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Covariant Integral Quantization of the Semi-Discrete SO(3)-Hypercylinder

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Abstract: Covariant integral quantization with rotational $\mathrm{SO}(3)$ symmetry is established for quantum motion on this group manifold. It can also be applied to Gabor signal analysis on this group. The corresponding phase space takes the form of a discrete-continuous hypercylinder. The central tool for implementing this procedure is the Weyl–Gabor operator, a non-unitary operator that operates on the Hilbert space of square-integrable functions on $\mathrm{SO}(3)$. This operator serves as the counterpart to the unitary Weyl or displacement operator used in constructing standard Schrödinger–Glauber–Sudarshan coherent states. We unveil a diverse range of properties associated with the quantizations and their corresponding semi-classical phase-space portraits, which are derived from different weight functions on the considered discrete-continuous hypercylinder. Certain classes of these weight functions lead to families of coherent states. Moreover, our approach allows us to define a Wigner distribution, satisfying the standard marginality conditions, along with its related Wigner transform.

Keywords: covariant Weyl–Heisenberg integral quantization; semi-discrete hypercylinder; coherent states; Weyl–Gabor operator; quantum models on $\mathrm{SO}(3)$; Wigner function; phase space portrait

MSC: 46L65; 81S10; 81S30; 81R30

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1. Introduction

The group $\mathrm{SO}(3)$ is of particular interest in physics as it serves as the configuration space for the motion of a rigid body fixed at a point. It is also relevant in signal analysis and processing. The rigid rotor is a classic problem in classical and quantum mechanics, describing the dynamics of a rigid body with its center of mass held fixed [1–4]. On the quantum level, it provides a consistent framework for describing the rotational spectra of molecules [2,5–16]. Furthermore, utilizing $\mathrm{SO}(3)$ as a configuration manifold has led to various applications, including texture analysis [17–19], protein–protein docking [20,21], air temperature control [22], the structure of interest rates in economics [23], the attitude determination of rigid bodies [24–29], quantum information [30], and spherical image analysis [31,32].

This work can be seen as a direct continuation of previous studies focused on the semi-discrete cylinder [33,34]. Our inspiration also stems from the insightful works of Mukunda et al. [35–37]. These authors were concerned with establishing a consistent Wigner function for compact Lie groups, and they illustrated their approach using the example of $\mathrm{SU}(2)$, the double covering of $\mathrm{SO}(3)$. For more recent and related works, please refer to, for instance, [38,39] and the references therein.

In [33], we implemented what is known as covariant quantization for the semi-discrete cylinder, considering it as the phase space for motion on a circle. Covariant integral quantizations linearly transform functions (referred to as "classical observables") on phase spaces, in a broader sense, into operators ("quantum observables") on specific Hilbert spaces of "quantum states". These quantization methods are based on resolving the identity using continuous or discrete families of normalized positive-operator-valued measures (POVMs) that transform covariantly under certain symmetry group actions. In simple cases (see, for instance, [40]), these symmetries can be described by the Weyl–Heisenberg group, which includes projective representations of translations in the Euclidean plane, or by the affine groups, which involve translations of a subset of variables combined with dilations of the remaining subset of variables bounded below.

Starting with (quasi-)probability distributions in the phase spaces where the Weyl–Heisenberg or affine groups act (representing classical Hamiltonian models), these quantization methods produce their corresponding quantum models and associated probabilities (e.g., Husimi) or quasi-probabilities (e.g., Wigner) distributions. In turn, tracing the involved POVM with the operators provides semi-classical portraits of the quantum models, which act as a regularization of the original classical model. These quantization methods, with origins traceable to Klauder, Berezin, and Toeplitz, are relatively easier to manipulate compared to geometric or deformation quantizations. Importantly, they enable us to circumvent issues related to ordering (canonical quantization) and those arising from the presence of singularities in classical models.

In this study, our objective is to implement and investigate the covariant integral quantization of functions or distributions in the phase space $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$, where $\widehat{\mathrm{SO}(3)}$ is a discrete set denoted as (l, m, n), $l \in \mathbb{N}\,, -l \leq m, n \leq l$, which labels the matrix elements of the unitary irreducible representations (UIR) of $\mathrm{SO}(3)$ with respect to the spherical harmonic Hilbertian basis of $\mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\Omega)$.

In Section 2, we briefly describe the general method of covariant integral quantization.

In Section 3, we apply this method to the quantum description of the motion of a particle on the $\mathrm{SO}(3)$ manifold.

Section 4 involves deriving the (non-unitary) Weyl–Gabor operator U acting on the Hilbert space $\mathcal{K} = L^2(\mathrm{SO}(3), \mathrm{d}\mathbf{x})$ of square-integrable functions on the $\mathrm{SO}(3)$ manifold equipped with its Haar measure. This operator leads to a decomposition of the identity on the space $\mathcal{K}$.

In Section 5, we first define our quantization tools, which include a weight function ϖ defined in the phase space and the related integral operator M^{ϖ} that acts on the representation space. We then define the quantization map, which transforms a function or distribution f in the phase space Γ into an operator A_f^{ϖ} acting on $\mathcal{K}$. We compute the quantization of separable functions in both position and momentum, only momentum, and only position.

In Section 6, we compute the so-called semi-classical portrait (or lower symbol) of the operator A_f^{ϖ} and study how closely they resemble the original function f.

Section 7 provides examples of quantum operators obtained through coherent state quantization.

In Section 8, we introduce a Wigner function built from what we define as the "squared rotation operator".

Finally, concluding with Section 9, we present some appealing investigations in the continuation of the present work and provide insights about the application of our formalism to the analysis of signals defined on the $\mathrm{SO}(3)$ manifold. Interesting formulas are included in Appendix A.

2. Resolution of the Identity as the Common Guideline

In this section, we provide an overview of integral quantization. For more comprehensive explanations, one can refer to [40,41] and, more recently, [42,43], along with their respective references.

2.1. Integral Quantization: The Essential

Given a measure space (X, μ) and a (separable) Hilbert space $\mathcal{K}$, an operator-valued function

$$X \ni x \mapsto \mathsf{M}(x) \text{ acting in } \mathcal{K}\,,$$

resolves the identity operator $\mathbb{1}$ in $\mathcal{K}$ with respect the measure μ if

$$\int_X \mathsf{M}(x)\,\mathrm{d}\mu(x) = \mathbb{1} \tag{1}$$

holds in a weak sense.

In signal analysis, *analysis* and *reconstruction* are grounded in the application of (1) on a signal, i.e., a vector in $\mathcal{K}$, where

$$\mathcal{K} \ni |s\rangle \overset{\text{reconstruction}}{=} \int_X \overbrace{\mathsf{M}(x)|s\rangle}^{\text{analysis}}\,\mathrm{d}\mu(x)\,.$$

In quantum formalism, *integral quantization* is grounded in the linear map of a function on X to an operator in $\mathcal{K}$:

$$f(x) \mapsto \int_X f(x)\mathsf{M}(x)\,\mathrm{d}\mu(x) = A_f\,, \quad 1 \mapsto \mathbb{1}\,.$$

2.2. Probabilistic Content of Integral Quantization: Semi-Classical Portraits

If the $\mathsf{M}(x)$ operators in

$$\int_X \mathsf{M}(x)\,\mathrm{d}\mu(x) = \mathbb{1} \tag{2}$$

are nonnegative, i.e., $\langle\phi|\mathsf{M}(x)|\phi\rangle \geq 0$ for all $x \in X$, one says that they form a (normalized) positive-operator-valued measure (POVM) on X.

If they are further unit trace-class, i.e., $\mathrm{tr}(\mathsf{M}(x)) = 1$ for all $x \in X$, i.e., if the $\mathsf{M}(x)$ operatorsare density operators, then the map

$$f(x) \mapsto \check{f}(x) := \mathrm{tr}(\mathsf{M}(x)A_f) = \int_X f(x')\,\mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x'))\,\mathrm{d}\mu(x') \tag{3}$$

is a local averaging of the original $f(x)$ (which can be very singular, like the Dirac defined in (8) below) with respect to the probability distribution on X,

$$x' \mapsto \mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x'))\,. \tag{4}$$

This averaging, or semi-classical portrait of the operator A_f, is, in general, a regularization. It depends, of course, on the topological nature of the measure space (X, μ) and the functional properties of the $\mathsf{M}(x)$ operators.

2.3. Classical Limit

Consider a set of parameters $\boldsymbol{\kappa}$ and corresponding families of POVM $\mathsf{M}_{\boldsymbol{\kappa}}(x)$ solving the identity

$$\int_X \mathsf{M}_{\boldsymbol{\kappa}}(x)\,\mathrm{d}\mu(x) = \mathbb{1}\,. \tag{5}$$

One says that the classical limit $f(x)$ holds at $\boldsymbol{\kappa}_0$ if

$$\check{f}_{\boldsymbol{\kappa}}(x) := \int_X f(x')\,\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x'))\,\mathrm{d}\mu(x') \to f(x) \quad \text{as} \quad \boldsymbol{\kappa} \to \boldsymbol{\kappa}_0\,, \tag{6}$$

where the convergence $\check{f} \to f$ is defined in the sense of a certain topology.

Otherwise said, $\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x'))$ tends to

$$\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x')) \to \delta_x(x')\,, \tag{7}$$

where δ_x is a Dirac measure with respect to μ,

$$\int_X f(x')\,\delta_x(x')\,\mathrm{d}\mu(x') = f(x)\,. \tag{8}$$

Of course, these definitions should be rigorously mathematically formulated, and there is no guarantee of the existence of such a limit.

3. Overview: Scalar Fields on the Rotation Group SO(3) and Fourier and Gabor Transforms

In this section, we first introduce (Section 3.1) all the necessary mathematical objects with which we work, including the group SO(3) and Hilbertian and harmonic analysis of it through its unitary irreducible representations (UIRs). Our primary source of reference for presenting this well-known material is Edmonds' book (1957) [44]. We then proceed to describe the semi-discrete phase space $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$ in Section 3.2. Finally, in Section 3.3, we delve into the core of our work with the definition of the Weyl–Gabor operator, associated coherent states, Gabor transform, and relevant properties and provide a few illustrative examples.

3.1. Quantum Formalism on SO(3)

An element $\mathbf{x}$ of SO(3) can be parametrized in several ways.

(a) In the Euler angles parametrization with ZYZ convention, α and γ are rotation angles about the third axis and β is rotation angle about the second axis, with $\alpha \in [0, 2\pi]$, $\beta \in [0, \pi]$, and $\gamma \in [0, 2\pi]$. The corresponding rotation matrix reads in terms of these one-dimensional matrices as

$$\mathrm{R}(\alpha, \beta, \gamma) = \mathrm{R}_3(\alpha)\mathrm{R}_2(\beta)\mathrm{R}_3(\gamma) \equiv \mathbf{x}(\alpha, \beta, \gamma)\,. \tag{9}$$

The related (non-normalized) Haar measure is given by

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}(\alpha, \beta, \gamma) = \sin\beta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\,, \tag{10}$$

which yields $\mathrm{Vol}(\mathrm{SO}(3)) = 8\pi^2$.

(b) In the axis-angle parametrization, ω belongs to the interval $[0, 2\pi)$ and represents the anticlockwise rotation angle (or follows the right-hand rule) about the oriented axis $\hat{\mathbf{n}}$, which is determined by the usual angular spherical coordinates (θ, φ). Here, θ ranges from 0 to π, and φ from 0 to 2π.

$$\begin{aligned} \mathbf{x}(\omega, \theta, \varphi) &\equiv \mathbf{x}(\omega, \hat{\mathbf{n}}(\theta, \varphi))\,, \\ \hat{\mathbf{n}}(\theta, \phi) &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \end{aligned} \tag{11}$$

The matrix representation of $\mathbf{x}(\omega, \hat{\mathbf{n}})$ is given by

$$\mathbf{x}(\omega, \hat{\mathbf{n}}) = \cos\omega\, \mathbb{1}_3 + (1 - \cos\omega)\, \hat{\mathbf{n}}\, {}^{\mathrm{t}}\hat{\mathbf{n}} + \sin\omega\, \hat{\mathbf{n}}\times = \exp(\omega \hat{\mathbf{n}}\times)\,, \tag{12}$$

where $\hat{\mathbf{n}}\, {}^{\mathrm{t}}\hat{\mathbf{n}}$ is the orthogonal projector on $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}\, {}^{\mathrm{t}}\hat{\mathbf{n}}\, \mathbf{v} = \hat{\mathbf{n}} \cdot \mathbf{v}$, and $\hat{\mathbf{n}}\times$ linearly acts on $\mathbb{R}^3$ as

$$\hat{\mathbf{n}} \times \mathbf{v} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_z & n_x & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}. \tag{13}$$

The related unnormalized Haar measure is

$$\mathrm{d}\mathbf{x}(\omega, \hat{\mathbf{n}}) = (1 - \cos\omega)\mathrm{d}\omega\, \mathrm{d}\hat{\mathbf{n}} = (1 - \cos\omega)\sin\theta\, \mathrm{d}\omega\, \mathrm{d}\theta\, \mathrm{d}\varphi\,. \tag{14}$$

We now consider the Hilbert space $\mathcal{K} = L^2(\mathrm{SO}(3), \mathrm{d}\mathbf{x})$ of square integrable functions ψ on the rotation group SO(3), that is, functions satisfying the condition,

$$\langle \psi | \psi \rangle \equiv \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\psi(\mathbf{x})|^2 < \infty\,. \tag{15}$$

The group multiplication on the left induces the unitary action of the operator L on $\mathcal{K}$:

$$\mathrm{SO}(3) \ni \mathbf{q} \mapsto \mathsf{L}(\mathbf{q})\,, \quad \psi \in \mathcal{K}\,, \quad (\mathsf{L}(\mathbf{q}))\psi)(\mathbf{x}) = \psi(\mathbf{q}^{-1}\mathbf{x})\,. \tag{16}$$

The three basic generators (angular momentum components) of this action in the Euler angles parametrization (9) are given by [44]:

$$L_x = -\mathrm{i}\left(-\cos\alpha\,\cot\beta\,\frac{\partial}{\partial\alpha} - \sin\alpha\,\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\,\frac{\partial}{\partial\gamma}\right), \tag{17}$$

$$L_y = -\mathrm{i}\left(-\sin\alpha\,\cot\beta\,\frac{\partial}{\partial\alpha} + \cos\alpha\,\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\,\frac{\partial}{\partial\gamma}\right), \tag{18}$$

$$L_z = -\mathrm{i}\frac{\partial}{\partial\alpha}\,. \tag{19}$$

One can expand any function $\psi \in \mathcal{K}$ in terms of the Wigner $\mathcal{D}$-functions $\mathcal{D}^l_{m\,n}$, with $l \in \mathbb{N}$, and $m, n = -l, -l+1, \ldots, 0, \ldots, l-1, l$. Hence, these functions form an Hilbertian basis of $\mathcal{K}$, and the set of triplets of integers

$$\{(l, m, n)\,,\, l \in \mathbb{N}\,,\, m, n = -l, -l+1, \ldots, 0, \ldots, l-1, l\} \equiv \widehat{SO(3)} \tag{20}$$

form the Fourier dual of SO(3). The Wigner $\mathcal{D}$-functions $\mathcal{D}^l_{m\,n}$ are matrix elements of the irreducible unitary representation of SO(3) with respect to the Hilbertian basis of normalized spherical harmonics $Y_{lm}(\theta, \varphi)$ in $\mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\hat{\mathbf{n}})$. Our convention concerning the latter is given by Edmonds [44]:

$$\begin{aligned} Y_{lm}(\theta, \varphi) &= (-1)^m \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} P^l_m(\cos\theta)\, e^{\mathrm{i}m\varphi}\,, \\ \int_{\mathbb{S}^2} &\overline{Y_{lm}(\theta, \varphi)} Y_{l'm'}(\theta, \varphi)\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi = \delta_{ll'}\,\delta_{mm'}\,, \end{aligned} \tag{21}$$

where $P^l_m(x)$ comprises the associated Legendre functions [45].

In the Euler angle parametrization, the Wigner $\mathcal{D}$-functions appear in the expansion [5,44]:

$$Y_{ln}(\mathbf{x}^{-1}(\alpha, \beta, \gamma) \cdot (\theta, \varphi)) = \sum_{m=-l}^{l} \mathcal{D}^l_{m\,n}(\mathbf{x}(\alpha, \beta, \gamma)) Y_{lm}(\theta, \varphi) \tag{22}$$

and are given by

$$\mathcal{D}^l_{m\,n}(\mathbf{x}(\alpha, \beta, \gamma)) = \int_{\mathbb{S}^2} \overline{Y_{lm}(\theta, \varphi)} Y_{l'n}(\mathbf{x}^{-1}(\alpha, \beta, \gamma) \cdot (\theta, \varphi))\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi = e^{\mathrm{i}m\alpha} d^l_{m\,n}(\beta) e^{\mathrm{i}n\gamma}\,. \tag{23}$$

In this expression, the functions $d^l_{m\,n}$ are expressed in terms of the Jacobi polynomials for $m - n \geq 0$ and $m + n \geq 0$ [45]:

$$d^l_{m\,n}(\beta) = \left[\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!}\right]^{1/2} \left(\cos\frac{\beta}{2}\right)^{m+n} \left(\sin\frac{\beta}{2}\right)^{m-n} P^{(m-n, m+n)}_{l-m}(\cos\beta)\,. \tag{24}$$

The other cases give similar expressions after using symmetries of indexes for these polynomials. More precisely, from the general expression of the Jacobi polynomials

$$P^{(\mu,\nu)}_n(x) = 2^{-n} \sum_r \binom{n+\mu}{r} \binom{n+\nu}{n-r} (x+1)^r (x-1)^{n-r}\,, \tag{25}$$

with

$$P^{(\mu,\nu)}_n(-x) = (-1)^n\, P^{(\nu,\mu)}_n(x)\,, \quad P^{(-l,\nu)}_n(x) = \frac{\binom{n+\nu}{l}}{\binom{n}{l}} \left(\frac{x-1}{2}\right)^l P^{(l,\nu)}_{n-l}(x)\,, \tag{26}$$

one can derive the Fourier series expansion of the Wigner d-functions (not trivial!):

$$d^l_{m\,n}(\beta) = \mathrm{i}^{n-m} \sum_{m'=-l}^{m'=l} \Delta^l_{m'm} \Delta^l_{m'n} e^{\mathrm{i}\, m'\beta}, \tag{27}$$

where $\Delta^l_{m\,n} = d^l_{m\,n}(\pi/2)$ [46,47]. In terms of its matrix elements, the unitarity of the $\mathcal{D}$-matrix at fixed l reads

$$\sum_n \mathcal{D}^l_{m\,n}(\mathbf{x})\, \mathcal{D}^l_{n\,m'}(\mathbf{x}^{-1}) = \sum_n \mathcal{D}^l_{m\,n}(\mathbf{x})\, \overline{\mathcal{D}^l_{m'\,n}(\mathbf{x})} = \delta_{mm'}\,, \tag{28}$$

and so,

$$\sum_n |\mathcal{D}^l_{m\,n}(\mathbf{x})|^2 = 1\,, \tag{29}$$

while the orthogonality relations obeyed by these $\mathcal{D}$-matrix elements read

$$\left\langle \mathcal{D}^l_{m\,n} | \mathcal{D}^{l'}_{m'\,n'} \right\rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\mathcal{D}^l_{m\,n}(\mathbf{x})} \mathcal{D}^{l'}_{m'\,n'}(\mathbf{x}) = \frac{8\pi^2}{2l+1} \delta_{l\,l'} \delta_{m\,m'} \delta_{n\,n'}\,, \tag{30}$$

Let us introduce the Dirac distribution $\delta_{\mathrm{e}}(\mathbf{x})$ on SO(3) as having its support at the group identity $\mathrm{e} \equiv \mathbb{1}_3$:

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \delta_{\mathrm{e}}(\mathbf{x}) = 1 \quad \text{and} \quad \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \delta_{\mathrm{e}}(\mathbf{x})\, f(\mathbf{x}) = f(\mathrm{e})\,, \tag{31}$$

for all test functions in some dense subspace of $\mathcal{K}$, e.g., infinitely differentiable. For any $\mathbf{q} \in \mathrm{SO}(3)$ and from the SO(3) invariance of the measure, we have

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \delta_{\mathrm{e}}(\mathbf{x})\, f(\mathbf{q}\mathbf{x}) = f(\mathbf{q}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \delta_{\mathrm{e}}(\mathbf{q}^{-1}\mathbf{x})\, f(\mathbf{x})\,, \tag{32}$$

which entails the definition of the Dirac distribution $\delta_{\mathbf{q}}$ with support at any point $\mathbf{q} \in \mathrm{SO}(3)$:

$$\delta_{\mathrm{e}}(\mathbf{q}^{-1}\mathbf{x}) \equiv \delta_{\mathbf{q}}(\mathbf{x}) \Rightarrow \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \delta_{\mathbf{q}}(\mathbf{x})\, f(\mathbf{x}) = f(\mathbf{q})\,. \tag{33}$$

Using Dirac notations, we introduce kets $|\mathbf{x}\rangle$ and their dual bras $\langle\mathbf{x}|$, both labeled by the points $\mathbf{x} \in \mathrm{SO}(3)$, as obeying the following orthogonality and normalization (in the distributional sense) and resolution of the unity in $\mathcal{K}$:

$$\langle \mathbf{x}|\mathbf{x}'\rangle = \delta_{\mathbf{x}'}(\mathbf{x}) = \delta_{\mathbf{x}}(\mathbf{x}') \equiv \delta(\mathbf{x},\mathbf{x}')\,, \tag{34}$$

$$\mathbb{1} = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\mathbf{x}\rangle\langle\mathbf{x}|. \tag{35}$$

From its construction, we derive the invariance property of the Dirac distribution on SO(3):

$$\delta(\mathbf{q}\mathbf{x}, \mathbf{q}\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}') \ \ \forall \mathbf{x} \in \mathrm{SO}(3)\,. \tag{36}$$

With these notations, one can write for any $\psi \in \mathcal{K}$ (or for suitably defined distributions)

$$\psi(\mathbf{x}) \equiv \langle \mathbf{x}|\psi\rangle\,. \tag{37}$$

With this formalism at hand, the completeness of the Hilbertian basis

$$\left\{ \mathcal{D}^l_{m\,n}\,,\ (l,m,n) \in \widehat{\mathrm{SO}(3)} \right\} \tag{38}$$

in $\mathcal{K}$ reads

$$\frac{2l+1}{8\pi^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \overline{\mathcal{D}^l_{m\,n}(\mathbf{x})} \mathcal{D}^l_{m\,n}(\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')\,. \tag{39}$$

On the other hand, as matrix elements of the unitary operator $\mathsf{L}(\mathbf{q})$ (irreducibly acting in the $(2l+1)$-dimensional subspace $\mathcal{H}_l$ of $\mathcal{H} = \oplus_{l=0}^{\infty}\mathcal{H}_l$), they are uniformly bounded by

$$|\mathcal{D}^l_{m\,n}(\mathbf{x})| \leq 1\,. \tag{40}$$

We now define the SO(3) *Fourier transform* of $\psi \in \mathcal{K}$ as the orthogonal projection of ψ on the basis $\left\{\mathcal{D}^l_{m\,n}\right\}$, which is its Fourier coefficient:

$$\hat{\psi}_{j\,m\,n} = \sqrt{\frac{2l+1}{8\pi^2}}\langle \mathcal{D}^l_{mn}|\psi\rangle = \sqrt{\frac{2l+1}{8\pi^2}}\int_{\mathrm{SO(3)}} \mathrm{d}\mathbf{x}\,\overline{\mathcal{D}^l_{mn}(\mathbf{x})}\psi(\mathbf{x})\,, \tag{41}$$

and its inverse is consistently the Fourier series expansion:

$$\psi(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}\mathcal{D}^l_{m\,n}(\mathbf{x})\hat{\psi}_{l\,m\,n}\,. \tag{42}$$

3.2. Phase-Space Formalism

Inspired by the Mukunda et al.'s approach [35,36], we now consider the rotation $\mathbf{x}$ as an element of the configuration space SO(3) and the triple (l,m,n) of its unitary ($\sim$ Fourier) dual $\widehat{\mathrm{SO}(3)}$ as *momentum* or *frequency* variables. Hence, we denote in the following:

$$\mathbf{p} \equiv (l,m,n) \in \widehat{\mathrm{SO}(3)}\,, \quad \langle \mathbf{p}|\psi\rangle \equiv \hat{\psi}_{l\,m\,n}\,, \tag{43}$$

with orthogonality relations and resolution of the identity

$$\langle \mathbf{p}|\mathbf{p}'\rangle = \delta_{\mathbf{p}\,\mathbf{p}'}\,, \quad \mathbb{1} = \sum_{\mathbf{p}}|\mathbf{p}\rangle\langle\mathbf{p}|\,. \tag{44}$$

With these shortened notations, we write the Hilbertian basis as

$$e_{\mathbf{p}}(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x})\,. \tag{45}$$

The completeness relation (39), Fourier transform (41), and its inverse (42) take the simplest forms:

$$\sum_{\mathbf{p}}\overline{e_{\mathbf{p}}(\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')\,, \tag{46}$$

$$\hat{\psi}_{j\,m\,n} \equiv \hat{\psi}(\mathbf{p}) = \int_{\mathrm{SO(3)}}\mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}}(\mathbf{x})}\psi(\mathbf{x}) \equiv \mathcal{F}[\psi](\mathbf{p})\,, \tag{47}$$

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}}e_{\mathbf{p}}(\mathbf{x})\hat{\psi}(\mathbf{p}) \equiv \overline{\mathcal{F}}\left[\hat{\psi}\right](\mathbf{x})\,. \tag{48}$$

3.3. SO(3)-Weyl–Gabor Operator, Coherent States, and Gabor Transform

In this subsection, we introduce the SO(3)-Weyl–Gabor operator, an essential tool for comprehending the integral quantization formalism that is expounded upon in the subsequent sections. It serves as a pivotal link between the introductory exploration of functions, whether in the form of wavefunctions or signals, within the Hilbert space $\mathcal{K} = L^2(\mathrm{SO}(3),\mathrm{d}\mathbf{x})$, as described above. It facilitates the transition from understanding these functions in their native Hilbert space to their representation in the phase space. This bridge is precisely established through the coherent states built from the action of the SO(3)-Weyl–Gabor operator on fiducial vectors. The journey from the abstract space of functions to their manifestation in the phase space paves the way for a more profound grasp of the integral quantization framework that is to follow.

3.3.1. SO(3)-Weyl–Gabor Operator

Besides the unitary representation operator L introduced in Equation (16), we define the non-unitary modulation operator by the momentum variable $\mathbf{p} = (l, m, n)$. This operator is a non-Hermitian, bounded multiplication operator:

$$(\mathsf{E}_{\mathbf{p}}\psi)(\mathbf{x}) = e_{\mathbf{p}}(\mathbf{x})\psi(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x})\psi(\mathbf{x}), \quad \|\mathsf{E}_{\mathbf{p}}\|_{\mathcal{K}} = \sqrt{\frac{2l+1}{8\pi^2}}. \tag{49}$$

Note that it is the sum of unitary operators due to (23) and (27):

$$(\mathsf{E}_{\mathbf{p}}\psi)(\mathbf{x}) = \mathrm{i}^{n-m}\sum_{m'=-l}^{m'=l}\Delta^l_{m'm}\Delta^l_{m'n}e^{\mathrm{i}\,(m\alpha+m'\beta+n\gamma)}\,\psi(\mathbf{x}(\alpha,\beta,\gamma)). \tag{50}$$

Its adjoint $\mathsf{E}^\dagger$ is defined by

$$(\mathsf{E}^\dagger_{\mathbf{p}}\psi)(x) = \overline{e_{\mathbf{p}}(\mathbf{x})}\psi(\mathbf{x}) == \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{n\,m}(\mathbf{x}^{-1})\psi(\mathbf{x}) = e_{\mathbf{p}^{\mathrm{t}}}(\mathbf{x}^{-1}), \tag{51}$$

where the transpose $\mathbf{p}^{\mathrm{t}}$ of $\mathbf{p}$ means

$$\mathbf{p}^{\mathrm{t}} = (l, m, n)^{\mathrm{t}} = (l, n, m). \tag{52}$$

Combining these operators leads to the (non-unitary) "SO(3)-Weyl-Gabor" operator

$$\mathsf{U}(\mathbf{q}, \mathbf{p}) := \mathsf{E}_{\mathbf{p}}\mathsf{L}(\mathbf{q}) = e_{\mathbf{p}}(\cdot)\mathsf{L}(\mathbf{q}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\cdot)\mathsf{L}(\mathbf{q}), \tag{53}$$

acting on $\mathcal{K}$ as

$$(\mathsf{U}(\mathbf{q}, \mathbf{p})\psi)(\mathbf{x}) = e_{\mathbf{p}}(\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{x}). \tag{54}$$

Its adjoint $\mathsf{U}^\dagger = \mathsf{L}^\dagger(\mathbf{q})\mathsf{E}^\dagger_{\mathbf{p}}$ acts on $\mathcal{K}$ as

$$(\mathsf{U}^\dagger(\mathbf{q}, \mathbf{p})\psi)(\mathbf{x}) = \overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})}\psi(\mathbf{q}\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\overline{\mathcal{D}^l_{m\,n}(\mathbf{q}\mathbf{x})}\psi(\mathbf{q}\mathbf{x}). \tag{55}$$

We then have the following actions on $\psi \in \mathcal{K}$:

$$\left(\mathsf{U}^\dagger(\mathbf{q}', \mathbf{p}')\mathsf{U}(\mathbf{q}, \mathbf{p})\psi\right)(\mathbf{x}) = \overline{e_{\mathbf{p}'}(\mathbf{q}'\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{q}'\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x}), \tag{56}$$

$$\left(\mathsf{U}(\mathbf{q}, \mathbf{p})\mathsf{U}(\mathbf{q}', \mathbf{p}')\mathsf{U}^\dagger(\mathbf{q}, \mathbf{p})\psi\right)(\mathbf{x}) = e_{\mathbf{p}}(\mathbf{x})e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})}\psi(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x}). \tag{57}$$

In particular, the lack of unitarity of U and $\mathsf{U}^\dagger$ is obvious from the fact that $\mathsf{U}\,\mathsf{U}^\dagger$ and $\mathsf{U}^\dagger\,\mathsf{U}$ are nontrivial bounded multiplication operators:

$$\left(\mathsf{U}^\dagger(\mathbf{q}, \mathbf{p})\,\mathsf{U}(\mathbf{q}, \mathbf{p})\,\psi\right)(\mathbf{x}) = |e_{\mathbf{p}}(\mathbf{q}\mathbf{x})|^2\,\psi(\mathbf{x}), \quad \left(\mathsf{U}(\mathbf{q}, \mathbf{p})\,\mathsf{U}^\dagger(\mathbf{q}, \mathbf{p})\,\psi\right)(\mathbf{x}) = |e_{\mathbf{p}}(\mathbf{x})|^2\,\psi(\mathbf{x}) \tag{58}$$

3.3.2. Coherent States

Let us pick a normalized vector ϕ in $\mathcal{K}$ and consider the family of family of states labeled by the elements of the phase space $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$:

$$|\mathbf{q}, \mathbf{p}\rangle_\phi := \mathsf{U}(\mathbf{q}, \mathbf{p})\phi, \quad \langle\mathbf{x}|\mathbf{q}, \mathbf{p}\rangle_\phi = e_{\mathbf{p}}(\mathbf{x})\phi(\mathbf{q}^{-1}\mathbf{x}). \tag{59}$$

These states are named Γ-coherent states with *fiducial* vector ϕ for the reason that they solve the identity in $\mathcal{K}$, as asserted in the following.

Proposition 1. *Let us equip the phase space* Γ *with the measure*

$$\int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p} := \sum_{\mathbf{p}=(lmn)} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,. \tag{60}$$

Then, the states $|\mathbf{q},\mathbf{p}\rangle_{\phi}$ *resolve the identity* $\mathbb{1}$ *in* $\mathcal{K}$ *with respect to this measure:*

$$\mathbb{1} = \int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,|\mathbf{q},\mathbf{p}\rangle_{\phi\phi}\langle \mathbf{q},\mathbf{p}|\,. \tag{61}$$

Proof. Pick ψ, $\psi' \in \mathcal{K}$, and compute

$$\begin{aligned}\langle\psi|\left[\int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,|\mathbf{q},\mathbf{p}\rangle_{\phi\phi}\langle \mathbf{q},\mathbf{p}|\right]|\psi'\rangle &= \int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,\langle\psi|\mathbf{q},\mathbf{p}\rangle_{\phi\phi}\langle \mathbf{q},\mathbf{p}|\psi'\rangle \\ &= \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\,\psi'(\mathbf{x}')\overline{e_{\mathbf{p}}(\mathbf{x})} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\phi\left(\mathbf{q}^{-1}\mathbf{x}\right)\overline{\phi(\mathbf{q}^{-1}\mathbf{x}')}\,.\end{aligned}$$

First, performing the sum on $\mathbf{p} = (l,m,n)$ yields $\delta(\mathbf{x},\mathbf{x}')$ by the application of (46). By integrating the latter and using the invariance of the Haar measure $\mathrm{d}\mathbf{q}$, we end with

$$\begin{aligned}&\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}\,\psi'(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q})}\,\phi(\mathbf{q}) \\ &= \langle\psi|\psi'\rangle\,\|\phi\|^2 = \langle\psi|\psi'\rangle\,.\end{aligned}$$

□

3.3.3. Gabor Transform

The Gabor transform, denoted by $\mathcal{L}_{\phi}$, maps $\psi \in \mathcal{K}$ to a function $\Psi(\mathbf{q},\mathbf{p})$ in the Hilbert space $\mathfrak{K} = L^2(\Gamma, \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p})$ of square integrable functions on the phase space Γ equipped with the measure $\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}$:

$$\mathcal{L}_{\phi} : \psi \mapsto \Psi\,, \quad \Psi(\mathbf{q},\mathbf{p}) = (\mathcal{L}_{\phi}\psi)(\mathbf{q},\mathbf{p}) = {}_{\phi}\langle \mathbf{p},\mathbf{q}|\psi\rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}}(\mathbf{x})}\,\overline{\phi(\mathbf{q}^{-1}\mathbf{x})}\psi(\mathbf{x}) \tag{62}$$

Proposition 2. *The map* $\mathcal{L}_{\phi}$ *satisfies the following properties:*

(i) *It is an isometry:*

$$||\psi||^2 = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,|\psi(\mathbf{x})|^2 = \int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,|\Psi(\mathbf{q},\mathbf{p})|^2 = ||\Psi||^2\,; \tag{63}$$

(ii) *It can be inverted on its range:*

$$\psi(\mathbf{x}) = \int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,\Psi(\mathbf{q},\mathbf{p})\langle \mathbf{x}|\mathbf{q},\mathbf{p}\rangle_{\phi}\,; \tag{64}$$

(iii) *The closure of the range of* $\mathcal{L}_{\phi}$ *is a reproducing kernel Hilbert space:*

$$\begin{aligned}(\mathcal{L}_{\phi}\psi)(\mathbf{q},\mathbf{p}) = \Psi(\mathbf{q},\mathbf{p}) &= \int_{\Gamma} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{p}'\,{}_{\phi}\langle \mathbf{q},\mathbf{p}|\mathbf{q}',\mathbf{p}'\rangle_{\phi}\Psi(\mathbf{q}',\mathbf{p}') \\ &\equiv \int_{\Gamma} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{p}'\,K_{\phi}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\,\Psi(\mathbf{q}',\mathbf{p}').\end{aligned} \tag{65}$$

Proof. All statements are straightforward consequences of the resolution of the identity (61).
□

Proposition 3. *We have the following trace formulas for the SO(3)-Weyl–Gabor operator:*

$$\mathrm{Tr}[\mathsf{U}(\mathbf{q},\mathbf{p})] = \delta_{\mathbf{p}\,\mathbf{0}}\,\delta_{\mathrm{e}}(\mathbf{q})\,,\quad \mathbf{p}=(l,m,n)\,,\quad \mathbf{0}=(0,0,0)\,, \tag{66}$$

$$\mathrm{Tr}\Big[\mathsf{U}^{\dagger}(\mathbf{q}',\mathbf{p}')\mathsf{U}(\mathbf{q},\mathbf{p})\Big] = \delta_{\mathbf{p}\mathbf{p}'}\delta(\mathbf{q},\mathbf{q}')\,. \tag{67}$$

Proof. For (66), using (46) and the orthonormality of $e_{\mathbf{p}}$,

$$\mathrm{Tr}[\mathsf{U}(\mathbf{q},\mathbf{p})] = \sum_{\mathbf{p}'}\int_{SO(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x})e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x}) = \delta_{\mathrm{e}}(\mathbf{q}^{-1})\delta_{\mathbf{p}\mathbf{0}} = \delta_{\mathrm{e}}(\mathbf{q})\delta_{\mathbf{p}\mathbf{0}}\,.$$

For (67),

$$\begin{aligned}
\mathrm{Tr}[\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\mathsf{U}(\mathbf{q}',\mathbf{p}')] &= \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\Big(\mathsf{E}^{\dagger}_{\mathbf{p}}\mathsf{L}(\mathbf{q}^{-1}\mathbf{q}')\mathsf{E}_{\mathbf{p}'}e_{\mathbf{b}}\Big)(\mathbf{x})\\
&= \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{b}(\mathbf{x})}\,\overline{e_{\mathbf{p}}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x})e_{\mathbf{b}}(\mathbf{q}'^{-1}\mathbf{q}\mathbf{x})\\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x})\delta(\mathbf{x}^{-1}\mathbf{q}'^{-1}\mathbf{q}\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x})\delta(\mathbf{q}'^{-1}\mathbf{q})\\
&= \delta_{\mathbf{p}\mathbf{p}'}\,\delta(\mathbf{q},\mathbf{q}')\,.
\end{aligned}$$

□

3.4. Example of Fiducial Vectors and Coherent States

As seen above, any square-integrable function on SO(3), including the completely non-localized function $\phi = 1$ on the manifold SO(3), can be viewed as a fiducial vector. Corresponding coherent states form the family $\{\langle\mathbf{x}|\mathbf{q},\mathbf{p}\rangle_{\phi}\}$ of transported ϕ through the SO(3)-Weyl–Gabor operator. It is, of course, interesting to consider fiducial vectors that are well "localized" in position and momentum. Although it is not the main purpose of this paper, we present a few fiducial vectors that can be of interest. Some of these examples are extracted from signal processing on SO(3) as related to probability densities.

1. Eigenfunctions of certain operators [48]. The first example is the free rotor fiducial vector, which is the eigenfunction of $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$.

$$e_{\mathbf{p}}(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,. \tag{68}$$

   The second example is the highest fiducial vector for SO(3). which is cancelled by $L_+ = L_x + \mathrm{i}\,L_y$, that is,

$$e_{l\,l\,l}(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^{l}_{l\,l}(\mathbf{x}(\alpha,\beta,\gamma))\,. \tag{69}$$

2. Some radial fiducial vectors. Below, we give examples of fiducial vectors depending only on

$$|\mathbf{x}(\alpha,\beta,\gamma)| := \arccos\left(\frac{\mathrm{Tr}[\mathbf{x}(\alpha,\beta,\gamma)]-1}{2}\right). \tag{70}$$

   The above expression defines a metric on SO(3), whose details can be found in [21].

   (a) The κ-dependent von Mises–Fisher Kernel fiducial vectors ϕ [21], their derivatives with respect to β and α, and their difference at two different κ:

$$\phi(\mathbf{x}(\alpha,\beta,\gamma)) = \frac{e^{\kappa\cos(|\,\mathbf{x}(\alpha,\beta,\gamma)\,|)}}{I_0(\kappa)-I_1(\kappa)} = \frac{e^{\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa)-I_1(\kappa)}\,,\quad \kappa>0\,; \tag{71}$$

$$\phi^{(1)}_{\beta}(\mathbf{x}(\alpha,\beta,\gamma)) = -\frac{K}{2}\sin\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}\frac{e^{\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa)-I_1(\kappa)}, \quad \kappa>0\,; \tag{72}$$

$$\phi^{(1)}_{\alpha}(\mathbf{x}(\alpha,\beta,\gamma)) = -\frac{K}{2}\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}\frac{e^{\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa)-I_1(\kappa)}, \quad \kappa>0\,. \tag{73}$$

$$\phi_{dov}(\mathbf{x}(\alpha,\beta,\gamma)) = \frac{e^{\kappa_1\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa_1)-I_1(\kappa_1)} - \frac{e^{\kappa_2\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa_2)-I_1(\kappa_2)}, \quad \kappa_1,\kappa_2>0\,. \tag{74}$$

where I_n, $n\in\mathbb{N}$, denotes the modified Bessel functions of the first kind.
In Appendix B, we give plots of these fiducial vectors in α and β variables at a fixed γ and for a few values of κ (Figures A1 and A2).

(b) The Abel–Poisson fiducial vector ϕ [21]:

$$\phi(\mathbf{x}(\alpha,\beta,\gamma)) = \frac{1}{2}\left[\frac{1-\kappa^2}{1+2\kappa\cos(|\,\mathbf{x}(\alpha,\beta,\gamma)\,|)+\kappa^2} - \frac{1-\kappa^2}{1-2\kappa\cos(|\,\mathbf{x}(\alpha,\beta,\gamma)\,|)+\kappa^2}\right]$$
$$= \frac{1}{2}\left[\frac{1-\kappa^2}{1+2\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})\,|)+\kappa^2} - \frac{1-\kappa^2}{1-2\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})+\kappa^2}\right], \tag{75}$$

with $\kappa>0$.

4. Quantization Operators and the Quantization Map

In line with prior research [42,43], we select a function, denoted as ϖ, to serve as a weight in the phase space Γ. However, this function is not necessarily positive. We then define the operator M^{ϖ} by

$$\mathsf{M}^{\varpi} = \sum_{\mathbf{p}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\varpi(\mathbf{q},\mathbf{p})\,\mathsf{U}(\mathbf{q},\mathbf{p})\,, \tag{76}$$

and we choose the weight such that the operator M^{ϖ} is bounded and symmetric, i.e., self-adjoint on the Hilbert $\mathcal{K}=L^2(\mathrm{SO}(3),\mathrm{d}\mathbf{x})$ of "physical states".

In what follows, we compute the kernel of this operator and the related trace.

Proposition 4. *With the assumption that the weight ϖ is chosen such that the operator M^{ϖ}, defined by (76), is bounded:*

(i) The operator M^{ϖ} is the integral operator:

$$(\mathsf{M}^{\varpi}\psi)(\mathbf{x}) = \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}'\,\mathcal{M}^{\varpi}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')\,, \tag{77}$$

where the kernel $\mathcal{M}^{\varpi}(\mathbf{x},\mathbf{x}')$ is given by

$$\mathcal{M}^{\varpi}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}}\varpi(\mathbf{x}\mathbf{x}'^{-1},\mathbf{p})e_{\mathbf{p}}(\mathbf{x}) = \widetilde{\varpi}_p(\mathbf{x}\mathbf{x}'^{-1},\mathbf{x}), \tag{78}$$

where $\widetilde{\varpi}_p$ is the partial inverse discrete Fourier transform (48) of ϖ with respect to the discrete variables.

(ii) The operator M^{ϖ} is symmetric if and only the weight satisfies

$$\mathsf{M}^{\varpi} = \mathsf{M}^{\varpi\dagger} \Leftrightarrow \widetilde{\varpi}_p(\mathbf{x}{\mathbf{x}'}^{-1},\mathbf{x}) = \overline{\widetilde{\varpi}}_p(\mathbf{x}'\mathbf{x}^{-1},\mathbf{x}')\,. \tag{79}$$

(iii) The trace of M^{ϖ} is given by

$$\mathrm{Tr}(\mathsf{M}^{\varpi}) = \varpi(\mathrm{e},\mathbf{0})\,. \tag{80}$$

Proof.

(i) The action of M^{ϖ} on ψ is given by

$$(\mathsf{M}^{\varpi}\psi)(\mathbf{x}) = \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \varpi(\mathbf{q},\mathbf{p}) e_{\mathbf{p}}(\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{x})\,.$$

Using the change of variable $\mathbf{q} \mapsto \mathbf{x}' = \mathbf{q}^{-1}\mathbf{x}$, the invariance of the Haar measure $\mathrm{d}\mathbf{q}$, and the partial inverse discrete Fourier transform (48), we obtain the expected kernel (78).

(ii) The action of $\mathsf{M}^{\varpi\dagger}$ on ψ is given by

$$(\mathsf{M}^{\varpi\dagger}\psi)(\mathbf{x}) = \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \overline{\varpi(\mathbf{q},\mathbf{p})}\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})}\psi(\mathbf{q}\mathbf{x})\,.$$

Using the change of variable $\mathbf{q} \mapsto \mathbf{x}' = \mathbf{q}\mathbf{x}$, the invariance of the Haar measure $\mathrm{d}\mathbf{q}$, and the partial discrete Fourier transform (47) we formally obtain (79) by comparison with (78).

(iii) The relation (80) trivially results from (66).

□

Furthermore, we demonstrate in the following proposition that the weight ϖ can be obtained from the quantization operator M^{ϖ} through a trace operation.

Proposition 5. *The trace of the operator* $\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\mathsf{M}^{\varpi}$ *is given by*

$$\mathrm{Tr}[\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\mathsf{M}^{\varpi}] = \varpi(\mathbf{q},\mathbf{p})\,. \tag{81}$$

Proof. This relation trivially results from (67). □

As a helpful example, let us examine the case $\varpi(\mathbf{q},\mathbf{p}) = e_{\mathbf{p}}(\mathbf{q})$. Then, the operator $\mathsf{M}^{e_{\mathbf{p}}}$ is determined through its action on basis elements $e_{\mathbf{p}'}(x)$:

$$\begin{aligned}
\left(\mathsf{M}^{e_{\mathbf{p}}} e_{\mathbf{p}'}\right)(\mathbf{x}) &= \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, e_{\mathbf{p}}(\mathbf{q})\, e_{\mathbf{p}}(\mathbf{x}) e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x}) \\
&= \sum_{l,m,n}\sum_{m''} \frac{2l+1}{8\pi^2}\sqrt{\frac{(2l'+1}{8\pi^2}}\left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathcal{D}^{l}_{mn}(\mathbf{q})\, \mathcal{D}^{l'}_{m'm''}(\mathbf{q}^{-1})\right] \mathcal{D}^{l}_{mn}(\mathbf{x})\, \mathcal{D}^{l'}_{m''n'}(\mathbf{x}) \\
&= \sum_{l,m,n}\sum_{m''} \frac{2l+1}{8\pi^2}\sqrt{\frac{(2l'+1}{8\pi^2}}\left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathcal{D}^{l}_{mn}(\mathbf{q})\, \overline{\mathcal{D}^{l'}_{m''m'}(\mathbf{q})}\right] \mathcal{D}^{l}_{mn}(\mathbf{x})\, \mathcal{D}^{l'}_{m''n'}(\mathbf{x}) \\
&= \sum_{l,m,n}\sum_{m''} \sqrt{\frac{(2l'+1}{8\pi^2}}\,\delta_{ll'}\,\delta_{mm''}\,\delta_{nm'}\,\mathcal{D}^{l}_{mn}(\mathbf{x})\, \mathcal{D}^{l'}_{m''n'}(\mathbf{x}) \\
&= \sum_{m''=-l'}^{l'} \mathcal{D}^{l'}_{m''m'}(\mathbf{x})\, e_{l'm''n'}(\mathbf{x}) = \sum_{m''=-l'}^{l'} \overline{\mathcal{D}^{l'}_{m'm''}(\mathbf{x}^{-1})}\, e_{l'm''n'}(\mathbf{x})\,.
\end{aligned} \tag{82}$$

Let us introduce the *squared rotation operator* $\hat{\mathsf{I}}_{\mathrm{sq}}$, defined by

$$\hat{\mathsf{I}}_{\mathrm{sq}} : \psi \in \mathcal{K} \mapsto \hat{\mathsf{I}}_{\mathrm{sq}}\psi\,, \quad (\hat{\mathsf{I}}_{\mathrm{sq}}\psi)(\mathbf{x}) = \psi(\mathbf{x}^2)\,. \tag{83}$$

With this definition, we precisely obtain from (82)

$$\mathsf{M}^{e_{\mathbf{p}}} = \hat{\mathsf{I}}^{\mathrm{t}}_{\mathrm{sq}} \Leftrightarrow \mathsf{M}^{e_{\mathbf{p}^{\mathrm{t}}}} = \hat{\mathsf{I}}_{\mathrm{sq}}\,, \tag{84}$$

where $\hat{\mathsf{I}}^{\mathrm{t}}_{\mathrm{sq}}$ is the transpose of $\hat{\mathsf{I}}_{\mathrm{sq}}$ (keep in mind that $\mathbf{p}^{\mathrm{t}} = (l,m,n)^{\mathrm{t}} = (l,n,m)$). This operator plays the central role in our definition of the Wigner-like function within the present context (see Section 8). Other examples of weights are considered in the rest of the paper.

5. SO(3)-Covariant Integral Quantization from Weight Function

We now introduce general formulae for quantizing a function in the phase space Γ, resulting in a semi-classical phase space portrait represented as a new function on Γ. The procedure's outcome is contingent upon the choice of a weight. Different weight selections can give rise to two distinct quantizations. In practice, one weight may facilitate the regularization of singularities present in the classical model, while another weight may preserve these singularities intact.

5.1. General Results

We now establish general formulae for the integral quantization issued from a weight function $\varpi(\mathbf{q},\mathbf{p})$ on $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$, yielding the bounded self-adjoint operator M^{ϖ} defined in (76). This allows us to build a family of operators obtained from the SO(3)-Weyl–Gabor operator transport of M^{ϖ}:

$$\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p}) = \mathsf{U}(\mathbf{q},\mathbf{p})\mathsf{M}^{\varpi}\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\,. \tag{85}$$

Then, the corresponding integral quantization is given by the linear map:

$$f \mapsto A_f^{\varpi} = \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f(\mathbf{q},\mathbf{p})\, \mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\,. \tag{86}$$

We have the following result.

Proposition 6. *A_f^{ϖ} is the integral operator on $L^2(\mathrm{SO}(3),\mathrm{d}\mathbf{x})$:*

$$(A_f^{\varpi}\psi)(\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\, \mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}')\, \psi(\mathbf{x}')\,, \tag{87}$$

and its kernel is given by

$$\mathcal{A}_f^{\varpi}(\mathbf{x},\mathbf{x}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \delta^f(\mathbf{x}\mathbf{q}^{-1};(\mathbf{x},\,\mathbf{x}'))\, \widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,, \tag{88}$$

with a weighted version of the completeness relation

$$\delta^f(\mathbf{q};(\mathbf{x},\mathbf{x}')) := \sum_{\mathbf{p}} f(\mathbf{q},\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\,, \quad \delta^1(\mathbf{q};(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')\,. \tag{89}$$

The requirement that $f = 1$ is mapped to the unit operator imposes the following normalization condition on ϖ:

$$\varpi(\mathrm{e},\mathbf{0}) = 1. \tag{90}$$

Proof. The calculation of the kernel of the integral operator A_f^{ϖ} proceeds through the following steps, which are derived from the expressions in (85) and (57).

$$\begin{aligned}
&(A_f^{\varpi})(\psi)(\mathbf{x}) \\
&= \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f(\mathbf{q},\mathbf{p})\, \big(\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\psi\big)(\mathbf{x}) \\
&= \sum_{\mathbf{p}}\sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, f(\mathbf{q},\mathbf{p})\varpi(\mathbf{q}',\mathbf{p}')\Big(U(\mathbf{q},\mathbf{p})U(\mathbf{q}',\mathbf{p}')U^{\dagger}(\mathbf{q},\mathbf{p})\Big)\psi(\mathbf{x}) \\
&= \sum_{\mathbf{p}}\sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, f(\mathbf{q},\mathbf{p})\varpi(\mathbf{q}',\mathbf{p}')e_{\mathbf{p}}(\mathbf{x})e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})}\psi(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})\,.
\end{aligned}$$

We then proceed with the change of variable

$$\mathbf{q}' \mapsto \mathbf{x}' = \mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x}$$

and use the SO(3) invariance of the measures $\mathrm{d}\mathbf{q}'$ to obtain the form

$$(A^{\varpi}_f)(\psi)(\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\, \mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}')\, \psi(\mathbf{x}')\,,$$

with

$$\mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}}\sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f(\mathbf{q},\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\varpi(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{p}')e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x})\,.$$

We then proceed with the change of variables $\mathbf{q} \mapsto \mathbf{z} = \mathbf{q}^{-1}\mathbf{x}$ to obtain

$$\begin{aligned}\mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}') &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z} \left[\sum_{\mathbf{p}} f(\mathbf{x}\mathbf{z}^{-1},\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\right] \left[\sum_{\mathbf{p}'} \varpi(\mathbf{z}\mathbf{x}'^{-1}\mathbf{x}\mathbf{z}^{-1},\mathbf{p}')e_{\mathbf{p}'}(\mathbf{z})\right] \\ &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z}\, \delta^f(\mathbf{x}\mathbf{z}^{-1};(\mathbf{x},\mathbf{x}'))\, \widetilde{\varpi}_p\left(\mathbf{z}\mathbf{x}'^{-1}\mathbf{x}\mathbf{z}^{-1},\mathbf{z}\right).\end{aligned}$$

This is Equation (88) with the notation (89).

Putting $f = 1$ in the above expression and using the completeness relation (46) give $\delta^1(\mathbf{x}\mathbf{z}^{-1};(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')$ and yield (90). □

5.2. Particular Quantizations

In the following discussion, we examine three simplified versions of functions, denoted as $f(\mathbf{q},\mathbf{p})$, on the phase space Γ to make analytical calculations feasible. In more general cases, numerical computations may be required. Therefore, we begin with a function that depends solely on the angle variables, denoted as $f(\mathbf{q},\mathbf{p}) = u(q)$, allowing us to explore situations where singularities occur in the position variable. Similarly, a function that depends only on the momentum variable, denoted as $f(\mathbf{q},\mathbf{p}) = v(p)$, highlights the fact that SO(3) is a curved manifold because of additional terms that appear in its quantized form. The next example involves the separable form, $f(\mathbf{q},\mathbf{p}) = u(q)v(p)$.

To facilitate the calculations, we begin by introducing the following function:

$$\Omega(\mathbf{x},\mathbf{x}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,. \tag{91}$$

It obeys

$$\Omega(\mathbf{x},\mathbf{x}) = \varpi(\mathrm{e},\mathbf{0}) = 1\,. \tag{92}$$

We also introduce the notations

$$\frac{\partial^j}{\partial\alpha'^j}(\Omega(\mathbf{x},\mathbf{x}'))\bigg|_{\mathbf{x}'=\mathbf{x}} = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \frac{\partial^j}{\partial\alpha'^j}\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\bigg|_{\mathbf{x}'=\mathbf{x}} \equiv \Omega^{(j)}_{\alpha}\,. \tag{93}$$

$$\frac{\partial^j}{\partial\beta'^j}(\Omega(\mathbf{x},\mathbf{x}'))\bigg|_{\mathbf{x}'=\mathbf{x}} \equiv \Omega^{(j)}_{\beta}\,, \tag{94}$$

$$\frac{\partial^j}{\partial\gamma'^j}(\Omega(\mathbf{x},\mathbf{x}'))\bigg|_{\mathbf{x}'=\mathbf{x}} \equiv \Omega^{(j)}_{\gamma}\,. \tag{95}$$

In Appendix A, we give examples of such calculations in the case of coherent states.

Of course, $\frac{\partial}{\partial\alpha'}\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})$ or, equivalently, $\frac{\partial}{\partial\alpha'}\varpi(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{p})$ should be understood as

$$\frac{\partial}{\partial\alpha'}\varpi(\underbrace{\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1}}_{\mathbf{z}},\mathbf{p}) = \nabla_{\mathbf{z}}\varpi\cdot\begin{pmatrix}\frac{\partial\alpha(\mathbf{z})}{\partial\alpha'}\\ \frac{\partial\beta(\mathbf{z})}{\partial\alpha'}\\ \frac{\partial\gamma(\mathbf{z})}{\partial\alpha'}\end{pmatrix}, \quad \nabla_{\mathbf{z}}\varpi := \begin{pmatrix}\frac{\partial\varpi}{\partial\alpha(\mathbf{z})}\\ \frac{\partial\varpi}{\partial\beta(\mathbf{z})}\\ \frac{\partial\varpi}{\partial\gamma(bz)}\end{pmatrix}, \tag{96}$$

and $\alpha(\mathbf{z})$ means the Euler angle α of the rotation $\mathbf{z}$, etc. We also need the integral formulae (with suitable conditions on functions appearing in the integrand)

$$\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\alpha}f_1(\mathbf{x})\right)f_2(\mathbf{x}) = -\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,f_1(\mathbf{x})\left(\frac{\partial}{\partial\alpha}f_2(\mathbf{x})\right), \tag{97}$$

$$\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\gamma}f_1(\mathbf{x})\right)f_2(\mathbf{x}) = -\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,f_1(\mathbf{x})\left(\frac{\partial}{\partial\gamma}f_2(\mathbf{x})\right), \tag{98}$$

$$\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\beta}f_1(\mathbf{x})\right)f_2(\mathbf{x}) = -\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,f_1(\mathbf{x})\left(\frac{\partial}{\partial\beta}f_2(\mathbf{x})\right) - \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,\cot\beta\,f_1(\mathbf{x})\,f_2(\mathbf{x})\,. \tag{99}$$

5.2.1. Separable Functions $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})v(\mathbf{p})$

In this case, δ^{uv} in the integral factorizes as

$$\delta^{uv}(\mathbf{x}\mathbf{q}^{-1};(\mathbf{x},\mathbf{x}')) = u(\mathbf{x}\mathbf{q}^{-1})\,\delta^{v}(\mathbf{x},\mathbf{x}')\,, \tag{100}$$

with the notation

$$\delta^{v}(\mathbf{x},\mathbf{x}')) = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}\,. \tag{101}$$

Hence,

$$\mathcal{A}^{\varpi}_{f}(\mathbf{x},\mathbf{x}') = \delta^{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,. \tag{102}$$

5.2.2. Univariate Function $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$

In this case the above, (102) simplifies to

$$\begin{aligned}\mathcal{A}^{\varpi}_{u}(\mathbf{x},\mathbf{x}') &= \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\\ &= \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{\varpi}_p(\mathrm{e},\mathbf{q})\\ &= \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,u(\mathbf{q})\,\widetilde{\varpi}_p(\mathrm{e},\mathbf{q}^{-1}\mathbf{x})\,.\end{aligned} \tag{103}$$

Hence, the quantization of $u(\mathbf{q})$ is the multiplication operator.

$$\left(A^{\varpi}_{u(\mathbf{q})}\psi\right)(\mathbf{x}) = \left(u *_{\mathrm{SO}(3)}\widetilde{\varpi}_p(\mathrm{e},\cdot)\right)(\mathbf{x})\psi(\mathbf{x})\,, \tag{104}$$

where the (noncommutative) convolution $*_{\mathrm{SO}(3)}$ on the group SO(3) is defined by

$$\left(f_1 *_{\mathrm{SO}(3)} f_2\right)(\mathbf{x}) = \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,f_1(\mathbf{x}\mathbf{q}^{-1})\,f_2(\mathbf{q}) = \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,f_1(\mathbf{q})\,f_2(\mathbf{q}^{-1}\mathbf{x})\,. \tag{105}$$

Let us give the quantizations of some basic Fourier or trigonometric functions $v(\mathbf{q})$. In the sequel, we put $\left(A^{\varpi}_{u(\mathbf{q})}\psi\right)(\mathbf{x}) \equiv B^{\varpi}_{u(\mathbf{q})}(\mathbf{x})\,\psi(\mathbf{x})$.

- For $u(\mathbf{q}) = e^{\mathrm{i}\alpha}$, we obtain

$$B^{\varpi}_{e^{i\alpha}}(\mathbf{x}) = e^{i\alpha}\left[\sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,0,n))\mathrm{D}^{l}_{1\,n}(0,\beta,\gamma)\mathrm{a}_l\right]. \tag{106}$$

where

$$\mathrm{a}_l = \int_0^{\pi}\mathrm{d}\beta_{\mathbf{q}}\,\sin(\beta_{\mathbf{q}})\mathrm{d}^{l}_{10}(\beta_{\mathbf{q}})\,. \tag{107}$$

- For $u(\mathbf{q}) = e^{i\gamma}$,

$$B^{\varpi}_{e^{i\gamma}}(\mathbf{x}) == \sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,1,n))\mathrm{D}^{l}_{0\,n}(\alpha,\beta,\gamma)\mathrm{b}_l\,. \tag{108}$$

where

$$\mathrm{b}_l = \int_0^{\pi}\mathrm{d}\beta\,\sin(\beta)\,\mathrm{d}^{l}_{01}(\beta) \tag{109}$$

- For $u(\mathbf{q}) = 1/\sin\beta$,

$$B^{\varpi}_{1/\sin\beta}(\mathbf{x}) = \sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,0,n))\mathrm{D}^{l}_{0\,n}(0,\beta,\gamma)\mathrm{c}_l\,. \tag{110}$$

where

$$\mathrm{c}_l = \int_0^{\pi}\mathrm{d}\beta\mathrm{d}^{l}_{00}(\beta) \tag{111}$$

- For $u(\mathbf{q}) = \cot\beta$,

$$B^{\varpi}_{\cot\beta}(\mathbf{x}) = \sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,0,n))\mathrm{D}^{l}_{0\,n}(0,\beta,\gamma)\mathrm{d}_l\,. \tag{112}$$

where

$$\mathrm{d}_l = \int_0^{\pi}\mathrm{d}\beta\cos(\beta)\mathrm{d}^{l}_{00}(\beta) \tag{113}$$

5.2.3. Univariate Function $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$

The integral kernel reads, in this case,

$$\mathcal{A}^{\varpi}_{f}(\mathbf{x},\mathbf{x}') = \delta^{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}). \tag{114}$$

We know that the values of $\mathbf{p} = (l,m,n)$ are constrained in a forward-rectangular discrete pyramid, which is the momentum space. We here work with Euler angle parameters $\mathbf{x}(\alpha,\beta,\gamma)$ and $\mathbf{x}'(\alpha',\beta',\gamma')$ (for the sake of simplicity, we omit them and explicitly include them when necessary). Let us present the quantization of a few elementary functions $v(\mathbf{p})$.

- $v(l,m,n) = m$

  We have

$$\begin{aligned}\delta^{v}(\mathbf{x},\mathbf{x}') &= \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l}\frac{2l+1}{8\pi^2}\,D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,m\,\overline{D^{l}_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma'))\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l}\frac{2l+1}{8\pi^2}D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,i\frac{\partial}{\partial\alpha'}\overline{D^{l}_{m\,n}(\mathbf{x}'(\alpha',\beta',\gamma'))}\\ &= i\frac{\partial}{\partial\alpha'}\delta(\mathbf{x}(\alpha,\beta,\gamma)),\mathbf{x}'^{-1}(\alpha',\beta',\gamma')\,.\end{aligned}$$

The kernel is then given by

$$\mathcal{A}^{\varpi}_{m}(\mathbf{x},\mathbf{x}') = \left[\mathrm{i}\frac{\partial}{\partial\alpha'}\delta(\mathbf{x},\mathbf{x}')\right]\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,.$$

The action of the quantum version of m on $\psi \in \mathcal{K}$ is then obtained through integration by part and the use of (92) and notation (93):

$$\begin{aligned}(A^{\varpi}_{m}\psi)(\mathbf{x}) &= \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}'\,\delta(\mathbf{x},\mathbf{x}')\left[\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right](-\mathrm{i})\frac{\partial}{\partial\alpha'}\,\psi(\mathbf{x}')\\ &\quad -\mathrm{i}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}'\,\delta(\mathbf{x},\mathbf{x}')\left[\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\frac{\partial}{\partial\alpha'}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right]\psi(\mathbf{x}')\\ &=\left(-\mathrm{i}\frac{\partial}{\partial\alpha}-\mathrm{i}\Omega^{(1)}_{\alpha}\right)\psi(\mathbf{x}) = \left(L_z-\mathrm{i}\Omega^{(1)}_{\alpha}\right)\psi(\mathbf{x})\,.\end{aligned} \tag{115}$$

Under mild conditions on the weight function, we have $\Omega^{(1)}_{\alpha} = 0$. Then, we exactly recover the angular momentum operator component L_z. A similar result holds with the quantization of $v(l,m,n) = n$:

$$A^{\varpi}_{m} = -\mathrm{i}\frac{\partial}{\partial\gamma} - \mathrm{i}\Omega^{(1)}_{\gamma}\,. \tag{116}$$

- $v(l,m,n) = m^2$
  We have

$$\begin{aligned}\delta^{v}(\mathbf{x},\mathbf{x}') &= \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l}\frac{2l+1}{8\pi^2}\,D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,m^2\overline{D^{l}_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma'))\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l}\frac{2l+1}{8\pi^2}\,D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\left[-\frac{\partial^2}{\partial\alpha'^2}\overline{D^{l}_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma'))\right]\\ &= -\frac{\partial^2}{\partial\alpha'^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l}\frac{2l+1}{8\pi^2}\,\overline{D^{l}_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma'))D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\\ &= -\frac{\partial^2}{\partial\alpha'^2}\delta(\mathbf{x}'^{-1}(\alpha',\beta',\gamma')\mathbf{x}(\alpha,\beta,\gamma)) = -\frac{\partial^2}{\partial\alpha'^2}\delta(\mathbf{x},\mathbf{x}')\,.\end{aligned}$$

The expression of the kernel then reads as follows:

$$\mathcal{A}^{\varpi}_{m^2}(\mathbf{x},\mathbf{x}') = \left[-\frac{\partial^2}{\partial\alpha'^2}\delta(\mathbf{x},\mathbf{x}')\right]\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,, \tag{117}$$

and for the quantum operator,

$$\begin{aligned}(A^{\varpi}_{m^2}\psi)(\mathbf{x}) &= \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\left[-\frac{\partial^2}{\partial\alpha'^2}\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right]_{\mathbf{x}'=\mathbf{x}}\psi(\mathbf{x})\\ &\quad +2\mathrm{i}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\left[\frac{\partial}{\partial\alpha'}\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right]_{\mathbf{x}'=\mathbf{x}}\left(-\mathrm{i}\frac{\partial}{\partial\alpha}\psi(\mathbf{x})\right)\\ &\quad +\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\widetilde{\varpi}(\mathrm{e},\mathbf{q})\left(-\frac{\partial^2}{\partial\alpha^2})\psi(\mathbf{x})\right).\end{aligned}$$

i.e.,

$$A^{\varpi}_{m^2} = L_z^2 + 2\mathrm{i}\Omega^{(1)}_{\alpha}(\mathrm{e})L_z - \Omega^{(2)}_{\alpha}(\mathrm{e})\,. \tag{118}$$

- $v(l,m,n) = l(l+1)$. We have to just use the eigenvalue property of the functions $D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))$ or, equivalently, of the functions $d^{l}_{m\,n}(\beta)$ [44],

$$\begin{aligned} &l(l+1)\, D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma)) \\ &= \left[-\frac{\partial^2}{\partial\beta^2} - \cot\beta\frac{\partial}{\partial\beta} + \frac{m^2+n^2-2mn\cos\beta}{\sin^2\beta}\right] D^{l}_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,, \end{aligned} \tag{119}$$

and the results given in the above examples. The corresponding kernel is given by

$$\mathcal{A}^{\varpi}_{l(l+1)}(\mathbf{x},\mathbf{x}') = \left(\mathbf{L}^{(\prime)}\cdot\mathbf{L}^{(\prime)}\delta\right)(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,, \tag{120}$$

where [44]

$$\mathbf{L}^{(\prime)}\cdot\mathbf{L}^{(\prime)} = \left[-\frac{\partial^2}{\partial\beta'^2} - \cot\beta'\frac{\partial}{\partial\beta'} - \frac{1}{\sin^2\beta'}\left(\frac{\partial^2}{\partial\alpha'^2} + \frac{\partial'^2}{\partial\gamma'2} - 2\cos\beta'\frac{\partial^2}{\partial\alpha'\partial\gamma'}\right)\right]. \tag{121}$$

6. Semi-Classical Portrait

Given a function $\varpi(\mathbf{q},\mathbf{p})$ in the phase space Γ, normalized at $\varpi(\mathrm{e},(0,0,0)) = 1$ and yielding a non-negative unit trace operator, i.e., a density operator, M^{ϖ}, the quantum phase space portrait of an operator A on $L^2(\Gamma,\mathrm{d}\gamma)$ is defined as

$$\check{A}(\mathbf{q},\mathbf{p}) := \mathrm{Tr}\Big(A\,U(\mathbf{q},\mathbf{p})\,\mathsf{M}^{\varpi}U^{\dagger}(\mathbf{q},\mathbf{p})\Big) = \mathrm{Tr}\big(A\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\big)\,. \tag{122}$$

The most interesting aspect of this notion in terms of probabilistic interpretation holds when the operator A is precisely the integral quantized version A^{ϖ}_f of a classical $f(\mathbf{q},\mathbf{p})$ with the same function ϖ (actually, we can define the transform with two different weights, one for the "analysis" and the other for the "reconstruction"). Then, with the use of the composition rule, let us compute the transform:

$$f(\mathbf{q},\mathbf{p}) \mapsto \check{f}(\mathbf{q},\mathbf{p}) \equiv \check{A}^{\varpi}_f(\mathbf{q},\mathbf{p}) = \mathrm{Tr}\Big(A^{\varpi}_f\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\Big)\,. \tag{123}$$

We successively have

$$\begin{aligned} &\mathrm{Tr}\Big(A^{\varpi}_f\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\Big) = \sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,f(\mathbf{q}',\mathbf{p}')\varpi(\mathbf{q}'',\mathbf{p}'')\varpi(\mathbf{q}''',\mathbf{p}''')\times \\ &\mathrm{Tr}\Big(U(\mathbf{q}',\mathbf{p}')U(\mathbf{q}'',\mathbf{p}'')U^{\dagger}(\mathbf{q}',\mathbf{p}')\,U(\mathbf{q},\mathbf{p})U(\mathbf{q}''',\mathbf{p}''')U^{\dagger}(\mathbf{q},\mathbf{p})\Big) \\ &= \sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,f(\mathbf{q}',\mathbf{p}')\varpi(\mathbf{q}'',\mathbf{p}'')\varpi(\mathbf{q}''',\mathbf{p}''')\times \\ &\sum_{\mathbf{b}}\langle\,U(\mathbf{q}',\mathbf{p}')U^{\dagger}(\mathbf{q}'',\mathbf{p}'')U^{\dagger}(\mathbf{q}',\mathbf{p}')\,e_{\mathbf{b}}|\,U(\mathbf{q},\mathbf{p})U(\mathbf{q}''',\mathbf{p}''')U^{\dagger}(\mathbf{q},\mathbf{p})e_{\mathbf{b}}\rangle \\ &= \sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,f(\mathbf{q}',\mathbf{p}')\varpi(\mathbf{q}'',\mathbf{p}'')\varpi(\mathbf{q}''',\mathbf{p}''')\times \\ &\sum_{\mathbf{b}}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}''}(\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\overline{e_{\mathbf{b}}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})}\times \\ &e_{\mathbf{p}}(\mathbf{x})\,e_{\mathbf{p}'''}(\mathbf{q}^{-1}\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})}e_{\mathbf{b}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})\,. \end{aligned}$$

Using the partial Fourier transform $\widetilde{\varpi}_p$ of ϖ, we obtain

$$\begin{aligned} \check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'',\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}''',\mathbf{q}^{-1}\mathbf{x})\times \\ \sum_{\mathbf{b}}\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\;\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})}\,\overline{e_{\mathbf{b}}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x})\,e_{\mathbf{b}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})\,. \end{aligned}$$

Summing on $\mathbf{b}$ gives

$$\check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'} \iiiint_{(\mathrm{SO}(3))^4} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,\mathrm{d}\mathbf{x}\, f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'',\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}''',\mathbf{q}^{-1}\mathbf{x})\times$$
$$\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})}\,\delta(\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{q}'\mathbf{q}''^{-1}\mathbf{q}'^{-1}\mathbf{q})$$
$$= \sum_{\mathbf{p}'} \iiint_{(\mathrm{SO}(3))^3} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{x}\, f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'',\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{q}'\mathbf{q}''^{-1}\mathbf{q}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times$$
$$\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})}$$
$$= \sum_{\mathbf{p}'} \iiint_{(\mathrm{SO}(3))^3} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\, f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{y},\mathbf{y}\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{q}'\mathbf{y}^{-1}\mathbf{q}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times$$
$$\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{q}'\mathbf{y}\mathbf{q}'^{-1}\mathbf{x})\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}'\mathbf{y}\mathbf{q}'^{-1}\mathbf{x})}\,,\quad (\mathbf{x}'=\mathbf{y})$$
$$= \sum_{\mathbf{p}'} \iiint_{(\mathrm{SO}(3))^3} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{y}'\,\mathrm{d}\mathbf{x}\, f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{y}'\mathbf{q},\mathbf{q}'^{-1}\mathbf{y}'\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{y}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times$$
$$\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{y}'\mathbf{x})\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{y}'\mathbf{x})}\,,\quad (\mathbf{y}'=\mathbf{q}'\mathbf{y}\mathbf{q}'^{-1})$$
$$= \sum_{\mathbf{p}'} \iiint_{(\mathrm{SO}(3))^3} \mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\, f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'\mathbf{x}^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x}')\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times$$
$$\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{x}')\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\,,\quad (\mathbf{x}'=\mathbf{y}'\mathbf{x})\,,\quad (\mathbf{y}'=\mathbf{x}'\mathbf{x}^{-1})\,.$$

Hence, we can write

$$\check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'} \int_{(\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\, f(\mathbf{q}',\mathbf{p}')\,,$$

where the kernel is given by

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')$$
$$= \iint_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\,\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'\mathbf{x}^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x}')\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{x}')\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\,.$$

Using the adjointness condition for M^{ϖ}, one obtains

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')$$
$$= \iint_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,,\mathrm{d}\mathbf{x}'\,\overline{\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x})}\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{x}')\;\; e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\,,$$

where

$$\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'\mathbf{x}^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x}') = \widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'(\mathbf{q}'^{-1}\mathbf{x})^{-1},\mathbf{q}'^{-1}\mathbf{x}') = \overline{\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x})}.$$

Hence, we can conclude with the following result:

Proposition 7. *The semi-classical portrait of the operator A^{ϖ}_f with respect to the weight ϖ is given by*

$$f \to A^{\varpi}_f \to \check{f}(\mathbf{q},\mathbf{p}) = \mathrm{Tr}(A^{\varpi}_f\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})) = \sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, f(\mathbf{q}',\mathbf{p}')K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\,, \tag{124}$$

where the kernel is given by

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') \tag{125}$$
$$= \int_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}\, e_{\mathbf{p}'}(\mathbf{x}')\,\overline{\widetilde{\varpi}(\mathbf{q}'^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x})}\,\widetilde{\varpi}(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x}).$$

This kernel satisfies the property

$$K(\mathbf{q}',\mathbf{p}';\mathbf{q},\mathbf{p}) = \overline{K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')}. \tag{126}$$

Below, we give the kernels and semi-classical portraits for two specific weights.

(i) For the unit weight $\varpi(\mathbf{q},\mathbf{p}) = 1$, the kernel reads

$$\begin{aligned} K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') &= \int_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\, \overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\, e_{\mathbf{p}'}(\mathbf{x}') \\ &\quad \times \left\{ \sum_{\mathbf{b}'} \overline{e_{\mathbf{b}'}(\mathbf{q}'^{-1}\mathbf{x})} \right\} \left\{ \sum_{\mathbf{b}} e_{\mathbf{b}}(\mathbf{q}^{-1}\mathbf{x}) \right\} \\ &= \delta_{\mathbf{p}\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\, e_{\mathbf{p}}(\mathbf{x})|^2 \left\{ \sum_{\mathbf{b}'} \overline{e_{\mathbf{b}'}(\mathbf{q}'^{-1}\mathbf{x})} \right\} \left\{ \sum_{\mathbf{b}} e_{\mathbf{b}}(\mathbf{q}^{-1}\mathbf{x}) \right\} \\ &= \delta_{\mathbf{p}\mathbf{p}'} \mathrm{h}(\mathbf{q},\mathbf{q}'), \end{aligned} \tag{127}$$

where

$$\mathrm{h}(\mathbf{q},\mathbf{q}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\, e_{\mathbf{p}}(\mathbf{x})|^2 \left\{ \sum_{\mathbf{b}'} \overline{e_{\mathbf{b}'}(\mathbf{q}'^{-1}\mathbf{x})} \right\} \left\{ \sum_{\mathbf{b}} e_{\mathbf{b}}(\mathbf{q}^{-1}\mathbf{x}) \right\}. \tag{128}$$

Finally,

$$\check{f}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, f(\mathbf{q}',\mathbf{p}) \mathrm{h}(\mathbf{q},\mathbf{q}')\,. \tag{129}$$

(ii) For the squared rotation map weight $\varpi(\mathbf{q},\mathbf{p}) = \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})} = e_{\mathbf{p}^t}(\mathbf{q})$,

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = \delta(\mathbf{q},\mathbf{q}') \int_G \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}'}(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x})\, e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}) \overline{e_{\mathbf{p}'}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})} \tag{130}$$

and

$$\check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'} f(\mathbf{q},\mathbf{p}') \{ \int_G \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})} e_{\mathbf{p}'}(\mathbf{x})\, e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}) \overline{e_{\mathbf{p}'}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})} \}. \tag{131}$$

For the univariate functions $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$ and $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$,

$$\begin{aligned} \check{u}(\mathbf{q}) &= u(\mathbf{q}), \\ \check{v}(\mathbf{p}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}) \left\{ \sum_{\mathbf{p}'} v(\mathbf{p}') e_{\mathbf{p}'}(\mathbf{x}) \overline{e_{\mathbf{p}'}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})} \right\}. \end{aligned} \tag{132}$$

A third example of weight, namely that one corresponding to coherent states, is given in the next section.

7. Quantization and Semi-Classical Portraits with Coherent States with Non-Unit Fiducial States

The entire procedure of covariant integral quantization and the explanation of semi-classical phase-space portraits provided above are well-implemented in cases where the operator M^{ϖ^ϕ} is a one-rank density operator, i.e., when we employ coherent-state quantization.

7.1. CS Quantization

Let us implement the quantization yielded by the weight ϖ^ϕ, which corresponds, through Proposition 5, to the one-rank density operator $|\phi\rangle\langle\phi| = \mathsf{M}^{\varpi^\phi}$, $\phi \in \mathcal{K}$, $\|\phi\|^2 = 1$. Particularizing Proposition 6 and subsequent derivations of various quantizations to the CS case allows us to state the following.

Let us implement the quantization defined by the weight ϖ^ϕ, which corresponds to the one-rank density operator $|\phi\rangle\langle\phi| = \mathsf{M}^{\varpi^\phi}$, where ϕ belongs to the Hilbert space $\mathcal{K}$ and $\|\phi\|^2 = 1$, as established in Proposition 5. By specializing Proposition 6 and subsequent quantization derivations to the coherent-state (CS) case, we can confidently present the following without going into a detailed proof:

Proposition 8. *Given a fiducial vector ϕ and the projector $|\phi\rangle\langle\phi|$, the trace of the operator $\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})|\phi\rangle\langle\phi|$ is given by (with the notations of (59))*

$$\varpi^\phi(\mathbf{q},\mathbf{p}) \equiv \mathrm{Tr}\Big[\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})|\phi\rangle\langle\phi|\Big] = {}_\phi\langle\mathbf{q},\mathbf{p}|\phi\rangle\,. \tag{133}$$

In addition, we have

(i) The partial inverse Fourier transform of $\varpi^\phi(\mathbf{q},\mathbf{p})$ with respect to $\mathbf{p}$ is given by

$$\widetilde{\varpi^\phi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) = \overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q})\,; \tag{134}$$

(ii) the kernel of the related quantum operator is given by

$$\mathcal{A}_f^{\varpi^\phi}(\mathbf{x},\mathbf{x}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\delta^f(\mathbf{x}\mathbf{q}^{-1};(\mathbf{x},\mathbf{x}')\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q})\,, \tag{135}$$

with the notations of (89).

In what follows, we compute the kernels $\mathcal{A}_f^{\varpi^\phi}$ or/and the corresponding operators for various simple cases of $f(\mathbf{q},\mathbf{p})$.

(a) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})v(\mathbf{p})$,

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{v}(\mathbf{x},\mathbf{x}')\,,\quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')},$$
$$\mathcal{A}_f^{\varpi^\phi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,u(\mathbf{x}\mathbf{q}^{-1})\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q})\,.$$

(b) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$,

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')\,u(\mathbf{x}\mathbf{q}^{-1}),$$
$$\mathcal{A}_f^{\varpi^\phi}(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,u(\mathbf{x}\mathbf{q}^{-1})|\phi(\mathbf{q})|^2\,.$$

Hence, the quantized of $u(\mathbf{q})$ is the multiplication operator.

$$\left(A_{u(\mathbf{q})}^{\varpi^\phi}\psi\right)(\mathbf{x}) = \left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,u(\mathbf{x}\mathbf{q}^{-1})\,|\phi(\mathbf{q})|^2\right]\psi(\mathbf{x})\,.$$

(c) For $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$,

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = \widetilde{v}(\mathbf{x},\mathbf{x}')\,,\quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')},$$
$$\mathcal{A}_f^{\varpi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')\phi(\mathbf{q})}\,.$$

7.2. Semi-Classical Portraits Through CS

For the coherent state weight $\varpi^\phi(\mathbf{q},\mathbf{p}) = {}_\phi\langle\mathbf{q},\mathbf{p}|\phi\rangle$, the kernel is the probability distribution in the phase space:

$$K^\phi(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = |\langle_\phi\langle\mathbf{q},\mathbf{p}|\mathbf{q}',\mathbf{p}'\rangle_\phi|^2\,. \tag{136}$$

Hence, $\check{f}(\mathbf{q},\mathbf{p})$ is the local averaging of the original $f(\mathbf{q},\mathbf{p})$:

$$\check{f}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}'\, f(\mathbf{q}',\mathbf{p}')\, |{}_{\phi}\langle \mathbf{q},\mathbf{p}|\mathbf{q}',\mathbf{p}'\rangle_{\phi}|^2\,. \tag{137}$$

8. Squaring Rotation Operator for the Wigner Function

Based on the previoussection, let us consider the following phase portrait of a state $\psi \in \mathcal{K}$:

$$W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) := \langle \psi | U(\mathbf{q},\mathbf{p}) \widehat{\mathcal{O}} U^{\dagger}(\mathbf{q},\mathbf{p}) |\psi\rangle\,, \tag{138}$$

where the operator $\widehat{\mathcal{O}}$ is requested to yield marginality properties 'a la Wigner for $W_{\psi}^{\widehat{\mathcal{O}}}$:

$$\sum_{\mathbf{p}} W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{q})|^2\,, \quad \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{p})|^2\,. \tag{139}$$

Let us assume that $\widehat{\mathcal{O}}$ acts on $\mathcal{K}$ through some differentiable transformation of the group manifold $\mathrm{SO}(3) \ni \mathbf{x} \mapsto \zeta(\mathbf{x}) \in \mathrm{SO}(3)$, namely,

$$(\widehat{\mathcal{O}}\psi)(\mathbf{x}) = \mathsf{m}(\mathbf{x})\psi(\zeta(\mathbf{x}))\,, \tag{140}$$

where the factor $\mathsf{m}(\mathbf{x})$ has to also be determined.

With (54), (55), and the above definitions, we have

$$\begin{aligned}
W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x}) \left(\widehat{\mathcal{O}}\, U^{\dagger}(\mathbf{q},\mathbf{p})\psi\right)\left(\mathbf{q}^{-1}\mathbf{x}\right) \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x})\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right) \left(U^{\dagger}(\mathbf{q},\mathbf{p})\psi\right)\left(\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right) \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x})\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right) \overline{e_{\mathbf{p}}(\mathbf{q}\zeta(\mathbf{q}^{-1}\mathbf{x}))}\psi\left(\mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right).
\end{aligned}$$

The completeness relation (46) combined with the above integral and change of variable $\mathbf{x} \mapsto \mathbf{y} = \mathbf{q}^{-1}\mathbf{x}$ allows us to write

$$\begin{aligned}
\sum_{\mathbf{p}} W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\psi(\mathbf{x})|^2\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right) \delta\left(\mathbf{x}, \mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right) \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{y}\, |\psi(\mathbf{q}\mathbf{y})|^2\, \mathsf{m}(\mathbf{y})\, \delta(\mathbf{q}\mathbf{y}, \mathbf{q}\zeta(\mathbf{y}))\,.
\end{aligned}$$

The condition for achieving marginality with regard to summing on $\mathbf{p}$, namely,

$$\sum_{\mathbf{p}} W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{q})|^2\,, \tag{141}$$

thus imposes on the action ζ and the function m the following conditions:

$$\delta(\mathbf{q}\mathbf{y}, \mathbf{q}\zeta(\mathbf{y})) = \delta(\mathbf{y}, \mathrm{e})\,, \quad \mathsf{m}(\mathrm{e}) = 1\,.$$

Besides the constraint $\mathsf{m}(\mathrm{e}) = 1$, possible solutions for ζ are

$$\zeta(\mathbf{y}) = \mathbf{y}^n\,, \quad n \in \mathbb{Z} \tag{142}$$

since the group structure imposes that $\mathbf{q}\mathbf{y} = \mathbf{q}\mathbf{y}^n \Rightarrow \mathbf{y}^{n-1} = \mathrm{e}$, i.e., $\mathbf{y}$ should be one of the $n-1$ roots of the unity in $\mathrm{SO}(3)$. The most natural choice for n is obviously $n = 2$:

$$\zeta(\mathbf{y}) = \mathbf{y}^2\,, \tag{143}$$

and we keep it in the sequel. Let us now examine the condition for getting marginality with regard to integrating on $\mathbf{q}$, namely,

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = |\widehat{\psi}(\mathbf{p})|^2\,, \tag{144}$$

where we recall that $\widehat{\psi}$ is the Fourier transform of ψ:

$$\widehat{\psi}(\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})}\psi(\mathbf{x}) \equiv \mathcal{F}[\psi](\mathbf{p})\,.$$

With the solution (143) at hand, let us evaluate the l.h.s. of (144), assuming that inverting the order of integrations is legitimate.

$$\begin{aligned}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})}e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right) \overline{e_{\mathbf{p}}(\mathbf{q}\zeta(\mathbf{q}^{-1}\mathbf{x}))}\psi\left(\mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right)\\ &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})}e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right) \overline{e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})}\psi\left(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}\right).\end{aligned}$$

Now, after changing the variable in the second integral $\mathbf{q} \mapsto \mathbf{q}' = \mathbf{x}\mathbf{q}^{-1}\mathbf{x}$ and using the invariance of the measure $\mathrm{d}\mathbf{q} \mapsto \mathrm{d}\mathbf{q}'$, one obtains

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, \mathsf{m}\left(\mathbf{x}^{-1}\mathbf{q}'\right) \overline{e_{\mathbf{p}}(\mathbf{q}')}\, \psi(\mathbf{q}')\,.$$

In order to achieve marginality with regard to integrating on $\mathbf{q}$, the only possibility is that $\mathsf{m}(\mathbf{x}) = 1$. Then, we obtain (144), and $\widehat{\mathcal{O}}$ is precisely the squared rotation operator introduced in (83),

$$\widehat{\mathcal{O}} \equiv \hat{\mathsf{I}}_{\mathrm{sq}}\,. \tag{145}$$

In summary and extending the above properties to mixed states in $\mathcal{K}$, we state the following.

Proposition 9. *To any density operator $\hat{\rho}$ in $\mathcal{K}$, the squared rotation operator $\hat{\mathsf{I}}_{\mathrm{sq}}$ associates its Wigner-like function defined by*

$$W_{\hat{\rho}}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p}) := \mathrm{Tr}\left(\hat{\rho}\, U(\mathbf{q},\mathbf{p})\hat{\mathsf{I}}_{\mathrm{sq}}U^{\dagger}(\mathbf{q},\mathbf{p})\right). \tag{146}$$

This function obeys the marginality properties:

(i)

$$\sum_{\mathbf{p}} W_{\hat{\rho}}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'\mathbf{p}''} \hat{\rho}_{\mathbf{p}'\mathbf{p}''}\, e_{\mathbf{p}'}(\mathbf{q})\, \overline{e_{\mathbf{p}''}(\mathbf{q})}\,, \tag{147}$$

where $\hat{\rho}_{\mathbf{p}'\mathbf{p}''}$ comprises the matrix elements of $\hat{\rho}$ on the basis $\{e_{\mathbf{p}}(\mathbf{q})\}$.

(ii)

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\hat{\rho}}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p}) = \hat{\rho}_{\mathbf{p}\mathbf{p}}\,. \tag{148}$$

(iii) For a pure state $\hat{\rho} = [\psi\rangle\langle\psi|$, these formulae simplify to

$$\sum_{\mathbf{p}} W_{\psi}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{q})|^2\,, \quad \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p}) = |\widehat{\psi}(\mathbf{p})|^2\,. \tag{149}$$

(iv) From these two marginal properties, the normalization of $W_{\psi}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p})$ results as a complex-valued quasi-distribution:

$$\sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\hat{\rho}}^{\hat{\mathsf{I}}_{\mathrm{sq}}}(\mathbf{q},\mathbf{p}) = 1\,. \tag{150}$$

Let us discuss more about the properties of the squared rotation operator.

Proposition 10.

(i) *The operator* $\hat{\mathsf{I}}_{\mathrm{sq}}$ *is unit trace.*

$$\mathrm{Tr}(\hat{\mathsf{I}}_{\mathrm{sq}}) = 1$$

(ii) *The weight function* $\varpi^{\mathrm{sq}}(\mathbf{q},\mathbf{p})$ *giving rise to* $\hat{\mathsf{I}}_{\mathrm{sq}}$ *through* (76)*, i.e.,* $\mathsf{M}^{\varpi^{sq}} = \hat{\mathsf{I}}_{\mathrm{sq}}$*, is given by the trace of the operator* $\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\hat{\mathsf{I}}_{\mathrm{sq}}$*:*

$$\varpi^{\mathrm{sq}}(\mathbf{q},\mathbf{p}) = \mathrm{Tr}(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\hat{\mathsf{I}}_{\mathrm{sq}}) = \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})}\,. \tag{151}$$

(iii) *The inverse partial Fourier transform of the weight with respect to the momentum is given by*

$$\widetilde{\varpi^{\mathrm{sq}}}_p(\mathbf{q},\mathbf{x}) = \sum_{\mathbf{p}} e_{\mathbf{p}}(\mathbf{x})\,\varpi^{\mathrm{sq}}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}} e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{q}^{-1})} = \delta(\mathbf{q}^{-1},\mathbf{x}) \tag{152}$$

and

$$\widetilde{\varpi^{\mathrm{sq}}}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) = \delta(\mathbf{x}'^{-1}\mathbf{x}\mathbf{q})\,, \tag{153}$$

and the kernel of the related quantum operator is given by

$$\mathcal{A}_f^{\varpi^{\phi}}(\mathbf{x},\mathbf{x}') = \delta^f(\mathbf{x}';(\mathbf{x},\mathbf{x}'))\,, \tag{154}$$

with the notations of (89).

Proof.

(i)

$$\mathrm{Tr}(\hat{\mathsf{I}}_{\mathrm{sq}}) = \sum_{\mathbf{b}}\langle e_{\mathbf{p}}|\,\hat{\mathsf{I}}_{\mathrm{sq}}|e_{\mathbf{p}}\rangle = \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\;\overline{e_{\mathbf{p}}(\mathbf{x})}\,(\hat{\mathsf{I}}_{\mathrm{sq}}e_{\mathbf{p}})(\mathbf{x}) = \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\;\overline{e_{\mathbf{b}}(\mathbf{x})}\,e_{\mathbf{b}}(\mathbf{x}^2)$$

Using the completeness property (46) and (36), we obtain

$$\mathrm{Tr}(\hat{\mathsf{I}}_{\mathrm{sq}}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\;\delta(\mathbf{x},\mathbf{x}^2) = 1\,.$$

(ii)

$$\begin{aligned}
\mathrm{Tr}\Big(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p}))\hat{\mathsf{I}}_{\mathrm{sq}}\Big) &= \sum_{\mathbf{b}}\langle e_{\mathbf{b}}|\,\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\,\hat{\mathsf{I}}_{\mathrm{sq}}|e_{\mathbf{b}}\rangle = \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\,\Big(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\,\hat{\mathsf{I}}_{\mathrm{sq}}\,e_{\mathbf{b}}\Big)(\mathbf{x}) \\
&= \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\Big(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\;(\hat{\mathsf{I}}_{\mathrm{sq}}e_{\mathbf{b}})\Big)(\mathbf{x}) = \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})}\,(\hat{\mathsf{I}}_{\mathrm{sq}}e_{\mathbf{b}})(\mathbf{q}\mathbf{x}) \\
&= \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})}e_{\mathbf{b}}((\mathbf{q}\mathbf{x})^2) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta\Big(\mathbf{x},(\mathbf{q}\mathbf{x})^2\Big)\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})} \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{y}\,\delta\Big(\mathbf{q}^{-1}\mathbf{y},\mathbf{y}^2\Big)\,\overline{e_{\mathbf{p}}(\mathbf{y})} = \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})} = e_{\mathbf{p}^t}(\mathbf{q})\,,
\end{aligned}$$

where we again use the completeness property (46) and (36) after changing the variable $\mathbf{x} \mapsto \mathbf{y} = \mathbf{q}\mathbf{x}$.

(iii) The proof is direct.

□

In the following, we calculate the kernels $\mathcal{A}_f^{\varpi^{\mathrm{sq}}}$ and/or the corresponding operators for various straightforward cases of $f(\mathbf{q},\mathbf{p})$. It is worth noting that we obtain our results quite easily, in contrast to the outcomes that would be obtained using alternative definitions of Wigner distributions Γ, such as those discussed in [36] and [39].

(a) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})v(\mathbf{p})$,

$$\widetilde{f}(\mathbf{x}',(\mathbf{x},\mathbf{x}')) = u(\mathbf{x}')\,\widetilde{v}(\mathbf{x},\mathbf{x}')\,, \quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')},$$
$$\mathcal{A}_f^{\varpi\phi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')u(\mathbf{x}').$$

(b) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$,

$$\widetilde{f}(\mathbf{x}',(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')\,u(\mathbf{x}') = \mathcal{A}_f^{\varpi\phi}(\mathbf{x},\mathbf{x}')\,.$$

Hence, the quantized of $u(\mathbf{q})$ is the multiplication operator.

$$\left(A^{\varpi\phi}_{u(\mathbf{q})}\psi\right)(\mathbf{x}) = u(\mathbf{x})\psi(\mathbf{x})\,.$$

(c) For $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$,

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = \widetilde{v}(\mathbf{x},\mathbf{x}')\,, \quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')},$$
$$\mathcal{A}_f^{\varpi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')\,.$$

9. Conclusions

In this paper, we introduce a covariant integral quantization approach for systems with a phase space, known as the semi-discrete hypercylinder $\mathrm{SO}(3)\times\widehat{\mathrm{SO}(3)}$, i.e., where the configuration space is $SO(3)$ and $\widehat{\mathrm{SO}(3)}$ is defined by $\widehat{\mathrm{SO}(3)} = \{(l,m,n), l\in\mathbb{N}, m=-l:1:l, n=-l:1:l\}$. This extends our prior work on the discrete cylinder $\mathrm{SO}(2)\times\mathbb{Z}$, which addressed the motion of a particle on the circle $\mathbb{S}^1 = \mathrm{SO}(2)$. The phase space $\mathrm{SO}(3)\times\widehat{\mathrm{SO}(3)}$ is not a coset arising from a group; we demonstrate that the Weyl–Gabor formalism is applicable.

First, we establish the concomitant resolution of the identity and its subsequent properties, including the Gabor transform on $\mathrm{SO}(3)$, its inversion, the reproducing kernel, and the crucial observation that any square-integrable function on the group $\mathrm{SO}(3)$ can be considered a fiducial vector. Leveraging the decomposition of the identity, we define an integral operator M^{ϖ} based on a weight function ϖ defined on the phase space.

There are noticeable results related to the quantization of a point

$$(\mathbf{q},\mathbf{p}) = ((\alpha,\beta,\gamma),(l,m,n))$$

in the phase space according to two standard choices of the weight.

- With the squared rotation weight, which yields the Wigner distribution, the quantization of the projection of momentum on the third axis is the expected angular momentum operator L_z.

$$m \mapsto \hat{m} = L_z\,, \quad L_z\psi(\alpha,\beta,\gamma) = -\mathrm{i}\frac{\partial}{\partial\alpha}\psi(\alpha,\beta,\gamma)\,, \tag{155}$$

  while the quantization of the angle yields the multiplication operator by the angle:

$$\theta_i \mapsto \hat{\theta}_i\,, \quad \hat{\theta}_i\psi(\theta) = \theta_i\psi(\theta)\,, \quad \theta = (\alpha,\beta,\gamma)\,. \tag{156}$$

  This is clearly not acceptable because of the discontinuity in the periodized angle function, as there is no regularization for this discontinuity.
- On the other hand, by using the squared rotation weight, we obtain a Wigner distribution that is more manageable than those derived in previous works [36,39]. We will demonstrate this through concrete examples in our future research.
- Using the coherent state weight, one can obtain the quantization of momentum, which includes the conventional L_z operator along with an additional term. This additional

term can be seen as a kind of covariant derivative along the circle $\mathbb{S}^1$ within SO(3), taking into account the topology of this manifold,

$$A_m^{\varpi} = -\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\Omega_\alpha^{(1)}\,. \tag{157}$$

Furthermore, the quantization of the periodized function in the α variable results in its smooth regularization.

In future studies, we intend to expand upon the findings of this work in various directions.

- We plan to extend this work to encompass all rotation groups $\mathrm{SO}(n)$, as well as the associated spheres $\mathbb{S}^n$. Additionally, we will explore the quantization of continuous phase space related to the Euclidean groups $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n), \geq 2$, as described in the references [49,50].
- We plan to extend this work to the full configuration space of the rigid body, that is, the Euclidean motion group in three dimensions $\mathrm{E}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$.
- We plan to apply our formalism to the case where the configuration space is a non-compact group, for example, $\mathrm{SO}_0(2,1) \simeq \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$.
- We plan to explore the possibility of the covariant integral quantization in the situation where the phase space is $T^*\mathrm{SO}(3)$. This phase space is used in the context of quantum loop gravity [51,52].
- Finally, we plan to apply our approach to signal analysis on SO(3). We will investigate the robustness of the phase space representation $(\mathbf{q},\mathbf{p}) \to S(\mathbf{q},\mathbf{p})$ of a signal $x \to s(\mathbf{x})$ in capturing salient futures in the signal. Various tools will be used. These tools include visualization of various partial energy densities of the Gabor transform, quantum operators related to various fiducial vectors, Husimi distributions, Wigner distribution, entropy, and sampling/frames on SO(3) [46,53].

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Abbreviations and Nomenclature

The following abbreviations are used in this manuscript:

POVM	Positive operator-valued measure
UIR	Unitary irreducible representation
CS	Coherent state
$\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$	Phase space
$\mathcal{H}$	Hilbert space $L^2(\mathbb{S}^2, \mathrm{d}\Omega)$
$\mathcal{K}$	Hilbert space $L^2(\mathrm{SO}(3), \mathrm{d}\mathbf{x})$
$\mathfrak{K}$	Hilbert space $L^2(\Gamma, \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p})$
ZYZ	Successive rotation about the z-axis, new x-axis, and new z-axis
$\mathsf{L}(\mathbf{q})$	Defined in (17)
$\delta_{\mathrm{e}}(\mathbf{x})$	Dirac distribution with support the neutral element $\mathrm{e} \equiv \mathbb{1}_3$ of SO(3), defined in (31)
$\delta_{\mathbf{q}}(\mathbf{x})$	Dirac distribution with support the element $\mathbf{q}$ of SO(3), defined in (33)
$\hat{\psi}_{j\,m\,n}$	Fourier transform of $\psi(\mathbf{x})$, defined in (41)

$\mathbf{p} \equiv (l,m,n)$	Discrete momentum, defined in (43)
$e_{\mathbf{p}}(\mathbf{x})$	Hilbertian basis element, defined in (45)
$\mathsf{E}_{\mathbf{p}}$	Multiplication operator defined in (49)
$\mathbf{p}^{\mathrm{t}}$	Transpose of $\mathbf{p}$, defined in (52)
$\mathsf{U}(\mathbf{q},\mathbf{p})$	SO(3)-Weyl–Gabor operator, defined in (53)
$\lvert\mathbf{q},\mathbf{p}\rangle_\phi$	Coherent state from fiducial vector ϕ, defined in (59)
$\mathcal{L}_\phi$	Gabor transform, defined in (62)
M^{ϖ}	Quantization operator defined in (76)
$\widetilde{\varpi}_p$	Partial inverse discrete Fourier transform (48) of ϖ with respect to the discrete variables
$\hat{\mathsf{I}}_{\mathrm{sq}}$	Squaring rotation operator defined in (83)
$\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})$	SO(3)-Weyl-Gabor operator transport of M^{ϖ}, defined in (85)
$\Omega(\mathbf{x},\mathbf{x}')$	Integral defined in (91)
$\Omega^{(j)}_\alpha$, $\Omega^{(j)}_\beta$, $\Omega^{(j)}_\gamma$	Defined in (93), (94), (95), respectively
ϖ^ϕ	Weight corresponding to the one-rank density operator $\lvert\phi\rangle\langle\phi\rvert$, defined in (10)

Appendix A. Some Formulas for CS Quantization

In this appendix, we compute $\Omega^{(j)}_\alpha(\mathbf{x},\mathbf{x}) = \frac{\partial^j}{\partial\alpha'^j}\Omega(\mathbf{x},\mathbf{x}')\Big|_{\mathbf{x}'=\mathbf{x}}$, for the coherent state weight $\varpi(\mathbf{q},\mathbf{p}) = \langle \mathsf{U}(\mathbf{q},\mathbf{p})\phi|\phi\rangle$ for $j=1,2$, using the following summation formulae [54]:

$$\sum_{m'} m' |\mathrm{d}^l_{m'\,m}(\beta)|^2 = m\cos(\beta);\sum_{m'} m'^2 |\mathrm{d}^l_{m'\,m}(\beta)|^2 = \frac{1}{2}\Big\{l(l+1)\sin^2\beta - m^2(3\cos\beta - 1)\Big\}.$$

$$\begin{aligned}
\Omega(\mathbf{x},\mathbf{x}') &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q}) = \sum_{(l,m,n)} \widehat{\phi}(l,m,n)\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^l_{m\,n}(\mathbf{q}) \\
&= \sum_{(l,m,n)} \widehat{\phi}(l,m,n)\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z}\,\overline{\phi(\mathbf{z})}\sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^l_{m\,n}(\mathbf{z}\mathbf{x}'^{-1}\mathbf{x}) \\
&= \sum_{(l,m,n)}\sum_k \widehat{\phi}(l,m,n)\mathrm{D}^l_{k\,n}(\mathbf{x}'^{-1}\mathbf{x})\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z}\,\overline{\phi(\mathbf{z})}\sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^l_{m\,k}(\mathbf{z}) \\
&= \sum_{(l,m,n)}\sum_k \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\mathrm{D}^l_{k\,n}(\mathbf{x}'^{-1}\mathbf{x}) \\
&= \sum_{(l,m,n)}\sum_k\sum_{k'} \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\mathrm{D}^l_{k\,k'}(\mathbf{x}'^{-1})\mathrm{D}^l_{k'\,n}(\mathbf{x}) \\
&= \sum_{(l,m,n)}\sum_k\sum_{k'} \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x}')}\mathrm{D}^l_{k'\,n}(\mathbf{x})
\end{aligned}$$

We therefore have

$$\Omega^{(j)}_\alpha(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_k\sum_{k'} (-\mathrm{i})^{\mathrm{j}} k'^{\mathrm{j}} \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x})}\mathrm{D}^l_{k'\,n}(\mathbf{x}). \tag{A1}$$

$$\Omega^{(j)}_\gamma(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_k\sum_{k'} (-\mathrm{i})^{\mathrm{j}} k^{\mathrm{j}} \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x})}\mathrm{D}^l_{k'\,n}(\mathbf{x}). \tag{A2}$$

Let us now two types of fiducial vectors.

- Free rotor and highest weight fiducial vectors.

  For the free rotor $\phi(\mathbf{x}) = \sqrt{\frac{2l_0+1}{8\pi^2}}\mathrm{D}^{l_0}_{m_0\,n_0}(\mathbf{x})$ with momentum (l_0,m_0,n_0), we obtain $\widehat{\phi}(l,m,n) = \frac{2l_0+1}{8\pi^2}\delta_{l\,l_0}\delta_{m\,m_0}\delta_{n\,n_0}$, $\widehat{\phi}(l,m,k) = \frac{2l_0+1}{8\pi^2}\delta_{l\,l_0}\delta_{m\,m_0}\delta_{k\,n_0}$ and

$$\Omega_{\alpha}^{(j)}(\mathbf{x},\mathbf{x}) = \delta_{l_0\,l_0}\delta_{m_0\,m_0}\frac{2l_0+1}{8\pi^2}\,(-\mathrm{i})^{\mathrm{j}}\sum_{k'}k'^j|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2 \quad \text{(A3)}$$
$$= \frac{2l_0+1}{8\pi^2}\,(-\mathrm{i})^{\mathrm{j}}\sum_{k'}k'^j|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2,$$

$$\Omega_{\gamma}^{(j)}(\mathbf{x},\mathbf{x}) = \delta_{l_0\,l_0}\delta_{m_0\,m_0}\frac{2l_0+1}{8\pi^2}\,(-\mathrm{i})^{\mathrm{j}}{n_0}^j\sum_{k'}|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2 \quad \text{(A4)}$$
$$= \frac{2l_0+1}{8\pi^2}\,(-\mathrm{i})^{\mathrm{j}}{n_0}^j\sum_{k'}|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2.$$

Using summation formula in (A1) for j = 1 and j = 2, we obtain

$$\Omega_{\alpha}^{(1)}(\mathbf{x},\mathbf{x}) = -\mathrm{i}\,\frac{2l_0+1}{8\pi^2}\,n_0\cos(\beta), \quad \text{(A5)}$$
$$\Omega_{\alpha}^{(2)}(\mathbf{x},\mathbf{x}) = -\frac{1}{2}\frac{2l_0+1}{8\pi^2}\Big\{l_0(l_0+1)\sin^2\beta - n_0^2(3\cos\beta - 1)\Big\},$$

$$\Omega_{\gamma}^{(1)}(\mathbf{x},\mathbf{x}) = -\mathrm{i}\,\frac{2l_0+1}{8\pi^2}\,n_0, \quad \text{(A6)}$$
$$\Omega_{\gamma}^{(2)}(\mathbf{x},\mathbf{x}) = -\frac{1}{2}\frac{2l_0+1}{8\pi^2}{n_0}^2.$$

For the highest weight state for SO(3), that is, $\phi(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^{l_0}_{l_0\,l_0}(\mathbf{x})$ with momentum (l_0, l_0, l_0), we obtain

$$\Omega_{\alpha}^{(1)}(\mathbf{x},\mathbf{x}) = -\mathrm{i}\left[\frac{2l_0+1}{8\pi^2}\right]l_0\cos(\beta), \quad \text{(A7)}$$
$$\Omega_{\alpha}^{(2)}(\mathbf{x},\mathbf{x}) = \frac{1}{2}\,\frac{2l_0+1}{8\pi^2}\Big\{\cos\beta(\cos\beta+3)l_0^2 - l_0\sin^2\beta)\Big\},$$

$$\Omega_{\gamma}^{(1)}(\mathbf{x},\mathbf{x}) = -\mathrm{i}\,\frac{2l_0+1}{8\pi^2}\,l_0, \quad \text{(A8)}$$
$$\Omega_{\gamma}^{(2)}(\mathbf{x},\mathbf{x}) = \frac{1}{2}\,\frac{2l_0+1}{8\pi^2}l_0^2.$$

- Radial fiducial vector.
  We consider a radial vector ϕ depending only on the distance $|\mathbf{x}|$ to the origin e. We have $\widehat{\phi}(l,m,n) = f(l)\delta_{m\,n}$ and $\widehat{\phi}(l,m,k) = f(l)\delta_{m\,k}$ [21].

$$\Omega_{\alpha}^{(j)}(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_{k}\sum_{k'}(-\mathrm{i})^{\mathrm{j}}k'^{\mathrm{j}}\widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x})}\mathrm{D}^l_{k'\,n}(\mathbf{x}). \quad \text{(A9)}$$

$$\Omega_{\alpha}^{(j)}(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_{k}\sum_{k'}(-\mathrm{i})^{\mathrm{j}}k'^{\mathrm{j}}|\,f(l)|^2\delta_{m\,n}\delta_{m\,k}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x})}\mathrm{D}^l_{k'\,n}(\mathbf{x}), \quad \text{(A10)}$$
$$= \sum_{l}(2l+1)|\,f(l)|^2\sum_{k}\sum_{k'}(-\mathrm{i})^{\mathrm{j}}k'^{\mathrm{j}}\;|\mathrm{d}^l_{k'\,k}(\beta)|^2.$$

Finally,

$$\Omega_{\alpha}^{(1)}(\mathbf{x},\mathbf{x}) = \left[-\mathrm{i}\sum_{l}(2l+1)|\,f(l)|^2\right](2l+1)\cos\beta \quad \text{(A11)}$$
$$= \left[-\mathrm{i}\sum_{l}(2l+1)^2|\,f(l)|^2\right]\cos\beta,$$
$$\Omega_{\alpha}^{(2)}(\mathbf{x},\mathbf{x}) = -\left[\frac{1}{2}\sum_{l}(2l+1)^2|\,f(l)|^2\right]\Big\{\cos\beta(\cos\beta+3)l_0^2 - l_0\sin^2\beta)\Big\}.$$

Appendix B. Plot of Von Mises Fiducial Vector and Derivative

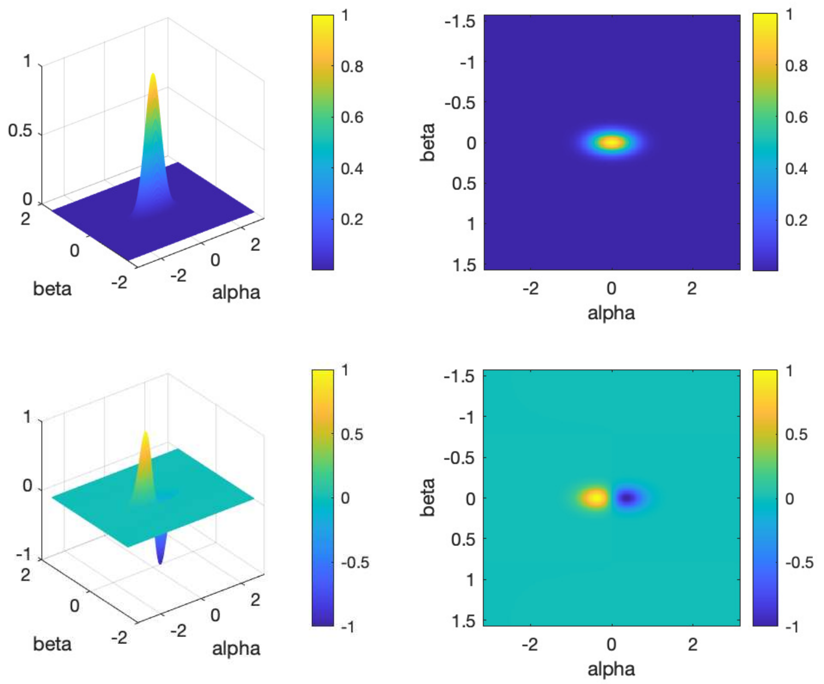

Figure A1. Top left and right: Von Mises fiducial vector as a function of the Euler angles (α, β) at parameter $\kappa = 30$ and the third Euler angle $\gamma = 0$. **Bottom left and right**: Derivative with respect to β of the Von Mises fiducial vector as a function of the Euler angles (α, β) at $\kappa = 30$ and $\gamma = 0$. Color bars indicate the values of these functions. The peaked and localized nature of these functions is evident.

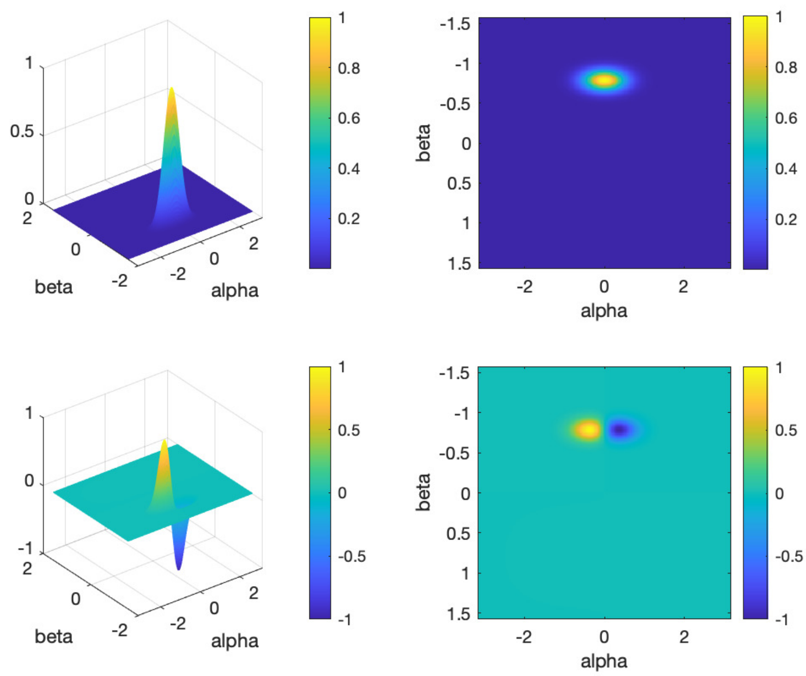

Figure A2. Top left and right: Von Mises fiducial vector as a function of the Euler angles (α, β) at parameter $\kappa = 30$ and the third Euler angle $\gamma = \pi$. **Bottom left and right**: Derivative with respect to β of the Von Mises fiducial vector as a function of the Euler angles (α, β) at $\kappa = 30$ and $\gamma = \pi$. The peaked and localized nature of these functions is also evident.

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