Algebraic Version of the Soliton Perturbation Theory

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Abstract : A scheme is proposed for calculating higher-order corrections to a soliton in the presence of perturbations. This approach based on the matrix Riemann problem has made it possible to avoid the use of integral equations.

1. Non-linear excitations of systems whose properties are close to those of real physical objects do not in general described by the completely integrable equations. In a number of cases, however, disturbances which break the exact integrability are small and we may take iteratively into account an influence of small perturbations on the soliton evolution. The existing variants of the soliton perturbation theory (SPT) (see papers [1-3] and review [4]) are based on treating a perturbation-induced evolution of the scattering data of an associated spectral problem with the subsequent solution of the Gel'fand–Levitan–Marchenko-type integral equations. Here we consider a version of SPT based on the matrix Riemann problem. This approach is technically more transparent and uses no any integral equation.

2. Let us consider the AKNS spectral problem $\Phi_x(x,\lambda) = (-i\lambda\sigma_3 + \mathbf{Q}(x))\Phi(x,\lambda)$, where $\mathbf{Q}(x) = i\begin{pmatrix} 0 & q(x) \\ r(x) & 0 \end{pmatrix}$, r and q tend to zero as $|x| \to \infty$. Let $\mathbf{T}_\pm(x,\lambda)$ are 2×2 matrix Jost solutions whose asymptotics are $\mathbf{T}_\pm(x,\lambda) \to J(\lambda x) = \exp(-i\lambda\sigma_3 x)$ as $x \to \pm\infty$. Now define new matrices ($\mathbf{T}^{(i)}$ is the ith column of $\mathbf{T}$) :

$$\Theta(x,\lambda) = \left(\mathbf{T}_-^{(1)} e^{i\lambda x}, \mathbf{T}_+^{(2)} e^{-i\lambda x}\right), \quad \bar{\Theta}(x,\lambda) = \left(\mathbf{T}_+^{(1)} e^{i\lambda x}, \mathbf{T}_-^{(2)} e^{-i\lambda x}\right) \qquad (1)$$

with the property $\Theta(x,\lambda) \underset{x\to\pm\infty}{\longrightarrow} J(\lambda x)\Theta_\pm(\lambda)J^{-1}(\lambda x)$ and similarly for $\bar{\Theta}$. Here $\mathbf{Q}_+(x) = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, $\mathbf{Q}_-(x) = \begin{pmatrix} 1 & \bar{b} \\ 0 & a \end{pmatrix}$ and the functions $a(\lambda)$ and $b(\lambda)$ enter the scattering matrix $\mathbf{S}(\lambda) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$. Besides, $\Theta_+ = \mathbf{S}\Theta_-$ and $\bar{\Theta}_+ = \mathbf{S}\bar{\Theta}_-$. We have also $\det\Theta = a(\lambda)$ and $\det\bar{\Theta} = \bar{a}(\lambda)$. Discrete spectrum characteristics are given by the quantities $\lambda_j, \bar{\lambda}_k, \gamma_j$ and $\bar{\gamma}_k (j = 1,\ldots,N\,, k = 1,\ldots,\bar{N})$ where λ_j

and $\bar{\lambda}_k$ are zeroes of $a(\lambda)$ and $\bar{a}(\lambda)$, respectively, while γ_j and and $\bar{\gamma}_k$ are defined by

$$\Theta^{(1)}(x,\lambda_j) = \gamma_j(x)\Theta^{(2)}(x,\lambda_j)\,,\ \gamma_j(x) = \gamma_j e^{2i\lambda_j x}\,,\ \mathrm{Im}\lambda_j > 0\,,\ \gamma_j \in \mathbb{C}\,,$$
$$\bar{\Theta}^{(2)}(x,\bar{\lambda}_k) = -\bar{\gamma}_k(x)\bar{\Theta}^{(1)}(x,\bar{\lambda}_k)\,,\ \bar{\gamma}_k(x) = \bar{\gamma}_k e^{-2i\bar{\lambda}_k x}\,,\ \mathrm{Im}\bar{\lambda}_k < 0\,,\ \bar{\gamma}_k \in \mathbb{C}\,.(2)$$

A reconstruction of the potential $\mathbf{Q}(x)$ from the scattering data is realized by means of the Riemann problem

$$\bar{\mathbf{Q}}^{+}(x,\xi)\,\mathbf{Q}(x,\xi) = \mathbf{G}(x,\xi)\,, \tag{3}$$

where $\xi = \mathrm{Re}\lambda$, $\mathbf{G}(x,\xi) = J(\xi x)\mathbf{G}(\xi)J^{-1}(\xi x)$, $\mathbf{G}(\xi) = \begin{pmatrix} 1 & \bar{b} \\ b & 1 \end{pmatrix}$, $\bar{\Theta}^{+} \equiv \det\bar{\Theta}\bar{\Theta}^{-1}$. A solution of (3) has the form [5,6]

$$\bar{\Theta}(x,\lambda) = I - \sum_{j=1}^{N}(\lambda_j - \lambda)^{-1}\,\frac{\Theta(x,\lambda_j)}{\dot{a}(\lambda_j)} -$$
$$- \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\,\Theta(x,\xi)\varrho^{+}(x,\xi)\,,\quad \mathrm{Im}\lambda < 0\,,$$
$$\Theta(x,\lambda) = I - \sum_{k=1}^{\bar{N}}(\bar{\lambda}_k - \lambda)^{-1}\,\frac{\bar{\Theta}(x,\bar{\lambda}_k)}{\dot{\bar{a}}(\bar{\lambda}_k)} +$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\lambda}\,\bar{\Theta}(x,\xi)\bar{\varrho}(x,\xi)\,,\quad \mathrm{Im}\lambda > 0\ . \tag{4}$$

Here $\dot{a}(\lambda_j) = \frac{d}{d\lambda}a(\lambda)\,|_{\lambda=\lambda_j}$, $\varrho = -a^{-1}J[\mathbf{G}(\xi) - I]J^{-1}$, $\bar{\varrho} = \bar{a}^{-1}J[\mathbf{G}(\xi) - I]J^{-1}$. As $|\lambda| \to \infty$, Im $\lambda < 0$, we have [7] $\bar{\Theta}(x,\lambda) = I + (2i\lambda)^{-1}\omega(x) + O(|\lambda|^{-1})$. Then $\mathbf{Q}(x)$ is expressed as $\mathbf{Q}(x) = \frac{1}{2}[\,\sigma_3\,,\,\omega(x)\,]$.

3. Let us consider a perturbed equation from the AKNS hierarchy. In this case we have "the ϵ–curvature representation" $U_t - V_x + [\,U\,,\,V\,] = i(\epsilon\widehat{R} - \lambda_t\sigma_3)$, where

$$U = -i\lambda\sigma_3 + \mathbf{Q}(x)\,,\quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\,,\quad \widehat{R}(r,q) = \begin{pmatrix} 0 & \bar{R}(r,q) \\ R(r,q) & 0 \end{pmatrix}\ . \tag{5}$$

Here ϵ is a small parameter, $R \to 0$ as $|x| \to \infty$. It can be shown that the perturbation–induced evolution law for the scattering matrix is expressed as

$$\mathbf{S}_t - [\,V_-\,,\,S\,] = i\epsilon\Theta_+\left(\int_{-\infty}^{\infty} J^{-1}\Theta^{-1}\widehat{R}\Theta J\,dx\right)\Theta_-^{-1} \tag{6}$$

with $V_- = \mathrm{diag}\,(A_-, -A_-)$ and $A_- = \lim_{x\to-\infty} A$. Then

$$a_t = \epsilon\langle\Theta^{(1)}|\sigma_2\widehat{R}|\Theta^{(2)}\rangle\,,$$
$$b_t + 4i\lambda^2 b = -\epsilon\langle\Theta^{(1)}|\sigma_2\widehat{R}e^{-2i\lambda x}|\bar{\Theta}^{(1)}\rangle\ . \tag{7}$$

Here the symbol $\langle \Theta^{(i)} | f(x) | \Theta^{(j)} \rangle$ stands for the integral $\int_{-\infty}^{\infty} dx^t\, \Theta^{(i)}(x) f(x)$ $\Theta^{(j)}(x)$. For the discrete spectrum we have

$$
\begin{aligned}
\Lambda_{j_t} &= -\frac{\epsilon}{\dot{a}_j} \langle \Theta^{(1)}(\lambda_j) | \sigma_2 \widehat{R} | \Theta^{(2)}(\lambda_j) \rangle\,, \quad \gamma_{j_t} + 4i\lambda_j^2 \gamma_j \\
&= -\epsilon \frac{\gamma_j}{\dot{a}_j} \left[\frac{\partial}{\partial \lambda} \langle \Theta^{(2)}(\lambda_j) | \sigma_2 \widehat{R} | \Theta^{(1)}(\lambda) - \gamma_j(x) \Theta^{(2)}(\lambda) \rangle_{\lambda=\lambda_j} - \right. \\
&\qquad \left. -2i \langle \Theta^{(1)}(\lambda_j) | \sigma_2 \widehat{R} x | \Theta^{(2)}(\lambda_j) \rangle \right]
\end{aligned} \tag{8}
$$

and the similar relations for $\bar{\lambda}_{k_t}$ and $\bar{\gamma}_{k_t}$.

It should be stressed that the equations (7) and (8) are exact ones, but the exact equations cannot be directly applied because their right-hand sides should be calculated for unknown solution of the perturbed nonlinear equation. Hence, an effective procedure is needed for practical calculation of the matrices Θ and $\bar{\Theta}$ in a relevant order of ϵ. Below we describe a method of obtaining the first and second-order corrections in the case of the perturbed nonlinear Schrödinger equation (NSE) $ir_t + r_{xx} + 2|r|^2 r = i\epsilon R(r)$, but this algorithm permits the advancement to higher orders, an actual possibility of such advancement being limited only by an amount of calculations.

4. We seek a solution of the perturbed NSE in the form $r = \overset{s}{r} + \overset{1}{r} + \overset{2}{r}$, where $\overset{s}{r}$ is a soliton solution, while the terms $\overset{i}{r}$ are of the order of ϵ^i. Thereby $\Theta = \overset{s}{\Theta} + \overset{1}{\Theta} + \overset{2}{\Theta}$ and $\bar{\Theta} = \overset{s}{\bar{\Theta}} + \overset{1}{\bar{\Theta}} + \overset{2}{\bar{\Theta}}$. Since $b(\lambda) = e^{-2i\lambda x} \det |\bar{\Theta}^{(1)}, \Theta^{(1)}|$ and $\overset{s}{b} = 0$, we get $b = \overset{1}{b} + \overset{2}{b}$, which gives $\varrho = \overset{1}{\varrho} + \overset{2}{\varrho}$ and $\bar{\varrho} = \overset{1}{\bar{\varrho}} + \overset{2}{\bar{\varrho}}$. As a consequence, the basic equations (4) are written in the form ($\lambda_1 = \xi_1 + i\eta_1$)

$$
\begin{aligned}
\overset{s}{\bar{\Theta}}(x,\lambda) + \overset{1}{\bar{\Theta}}(x,\lambda) + \overset{2}{\bar{\Theta}}(x,\lambda) &= I + \frac{2i\eta_1}{\lambda - \lambda_1} \left(1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} |\overset{1}{b}(\mu)|^2 \frac{d\mu}{\mu \lambda_1} \right) \times \\
&\times \left[\overset{s}{\Theta}(x,\lambda_1) + \overset{1}{\Theta}(x,\lambda_1) + \overset{2}{\Theta}(x,\lambda_1) \right] \mathbf{F}_1(x) - \\
&- \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \lambda} \left(\overset{s}{\Theta}(x,\xi) + \overset{1}{\Theta}(x,\xi) \right) \left(\overset{1}{\varrho}{}^{+}(x,\xi) + \overset{2}{\varrho}{}^{+}(x,\xi) \right)\,,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\overset{s}{\Theta}(x,\lambda) + \overset{1}{\Theta}(x,\lambda) + \overset{2}{\Theta}(x,\lambda) &= I - \frac{2i\eta_1}{\lambda - \bar{\lambda}_1} \left(1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} |\overset{1}{b}(\mu)|^2 \frac{d\mu}{\mu \bar{\lambda}_1} \right) \times \\
&\times \left[\overset{s}{\bar{\Theta}}(x,\bar{\lambda}_1) + \overset{1}{\bar{\Theta}}(x,\bar{\lambda}_1) + \overset{2}{\bar{\Theta}}(x,\bar{\lambda}_1) \right] \bar{\mathbf{F}}_1(x) + \\
&+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \lambda} \left(\overset{s}{\bar{\Theta}}(x,\xi) + \overset{1}{\bar{\Theta}}(x,\xi) \right) \left(\overset{1}{\bar{\varrho}}{}^{+}(x,\xi) + \overset{2}{\bar{\varrho}}{}^{+}(x,\xi) \right)\,,
\end{aligned} \tag{10}
$$

where we employ the matrices

$$
\mathbf{F}_1(x) = \begin{pmatrix} 0 & \gamma_1^{-1}(x) \\ \gamma_1(x) & 0 \end{pmatrix}\,, \quad \bar{\mathbf{F}}_1(x) = -\begin{pmatrix} 0 & \bar{\gamma}_1(x) \\ \bar{\gamma}_1^{-1}(x) & 0 \end{pmatrix} \tag{11}
$$

for taking into account the conditions (2) and use the formula $\dot{a}(\lambda_1) = (2i\eta_1)^{-1} \exp\left[(2\pi i)^{-1} \int_{-\infty}^{\infty} (\mu - \lambda_1)^{-1} \ln(1 - |b(\mu)|^2)\, d\mu\right]$ up to the second order of ϵ. The subsequent manipulations consist in the following. Collecting the terms in (9) without ϵ we get a system of algebraic equations

$$\overset{0}{\Theta} (x,\lambda) = I - \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \bar{\lambda}_1}\ \overset{0}{\bar{\Theta}} (x,\bar{\lambda}_1)\bar{\mathbf{F}}_1(x)\,,$$
$$\overset{0}{\bar{\Theta}} (x,\lambda) = I - \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}\ \overset{0}{\Theta} (x,\lambda_1)\mathbf{F}_1(x)\,. \tag{12}$$

Solving this system with respect to $\overset{0}{\bar{\Theta}}$, we find a non-perturbative soliton $\overset{0}{r}$. Solving then (8) with $\Theta = \overset{0}{\Theta}$ and $\bar{\Theta} = \overset{0}{\bar{\Theta}}$ and substituting the obtained perturbation-dependent parameters in $\overset{0}{r}$, we reach the adiabatic approximation [2].

Now collect the terms in (9) of the order of ϵ. This yields

$$\overset{1}{\bar{\Theta}} (x,\lambda) = \frac{2i\eta_1}{\lambda - \lambda_1}\ \overset{1}{\Theta} (x,\lambda_1)\, \mathbf{F}_1(x) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \lambda}\ \overset{0}{\Theta} (x,\xi)\, \overset{1}{\varrho}{}^{+}(x,\xi)\,,$$
$$\overset{1}{\Theta} (x,\lambda) = -\frac{2i\eta_1}{\lambda - \bar{\lambda}_1}\ \overset{1}{\bar{\Theta}} (x,\bar{\lambda}_1)\, \bar{\mathbf{F}}_1(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \lambda}\ \overset{0}{\bar{\Theta}} (x,\xi)\, \overset{1}{\bar{\varrho}} (x,\xi) \tag{13}$$

with parameters satisfying the equations of the adiabatic approximation. It should be noted that the integrands in (13) include $\overset{0}{\Theta}$ and $\overset{0}{\bar{\Theta}}$ obtained within the previous step. Thereby, equations (13) are again algebraical and not integral ones. Solving (13) with respect to $\overset{1}{\bar{\Theta}}$ we get a matrix $\overset{1}{\omega} (x)$ which gives a first-order correction to the soliton shape in the form $\overset{1}{\mathbf{Q}} (x) = \frac{1}{2}\left[\sigma_3\, ,\, \overset{1}{\omega} (x)\right]$.

For finding second-order corrections we collect in (9) the term of the order of ϵ^2 :

$$\overset{2}{\bar{\Theta}} (x,\lambda) = \frac{2i\eta_1}{\lambda - \lambda_1}\left[\overset{2}{G} (\lambda_1)\ \overset{0}{\Theta} (x,\lambda_1) + \overset{2}{\Theta} (x,\lambda_1)\right] \mathbf{F}_1 -$$
$$- \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \lambda}\left[\overset{0}{\Theta} (\xi)\ \overset{2}{\varrho}{}^{+}(x,\xi) + \overset{1}{\Theta} (\xi)\ \overset{1}{\varrho}{}^{+}(x,\xi)\right],$$
$$\overset{2}{\Theta} (x,\lambda) = -\frac{2i\eta_1}{\lambda - \bar{\lambda}_1}\left[\overset{2}{G} (\bar{\lambda}_1)\ \overset{0}{\bar{\Theta}} (x,\bar{\lambda}_1) + \overset{2}{\bar{\Theta}} (x,\bar{\lambda}_1)\right] \bar{\mathbf{F}}_1 +$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \lambda}\left[\overset{0}{\bar{\Theta}} (\xi)\ \overset{2}{\bar{\varrho}} (\xi) + \overset{1}{\bar{\Theta}} (\xi)\ \overset{1}{\bar{\varrho}} (\xi)\right]\ . \tag{14}$$

Here $\overset{2}{G} (\nu) = (2\pi i)^{-1} \int_{-\infty}^{\infty} |\ \overset{1}{b} (\mu)|^2 (\mu - \nu)^{-1}\, d\mu$. We point out that the first term in the right-hand side of (14) which corresponds to the discrete spectrum part includes $\overset{2}{G}$ which is determined by the continuous spectrum. thereby, in the second-order approximation there is no complete separation between the contributions of both kinds of the spectrum.

As an example of a possibility to overcome calculational barriers we give a second–order correction to the NSE soliton in the case of a dissipation $R = -r$:

$$\overset{2}{r}(x,t) = -\frac{i\epsilon^2}{32\eta_1^3}\Bigg[2(1 - z\tanh z)I_2 - 4\nu I_1(z)\tanh z - \Big(4\nu - 4\nu z\tanh z$$
$$+ 6\tanh z\Big)I_0(z) + \frac{3}{4}\zeta(3)(1 - z\tanh z)2(2 - \sinh^2 z)\times$$
$$\times(\ln 2\cosh z - z\tanh z) + \frac{\pi^2}{12}(1 - 4z\tanh z + 3\tanh z) - \frac{2}{3}z^3\tanh z$$
$$+ \frac{2}{3}z^3\tanh z + \nu\left(z^2 + \frac{\pi^2}{12}\right)(2 + \tanh z) + z^2\left(2 + 3\tanh z + e^{-z}\sinh z\right)$$
$$- \nu\left(\frac{\pi^2}{6} + \frac{2}{3}z^2\right)z\tanh z - \nu\sinh 2z\,\ln 2\cosh z + 2\nu z\sinh^2 z + \frac{\pi^2}{4}$$
$$- \frac{1}{2}\left(z^2 + \frac{\pi^2}{12}\right)^2\Bigg]e^{i\vartheta}\,\mathrm{sech}\, z\ .$$

Here $z = 2\eta_1(x - X(t))$, $\vartheta = \xi_1\eta_1^{-1}z + \Delta(t)$, $\gamma_1(t) = \exp i\,(\Delta(t) - 2\lambda_1 X(t))$, $\nu =$ const., $I_0(z) = \int_0^{e^z}\ln(1+y^2)\frac{dy}{y}$, $I_1(z) = \int_0^{e^z}\ln y\ln(1+y^2)\frac{dy}{y}$, $I_2(z) = \int_0^1 \ln^2(1+y^2)\frac{dy}{y}$. The leading term of the asymptotics $\overset{2}{r}(x,t)_{|x|\longrightarrow\infty}(i\epsilon^2)/(64\eta_1^3)z^4e^{i\vartheta}\mathrm{sech}\,z$ indicates the absence of a tail behind the soliton.

References

1. D. J. Kaup : SIAM J. Appl. Math. **31**, 121 (1976).
2. V. I. Karpmann and E. M. Maslov : Zh. Eksp. Teor, Fiz. **73**, 537 (1977).
3. D. J. Kaup and A. C. Newell : Proc. Roy. Soc. London A **361**, 413 (1978).
4. Yu. S. Kivshar and B. A. Malomed : Rev. Mod. Phys. **61**, 763 (1989).
5. V. E. Zakharov and A. B. Shabat : Sov. Phys. JETP **34**, 62 (1972).
6. T. Kawata : J. Phys. Soc. Japan **51**, 3381 (1982).
7. L. D. Faddeev and L. A. Takhtadjan : Hamiltonian Method in the Theory of Solitons (Springer–Verlag, Amsterdam, 1987).

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