_n\right). \tag{3.40}$$

For illustration it is useful to give two examples. The three-point conformal correlator of two spin one-half and one scalar operator is given, schematically, by

$$\langle \Psi_1\left(X_1, S_1\right) \Psi_2\left(X_2, S_2\right) \Phi(X_3, S_3) \rangle = \sum_k \frac{\mathbb{T}_k \lambda_{123}^k}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} X_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}, \tag{3.41}$$

where λ_{123}^k are the OPE coefficients of the components of this three-point correlator. k labels the independent tensor structures. In the same way, any four-point conformal correlator has the generic form

$$\langle \mathcal{O}_1\left(X_1, S_1\right) \cdots \mathcal{O}_4\left(X_4, S_4\right) \rangle = \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}} \frac{\sum_k \mathbb{T}_k \mathcal{A}_k(u,v)}{\left(X_{12}\right)^{\frac{\Delta_1+\Delta_2}{2}} \left(X_{34}\right)^{\frac{\Delta_3+\Delta_4}{2}}} \tag{3.42}$$

where $\mathcal{A}_k(u,v)$ are components of the reduced correlator which is a function of the cross-ratios u and v.

The three-point (3.41) and the four-point correlator (3.42) do not seem to explicitly depend on the spin of the operators. The reason is that in this thesis the tensor structures are defined such that they include the entire spin dependence. Obviously, this follows from the normalization of the tensor structures which is given in (3.39) and (3.40). The spin dependence $\prod_{i=1}^{n} \frac{b_i^{2l_i}}{a_i^{l_i}}$ of the conformal correlator is absorbed into $\mathbb{T}_k$.

3.2 Tensor Structures

3.2.1 Constructing an Independent Basis of Tensor Structures

It has already been stressed in section 3.1.3 that the embedding space formalism is very handy to construct tensor structures for correlation functions which contain fermionic operators. In embedding space one has to build Lorentz invariant quantities only which also satisfy the homogeneity property (3.34) and the transversality condition (3.36). However, the problem is that not all of these tensor structures are linearly independent, because there are algebraic relations between different tensor structures. One option to resolve this redundancy is to make use of Fierz identities or γ-matrix relation to relate the dependent tensor structures to each other, but in general it is difficult to know which identities should be applied.

A different way to count independent tensor structures and construct the web of relations among them is to gauge fix the conformal symmetry and to analyze the correlation function in a conformal frame. A **conformal frame** for an n-point function is any fixed configuration of points to which one can always map any n points using conformal transformations and exhausting all the symmetry [207]. A common conformal frame for a four-point correlator is drawn in 3.5. This removes all the ambiguity and a unique basis of tensor structures can be built. The following analysis is independent of the particular choice of the conformal frame.

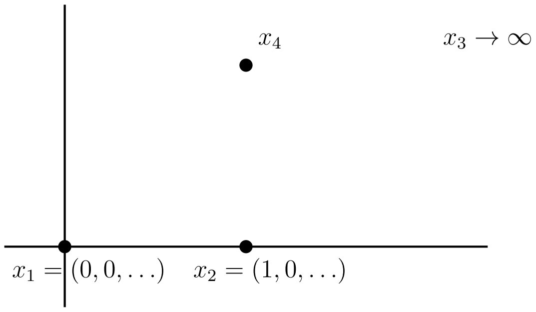


Figure 3.5: A conformal frame for a four-point correlator can be chosen such that the first three points are fixed by conformal symmetry and the remaining point x_4 can be rotated such that the plane spanned by x_1, x_2, x_4 is at the appropriate position.

The authors of [207] prove that there is a relationship between singlets of the stabilizer group and independent tensor structures. In particular they show that the singlets of the stabilizer group (or little group) $G_x = \{g \in G \,|\, g.x = x\}$ that leave the configuration of points x_i invariant are in one-to-one correspondence to the independent tensor structures of an n-point function. x_i are the positions of the operators of the conformal correlator for a chosen conformal frame. Therefore, to construct the basis of tensor structures for a correlator $\langle \mathcal{O}_1(X_1,S_1)\ldots\mathcal{O}_n(X_n,S_n)\rangle$ explicitly, the singlets of a group G under the stabilizer subgroup $H \subseteq G$ can be studied. G is the rotation group under which the operators $\mathcal{O}_i$ transform, e.g. for integer-spin operators G is given by $\mathrm{O}(\cdot)$ (if parity is conserved) or $\mathrm{SO}(\cdot)$ and for half-integer-spin operators the corresponding double cover has to be chosen, i.e. $\mathrm{Pin}(\cdot)$ or $\mathrm{Spin}(\cdot)$. This shall be denoted by

$$\left(\mathrm{Res}^G_H \bigotimes_{i=1}^{n} \rho_i\right)^H, \tag{3.43}$$

where Res^G_H denotes the restriction of a representation of G to $H \subseteq G$. The operator $\mathcal{O}_i$ transforms in the representation ρ_i of the Lorentz group and ρ_i^H denotes the H-singlets in ρ_i. In this thesis fermionic correlation functions of a parity symmetric theory in three dimensions are studied which specifies the groups of (3.43) to be $G = \mathrm{Pin}(3)$ and $H = \mathrm{Pin}(5-n)$.

Further, in the following analysis it is assumed that all operators have different scaling dimensions such that there are no further reductions in the number of independent tensor structures. Because, in general for identical operators the number of independent tensor structures is further reduced due to permutation symmetry.

The authors of [207] calculated the number of independent tensor structures in three dimensions for a theory of definite parity. For three-point functions it is given by

$$\begin{aligned} N^{\pm}_{3d} &= \frac{N_{3d}\,(l_1,l_2,l_3) \pm \kappa}{2}, \quad \text{with} \quad \kappa = \begin{cases} 1, & \text{if all operators are bosonic,} \\ 0, & \text{otherwise.} \end{cases} \\ N_{3d} &= (2l_1+1)(2l_2+1) - p(p+1), \quad p = \max\left(l_1+l_2-l_3, 0\right), \quad l_1 \le l_2 \le l_3. \end{aligned} \tag{3.44}$$

where l_i is the spin of the operator $\mathcal{O}_i$. The superscript $\pm$ denotes the parity of the corresponding tensor structure. The number of independent n-point tensor structures for $n \geq 4$

reads

$$N_{3d}\left(l_1,l_2,l_3,l_4\right)=\prod_{i=1}^{n}(2l_i+1).$$

It can be seen from (3.44) that if there is at least one half-integer spin operator, one can take an equal number of parity odd and parity even tensor structures.

The aforementioned method shall be illustrated by the example of a three-point function $\langle\mathcal{O}_1(X_1,S_1)\mathcal{O}_2(X_2,S_2)\mathcal{O}_3(X_3,S_3)\rangle$. This example is also chosen in [207].

1. The first step is to gauge fix the conformal correlation function. One possible choice is given by

$$g_0(S_1,S_2,S_3):=\lim_{L\to\infty}L^{2\Delta_3}\left\langle\mathcal{O}_1(X_1,S_1)\mathcal{O}_2(X_2,S_2)\mathcal{O}_3(X_3,S_3)\right\rangle \tag{3.45}$$

   at the positions

$$x_1=(0,0,0),\ x_2=(0,0,1),\ x_3=(0,0,L). \tag{3.46}$$

2. Clearly, the set of points (3.46) is invariant under the boost K_1. The explicit representation of this boost is given in the appendix in equ. (B.12). This forms the stabilizer subgroup of this configuration. Hence, the correlator (3.45) is invariant under

$$\begin{aligned} s_i\to e^{-i\lambda\bar{k}_1}s_i=&\left[\cosh\left(\frac{\lambda}{2}\right)\mathbb{1}_2+\sinh\left(\frac{\lambda}{2}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]s_i\\ =&\begin{pmatrix}\left(\cosh\left(\frac{\lambda}{2}\right)+\sinh\left(\frac{\lambda}{2}\right)\right)\xi_i\\ \left(\cosh\left(\frac{\lambda}{2}\right)-\sinh\left(\frac{\lambda}{2}\right)\right)\bar{\xi}_i\end{pmatrix}\quad\text{with}\quad(s_i)_\alpha=\begin{pmatrix}\xi_i\\ \bar{\xi}_i\end{pmatrix}. \end{aligned} \tag{3.47}$$

   It can be seen that ξ_i has charge $1/2$ whereas $\bar{\xi}_i$ is negatively charged with $-1/2$.

3. According to [207] the independent tensor structures are in one-to-one correspondence to the singlets of the stabilizer, i.e. to functions which are invariant under (3.47). A basis of such functions is given by the product of the monomials ξ_i and $\bar{\xi}_i$:

$$[q_1,q_2,q_3]=\prod_{i=1}^{3}\xi_i^{l_i+q_i}\bar{\xi}_i^{l_i-q_i}\quad\text{with}\quad\sum_{i=1}^{3}q_i=0 \tag{3.48}$$

   and $q_i\in\{-l_i,\dots,l_i\}$. The last two constraints have to be imposed, because there cannot be more than $2l_i$ polarization vectors for each operator. It is easy to see that (3.48) is invariant under (3.47). For brevity the abbreviation $\cosh(\lambda/2):=\mathrm{ch}$ and $\sinh(\lambda/2)=\mathrm{sh}$ is used.

$$\begin{aligned}[q_1,q_2,q_3]\to&(\mathrm{ch}+\mathrm{sh})^{l_1+l_2+l_3+q_1+q_2+q_3}(\mathrm{ch}-\mathrm{sh})^{l_1+l_2+l_3-q_1-q_2-q_3}[q_1,q_2,q_3]\\ =&\left(\mathrm{ch}^2-\mathrm{sh}^2\right)^{l_1+l_2+l_3}[q_1,q_2,q_3]=[q_1,q_2,q_3]\end{aligned}$$

   Thus in the conformal frame (3.46) the basis of tensor structures for three-point correlation functions is given by (3.48).

3.2.2 Basis of Tensor Structures

To analyze the factorization properties of the four-point conformal correlation functions $\langle\Psi_1\Psi_2\Phi_3\Phi_4\rangle$ and $\langle\Psi_1\Psi_2\Psi_3\Psi_4\rangle$ in three dimensions it is necessary to construct a basis of their tensor structures as well as the basis of the tensor structures of the three-point functions $\langle\Psi_1\Psi_2\Phi_3\rangle$, $\langle\Psi_1\Phi_2\Psi_{3,l}\rangle$ and $\langle\Psi_1\Psi_2\mathcal{O}_{3,l}\rangle$ with $l \geq 0$ onto which they factorize. This is established using the formalism described in section 3.2.1. The basis of tensor structures is of definite parity which is indicated by the superscript $\pm$. In addition, all the tensor structures shall be presented in embedding space and they are contracted with the polarization vectors; hence they are conformally invariant. The expression in physical space can be determined by the formulae presented in section 3.1.3. In particular, the following formulae are useful

$$\begin{aligned}\langle S_1 X_2 X_3 \cdots X_{k-1} S_k\rangle &\underset{3d}{\longrightarrow} \not{x}_{12}\not{x}_{23}\cdots\not{x}_{k-1,k},\\ \langle S_i X_a \cdots X_b S_m\rangle\langle S_k X_u \cdots X_v S_l\rangle &\underset{3d}{\longrightarrow} [\not{x}_{ia}\cdots\not{x}_{bm}]\,[\not{x}_{ku}\cdots\not{x}_{vl}]\,.\end{aligned}$$

The spinor indices on the r.h.s. have been suppressed and a bracket is used if more than one concatenation of spinor variables occurs such that it is evident how to interpret the expression in physical space.

Tensor Structures for Three-Point Functions

The conformal correlator $\langle\Psi_1\Psi_2\Phi_3\rangle$ of two spin one-half fermions and one scalar has two independent tensor structures which are chosen in the following way

$$r^+_{di} = \frac{\langle S_1 S_2\rangle}{\sqrt{X_{12}}}, \qquad r^-_{di} = \frac{\langle S_1 X_3 S_2\rangle}{\sqrt{X_{13}X_{32}}}.$$

According to (3.44) the conformal correlator $\langle\Psi_1\Psi_2\mathcal{O}_{3,l}\rangle$ with spin $0 < l \in \mathbb{N}^*$ has four independent tensor structures.[10] They can be defined to be

$$\begin{aligned} r^+_{di,1} &= \frac{\langle S_1S_2\rangle\langle S_3X_1X_2S_3\rangle^l}{X_{12}^{\frac{l+1}{2}}X_{13}^{\frac{l}{2}}X_{23}^{\frac{l}{2}}}, \qquad r^+_{di,2} = \frac{\langle S_1S_3\rangle\langle S_2S_3\rangle\langle S_3X_1X_2S_3\rangle^{l-1}}{X_{12}^{\frac{l-1}{2}}X_{13}^{\frac{l}{2}}X_{23}^{\frac{l}{2}}},\\ r^-_{di,3} &= \frac{\langle S_3X_1X_2S_3\rangle^{l-1}}{X_{12}^{\frac{l}{2}}X_{13}^{\frac{l+1}{2}}X_{23}^{\frac{l+1}{2}}}\left[X_{23}\langle S_1S_3\rangle\langle S_2X_1S_3\rangle + X_{13}\langle S_2S_3\rangle\langle S_1X_2S_3\rangle\right],\\ r^-_{di,4} &= \frac{\langle S_3X_1X_2S_3\rangle^{l-1}}{X_{12}^{\frac{l}{2}}X_{13}^{\frac{l+1}{2}}X_{23}^{\frac{l+1}{2}}}\left[X_{23}\langle S_1S_3\rangle\langle S_2X_1S_3\rangle - X_{13}\langle S_2S_3\rangle\langle S_1X_2S_3\rangle\right]. \end{aligned} \tag{3.49}$$

For the three-point function $\langle\Psi_1\Phi_2\Psi_{3,l}\rangle$ of a scalar Φ_2, a spin one-half Ψ_1 and a higher non-integer operator $\Psi_{3,l}$ with $0 < l \in \mathbb{N} + \frac{1}{2}$ the basis can be chosen such that

$$r^+_{cr} = \frac{\langle S_1S_3\rangle\langle S_3X_1X_2S_3\rangle^{l-\frac{1}{2}}}{X_{12}^{\frac{l}{2}-\frac{1}{4}}X_{13}^{\frac{l}{2}+\frac{1}{4}}X_{23}^{\frac{l}{2}-\frac{1}{4}}}, \quad r^-_{cr} = \frac{\langle S_1X_2S_3\rangle\langle S_3X_1X_2S_3\rangle^{l-\frac{1}{2}}}{X_{12}^{\frac{l}{2}+\frac{1}{4}}X_{13}^{\frac{l}{2}-\frac{1}{4}}X_{23}^{\frac{l}{2}+\frac{1}{4}}}. \tag{3.50}$$

Note that the basis which has been chosen in the associated publication [1] is the same as in [198, 199] but with different normalization.

[10] For $l = 0$, $r^+_{di,1}$ goes to r^+_{di} and $r^-_{di,3}$ goes to r^-_{di}.

Tensor Structures for Four-Point Functions

The conformal correlator $\langle \Psi_1 \Psi_2 \Phi_3 \Phi_4 \rangle$ has four independent tensor structures. As in [1, 198], they are taken to be

$$\begin{aligned} t_1^+ &= \frac{\langle S_1 S_2 \rangle}{\sqrt{X_{12}}}, & t_2^+ &= \frac{\langle S_1 X_3 X_4 S_2 \rangle}{\sqrt{X_{13} X_{34} X_{42}}}, \\ t_3^- &= \frac{\langle S_1 X_3 S_2 \rangle}{\sqrt{X_{13} X_{32}}}, & t_4^- &= \frac{\langle S_1 X_4 S_2 \rangle}{\sqrt{X_{14} X_{42}}}. \end{aligned} \tag{3.51}$$

For the conformal correlator $\langle \Psi_1 \Psi_2 \Psi_3 \Psi_4 \rangle$ of four spin one-half fermions the basis of independent tensor structures consists of sixteen elements. According to [1] they are given by eight parity even tensor structures

$$\begin{aligned} p_1^+ &= \frac{\langle S_1 S_2 \rangle \langle S_3 S_4 \rangle}{\sqrt{X_{12} X_{34}}}, & p_2^+ &= \frac{\langle S_1 S_2 \rangle \langle S_3 X_1 X_2 S_4 \rangle}{\sqrt{X_{12}^2 X_{13} X_{24}}}, \\ p_3^+ &= \frac{\left\langle S_1 X_3 \Gamma^A S_2 \right\rangle \langle S_3 X_1 \Gamma_A S_4 \rangle}{\sqrt{X_{13} X_{32} X_{31} X_{14}}}, & p_4^+ &= \frac{\left\langle S_1 \Gamma^A \Gamma^B S_2 \right\rangle \langle S_3 \Gamma_A \Gamma_B S_4 \rangle}{\sqrt{X_{12} X_{34}}}, \\ p_5^+ &= \frac{\langle S_1 X_3 S_2 \rangle \langle S_3 X_1 S_4 \rangle}{\sqrt{X_{13}^2 X_{14} X_{23}}}, & p_6^+ &= \frac{\langle S_1 X_3 S_2 \rangle \langle S_3 X_2 S_4 \rangle}{\sqrt{X_{23}^2 X_{13} X_{24}}}, \\ p_7^+ &= \frac{\langle S_1 X_4 S_2 \rangle \langle S_3 X_1 S_4 \rangle}{\sqrt{X_{13} X_{14}^2 X_{24}}}, & p_8^+ &= \frac{\langle S_1 X_4 S_2 \rangle \langle S_3 X_2 S_4 \rangle}{\sqrt{X_{14} X_{23} X_{24}^2}}, \end{aligned} \tag{3.52}$$

and eight parity odd tensor structures

$$\begin{aligned} p_9^- &= \frac{\langle S_1 S_2 \rangle \langle S_3 X_1 S_4 \rangle}{\sqrt{X_{12} X_{13} X_{14}}}, & p_{10}^- &= \frac{\langle S_1 S_2 \rangle \langle S_3 X_2 S_4 \rangle}{\sqrt{X_{12} X_{23} X_{24}}}, \\ p_{11}^- &= \frac{\langle S_1 X_3 S_2 \rangle \langle S_3 S_4 \rangle}{\sqrt{X_{13} X_{23} X_{34}}}, & p_{12}^- &= \frac{\langle S_1 X_4 S_2 \rangle \langle S_3 S_4 \rangle}{\sqrt{X_{14} X_{24} X_{34}}}, \\ p_{13}^- &= \frac{\left\langle S_1 \Gamma^A S_2 \right\rangle \langle S_3 \Gamma_A X_1 S_4 \rangle}{\sqrt{X_{12} X_{13} X_{14}}}, & p_{14}^- &= \frac{\left\langle S_1 \Gamma^A S_2 \right\rangle \langle S_3 \Gamma_A X_2 S_4 \rangle}{\sqrt{X_{12} X_{23} X_{24}}}, \\ p_{15}^- &= \frac{\left\langle S_1 \Gamma^A X_3 S_2 \right\rangle \langle S_3 \Gamma_A S_4 \rangle}{\sqrt{X_{13} X_{23} X_{34}}}, & p_{16}^- &= \frac{\left\langle S_1 \Gamma^A X_4 S_2 \right\rangle \langle S_3 \Gamma_A S_4 \rangle}{\sqrt{X_{14} X_{42} X_{34}}}. \end{aligned} \tag{3.53}$$

3.3 Mellin Amplitudes

In the following Mellin amplitudes in Euclidean spacetime are discussed. The Lorentzian expression can be obtained by adding the correct $i\epsilon$-prescription which is given by the map $X_{ij} \to (x_i - x_j)^2 + i\epsilon$.

3.3.1 Scalar Mellin Amplitudes

The study of Poincaré covariant correlation functions simplifies drastically in momentum space. Further, to calculate relevant cross sections of scattering processes the amputated correlation function is used which can be obtained by the LSZ procedure (2.7). This procedure is also naturally phrased in momentum space. Hence, for Poincaré covariant theories it seems natural to formulate the mathematical objects in momentum space.

Conformally covariant correlation functions do not exhibit such a nice form in momentum space. However, the Mellin-Barnes representation for scalar conformal correlators defined by Mack [99, 100] seems to be the analogue of momentum representation for conformal correlators. Consider scalar operators $\Phi_i(X_i)$ with scaling dimension Δ_i. Then the connected part of a correlation function can be written as

$$\langle \Phi_1(X_1)\dots\Phi_n(X_n)\rangle_c := \int_{-i\infty}^{i\infty} [ds_{ij}]\,\mathcal{M}_c(\{s_{ij}\}) \prod_{1\leq i<j\leq n} \Gamma(s_{ij}) X_{ij}^{-s_{ij}}. \tag{3.54}$$

To represent the correlation function, embedding space coordinates (3.28) with the shorthand notation $X_{ij} = -2X_i{\cdot}X_j$ have been used. The integration measure $[ds_{ij}]$ is over the $n(n-3)/2$ independent symmetric Mellin variables $s_{ij} = s_{ji}$ each weighted by $(2\pi i)^{-1}$ for $n \leq d$. If $n > d$ there are $nd - (d+1)(d+2)/2$ independent variables. Indeed the configuration space of n points in d dimensions is nd-dimensional and the conformal group $\mathrm{SO}(d+1,1)$ reduces the number of free parameters by $(d+1)(d+2)/2$. The correct integration contour is parallel to the imaginary axis such that the semi-infinite sequences of poles generated by the Γ-functions are not split like it is shown in figure 3.6. The exact measure of the integral is defined in the appendix B.3. $\mathcal{M}_c(\{s_{ij}\})$ is the **Mellin amplitude** which is a conformal invariant function of the Mellin variables only. Implicitly it depends on the scaling dimensions and the OPE coefficients of the operators included in the theory. Actually, it contains the entire information about the scaling dimensions and the OPE coefficients of the operators. To obtain simple growth conditions on the Mellin amplitude, the Γ-functions in (3.54) have been explicitly included. They fall of exponentially fast in the imaginary direction. The Γ-functions keep the integrand bounded as long as $\mathcal{M}_c$ does not grow exponentially.

Homogeneity of each embedding field $\Phi_j(X_j)$ in (3.54) yields n constraints and demands

$$\sum_{i\neq j} s_{ij} = \Delta_j, \quad s_{ij} = s_{ji}, \quad s_{ii} = 0. \tag{3.55}$$

which shows that there are $n(n-1)/2 - n = n(n-3)/2$ independent Mellin variables. However, (3.55) can be written in a more suggestive way which resembles the structure of amplitudes as functions of Mandelstam variables. Defining $s_{ij} = p_i \cdot p_j$ and $-\Delta_j = p_j \cdot p_j$ the constraints (3.55) can be rewritten as momentum conservation

$$\sum_{i=1}^{n} p_i = 0 \quad \text{because} \quad 0 = \sum_{i=1}^{n} p_i \cdot p_j = \sum_{i=1}^{n} s_{ij} = -\Delta_j + \sum_{i\neq j} s_{ij}. \tag{3.56}$$

Perturbatively a Mellin-Barnes representation always exists and can be evaluated in flat space by Symanzik's star formula [101] or by the generalization thereof in hyperbolic spaces [106]. Two simple examples are calculated in the appendix B.3. In perturbation theory it is also possible to derive some sort of "Feynman rules" in Mellin space for weakly coupled theories [104] and strongly coupled theories [102, 103]. However, there are certain limitations, since these Mellin space Feynman rules work at tree level only.[11] These rules suggest that one

[11] Progress on loops as been very limited [106, 208, 209].

should think of the Mellin amplitude as an amputated correlation function in Mellin-Barnes representation, because the external propagators do not contribute to it. Thus, extracting the Mellin amplitude from (3.54) can be thought of as an LSZ like procedure in Mellin space [33].

Physical sensible information of the CFT is encoded in the pole structure of the Mellin amplitude:

1. The location of its poles yields the information of the scaling dimensions of the exchanged operators.
2. And the residues of these poles contain the information about the OPE coefficients.

Hence, in principle the entire spectrum of the CFT can be derived from the knowledge of the Mellin amplitude. To prove this statement it is useful to study the Mellin amplitude of a four-point correlator, which can be written in the following form after solving the constraints (3.55).

$$\begin{aligned}
\langle \Phi_1(X_1)\Phi_2(X_2)\Phi_3(X_3)\Phi_4(X_4)\rangle_c &= \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}\frac{\mathcal{A}(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}}(X_{34})^{\frac{\Delta_3+\Delta_4}{2}}},\\
u = \frac{X_{12}X_{34}}{X_{13}X_{24}},&\qquad v=\frac{X_{14}X_{23}}{X_{13}X_{24}},\\
\mathcal{A}(u,v) &= \int_{c_s-i\infty}^{c_s+i\infty}\frac{ds}{4\pi i}\int_{c_t-i\infty}^{c_t+i\infty}\frac{dt}{4\pi i}\,\mathcal{M}(s,t)u^{\frac{s}{2}}v^{-\frac{s+t-\Delta_1-\Delta_4}{2}}\rho_{\Delta_i}(s,t)\\
\text{with}\quad s=-(p_1+p_2)^2 &= \Delta_1+\Delta_2-2s_{12},\qquad t=-(p_1+p_3)^2=\Delta_1+\Delta_3-2s_{13}.
\end{aligned}\tag{3.57}$$

and

$$\begin{aligned}
\rho_{\Delta_i}(s,t) = {}&\Gamma\left(\frac{\Delta_1+\Delta_2-s}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-s}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-t}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_4-t}{2}\right)\\
&\times\Gamma\left(\frac{s+t-\Delta_1-\Delta_4}{2}\right)\Gamma\left(\frac{s+t-\Delta_2-\Delta_3}{2}\right).
\end{aligned}$$

with Δ_i being the scaling dimensions of the external operators Φ_i. As expected the four-point Mellin amplitude is only a function of $\frac{4(4-3)}{2}=2$ Mellin variables.

In a theory that admits a large N-expansion, the poles of the Γ-function from (3.57) correspond to multi-trace operators that contribute to the conformal block expansion and have the said values of twist in a regime where anomalous dimensions are suppressed. The Mellin amplitude then accounts for the contributions from only single trace operators and is a meromorphic function [33]. However, in a generic CFT operators of dimension $\Delta_1+\Delta_2+2m$ do not appear. Thus the Mellin amplitude has to have zeros at the pole position of the Γ-functions to cancel these. This observation is useful and the program of Mellin-Polyakov bootstrap obtains information about the spectrum of the CFT by using the cancellation of these spurious poles as a consistency conditions [114–116].

In general, for every conformal primary with twist τ contributing to the conformal block expansion of $\mathcal{A}(u,v)$ in the s-channel, $\mathcal{M}(s,t)$ has poles at $s=\tau+2m$, $m\in\mathbb{N}$ where $m=0$ corresponds to the primary and the leading twist descendants (and similarly for the

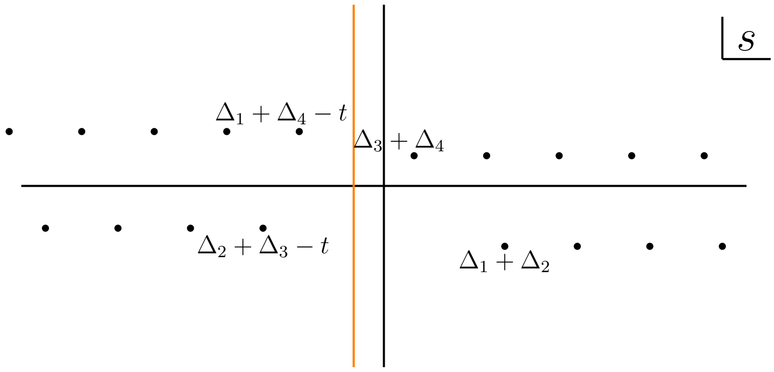


Figure 3.6: In the complex s-plane the contour has to be chosen such that no series of poles generated by the Γ-function is separated. The poles of the Γ-function are depicted by black dots. Hence, the contour (orange line) is a valid choice.

other channels). To prove this, consider the conformal block expansion (3.27) of the reduced correlator

$$\mathcal{A}(u,v) = \sum_k \lambda_{12k}\lambda_{34k} g_{\Delta_k,l_k}(u,v).$$

The authors of [192] have shown that the conformal block $g_{\Delta_k,l_k}(u,v)$ admits a small u expansion in the limit $(u,v) \to (0,1)$.

$$\begin{aligned} g_{\Delta,l}(u,v) &= u^{\frac{\Delta-l}{2}} \sum_{m=0}^{\infty} u^m g_m(v) \quad \text{where the first term reads} \\ g_0(v) &= \left(\frac{v-1}{2}\right)^l {}_2F_1\left(\frac{\Delta+l-\Delta_{12}}{2}, \frac{\Delta+l+\Delta_{34}}{2}, \Delta+l, 1-v\right). \end{aligned} \tag{3.58}$$

Here the abbreviation $\Delta_{ij} = \Delta_i - \Delta_j$ has been used. Since both expression of the reduced correlator (3.57) and (3.27) have to coincide, the Mellin amplitude has to have the following pole structure

$$\sum_k \sum_{m=0}^{\infty} \frac{\lambda_{12k}\lambda_{34k}\mathcal{Q}_{m,l_k}(t)}{s-\Delta_k+l_k+2m} \qquad \text{with} \quad m \in \mathbb{N}. \tag{3.59}$$

to reproduce the series expansion in u. Hence, the location of the poles encodes the twist $\tau_k = \Delta_k - l_k$ of the exchanged operator. The spin l_k of the exchanged operator can be deduced from the polynomial $\mathcal{Q}_{m,l_k}(t)$ which implies that the scaling dimension Δ_k can be obtained. Further, if the polynomial is known, the OPE coefficients $\lambda_{12k}\lambda_{34k}$ can be determined from the residue of the pole. Thus, the spectrum of the CFT is encoded in the Mellin amplitude as claimed [105].

This also shows another property of Mellin amplitudes - they factorize onto lower-point Mellin amplitudes. The Mellin amplitude of (3.57) factorizes onto two three-point Mellin amplitudes which are proportional to the OPE coefficients. These are constants since the conformality constraint fixes their form completely such that no integration parameter is left. This statement is verified in equ. (3.59). Therefore the existence of an OPE expansion implies that the Mellin amplitude factorizes [99, 105].

Properties	**(scalar) Mellin Amplitudes**	**Amplitudes**
Factorization	Mellin amplitudes factorize onto lower-point Mellin amplitudes due to the existence of a convergent OPE.	Tree-level amplitudes factorize onto lower-point amplitudes due to Feynman diagrammatic expansion.[12]
Conservation conditions	Conformal invariance requires $\sum_{i\neq j} s_{ij} = \Delta_j$ and $p_i \cdot p_j = s_{ij}$ and $-p_i^2 = \Delta_i$ implies $-(p_{i_1} + \dots p_{i_a})^2 = \sum_{j=1}^a \Delta_j - 2\sum_{l<k}^a p_{i_l} \cdot p_{i_k}$.	Momentum conservation implies $\sum_{i=1}^a p_i = 0$. Hence $-(p_{i_1} + \dots p_{i_a})^2 = \sum_{j=1}^a \Delta_j - 2\sum_{l<k}^a p_{i_l} \cdot p_{i_k}$.

Table 3.1: Comparison of Mellin amplitudes to amplitudes.

After this analysis it is useful to compare the properties of Mellin amplitudes to (usual) amplitudes which is done in table 3.1.

3.3.2 Spinning Mellin Amplitudes

Mellin amplitudes for operators which transform under a non-trivial representation of the Lorentz group can be defined in a similar way. However, since the correlation functions transform covariantly under Lorentz transformations they are given by a sum over the different tensor structures like in (3.41) and (3.42).

Considering a correlation function of $2K$ fermions and M scalars ($2K + M = n$), the Mellin amplitude can be defined with the following set of Mellin-Barnes integrals,

$$\begin{aligned}
&\langle \Psi_1 (X_1, S_1) \cdots \Psi_{2K} (X_{2K}, S_{2K}) \, \Phi_{2K+1} (X_{2K+1}) \cdots \Phi_n (X_n) \rangle \\
&:= \sum_k \tilde{\mathbb{T}}_k \int_{-i\infty}^{i\infty} [ds_{ij}] \prod_{1\leq i<j}^{n} X_{ij}^{-s_{ij}-a_{ij;k}} \Gamma\left(s_{ij} + a_{ij;k} + n_{ij;k} + \frac{1}{2}\sum_{m=1}^{K} \delta_{i,2m-1}\delta_{j,2m}\right) \\
&\times \mathcal{M}_k (\{s_{ij}\}) \prod_{m=1}^{K} \frac{1}{\sqrt{X_{2m-1,2m}}}.
\end{aligned} \tag{3.60}$$

The set $\{\tilde{\mathbb{T}}_k\}$ furnishes a basis of tensor structures. A basis for three- and four-point correlators in three dimensions has been given in section 3.2. The component of the Mellin amplitude $\mathcal{M}_k$ is associated to the tensor component $\tilde{\mathbb{T}}_k$.[13] Hence, for spinning operators the Mellin amplitude has several components. The definition of the fermionic Mellin amplitude is chosen such that the tensor structures are still represented in position space. Therefore (3.60) is a straightforward generalization of (3.54). The Mellin amplitude has been defined

[12] A Feynman diagrammatic analysis of tree-level amplitudes reveals that the only simple poles of the amplitude appear at the position of the propagators. Setting a propagator on-shell separates the l.h.s. and the r.h.s. of the diagram into two subdiagrams. Hence the tree-level amplitude factorizes onto lower-point amplitudes.
But an even stronger statement holds for amplitudes. The pole structure of momentum space correlators, called polology, is discussed in [8, 125]. There it is shown that momentum space correlators always have poles when an intermediate (fundamental or composite) particle goes on-shell and that at this pole the correlator factorizes onto lower-point matrix elements. This is a non-perturbative statement.

[13] In the following the subscript c for the connected part of the correlator is not written any more.

such that

$$\tau_i - \sum_{l \neq i} s_{li} = 0 \quad \forall i \quad \text{with the twist} \quad \tau_i = \Delta_i - l_i \tag{3.61}$$

generalization the case of scalar operators for which the twist coincides with the scaling dimension. This choice leads to a shift of the Γ-function and the additional terms $\prod_m (X_{2m-1,2m})^{-\frac{1}{2}}$. The conformality constraints imposed by (3.61) in (3.60) can be interpreted in terms of fictitious Mellin momenta p_i which satisfy $p_i \cdot p_j = s_{ij}$ and an on-shell condition $p_i^2 = -\tau_i$ as the overall conservation of Mellin momentum $\sum_i p_i = 0$. This extends the definitions of the corresponding scalar conformal correlator as discussed in 3.3.1 to spinning conformal correlators [1].

Further it is required that the tensor structures are normalized as described in (3.40). This fixes the numbers $a_{ij;k}$

$$\mathbb{T}_k = \tilde{\mathbb{T}}_k \prod_{1 \leq i < j}^{n} X_{ij}^{-a_{ij;k}}$$

by demanding

$$\mathbb{T}_k \left(\lambda_1 S_1, \cdots, \lambda_{2K} S_{2K}; \sigma_1 X_1, \cdots, \sigma_n X_n\right) = \mathbb{T}_k \left(S_1, \cdots, S_{2K}; X_1, \cdots, X_n\right). \tag{3.62}$$

In fact, the coefficients $a_{ij;k}$ are determined by setting $\lambda_i = \sqrt{\sigma_i}$ which follows from the comparison of (3.40) and (3.62).

The precise value of $n_{ij;k}$ is given in the next paragraph. Their values are chosen such that the Mellin amplitudes for the contact interactions are polynomials in the Mellin variable s_{ij} or constant in the perturbative regime. This ensures that the Mellin amplitudes in the large N limit of the strongly coupled CFT describe only the bulk dynamics. Hence, in perturbation theory the pole structure of the Mellin amplitudes does not include any information of the trivial composite operators.

Furthermore, the Mellin amplitude has only been defined for the connected part of the conformal correlator. Thus for simplicity, it is assumed that all operators of the same spin have different scaling dimensions Δ which implies that there is no disconnected part of the correlator.

With these remarks the Mellin amplitude of the correlator of two scalar and two spin one-half operators can be defined by

$$\langle \Psi_1 \Psi_2 \Phi_3 \Phi_4 \rangle = \int [ds_{ij}] \prod_{i<j} (X_{ij})^{-s_{ij}} \frac{1}{\sqrt{X_{12}}} \left[\sum_{i=1}^{4} t_i \bar{M}_i \left(\{s_{ab}\}\right) \right]. \tag{3.63}$$

It can be checked that the Mellin variables s_{ij} obey the conformality constraint stated in (3.61). The basis of tensor structures t_i is given in (3.51).[14] In addition, $n_{ij;k} = 0$ has been set. For readability, (3.63) has been defined with the components of the reduced Mellin amplitude $\bar{M}_i$ which also include the Γ-functions. These components are defined by the

[14]The superscript has been neglected.

relations:

$$\begin{aligned}
\mathcal{M}_1 &= \bar{M}_1 \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}\right)\right]^{-1}, \\
\mathcal{M}_2 &= \bar{M}_2 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1}, \\
\mathcal{M}_3 &= \bar{M}_3 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}\right)\right]^{-1}, \\
\mathcal{M}_4 &= \bar{M}_4 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}\right)\right]^{-1}.
\end{aligned} \tag{3.64}$$

In the same way, the Mellin amplitude of the four-point correlator of four spin one-half fermions is defined

$$\langle \Psi_1\Psi_2\Psi_3\Psi_4 \rangle = \int [ds_{ij}] \prod_{i<j} \left(X_{ij}\right)^{-s_{ij}} \frac{1}{\sqrt{X_{12}X_{34}}} \left[\sum_{i=1}^{16} p_i \bar{M}_i\left(\{s_{ab}\}\right)\right]. \tag{3.65}$$

The basis of tensor structures for the parity even part is given in (3.52) and for the parity odd part in (3.53). The parameters $n_{ij;k}$ are zero with the exception

$$n_{12;2} = n_{13;3} = n_{13;5} = n_{23;6} = n_{14;7} = n_{24;8} = -1. \tag{3.66}$$

The precise relation between the reduced Mellin amplitude and the Mellin amplitude is given in appendix B.4.1

3.3.3 Analyzing the Pole Structure of Spinning Mellin Amplitudes

In section 3.3.1 for scalar Mellin amplitudes the pole structure has been analyzed by comparing the Mellin-Barnes representation of the reduced correlator to its conformal block expansion in the limit $(u, v) \to (0, 1)$. Consistency requires that both expression have to coincide and this dictates the position of the poles. To obtain the location of the poles for fermionic Mellin amplitudes, the same method can be used [1].

Since the nature of the tensor structures depends on the dimension of spacetime, the following analysis is restricted to three dimensions. However, the method can be applied to any dimension.

Location of the Poles for the Fermion-Scalar Four-Point Correlator

There are two different channels to study: the direct channel (s-channel) and the crossed channel (t-channel). The u-channel is not independent and can be obtained by the other two using the equation $u+t+s = \sum_i \tau_i$. This relation follows from Mellin momenta conservation (3.56) for fermionic Mellin amplitudes (3.61).

Poles in the Direct Channel

In the direct channel the reduced correlator $\mathcal{A}(u,v)$ for $\langle\Psi_1\Psi_2\Phi_3\Phi_4\rangle$ is given by

$$\begin{aligned}\langle\Psi_1\Psi_2\Phi_3\Phi_4\rangle &= \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}\frac{\mathcal{A}(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}}(X_{34})^{\frac{\Delta_3+\Delta_4}{2}}},\\ \mathcal{A}(u,v) &= \int_{c_s-i\infty}^{c_s+i\infty}\frac{ds}{4\pi i}\int_{c_t-i\infty}^{c_t+i\infty}\frac{dt}{4\pi i}\left[\sum_{i=1}^{4}t_i\bar{M}_i(s,t)\right]u^{\frac{s}{2}}v^{-\frac{s+t-\tau_1-\tau_4}{2}},\\ s = -(p_1+p_2)^2 &= \tau_1+\tau_2-2s_{12},\quad t=-(p_1+p_3)^2=\tau_1+\tau_3-2s_{13}.\end{aligned} \tag{3.67}$$

$\bar{M}_i$ are the components of the reduced Mellin amplitude which is related to the Mellin amplitude by (3.64). Each component of the Mellin amplitude is associated to a certain tensor t_i, which are given in (3.51). Now, for each component of the Mellin amplitude the same analysis is done as in the scalar case; hence, it is instructive to write the reduced correlator as the following linear combination: $\mathcal{A}(u,v)=\sum_i t_i\mathcal{A}_i(u,v)$. Since all the components $\mathcal{A}_i$ are linearly independent, they have their own OPE series and OPE coefficients, respectively.

The corresponding components of the conformal block for each component $\mathcal{A}_i$ can be determined from the conformal block of the scalar conformal correlator $\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle$ by applying the differential operators

$$\begin{aligned}\mathcal{D}_1 &:= \langle S_1S_2\rangle\,\Pi_{\frac{1}{2},\frac{1}{2}},\\ \mathcal{D}_2 &:= \frac{-1}{4l(\Delta-1)}\left\langle S_1\frac{\delta}{\delta X_1}\frac{\delta}{\delta X_2}S_2\right\rangle\Pi_{-\frac{1}{2},-\frac{1}{2}}+\frac{(\Delta+\Delta_1+\Delta_2-l-4)(\Delta-\Delta_1-\Delta_2-l+1)}{4l(\Delta-1)}\mathcal{D}_1,\\ \mathcal{D}_3 &:= \frac{1}{2(\Delta-1)}\left[\left\langle S_1\frac{\delta}{\delta X_1}S_2\right\rangle\Pi_{-\frac{1}{2},\frac{1}{2}}-\left\langle S_2\frac{\delta}{\delta X_2}S_1\right\rangle\Pi_{\frac{1}{2},-\frac{1}{2}}\right],\\ \mathcal{D}_4 &:= \frac{1}{2l}\left[\left\langle S_1\frac{\delta}{\delta X_1}S_2\right\rangle\Pi_{-\frac{1}{2},\frac{1}{2}}+\left\langle S_2\frac{\delta}{\delta X_2}S_1\right\rangle\Pi_{\frac{1}{2},-\frac{1}{2}}\right]-\frac{\Delta_1-\Delta_2}{l}\mathcal{D}_3,\end{aligned}$$

to the scalar conformal block (3.58). The operator $\Pi_{a,b}$ applies a shift to the scaling dimension: $\Pi_{a,b}:(\Delta_1,\Delta_2)\to(\Delta_1+a,\Delta_2+b)$ and the derivative is defined as $\frac{\delta}{\delta X_a}:=\Gamma^A\frac{\partial}{\partial X_a^A}$ with the Γ-matrices given in (B.15). The differential operators generate the tensor structure of the fermionic correlator $\langle\Psi_1\Psi_2\mathcal{O}_l\rangle$ from the scalar correlator $\langle\Phi_1\Phi_2\mathcal{O}_l\rangle$. This works since in either case the exchanged operator in the direct channel are the same (symmetric traceless tensors) and the differential operators are constructed such that they produce the correct tensor structures. This has been shown in [199] by studying the conformal partial wave expansion of the fermionic correlation function. Explicitly, the relation between the components of the fermionic conformal block $g^{i,a}_{\Delta,l}$ and the scalar conformal block $g_{\Delta,l}$ reads[15]

$$\begin{aligned}&\left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}\frac{\sum_i t_ig^{i,a}_{\Delta,l}(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}}(X_{34})^{\frac{\Delta_3+\Delta_4}{2}}}\\ =&\mathcal{D}_a\left[\left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}\frac{g_{\Delta,l}(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}}(X_{34})^{\frac{\Delta_3+\Delta_4}{2}}}\right].\end{aligned} \tag{3.68}$$

[15]Note that the tensor structures t_i are differently normalized compared to the tensor structures in [199].

Due to the parity symmetric basis of tensor structures, only the following components of the fermionic blocks are non-zero: $g^{1,1}_{\Delta,l}$, $g^{1,2}_{\Delta,l}$, $g^{2,2}_{\Delta,l}$, $g^{3,3}_{\Delta,l}$, $g^{3,4}_{\Delta,l}$, $g^{4,3}_{\Delta,l}$ and $g^{4,4}_{\Delta,l}$. For the scalar exchange ($l=0$) the non-zero components are $g^{1,1}_{\Delta,0} \equiv g^{1,+}_{\Delta,0}$, $g^{3,3}_{\Delta,0} \equiv g^{3,-}_{\Delta,0}$ and $g^{4,3}_{\Delta,0} \equiv g^{4,-}_{\Delta,0}$.

For spin $l>0$ and in the limit $(u,v)\to(0,1)$ the behaviour of $g^{i,a}_{\Delta,l}$ is as follows

$$\begin{aligned}
\mathcal{A}_1 &\supset \lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{1,1}_{s,\Delta,l} + \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{1,2}_{s,\Delta,l} \\
&\approx u^{\frac{\Delta}{2}} \sum_{k=0}^{\lfloor\frac{l-2}{2}\rfloor} \left(\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{1,k}_1 + \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{2,k}_1\right)\left(\frac{v-1}{2\sqrt{u}}\right)^{l-2k} + \cdots ,
\end{aligned} \tag{3.69}$$

$$\mathcal{A}_2 \supset \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{2,2}_{s,\Delta,l} \approx \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} u^{\frac{\Delta}{2}} \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} \mathcal{K}^k_2 \left(\frac{v-1}{2\sqrt{u}}\right)^{l-1-2k} + \cdots , \tag{3.70}$$

$$\begin{aligned}
\mathcal{A}_3 &\supset \lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{3,3}_{s,\Delta,l} + \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{3,4}_{s,\Delta,l} \\
&\approx u^{\frac{\Delta}{2}} \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} \lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{3,k}_3 \left(\frac{v-1}{2\sqrt{u}}\right)^{l-2k} \\
&\quad + u^{\frac{\Delta-1}{2}} \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{4,k}_3 \left(\frac{v-1}{2\sqrt{u}}\right)^{l-1-2k} + \cdots ,
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
\mathcal{A}_4 &\supset \lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{4,3}_{s,\Delta,l} + \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4} g^{4,4}_{s,\Delta,l} \\
&\approx u^{\frac{\Delta}{2}} \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} \lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{3,k}_4 \left(\frac{v-1}{2\sqrt{u}}\right)^{l-2k} \\
&\quad + u^{\frac{\Delta-1}{2}} \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{4,k}_4 \left(\frac{v-1}{2\sqrt{u}}\right)^{l-1-2k} + \cdots .
\end{aligned} \tag{3.72}$$

$\lambda^a_{\psi_1\psi_2\mathcal{O}_l}$ are the structure constants of the fermionic three-point correlator $\langle\Psi_1\Psi_2\mathcal{O}_l\rangle$ associated to the tensor structure $r_{di,a}$ as in (3.49) and $\mathcal{K}^{j,k}_a$ are constants. The value $k=0$ corresponds to the exchange of a primary or leading twist descendant whereas the remaining ones $k>0$ account for the exchange of descendants with higher value of twist.

For a scalar exchange ($l=0$) denoted by ϕ, there are the two independent tensor structures (3.50); hence there can only be two OPE coefficients, i.e. $\lambda^1_{\psi_1\psi_2\mathcal{O}_{3,0}} \equiv \lambda^+_{\psi_1\psi_2\phi}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_{3,0}} \equiv \lambda^-_{\psi_1\psi_2\phi}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_{3,0}} \equiv 0$ and $\lambda^4_{\psi_1\psi_2\mathcal{O}_{3,0}} \equiv 0$. Thus the limiting behaviour reads

$$\begin{aligned}
\mathcal{A}_1 &\supset \lambda^+_{\psi_1\psi_2\phi}\lambda_{\phi\phi_3\phi_4}\mathcal{K}^{1,0}_1 u^{\frac{\Delta}{2}} + \cdots , \\
\mathcal{A}_3 &\supset \lambda^-_{\psi_1\psi_2\phi}\lambda_{\phi\phi_3\phi_4}\mathcal{K}^{3,0}_3 u^{\frac{\Delta}{2}} + \cdots , \\
\mathcal{A}_4 &\supset \lambda^-_{\psi_1\psi_2\phi}\lambda_{\phi\phi_3\phi_4}\mathcal{K}^{3,0}_4 u^{\frac{\Delta}{2}} + \cdots .
\end{aligned} \tag{3.73}$$

Comparing (3.69) - (3.72) and (3.73) with (3.67), the pole structure of the Mellin amplitude can be obtained. The results are listed in table 3.2.

Therefore, each component of the Mellin amplitude needs to have the following pole

Component of M.A.	**Location of Poles**	**Residues ~**
$\mathcal{M}_1$	$\tau + 2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4},\ \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$
$\mathcal{M}_2$	$\tau + 1 + 2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$
$\mathcal{M}_3$	$\tau + 2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$
$\mathcal{M}_4$	$\tau + 2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4},\ \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$

Table 3.2: s-channel poles of the Mellin amplitude of the two scalar and two fermion conformal correlator for $l > 0$. $\tau = \Delta - l$ is the twist of the exchanged operator. For a scalar exchange one should set $\lambda^2_{\psi_1\psi_2\mathcal{O}_0} \equiv 0$ and $\lambda^4_{\psi_1\psi_2\mathcal{O}_0} \equiv 0$.

structure

$$\sum_k \sum_{m=0}^{\infty} \frac{\lambda^a_{\psi_1\psi_2\mathcal{O}_k}\lambda_{\mathcal{O}_k\phi_3\phi_4}\tilde{\mathcal{Q}}^{(s)}_{m,l_k}(t)}{s - \Delta_k + l_k + b(a) + 2m} \qquad m \in \mathbb{N} \tag{3.74}$$

and the superscript s refers to the direct channel. Note that $b(2) = 1$ and for the other three components $b(a) = 0$. The exchange of the primary and leading twist descendants in the conformal multiplet corresponds to $m = 0$ and the exchange of the descendants corresponds to $m > 0$. The polynomial $\tilde{\mathcal{Q}}^{(s)}_{m,l_k}$, which is necessary to extract the OPE coefficients from the residue of the pole, can be determined in a similar way as for the scalar case.

Poles in the Crossed Channel

In the crossed channel (t-channel) the OPE between a fermionic and a scalar operator is taken (13)-(24); hence the exchanged operator has to be fermionic. The correlator can be written in the following way.[16]

$$\begin{aligned}\langle \Psi_1\Psi_2\Phi_3\Phi_4\rangle &= \left(\frac{X_{34}}{X_{14}}\right)^{\frac{\Delta_{13}}{2}}\left(\frac{X_{14}}{X_{12}}\right)^{\frac{\Delta_{24}}{2}} \frac{\tilde{v}^{\frac{\Delta_{13}}{2}}}{X_{13}^{\frac{\Delta_1+\Delta_3}{2}} X_{24}^{\frac{\Delta_2+\Delta_4}{2}}} \sum_{i=1}^{4} t_i \tilde{\mathcal{A}}_i(\tilde{u},\tilde{v}),\\ \tilde{u} &= \frac{X_{13}X_{24}}{X_{12}X_{34}}, \qquad \tilde{v} = \frac{X_{14}X_{23}}{X_{12}X_{34}}, \qquad \Delta_{ij} = \Delta_i - \Delta_j,\\ \tilde{\mathcal{A}}_i(\tilde{u},\tilde{v}) &= \int \frac{dt}{4\pi i}\int \frac{ds}{4\pi i} \bar{M}_i(s,t)\, \tilde{u}^{\frac{t+\frac{1}{2}}{2}}\, \tilde{v}^{\frac{s+t+\frac{1}{2}-\Delta_1-\Delta_4}{2}}.\end{aligned} \tag{3.75}$$

To obtain the pole structure of each component of the Mellin amplitude given in (3.75), the expression shall be compared with the leading behaviour of the conformal block in the limit $\tilde{u}, \tilde{v} \to (0, 1)$. The authors of [198, 202] have computed this conformal block in three dimensions and given its leading behaviour in the variables r and η which were first introduced in [210].

$$\tilde{u} = \frac{16r^2}{(1 + r^2 - 2r\eta)^2}, \qquad \tilde{v} = \frac{(1 + r^2 + 2r\eta)^2}{(1 + r^2 - 2r\eta)^2}.$$

[16] Note that (3.75) coincides with equation 2.12 of [198] with the following relabelling of the indices: (3.75) $\to$ [198]: $1 \to 1, 2 \to 3, 3 \to 4, 4 \to 2$ and renaming $g^I \to \mathcal{A}_i$.

Asymptotically for $r \to 0$ these coordinates are given by $\tilde{u} \approx r^2$ and $\eta \approx -\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}$ with η held constant.

As for the direct channel, the components of the fermionic conformal block $g^{i,jk}_{\Delta,l}$ are labelled by the associated tensor structures of the four-point correlator i as well as by j, $k = \pm$ which state into which three-point tensor structures these decompose in the OPE. Parity conservation demands that only $g^{1,++}_{\Delta,l}$, $g^{1,--}_{\Delta,l}$, $g^{2,++}_{\Delta,l}$, $g^{2,--}_{\Delta,l}$, $g^{3,+-}_{\Delta,l}$,$g^{3,-+}_{\Delta,l}$, $g^{4,+-}_{\Delta,l}$ and $g^{4,-+}_{\Delta,l}$ are non-zero. According to [198] the leading behaviour of the components in the limit $r \to 0$ is

$$
\begin{aligned}
g^{(1,++)}_{\Delta,l}(r,\eta) &= -r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(1,--)}_{\Delta,l}(r,\eta) &= -r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(2,++)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(2,--)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(3,+-)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(3,-+)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(4,+-)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),\\
g^{(4,-+)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right),
\end{aligned}
$$

with $P^{(\alpha,\beta)}_n(z)$ being the Jacobi polynomials. Note that they have the following symmetry property

$$
P^{(\alpha,\beta)}_n(-z) = (-1)^n P^{(\beta,\alpha)}_n. \tag{3.76}
$$

which implies that $P^{(\alpha,\beta)}_n(z) + P^{(\beta,\alpha)}_n(z)$ has only even powers of z for even n and only odd powers of z for odd n. On the other hand the difference $P^{(\alpha,\beta)}_n(z) - P^{(\beta,\alpha)}_n(z)$ has only odd powers of z for even n and even powers of z for odd n. Using this insight the series expansion of these polynomials leads to the following asymptotic behaviour (for $l > \frac{1}{2}$):

$$
\begin{aligned}
g^{(1,++)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx -\tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\lfloor \frac{l}{2} \rfloor} H^{+(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k} + \cdots,\\
g^{(1,--)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx -\tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\lceil \frac{l}{2} \rceil -1} H^{-(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k-1} + \cdots,\\
g^{(2,++)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\lceil \frac{l}{2} \rceil -1} H^{-(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k-1} + \cdots,\\
g^{(2,--)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\lfloor \frac{l}{2} \rfloor} H^{+(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k} + \cdots,
\end{aligned} \tag{3.77}
$$

Component of M.A.	**Location of Poles**	**Residues $\sim$**
$\mathcal{M}_1$	$t=\tau+2k$	$\lambda^{+}_{\psi_1\phi_3\psi_l}\lambda^{+}_{\psi_l\phi_4\psi_2}$
	$t=\tau+1+2k$	$\lambda^{-}_{\psi_1\phi_3\psi_l}\lambda^{-}_{\psi_l\phi_4\psi_2}$
$\mathcal{M}_2$	$t=\tau+1+2k$	$\lambda^{+}_{\psi_1\phi_3\psi_l}\lambda^{+}_{\psi_l\phi_4\psi_2}$
	$t=\tau+2k$	$\lambda^{-}_{\psi_1\phi_3\psi_l}\lambda^{-}_{\psi_l\phi_4\psi_2}$
$\mathcal{M}_3$	$t=\tau+1+2k$	$\lambda^{+}_{\psi_1\phi_3\psi_l}\lambda^{-}_{\psi_l\phi_4\psi_2}$
	$t=\tau+2k$	$\lambda^{-}_{\psi_1\phi_3\psi_l}\lambda^{+}_{\psi_l\phi_4\psi_2}$
$\mathcal{M}_4$	$t=\tau+2k$	$\lambda^{+}_{\psi_1\phi_3\psi_l}\lambda^{-}_{\psi_l\phi_4\psi_2}$
	$t=\tau+1+2k$	$\lambda^{-}_{\psi_1\phi_3\psi_l}\lambda^{+}_{\psi_l\phi_4\psi_2}$

Table 3.3: t-channel poles of the Mellin amplitude from the two fermion and two scalar correlator. $\tau=\Delta-l$ is the twist of the exchanged operator.

$$\begin{aligned}
g^{(3,+-)}_{\Delta,l}(\tilde u,\tilde v) &\approx \tilde u^{\frac{\Delta}{2}}\sum_{k=0}^{\lceil\frac{l}{2}\rceil-1} H^{-(0,1)}_{l,k}\left(\frac{1-\tilde v}{2\sqrt{\tilde u}}\right)^{l-2k-1}+\cdots\,,\\
g^{(3,-+)}_{\Delta,l}(\tilde u,\tilde v) &\approx \tilde u^{\frac{\Delta}{2}}\sum_{k=0}^{\lfloor\frac{l}{2}\rfloor} H^{+(0,1)}_{l,k}\left(\frac{1-\tilde v}{2\sqrt{\tilde u}}\right)^{l-2k}+\cdots\,,\\
g^{(4,+-)}_{\Delta,l}(\tilde u,\tilde v) &\approx \tilde u^{\frac{\Delta}{2}}\sum_{k=0}^{\lfloor\frac{l}{2}\rfloor} H^{+(0,1)}_{l,k}\left(\frac{1-\tilde v}{2\sqrt{\tilde u}}\right)^{l-2k}+\cdots\,,\\
g^{(4,-+)}_{\Delta,l}(\tilde u,\tilde v) &\approx \tilde u^{\frac{\Delta}{2}}\sum_{k=0}^{\lceil\frac{l}{2}\rceil-1} H^{-(0,1)}_{l,k}\left(\frac{1-\tilde v}{2\sqrt{\tilde u}}\right)^{l-2k-1}+\cdots
\end{aligned}\tag{3.78}$$

with $H^{\pm(\alpha,\beta)}_{n,k}$ being the coefficients of the series expansions of $P^{(\alpha,\beta)}_n(z)\pm P^{(\beta,\alpha)}_n(z)$. For $l=\frac{1}{2}$ the expansion reads

$$\begin{aligned}
g^{(1,++)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx -2\tilde u^{\frac{\Delta}{2}}+\cdots\,, & g^{(1,--)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx -2\tilde u^{\frac{\Delta+1}{2}}+\cdots\,,\\
g^{(2,++)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx 2\tilde u^{\frac{\Delta+1}{2}}+\cdots\,, & g^{(2,--)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx 2\tilde u^{\frac{\Delta}{2}}+\cdots\,,\\
g^{(3,+-)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx 2\tilde u^{\frac{\Delta+1}{2}}+\cdots\,, & g^{(3,-+)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx 2\tilde u^{\frac{\Delta}{2}}+\cdots\,,\\
g^{(4,+-)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx 2\tilde u^{\frac{\Delta}{2}}+\cdots\,, & g^{(4,-+)}_{\Delta,\frac{1}{2}}(\tilde u,\tilde v) &\approx 2\tilde u^{\frac{\Delta+1}{2}}+\cdots\,.
\end{aligned}\tag{3.79}$$

Comparing (3.77), (3.78) and (3.79) with (3.75) gives rise to the poles listed in table 3.3. The OPE coefficient associated to the tensor structure $r^{\pm}_{cr}$ from (3.50) for the three-point correlator $\langle\Psi_1\Phi_2\Psi_{3,l}\rangle$ is written in the following way $\lambda^{\pm}_{\psi\phi\psi_l}$.

Actually, it is interesting that in this case each component of the Mellin amplitude has two series of poles. This is significantly different from the scalar pole structure, where simply one series of poles appears. However, in a theory of definite parity merely one series shall appear, because the four-point correlator can only factorize onto one pair of OPE coefficients. This is verified in the perturbative calculations in the next section. The Mellin amplitudes of the u-channel are not independent and their pole structure is stated in the appendix B.4.2.

Pole Structure: Four Fermion Correlator

The fermionic conformal block for the four-point correlator $\langle \Psi_1 \Psi_2 \Psi_3 \Psi_4 \rangle$ can be obtained from the scalar conformal block by the method explained for the direct channel. The formula (3.68) can be used again, however, for $\langle \Psi_1 \Psi_2 \Psi_3 \Psi_4 \rangle$ two differential operators have to be applied, because both three-point functions onto which the four-point correlator factorizes, include a pair of fermionic operators [199]. This method works for the s- and t-channel, since only symmetric traceless operators are exchanged in both channels.

There is a further subtlety. A generic basis of tensor structures might lead to spurious singularities in the conformal block; hence a basis of these is chosen such that each component of the conformal block can be expanded in the limit $(u, v) \to (0, 1)$ in the following way

$$\sum_{i=1}^{I} u^{\frac{\tau - a_i}{2}} \sum_{k=0}^{\infty} u^k \tilde{g}_k(v).$$

$a_i < \tau$ and $I > 0$ are integer valued and $\tilde{g}_k$ has a power series expansion in $1 - v$. Such a choice ensures that each component of the Mellin amplitude has finitely many series of poles and the residue of them yields the OPE coefficients times a polynomial. Again, if the polynomial is known the OPE coefficient can be determined from the Mellin amplitude. As for the scalar case the degree of the polynomial is determined by the spin l of the exchanged operator. The basis given in (3.52), (3.53) is a good choice since it does not yield spurious poles.

The result for the s-channel poles is given in table 3.4. For a scalar exchange $l = 0$ the number of independent tensor structures is reduced. Therefore all OPE coefficients apart from $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}$, $\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$ and $\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$ should be set to zero. The t-channel and u-channel poles are listed in the appendix B.4.3.

Component of M.A.	**Location of Poles**	**Residues $\sim$**
$\mathcal{M}_1$	$s=\tau+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_2$	$s=\tau+1+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_7$, $\mathcal{M}_8$	$s=\tau-1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_4$	$s=\tau+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_9$, $\mathcal{M}_{10}$	$s=\tau+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_{11}$, $\mathcal{M}_{12}$	$s=\tau+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_{13}$, $\mathcal{M}_{14}$	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+2+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_{15}$, $\mathcal{M}_{16}$	$s=\tau+1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+2+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$

Table 3.4: s-channel poles for the fermionic four-point correlator corresponding to an integer spin l exchanged primary $\mathcal{O}_l$ of twist $\tau=\Delta-l$.

3.4 Perturbative Calculations

To illustrative the pole structure of the Mellin amplitudes described in section 3.3.3, perturbative tree-level calculations at strong and weak coupling are performed in this section. Although the computations are done for arbitrary dimensions, to analyze the pole structure of the Mellin amplitude the discussion is restricted to three dimensions because in subsection 3.3.3 the pole structure has been deduced in three dimensions only. However, a generalization to arbitrary dimensions is straightforward. All tree-level calculations are done in Euclidean signature and the final result is Wick rotated to Minkowskian signature. As usual the algebraic term of the perturbative expression is also presented in forms of diagrams. In the depicted diagrams solid lines with an arrow shall denote fermionic propagators whereas solid lines without an arrow shall represent scalar propagators. In the following the scaling dimension of the i^{th} external operator is written as Δ_i and Δ denotes the scaling dimension of the exchanged operator.

At first the calculations at strong coupling are done using **Witten diagrams**. The theoretical principle underlying these computations is the **AdS/CFT correspondence** which states that there is a duality between string theories in $d+1$ dimensions on anti de Sitter space (AdS) and conformal field theories defined on its d-dimensional boundary (∂AdS). Therefore a short review about the correspondence shall be given in section 3.4.1 to outline how the calculations at strong coupling, done in section 3.4.2, can be performed. Afterwards in section 3.4.3 Mellin amplitudes in the weak coupling regime are determined starting from standard position space Feynman rules.

3.4.1 Basics of AdS/CFT

The paradigm of AdS/CFT [30–32] states there is a one-to-one correspondence between string theory (quantum gravity) with asymptotically AdS boundary conditions and a CFT defined on its boundary. To explain this correspondence it is useful to introduce $d+1$-dimensional Euclidean AdS spacetime first. It can be defined by an embedding into $\mathbb{R}^{d+2}$

$$-x_0^2 + x_1^2 + \ldots + x_{d+1}^2 = -L^2$$

with the AdS radius L. Originally, it has been proposed [30] that $\mathcal{N}=4$ super Yang-Mills (SYM) theory with gauge group SU(N) in four dimensions is dual, i.e. produces the same observables, to string theory defined on the geometry $\text{AdS}_5 \times S^5$. However, in the pointlike limit of the string the massive states decouple and string theory can be approximated by (super)-gravity. If the gravity theory is weakly coupled, quantum corrections do not have to be included; hence, it can be described by classical gravity. For the first approximation to hold it is necessary to study a strongly coupled system on the boundary and the second statement demands that the system has a large number of degrees of freedom which is equal to the fact that the central charge of the CFT is large [211].[17]

[17] In the original proposal the coupling constants on $\mathcal{N}=4$ SYM theory and string theory are related by

$$\frac{l_s^2}{L^2} = \frac{1}{\sqrt{\lambda}} \quad \text{and} \quad \frac{l_p^8}{L^8} = \frac{\pi^4}{2N^2},$$

where l_p is the Planck length, l_s the string length and L the radius of AdS space. The 't Hooft coupling $\lambda = N g_{\text{YM}}^2$ is a product of the Yang-Mills coupling constant g_{YM} and the degree of the gauge group SU(N).

For the purpose of this thesis a scalar and a fermionic field (which source the corresponding operators in the CFT) in AdS shall be studied. Consider a scalar field ϕ in AdS which obeys the AdS-Klein-Gordon equation. The duality states that an operator $\mathcal{O}$ of the CFT is sourced by an appropriately defined boundary value ϕ_0 of the dual field ϕ which lives in AdS. In terms of the partition function [31, 32] the AdS/CFT correspondence reads

$$Z_{\text{QFT}}\left[\phi_0\right] := \left\langle \exp\left(\int d^d x \phi_0 \mathcal{O}\right)\right\rangle_{\text{QFT}} = Z_{\text{gravity}}\left[\phi_0\right], \tag{3.80}$$

where $Z_{\text{gravity}}\left[\phi_0\right]$ is the gravity path integral over all fields ϕ which allow ϕ_0 as at the AdS boundary value [32, 211]

$$Z_{\text{gravity}}\left[\phi_0\right] = \int_{\phi=\phi_0} \mathcal{D}\left[\phi\right] e^{S_{\text{gravity}}[\phi]}.$$

The bulk field approaches the boundary field like

$$\lim_{z_0\to 0} z_0^{\Delta-d}\phi(z) = \frac{1}{2\Delta - d}\phi_0(\mathbf{z}) \quad \text{and} \quad z = (z_0, \mathbf{z}), \tag{3.81}$$

where the normalization $z_0^{\Delta-d}$ is necessary to obtain a convergent boundary field [33].[18] Hence, the CFT operator $\mathcal{O}$ is sourced by the boundary value of the bulk field. If the operator $\mathcal{O}$ has scaling dimension Δ the source ϕ_0 has to have scaling dimension $d - \Delta$ such that the integral $\int d^d x \phi_0 \mathcal{O}$ is invariant under conformal transformations. In the regime where the classical solution of gravity is appropriate one can replace the r.h.s. of (3.80) by its on-shell action such that the correspondence simplifies to

$$Z_{\text{QFT}}\left[\phi_0\right] = \exp\left(S^{\text{on-shell}}_{\text{gravity}}\left[\phi\right]\Big|_{\phi=\phi_0}\right) \quad \text{with} \quad Z_{\text{gravity}}\left[\phi_0\right] = \exp\left(S^{\text{on-shell}}_{\text{gravity}}\left[\phi\right]\Big|_{\phi=\phi_0}\right).$$

Therefore, the connected n-point conformal correlation function can be computed by the generating functional $\log Z_{\text{gravity}}\left[\phi_0\right] = S^{\text{on-shell}}_{\text{gravity}}\left[\phi\right]\Big|_{\phi=\phi_0}$ of the gravity theory

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \left.\frac{\delta^{(n)} S^{\text{on-shell}}_{\text{gravity}}}{\delta\phi_0(x_1)\dots\delta\phi_0(x_n)}\right|_{\phi_0=0}. \tag{3.82}$$

The action $S^{\text{on-shell}}_{\text{gravity}}$ should be understood as the renormalized action with counterterms included. Naively, the on-shell action diverges because on the one hand the volume of AdS is infinite and on the other hand the induced metric on the boundary ∂AdS receives a double pole (as can be seen in (3.83) for $z_0 \to 0$). One way to regulate the gravity action is by restricting the bulk integral to the domain $z_0 > \epsilon > 0$ and evaluating the boundary integral at $z_0 = \epsilon$. After adding the counterterms the limit $\epsilon \to 0$ can be taken safely [212, 213].

Further, the Ricci scalar $R = -\frac{d(d+1)}{2L^2}$ (i.e. the curvature) is small if the radius L of AdS is large. To obtain a reliable gravity description the regime

$$N \gg 1 \quad \text{(planar limit)} \quad \text{and} \quad \lambda \gg 1 \quad \text{(at strong coupling)}$$

has to be studied [211].

[18] Notice that the additional factor $2\Delta - d$ is due to the normalization of the bulk-to-boundary propagator. In this section, boundary coordinates shall be denoted by $\mathbf{z}$ whereas bulk coordinates are given by $z = (z_0, \mathbf{z})$.

For the following discussion the Poincaré patch for (Euclidean) AdS is used:

$$ds^2 = \frac{L^2}{z_0^2}\left(dz_0^2 + d\mathbf{z}^2\right) = \frac{1}{z_0^2}dz^\mu dz_\mu. \tag{3.83}$$

In the following length scales are measured in terms of the AdS radius L which implies $L = 1$. It can be seen that the metric diverges at $z_0 \to 0$ and does not induce a metric at the boundary. However, it induces a metric up to conformal transformations which enables one to remove the double pole. (For instance the function $f(z) = z_0^2$ removes the double pole.) Since any conformal transformation which removes the pole is equally fine the metric can be constructed up to conformal transformations only [213].

A solution for the bulk field ϕ in terms of the boundary field ϕ_0 can be established with a Green's function $K_\Delta(z, \mathbf{x})$ such that $K_\Delta(z, \mathbf{x})$ solves the AdS Klein-Gordon equation $\nabla^2\phi = M^2\phi$ and gives rise to the correct boundary value $\mathbf{x}$. The solution reads

$$\phi(z_0, \mathbf{z}) = \int d^d y K_\Delta(z, \mathbf{y})\phi_0(\mathbf{y})$$

with the bulk-to-boundary propagator

$$K_\Delta(z, \mathbf{x}) = C_\Delta \frac{z_0^\Delta}{\left(z_0^2 + (\mathbf{x} - \mathbf{z})^2\right)^\Delta} \quad \text{with} \quad C_\Delta = \frac{\Gamma(\Delta)}{2\pi^h \Gamma(\Delta - h + 1)} \tag{3.84}$$

and $h = \frac{d}{2}$. K_Δ is constructed such that its boundary value is a δ-distribution with the normalization

$$\lim_{z_0 \to 0} z_0^{d-\Delta} K_\Delta(z, \mathbf{x}) = \delta^d(\mathbf{z} - \mathbf{x}).$$

In embedding space coordinates the propagator is simply given by

$$K_\Delta(Z, X) = \frac{C_\Delta}{(-2X \cdot Z)^\Delta} = \frac{C_\Delta}{\Gamma(\Delta)} \int_0^\infty \frac{dt}{t} t^\Delta e^{2tZ\cdot X}, \tag{3.85}$$

where $Z \in \mathrm{AdS}_{d+1}$ is a point in the bulk and $X \in \partial\mathrm{AdS}_d$ is a point at the boundary. The propagation between two bulk points is established by the usual AdS propagator which reads, in the harmonic space representation, as follows

$$\begin{aligned} G_\Delta(Z_1, Z_2) &= \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{\Omega_c(Z_1, Z_2)}{(\Delta - h)^2 - c^2} \\ \Omega_c(Z_1, Z_2) &= \frac{(c)_h (-c)_h}{2\pi^d} \int_{\partial\mathrm{AdS}} K_{h+c}(Z_1, P) K_{h-c}(Z_2, P) \end{aligned} \tag{3.86}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$ is the Pochhammer symbol [106]. Hence, the scalar bulk-to-bulk propagator (3.86) can be expressed as a convolution of two scalar bulk-to-boundary propagators (3.85) [32, 33, 211].

Similarly the propagators for a spinor field can be obtained [214–216]. For concreteness, assume that the mass $m \geq 0$ of the fermionic bulk field $\psi(z) = \psi^+(z) + \psi^-(z)$ is non-negative. $\psi^+(z)$ and $\psi^-(z)$ are eigenfunctions of Γ^0: $\Gamma^0\psi^\pm(z) = \pm\psi^\pm(z)$. If the classical bulk field $\psi(z)$ approaches the boundary, the field scales as

$$\psi^-(z) = z_0^{\frac{d}{2}-m}\psi_0^-(\mathbf{z}) + O\left(z_0^{\frac{d}{2}-m+1}\right) \quad \text{and} \quad \psi^+(z) = z_0^{\frac{d}{2}+m}\psi_0^+(\mathbf{z}) + O\left(z_0^{\frac{d}{2}+m+1}\right),$$

which is similar to the scalar field (3.81). Hence, for $m > 0$ and close to the boundary ψ_0^- dominates. Further, the objects ψ_0^- and ψ_0^+ and $\bar{\psi}_0^+$ and $\bar{\psi}_0^-$ are related, this can be proven by demanding regular solutions deep in the bulk $z_0 \to \infty$. This analysis shows that the boundary data is encoded in ψ_0^- and $\bar{\psi}_0^+$ only. Hence, in even-dimensional spacetime (odd-dimensional boundary) the boundary term of the bulk spinor is a Dirac spinor of the boundary and in odd-dimensional spacetime the boundary term is a Weyl spinor of the boundary.[19]

In addition, it is necessary to demand that the action is stationary at the classical path to obtain the correct classical theory from the quantum theory. This requires to add a boundary term to the Dirac action [216]. For the Klein-Gordon action no boundary term is required if Dirichlet boundary conditions $\delta\phi = 0$ are used, because the action does already vanish at the classical path [217]. Hence, the AdS action with a Yukawa interaction can be written as

$$\begin{aligned} S[\psi, \bar{\psi}, \phi] = &\int_M d^{d+1}z \, \sqrt{g} \left[\bar{\psi} \left(\not{D} - m \right) \psi + \frac{1}{2} \left((\nabla_\mu \phi)^2 + M^2 \phi^2 \right) + \lambda \phi \bar{\psi} \psi \right] \\ &+ \int_{\partial M_\epsilon} d^d x \, \sqrt{h_\epsilon} \, \bar{\psi} \psi, \end{aligned} \tag{3.87}$$

$h_{\epsilon;ij}$ is the induced metric on the surface ∂M_ϵ. ∂M_ϵ is the regularized boundary of the AdS space M, which approaches the boundary for $z_0 = \epsilon \to 0$ [214–216, 218].

In general it is hard to find an exact solution to (3.87), but a perturbative solution can be constructed by the following recursion relations of the fields

$$\begin{aligned} \phi(z) &= \phi_\epsilon^{(0)}(z) - \lambda \int d^{d+1}w \sqrt{g(w)} G_\epsilon(z, w) \bar{\psi}(w) \psi(w), \\ \psi(z) &= \psi_\epsilon^{(0)}(z) - \lambda \int d^{d+1}w \sqrt{g(w)} S_\epsilon(z, w) \phi(w) \psi(w), \\ \bar{\psi}(z) &= \bar{\psi}_\epsilon^{(0)}(z) - \lambda \int d^{d+1}w \sqrt{g(w)} \bar{\psi}(w) \phi(w) S_\epsilon(z, w). \end{aligned} \tag{3.88}$$

Here $\phi_\epsilon^{(0)}$, $\psi_\epsilon^{(0)}$ and $\bar{\psi}_\epsilon^{(0)}$ denote the regularized solutions to the e.o.m. in free theory. Further, $G_\epsilon(z, w)$ and $S_\epsilon(z, w)$ are the regularized scalar and spinorial bulk-to-bulk propagators [215, 219]. Eventually one takes the limit $\epsilon \to 0$ such that the regularized free theory solutions can be expressed in terms of the boundary values:[20]

$$\begin{aligned} \phi^{(0)} &= \lim_{\epsilon \to 0} \phi_\epsilon^{(0)}(z) = \int d^d x \, K_\Delta(z, \mathbf{x}) \phi_0(\vec{x}), \\ \psi^{(0)} &= \lim_{\epsilon \to 0} \psi_\epsilon^{0}(z) = \int d^d x \, \Sigma_\Delta \left(z, \mathbf{x}\right) \psi_0^- \left(\mathbf{x}\right) \\ \bar{\psi}^{(0)} &= \lim_{\epsilon \to 0} \bar{\psi}_\epsilon^{0}(z) = \int d^d x \, \bar{\psi}_0^+ \left(\mathbf{x}\right) \bar{\Sigma}_\Delta \left(z, \mathbf{x}\right) \end{aligned} \tag{3.89}$$

where

$$\begin{aligned} \Sigma_\Delta \left(z, \mathbf{x}\right) &= \frac{\Gamma_\mu \left(z^\mu - x^\mu\right)}{\sqrt{z_0}} K_{\Delta + \frac{1}{2}} \left(z, \mathbf{x}\right) \mathcal{P}^-, \\ \bar{\Sigma}_\Delta \left(z, \mathbf{x}\right) &= \mathcal{P}^+ \frac{\Gamma_\mu \left(z^\mu - x^\mu\right)}{\sqrt{z_0}} K_{\Delta + \frac{1}{2}} \left(z, \mathbf{x}\right) \end{aligned}$$

[19] This follows from the fact that in even dimensions the Dirac spinor transforms in a reducible representation while the Weyl spinor transforms in an irreducible representation of the algebra of the spin group. However, in odd dimensions the Dirac spinor transforms in an irreducible representation too. Further, in a d-dimensional spacetime the spinor representation is realized in a $2^{\lfloor d/2 \rfloor}$-dimensional vector space.

[20] The conformal dimension of the scalar field satisfies $\Delta(\Delta - d) = M^2$ [32] and for the spinor fields $\Delta = m + \frac{d}{2}$ [214, 215].

are the fermionic bulk-to-boundary propagators, respectively [218]. Γ^μ are the Dirac-matrices of the bulk theory and $\mathcal{P}^\pm = (\mathbb{1} \pm \Gamma^0)/2$ are projection operators. It is easy to verify that the spinor product of the two fermionic bulk-to-boundary propagators $\bar{\Sigma}_{\Delta_1}$ and Σ_{Δ_2} can be written as a product of two scalar bulk-to-boundary propagators $K_{\Delta_1+\frac{1}{2}}$ and $K_{\Delta_2+\frac{1}{2}}$ with an additional tensor structure:

$$\bar{\Sigma}_{\Delta_1}(z,\mathbf{x}_1)\Sigma_{\Delta_2}(z,\mathbf{x}_2) = \left(\mathbf{x}_{12}^\mu \Gamma_\mu \mathcal{P}^-\right) K_{\Delta_1+\frac{1}{2}}(z,\mathbf{x}_1) K_{\Delta_2+\frac{1}{2}}(z,\mathbf{x}_2). \tag{3.90}$$

If the tensor structure $\mathbf{x}_{12}^\mu \Gamma_\mu \mathcal{P}^-$ is contracted with polarization vectors of the boundary S_i it is equivalent to $\mathbf{x}_{12}^a \gamma_a \equiv \not{x}_{12}$ where γ_a are the Dirac-matrices of the boundary theory.

Plugging back the recursive definition of the fields (3.88) into the action (3.87), it can be written as an expansion in the boundary fields ϕ_0, ψ_0^- and $\bar{\psi}_0^+$.[21] Taking the functional derivative w.r.t. the boundary fields according to (3.82) shall give the correlator in the planar limit in the boundary CFT. This method can be depicted diagrammatically by tree-level Witten diagrams [32, 220].

3.4.2 Strong Coupling - Witten Diagrams

The position space representation of Witten diagrams is quite cumbersome, but in the Mellin-Barnes representation, they are simplified greatly and the relevant (physical) data is concisely encoded in the Mellin amplitude [103, 106, 221, 222]. In addition the so-obtained Mellin amplitude can be used to compute scattering amplitudes in QFTs of one dimension higher by the flat-space limit [106, 107, 113].

In this thesis tree-level Witten diagrams with external fermions and scalars are evaluated to exemplify the properties of the Mellin amplitudes studied in section 3.3.3. The strategy to evaluate these diagrams is the following:

1. The position space expression of the fermionic Witten diagram is reduced to a scalar Witten diagram attached with the tensor structures defined in section 3.2.2 [218, 223].
2. For these scalar Witten diagrams the Mellin-Barnes representation is known [103, 106]. Hence, the Mellin amplitude for the fermionic Witten diagram can be constructed from these results.

Contact Witten Diagram

The position space representation of a tree-level Witten diagram involving two fermions and two scalars is merely given by the scalar Witten diagram

$$B^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} = \langle S_1 S_2 \rangle \int_{AdS} dZ \prod_{i=1}^{2} K_{\Delta_i+\frac{1}{2}}(Z,X_i) \prod_{i=3}^{4} K_{\Delta_i}(Z,X_i), \tag{3.91}$$

which can be checked using the spinor product relation (3.90). The result is written in embedding space coordinates. The measure dZ integrates over AdS-space and X_i are points

[21] Since the bulk integrals might diverge, the limit $\epsilon \to 0$ has to be taken after integration. However, in most cases a careful treatment of the cutoff ϵ is needed for the two-point correlator only [220].

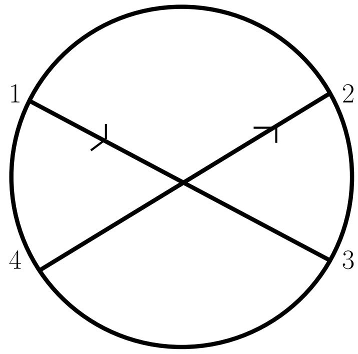


Figure 3.7: Four point contact Witten diagram with two fermions and two scalars.

on the conformal boundary of AdS. In figure 3.7 the diagram is drawn. The integral in (3.91) is a contact Witten diagram of four scalars expressed in terms of the scalar bulk-to-boundary propagator (3.85). This can be expressed in Mellin space [106] to obtain

$$\begin{aligned} B^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} &= \langle S_1 S_2\rangle \prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}} \Gamma\left(s_{il}\right) \mathbb{M}_{2,2} \prod_{i=1}^{4} \hat{\delta}\left(\Delta_i + \frac{1}{2}(\delta_{i1}+\delta_{i2}) - \sum_{j\neq i} s_{ij}\right) \\ &= \frac{\langle S_1 S_2\rangle}{\sqrt{X_{12}}} \prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}} \Gamma(s_{il}+\delta_{1i}\delta_{2l}) \mathbb{M}_{2,2} \prod_{i=1}^{4} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right), \end{aligned}$$

where the normalization for the delta-distribution reads $\hat{\delta}(x) = 2\pi i\; \delta(x)$ and the integration measure is given by $(ds_{il}) = \frac{ds_{il}}{2\pi i}$. It is trivial to generalize this result to $2n$ boundary fermions and m boundary scalars

$$\mathcal{M}_1 = \mathbb{M}_{2n,m} = \pi^h \Gamma\left(\frac{1}{2}\sum_{i=1}^{2n+m} \Delta_i + \frac{n}{2} - h\right) \prod_{i=1}^{2n} \left[\frac{C_{\Delta_i+\frac{1}{2}}}{\Gamma(\Delta_i+\frac{1}{2})}\right] \prod_{i=2n+1}^{2n+m} \left[\frac{C_{\Delta_i}}{\Gamma(\Delta_i)}\right] \tag{3.92}$$

with $h = \frac{d}{2}$ and C_Δ is given in (3.84). The only non-zero component of the Mellin amplitude is $\mathcal{M}_1 = \mathbb{M}_{2,2}$ which is a constant as a function of the Mellin variables.

Fermion-Scalar Four-point Witten Diagram: Scalar and Fermionic Exchange

The scalar exchange Witten diagram with two fermions and two scalars, sketched in figure 3.8, is in position space given by

$$\int_{AdS} dZ_1 \int_{AdS} dZ_2\; \bar{\Sigma}_{\Delta_1}(Z_1,X_1)\Sigma_{\Delta_2}(Z_1,X_2) G_\Delta(Z_1,Z_2) \prod_{i=3}^{4} K_{\Delta_i}(Z_2,X_i).$$

$G_\Delta(z_1,z_2)$ is the scalar bulk-to-bulk propagator (3.86). Using the spinor product reduction (3.90) the scalar exchange Witten diagram is

$$A^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} = \langle S_1 S_2\rangle \int_{AdS} dZ_1 \int_{AdS} dZ_2 \prod_{i=1}^{2} K_{\Delta_i+\frac{1}{2}}(Z_1,X_i) G_\Delta(Z_1,Z_2) \prod_{i=3}^{4} K_{\Delta_i}(Z_2,X_i).$$

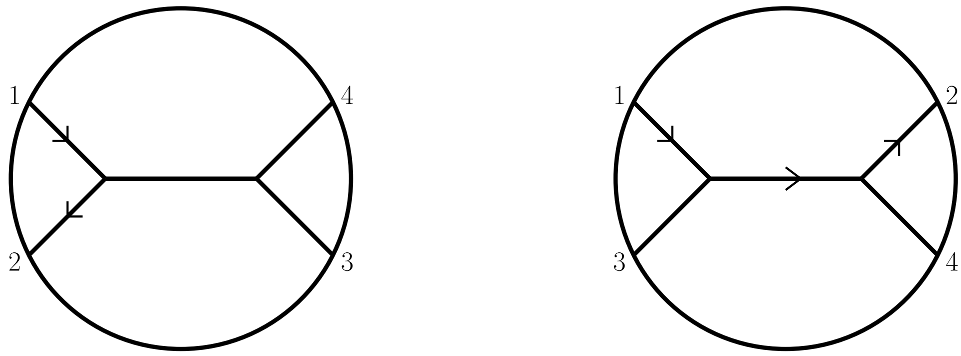


Figure 3.8: Fermionic-scalar exchange Witten diagrams with scalar and spinor exchange.

The above integral has been evaluated in [106].[22] The final result reads

$$A^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} = \frac{\langle S_1 S_2\rangle}{\sqrt{X_{12}}} \prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}} \Gamma\left(s_{il}+\delta_{1i}\delta_{2l}\right) \mathbb{N}^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4}(s_{il}) \prod_{i=1}^{4} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right).$$

Thus, as a function of the Mandelstam variable $s = \tau_1 + \tau_2 - 2s_{12}$ the non-zero component of the Mellin amplitude is given by

$$\begin{aligned}
\mathcal{M}_1 =& \mathbb{N}^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4}(s_{il}) = \frac{\mathbb{M}_{2,2}}{\Gamma\left(\frac{\sum_i \Delta_i}{2} + \frac{1}{2} - h\right)\Gamma\left(\frac{\Delta_1+\Delta_2+1-s}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-s}{2}\right)} \\
&\times \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{l(c)l(-c)}{(\Delta-h)^2 - c^2}, \\
l(c) =& \frac{\Gamma\left(\frac{h+c-s}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-h+c}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-h+c}{2}\right)}{2\Gamma(c)},
\end{aligned} \tag{3.93}$$

where Δ is the conformal dimension of the exchanged operator.

The physical poles are located at $s = \Delta + 2m$ with $m \in \mathbb{N}$ and they can only occur in (3.93) if the contour is pinched between two colliding poles of the integrand as explained in [99, 105]. On the other hand there are also several spurious poles which are generated by the integrand, but these are cancelled by the zeroes of the Γ-function in the prefactor. Hence, the location of the poles of the component $\mathcal{M}_1$ are exactly given as predicted in section 3.3.3. Further, the Mellin amplitude can be expressed as a series over these poles as has been shown in [103, 106].

The calculation of the four-point spinor exchange diagram 3.8 is quite tedious.[23] Therefore it is presented in the appendix B.5. However, the idea how to compute it is simple. To find the Mellin-Barnes representation for this diagram the same strategy is used as for the previous computations. The first step is to write the position space expression of the spinor exchange diagram as a scalar exchange diagram with some prefactor. This has been done in [218]. The Mellin-Barnes representation is known [106] for this integral and the result can

[22] The notation in this thesis differs from [106]. In [106] Mellin variables are denoted by δ_{ij} and Mandelstam variables as $s_{i_1\cdots i_k}$.

[23] Note the non-standard labelling of the external legs.

be written as

$$
\begin{aligned}
A^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4} =& \frac{\langle S_1 S_2\rangle}{\sqrt{X_{12}}} \prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}} \Gamma\left(s_{il}+\delta_{1i}\delta_{2l}\right) \\
&\times \left(\Delta_1+\Delta_3+\Delta+1-d-2s_{13}\right) \mathbb{N}^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4}\left(s_{il}\right) \prod_{i=1}^{4} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right) \\
&+ 2\frac{\langle S_1 X_3 X_4 S_2\rangle}{\sqrt{X_{13}X_{34}X_{42}}} \prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}} \bar{\mathbb{N}}^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4}\left(s_{il}\right) \\
&\times \Gamma\left(s_{il}+\frac{1}{2}\left(\delta_{i1}\delta_{l2}+\delta_{i1}\delta_{l3}+\delta_{i3}\delta_{l4}+\delta_{i2}\delta_{l4}\right)\right) \prod_{i=1}^{4} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right),
\end{aligned}
\tag{3.94}
$$

The two parity even independent tensor structures t_1^+ and t_2^+ given in (3.51) appear in (3.94), which implies that the Mellin amplitudes has the two non-zero components $\mathcal{M}_1$ and $\mathcal{M}_2$. In terms of the Mandelstam variable $t = \tau_1 + \tau_3 - 2s_{13}$, the first component $\mathcal{M}_1$ is given by

$$
\begin{aligned}
\mathcal{M}_1 =& (t+\tau+2-d)\, \mathbb{N}^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4}\left(s_{il}\right) \\
=& \frac{(t+\tau+2-d)\, \mathbb{M}_{2,2}}{\Gamma\left(\frac{\sum_i \Delta_i}{2}+\frac{1}{2}-h\right)\Gamma\left(\frac{\tau_1+\tau_3-t}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-t}{2}\right)} \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{l(c)l(-c)}{(\tau+1-h)^2-c^2}, \\
l(c) =& \frac{\Gamma\left(\frac{h+c-t-1}{2}\right)\Gamma\left(\frac{\tau_1+\tau_3-h+c+1}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-h+c+1}{2}\right)}{2\Gamma(c)}.
\end{aligned}
\tag{3.95}
$$

The position of the poles is at $t = \tau + 2m$. τ is the twist of the exchanged spinor. Considering that the relevant three-point function is parity even, these poles match with the predictions stated in section 3.3.3. The second component of the Mellin amplitude is of the form

$$
\begin{aligned}
\mathcal{M}_2 =& 2\, \bar{\mathbb{N}}^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4}\left(s_{il}\right) \\
=& \frac{2\, \mathbb{M}_{2,2}}{\Gamma\left(\frac{\sum_i \Delta_i}{2}+\frac{1}{2}-h\right)\Gamma\left(\frac{\tau_1+\tau_3-t+1}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-t+1}{2}\right)} \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{l(c)l(-c)}{(\tau+1-h)^2-c^2}, \\
l(c) =& \frac{\Gamma\left(\frac{h+c-t}{2}\right)\Gamma\left(\frac{\tau_1+\tau_3-h+c+1}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-h+c+1}{2}\right)}{2\Gamma(c)}
\end{aligned}
\tag{3.96}
$$

with poles located at $t = \tau + 1 + 2m$. For a parity even three-point correlator, this is also consistent with the analysis done in section 3.3.3.

Scalar Exchange Witten Diagram with Four External Fermions

The computation of the four-fermion Witten diagram with a scalar exchange, presented in diagram 3.9, is very similar to the scalar exchange of the mixed scalar-fermion Witten diagram. The position space expression reads

$$
A^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4} = \langle S_1 S_2\rangle \langle S_3 S_4\rangle \int_{AdS} dZ_1 \int_{AdS} dZ_2 \prod_{i=1}^{2} K_{\Delta_i+\frac{1}{2}}(Z_1, X_i) G_\Delta(Z_1, Z_2) \prod_{i=3}^{4} K_{\Delta_i+\frac{1}{2}}(Z_2, X_i). \tag{3.97}
$$

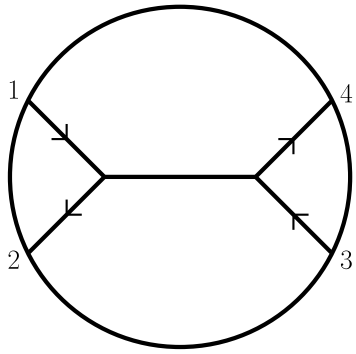


Figure 3.9: Fermionic four-point Witten diagram with scalar exchange.

It follows from the decomposition of the Mellin amplitude done in section 3.3.3 that the only non-zero component is given by

$$\begin{aligned}\mathcal{M}_1 = \mathbb{N}^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4}(s_{il}) =& \frac{\mathbb{M}_{4,0}}{\Gamma\left(\frac{\sum_i \Delta_i}{2}+1-h\right)\Gamma\left(\frac{\Delta_1+\Delta_2-s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-s}{2}+\frac{1}{2}\right)}\\ &\times\int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{l(c)l(-c)}{(\Delta-h)^2-c^2},\\ l(c) =& \frac{\Gamma\left(\frac{h+c-s}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-h+c}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-h+c}{2}+\frac{1}{2}\right)}{2\Gamma(c)}.\end{aligned} \tag{3.98}$$

with the location of the poles at $s = \Delta + 2m$. The other series of poles at $s = \Delta + 1 + 2m$, predicted in section 3.3.3, is absent because the three-point correlator is parity even.

3.4.3 Weak Coupling - Conformal Feynman Integrals

Like Witten diagrams, conformal Feynman integrals take a very simple form in Mellin-Barnes representation [104, 224, 225]. In [104] Mellin space Feynman rules for tree-level interactions in the weak coupling regime were derived for scalar operators. The diagrammatic rules for the Mellin-Barnes representation showed that assuming an interaction without derivatives, the Mellin amplitude associated with a tree-level diagram is given by a product of Euler-beta-functions (B-functions), each of which is associated with an internal propagator. Each vertex yields the trivial contribution 1. The B-function propagator is a function of the Mandelstam variables composed of the fictitious Mellin momenta with the right kind of poles as expected from the Mellin amplitude.

In this section these calculations are extended to Mellin amplitudes associated to tree-level interactions with two or four external fermions. The interaction Lagrangian is of Yukawa type without derivatives acting on the fields.

The conformal Feynman integrals are evaluated with four external legs. However, it is straightforward to generalize these results by adding any number of external scalars. The challenge is to add more pairs of fermionic legs. It turns out that a recursive method which reduces all computations to contact diagram calculations effectively is quite useful in these calculations.[24]

Recursive Method

In [104] the authors derived the Mellin space Feynman rules in the weak coupling regime using a method which included nested Schwinger parameter integrals. These integrals simplified drastically because conformal covariance of the correlation function requires that a certain conformality condition (see (3.100)) has to hold. However, if spinning particles are included the conformality condition is not as useful as in the scalar case any more. Therefore, it is easier to use a recursive method in which only contact diagrams have to be calculated in Mellin space. These can be evaluated using Symanzik's formula derived in [101]. Even though this method introduces some nested Mellin-Barnes integrals, these are easier to handle than the nested Schwinger parameter integrals in the present case.

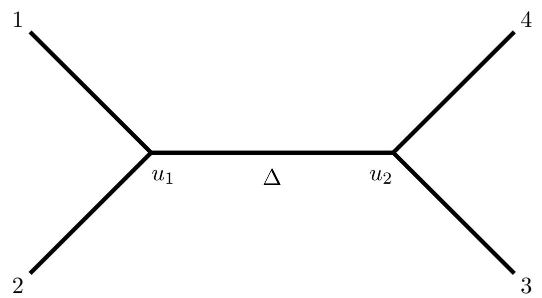


Figure 3.10: Scalar four-point diagram with scalar exchange.

The recursive method shall be illustrated on an easy example. Consider the scalar four-point diagram (3.10) with a scalar exchange. Using position space Feynman rules the conformal integral is given by

$$I^{\phi_1\phi_2}_{\phi_3\phi_4} = \int \mathcal{D}u_1 \int \mathcal{D}u_2 \prod_{i=1}^{2} \frac{\Gamma(\Delta_i)}{|x_i - u_1|^{2\Delta_i}} \prod_{i=3}^{4} \frac{\Gamma(\Delta_i)}{|x_i - u_2|^{2\Delta_i}} \frac{1}{|u_1 - u_2|^{2\Delta}} \text{ with } \mathcal{D}u = \frac{1}{2}\frac{d^d u}{\pi^{d/2}}. \tag{3.99}$$

Note that the external propagators have been normalized by additional Euler-gamma functions to simplify the final result. Using conformal covariance of the integral, it can be deduced that the conformality condition

$$\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4 = d - \Delta \tag{3.100}$$

has to hold [104].[25] The recursive method treats the internal line as an external leg, like it would be independent of the interaction vertex u_1. This is sketched in the diagram 3.11.

Hence, Symanzik's formula (B.19) can be applied to it, which leads to the partial Mellin-

[24] Arnab Rudra developed this method while working on [104].

[25] This is easy to prove using the homogeneity property (3.30) if the integral is rewritten in embedding space coordinates.

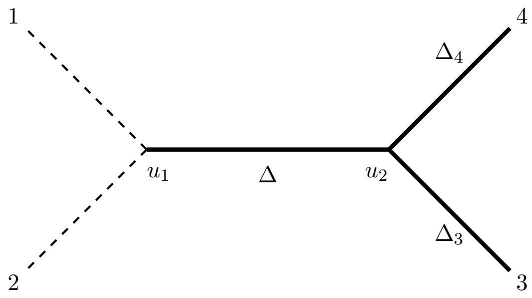


Figure 3.11: Recursive method: First step

Barnes representation

$$\int \mathcal{D}u_2 \prod_{i=3}^{4} \frac{\Gamma(\Delta_i)}{|x_i - u_2|^{2\Delta_i}} \frac{1}{|u_1 - u_2|^{2\Delta}} = \left(\prod_{i=3}^{4} \int_{c_{iu}-i\infty}^{c_{iu}+i\infty} (ds_{iu}) \right) \frac{1}{\Gamma(\Delta)} \int_{\bar{c}_{34}-i\infty}^{\bar{c}_{34}+i\infty} (d\bar{s}_{34})$$
$$\times \frac{\Gamma(s_{iu})}{|x_i - u_1|^{2s_{iu}}} \frac{\Gamma(\bar{s}_{34})}{|x_{34}|^{2\bar{s}_{34}}} \prod_{i=3}^{4} \hat{\delta}(\Delta_i - \bar{s}_{34} - s_{iu})\, \hat{\delta}(\Delta - s_{3u} - s_{4u}) \tag{3.101}$$

with the integration measure $(ds_{il}) = \frac{ds_{il}}{2\pi i}$ of the Mellin variables. The correct contours of the Mellin-Barnes integrals are such that the series of poles generated by the Γ-functions are not separated. Plugging (3.101) back into (3.99) the integral over the measure $\mathcal{D}u_1$ can be performed using Symanzik's formula (B.19). In this case the "external legs" are given by the edges (x_1, u_1), (x_2, u_1), (x_3, u_1) and (x_4, u_1) with "scaling dimension" Δ_1, Δ_2, s_{3u} and s_{4u}, respectively. This is sketched in diagram 3.12. Using the δ-distribution constraint $2\pi i\ \delta(\Delta - s_{3u} - s_{4u}) = \hat{\delta}(\Delta - s_{3u} - s_{4u})$ in (3.101) the required conformality condition for this integral $\Delta_1 + \Delta_2 + s_{3u} + s_{4u} = d$ is also satisfied.

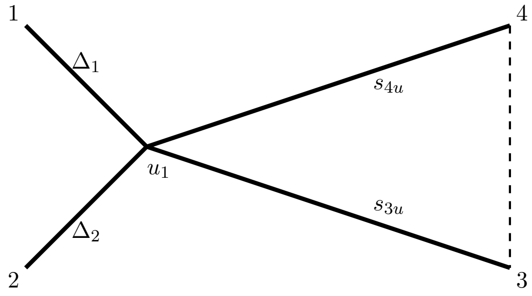


Figure 3.12: Recursive method: Second step

The result reads

$$
\begin{aligned}
I^{\phi_1\phi_2}_{\phi_3\phi_4} = & \prod_{1\leq i<l}^{4} \int_{\tilde{c}_{il}-i\infty}^{\tilde{c}_{il}+i\infty} (d\tilde{s}_{il}) \frac{\Gamma(\tilde{s}_{il})}{|x_{il}|^{2\tilde{s}_{il}}} \int_{\bar{c}_{34}-i\infty}^{\bar{c}_{34}+i\infty} (d\bar{s}_{34}) \frac{\Gamma(\bar{s}_{34})}{|x_{34}|^{2\bar{s}_{34}}} \frac{1}{\Gamma(\Delta)} \\
& \times \left(\prod_{i=3}^{4} \int_{c_{iu}-i\infty}^{c_{iu}+i\infty} (ds_{iu}) \right) \hat{\delta}(s_{3u} - \tilde{s}_{13} - \tilde{s}_{23} - \tilde{s_{34}}) \, \hat{\delta}(s_{4u} - \tilde{s}_{14} - \tilde{s}_{24} - \tilde{s_{34}}) \\
& \times \hat{\delta}(\Delta_1 - \tilde{s}_{12} - \tilde{s}_{13} - \tilde{s}_{14}) \, \hat{\delta}(\Delta_2 - \tilde{s}_{12} - \tilde{s}_{23} - \tilde{s}_{24}) \\
& \times \prod_{i=3}^{4} \hat{\delta}(\Delta_i - \bar{s}_{34} - s_{iu}) \, \hat{\delta}(\Delta - s_{3u} - s_{4u}) \, .
\end{aligned}
$$

To distinguish the Mellin variables introduced in the last step from the previous one they are decorated with a tilde. The δ-distributions can be used to integrate out s_{iu}. Renaming $\tilde{s}_{ij} = s_{ij}$ for $(i,j) \neq (3,4)$ and taking $\bar{s}_{34} = s_{34} - \tilde{s}_{34}$ yields

$$
\begin{aligned}
I^{\phi_1\phi_2}_{\phi_3\phi_4} = & \prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il}) \frac{\Gamma(s_{il})}{|x_{il}|^{2s_{il}}} \prod_{i=1}^{4} \hat{\delta} \left(\Delta_i - \sum_{k=1,k\neq i}^{4} s_{ik} \right) \\
& \times \frac{1}{\Gamma(\Delta)} \int_{\tilde{c}_{34}-i\infty}^{\tilde{c}_{34}+i\infty} (d\tilde{s}_{34}) \frac{\Gamma(\tilde{s}_{34}) \, \Gamma(s_{34} - \tilde{s}_{34})}{\Gamma(s_{34})} \hat{\delta}(\Delta - K_{12,34} - 2\tilde{s}_{34}) \, .
\end{aligned}
$$

To make the result more transparent the object $K_{ij,kl} = s_{ik} + s_{il} + s_{jk} + s_{jl}$ has been introduced. The last integral over $(d\tilde{s_{34}})$ gives finally

$$
\begin{aligned}
I^{\phi_1\phi_2}_{\phi_3\phi_4} = & \prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il}) \frac{\Gamma(s_{il})}{|x_{il}|^{2s_{il}}} \prod_{i=1}^{4} \hat{\delta} \left(\Delta_i - \sum_{k=1,k\neq i}^{4} s_{ik} \right) \frac{1}{2\Gamma(\Delta)} B\left(\frac{\Delta - K_{12,34}}{2}, \frac{d-2\Delta}{2} \right) \\
& \text{with} \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} . \qquad (3.102)
\end{aligned}
$$

The second argument of the B-function has been rewritten using the constraints from the conformality condition and the δ-distributions:

$$
d = \Delta + \Delta_3 + \Delta_4 \qquad \text{and} \qquad \Delta_i = \sum_{k=1,i\neq k}^{4} s_{ik} .
$$

The same result (3.102) has been calculated in [104, 224] before.

Contact Diagrams

To apply the recursive method for diagrams with fermionic legs, the results for the Mellin amplitude associated to the corresponding contact interaction diagrams have to be known. In this section, the Mellin-Barnes representation of the contact interaction with two and four fermions is presented. The corresponding diagrams are drawn in 3.13.

This calculation has been done by Symanzik [101]. The conformal integral for the contact interaction of two fermions and two scalars is given by

$$
C^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} = \int \mathcal{D}u \frac{\not{x}_1 - \not{u}}{|x_1 - u|^{2\Delta_1+1}} \Gamma\left(\Delta_1 + \frac{1}{2}\right) \frac{\not{u} - \not{x}_2}{|u - x_2|^{2\Delta_2+1}} \Gamma\left(\Delta_2 + \frac{1}{2}\right) \frac{\Gamma(\Delta_3)}{|x_1 - u|^{2\Delta_3}} \frac{\Gamma(\Delta_4)}{|x_4 - u|^{2\Delta_4}} ,
$$

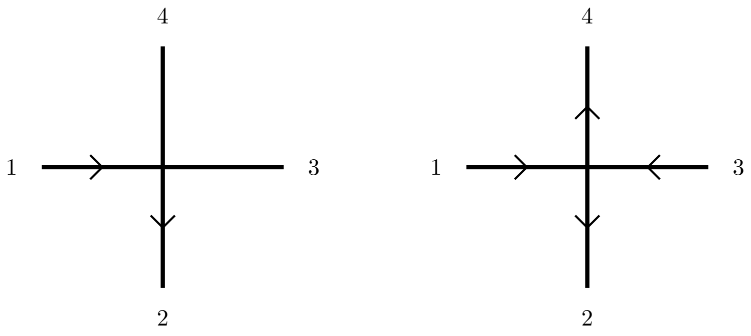


Figure 3.13: Contact diagrams with two and four fermions.

where spinor indices on $\not{x} = x^\mu \gamma_\mu$ have been suppressed. The conformality constraint requires that the sum of all the scaling dimensions has to be equal to the spacetime dimension d. In embedding space notation, and in accordance with the definition (3.63), the Mellin-Barnes representation of this conformal integral reads

$$\begin{aligned}&\sum_{j=3}^{4} \frac{\langle S_1 X_j S_2 \rangle}{\sqrt{X_{1j} X_{j2}}} \prod_{1 \leq i < l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{l2}} \\ &\times \Gamma\left(s_{il} + \frac{1}{2}\left(\delta_{i1}+\delta_{i2}\right)\delta_{jl} + \frac{1}{2}\delta_{i1}\delta_{l2}\right) \prod_i \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right),\end{aligned} \tag{3.103}$$

which implies that the Mellin amplitude is given by

$$\mathcal{M}_3 = \mathcal{M}_4 = 1. \tag{3.104}$$

The Mellin amplitude associated to the fermionic contact diagram with four legs

$$C^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4} = \int \mathcal{D}u \prod_{i=1}^{4} \Gamma\!\left(\Delta_i + \frac{1}{2}\right) \left[\frac{\not{x}_1 - \not{u}}{|x_1-u|^{2\Delta_1+1}} \frac{\not{u} - \not{x}_2}{|u-x_2|^{2\Delta_2+1}}\right] \left[\frac{\not{x}_3 - \not{u}}{|x_3-u|^{2\Delta_3+1}} \frac{\not{u} - \not{x}_4}{|u-x_4|^{2\Delta_4+1}}\right]$$

has been evaluated in [101] to be

$$\begin{aligned}&\prod_{1\leq i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{l2}-\frac{1}{2}\delta_{i3}\delta_{l4}} \left[\frac{1}{2}\frac{\left\langle S_1 \Gamma^A S_2\right\rangle \langle S_3 \Gamma_A S_4\rangle}{\sqrt{X_{12}X_{34}}} \Gamma\left(s_{il} + \delta_{i1}\delta_{l2} + \delta_{i3}\delta_{l4}\right)\right. \\ +\;& \sum_{j=3}^{4}\sum_{k=1}^{2} \frac{\langle S_1 X_j S_2\rangle \langle S_3 X_k S_4\rangle}{\sqrt{X_{1j}X_{j2}X_{3k}X_{k4}}} \\ \times\;& \left.\Gamma\!\left(s_{il} + \frac{1}{2}\left(\delta_{i1}+\delta_{i2}\right)\delta_{lj} + \frac{1}{2}\left(\delta_{3l}+\delta_{4l}\right)\delta_{ik} + \frac{1}{2}\delta_{i1}\delta_{l2} + \frac{1}{2}\delta_{i3}\delta_{l4}\right)\right] \prod_i \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right),\end{aligned} \tag{3.105}$$

where the result has been given in embedding space.[26]

[26]To relate this expression to the one stated in [101], note that $\left\langle S_1 \Gamma^A S_2\right\rangle \langle S_3 \Gamma_A S_4\rangle \xrightarrow[\text{physical space}]{} \left[\not{x}_1 \gamma^\mu + \gamma^\mu \not{x}_2\right] \left[\not{x}_3 \gamma_\mu + \gamma_\mu \not{x}_4\right] - 2\left[\not{x}_1 \not{x}_2\right][\mathbb{1}] - 2[\mathbb{1}]\left[\not{x}_3 \not{x}_4\right]$ holds.

Expanding the tensor structures appearing in (3.105) into the basis (3.52) and (3.53)

$$\frac{\left\langle S_1 \Gamma^A S_2 \right\rangle \left\langle S_3 \Gamma_A S_4 \right\rangle}{\sqrt{X_{12} X_{34}}} = \frac{1}{2} p_1 + 2\sqrt{\frac{v}{u}} p_3 + \frac{1}{2} p_4 - 2\sqrt{\frac{v}{u}} p_5. \tag{3.106}$$

gives the following non-vanishing Mellin amplitudes

$$\begin{aligned} &\mathcal{M}_1 = \frac{1}{4}, \qquad \mathcal{M}_3 = 1, \qquad \mathcal{M}_4 = \frac{1}{4}, \qquad \mathcal{M}_5 = s_{13} - 1, \\ &\mathcal{M}_6 = s_{23}, \qquad \mathcal{M}_7 = s_{14}, \qquad \mathcal{M}_8 = s_{24}. \end{aligned} \tag{3.107}$$

Further, the results (3.104) and (3.107) are consistent with the fact that in three dimensions the Yukawa interaction in Minkowski spacetime with signature $(-,+,+)$ is parity odd. Because the Mellin amplitudes given in (3.104) and (3.107) decompose in the OPE into parity odd three-point functions as can be seen in the tables 3.2 and 3.4.[27]

Fermion-Scalar Four-Point Functions: Scalar and Fermionic Exchange.

The conformal Feynman integrals of the fermion-scalar four-point correlator are drawn in figure 3.14. They are evaluated with the recursive method described at the beginning of this section and the known result of the contact interaction (3.103).

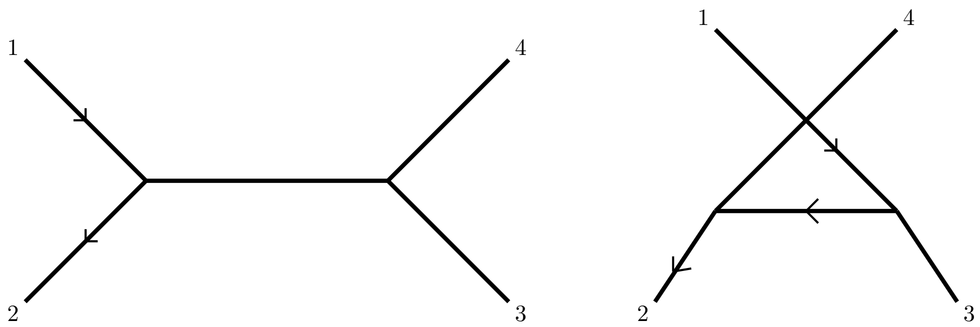


Figure 3.14: Fermion scalar four point diagrams with scalar and fermionic exchange.

The conformal integral with a scalar exchange is given by

$$\begin{aligned} I^{\bar\psi_1\psi_2}_{\phi_3\phi_4} = &\int \mathcal{D}u_1 \int \mathcal{D}u_2 \frac{\not{x}_1 - \not{u}_1}{|x_1 - u_1|^{2\Delta_1+1}} \Gamma\left(\Delta_1 + \frac{1}{2}\right) \frac{\not{u}_1 - \not{x}_2}{|u_1 - x_2|^{2\Delta_2+1}} \Gamma\left(\Delta_2 + \frac{1}{2}\right) \\ &\times \prod_{i=3}^{4} \frac{\Gamma(\Delta_i)}{|x_i - u_2|^{2\Delta_i}} \frac{1}{|u_1 - u_2|^{2\Delta}}. \end{aligned}$$

[27]To clarify contact diagrams are polynomials (or constants) in the Mellin variable, they do not factorize onto corresponding three-point functions. (Their residue is zero.) However, it is obvious from the recursive method that for the exchange diagrams the tensor structures from the contact diagrams can appear only; hence, it can already been deduced that all exchange diagrams will factorize consistently with the fact that the three-point correlator is parity odd. In particular, this implies that merely one series of poles from table 3.3 is relevant in the t-channel for the two-fermion four-point correlator. The same holds for the fermionic four-point correlator as can be seen in the tables 3.4, B.2.

Conformality of this integral demands $\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4 = d - \Delta$. The Mellin-Barnes representation of this integral reads

$$\sum_{j=3}^{4} \frac{\langle S_1 X_j S_2 \rangle}{\sqrt{X_{1j} X_{j2}}} \prod_{1 \le i < l}^{4} \int_{c_{il} - i\infty}^{c_{il} + i\infty} (ds_{il})\, X_{il}^{-s_{il} - \frac{1}{2}\delta_{i1}\delta_{l2}} \Gamma\left(s_{il} + \frac{1}{2}\left(\delta_{1i} + \delta_{2i}\right)\delta_{lj} + \frac{1}{2}\delta_{i1}\delta_{l2}\right)$$
$$\times \frac{1}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right) \prod_i \hat{\delta}\left(\tau_i - \sum_{j \neq i} s_{ij}\right),$$

which yields the non-vanishing components of the Mellin amplitude

$$\mathcal{M}_3 = \mathcal{M}_4 = \frac{1}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right). \tag{3.108}$$

The poles are located at $-(p_1 + p_2)^2 = s = \Delta + 2m$ with $m \in \mathbb{N}$ which matches the predictions made in section 3.3.3.

The conformal Feynman integral with an internal spin one-half fermion is given by

$$I^{\bar{\psi}_1 \phi_3}_{\psi_2 \phi_4} = \int \mathcal{D}u_1 \int \mathcal{D}u_2 \frac{\not{x}_1 - \not{u}_1}{|x_1 - u_1|^{2\Delta_1 + 1}} \Gamma\left(\Delta_1 + \frac{1}{2}\right) \frac{\not{u}_1 - \not{u}_2}{|u_1 - u_2|^{2\Delta + 1}}$$
$$\times \frac{\not{u}_2 - \not{x}_2}{|u_2 - x_2|^{2\Delta_2 + 1}} \Gamma\left(\Delta_2 + \frac{1}{2}\right) \frac{\Gamma(\Delta_3)}{|x_3 - u_1|^{2\Delta_3}} \frac{\Gamma(\Delta_4)}{|x_4 - u_2|^{2\Delta_4}}.$$

The corresponding Mellin-Barnes representation reads

$$\prod_{1 \le i < l}^{4} \int_{c_{il} - i\infty}^{c_{il} + i\infty} (ds_{il})\, X_{il}^{-s_{il} - \frac{1}{2}\delta_{1i}\delta_{2l}} \prod_i \hat{\delta}\left(\tau_i - \sum_{j \neq i} s_{ij}\right) \left[\frac{\langle S_1 X_3 X_4 S_2 \rangle}{\sqrt{X_{13} X_{34} X_{42}}} \frac{1}{2\Gamma(\tau + 1)} \right.$$
$$\times B\left(\frac{\tau - t}{2}, \frac{d}{2} - \tau\right) \prod_{1 \le i < l} \Gamma\left(s_{il} + \frac{1}{2}\left(\delta_{i1}\delta_{2l} + \delta_{i1}\delta_{l3} + \delta_{i3}\delta_{l4} + \delta_{i2}\delta_{l4}\right)\right)$$
$$\left. - \frac{\langle S_1 S_2 \rangle}{\sqrt{X_{12}}} \frac{s_{13}}{2\Gamma(\tau + 1)} B\left(\frac{\tau + 1 - t}{2}, \frac{d}{2} - \tau\right) \prod_{1 \le i < l} \Gamma\left(s_{il} + \delta_{i1}\delta_{l2}\right) \right].$$

Hence, the only non-vanishing components of the Mellin amplitude are

$$\mathcal{M}_1 = -\frac{\tau_1 + \tau_3 - t}{4\Gamma(\tau + 1)} B\left(\frac{\tau + 1 - t}{2}, \frac{d}{2} - \tau\right), \quad \mathcal{M}_2 = \frac{1}{2\Gamma(\tau + 1)} B\left(\frac{\tau - t}{2}, \frac{d}{2} - \tau\right). \tag{3.109}$$

$\mathcal{M}_1$ has poles at $-(p_1 + p_3)^2 = t = \tau + 1 + 2m$ while $\mathcal{M}_2$ has poles at $t = \tau + 2m$, where $\tau = \Delta - \frac{1}{2}$ is the twist of the exchanged operator. This again is consistent with the general structure stated in section 3.3.3.

Fermionic Four-Point Function: Scalar Exchange

The position space expression for the four-fermion Feynman diagram with a scalar exchange (s-channel) is

$$\int \mathcal{D}u_1 \int \mathcal{D}u_2 \left[\frac{\not{x}_1 - \not{u}_1}{|x_1 - u_1|^{2\Delta_1 + 1}} \frac{\not{u}_1 - \not{x}_2}{|u_1 - x_2|^{2\Delta_2 + 1}} \right] \frac{1}{|u_1 - u_2|^{2\Delta}}$$
$$\times \left[\frac{\not{x}_3 - \not{u}_2}{|x_3 - u_2|^{2\Delta_3 + 1}} \frac{\not{u}_2 - \not{x}_4}{|u_2 - x_4|^{2\Delta_4 + 1}} \right] \prod_{i=1}^{4} \Gamma\left(\Delta_i + \frac{1}{2}\right).$$

The Mellin-Barnes representation of this conformal integral reads

$$
\begin{aligned}
I^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4} = &\prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{j2}-\frac{1}{2}\delta_{i3}\delta_{j4}} \prod_i \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right) \\
&\times \left[\frac{1}{2}\frac{\left\langle S_1\Gamma^A S_2\right\rangle \langle S_3\Gamma_A S_4\rangle}{\sqrt{X_{12}X_{34}}} \frac{1}{2\Gamma(\Delta)} B\left(\frac{\Delta - s + 1}{2}, \frac{d}{2} - \Delta\right) \prod_{i<l} \Gamma\left(s_{il} + \delta_{1i}\delta_{2j} + \delta_{3i}\delta_{4j}\right)\right. \\
&+ \sum_{j=3}^{4}\sum_{k=1}^{2} \frac{\langle S_1 X_j S_2\rangle \langle S_3 X_k S_4\rangle}{\sqrt{X_{1j}X_{j2}X_{3k}X_{k4}}} \frac{1}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right) \\
&\left.\times \prod_{i<l} \Gamma\left(s_{il} + \frac{1}{2}\left(\delta_{1i} + \delta_{2i}\right)\delta_{jl} + \frac{1}{2}\delta_{ik}\left(\delta_{3l} + \delta_{4l}\right) + \frac{1}{2}\left(\delta_{1i}\delta_{2j} + \delta_{3i}\delta_{4j}\right)\right)\right].
\end{aligned}
$$

Rewriting the tensor structure which appears in $I^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4}$ into the basis (3.52) and (3.53) using the decomposition (3.106) gives the following non-vanishing components for the Mellin amplitude

$$
\begin{aligned}
\mathcal{M}_1 =& \mathcal{M}_4 = \frac{1}{8\Gamma(\Delta)} B\left(\frac{\Delta + 1 - s}{2}, \frac{d}{2} - \Delta\right), & \mathcal{M}_3 =& \frac{1}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right), \\
\mathcal{M}_5 =& \frac{s_{13} - 1}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right), & \mathcal{M}_6 =& \frac{s_{23}}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right), \\
\mathcal{M}_7 =& \frac{s_{14}}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right), & \mathcal{M}_8 =& \frac{s_{24}}{2\Gamma(\Delta)} B\left(\frac{\Delta - s}{2}, \frac{d}{2} - \Delta\right).
\end{aligned}
\tag{3.110}
$$

The poles of all the non-zero $\mathcal{M}_i$ are exactly as predicted in section 3.3.3, since the three-point interaction is parity odd.

3.5 Conclusions

It has been shown that a generalization to Mellin amplitudes containing spin one-half fermions is straight forward and it exhibits the expected pole structure, i.e. the location of the pole is completely fixed by the twist τ of the exchanged operator. This analysis has also been confirmed by computing different four-point correlators at strong and weak coupling using Witten diagrams and Feynman diagrams respectively. Especially it could be shown that the positions of the poles predicted in section 3.3.3 agree with the perturbative analysis performed in section 3.4.

However, spinning conformal correlation functions are always equipped with some additional tensor structure. There is no canonical choice for a basis of tensor structures and not every basis is suitable, because there can appear spurious poles in the corresponding conformal block which is also reflected in the Mellin amplitude. In addition, contrary to scalar Mellin amplitudes, fermionic Mellin amplitudes can exhibit more than one series of poles in a particular channel like it appeared in the t-channel as it is shown in table 3.3. This might be related to the chosen basis of tensor structures. It would be interesting to study the exact relationship between the pole structure of the Mellin amplitude and the choice of basis. In this thesis a basis of definite parity has been chosen which does not include any spurious poles and makes the analytic properties of the Mellin amplitude manifest. Further, this basis is rather well suited to study theories of definite parity because only one series of poles will appear in this case like the perturbative examples have shown.

The location of the pole contains only part of the CFT data. To compute the OPE coefficients one has to determine the analogue of the continuous Hahn polynomial $\mathcal{Q}_{m,l}(t)$ from equation (3.59). It is to be expected that these can be obtained in a similar way as they have been obtained in the literature for scalar Mellin amplitudes:

1. In one approach it has been deduced from the Mack polynomial which is the Mellin amplitude of the conformal partial wave [99]. Hence, it depends on both Mellin variables s, t. The residue of the Mack polynomial in one of the variables gives the continuous Hahn polynomial $\mathcal{Q}_{m,l}(t)$ in the other variable.

2. A second approach is to use the conformal Casimir equation which $\mathcal{Q}_{m,l}(t)$ has to obey. The Casimir equation turns into difference equation in Mellin space and solving this difference equation shall give $\mathcal{Q}_{m,l}(t)$, which is an orthogonal polynomial with respect to the measure dictated by the Mellin-Barnes transformation [105].

This part is still work in progress and shall be presented in another publication.

If the polynomial residues of the Mellin amplitudes are known the complete CFT data can be obtained from the Mellin amplitude in principle. To access this data one can apply Polyakov-Mellin bootstrap techniques to fermionic CFTs. In three dimensions potential candidates are the Gross-Neveu theory [226] or the Gross-Neveu-Yukawa theory [227].

Another way to pursue is the derivation of Feynman rules in Mellin space for fermionic operators. Mellin amplitudes at strong coupling have a simple form even for theories including fermions. It has been shown that diagrams which only contain fermions on their external legs can be reduced to the calculation of scalar Mellin amplitudes. Therefore it might be possible to derive certain Feynman rules in Mellin space for fermionic Witten diagrams too using the results obtained for scalar Mellin amplitudes [102, 103]. It seems also possible to generalize this technique to higher spin Mellin amplitudes. In contrast to the strong coupling regime, at weak coupling it might be more difficult to derive general fermionic Feynman rules, because the complexity of the tensor structures increases rapidly with the number of external fermions [101]. So far it is not obvious how to generalize the computation of Feynman diagrams in Mellin space with more than four external legs except by a straight forward calculation.

Appendix A

Amplitudes

A.1 Spinor-Helicity Formalism

The spinor-helicity formalism maps Lorentz vectors to 2×2 Hermitian matrices. The basis of this space is spanned by

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

which is conveniently packaged into the four vectors $(\sigma^\mu)_{\alpha\dot{\alpha}} = (\mathbb{1}_2, \sigma)_{\alpha\dot{\alpha}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (\mathbb{1}_2, -\sigma)^{\dot{\alpha}\alpha}$. The $\mathrm{SL}(2,\mathbb{C})$ indices are $\alpha, \dot{\alpha} = 1, 2$. To raise and lower indices the invariant antisymmetric tensors $\epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{12} = -\epsilon_{\dot{1}\dot{2}} = 1$ are used which read $\epsilon = i\sigma_2$. Thus to each object which carries $\mathrm{SL}(2,\mathbb{C})$ indices their duals can be defined by

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = -\left(\epsilon\bar{\sigma}^\mu\epsilon\right)_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\bar{\sigma}^\mu\right)^{\dot{\beta}\beta}, \quad \chi^\alpha = (\epsilon\chi)^\alpha = \epsilon^{\alpha\beta}\chi_\beta, \quad \bar{\chi}_{\dot{\alpha}} = (\epsilon\bar{\chi})_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}}.$$

Note that for real momenta the fundamental χ_α and antifundamental $\bar{\chi}^{\dot{\alpha}}$ Weyl spinors are related by $(\chi_\alpha)^* = \bar{\chi}_{\dot{\alpha}}$.

It is instructive to show the relation between the Lorentz group $\mathrm{SO}_+(1,3)$ and its double cover $\mathrm{SL}(2,\mathbb{C})$. A general Dirac spinor ψ transforms under Lorentz transformations

$$\Lambda : \psi(x) \to \psi'(x') = S_\Lambda \psi(\Lambda^{-1}x),$$

where

$$S_\Lambda = \exp\left(\frac{1}{8} w_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right]\right) = \begin{pmatrix} \exp\left(\frac{1}{8}w_{\mu\nu}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)\right) & 0 \\ 0 & \exp\left(\frac{1}{8}w_{\mu\nu}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)\right) \end{pmatrix}.$$

In the last step the Weyl representation (A.2) of the γ-matrices has been used. This shows that the Dirac spinor is reducible. In fact, separating the totally antisymmetric constant matrix $w_{\mu\nu}$ into boosts $\beta_i = w_{0i}$ and rotations $w_{ij} = \epsilon_{ijk}\Theta_k$, the transformation behaviour of the Weyl spinors can be deduced:

$$\Lambda : \psi_{L,\alpha}(x) \to M_\alpha^{\ \beta}(\beta, \mathbf{\Theta})\psi_{L,\beta}(\Lambda^{-1}x) \qquad \text{with} \quad M(\beta,\mathbf{\Theta}) = \exp\left(-\frac{1}{2}\left(\beta + i\mathbf{\Theta}\right)\cdot\sigma\right),$$

$$\Lambda : \psi_R^{\dot{\alpha}}(x) \to \bar{M}^{\dot{\alpha}}_{\ \dot{\beta}}(\beta, \mathbf{\Theta})\psi_R^{\dot{\beta}}(\Lambda^{-1}x) \qquad \text{with} \quad \bar{M}(\beta,\mathbf{\Theta}) = \exp\left(\frac{1}{2}\left(\beta + i\mathbf{\Theta}\right)^*\cdot\sigma\right).$$

Since the matrices M and $\bar{M}$ are generated by the six (anti-)Hermitian generators σ_j and $i\sigma_j$ which are traceless, it follows that $M \in \mathrm{SL}(2,\mathbb{C})$. $\bar{M}^{\dot{\alpha}}_{\ \dot{\beta}} = \left(M^{\alpha}_{\ \beta}\right)^*$ is the complex conjugate representation of $M^{\ \alpha}_{\beta} = \epsilon^{\alpha\alpha_1} M^{\ \beta_1}_{\alpha_1} \epsilon_{\beta_1\beta}$, which can be proven using $\epsilon\,\sigma_j \epsilon = -\sigma_j^*$. The natural metric for the Weyl representation is given by ϵ, because it obeys the same relation as η does for the vector representation.

$$\eta_{\mu\nu} = \left(\Lambda^T \eta \Lambda\right)_{\mu\nu}, \quad \epsilon_{\alpha\beta} = \left(M^T \epsilon M\right)_{\alpha\beta}.$$

The power of the spinor-helicity formalism is rooted in the universal usage of the Fourier transformed Weyl spinors

$$u_+(p) = v_-(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \quad \text{and} \quad u_-(p) = v_+(p) = \begin{pmatrix} 0 \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix},$$

here given in the helicity basis. They can represent the momenta p as well as the polarization vectors $\varepsilon_\pm$:

$$\begin{aligned} p^{\alpha\dot{\alpha}} &= \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}, \\ \varepsilon_+^{\alpha\dot{\alpha}}(p_i) &= -\sqrt{2}\frac{\bar{\lambda}_i^{\dot{\alpha}} r_i^\alpha}{\langle \lambda_i r_i \rangle}, \qquad & \varepsilon_-^{\alpha\dot{\alpha}}(p_i) &= \sqrt{2}\frac{\lambda_i^\alpha \bar{r}_i^{\dot{\alpha}}}{[\lambda_i r_i]}, \\ \langle \lambda r \rangle &= \lambda^\alpha r_\alpha, & [\lambda r] &= \bar{\lambda}_{\dot{\alpha}} \bar{r}^{\dot{\alpha}}. \end{aligned}$$

The spinors r^α and $\bar{r}^{\dot{\alpha}}$ are auxiliary variables, which are associated to the reference momentum $q^{\alpha\dot{\alpha}} := r^\alpha \bar{r}^{\dot{\alpha}}$. $q \neq p$ can be chosen arbitrarily. This freedom is related to performing a gauge transformation.

In the Weyl representation of the γ-matrices the following relation holds

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_{\alpha\dot{\alpha}} \\ p^{\dot{\alpha}\alpha} & 0 \end{pmatrix} \quad \text{with} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{A.2}$$

This representation can be used to show several identities of the spinor products

$$\langle i\,|\gamma^\mu|\,j] = [j\,|\gamma^\mu|\,i\rangle, \quad \langle p\,|\gamma^\mu|\,p] = \lambda^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} = 2p^\mu, \quad \langle i\,|\gamma^\mu|\,j]\,[l\,|\gamma_\mu|\,k\rangle = 2\,[ij]\,\langle lj\rangle \tag{A.3}$$

and the Shouten identity

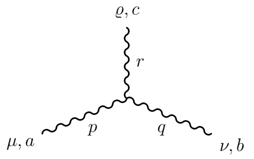
$$\langle i_1 i_2\rangle\langle i_3 l\rangle + \langle i_2 i_3\rangle\langle i_1 l\rangle + \langle i_3 i_1\rangle\langle i_2 l\rangle = 0. \tag{A.4}$$

For concrete calculations involving $\mathrm{SO}_+(1,3)$ and $\mathrm{SL}(2,\mathbb{C})$ the following identities are helpful:

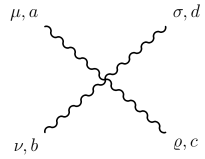
$$\begin{aligned} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu &= 2\eta^{\mu\nu}\mathbb{1}_2, \qquad & \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu &= 2\eta^{\mu\nu}\mathbb{1}_2, \\ \mathrm{Tr}\left(\bar{\sigma}^\mu \sigma^\nu\right) &= 2\eta^{\mu\nu}, & (\sigma_\mu)_{\alpha\dot{\alpha}} \left(\bar{\sigma}^\mu\right)^{\dot{\beta}\beta} &= 2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}}. \end{aligned}$$

A.2 Details to Einstein-Yang-Mills Amplitudes

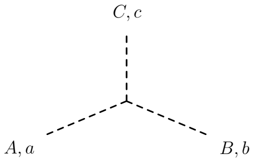
A.2.1 Feynman Rules



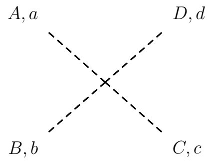
$$-gf^{abc}\left[(r_\mu - q_\mu)\,\eta_{\nu\varrho} + (p_\nu - r_\nu)\,\eta_{\varrho\mu} + (q_\varrho - p_\varrho)\,\eta_{\mu\nu}\right]$$



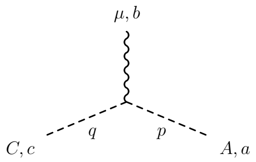
$$\begin{aligned}-ig^2\Big[&f^{abe'}f^{e'cd}\left(\eta_{\mu\varrho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\varrho}\right)\\ +&f^{ace'}f^{e'db}\left(\eta_{\mu\sigma}\eta_{\varrho\nu} - \eta_{\mu\nu}\eta_{\varrho\sigma}\right)\\ +&f^{ade'}f^{e'bc}\left(\eta_{\mu\nu}\eta_{\sigma\varrho} - \eta_{\mu\varrho}\eta_{\sigma\nu}\right)\Big]\end{aligned}$$



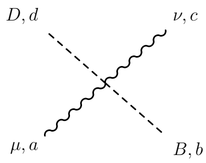
$$i\lambda g f^{abc}F^{ABC}$$



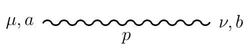
$$\begin{aligned}-ig^2\Big[&f^{abe'}f^{e'cd}\left(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}\right)\\ +&f^{ace'}f^{e'db}\left(\delta^{AD}\delta^{BC} - \delta^{AB}\delta^{CD}\right)\\ +&f^{ade'}f^{e'bc}\left(\delta^{AB}\delta^{CD} - \delta^{AC}\delta^{BD}\right)\Big]\end{aligned}$$



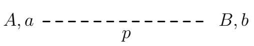
$$gf^{abc}\delta^{AC}\left(p_\mu - q_\mu\right)$$



$$ig^2\left(f^{abe'}f^{e'cd} + f^{cbe'}f^{e'ad}\right)\delta^{BD}\eta_{\mu\nu}$$



$$\frac{-i\;\eta^{\mu\nu}\,\delta^{ab}}{p^2 + i\epsilon}$$



$$\frac{i\;\delta^{AB}\,\delta^{ab}}{p^2 + i\epsilon}$$

Table A.1: Feynman rules for YM+ϕ^3 derived from the Lagrangian (2.66). In this convention all momenta are outgoing. For the gluon propagator the Feynman gauge $\xi = 1$ is used.

A.2.2 Integrands

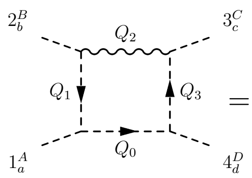
All integrands are presented in $4-2\epsilon$ dimensions. For the colour factors, the notation from (2.70) is used. The inverse of the stripped propagators is given by $D_j = Q_j^2 + i\epsilon = (q_j + L)^2 + i\epsilon = (\sum_{k=1}^{j} p_k + l)^2 - \mu^2 + i\epsilon$.

Integrands which are proportional to the colour ordering c^{adbc} shall be written with permuted external momenta p_i, i.e. $\tilde{Q}_0 = L$, $\tilde{Q}_1 = L + p_2$, $\tilde{Q}_2 = L + p_2 + p_3$, $\tilde{Q}_3 = Q_3$. The corresponding stripped propagators are given by $(\tilde{D}_j)^{-1} = (\tilde{Q}_j^2 + i\epsilon)^{-1}$.

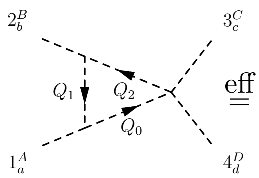
To present the results clearly only the parts of the integrand in YM+ϕ^3 are written which contribute to the gravity amplitude in EYM, hence all terms that are not proportional to the colour structure (2.70) are not displayed. To simplify the expressions the explicit value of the quadratic Casimir of the adjoint representation c_A is used, which for SU(N) is given by $f^{a'ab'}f^{b'ba'} = c_A\delta^{ab} = N\delta^{ab}$.

Integrands for $\langle 1_a^A\, 2_b^B\, 3_c^C\, 4_d^D \rangle$

These are the two types of integrands that appear in the computation of $\langle 1_a^A\, 2_b^B\, 3_c^C\, 4_d^D \rangle\big|_{\lambda^2 g^4}$. The remaining integrands can be obtained by permuting the external legs. In total, twelve non-equal box integrands and six non-equal triangle integrands contribute.

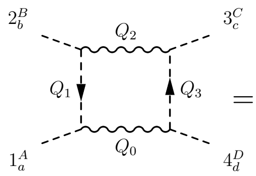


$$= \frac{\lambda^2 g^4 c^{abcd}}{D_0 D_1 D_2 D_3} F^{ABE'} F^{E'CD} \left[(p_3 + q_3 + l) \cdot (q_1 + l - p_2) - \mu^2 \right]$$

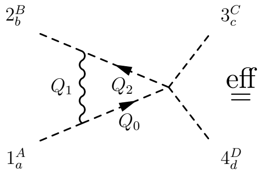


$$\overset{\text{eff}}{=} -\frac{g^4 \lambda^2 c^{abdc}}{D_0 D_1 D_2} \left(F^{ADE'} F^{E'BC} + F^{ACE'} F^{E'BD} + 2\delta^{CD} F^{A'AB'} F^{B'BA'} \right)$$

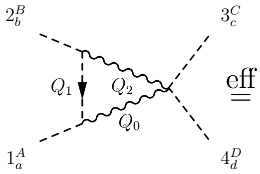
For the computation of $\langle 1_a^A\, 2_b^B\, 3_c^C\, 4_d^D \rangle\big|_{g^4}$ the following diagrams have to be evaluated. $N_g = N^2 - 1$ is the number of adjoint generators of SU(N). By permuting the external legs the remaining non-equal integrands can be obtained (six box graphs, twelve triangles and six bubbles).



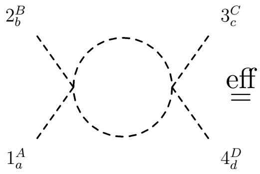
$$= \frac{g^4 c^{abcd}}{D_0 D_1 D_2 D_3} \delta^{AB} \delta^{CD} \left[(Q_1 - p_2) \cdot (p_3 + Q_3) \right] \left[(p_1 + Q_1) \cdot (Q_3 - p_4) \right]$$



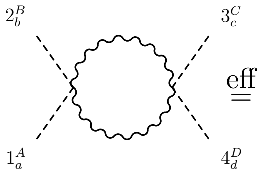
$$\stackrel{\text{eff}}{=} \frac{g^4 c^{abdc}}{D_0 D_1 D_2} \left(\delta^{AC}\delta^{BD} + \delta^{AD}\delta^{BC} - 2\delta^{AB}\delta^{CD}\right)(Q_0 - p_1)\cdot(Q_2 + p_2)$$



$$\stackrel{\text{eff}}{=} -2\frac{g^4 c^{abdc}}{D_0 D_1 D_2}\delta^{AB}\delta^{CD}\left[(q_1 + l - p_2)\cdot(p_1 + q_1 + l) - \mu^2\right]$$



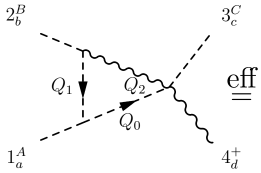
$$\stackrel{\text{eff}}{=} 2\frac{g^4}{D_0 D_2}c^{abdc}\left[2\left(N_g - 2\right)\delta^{AB}\delta^{CD} + \delta^{AD}\delta^{BC} + \delta^{AC}\delta^{BD}\right]$$



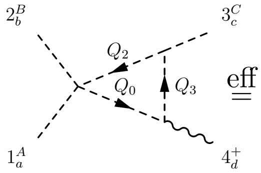
$$\stackrel{\text{eff}}{=} 16\frac{g^4}{D_0 D_2}c^{abdc}\delta^{AB}\delta^{CD}$$

Integrand for $\langle 1_a^A\, 2_b^B\, 3_c^C\, 4_d^+\rangle$

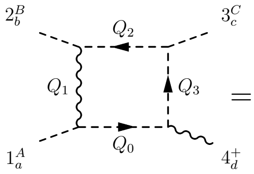
These four types of integrands can appear in general for the computation of $\langle 1_a^A\, 2_b^B\, 3_c^C\, 4_d^+\rangle\Big|_{g^4\lambda}$. However, once the double copy is performed the second type of triangle graphs vanishes. For the first type of box and triangle graphs there are five additional permutations whereas for the other box graph there are three in total.



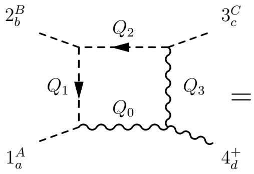
$$\stackrel{\text{eff}}{=} i\frac{g^4\lambda\, c^{abdc}}{D_0 D_1 D_2}F^{ABC}\frac{\langle r_4\, |q_1 + l - p_2|\, 4]}{\sqrt{2}\,\langle r_4 4\rangle}$$



$$\stackrel{\text{eff}}{=} 0$$



$$= \; -i\frac{g^4\lambda\, c^{abcd}}{D_0D_1D_2D_3}F^{ABC}\,(p_2+Q_2)\cdot(Q_0-p_1)\,\frac{\langle r_4\,|q_0+q_3+2l|\,4]}{\sqrt{2}\,\langle r_4 4\rangle}$$

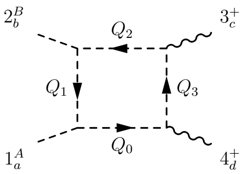


$$= \; i\frac{g^4\lambda\, c^{abcd}}{D_0D_1D_2D_3}F^{ABC}\left[(p_1+Q_1)\cdot(Q_3-p_4)\,\frac{\langle r_4\,|q_2+l-p_3|\,4]}{\sqrt{2}\,\langle r_4 4\rangle}\right.$$

$$\left.+\,(Q_2-p_3)\cdot(p_4+Q_0)\,\frac{\langle r_4\,|p_1+q_1+l|\,4]}{\sqrt{2}\,\langle r_4 4\rangle}-(Q_2-p_3)\cdot(p_1+Q_1)\,\frac{\langle r_4\,|q_0+q_3+2l|\,4]}{\sqrt{2}\,\langle r_4 4\rangle}\right]$$

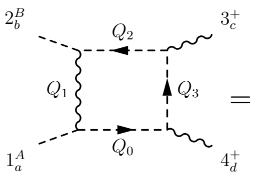
Integrands for $\langle 1_a^A\,2_b^B\,3_c^+\,4_d^+\rangle$

The gauge choice $r_3 = r_4$ reduces the amount of diagrams to compute. This gauge choice implies that only the three box graphs contribute to $\langle 1_a^A\,2_b^B\,3_c^+\,4_d^+\rangle\big|_{g^2\lambda^2}$.

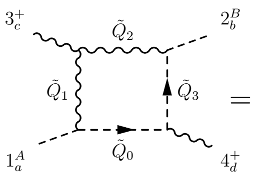


$$= \; -\frac{g^4\lambda^2\, c^{abcd}}{D_0D_1D_2D_3}F^{A'BB'}F^{B'AA'}\,\frac{\langle r_4\,|Q_2+Q_3|\,3]\,\langle r_4\,|Q_0+Q_3|\,4]}{2\,\langle r_4 3\rangle\langle r_4 4\rangle}$$

For $\langle 1_a^A\,2_b^B\,3_c^+\,4_d^+\rangle\big|_{g^4}$ this gauge choice sets all bubble graphs to zero. The remaining graphs give a contribution to $\langle 1_a^A\,2_b^B\,3_c^+\,4_d^+\rangle\big|_{g^4}$. To all graphs with box topology one other distinguished integrand is obtained in addition. The next three triangles are the only diagrams which can be drawn with this topology. The last two graphs can be drawn in four inequivalent ways.

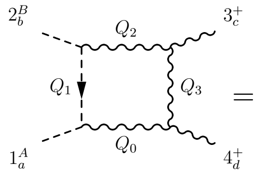


$$= \; -\frac{g^4\, c^{abcd}\,\delta^{AB}}{D_0D_1D_2D_3}\,(p_2+Q_2)\cdot(Q_0-p_1)\,\frac{\langle r_4\,|Q_2+Q_3|\,3]\,\langle r_4\,|Q_0+Q_3|\,4]}{2\,\langle r_4 3\rangle\langle r_4 4\rangle}$$



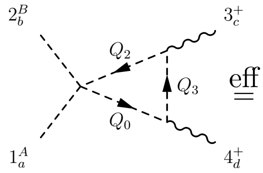
$$= \; \frac{g^4\, c^{adbc}\,\delta^{AB}}{\tilde{D}_0\tilde{D}_1\tilde{D}_2\tilde{D}_3}\frac{\left\langle r_4\left|\tilde{Q}_0+\tilde{Q}_3\right|4\right]}{2\,\langle r_4 3\rangle\langle r_4 4\rangle}\Big[\left\langle r_4\left|p_1+\tilde{Q}_3\right|3\right]\left(p_3+\tilde{Q}_2\right)\cdot\left(\tilde{Q}_0-p_2\right)$$

$$+\left\langle r_4\left|\tilde{Q}_0-p_2\right|3\right]\left(\tilde{Q}_1-p_3\right)\cdot\left(\tilde{Q}_3+p_1\right)-\left\langle r_4\left|\tilde{Q}_1+\tilde{Q}_2\right|3\right]\left(p_1+\tilde{Q}_3\right)\cdot\left(\tilde{Q}_0-p_2\right)\Big]$$

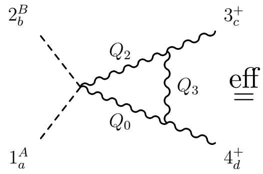


$$= \quad -\frac{g^4\, c^{abcd}\, \delta^{AB}}{2\, D_0 D_1 D_2 D_3}\Big[\frac{\langle r_4\, |Q_1 - p_2|\, 3]\, \langle r_4\, |Q_2 - p_3|\, 4] - \langle r_4\, |Q_3 + Q_2|\, 3]\, \langle r_4\, |Q_1 - p_2|\, 4]}{\langle r_4 3\rangle \langle r_4 4\rangle}$$

$$\times (Q_3 - p_4)\cdot(p_1 + Q_1) + \frac{\langle r_4\, |p_1 + Q_1|\, 4]}{\langle r_4 3\rangle\langle r_4 4\rangle}\big[\, \langle r_4\, |p_4 + Q_0|\, 3]\, (Q_1 - p_2)\cdot(Q_3 + p_3)$$
$$+ \langle r_4\, |Q_1 - p_2|\, 3]\, (Q_2 - p_3)\cdot(Q_0 + p_4) - \langle r_4\, |Q_2 + Q_3|\, 3]\, (Q_1 - p_2)\cdot(Q_0 + p_4)\, \big]$$
$$- \frac{\langle r_4\, |Q_0 + Q_3|\, 4]}{\langle r_4 3\rangle\langle r_4 4\rangle}\big[\, \langle r_4\, |Q_1 - p_2|\, 3]\, (Q_2 - p_3)\cdot(Q_1 + p_1)$$
$$+ \langle r_4\, |p_1 + Q_1|\, 3]\, (Q_1 - p_2)\cdot(Q_3 + p_3) - \langle r_4\, |Q_2 + Q_3|\, 3]\, (Q_1 - p_2)\cdot(Q_1 + p_1)\, \big]\Big]$$

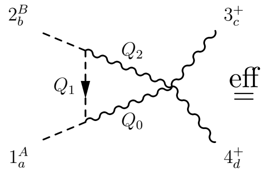


$$\overset{\text{eff}}{=} \quad -\frac{g^4\, c^{abcd}}{D_0 D_2 D_3}\delta^{AB}\,\frac{\langle r_4\, |Q_2 + Q_3|\, 3]\, \langle r_4\, |Q_0 + Q_3|\, 4]}{\langle r_4 3\rangle\langle r_4 4\rangle}\,(1 - N_g)$$

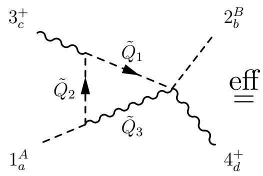


$$\overset{\text{eff}}{=} \quad \frac{g^4\, c^{abcd}}{D_0 D_2 D_3}\,\frac{\delta^{AB}}{\langle r_4 3\rangle\langle r_4 4\rangle}\Big[2\,\langle r_4\, |Q_2 + Q_3|\, 3]\, \langle r_4\, |Q_0 + Q_3|\, 4]$$

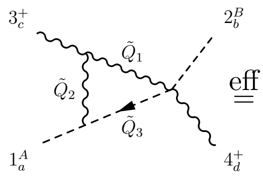
$$+ \langle r_4\, |Q_0 + p_4|\, 3]\, \langle r_4\, |p_3 + Q_3|\, 4] + \langle r_4\, |Q_3 - p_4|\, 3]\, \langle r_4\, |Q_2 - p_3|\, 4]\,\Big]$$



$$\overset{\text{eff}}{=} \quad -\frac{g^4\, c^{abcd}}{D_0 D_1 D_2}\delta^{AB}\,\left[\frac{\langle r_4\, |Q_1 - p_2|\, 3]\, \langle r_4\, |p_1 + Q_1|\, 4]}{2\,\langle r_4 3\rangle\langle r_4 4\rangle} + \frac{\langle r_4\, |p_1 + Q_1|\, 3]\, \langle r_4\, |Q_1 - p_2|\, 4]}{2\,\langle r_4 3\rangle\langle r_4 4\rangle}\right]$$



$$\overset{\text{eff}}{=} \quad -\frac{g^4\, c^{adbc}}{\tilde{D}_1\tilde{D}_2\tilde{D}_3}\delta^{AB}\,\frac{\left\langle r_4\, \left|\tilde{Q}_1 + \tilde{Q}_2\right|\, 3\right]\, \left\langle r_4\, \left|p_1 - \tilde{Q}_2\right|\, 4\right]}{2\,\langle r_4 3\rangle\langle r_4 4\rangle}$$



$$\overset{\text{eff}}{=} \quad -\frac{g^4\, c^{adbc}}{\tilde{D}_1\tilde{D}_2\tilde{D}_3}\delta^{AB}$$

$$\times\Big[\frac{\left\langle r_4\, \left|\tilde{Q}_3 + p_1\right|\, 3\right]\, \left\langle r_4\, \left|p_3 + \tilde{Q}_2\right|\, 4\right]}{2\langle r_4 3\rangle\langle r_4 4\rangle} - \frac{\left\langle r_4\, \left|\tilde{Q}_1 + \tilde{Q}_2\right|\, 3\right]\, \left\langle r_4\, \left|p_1 + \tilde{Q}_3\right|\, 4\right]}{2\langle r_4 3\rangle\langle r_4 4\rangle}\Big]$$

Appendix B

Mellin Amplitudes

B.1 Conformal Algebra

In this appendix it is shown that conformal primary operators transform as

$$\mathcal{O}^a(x) \to \mathcal{O}'^a(x') = \Omega^{-\Delta}(x) D\left(\Lambda^\mu_{\ \nu}(x)\right)^a_{\ b} \mathcal{O}^b(x), \tag{B.1}$$

where $\Omega(x)$ is the scale factor from (3.6) and $D\left(\Lambda^\mu_{\ \nu}(x)\right)^a_{\ b}$ is the $\mathrm{SO}(1, d-1)$ representation of the operator $\mathcal{O}^a$. For example a scalar operator transforms in the trivial representation $D\left(\Lambda^\mu_{\ \nu}(x)\right) = 1$, a vector operator transform in the vector representation $D\left(\Lambda^\mu_{\ \nu}(x)\right)^\mu_{\ \nu} = \Lambda^\mu_{\ \nu}(x)$ etc. In the following the $\mathrm{SO}(1, d-1)$ representation indices $a, b, \ldots$ are not written out explicitly.

To derive (B.1) from the action of the conformal generators on the operators, the Hausdorff formula is useful

$$e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}\left[[B, A], A\right] + \frac{1}{3!}\left[\left[[B, A], A\right], A\right] + \ldots \tag{B.2}$$

where A and B are two operators. The Hausdorff formula permits to calculate the action of a generator on a operator at any position x if the action at $x = 0$ is known.

Action of translations
It follows from $[P_\mu, \mathcal{O}(0)] = -i\partial_\mu \mathcal{O}(0)$ that the translated generator P_μ is given by the same formula, since

$$e^{ix\cdot P} P_\mu e^{-ix\cdot P} = P_\mu. \tag{B.3}$$

Action of Lorentz transformations
The Lorentz generator acts as $[M^\mu_{\ \nu}, \mathcal{O}(0)] = S^\mu_{\ \nu}\mathcal{O}(0)$, where $S^\mu_{\ \nu}$ is a matrix-valued representation of it. Translating this generator to any position x yields

$$e^{ix\cdot P} M_{\mu\nu} e^{-ix\cdot P} = M_{\mu\nu} - ix^\rho \left[M_{\mu\nu}, P_\rho\right] = M_{\mu\nu} - \left(x_\mu P_\nu - x_\nu P_\mu\right). \tag{B.4}$$

Action of dilatation

In a scale invariant theory it is natural to chose the primary operator $\mathcal{O}(0)$ to be an eigenfunction of the dilatation operator, i.e. $[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0)$.[1] Δ is called the scaling dimension of the operator.

$$e^{ix\cdot P} D e^{-ix\cdot P} = D - ix^\rho [D, P_\rho] = D + x^\rho P_\rho. \tag{B.5}$$

Action of special conformal transformation
The matrix representation of the dilatation operator is proportional to the identity. Hence, all other matrix representations of the generators of the conformal algebra have to commute with it. In particular, this implies that $[K_\mu, \mathcal{O}(0)] = 0$, since $[D, K_\mu] = -iK_\mu$.

$$\begin{aligned} e^{ix\cdot P} K_\mu e^{-ix\cdot P} &= K_\mu - ix^\nu [K_\mu, P_\nu] - \frac{1}{2} x^\nu x^\rho [[K_\mu, P_\nu], P_\rho] \\ &= K_\mu + 2x_\mu D - 2x^\nu M_{\mu\nu} + 2x_\mu x^\nu P_\nu - x^2 P_\mu. \end{aligned} \tag{B.6}$$

This analysis yields the final transformation formulae for an operator under infinitesimal conformal transformations

$$\begin{aligned} [P_\mu, \mathcal{O}(x)] &= -i\partial_\mu \mathcal{O}(x), \\ [M_{\mu\nu}, \mathcal{O}(x)] &= (S_{\mu\nu} + i(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathbb{1})\mathcal{O}(x), \\ [D, \mathcal{O}(x)] &= -i(\Delta + x^\rho\partial_\rho\mathbb{1})\mathcal{O}(x), \\ [K_\mu, \mathcal{O}(x)] &= \left(-i2x_\mu\Delta\mathbb{1} - 2x^\nu S_{\mu\nu} - i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)\mathbb{1}\right)\mathcal{O}(x). \end{aligned} \tag{B.7}$$

The finite conformal transformations are related to the infinitesimal by the formulae

$$x'^\mu = x^\mu + c_a \tfrac{\partial\epsilon^\mu}{\partial c_a}, \quad \mathcal{O}'(x') = \mathcal{O}(x) + c_a \tfrac{\partial\mathcal{F}}{\partial c_a} \tag{B.8}$$

with $c_a = \{i\lambda, ia^\mu, ib^\mu, \frac{i}{2}w^{\mu\nu}\}$. This transformation changes the coordinate from x to x' as well as the operator from $\mathcal{O}(x)$ to $\mathcal{O}'(x') = \mathcal{F}(\mathcal{O})(x)$. Using (B.7) and (B.8) the transformation for the operator reads explicitly

$$\begin{aligned} \mathcal{O}'(x') &= \left(\mathbb{1} + \frac{i}{2}w^{\mu\nu}S_{\mu\nu} + \lambda\Delta\mathbb{1} + b^\mu(2x_\mu\Delta\mathbb{1} - i2x^\nu S_{\mu\nu})\right)\mathcal{O}(x) \\ &= \left(\mathbb{1} + \frac{\Delta}{d}\partial_\mu\epsilon^\mu\mathbb{1} - \frac{i}{2}\partial^\mu\epsilon^\nu S_{\mu\nu}\right)\mathcal{O}(x) \\ &= \left(1 + \frac{\Delta}{d}\partial_\mu\epsilon^\mu\right)\left(\mathbb{1} - \frac{i}{2}\partial^\mu\epsilon^\nu S_{\mu\nu}\right)\mathcal{O}(x) \\ &\text{with}\quad \partial_\mu\epsilon^\mu = d(\lambda + 2x^\mu b_\mu), \quad\text{and}\quad \partial_\mu\epsilon_\nu S^{\mu\nu} = i\left(\frac{i}{2}w^{\mu\nu} - i2b^\mu x^\nu\right)S_{\mu\nu}, \end{aligned} \tag{B.9}$$

in leading order of the parameters c_a. Here $\epsilon = \epsilon^\mu\partial_\mu$ is the conformal Killing vector (3.8).

On the other hand the metric transforms under a finite conformal transformation as

$$\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} = \Omega^2(x)\eta_{\mu\nu}$$

[1] Alternatively, one can also demand that $\mathcal{O}(0)$ transforms in an irreducible representation of the Lorentz group. Since the dilatation operator commutes with Lorentz operators it has to be proportional to the identity matrix by Schur's lemma.

according to (3.3) and (3.6). This finite transformation is established by the coordinate transformation

$$\frac{\partial x^\nu}{\partial x'^\mu} = \Omega(x)\Lambda^\nu{}_\mu(x) \quad \text{with} \quad \Lambda^T\eta\Lambda = \eta.$$

The infinitesimal version is obtained by considering the infinitesimal change $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$ which yields

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\epsilon^\nu = \left(1 - \frac{1}{d}\partial_\mu\epsilon^\mu\right)\left(\delta^\nu_\mu - \frac{1}{2}\left(\partial_\mu\epsilon^\nu - \partial^\nu\epsilon_\mu\right)\right). \tag{B.10}$$

In the last step the conformal Killing equation (3.7) has been used. This implies

$$\Omega(x) = 1 - \frac{1}{d}\partial_\mu\epsilon^\mu, \quad \text{and} \quad \Lambda^\nu{}_\mu(x) = \delta^\nu_\mu - \frac{1}{2}\left(\partial_\mu\epsilon^\nu - \partial^\nu\epsilon_\mu\right).$$

Hence, a conformal transformation is locally a combination of a scale transformation and a rotation. Comparing (B.9) and (B.10) yields the desired expression (B.1) [29, 124].

B.2 Vector and Spinor Representation of the Lorentz Algebra in Three Dimensions

The fundamental (vector) representation of the Lorentz generator reads

$$(M^{\mu\nu})^\rho{}_\sigma = i\left(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma\right). \tag{B.11}$$

It can be easily checked that this construction obeys the commutation relations of (3.10). In three dimensions the generators are given by one generator $M^{12} = J$ of rotation in the x_1-x_2-plane and two boosts $M^{0i} = M^0_i = K_i$. Explicitly they read

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{B.12}$$

The three-dimensional spinor representation of the Lorentz group is given by $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1) \cong \mathrm{Sp}(2,\mathbb{R})$ which is the double cover of SO(1,2). The explicit mapping between the spin group (which is labelled by the indices $\alpha, \beta, \ldots$) and the Lorentz group (which is labelled by the indices $\mu, \nu, \ldots$) is given by the γ-matrices

$$\left(\gamma^0\right)^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \left(\gamma^1\right)^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \left(\gamma^2\right)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.13}$$

which obey the Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \eta^{\mu\nu}\mathbb{1}_2 \quad \text{with} \quad \eta^{\mu\nu} = \mathrm{diag}(-1,1,1).$$

The spin representation of the Lorentz algebra $\mathfrak{sp}(2,\mathbb{R}) = \{M \in \mathrm{GL}(2,\mathbb{R}) | M^T\omega + \omega M = 0\}$ can be constructed from (B.13) and the generators are of the form

$$(S_2^{\mu\nu})^\alpha{}_\beta = \frac{i}{4}\left([\gamma^\mu, \gamma^\nu]\right)^\alpha{}_\beta,$$

which obey the Lorentz algebra of (3.10). The indices of the generators are lowered and raised by the symplectic form ω given in (3.35), e.g. $\gamma^{\mu}_{\alpha\beta} = \omega_{\alpha\sigma} \left(\gamma^{\mu}\right)^{\sigma}_{\ \beta}$ and $\left(\gamma^{\mu}\right)^{\alpha\beta} = \left(\gamma^{\mu}\right)^{\alpha}_{\ \sigma} \omega^{\sigma\beta}$. Hence, the three generators $j = M^{12}$, $k_i = M^{0}_{\ i}$ are

$$(j)^{\alpha}_{\ \beta} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (k_1)^{\alpha}_{\ \beta} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (k_2)^{\alpha}_{\ \beta} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{B.14}$$

The anti-fundamental generators are obtained from (B.14) by acting with the symplectic form: $\bar{j} = \omega j \omega^{-1} = j$ and $\bar{k}_i = \omega k_i \omega^{-1} = -k_i$.

The algebra in five dimensions is constructed analogously. The double cover of SO(3,2) is $\mathrm{Sp}(4,\mathbb{R})$ and the corresponding generators transform under $\mathfrak{sp}(4,\mathbb{R}) = \{M \in \mathrm{GL}(4,\mathbb{R}) | M^T\Omega + \Omega M = 0\}$ where the symplectic form Ω_{IJ} is given in (3.35). The spin representation is explicitly constructed by

$$\left(S_4^{MN}\right)^{I}_{\ J} = \frac{i}{4}\left(\left[\Gamma^M, \Gamma^N\right]\right)^{I}_{\ J},$$

where the Γ-matrices in five dimensions $\left(\Gamma^N\right)^{I}_{\ J}$ are realized by

$$\begin{aligned} \Gamma^0 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ \Gamma^3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.15}$$

They obey the algebra

$$\Gamma^M\Gamma^N + \Gamma^N\Gamma^M = \eta^{NM}\mathbb{1}_4 \quad \text{with} \quad \eta^{NM} = \mathrm{diag}(-1,1,1,-1).$$

For more information see [199].

B.3 Perturbative Calculation of Mellin Ampitudes

The existence of a Mellin represenation for perturbative calculations follows from Symanzik's star formula which shall be derived for the scalar case in this appendix [101]. The star formula allows to calculate the contact terms of the Mellin space Feynman rules in the weak coupling regime for scalar operators. These Feynman rules have been derived in [104].

Weak Coupling Regime

Consider the n-point correlator $\langle \phi(x_1) \dots \phi(x_n) \rangle := I$. The Mellin amplitude can be determined by the following algorithm:

1. Represent the correlation function in position space. The integration variable is labelled by u.

2. Perform a Schwinger parametrization for each propagator:

$$\frac{\Gamma(\Delta)}{\left((x-y)^2\right)^{\Delta}} = \int_0^\infty \frac{d\alpha}{\alpha}\alpha^{\Delta} e^{-\alpha(x-y)^2}. \tag{B.16}$$

3. Perform the integration over the vertex u with the Gaussian integral

$$\int_{\mathbb{R}} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}. \tag{B.17}$$

4. Insert a partition of unity $1 = \int_0^\infty dv\, \delta(v - \sum_i \alpha_i)$ and rescale your Schwinger parameters $\alpha_i \to \sqrt{v}\alpha_i$.

5. Use the inverse Mellin transform for each function of the type

$$e^{-x} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\Gamma(s)x^{-s}. \tag{B.18}$$

For this calculation each external leg is normalized by $\Gamma(\Delta_i)$ and the vertex by $\frac{1}{2}\pi^{-h}$ with $h = \frac{d}{2}$ to simplify the final result. Further, the vertex is proportional to the coupling constant g. Hence,

$$I = g\int_{\mathbb{R}^d} \mathcal{D}u \prod_{i=1}^{n} \frac{\Gamma\left(\Delta_i\right)}{(x_i-u)^{2\Delta_i}} \quad \text{with} \quad \mathcal{D}u = \frac{1}{2}\frac{d^d u}{\pi^h}.$$

Conformality of the integral demands $\sum_i \Delta_i = d$. Using the algorithm stated above the Mellin representation can be obtained by the following steps

$$\begin{aligned} I &= g\int_{\mathbb{R}^d} \mathcal{D}u \prod_{i=1}^{n} \int_0^\infty \frac{d\alpha_i}{\alpha_i}\alpha_i^{\Delta_i} e^{-\alpha_i(x_i-u)^2} \\ &= g\prod_{i=1}^{n} \int_0^\infty \frac{d\alpha_i}{\alpha_i}\alpha_i^{\Delta_i} e^{-\sum_{i<j}\alpha_i\alpha_j x_{ij}^2} \quad \text{with} \quad x_{ij} = x_i - x_j \\ &= g\prod_{i=1}^{n} \int_0^\infty \frac{d\alpha_i}{\alpha_i}\alpha_i^{\Delta_i} \prod_{i<j}\int_{c_{ij}-i\infty}^{c_{ij}+i\infty} \frac{ds_{ij}}{2\pi i}\Gamma(s_{ij})\left(\alpha_i\alpha_j x_{ij}^2\right)^{-s_{ij}} \\ &= g\prod_{i<j}\int_{c_{ij}-i\infty}^{c_{ij}+i\infty} \frac{ds_{ij}}{2\pi i}\Gamma(s_{ij})\left(x_{ij}^2\right)^{-s_{ij}} \prod_{i=1}^{n}\hat{\delta}\left(\Delta_i - \sum_{j\neq i} s_{ij}\right). \end{aligned} \tag{B.19}$$

In the last step $\int_0^\infty \frac{d\alpha}{\alpha}\alpha^a = \hat{\delta}(a)$ has been used with $\hat{\delta}(a) = 2\pi i\,\delta(a)$. This holds if the real part of the exponent a along the contour vanishes [104]. Comparing this equation with (3.54) shows that the Mellin amplitude is constant, i.e. $\mathcal{M}_c(\{s_{ij}\}) = g$.

However, Symanzik wrote the solution of (B.19) in the general form

$$2\prod_{i=1}^{n}\int_0^\infty \frac{d\alpha_i}{\alpha_i}\alpha_i^{\Delta_i} e^{-\sum_{i<j}\alpha_i\alpha_j x_{ij}^2} = \frac{1}{(2\pi i)^{\frac{n(n-3)}{2}}}\int_{-i\infty}^{i\infty} ds_1 \dots ds_{\frac{n(n-3)}{2}} \prod_{i<j}\frac{\Gamma(s_{ij})}{\left(x_{ij}^2\right)^{s_{ij}}}, \tag{B.20}$$

where the new integration variables s_i are given by

$$s_{ij} = s_{ij}^0 + \sum_{k=1}^{\frac{n(n-3)}{2}} c_{ij,k}\, s_k.$$

s_{ij}^0 is a particular solution with positive real parts which satisfies $\sum_{i\neq j} s_{ij}^0 = \Delta_j$ and the coefficients $c_{ij,k} = c_{ji,k} \in \mathbb{R}$ obey

$$c_{ii,k} = 0 \quad \text{and} \quad \sum_{j\neq i}^{n} c_{ij,k} = 0.$$

Not all $c_{ij,k}$ can be independent, but only $\left(\frac{n(n-3)}{2}\right)^2$ of these coefficients are independent. In [101] the independent coefficients $c_{ij,k}$ range from $2 \leq i < j \leq n$ and $1 \leq k \leq n(n-3)/2$ without $c_{23,k}$. To obtain a unit Jacobian, it has to be demanded that $|\det c_{ij,k}| = 1$. Note that (B.19) and (B.20) differ by an overall factor of 2. This factor arises from the Jacobian of one of the δ-distribution, the remaining δ-distributions have unit Jacobian.

Strong Coupling Regime

A contact Witten diagram can be evaluated in a similar way. Let the bulk coordinate (in lightcone coordinates) be parametrized in Poincaré coordinates $Z = (Z^+, Z^-, Z^\mu) = \frac{1}{z}(1, z^2 + y^2, y^\mu)$ and the boundary coordinate be given by $X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu)$. The scalar product of these coordinates reads $-2X \cdot Z = \frac{1}{z}(z^2 + (x-y)^2)$. The bulk to boundary propagator can be written as

$$K_\Delta(X, Z) = \frac{C_\Delta}{\Gamma(\Delta)} \int_0^\infty \frac{dt}{t} t^\Delta e^{-2tZ\cdot X} \quad \text{with} \quad C_\Delta = \frac{\Gamma(\Delta)}{2\pi^h \Gamma(\Delta - h + 1)} \quad \text{and} \quad h = \frac{d}{2}.$$

Consider $Q = \sum_{i=1}^n t_i X_i$ and $-2Q\cdot Z = z^{-1} \sum_{i=1}^n t_i \left(z^2 + (x_i - y)^2\right)$ and the following integral in AdS evaluated in Poincaré coordinates

$$\begin{aligned}\int_{\text{AdS}} e^{2Q\cdot Z} &= \int_0^\infty \frac{dz}{z} z^{-d} \int_{\mathbb{R}^d} d^d y e^{-\sum_i t_i z} e^{-\frac{1}{z}\sum_i t_i (x_i - y)^2} \\ &= \int_0^\infty \frac{dz}{z} e^{-\sum_i t_i z} \left(\frac{\pi}{z \sum_i t_i}\right)^h e^{-\left(z\sum_i t_i\right)^{-1} \sum_{i<j} t_i t_j x_{ij}^2} \\ &= \pi^h \int_0^\infty \frac{dz}{z} z^{-h} e^{-z} e^{-\frac{1}{z}\sum_{i<j} t_i t_j x_{ij}^2} = \pi^h \int_0^\infty \frac{dz}{z} z^{-h} e^{-z} e^{-\frac{1}{z}Q^2},\end{aligned} \tag{B.21}$$

where the variables have been rescaled by $z\sum_i t_i \to z$. Using (B.21) the Mellin amplitude of the contact Witten diagram $A = \langle \phi_1(x_1) \ldots \phi_n(x_n)\rangle$ of n-scalar fields can be determined.

$$\begin{aligned}A &= g \int_{\text{AdS}} dZ \prod_{i=1}^n K_{\Delta_i}(Z, X_i) = g \prod_{i=1}^n \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} \int_{\text{AdS}} e^{2Q\cdot Z} \\ &= g\pi^h \prod_{i=1}^n \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} \int_0^\infty \frac{dz}{z} z^{-h} e^{-z} e^{-\frac{1}{z}Q^2} \\ &= g\pi^h \prod_{i=1}^n \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} e^{-Q^2} \int_0^\infty \frac{dz}{z} z^{-h+\frac{1}{2}\sum_i \Delta_i} e^{-z} \\ &= g\pi^h \Gamma\!\left(\frac{1}{2}\sum_{i=1}^n \Delta_i - h\right) \prod_{i=1}^n \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} e^{-\sum_{i<j} t_i t_j x_{ij}^2} \\ &= 2g\pi^h \Gamma\!\left(\frac{1}{2}\sum_{i=1}^n \Delta_i - h\right) \prod_{i=1}^n \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \prod_{i<j}^n \int_{c_{ij}-i\infty}^{c_{ij}+i\infty} \frac{ds_{ij}}{2\pi i} \frac{\Gamma(s_{ij})}{\left(x_{ij}^2\right)^{s_{ij}}} \prod_{i=1}^n \hat{\delta}\left(\Delta_i - \sum_{j\neq i} s_{ij}\right).\end{aligned}$$

In the third line the rescaling $t_i \to t_i\sqrt{z}$ has been performed and afterwards the integral over dz has been written as a Γ-function. In the last line (B.19) has been used.

B.4 More Results on Mellin Amplitudes

B.4.1 Relation between the Reduced Mellin Amplitude and Mellin Amplitude of the Four-Fermion Correlator

The components of the Mellin amplitude for the four-fermion conformal correlator (3.65) read explicitly as follows

$$\mathcal{M}_1 = \bar{M}_1 \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$
$$\mathcal{M}_2 = \bar{M}_2 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_3 = \bar{M}_3 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_4 = \bar{M}_4 \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$
$$\mathcal{M}_5 = \bar{M}_5 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_6 = \bar{M}_6 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_7 = \bar{M}_7 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_8 = \bar{M}_8 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_9 = \bar{M}_9 \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_{10} = \bar{M}_{10} \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_{11} = \bar{M}_{11} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$
$$\mathcal{M}_{12} = \bar{M}_{12} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$
$$\mathcal{M}_{13} = \bar{M}_{13} \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_{14} = \bar{M}_{14} \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$
$$\mathcal{M}_{15} = \bar{M}_{15} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$
$$\mathcal{M}_{16} = \bar{M}_{16} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+1\right)\right]^{-1}.$$

B.4.2 u-Channel Poles of the Mellin Amplitude of the Fermion-Scalar Four-Point Correlator

The u-channel poles of $\langle\Psi_1\Psi_2\Phi_3\Phi_4\rangle$ are summarized in table B.1.

Component of M.A.	Location of Poles	Residues $\sim$
$\mathcal{M}_1$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^+_{\psi_1\phi_4\psi_l}\lambda^+_{\psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^-_{\psi_1\phi_4\psi_l}\lambda^-_{\psi_l\phi_3\psi_2}$
$\mathcal{M}_2$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^+_{\psi_1\phi_4\psi_l}\lambda^+_{\psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^-_{\psi_1\phi_4\psi_l}\lambda^-_{\psi_l\phi_3\psi_2}$
$\mathcal{M}_3$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^+_{\psi_1\phi_4\psi_l}\lambda^-_{\psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^-_{\psi_1\phi_4\psi_l}\lambda^+_{\psi_l\phi_3\psi_2}$
$\mathcal{M}_4$	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^+_{\psi_1\phi_4\psi_l}\lambda^-_{\psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^-_{\psi_1\phi_4\psi_l}\lambda^+_{\psi_l\phi_3\psi_2}$

Table B.1: u-channel poles of fermion-scalar four-point correlator.

B.4.3 t- and u-Channel Poles of the Mellin Amplitude of the Fermionic Four-Point Correlator

The t-channel poles of $\langle\Psi_1\Psi_2\Psi_3\Psi_4\rangle$ are summarized in table B.2.

The u-channel poles of $\langle\Psi_1\Psi_2\Psi_3\Psi_4\rangle$ are summarized in table B.3. In the case of a scalar exchange $l=0$, all OPE coefficients apart from λ^1, λ^3 should be taken to zero.

Component of M.A.	**Location of Poles**	**Residues $\sim$**
$\mathcal{M}_1$, $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_5$	$t=\tau-1+2k$	$\lambda^1_{\psi_1\psi_3\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^1_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$
	$t=\tau+2k$	$\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_2$, $\mathcal{M}_6$, $\mathcal{M}_7$	$t=\tau+2k$	$\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$
	$t=\tau+1+2k$	$\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_8$	$t=\tau+1+2k$	$\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$
	$t=\tau+2+2k$	$\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_9$, $\mathcal{M}_{11}$, $\mathcal{M}_{13}$, $\mathcal{M}_{15}$	$t=\tau+2k$	$\lambda^1_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^1_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$
	$t=\tau+1+2k$	$\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_{10}$, $\mathcal{M}_{12}$, $\mathcal{M}_{14}$, $\mathcal{M}_{16}$	$t=\tau+2k$	$\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^4_{\psi_1\psi_3\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_2\psi_4}$
	$t=\tau+1+2k$	$\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^2_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_2\psi_4}$

Table B.2: t-channel poles of the four-fermion correlator.

Component of M.A.	**Location of Poles**	**Residues** $\sim$
$\mathcal{M}_1$	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_2$, $\mathcal{M}_4$	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_3$, $\mathcal{M}_5$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_6$, $\mathcal{M}_7$	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_8$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_9$, $\mathcal{M}_{12}$, $\mathcal{M}_{13}$, $\mathcal{M}_{16}$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_{10}$, $\mathcal{M}_{11}$, $\mathcal{M}_{14}$, $\mathcal{M}_{15}$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$

Table B.3: u-channel poles of the four-fermion correlator.

B.5 Spinor Exchange in AdS

In this section the spinor exchange diagram is calculated. Note that in this calculation the two scalars are switched, i.e. the quantity $A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3}$ is computed.

Plugging the perturbative solution (3.88) into the action (3.87) one obtains the on-shell action

$$S_{\bar{\psi}\phi\psi\phi} = -2\lambda^2 G \prod_{i=1}^{4} \int_{-\infty}^{\infty} d^d\mathbf{x}_i \; \bar{\psi}_0^+ (\mathbf{x}_1)\, \phi_0 (\mathbf{x}_4)\, A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\, \psi_0^- (\mathbf{x}_2)\, \phi_0 (\mathbf{x}_3)\,.$$

The diagram $A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3}$ is given by

$$A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3} = -\int d^{d+1}z\sqrt{g(z)}d^{d+1}w\sqrt{g(w)}K_{\Delta_4}(z,\mathbf{x}_4)\bar{\Sigma}_{\Delta_1}(z,\mathbf{x}_1)S(z,w)\Sigma_{\Delta_2}(w,\mathbf{x}_2)K_{\Delta_3}(w,\mathbf{x}_3).$$

To find the Mellin-Barnes representation of this diagram, $A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3}$ is rewritten as a scalar exchange Witten diagram [218]. The first step is to use the conformal symmetry on the boundary to translate all coordinates by $\mathbf{x}_2$ such that the new coordinates on the boundary are given by $\mathbf{y}_i = \mathbf{x}_i - \mathbf{x}_2$ for $i \neq 2$. Afterwards these coordinates are inverted $\mathbf{y}'_i = \mathbf{y}_i / |\mathbf{y}_i|^2$. Since the bulk measure is invariant under inversion and due to the definite transformation behaviour of the propagators, the amplitude can be rewritten as

$$A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3} = \frac{\not{\mathbf{y}}_1}{|\mathbf{y}_1|^{2\Delta_1+1}|\mathbf{y}_3|^{2\Delta_3}|\mathbf{y}_4|^{2\Delta_4}}\left[-\not{\mathbf{y}}'_{14}\not{\partial}'_4 + \left(\Delta_1 + \frac{1}{2} + \Delta_4 + \Delta_+ - d\right)\right] I(\mathbf{y}'_1, \mathbf{y}'_3, \mathbf{y}'_4) \tag{B.22}$$

with $\Delta_+ = d/2 + m + 1/2$ and m being the mass of the exchanged fermion. In [218] the explicit expression for I is given by

$$\begin{aligned} I(\mathbf{y_1}', \mathbf{y}'_3, \mathbf{y}'_4) = &\int d^{d+1}z\sqrt{g(z)}\, d^{d+1}w\sqrt{g(w)}K_{\Delta_4}(z,\mathbf{y}'_4)K_{\Delta_1+\frac{1}{2}}(z,\mathbf{y}'_1)G_{\Delta_+}(z,w) \\ &\times K_{\Delta_2+\frac{1}{2}}(w',0)\, K_{\Delta_3}(w,\mathbf{y}'_3). \end{aligned}$$

Further, the AdS measure is invariant under inversion and the scalar bulk-to-boundary propagator transforms covariantly under inversion: $K_\Delta(z',\mathbf{x}') = |\mathbf{x}|^{2\Delta} K_\Delta(z,\mathbf{x})$. In addition, the scalar bulk-to-bulk propagator only depends on the chordal distance $u = \frac{(z-w)^2}{z_0^2 w_0^2}$ and therefore is invariant under inversion $G_\Delta(z',w') = G_\Delta(z,w)$. These properties allow to rewrite I as a scalar exchange diagram with four external scalars:

$$\begin{aligned} I = &|\mathbf{y}_1|^{2\Delta_1+1}|\mathbf{y}_3|^{2\Delta_3}|\mathbf{y}_4|^{2\Delta_4}\int d^{d+1}z\sqrt{g(z)}d^{d+1}w\sqrt{g(w)}K_{\Delta_4}(z,\mathbf{y}_4)K_{\Delta_1+\frac{1}{2}}(z,\mathbf{y}_1) \\ &\times G_{\Delta_+}(z,w)K_{\Delta_2+\frac{1}{2}}(w,0)\, K_{\Delta_3}(w,\mathbf{y}_3) \\ = &|\mathbf{y}_1|^{2\Delta_1+1}|\mathbf{y}_3|^{2\Delta_3}|\mathbf{y}_4|^{2\Delta_4}\int d^{d+1}z\sqrt{g(z)}d^{d+1}w\sqrt{g(w)}K_{\Delta_4}(z,\mathbf{x}_4)K_{\Delta_1+\frac{1}{2}}(z,\mathbf{x}_1) \\ &\times G_{\Delta_+}(z,w)K_{\Delta_2+\frac{1}{2}}(w,\mathbf{x}_2)\, K_{\Delta_3}(w,\mathbf{x}_3). \end{aligned}$$

In the last step the bulk coordinates $\mathbf{z} \to \mathbf{z} - \mathbf{x}_2$ and $\mathbf{w} \to \mathbf{w} - \mathbf{x}_2$ have been translated. The Mellin-Barnes representation of this expression is known [106]. Further, I depends on the unprimed coordinates only. Thus one can define a new quantity $\tilde{I}\left(\mathbf{y_1},\mathbf{y}_2,\mathbf{y}_4\right) := I\left(\mathbf{y_1}',\mathbf{y}_2',\mathbf{y}_4'\right)$ such that

$$\tilde{I} = |\mathbf{y}_1|^{2\Delta_1+1}|\mathbf{y}_3|^{2\Delta_3}|\mathbf{y}_4|^{2\Delta_4}\prod_{1\leq i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,\Gamma\left(s_{il}\right)\mathcal{M}(s_{il})\frac{1}{|\mathbf{y}_{13}|^{2s_{13}}|\mathbf{y}_{14}|^{2s_{14}}|\mathbf{y}_{34}|^{2s_{34}}}$$
$$\times\frac{1}{|\mathbf{y}_1|^{2s_{12}}|\mathbf{y}_3|^{2s_{23}}|\mathbf{y}|_4^{2s_{24}}}\prod_{i=1}^{4}\hat{\delta}\left(\Delta_i+\frac{1}{2}\left(\delta_{1i}+\delta_{2i}\right)-\sum_{k=1,k\neq i}^{4}s_{ik}\right)$$

is obtained. To evaluate (B.22) note that the tensor structure is generated by the derivative and $\not{\mathbf{y}}_{14}$. After inverting these

$$\not{\mathbf{y}}'_{14} = \frac{\not{\mathbf{y}}_1}{|\mathbf{y}_1|^2}-\frac{\not{\mathbf{y}}_4}{|\mathbf{y}_4|^2}\qquad\text{and}$$
$$\not{\partial}_{\mathbf{y}'_4} = \gamma^\mu\frac{\partial}{\partial\mathbf{y}'_{4,\mu}}\frac{\mathbf{y}_4'^{\nu}}{|\mathbf{y}'_4|^2}\frac{\partial}{\partial\mathbf{y}_4^\nu} = \mathbf{y}'_{4\mu}\frac{\partial}{\partial\mathbf{y}_4^\nu} = |\mathbf{y}_4|^2\not{\partial}_{\mathbf{y}_4}-2\not{\mathbf{y}}_4\mathbf{y}_4\cdot\frac{\partial}{\partial\mathbf{y}_4},$$

the following three types of terms are obtained:

$$\not{\mathbf{y}}_1\left(\frac{\not{\mathbf{y}}_1}{|\mathbf{y}_1|^2}-\frac{\not{\mathbf{y}}_4}{|\mathbf{y}_4|^2}\right)\left(|\mathbf{y}_4|^2\gamma^\mu-2\not{\mathbf{y}}_4\mathbf{y}_4^\mu\right)\frac{\mathbf{y}_{4\mu}}{|\mathbf{y}_4|^2} = \not{\mathbf{y}}_{14}\quad\text{with coefficient}\quad 2\Delta_4-2s_{24},$$
$$\not{\mathbf{y}}_1\left(\frac{\not{\mathbf{y}}_1}{|\mathbf{y}_1|^2}-\frac{\not{\mathbf{y}}_4}{|\mathbf{y}_4|^2}\right)\left(|\mathbf{y}_4|^2\gamma^\mu-2\not{\mathbf{y}}_4\mathbf{y}_4^\mu\right)\frac{-\mathbf{y}_{14\mu}}{|\mathbf{y}_{14}|^2} = \not{\mathbf{y}}_4\quad\text{with coefficient}\quad -2s_{14},$$
$$\not{\mathbf{y}}_1\left(\frac{\not{\mathbf{y}}_1}{|\mathbf{y}_1|^2}-\frac{\not{\mathbf{y}}_4}{|\mathbf{y}_4|^2}\right)\left(|\mathbf{y}_4|^2\gamma^\mu-2\not{\mathbf{y}}_4\mathbf{y}_4^\mu\right)\frac{\mathbf{y}_{43\mu}}{|\mathbf{y}_{34}|^2} = \frac{\not{\mathbf{y}}_{14}\not{\mathbf{y}}_{43}\not{\mathbf{y}}_4}{|\mathbf{y}_{34}|^2}\quad\text{with coefficient}\quad -2s_{34}.$$

This gives

$$A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3} = \prod_{1\leq i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\frac{\Gamma\left(s_{il}\right)}{|\mathbf{x}_{il}|^{2s_{il}}}\mathcal{M}(s_{il})\prod_{i=1}^{4}\hat{\delta}\left(\Delta_i+\frac{1}{2}\left(\delta_{1i}+\delta_{2i}\right)-\sum_{k=1,k\neq i}^{4}s_{ik}\right)$$
$$\times\left(\not{\mathbf{x}}_{12}\left(\Delta_1+\frac{1}{2}+\Delta_4+\Delta_+-d\right)+2\not{\mathbf{x}}_{14}\left(s_{24}-\Delta_4\right)+2\not{\mathbf{x}}_{42}s_{14}+2\frac{\not{\mathbf{x}}_{14}\not{\mathbf{x}}_{43}\not{\mathbf{x}}_{42}}{|\mathbf{x}_{43}|^2}s_{34}\right)$$
$$= \prod_{1\leq i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\frac{\Gamma\left(s_{il}\right)}{|\mathbf{x}_{il}|^{2s_{il}}}\mathcal{M}(s_{il})\prod_{i=1}^{4}\hat{\delta}\left(\Delta_i+\frac{1}{2}\left(\delta_{1i}+\delta_{2i}\right)-\sum_{k=1,k\neq i}^{4}s_{ik}\right)$$
$$\times\left(\not{\mathbf{x}}_{12}\left(\Delta_1+\frac{1}{2}+\Delta_4+\Delta_+-d-2s_{14}\right)+2\frac{\not{\mathbf{x}}_{14}\not{\mathbf{x}}_{43}\not{\mathbf{x}}_{32}}{|\mathbf{x}_{43}|^2}s_{34}\right)$$

after using $s_{14}+s_{24}+s_{34}=\Delta_4$. Rearranging the delta constraints in canonical form and applying $\Delta_1+\Delta_4-2s_{14}+\Delta_+-d+\frac{1}{2}=t+\tau-d+2$[2] $A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3}$ finally yields

$$A^{\bar{\psi}_1\phi_4}_{\psi_2\phi_3} = \Bigg[\frac{\not{\mathbf{x}}_{12}}{|\mathbf{x}_{12}|}\left(t+\tau-d+2\right)\prod_{1\leq i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\frac{\Gamma\left(s_{il}+\delta_{i1}\delta_{2l}\right)}{|\mathbf{x}_{il}|^{2s_{il}+\delta_{i1}\delta_{2l}}}\mathcal{M}(s_{il}+\delta_{i1}\delta_{2l})$$
$$+2\frac{\not{\mathbf{x}}_{14}\not{\mathbf{x}}_{43}\not{\mathbf{x}}_{32}}{|\mathbf{x}_{14}||\mathbf{x}_{43}||\mathbf{x}_{32}|}\prod_{1\leq i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,\Gamma\left(s_{il}+\frac{1}{2}\left(\delta_{i1}\delta_{2l}+\delta_{i1}\delta_{4l}+\delta_{2i}\delta_{3l}+\delta_{i3}\delta_{4l}\right)\right)$$
$$\times\frac{1}{|\mathbf{x}_{il}|^{2s_{il}+\delta_{i1}\delta_{i2}}}\mathcal{M}\left(s_{il}+\frac{1}{2}\left(\delta_{i1}\delta_{2l}+\delta_{i1}\delta_{4l}+\delta_{2i}\delta_{3l}+\delta_{i3}\delta_{4l}\right)\right)\Bigg]\prod_{i=1}^{4}\hat{\delta}\left(\tau_i-\sum_{k=1,k\neq i}^{4}s_{ik}\right).$$

[2]In this calculation the 't-channel' is given by (14)-(23).

The actual definition of the Mellin amplitude $\mathcal{M}(s_{il})$ is given in the main text in equations (3.95) and (3.96).

Bibliography

[1] J. Faller, S. Sarkar and M. Verma, *"Mellin Amplitudes for Fermionic Conformal Correlators"*, JHEP 1803, 106 (2018), arxiv:1711.07929.

[2] J. Faller and J. Plefka, *"Positive helicity Einstein-Yang-Mills amplitudes from the double copy method"*, Phys. Rev. D99, 046008 (2019), arxiv:1812.04053.

[3] V. Shtabovenko, R. Mertig and F. Orellana, *"New Developments in FeynCalc 9.0"*, Comput. Phys. Commun. 207, 432 (2016), arxiv:1601.01167.

[4] R. Mertig, M. Bohm and A. Denner, *"FEYN CALC: Computer algebraic calculation of Feynman amplitudes"*, Comput. Phys. Commun. 64, 345 (1991).

[5] D. Maitre and P. Mastrolia, *"S@M, a Mathematica Implementation of the Spinor-Helicity Formalism"*, Comput. Phys. Commun. 179, 501 (2008), arxiv:0710.5559.

[6] T. Kuhn, *"The Structure of Scientific Revolutions"*, Chicago Press, Chicago, US.

[7] R. Feynman, *"The Character of Physical Law"*, The MIT Press, Cambridge, US.

[8] S. Weinberg, *"The Quantum Theory of Fields - Volume 1"*, Cambridge University Press, Cambridge, UK.

[9] S. Weinberg, *"What is quantum field theory, and what did we think it is?"*, hep-th/9702027, in: *"Conceptual foundations of quantum field theory. Proceedings, Symposium and Workshop, Boston, USA, March 1-3, 1996"*, pp. 241-251.

[10] J. A. Wheeler, *"On the Mathematical Description of Light Nuclei by the Method of Resonating Group Structure"*, Phys. Rev. 52, 1107 (1937).

[11] W. Heisenberg, *"Die „beobachtbaren Größen" in der Theorie der Elementarteilchen"*, Zeitschrift für Ph 120, 513 (1943), https://doi.org/10.1007/BF01329800.

[12] W. Heisenberg, *"Die beobachtbaren Größen in der Theorie der Elementarteilchen. II"*, Zeitschrift für Ph 120, 673 (1943), https://doi.org/10.1007/BF01336936.

[13] W. Heisenberg, *"Die beobachtbaren Größen in der Theorie der Elementarteilchen. III"*, Zeitschrift für Physik A Hadrons and Nu 123, 93 (1944), https://doi.org/10.1007/BF01375146.

[14] S. L. Glashow, *"Partial Symmetries of Weak Interactions"*, Nucl. Phys. 22, 579 (1961).

[15] P. W. Higgs, *"Broken Symmetries and the Masses of Gauge Bosons"*, Phys. Rev. Lett. 13, 508 (1964), [,160(1964)].

[16] F. Englert and R. Brout, *"Broken Symmetry and the Mass of Gauge Vector Mesons"*, Phys. Rev. Lett. 13, 321 (1964), [,157(1964)].

[17] A. Salam, *"Weak and Electromagnetic Interactions"*, Conf. Proc. C680519, 367 (1968), in: *"8th Nobel Symposium Lerum, Sweden, May 19-25, 1968"*, pp. 367-377.

[18] S. Weinberg, *"A Model of Leptons"*, Phys. Rev. Lett. 19, 1264 (1967).

[19] CDF Collaboration, F. Abe et al., *"Observation of top quark production in $\bar{p}p$ collisions"*, Phys. Rev. Lett. 74, 2626 (1995), hep-ex/9503002.

[20] DONUT Collaboration, K. Kodama et al., *"Observation of tau neutrino interactions"*, Phys. Lett. B504, 218 (2001), hep-ex/0012035.

[21] ATLAS Collaboration, G. Aad et al., *"Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC"*, Phys. Lett. B716, 1 (2012), arxiv:1207.7214.

[22] CMS Collaboration, S. Chatrchyan et al., *"Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC"*, Phys. Lett. B716, 30 (2012), arxiv:1207.7235.

[23] K. G. Wilson, *"Quantum field theory models in less than four-dimensions"*, Phys. Rev. D7, 2911 (1973).

[24] K. G. Wilson and J. B. Kogut, *"The Renormalization group and the epsilon expansion"*, Phys. Rept. 12, 75 (1974).

[25] K. G. Wilson, *"The renormalization group and critical phenomena"*, Rev. Mod. Phys. 55, 583 (1983).

[26] B. Delamotte, *"An Introduction to the nonperturbative renormalization group"*, Lect. Notes Phys. 852, 49 (2012), cond-mat/0702365.

[27] M. Le Bellac, *"Quantum and statistical field theory"*.

[28] A. M. Polyakov, *"Conformal symmetry of critical fluctuations"*, JETP Lett. 12, 381 (1970), [Pisma Zh. Eksp. Teor. Fiz.12,538(1970)].

[29] P. Di Francesco, P. Mathieu and D. Sénéchal, *"Conformal Field Theory"*, Springer, New York, USA.

[30] J. M. Maldacena, *"The Large N limit of superconformal field theories and supergravity"*, Int. J. Theor. Phys. 38, 1113 (1999), hep-th/9711200, [Adv. Theor. Math. Phys.2,231(1998)].

[31] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"Gauge theory correlators from noncritical string theory"*, Phys. Lett. B428, 105 (1998), hep-th/9802109.

[32] E. Witten, *"Anti-de Sitter space and holography"*, Adv. Theor. Math. Phys. 2, 253 (1998), hep-th/9802150.

[33] J. Penedones, *"TASI lectures on AdS/CFT"*, arxiv:1608.04948, in: *"Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015"*, pp. 75-136.

[34] J. Kaplan, *"Lectures on AdS/CFT from the Bottom Up"*, Accessed: 2019-07-11.

[35] S. D. Mathur, *"The Information paradox: A Pedagogical introduction"*, Class. Quant. Grav. 26, 224001 (2009), arxiv:0909.1038, in: *"Strings, Supergravity and Gauge Theories. Proceedings, CERN Winter School, CERN, Geneva, Switzerland, February 9-13 2009"*, pp. 224001.

[36] D. Marolf and J. Polchinski, *"Gauge/Gravity Duality and the Black Hole Interior"*, Phys. Rev. Lett. 111, 171301 (2013), arxiv:1307.4706.

[37] G. 't Hooft, *"Dimensional reduction in quantum gravity"*, Conf. Proc. C930308, 284 (1993), gr-qc/9310026, in: *"Conference on Highlights of Particle and Condensed Matter Physics (SALAMFEST) Trieste, Italy, March 8-12, 1993"*, pp. 284-296.

[38] L. Susskind, *"The World as a hologram"*, J. Math. Phys. 36, 6377 (1995), hep-th/9409089.

[39] E. Witten, *"Symmetry and Emergence"*, Nature Phys. 14, 116 (2018), arxiv:1710.01791.

[40] R. J. Riddell, *"The Number of Feynman Diagrams"*, Phys. Rev. 91, 1243 (1953), https://link.aps.org/doi/10.1103/PhysRev.91.1243.

[41] M. L. Mangano, S. J. Parke and Z. Xu, *"Duality and Multi-Gluon Scattering"*, Nucl. Phys. B298, 653 (1988).

[42] Z. Bern and D. A. Kosower, *"Color decomposition of one loop amplitudes in gauge theories"*, Nucl. Phys. B362, 389 (1991).

[43] P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, *"Multiple Bremsstrahlung in Gauge Theories at High-Energies. 1. General Formalism for Quantum Electrodynamics"*, Nucl. Phys. B206, 53 (1982).

[44] R. Kleiss and W. J. Stirling, *"Spinor Techniques for Calculating p anti-p —> W+- / Z0 + Jets"*, Nucl. Phys. B262, 235 (1985).

[45] M. T. Grisaru and H. N. Pendleton, *"Some Properties of Scattering Amplitudes in Supersymmetric Theories"*, Nucl. Phys. B124, 81 (1977).

[46] S. J. Parke and T. R. Taylor, *"Perturbative QCD Utilizing Extended Supersymmetry"*, Phys. Lett. 157B, 81 (1985), [Erratum: Phys. Lett.B174,465(1986)].

[47] F. A. Berends and W. T. Giele, *"Recursive Calculations for Processes with n Gluons"*, Nucl. Phys. B306, 759 (1988).

[48] R. E. Cutkosky, *"Singularities and discontinuities of Feynman amplitudes"*, J. Math. Phys. 1, 429 (1960).

[49] J. L. Bourjaily, E. Herrmann and J. Trnka, *"Prescriptive Unitarity"*, JHEP 1706, 059 (2017), arxiv:1704.05460.

[50] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *"One loop n point gauge theory amplitudes, unitarity and collinear limits"*, Nucl. Phys. B425, 217 (1994), hep-ph/9403226.

[51] R. Penrose and M. A. H. MacCallum, *"Twistor theory: An Approach to the quantization of fields and space-time"*, Phys. Rept. 6, 241 (1972).

[52] E. Witten, *"Perturbative gauge theory as a string theory in twistor space"*, Commun. Math. Phys. 252, 189 (2004), hep-th/0312171.

[53] R. Britto, F. Cachazo and B. Feng, *"Generalized unitarity and one-loop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills"*, Nucl. Phys. B725, 275 (2005), hep-th/0412103.

[54] R. Britto, F. Cachazo, B. Feng and E. Witten, *"Direct proof of tree-level recursion relation in Yang-Mills theory"*, Phys. Rev. Lett. 94, 181602 (2005), hep-th/0501052.

[55] N. Arkani-Hamed and J. Trnka, *"The Amplituhedron"*, JHEP 1410, 030 (2014), arxiv:1312.2007.

[56] N. Arkani-Hamed and J. Trnka, *"Into the Amplituhedron"*, JHEP 1412, 182 (2014), arxiv:1312.7878.

[57] F. Cachazo, S. He and E. Y. Yuan, *“Scattering equations and Kawai-Lewellen-Tye orthogonality”*, Phys. Rev. D90, 065001 (2014), arxiv:1306.6575.

[58] F. Cachazo, S. He and E. Y. Yuan, *“Scattering of Massless Particles in Arbitrary Dimensions”*, Phys. Rev. Lett. 113, 171601 (2014), arxiv:1307.2199.

[59] F. Cachazo, S. He and E. Y. Yuan, *“Scattering of Massless Particles: Scalars, Gluons and Gravitons”*, JHEP 1407, 033 (2014), arxiv:1309.0885.

[60] Z. Bern, J. Carrasco and H. Johansson, *“New Relations for Gauge-Theory Amplitudes”*, Phys. Rev. D78, 085011 (2008), arxiv:0805.3993.

[61] Z. Bern, J. J. M. Carrasco and H. Johansson, *“Perturbative Quantum Gravity as a Double Copy of Gauge Theory”*, Phys. Rev. Lett. 105, 061602 (2010), arxiv:1004.0476.

[62] B. S. DeWitt, *“Quantum Theory of Gravity. 2. The Manifestly Covariant Theory”*, Phys. Rev. 162, 1195 (1967), [,298(1967)].

[63] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *“Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity”*, JHEP 1501, 081 (2015), arxiv:1408.0764.

[64] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *“Explicit Formulae for Yang-Mills-Einstein Amplitudes from the Double Copy”*, JHEP 1707, 002 (2017), arxiv:1703.00421.

[65] H. Johansson and A. Ochirov, *“Pure Gravities via Color-Kinematics Duality for Fundamental Matter”*, JHEP 1511, 046 (2015), arxiv:1407.4772.

[66] H. Johansson, G. Mogull and F. Teng, *“Unraveling conformal gravity amplitudes”*, JHEP 1809, 080 (2018), arxiv:1806.05124.

[67] Z. Bern, J. J. M. Carrasco, W.-M. Chen, H. Johansson, R. Roiban and M. Zeng, *“Five-loop four-point integrand of $N = 8$ supergravity as a generalized double copy”*, Phys. Rev. D96, 126012 (2017), arxiv:1708.06807.

[68] Z. Bern, J. J. Carrasco, W.-M. Chen, H. Johansson and R. Roiban, *“Gravity Amplitudes as Generalized Double Copies of Gauge-Theory Amplitudes”*, Phys. Rev. Lett. 118, 181602 (2017), arxiv:1701.02519.

[69] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *“Non-Abelian gauged supergravities as double copies”*, JHEP 1906, 099 (2019), arxiv:1812.10434.

[70] Z. Bern, D. Kosower and J. Parra-Martinez, *“Two-loop n-point anomalous amplitudes in $N = 4$ supergravity”*, arxiv:1905.05151.

[71] R. Monteiro, D. O’Connell and C. D. White, *“Black holes and the double copy”*, JHEP 1412, 056 (2014), arxiv:1410.0239.

[72] A. Luna, R. Monteiro, D. O’Connell and C. D. White, *“The classical double copy for Taub–NUT spacetime”*, Phys. Lett. B750, 272 (2015), arxiv:1507.01869.

[73] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O’Connell, N. Westerberg and C. D. White, *“Perturbative spacetimes from Yang-Mills theory”*, JHEP 1704, 069 (2017), arxiv:1611.07508.

[74] W. D. Goldberger and A. K. Ridgway, *“Radiation and the classical double copy for color charges”*, Phys. Rev. D95, 125010 (2017), arxiv:1611.03493.

[75] C.-H. Shen, *"Gravitational Radiation from Color-Kinematics Duality"*, JHEP 1811, 162 (2018), arxiv:1806.07388.

[76] J. Plefka, J. Steinhoff and W. Wormsbecher, *"Effective action of dilaton gravity as the classical double copy of Yang-Mills theory"*, Phys. Rev. D99, 024021 (2019), arxiv:1807.09859.

[77] J. Plefka, C. Shi, J. Steinhoff and T. Wang, *"Breakdown of the classical double copy for the effective action of dilaton-gravity at NNLO"*, arxiv:1906.05875.

[78] N. Arkani-Hamed, Y.-t. Huang and D. O'Connell, *"Kerr Black Holes as Elementary Particles"*, arxiv:1906.10100.

[79] H. Johansson and A. Ochirov, *"Double copy for massive quantum particles with spin"*, arxiv:1906.12292.

[80] D. Nandan, J. Plefka and G. Travaglini, *"All rational one-loop Einstein-Yang-Mills amplitudes at four points"*, JHEP 1809, 011 (2018), arxiv:1803.08497.

[81] D. Freeman, *"Is a Graviton Detectable?"*, https://publications.ias.edu/dyson/poincare2012.

[82] T. Rothman and S. Boughn, *"Can gravitons be detected?"*, Found. Phys. 36, 1801 (2006), gr-qc/0601043.

[83] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, *"General Relativity from Scattering Amplitudes"*, Phys. Rev. Lett. 121, 171601 (2018), arxiv:1806.04920.

[84] T. Damour, *"High-energy gravitational scattering and the general relativistic two-body problem"*, Phys. Rev. D97, 044038 (2018), arxiv:1710.10599.

[85] C. Cheung, I. Z. Rothstein and M. P. Solon, *"From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion"*, Phys. Rev. Lett. 121, 251101 (2018), arxiv:1808.02489.

[86] D. A. Kosower, B. Maybee and D. O'Connell, *"Amplitudes, Observables, and Classical Scattering"*, JHEP 1902, 137 (2019), arxiv:1811.10950.

[87] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *"Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order"*, Phys. Rev. Lett. 122, 201603 (2019), arxiv:1901.04424.

[88] Y. F. Bautista and A. Guevara, *"From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems"*, arxiv:1903.12419.

[89] B. Maybee, D. O'Connell and J. Vines, *"Observables and amplitudes for spinning particles and black holes"*, arxiv:1906.09260.

[90] Y. Iwasaki, *"Quantum theory of gravitation vs. classical theory. - fourth-order potential"*, Prog. Theor. Phys. 46, 1587 (1971).

[91] M. J. Duff, *"Quantum Tree Graphs and the Schwarzschild Solution"*, Phys. Rev. D7, 2317 (1973).

[92] B. R. Holstein and J. F. Donoghue, *"Classical physics and quantum loops"*, Phys. Rev. Lett. 93, 201602 (2004), hep-th/0405239.

[93] LIGO Scientific, Virgo Collaboration, B. P. Abbott et al., *"Observation of Gravitational Waves from a Binary Black Hole Merger"*, Phys. Rev. Lett. 116, 061102 (2016), arxiv:1602.03837.

[94] LIGO Scientific, Virgo Collaboration, B. P. Abbott et al., *"GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence"*, Phys. Rev. Lett. 116, 241103 (2016), arxiv:1606.04855.

[95] S. Ferrara, A. F. Grillo and R. Gatto, *"Tensor representations of conformal algebra and conformally covariant operator product expansion"*, Annals Phys. 76, 161 (1973).

[96] A. M. Polyakov, *"Nonhamiltonian approach to conformal quantum field theory"*, Zh. Eksp. Teor. Fiz. 66, 23 (1974), [Sov. Phys. JETP39,9(1974)].

[97] G. Mack, *"Duality in quantum field theory"*, Nucl. Phys. B118, 445 (1977).

[98] S. Rychkov, *"EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions"*.

[99] G. Mack, *"D-independent representation of Conformal Field Theories in D dimensions via transformation to auxiliary Dual Resonance Models. Scalar amplitudes"*, arxiv:0907.2407.

[100] G. Mack, *"D-dimensional Conformal Field Theories with anomalous dimensions as Dual Resonance Models"*, Bulg. J. Phys. 36, 214 (2009), arxiv:0909.1024, in: *"Algebraic methods in quantum field theory. Proceedings, Conference, 50 Years Mathematical Physics in Bulgaria on the Occasion of the 75th Anniversary of Ivan Todorov, Sofia, Bulgaria, May 15-16, 2009"*, pp. 214-226.

[101] K. Symanzik, *"On Calculations in conformal invariant field theories"*, Lett. Nuovo Cim. 3, 734 (1972).

[102] A. L. Fitzpatrick, J. Kaplan, J. Penedones, S. Raju and B. C. van Rees, *"A Natural Language for AdS/CFT Correlators"*, JHEP 1111, 095 (2011), arxiv:1107.1499.

[103] M. F. Paulos, *"Towards Feynman rules for Mellin amplitudes"*, JHEP 1110, 074 (2011), arxiv:1107.1504.

[104] A. A. Nizami, A. Rudra, S. Sarkar and M. Verma, *"Exploring Perturbative Conformal Field Theory in Mellin space"*, JHEP 1701, 102 (2017), arxiv:1607.07334.

[105] M. S. Costa, V. Goncalves and J. Penedones, *"Conformal Regge theory"*, JHEP 1212, 091 (2012), arxiv:1209.4355.

[106] J. Penedones, *"Writing CFT correlation functions as AdS scattering amplitudes"*, JHEP 1103, 025 (2011), arxiv:1011.1485.

[107] A. L. Fitzpatrick and J. Kaplan, *"Analyticity and the Holographic S-Matrix"*, JHEP 1210, 127 (2012), arxiv:1111.6972.

[108] A. L. Fitzpatrick and J. Kaplan, *"Unitarity and the Holographic S-Matrix"*, JHEP 1210, 032 (2012), arxiv:1112.4845.

[109] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees and P. Vieira, *"The S-matrix bootstrap. Part I: QFT in AdS"*, JHEP 1711, 133 (2017), arxiv:1607.06109.

[110] A. L. Fitzpatrick, J. Kaplan, D. Poland and D. Simmons-Duffin, *"The Analytic Bootstrap and AdS Superhorizon Locality"*, JHEP 1312, 004 (2013), arxiv:1212.3616.

[111] Z. Komargodski and A. Zhiboedov, *"Convexity and Liberation at Large Spin"*, JHEP 1311, 140 (2013), arxiv:1212.4103.

[112] L. Rastelli and X. Zhou, *"How to Succeed at Holographic Correlators Without Really Trying"*, JHEP 1804, 014 (2018), arxiv:1710.05923.

[113] V. Gonçalves, J. Penedones and E. Trevisani, *"Factorization of Mellin amplitudes"*, JHEP 1510, 040 (2015), arxiv:1410.4185.

[114] R. Gopakumar, A. Kaviraj, K. Sen and A. Sinha, *"Conformal Bootstrap in Mellin Space"*, Phys. Rev. Lett. 118, 081601 (2017), arxiv:1609.00572.

[115] R. Gopakumar, A. Kaviraj, K. Sen and A. Sinha, *"A Mellin space approach to the conformal bootstrap"*, JHEP 1705, 027 (2017), arxiv:1611.08407.

[116] R. Gopakumar and A. Sinha, *"On the Polyakov-Mellin bootstrap"*, JHEP 1812, 040 (2018), arxiv:1809.10975.

[117] P. Dey, A. Kaviraj and A. Sinha, *"Mellin space bootstrap for global symmetry"*, JHEP 1707, 019 (2017), arxiv:1612.05032.

[118] P. Dey, K. Ghosh and A. Sinha, *"Simplifying large spin bootstrap in Mellin space"*, JHEP 1801, 152 (2018), arxiv:1709.06110.

[119] P. Dey and A. Kaviraj, *"Towards a Bootstrap approach to higher orders of epsilon expansion"*, JHEP 1802, 153 (2018), arxiv:1711.01173.

[120] J. Golden and D. R. Mayerson, *"Mellin Bootstrap for Scalars in Generic Dimension"*, JHEP 1806, 066 (2018), arxiv:1711.03980.

[121] C. Sleight and M. Taronna, *"Spinning Mellin Bootstrap: Conformal Partial Waves, Crossing Kernels and Applications"*, Fortsch. Phys. 66, 1800038 (2018), arxiv:1804.09334.

[122] K. Ghosh, *"Polyakov-Mellin Bootstrap for AdS loops"*, arxiv:1811.00504.

[123] H.-Y. Chen, E.-J. Kuo and H. Kyono, *"Towards Spinning Mellin Amplitudes"*, Nucl. Phys. B931, 291 (2018), arxiv:1712.07991.

[124] D. Simmons-Duffin, *"The Conformal Bootstrap"*, arxiv:1602.07982, in: *"Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015"*, pp. 1-74.

[125] M. D. Schwartz, *"Quantum Field Theory and the Standard Model"*, Cambridge University Press, Cambridge, UK.

[126] J. Polchinski, *"String Theory - Volume 1"*, Cambridge University Press, Cambridge, UK.

[127] M. Srednicki, *"Quantum Field Theory"*, Cambridge University Press, Cambridge, UK.

[128] M. E. Peskin and D. V. Schroeder, *"An Introduction to Quantum Field Theory"*, Perseus Books, Massachusetts, USA.

[129] E. P. Wigner, *"On Unitary Representations of the Inhomogeneous Lorentz Group"*, Annals Math. 40, 149 (1939), in: *"Proceedings, International Symposium on Spacetime Symmetries: College Park, Maryland, May 24-28, 1988"*, pp. 149-204, [Reprint: Nucl. Phys. Proc. Suppl.6,9(1989)].

[130] H. Lehmann, K. Symanzik and W. Zimmermann, *"On the formulation of quantized field theories"*, Nuovo Cim. 1, 205 (1955).

[131] T. R. Taylor, *"A Course in Amplitudes"*, Phys. Rept. 691, 1 (2017), arxiv:1703.05670.

[132] L. E. Landau, L.D., *"The Classical Theory of Fields. Course of Theoretical Physics. (4th ed."*, Butterworth-Heinemann, Oxford, UK.

[133] C. Moeller, *"Central Properties of the Characterstic Matrix in the Theory of Elementary Particles 1"*, D. Kgl Danske Vidensk. Selsk. Mat.-Fys. Medd. 23 (1) (1945).

[134] R. J. Eden, P. V. Landshoff, O. D. I. and P. J. C., *"The Analytic S-Matrix"*, Cambridge University Press, Cambridge, UK.

[135] M. Veltman, *"Diagrammatica"*, Cambridge University Press, Cambridge, UK.

[136] Z. Bern and Y.-t. Huang, *"Basics of Generalized Unitarity"*, J. Phys. A44, 454003 (2011), `arxiv:1103.1869`.

[137] Z. Bern and A. Morgan, *"Massive loop amplitudes from unitarity"*, Nucl. Phys. B467, 479 (1996), `hep-ph/9511336`.

[138] J. M. Henn and J. C. Plefka, *"Scattering Amplitudes in Gauge Theories"*, Lect. Notes Phys. 883, J. C. Plefka (2014).

[139] L. J. Dixon, *"Calculating scattering amplitudes efficiently"*, `hep-ph/9601359`, in: *"QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95, Boulder, USA, June 4-30, 1995"*, pp. 539-584.

[140] J. C. Collins, *"Renormalization"*, Cambridge University Press, Cambridge, UK.

[141] Z. Bern and D. A. Kosower, *"The Computation of loop amplitudes in gauge theories"*, Nucl. Phys. B379, 451 (1992).

[142] Z. Bern, A. De Freitas, L. J. Dixon and H. Wong, *"Supersymmetric regularization, two loop QCD amplitudes and coupling shifts"*, Phys. Rev. D66, 085002 (2002), `hep-ph/0202271`.

[143] G. 't Hooft and M. J. G. Veltman, *"Regularization and Renormalization of Gauge Fields"*, Nucl. Phys. B44, 189 (1972).

[144] Z. Kunszt, A. Signer and Z. Trocsanyi, *"One loop helicity amplitudes for all 2 —> 2 processes in QCD and N=1 supersymmetric Yang-Mills theory"*, Nucl. Phys. B411, 397 (1994), `hep-ph/9305239`.

[145] W. Siegel, *"Supersymmetric Dimensional Regularization via Dimensional Reduction"*, Phys. Lett. B84, 193 (1979).

[146] D. M. Capper, D. R. T. Jones and P. van Nieuwenhuizen, *"Regularization by Dimensional Reduction of Supersymmetric and Nonsupersymmetric Gauge Theories"*, Nucl. Phys. B167, 479 (1980).

[147] G. Passarino and M. Veltman, *"One Loop Corrections for $e^+ e^-$ Annihilation Into $\mu^+ \mu^-$ in the Weinberg Model"*, Nucl. Phys. B160, 151 (1979).

[148] G. 't Hooft and M. J. G. Veltman, *"Scalar One Loop Integrals"*, Nucl. Phys. B153, 365 (1979).

[149] A. Denner, *"Techniques for calculation of electroweak radiative corrections at the one loop level and results for W physics at LEP-200"*, Fortsch. Phys. 41, 307 (1993), `arxiv:0709.1075`.

[150] H. Kawai, D. C. Lewellen and S. H. H. Tye, *"A relation between tree amplitudes of closed and open strings"*, Nucl. Phys. B269, 1 (1986).

[151] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, *"Multileg one loop gravity amplitudes from gauge theory"*, Nucl. Phys. B546, 423 (1999), `hep-th/9811140`.

[152] S. He and O. Schlotterer, *"New Relations for Gauge-Theory and Gravity Amplitudes at Loop Level"*, Phys. Rev. Lett. 118, 161601 (2017), arxiv:1612.00417.

[153] S. He, O. Schlotterer and Y. Zhang, *"New BCJ representations for one-loop amplitudes in gauge theories and gravity"*, Nucl. Phys. B930, 328 (2018), arxiv:1706.00640.

[154] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *"Gravity as the Square of Gauge Theory"*, Phys. Rev. D82, 065003 (2010), arxiv:1004.0693.

[155] S. H. Henry Tye and Y. Zhang, *"Dual Identities inside the Gluon and the Graviton Scattering Amplitudes"*, JHEP 1006, 071 (2010), arxiv:1003.1732, [Erratum: JHEP04,114(2011)].

[156] C. R. Mafra, *"Simplifying the Tree-level Superstring Massless Five-point Amplitude"*, JHEP 1001, 007 (2010), arxiv:0909.5206.

[157] C. R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, *"Six Open String Disk Amplitude in Pure Spinor Superspace"*, Nucl. Phys. B846, 359 (2011), arxiv:1011.0994.

[158] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, *"Minimal Basis for Gauge Theory Amplitudes"*, Phys. Rev. Lett. 103, 161602 (2009), arxiv:0907.1425.

[159] C. R. Mafra, O. Schlotterer and S. Stieberger, *"Explicit BCJ Numerators from Pure Spinors"*, JHEP 1107, 092 (2011), arxiv:1104.5224.

[160] J. H. Schwarz, *"Dilaton - axion symmetry"*, hep-th/9209125, in: *"International Workshop on String Theory, Quantum Gravity and the Unification of Fundamental Interactions Rome, Italy, September 21-26, 1992"*, pp. 503-520.

[161] M. J. Duff and P. van Nieuwenhuizen, *"Quantum Inequivalence of Different Field Representations"*, Phys. Lett. 94B, 179 (1980).

[162] W. Siegel, *"Quantum Equivalence of Different Field Representations"*, Phys. Lett. 103B, 107 (1981).

[163] E. S. Fradkin and A. A. Tseytlin, *"Quantum Equivalence of Dual Field Theories"*, Annals Phys. 162, 31 (1985).

[164] M. T. Grisaru, N. K. Nielsen, W. Siegel and D. Zanon, *"Energy Momentum Tensors, Supercurrents, (Super)traces and Quantum Equivalence"*, Nucl. Phys. B247, 157 (1984).

[165] E. S. Fradkin and A. A. Tseytlin, *"One Loop Infinities in Dimensionally Reduced Supergravities"*, Phys. Lett. 137B, 357 (1984).

[166] Z. Bern, C. Cheung, H.-H. Chi, S. Davies, L. Dixon and J. Nohle, *"Evanescent Effects Can Alter Ultraviolet Divergences in Quantum Gravity without Physical Consequences"*, Phys. Rev. Lett. 115, 211301 (2015), arxiv:1507.06118.

[167] F. Bastianelli, P. Benincasa and S. Giombi, *"Worldline approach to vector and antisymmetric tensor fields"*, JHEP 0504, 010 (2005), hep-th/0503155.

[168] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *"One loop self-dual and $\mathcal{N} = 4$ super Yang-Mills"*, Phys. Lett. B394, 105 (1997), hep-th/9611127.

[169] A. Brandhuber, S. McNamara, B. J. Spence and G. Travaglini, *"Loop amplitudes in pure Yang-Mills from generalised unitarity"*, JHEP 0510, 011 (2005), hep-th/0506068.

[170] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang and J. Nohle, *"Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops"*, Phys. Rev. D92, 045041 (2015), arxiv:1303.6605.

[171] D. C. Dunbar and P. S. Norridge, *"Calculation of graviton scattering amplitudes using string based methods"*, Nucl. Phys. B433, 181 (1995), hep-th/9408014.

[172] K. Osterwalder and R. Schrader, *"AXIOMS FOR EUCLIDEAN GREEN'S FUNCTIONS"*, Commun. Math. Phys. 31, 83 (1973).

[173] K. Osterwalder and R. Schrader, *"Axioms for Euclidean Green's Functions. 2."*, Commun. Math. Phys. 42, 281 (1975).

[174] M. Luscher and G. Mack, *"Global Conformal Invariance in Quantum Field Theory"*, Commun. Math. Phys. 41, 203 (1975).

[175] T. Hartman, S. Jain and S. Kundu, *"Causality Constraints in Conformal Field Theory"*, JHEP 1605, 099 (2016), arxiv:1509.00014.

[176] D. Simmons-Duffin, *"TASI Lectures on Conformal Field Theory in Lorentzian Signature"*, Accessed: 2019-07-05.

[177] R. F. Streater and A. S. Wightman, *"PCT, spin and statistics, and all that"*.

[178] V. K. Dobrev, V. B. Petkova, S. G. Petrova and I. T. Todorov, *"Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory"*, Phys. Rev. D13, 887 (1976).

[179] G. Mack, *"Convergence of Operator Product Expansions on the Vacuum in Conformal Invariant Quantum Field Theory"*, Commun. Math. Phys. 53, 155 (1977).

[180] S. Ferrara, P. Gatto and A. F. Grilla, *"Conformal algebra in space-time and operator product expansion"*, Springer Tracts Mod. Phys. 67, 1 (1973).

[181] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, *"Bounding scalar operator dimensions in 4D CFT"*, JHEP 0812, 031 (2008), arxiv:0807.0004.

[182] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *"Solving the 3D Ising Model with the Conformal Bootstrap"*, Phys. Rev. D86, 025022 (2012), arxiv:1203.6064.

[183] S. El-Showk and M. F. Paulos, *"Bootstrapping Conformal Field Theories with the Extremal Functional Method"*, Phys. Rev. Lett. 111, 241601 (2013), arxiv:1211.2810.

[184] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *"Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents"*, J. Stat. Phys. 157, 869 (2014), arxiv:1403.4545.

[185] F. Kos, D. Poland and D. Simmons-Duffin, *"Bootstrapping Mixed Correlators in the 3D Ising Model"*, JHEP 1411, 109 (2014), arxiv:1406.4858.

[186] D. Simmons-Duffin, *"A Semidefinite Program Solver for the Conformal Bootstrap"*, JHEP 1506, 174 (2015), arxiv:1502.02033.

[187] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, *"Bootstrapping the O(N) Archipelago"*, JHEP 1511, 106 (2015), arxiv:1504.07997.

[188] F. Gliozzi, *"More constraining conformal bootstrap"*, Phys. Rev. Lett. 111, 161602 (2013), arxiv:1307.3111.

[189] F. Gliozzi and A. Rago, *"Critical exponents of the 3d Ising and related models from Conformal Bootstrap"*, JHEP 1410, 042 (2014), arxiv:1403.6003.

[190] D. Simmons-Duffin, *"The Lightcone Bootstrap and the Spectrum of the 3d Ising CFT"*, JHEP 1703, 086 (2017), arxiv:1612.08471.

[191] S. Ferrara, A. F. Grillo, G. Parisi and R. Gatto, *"The shadow operator formalism for conformal algebra. Vacuum expectation values and operator products"*, Lett. Nuovo Cim. 4S2, 115 (1972), [Lett. Nuovo Cim.4,115(1972)].

[192] F. A. Dolan and H. Osborn, *"Conformal four point functions and the operator product expansion"*, Nucl. Phys. B599, 459 (2001), hep-th/0011040.

[193] F. A. Dolan and H. Osborn, *"Conformal partial waves and the operator product expansion"*, Nucl. Phys. B678, 491 (2004), hep-th/0309180.

[194] F. A. Dolan and H. Osborn, *"Conformal Partial Waves: Further Mathematical Results"*, arxiv:1108.6194.

[195] D. Simmons-Duffin, *"Projectors, Shadows, and Conformal Blocks"*, JHEP 1404, 146 (2014), arxiv:1204.3894.

[196] M. Hogervorst and S. Rychkov, *"Radial Coordinates for Conformal Blocks"*, Phys. Rev. D87, 106004 (2013), arxiv:1303.1111.

[197] J. Penedones, E. Trevisani and M. Yamazaki, *"Recursion Relations for Conformal Blocks"*, JHEP 1609, 070 (2016), arxiv:1509.00428.

[198] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin and R. Yacoby, *"Fermion-Scalar Conformal Blocks"*, JHEP 1604, 074 (2016), arxiv:1511.01497.

[199] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin and R. Yacoby, *"Bootstrapping 3D Fermions"*, JHEP 1603, 120 (2016), arxiv:1508.00012.

[200] A. Castedo Echeverri, E. Elkhidir, D. Karateev and M. Serone, *"Seed Conformal Blocks in 4D CFT"*, JHEP 1602, 183 (2016), arxiv:1601.05325.

[201] P. Kravchuk, *"Casimir recursion relations for general conformal blocks"*, JHEP 1802, 011 (2018), arxiv:1709.05347, [,164(2017)].

[202] D. Karateev, P. Kravchuk and D. Simmons-Duffin, *"Weight Shifting Operators and Conformal Blocks"*, JHEP 1802, 081 (2018), arxiv:1706.07813, [,91(2017)].

[203] P. A. M. Dirac, *"Wave equations in conformal space"*, Annals Math. 37, 429 (1936).

[204] S. Weinberg, *"Six-dimensional Methods for Four-dimensional Conformal Field Theories"*, Phys. Rev. D82, 045031 (2010), arxiv:1006.3480.

[205] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *"Spinning Conformal Correlators"*, JHEP 1111, 071 (2011), arxiv:1107.3554.

[206] M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *"Spinning Conformal Blocks"*, JHEP 1111, 154 (2011), arxiv:1109.6321.

[207] P. Kravchuk and D. Simmons-Duffin, *"Counting Conformal Correlators"*, JHEP 1802, 096 (2018), arxiv:1612.08987.

[208] E. Y. Yuan, *"Loops in the Bulk"*, arxiv:1710.01361.

[209] E. Y. Yuan, *"Simplicity in AdS Perturbative Dynamics"*, arxiv:1801.07283.

[210] D. Pappadopulo, S. Rychkov, J. Espin and R. Rattazzi, *"OPE Convergence in Conformal Field Theory"*, Phys. Rev. D86, 105043 (2012), arxiv:1208.6449.

[211] A. V. Ramallo, *"Introduction to the AdS/CFT correspondence"*, Springer Proc. Phys. 161, 411 (2015), arxiv:1310.4319, in: *"Proceedings, 3rd IDPASC School: Santiago de Compostela, Spain, January 21-February 2, 2013"*, pp. 411-474.

[212] M. Henningson and K. Skenderis, *"The Holographic Weyl anomaly"*, JHEP 9807, 023 (1998), hep-th/9806087.

[213] S. de Haro, S. N. Solodukhin and K. Skenderis, *"Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence"*, Commun. Math. Phys. 217, 595 (2001), hep-th/0002230.

[214] M. Henningson and K. Sfetsos, *"Spinors and the AdS / CFT correspondence"*, Phys. Lett. B431, 63 (1998), hep-th/9803251.

[215] W. Mueck and K. S. Viswanathan, *"Conformal field theory correlators from classical scalar field theory on AdS(d+1)"*, Phys. Rev. D58, 041901 (1998), hep-th/9804035.

[216] M. Henneaux, *"Boundary terms in the AdS / CFT correspondence for spinor fields"*, hep-th/9902137, in: *"Mathematical methods in modern theoretical physics. Proceedings, International Meeting, School and Workshop, ISPM'98, Tbilisi, Georgia, September 5-18, 1998"*, pp. 161-170.

[217] P. Minces and V. O. Rivelles, *"Scalar field theory in the AdS / CFT correspondence revisited"*, Nucl. Phys. B572, 651 (2000), hep-th/9907079.

[218] T. Kawano and K. Okuyama, *"Spinor exchange in AdS(d+1)"*, Nucl. Phys. B565, 427 (2000), hep-th/9905130.

[219] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *"Correlation functions in the CFT(d) / AdS(d+1) correspondence"*, Nucl. Phys. B546, 96 (1999), hep-th/9804058.

[220] D. Z. Freedman and A. Van Proeyen, *"Supergravity"*, Cambridge Univ. Press (2012), Cambridge, UK.

[221] D. Nandan, A. Volovich and C. Wen, *"On Feynman Rules for Mellin Amplitudes in AdS/CFT"*, JHEP 1205, 129 (2012), arxiv:1112.0305.

[222] S. Kharel and G. Siopsis, *"Tree-level Correlators of scalar and vector fields in AdS/CFT"*, JHEP 1311, 159 (2013), arxiv:1308.2515.

[223] A. M. Ghezelbash, K. Kaviani, S. Parvizi and A. H. Fatollahi, *"Interacting spinors - scalars and AdS / CFT correspondence"*, Phys. Lett. B435, 291 (1998), hep-th/9805162.

[224] M. F. Paulos, M. Spradlin and A. Volovich, *"Mellin Amplitudes for Dual Conformal Integrals"*, JHEP 1208, 072 (2012), arxiv:1203.6362.

[225] D. Nandan, M. F. Paulos, M. Spradlin and A. Volovich, *"Star Integrals, Convolutions and Simplices"*, JHEP 1305, 105 (2013), arxiv:1301.2500.

[226] D. J. Gross and A. Neveu, *"Dynamical Symmetry Breaking in Asymptotically Free Field Theories"*, Phys. Rev. D10, 3235 (1974).

[227] J. Zinn-Justin, *"Four fermion interaction near four-dimensions"*, Nucl. Phys. B367, 105 (1991), in: *"2nd JINR Conference on Renormalization Group Dubna, USSR, September 3-6, 1991"*, pp. 105-122.