Eur. Phys. J. Plus (2024) 139:592
https://doi.org/10.1140/epjp/s13360-024-05373-y

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Hypergeometric Gevrey-0 approximation for the Gevrey-*k* divergent series with application to eight-loop renormalization group functions of the $O(N)$-symmetric field model

Abouzeid M. Shalaby[1,a], **Hamdi M. Abdelhamid**[2,3], **I. S. Elkamash**[2]

[1] Physics and Materials Sciences Department, College of Arts and Sciences, Qatar University, P.O. Box 2713, Doha, Qatar
[2] Physics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[3] Physics Department, Faculty of Science, New Mansoura University, New Mansoura City, Egypt

Received: 14 December 2023 / Accepted: 14 June 2024

Abstract Mera et al. (Phys Rev Lett 115:143001, 2015) discovered that the hypergeometric function ${}_2F_1(a_1, a_2; b_1; \omega g)$ can serve as an accurate approximant for a divergent Gevrey-1 type of series with an asymptotic large-order behavior of the form $n!\, n^b \sigma^n$. What is strange about this approximant is that it has a series expansion with the wrong large-order behavior (Gevrey-0 type). In this work, we extend this discovery to Gevrey-k series where we show that the hypergeometric approximants and its extension to the generalized hypergeometric approximants are not only able to approximate divergent (Gevrey-1) series but also able to approximate strongly-divergent series of Gevrey-k type with $k = 2, 3, \ldots$. Moreover, we show that these hypergeometric approximants are able to predict accurate results for the non-perturbative strong-coupling and large-order parameters from weak-coupling data as input. Examples studied here are the ground-state energy for the x^n anharmonic oscillators. The hypergeometric approximants are also used to approximate the recent eight-loop series (g-expansion) of the renormalization group functions for the $O(N)$-symmetric ϕ^4 scalar field model. Form these functions for $N = 0, 1, 2,$ and 3, critical exponents are extracted which are very competitive to results from more sophisticated approximation techniques.

1 Introduction

Exact solutions for different problems in physics can barely be found for realistic models. Physicists frequently resort to perturbative calculations which are the prominent tool in treating a problem in physics. In many situations, the perturbative expansion is carried out around an essential singularity that leads the series to be divergent. In such cases, the perturbative calculations are useless by their own, and resummation procedures are recommended to follow the perturbative calculation if one needs to obtain reliable results. It was Dyson who draw the attention of the physics community to this fact by analyzing the QED perturbation series and concluded that the series must be divergent [1]. His argument was based on the realization that when one crosses the origin of the coupling space from positive to negative coupling (say), the theory is turned out to be unstable. However, divergent series like those in QED are asymptotic and because of the smallness of the fine-structure constant, one can realize the divergence in the series at a relatively high order of loop calculations [2]. On the other hand, one can realize this fact very clearly for relatively low orders in models like the $O(n)$-symmetric ϕ^4 scalar field theory where it can be used to predict the critical behavior for different models that lie in the same class of universality ranging from polymers, ${}^4He$ superfluidity to QCD [2–5]. In quantum mechanics, one can also find such type of divergent series in examples such as the ground-state energies for the x^4 and the $\mathcal{PT}$-symmetric ix^3 anharmonic oscillators (for instance). Accordingly, to be able to get reliable results from divergent series, resummation is a necessary tool to get the analytic continuation of divergent series that are existing in many branches in physics.

One of the most successful approximation techniques that deal with divergent series is the Borel algorithm [2, 6–11]. In fact, this powerful algorithm uses a kind of Laplace (Borel) transformation. For the choice of the right transformation, one needs a priory knowledge of the large-order asymptotic behavior of the given divergent series. Of course the transformation needed for a divergent series with an asymptotic large-order behavior of the form $n!\,\sigma^n n^b$ is different from that is needed for a strongly-divergent series with an asymptotic behavior of the form $(2n)!\,\sigma^n n^b$. Another issue with Borel algorithm is the slow convergence when applied to a strongly-divergent series. For instance, for the series of the ground-state energy of the octic oscillator, it can give reliable results only for very small values of the coupling [12–14].

Another successful approximation technique was developed by us in Ref. [14]. In that technique, the selection of the suitable approximant depends on the large-order asymptotic behavior of the given series too. Therefore, we used the divergent hypergeometric

[a] e-mail: amshalab@qu.edu.qa (corresponding author)

Published online: 07 July 2024

series ${}_pF_{p-2}(a_1, \ldots a_p; b_1 \ldots .b_{p-2}; \sigma g)$ to approximate the ground-state energy of the x^4 anharmonic oscillator. The given series and the suggested hypergeometric approximant have the same form of the large-order behavior which takes the form:

$$c_n \sim \gamma n!\,(-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right). \tag{1}$$

For the sextic oscillator, we used another hypergeometric approximants ${}_pF_{p-3}(a_1, \ldots a_p; b_1 \ldots .b_{p-3}; \sigma g)$ which is a strongly-divergent series like the given series with a large-order behavior given by:

$$c_n \sim \gamma (2n)!\,(-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right). \tag{2}$$

In the same reference, we stressed also the case of the octic oscillator where we used the strongly-divergent (even stronger than the sextic case) approximant ${}_pF_{p-4}(a_1, \ldots a_p; b_1 \ldots .b_{p-4}; \sigma g)$ which shares the same large order behavior as the given series

$$c_n \sim \gamma (3n)!\,(-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right). \tag{3}$$

All the above (different) hypergeometric approximants are divergent and should be followed by an analytic continuation process via using the Mellin–Barnes integral representation of the form:

$${}_pF_q(a_1, \ldots a_p; b_1 \ldots .b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{j=1}^{p} \Gamma(a_j)} \frac{1}{2\pi i} \int_C \frac{\Gamma(s)\prod_{j=1}^{p} \Gamma(a_j - s)}{\prod_{k=1}^{q} \Gamma(b_k - s)} (-z)^{-s} ds. \tag{4}$$

In fact, to apply such approximants (as in the case of applying Borel algorithm), one needs in advance to know the large-order asymptotic behavior of the given series to predict which approximant is suitable to treat the given series. This asymptotic behavior is in fact known but for a limited number of series, and thus, one needs an algorithm that can be applied even in case, the large-order behavior is not explicitly known.

Bearing all of these facts in mind, it will be more than important to seek an approximating algorithm that is not in a need for a priory knowledge of the asymptotic behavior of the given series as most of the known series do not have such behavior obtained yet. It is noteworthy to mention that the large-order and strong-coupling behaviors (if known) might be used to accelerate the convergence of approximation algorithms [2, 6–10]. So even in case, the large-order behavior for the given series is known, the suggested algorithm should be capable to accommodate such parameters to accelerate the convergence like the traditional Padé [15–17] and Borel algorithms. The aimed algorithm has been discovered by Mera et al. in Ref. [9] but for a Gevrey-1 type of series only. In this work, we extend the idea to include any Gevrey-k type of series plus showing how one can extract the non-perturbative strong-coupling and large-order behaviors of the given series from perturbative terms as input.

In Ref. [9], Mera et al. discovered that a divergent series with coefficients c_n having asymptotic large-order behavior of the form $c_n \sim n!\,\sigma^n n^b$ can be approximated by a hypergeometric function of the form ${}_2F_1(a_1, a_2; b_1\ \omega z)$. What is strange about such approximant is that it has a series expansion of a wrong large-order behavior which can be explicitly shown as follows:

$${}_2F_1(a_1, a_2; b_1\ \omega z) = \sum_{n=0}^{\infty} g_n z^n, \tag{5}$$

where

$$g_n = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(a_2)} \frac{\Gamma(a_1 + n)\,\Gamma(a_2 + n)}{n!\,\Gamma(b_1 + n)} \omega^n.$$

It is easy to show that for large n, the coefficients behave as $g_n \sim \omega^n n^b$, where $b = a_1 + a_2 - b_1 - 1$ [3, 5, 18–21]. The hypergeometric approximant above does not have explicitly the asymptotic large-order behavior that the given series has. Nevertheless, it can give a good approximation for that series. The question that may arise is that, can such approximant works also for strongly-divergent series with coefficients behaving (for large n) like $(2n)!\,\sigma^n n^b$ or $(3n)!\,\sigma^n n^b$ (for instance)? If yes, then the hypergeometric approximant is not in a need for a priory knowledge of the asymptotic large-order behavior as it can approximate divergent series with different large-order behaviors. In other words, the approximant does not care about the large-order behavior of the given divergent series. Instead, its parameters will take suitable values that make the approximant asymptotically approaches the needed features. In fact, for most of divergent series, the large-order behavior is not known in advance, and thus, such features of the approximant are beneficial. Apart from this, note that the hypergeometric approximant ${}_2F_1(a_1, a_2; b_1\ \omega z)$ has an asymptotic large-order behavior of the form $\omega^n n^b$ and thus by default is suitable to approximate a series with finite radius of convergence too.

In Ref. [22], we have shown that the set $\{-a_1, -a_2\}$ is representing the asymptotic strong-coupling parameters for the hypergeometric series ${}_2F_1(a_1, a_2; b_1\ \omega z)$. Needless to say that the fourth-order approximant ${}_2F_1(a_1, a_2; b_1\ \omega z)$ can be extended to any order by using the generalized hypergeometric function ${}_pF_{p-1}(a_1, a_2, \ldots .a_p; b_1, b_2, \ldots .b_{p-1}\ \omega z)$ where the set $\{-a_1, -a_2, \ldots ..\}$ predicts the strong-coupling parameters. If the hypergeometric approximant can approximate a series regardless of its large-order behavior, then one has to find the reason behind that. For that, consider the fourth-order hypergeometric approximant ${}_2F_1(a_1,$

$a_2; b_1\ \omega z$). In the parametrization of this approximant for a divergent series with zero-radius of convergence and asymptotic large-order behavior of the form $n!\,\sigma^n n^b$, we realized that the two parameters b_1 and ω are taking relatively large values. When extending to higher orders and using the generalized hypergeometric approximant ${}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots.b_{p-1}\ \omega z)$, then ω and at least one of the b_j denominator parameters are taking larger values. For instance, let us consider the ground-state energy of the x^4 anharmonic oscillator of the form [23]

$$E_0(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + O(g^5). \tag{6}$$

This series is divergent and has a large-order asymptotic form of the coefficients $c_n \sim n!\,\sigma^n n^b$, where $\sigma = -3$ and $b = -\frac{1}{2}$ [23]. Let us approximate the 4^{th} order using the series ${}_2F_1(a_1, a_2; b_1\ \omega z)$ which has a wrong (explicit) large-order behavior of the form $\omega^n n^b$. In fact, the hypergeometric function has the series expansion of the form

$$\frac{1}{2}\,{}_2F_1(a_1, a_2; b_1\ \omega g) = \sum_{i=0}^{\infty} h_i g^i,$$

where

$$h_n = \frac{1}{2}\frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(a_2)}\frac{\Gamma(a_1+n)\,\Gamma(a_2+n)}{n!\,\Gamma(b_1+n)}\omega^n.$$

Solving the nonlinear relations $c_i = h_i$ $(i = 0, 1, \ldots, 4)$, where c_i are the coefficients in Eq. (6), we obtain the parameter values as: $a_1 = -0.30337$, $a_2 = 0.75632$, $b_1 = 7.0098$, and $\omega = -45.8265$. Note that a_1 here is representing the strong-coupling parameter such that $E_0 \sim Ag^{-a_1}$ which is close to its exact value of $-\frac{1}{3} \approx -0.33333$. With no loss of generality, let $\omega = \sigma b_1$ where then $\sigma = \frac{-45.8265}{7.0098} = -6.5375$. One can realize that the parameter values b_1 and ω are taking relatively high values when compared with the other parameters. To understand why such hypergeometric approximant with wrong asymptotic large-order behavior is capable of approximating a divergent series with zero-radius of convergence, we consider the limit

$$\lim_{b_1\to\infty}\left({}_2F_1(a_1, a_2; b_1; -b_1\sigma x)\right) = {}_2F_0(a_1, a_2; -\sigma x). \tag{7}$$

In fact, the hypergeometric series ${}_2F_0(a_1, a_2; -\sigma x)$ has the correct form of the asymptotic large-order behavior which takes the form $n!\,\sigma^n n^b$ [3, 5, 18–21]. However, the large-order parameter σ predicted from the fourth order approximant is almost twice its exact value. We expect that accurate results for the non-perturbative parameters can be obtained at high-orders parametrization. For instance, when considering the 10^{th} order approximant ${}_5F_4(a_1, a_2, \ldots..a_5; b_1, b_2, \ldots.b_4\ \omega g)$, we get the results $a_1 = -0.33172$, $a_2 = 0.38680$, $a_3 = 0.8985 - 11.2519i$, $a_4 = 0.8985 + 11.2519i$, $a_5 = 3.7185$ while $b_1 = 1.1680$, $b_2 = 5.5611 - 12.4002i$, $b_3 = 5.5611 + 12.4002i$, $b_4 = 67.880$, $\omega = -361.30$. The strong-coupling asymptotic parameter $a_1 = -0.33172$ is now more accurate and also one can realize that both b_4 and ω are taking larger values than in the fourth order. Those large values make the limit that takes the approximant ${}_5F_4(a_1, a_2, \ldots..a_5; b_1, b_2, \ldots.b_4\ \omega g)$ to ${}_5F_3(a_1, a_2, \ldots..a_5; b_1, b_2, b_3\ \ \sigma g)$ more accessible. The predicted value of σ is $\sigma = \frac{\omega}{b_4} = -5.3226$ which is still not accurate enough but it is better than the fourth-order prediction. This analysis shows why the hypergeometric approximant ${}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots.b_{p-1}\ \omega g)$ which has the wrong large-order asymptotic behavior can be parametrized to approximate a divergent series with zero-radius of convergence. Not only this but also it can predict accurate values (with sufficient weak-coupling terms as input) for the non-perturbative strong-coupling and large-order parameters. As we explained above this is because of the existence of the following limit:

$$\lim_{b_s\to\infty}\left({}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, .., b_s, ..b_{p-1}\ b_s\sigma g)\right) = {}_pF_{p-2}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots.b_{p-2}\ \sigma g), \tag{8}$$

where $|b_s|$ is the largest parameter among the parameters $|b_1|, |b_2|, .., |b_s|..|b_{p-1}|$.

Divergent series with zero-radius of convergence can have different types of asymptotic large-order behavior. For instance, the ground-state energies of the sextic and octic anharmonic oscillators are strongly-divergent in such a way that the coefficients in the weak-coupling series expansions behave as (for large n) $(2n)!\,\sigma^n n^b$ (Gevrey-2 type) and $(3n)!\,\sigma^n n^b$ (Gevrey-3 type), respectively. However, the hypergeometric approximant ${}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots.b_{p-1}\ \omega g)$ can be parametrized to give accurate results for such cases too. In such cases and for the sextic oscillator (for instance), two of the b_j parameters (b_s and b_t say) will take large values. In view of the iterated limit of the form:

$$\lim_{b_s\to\infty}\left(\lim_{b_t\to\infty}\ \left({}_pF_{p-1}(a_1, a_2, \ldots a_p; b_1, b_2, \ldots, b_s, b_t..b_{p-1}\ ; b_s b_t\sigma g)\right)\right)$$
$$= {}_pF_{p-3}(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_{p-3}\ ; \sigma g), \tag{9}$$

one finds that the approximant ${}_pF_{p-3}(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots.b_{p-3}\ \sigma g)$ has the right large-order asymptotic behavior as the given series.

For the octic oscillator, we have

$$\lim_{b_s\to\infty}\left(\lim_{b_t\to\infty}\left(\lim_{b_u\to\infty}\ \left({}_pF_{p-1}\big(a_1, a_2, \ldots..a_p; b_1, b_2, .., b_s, b_t, b_u..b_{p-1}\ ; b_s b_t b_u\sigma g\big)\right)\right)\right)$$

$$= {}_pF_{p-4}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots .b_{p-4}\ ; \sigma g). \tag{10}$$

Again the hypergeometric series ${}_pF_{p-4}$ has the same large-order behavior as the given series. This analysis explains why the hypergeometric approximant ${}_pF_{p-1}$ is capable to approximate Gevrey-k type of divergent series for $k \geq 1$.

Although the hypergeometric approximant ${}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots .b_{p-1}\ \omega g)$, which is a Gevrey-0 type, is expected to approximate divergent series of a Gevrey-k type ($k = 1, 2, 3, \ldots$), there exist technical issues when parametrizing such approximant to approximate divergent series. For instance, it is well-known that the approximant ${}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots .b_{p-1}\ \omega g)$ has a branch cut starting from $g = \frac{1}{\omega}$ to $\omega g \to \infty$. Accordingly, if the parameter ω takes a positive value, the approximant fails to give reliable results for the region of interest (for positive g values). If for some order, this is the case, we suggest to parametrize the series expansion for the reciprocal of the quantity which may result in a negative ω value. If for some order, the parametrization for both series fails to give a negative ω value, then the approximant can be taken to be the right-hand side of Eqs. (8, 9, and 10) while the analytic continuation is then go through the representation [14, 24]:

$$\begin{aligned} &{}_pF_q(a_1, \ldots a_p; b_1 \ldots .b_q; \sigma\ z) \\ &= \frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(a_i)} \sum_{k=1}^{p} (-\sigma\ z)^{-a_k} \frac{\Gamma(a_k)}{\prod_{j=1}^{q} \Gamma\left(b_j - a_k\right)} \prod_{j=1, j\neq k}^{p} \Gamma\left(a_j - a_k\right) \\ &\times\ {}_{q+1}F_{p-1}\left(a_k, a_k - b_1 + 1, \ldots..., a_k - b_q + 1; -a_1 + a_k + 1, \underbrace{\ldots..}_{*}, -a_p + a_k + 1; \frac{(-1)^{p-q+1}}{\sigma\ z}\right). \end{aligned} \tag{11}$$

Here the asterisk means that we exclude terms of the form $1 - a_i + a_j$ for $i = j$.

Based on the above realizations, in this work, we use only one type of hypergeometric approximants for divergent series of different large-order behaviors. In other words, we show that we can use the approximants ${}_pF_{p-1}(a_1, \ldots a_p; b_1 \ldots .b_{p-1}; \sigma g)$ as all-in-one approximation technique. In fact either in our previous work in [14] or the traditional Borel resummation, one has to treat the above-mentioned different types of series differently which means that it will take longer time and more efforts to approximate the series under consideration. Plus for such techniques, we need to know the large-order behavior in advance.

The structure of this paper is as follows. In Sec. 2, we use the hypergeometric approximants ${}_pF_{p-1}$ to approximate the ground-state energy of the quartic oscillator. In this section, accurate results are obtained for the ground-state energy and the non-perturbative strong-coupling and large-order parameters. In Sec. 3, we study the strongly-divergent Gevrey-2 series of the ground-state energy of the sextic oscillator where again we obtained accurate results for both the ground-state energy and the non-perturbative parameters. The strongly-divergent Gevrey-3 series for the ground-state energy of the octic oscillator is also studied in Sec. 4. Critical exponents for the $O(N)$-symmetric ϕ^4 scalar field model are also stressed in Sec. 5. In this section, accurate results are obtained for the critical coupling as well as critical exponents using the eight-loop of weak-coupling series that recently appeared in the literature. Summary and conclusions follow in Sec. 7.

2 Hypergeometric parametrization for the ground-state energy of the quartic anharmonic oscillator

The Hamiltonian for the quartic oscillator is given by:

$$H = \frac{p^2}{2} + \frac{1}{2}x^2 + gx^4. \tag{12}$$

The weak-coupling perturbation series for the ground-state energy is given in Eq. (6) above as

$$E_0(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + O(g^5).$$

As explained in the introduction, this is a divergent series with zero-radius of convergence as its asymptotic large-order behavior has the form $n!\,\sigma^n n^b$. Nevertheless, the hypergeometric approximant ${}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots .b_{p-1}\ \omega g)$, which has a wrong asymptotic large-order behavior ($\omega^n n^b$), can be parametrized to give accurate predictions. The point is that one of the b_j parameters and the ω ($\omega = \sigma b_s$) parameter shall take large values in such a way that the following limit

$$\lim_{b_s \to \infty} ({}_pF_{p-1}(a_1, a_2, \ldots..a_p; b_1, b_2, .., b_s..b_{p-1}\ b_s\sigma g)) = {}_pF_{p-2}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots .b_{p-2}\ \sigma g),$$

is made possible. The hypergeometric function ${}_pF_{p-2}(a_1, a_2, \ldots..a_p; b_1, b_2, \ldots .b_{p-2}\ \sigma g)$, on the other hand, has an expansion which has the right large-order asymptotic behavior. This explanation rationalizes the discovery by Mera. et al. in Ref. [9] that the fourth-order hypergeometric function ${}_2F_1(a_1, a_2; b_1; \omega g)$ can give accurate approximation for a series of zero-radius of convergence.

Table 1 The hypergeometric approximation for the ground-state energy of the x^4 anharmonic oscillator for different orders of weak-coupling data is listed
The exact (numerical) results from Ref. [26] are shown for comparison

g	${}_1F_0$	${}_3F_2$	${}_5F_4$	${}_{12}F_{11}$	Exact
0.5	0.669972	0.696097	0.696175	0.696176	0.696176
1	0.743891	0.803389	0.803763	0.803771	0.803771
2	0.832700	0.950177	0.951533	0.951568	0.951568
50	1.45540	2.43906	2.49657	2.49970	2.49971

In going to higher orders, one might face a problematic parametrization in which the parameter ω is positive. In this case, one might consider the expansion of $\frac{1}{E_0(g)}$ which is then:

$$\frac{1}{E_0(g)} = 2 - 3g + 15g^2 - \frac{243}{2}g^3 + \frac{43233}{32}g^4 + O\left(g^5\right), \tag{13}$$

where one can realize that the sign of odd orders in this equation is negative while they were positive in Eq. (6). This shows that in view of the large-order behavior $\omega^n n^b$, the parametrization of $\frac{1}{E_0(g)}$ might lead to an ω value of opposite sign. However, since the sign of a specific order in the expansion of the hypergeometric approximant depends also on numerator and denominator parameters, one might have for some orders that both parametrizations lead to positive ω values, and thus, the branch cut lies in the region of interest. In this case, one can resort to the formula in Eq. (8) as an approximation for the given order.

The hypergeometric approximant can also predict the non-perturbative parameters. For instance, the strong-coupling parameter s^* can be determined from [14]

$$E_0(g) \simeq \frac{\prod_{i=1}^{q}\Gamma(b_i)}{2\prod_{i=1}^{p}\Gamma(a_i)} \sum_{k=1}^{p} (-\omega\, g)^{-a_k} \frac{\Gamma(a_k)}{\prod_{j=1}^{p-2}\Gamma\left(b_j - a_k\right)} \prod_{j=1, j\neq k}^{p} \Gamma\left(a_j - a_k\right).$$
$$\simeq A g^{s^*}, \tag{14}$$

where $s^* = \max\{Re(-a_1),\ Re(-a_2),\ \ldots.Re(-a_p)\}$. Also the limits in Eqs. (8, 9, and 10) suggest that the large-order parameter b can be predicted from the relation:

$$b = \sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i - \left(\frac{p - q + n_l + 2}{2}\right) + b_s + b_t\ + \ldots.. \tag{15}$$

here $b_s,\ b_t,\ \ldots\ldots$ are representing the largest denominator parameters (large absolute values) in the approximants, and n_l is the number of these largest denominator parameters (one parameter for Gevrey-1, two for Gevrey-2, and so on).

For relatively high orders, there might be some kind of saturation in which one (or more) of the numerator parameters a_i equals one (or more) of denominator parameters b_j. In such case, the form in Eq. (11) shows that the approximant is singular. However, one can exclude such singularity by using the fact that:

$${}_{p+1}F_{q+1}(a,\, a_2,\, \ldots a_p; a,\, b_2 \ldots .b_q; \omega\, g) = {}_pF_q(a_2,\, \ldots a_p; b_2 \ldots .b_q; \omega\, g).$$

We applied the above detailed approximation technique to approximate the ground-state energy of the x^4 anharmonic oscillator with the results as shown in Table 1. In this table, one can realize how accurate is that simple generalized hypergeometric approximant. For instance, for a relatively strong-coupling value ($g = 50$), the hypergeometric approximant gives the result $E_0(50) \approx 2.49970$ which shares the first five digits with the exact result shown in Table 1. This result is very competitive to the predictions from more sophisticated algorithms [10] with larger size of weak-coupling data as input.

It is a matter of fact that most of the well-known resummation algorithms fail to give reliable results for large coupling values [12, 13]. The generalized hypergeometric algorithm, on the other hand, predicts very accurate results for the strong-coupling parameter s^* as shown from Table 2. The strong-coupling asymptotic behavior takes the form $E_0(g) \sim A g^{s^*}$. Our predictions for A_β can be compared to the exact ones from Ref. [25], where A_β is gotten from A via a scale transformation ($A_\beta = 4^{s^*} A$).

The hypergeometric approximants we use in this paper have explicit wrong large-order behavior. However, in view of the iterated limits in Eqs. (8, 9, and 10) and the formula for the large-order parameter b in Eq. (15), one can predict approximate values for the large-order parameters b and σ. Our predictions are listed in Table 2 where one can see accurate results especially for relatively high orders.

3 Hypergeometric parametrization for the ground-state energy of the sextic anharmonic oscillator

The Hamiltonian model for the sextic anharmonic oscillator is given by:

$$H = \frac{p^2}{2} + \frac{1}{2}x^2 + g x^6. \tag{16}$$

Table 2 For different orders of input weak-coupling data, we list the hypergeometric prediction for the strong-coupling parameters s^* and A_β for the ground-state energy of the x^4 anharmonic oscillator

Parameter	${}_1F_0$	${}_2F_1$	${}_5F_4$	${}_{10}F_9$	${}_{12}F_{11}$	${}_{25}F_{24}$	Exact
s^*	0.176471	0.303372	0.331716	0.333256	0.333310	0.333333	$\frac{1}{3} \approx 0.333333$
A_β	0.931603	1.08778	1.065470	1.060743	1.060492	1.06036	1.06036
σ	−8.50000	−6.53745	−5.32275	−3.04634	−3.01323	−3.00000	−3.00000
b	−2.17647	−1.54704	−8.71955	−1.53812	−1.05166	−0.500044	$-\frac{1}{2} \equiv -0.500000$

The exact result for A_β is taken from Ref. [25] (Table 7, m=2,$n = 0$). Also, the predictions for both b and σ parameters are listed and compared to their exact values

The weak-coupling perturbation series expansion of the ground-state energy is listed in Ref. [27] as:

$$E_0(g) = \frac{1}{2} + \frac{15}{8}g - \frac{3495}{64}g^2 + \frac{1239675}{256}g^3 - \frac{3342323355}{4096}g^4 + O(g^5). \tag{17}$$

Also, the expansion of $1/E_0(g)$ goes like:

$$\frac{1}{E_0(g)} = 2 - \frac{15}{4}g + \frac{3945}{64}g^2 - \frac{1351275}{512}g^3 + \frac{3525355905}{16384}g^4 + O(g^5). \tag{18}$$

The series is strongly-divergent (Gevrey-2) in the sense that its large-order asymptotic behavior goes like:

$$c_n \sim \gamma(2n)!(-\sigma_h)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right). \tag{19}$$

here $\sigma_h = \frac{32}{\pi^2}$ [28]. The hypergeometric approximant (with wrong large-order behavior) used to approximate this series is

$$\frac{1}{2}\ {}_pF_{p-1}(a_1, a_2, \ldots\ldots.a_p; b_1, b_2, \ldots, b_{p-1}; \omega g). \tag{20}$$

However, the iterated limit in Eq. (9) shows that the parameters in this approximant can take the proper values that can lead to the expected large-order behavior. This is why it is expected that the approximant shall predict accurate values for the ground-state energy as well as the non-perturbative parameters associated with the given series.

In most of the orders we considered, the parametrization of the hypergeometric approximants results in a positive ω parameter. Accordingly, the branch cut covers most of the positive g axis. In fact, we realized that the ω parameter and one (for relatively low order) or two (for relatively high orders) of the b_i parameters are taking relatively large values which means that one can resort to the limit in Eqs. (8) or (9). We choose to work with the limit in Eq. (8) which gives the suitable branch cut and choose to work with the largest b_i parameter. This step, however, needs analytic continuation process to follow the limit step which can be carried out via the representation in Eq. (11). Note that the limit in Eq. (8) will lead to an ${}_pF_{p-2}$ which can be represented via entire hypergeometric functions in Eq. (11). For more clarifications, let us consider the $8th$ order approximant:

$$E_0(g) \simeq \frac{1}{2}\ {}_4F_3(a_1, a_2, \ldots a_4; b_1, b_2, b_3\ \omega g) \tag{21}$$

Using the weak-coupling data, one can get the parameter values as $a_1 = -0.248382$, $a_2 = 0.12755 - 5.34046i$, $a_3 = 0.12755 + 5.34046i$, $a_4 = 0.257737$ and $b_1 = -34.3930$, $b_2 = 0.34298 - 8.70971i$, $b_3 = 0.34298 + 8.70971i$ while $\omega = 5363.89$. The limit in Eq. (8) gives

$$\begin{aligned} E_0(g) &\simeq \frac{1}{2}\ {}_4F_3(a_1, a_2, \ldots a_4; b_1, b_2, b_3\ ; \omega g) \\ &\simeq \frac{1}{2}\ {}_4F_2(a_1, a_2, \ldots a_4; b_2, b_3\ ; \frac{\omega}{b_1} g). \end{aligned} \tag{22}$$

The prediction of this order is listed in Table 3 which shows accurate results. For different orders, we also listed the results in the same table, and it is very clear that more accurate results are obtained with bigger size of weak-coupling input data. Note that, traditional algorithms like Borel as well as Padé approximants fail to produce reliable results for such model especially for strong couplings [12, 13]. Contrary to this, we find the important feature of the hypergeometric approximant which is its capability to predict accurate results for the non-perturbative parameters. For instance, the $20th$ order approximant results in $s^* = 0.249922$ compared to its exact result 0.25. Also $A_\beta = 1.145001$ which is very close to its exact value 1.144802 [25]. This order prediction for the large-order parameter σ is −13.0031 compared to its exact value of −12.96911 and $b = -0.629353$ compared to exact value of −0.5. More accurate results especially for large-order parameters can be obtained by using higher orders. For instance, our $28th$ order prediction for b is −0.502320, while for σ, it is −12.9694 which are close to the exact ones. For more orders like the $50th$ order, one finds that b is −0.5, and for σ, it is −12.9691 which are almost exact [12, 13].

Table 3 The hypergeometric approximation for the ground-state energy of the x^6 anharmonic oscillator for different orders of weak-coupling data as input

g	${}_1F_0$	${}_2F_1$	${}_4F_3$	${}_{10}F_9$	Exact
0.1	1.12682	1.17636	1.17470	1.17390	1.173887
0.5	1.2332	1.44018	1.43750	1.43565	1.435625
2.0	1.3391	1.83322	1.83314	1.83047	1.830437
50	1.6262	3.66197	3.71548	3.71673	3.716974
1000	1.9492	7.39724	7.66973	7.69993	7.701738

Exact results taken from Ref. [29] are listed for comparison

Table 4 The hypergeometric approximation for the ground-state energy of the x^8 anharmonic oscillator for different orders of weak-coupling data

g	${}_1F_0$	${}_3F_2$	${}_4F_3$	${}_8F_7$	Exact
0.1	1.08762	1.23391	1.24591	1.2405	1.241028
0.5	1.12294	1.45902	1.51305	1.49219	1.491020
2.0	1.15446	1.73548	1.88695	1.82994	1.822180
50	1.23122	2.72077	3.62765	3.27012	3.188654

Exact results taken from Ref. [29] are listed for comparison too

4 Hypergeometric parametrization for the ground-state energy of the octic anharmonic oscillator

The potential term for the octic anharmonic oscillator is x^8 which results in the weak-coupling series expansion for the ground-state energy to take the form [27]:

$$E_0(g) = \frac{1}{2} + \frac{105}{16}g - \frac{67515}{32}g^2 + \frac{401548875}{128}g^3 - \frac{25424096867715}{2048}g^4 + O(g^5). \tag{23}$$

This series is strongly-divergent (stronger than x^6 and is a Gevrey-3 type) as the large-order asymptotic behavior of the coefficients takes the form:

$$c_n \sim \gamma(3n)!(-\sigma_h)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right), \tag{24}$$

where $\sigma_h = \frac{3375}{16\pi^6}\left(\Gamma\left(\frac{2}{3}\right)\right)^9$ [28]. From Eq. (23), one can deduce the expansion of $\frac{1}{E_0(g)}$ as:

$$\frac{1}{E_0(g)} = 2 - \frac{105}{8}g + \frac{281085}{128}g^2 - \frac{3270261225}{2048}g^3 + \frac{102452975769585}{32768}g^4 + O(g^5). \tag{25}$$

The approximants for the series in Eq. (23) are

$$\frac{1}{2}\ {}_pF_{p-1}(a_1, a_2, \ldots\ldots.a_p; b_1, b_2, \ldots.b_{p-1}; \omega g), \tag{26}$$

while for that in Eq. (25), it takes the form:

$$2\ {}_pF_{p-1}(a_1, a_2, \ldots\ldots.a_p; b_1, b_2, \ldots.b_{p-1}; \omega g). \tag{27}$$

The prediction of the non-perturbative parameters goes as:

$$b = \sum_{i=1}^{p} a_i - \sum_{i=1}^{p-1} b_i - (p - q + 2) + b_{p-3} + b_{p-2} + b_{p-1},$$

where b_{p-3}, $b_{p-2}\ b_{p-1}$are the denominator parameters with largest absolute values. Also, the large-order parameter σ is determined as $\sigma = \frac{\omega}{b_{p-3}\ b_{p-2}\ b_{p-1}}$ while the strong-coupling parameters A and s^* are predicted from Eq. (14).The results for $E_0(g)$ are listed in Table 4 and are compared to the exact results.

For the strong-coupling values for the 8*th* order, we have $s^* = 0.256960$ compared to 0.200000 exact value while $A_\beta = 1.205983$ with its exact prediction as 1.225820 [29]. Also for this order, we get $\sigma = -111.3352$ compared to the exact one as -90.68318 while $b = -2.28660$ which is not in good agreement with the exact result of $b = -\frac{1}{2}$ [12, 13]. The predictions are more accurate for higher orders. For instance, the 24*th* order gives $A_\beta = 1.21050$, $s^* = 0.206921$, $\sigma = -90.6848$ and $b = -0.500762$.

5 Eight-loop renormalization group functions of the $O(N)$-symmetricϕ^4 model

In the previous sections, we showed that the hypergeometric approximant:

$$_pF_{p-1}(a_1, a_2, \ldots a_p; b_1, b_2 \ldots .b_{p-1}; \omega g),$$

which is of explicit wrong large-order behavior, can approximate not only divergent Gevrey-1 type of series but also the strongly-divergent Gevrey-2 and Gevrey-3 series. Specially for the last two series, resummation techniques cannot give reliable results for relatively large couplings. Accordingly, these simple approximants should be taken seriously for the approximation of a divergent series. In this section, we need to test its results for more realistic cases. Here, we show how one can extract the critical exponents for the second-order phase transition of the $O(N)$-symmetric ϕ^4 field theory. Very recently, the eight-loop (g-expansion) for this model has been obtained [30] for the anomalous field dimension γ_ϕ. In fact, the seven-loop β-function and the eight-loop γ_ϕ are up g^8 order but cannot both lead to the 8th order in $\varepsilon = 4 - d$, where d is the dimension of the space–time. It is a well-known fact that working with ε-expansion leads to more accurate results than working with g-expansion [2], and thus, it would be more illustrative to test the accuracy of the g-expansion within the new 8th order.

The Lagrangian density of the $O(N)-$symmetric model is given by:

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4,$$

where $\Phi = (\phi_1, \phi_2, \phi_3, \ldots\ldots\ldots.\phi_N)$ is an N-component field with $O(N)$ symmetry such that $\Phi^4 = \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \ldots\ldots\ldots.\phi_N^2\right)^2$. The renormalization group function γ_ϕ is related to the η critical exponent as $\eta(g_c) = 2\gamma_\phi$ (g_c) where g_c is the critical coupling. The series for both $\gamma_\phi$and β renormalization group functions are divergent and have an asymptotic large-order behavior like the one discussed in the x^4 example. This means that both series can be approximated by the hypergeometric approximant:

$$_pF_{p-1}(a_1, a_2, \ldots a_p; b_1, b_2 \ldots .b_{p-1}; \omega g).$$

The critical coupling is predicted from the condition $\beta(g_c) = 0$ while the correction to scaling critical exponent is taken as $\omega = \beta'$ (g_c). The series obtained in Refs. [30–32] is within the minimal subtraction technique $\overline{MS}$. In the following, we list the results for the hypergeometric approximation of the $\beta-$ function from which we extract the critical coupling g_c and the correction to scaling critical exponent ω. Also, we list the results of the resummation of the field anomalous dimension γ_ϕ from which we deduce the critical exponent η. The hypergeometric approximation will be carried out for $N = 0, 1, 2, 3$.

5.1 Hypergeometric approximation for β and γ_ϕ functions for the $N = 0$ case

The $N = 0$ case lies in the same class of universality with some polymers [33]. In three dimensions ($\varepsilon = 1$), the seven-loop $\beta-$function is given by:

$$\beta(g) \approx -g + 2.6667g^2 - 4.6667g^3 + 25.457g^4 - 200.93g^5 + 2004.0g^6 - 23315g^7 + 303869g^8.$$

The suitable hypergeometric approximant for this order is

$$\beta(g) = {}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; g\sigma) - 1.$$

This approximant predicts the critical coupling value of $g_c = 0.52775$ compared to 0.5408(83) from Janke–Kleinert resummation algorithm [2] and the recent Borel with conformal mapping (BCM) (ε-expansion) prediction of 0.52988 ± 0.00225 [7]. This result leads to the correction to scaling critical exponent $\omega = \beta'(g_c) = 0.87328$ where it can be compared to 0.841(13) in Ref. [7], 0.904(5) [34] and our previous seven-loop $\varepsilon-$ expansion of 0.8484(17) [5].

For the critical exponent $\eta = 2\gamma_\phi(g_c)$, we have the eight-loop order for γ_ϕ given by

$$\gamma_\phi(g) = 0.05556g^2 - 0.03704g^3 + 0.19290g^4 - 1.0060g^5 + 7.0946g^6 - 57.739g^7 + 515.12g^8. \quad (28)$$

The series $\frac{\gamma_\phi(g)}{g^2}$ can be approximated by

$$\frac{\gamma_\phi(g)}{g^2} = 0.05556\ {}_3F_2(a_1, a_2, a_3; b_1, b_2; \sigma g),$$

from which we get the critical exponent $\eta = 2\gamma_\phi(g_c) = 0.0302696$. Our seven-loop ε-expansion in Ref. [5] gives the result 0.03121(70) while Monte Carlo simulation (MC) in Ref. [34] gives 0.031043(3) and conformal bootstrap (CB) predicts the result 0.0282(4) [35]. Our results are listed in Table 5 where we added predictions from other algorithms for comparison.

Table 5 The hypergeometric approximation for the critical coupling g_c and critical exponents η and ω for the self-avoiding walks model ($N = 0$) where predictions from other computational techniques are added for comparison

g_c	ω	η
$0.52775^{[This\ work]}$	$0.87328^{[This\ work]}$	$0.0302696^{[This\ work]}$
0.5408(83)[2]	0.8484(17)[5]	0.03121(70)[5]
0.52988 ± 0.00225[7]	0.846(15) [36]	0.031043(3) [37]
–	0.899(12) [34]	0.0282(4) [35]

Table 6 The hypergeometric approximation for the critical coupling g_c and critical exponents η and ω for Ising-like model ($N = 1$)

g_c	ω	η
$0.475262^{[This\ work]}$	$0.841012^{[This\ work]}$	$0.035597^{[This\ work]}$
0.4810(91) [2]	0.82311(50) [5]	0.03653(65) [5]
0.47033 ± 0.001 [7]	0.832(6) [38]	0.03627(10) [38]
–	0.8303(18) [39]	0.03631(3) [39]

5.2 The Ising-like model $N = 1$

The case $N = 1$ lies in the same class of universality with the Ising model. The seven-loop β-function is given by:

$$\beta \approx -g + 3g^2 - 5.66667g^3 + 32.5497g^4 - 271.606g^5 + 2848.57g^6 - 34776.1g^7 + 474651g^8. \tag{29}$$

As in the $N = 0$ case, it can be approximated by the hypergeometric approximant:

$$\beta(g) = {}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; g\sigma) - 1,$$

which leads to $g_c = 0.475262$. In Ref. [2], the Janke–Kleinert resummation algorithm gives the result $g_c = 0.4810(91)$ while a recent Borel with conformal mapping calculations gives 0.47033 ± 0.001 [7]. Our prediction for g_c also leads to the result $\omega = 0.841012$ which can be compared to the Monte Carlo simulation prediction of 0.832(6) [38] while conformal bootstrap calculations give the result 0.8303(18) [39].

The eight-loop series for the field anomalous dimension γ_ϕ is listed also in Ref. [30] as:

$$\gamma_\phi \approx 0.08333g^2 - 0.06250g^3 + 0.33854g^4 - 1.9256g^5 + 14.384g^6 - 124.16g^7 + 1171.88g^8. \tag{30}$$

This series can be approximated by:

$$\frac{\gamma_\phi(g)}{g^2} = 0.08333\ {}_3F_2(a_1, a_2, a_3; b_1, b_2; g\sigma),$$

from which we get η as

$$\begin{aligned}\eta &= 2g_c^2\ (0.08333 {}_3F_2(a_1, a_2, a_3; b_1, b_2; \sigma g_c)) \\ &= 0.035597.\end{aligned}$$

For comparison, we list the result for MC as $\eta = 0.03627(10)$ [38] while conformal bootstrap calculations give the result 0.03631(3) [39]. In Table 6, our predictions for g_c, ω, and η are listed for $N = 1$ and compared to the predictions from other computational tools.

5.3 Hypergeometric approximation for the case $N = 2$ (XY universality class)

This model can also describe the super-fluid ^4He phase transition. The seven-loop β-function is given by:

$$\beta(g) \approx -g + 3.3333g^2 - 6.6667g^3 + 39.948g^4 - 350.51g^5 + 3844.5g^6 - 48999g^7 + 696998g^8. \tag{31}$$

These data result in $g_c = 0.427969$ which can be compared to the result of Janke–Kleinert resummation algorithm which gives $g_c = 0.5032(239)$ [2] and Borel with conformal mapping result of 0.4209 ± 0.001 [7]. Based on our result for g_c, we obtained the result $\omega = 0.84161$ while MC gives the result 0.789(4) [40] and CB predicts the result $\omega = 0.811(10)$ [7, 41].

For the field anomalous dimension γ_ϕ, we have: [30]

$$\gamma_\phi \approx 0.11111g^2 - 0.09259g^3 + 0.50926g^4 - 3.1481g^5 + 24.706g^6 - 224.57g^7 + 2226.9g^8. \tag{32}$$

This gives the result $\eta = 0.036963$ while MC gives 0.03810(8) [40] and CB gives $\eta = 0.03852(64)$ [42]. We list our results in Table 7 and compare them to predictions from other algorithms.

Table 7 The hypergeometric approximation for the critical coupling g_c and critical exponents η and ω for the XY universality class ($N = 2$)

g_c	ω	η
$0.427969^{[This\ work]}$	$0.841606^{[This\ work]}$	$0.0369631^{[This\ work]}$
0.5032(239)[2]	0.789(13) [5]	0.03810(56) [5]
0.4209 ± 0.001 [7]	0.789(4) [40]	0.03810(8) [40]
–	0.811(10) [7, 41]	0.03852(64) [7, 41]

Table 8 The hypergeometric approximation for the critical coupling g_c and critical exponents η and ω for the Heisenberg universality class ($N = 3$)

g_c	ω	η
$0.39443^{[This\ work]}$	$0.78231^{[This\ work]}$	$0.03768^{[This\ work]}$
0.3895(71) [2]	0.764(18) [5]	0.03809(62) [5]
0.37936 ± 0.001 [7]	0.773 [43]	0.0378(3) [44]
–	0.791(22) [7, 41]	0.0386(12) [42]

6 Hypergeometric approximation for Heisenberg universality class $N = 3$

The seven-loop β-function for $N = 3$ is given by:

$$\beta(g) \approx -g + 3.66667g^2 - 7.66667g^3 + 47.6514g^4 - 437.646g^5 + 4998.62g^6 - 66242.7g^7 + 978330g^8. \tag{33}$$

It results in $g_c = 0.39443$ compared to $g_c = 0.3895(71)$ from Janke–Kleinert resummation algorithm [2] and Borel resummation of $g_c = 0.37936 \pm 0.001$ [7]. The corresponding ω exponent is predicted to be 0.78231 while the MC result is 0.773 [43] and the CB result of 0.791(22) [7, 41].

The eight-loop series for γ_ϕ is also given by:

$$\frac{\gamma_\phi}{g^2} = 0.138889 - 0.127315g + 0.699267g^2 - 4.68924g^3 + 38.4364g^4 - 365.900g^5 + 3792.05g^6, \tag{34}$$

which results in $\eta = 0.03768$. This result is very competitive to the prediction from more sophisticated techniques such as the MC result of $\eta = 0.0378(3)$ [44] and the CB prediction of 0.0386(12) [42]. More comparisons can be found in Table 8.

To value the above results, one should take into account the fact that the ε-series gives better results than the g-series. The above results have stressed the g-series, and thus, one can confidently say that the simple hypergeometric approximants are very competitive to more sophisticated algorithms like the BCM.

7 Summary and conclusions

The hypergeometric approximant ${}_2F_1$ has been used in Ref. [9] to approximate a divergent series with zero-radius of convergence with a large-order behavior of the form $n!\sigma^n n^b$ (Gevrey-1 type). What is strange about this approximant and its extension to generalized hypergeometric functions ${}_pF_{p-1}$ is that they do have a series expansion of finite radius of convergence while it has been used (successfully) to approximate a series with zero-radius of convergence. We explained why this is happening by realizing that the parameters in the hypergeometric approximant are taking values such that its asymptotic large-order behavior approaches the one for the given series. Accordingly, one can even predict the non-perturbative parameters from weak-coupling data as input to this simple approximant. In fact, especially for the large-order parameters, the extracted values are more and more accurate when feeding the approximants with larger size of input weak-coupling information. We applied this approximation to the divergent series of the x^4 anharmonic oscillator and were able to get accurate results for both the ground-state energy and the non-perturbative parameters, namely, the strong-coupling and the large-order ones.

Divergent series can take different shapes in their asymptotic large-order behavior. For instance, the ground-state energy of the x^6 anharmonic oscillator has a series expansion which reflects strong divergence in the sense that the coefficients at large orders behave like $c_n \sim (2n)!\sigma^n n^b$ (Gevrey-2 type). Also, the series for the ground-state energy of the x^8 anharmonic oscillator is a Gevrey-3 type with a large-order behavior as $c_n \sim (3n)!\sigma^n n^b$. These kinds of series are strongly divergent, and traditional resummation techniques cannot give reliable results especially for strong couplings. We showed that the hypergeometric approximants ${}_pF_{p-1}$ though of wrong large-order behavior can give accurate results for such kinds of series too.

Our experience in working with the approximants ${}_pF_{p-1}$ tells us that more accurate results are obtained for a relatively high orders (eight or greater). Very recently, Oliver Schnetz obtained the eight-loop (g-expansion) for the field anomalous dimension γ_ϕ of the $O(n)$-symmetric ϕ^4 scalar field theory [30]. Both the seven-loop β-function and the eight-loop γ_ϕ are up to $8th$ order in g. So we can resum them, and from the approximants, we can get approximate values for the critical coupling g_c and critical exponents ω and η. We have stressed the mentioned renormalization group functions and obtained these critical quantities for $N = 0,\ 1,\ 2,\ and 3$.

Our results using the simple hypergeometric approximants are very competitive to predictions from sophisticated resummation algorithms as well as numerical and conformal field predictions. Note that, within the minimal subtraction technique, it is well-known that resummation of the g-series is less accurate than the ε-series [2]. Up to the best of our knowledge, the new order of the g-series stressed in this work has not been treated yet using any resummation method, and we expect that such resummation techniques though are more sophisticated may at most be competitive to the simple hypergeometric approximants used in this work. Moreover, the 8*th* order of the ε-series cannot be obtained from the available seven-loop β-function and the eight-loop anomalous field dimension as one should have the eight-loop order for both.

Acknowledgements Open Access funding provided by the Qatar National Library.

Funding Open Access funding provided by the Qatar National Library.

Data Availability Statement No data associated in the manuscript.

References

1. F.J. Dyson, Divergence of perturbation theory in quantum electrodynamics. Phys. Rev. **85**, 631 (1952). https://doi.org/10.1103/PhysRev.85.631
2. H. Kleinert, V. Schulte-Frohlinde, Critical Properties of ϕ^4-Theories (publisher World Scientific, 2001) p. 311. https://doi.org/10.1142/4733
3. A.M. Shalaby, λ-point anomaly in view of the seven-loop hypergeometric resummation for the critical exponent ν of the O(2) ϕ^4 model. Phys. Rev. D **102**, 105017 (2020a). https://doi.org/10.1103/PhysRevD.102.105017
4. A.M. Shalaby, Critical exponents from the weak-coupling, strong-coupling and large-order parametrization of the hypergeometric ${}_{k+1}\mathrm{F}_k$ approximants. Ann. Phys. (N.Y.) **427**, 168404 (2021a). https://doi.org/10.1016/j.aop.2021.168404
5. A.M. Shalaby, Critical exponents of the O(N)-symmetric ϕ^4 model from the ε^7 hypergeometric-Meijer resummation. Eur. Phys. J. C **81**, 87 (2021b). https://doi.org/10.1140/epjc/s10052-021-08884-5
6. R. Guida, J. Zinn-Justin, Critical exponents of the N-vector model. J. Phys. A: Math. Gen. **31**, 8103 (1998). https://doi.org/10.1088/0305-4470/31/40/006
7. M.V. Kompaniets, E. Panzer, Minimally subtracted six-loop renormalization of $O(n)$-symmetric ϕ^4 theory and critical exponents. Phys. Rev. D **96**, 036016 (2017). https://doi.org/10.1103/PhysRevD.96.036016
8. L. Epele, H. Fanchiotti, C. Garcí́a Canal, M. Marucho, Generalized Borel transform technique in quantum mechanics. Phys. Lett. B **556**, 87 (2003). https://doi.org/10.1016/S0370-2693(03)00106-0
9. H. Mera, T.G. Pedersen, B.K. Nikolić, Nonperturbative quantum physics from low-order perturbation theory. Phys. Rev. Lett. **115**, 143001 (2015). https://doi.org/10.1103/PhysRevLett.115.143001
10. H. Mera, T.G. Pedersen, B.K. Nikolić, Fast summation of divergent series and resurgent transseries from Meijer-G approximants. Phys. Rev. D **97**, 105027 (2018). https://doi.org/10.1103/PhysRevD.97.105027
11. O. Costin, G.V. Dunne, Conformal and uniformizing maps in Borel analysis. Eur. Phys. J. Spec. Top. **230**, 2679 (2021). https://doi.org/10.1140/epjs/s11734-021-00267-x
12. E.J. Weniger, Construction of the strong coupling expansion for the ground state energy of the quartic, sextic, and octic anharmonic oscillator via a renormalized strong coupling expansion. Phys. Rev. Lett. **77**, 2859 (1996). https://doi.org/10.1103/PhysRevLett.77.2859
13. E.J. Weniger, A convergent renormalized strong coupling perturbation expansion for the ground state energy of the quartic, sextic, and octic anharmonic oscillator. Ann. Phys. (N.Y). **246**, 133 (1996b). https://doi.org/10.1006/aphy.1996.0023
14. I. Elkamash, H.M. Abdelhamid, A.M. Shalaby, Entire hypergeometric approximants for the ground state energy perturbation series of the quartic, sextic and octic anharmonic oscillators. Ann. Phys. (N. Y). **457**, 169427 (2023). https://doi.org/10.1016/j.aop.2023.169427
15. J.L. Basdevant, The Padé approximation and its physical applications. Fortschritte der Phys. **20**, 283 (1972). https://doi.org/10.1002/prop.19720200502
16. G.A. Baker, P. Graves-Morris, *Padé Approximants Second Edition* (Cambridge University Press, Cambridge, 1996). https://doi.org/10.1017/CBO9780511530074
17. I. Andrianov, A. Shatrov, Padé approximants, their properties, and applications to hydrodynamic problems. Symmetry (Basel). **13**, 1869 (2021). https://doi.org/10.3390/sym13101869
18. A.M. Shalaby, Weak-coupling, strong-coupling and large-order parametrization of the hypergeometric-Meijer approximants. Results Phys. **19**, 103376 (2020). https://doi.org/10.1016/j.rinp.2020.103376
19. A.M. Shalaby, Precise critical exponents of the $O(n)$-symmetric quantum field model using hypergeometric-Meijer resummation. Phys. Rev. D **101**, 105006 (2020). https://doi.org/10.1103/PhysRevD.101.105006
20. A.M. Shalaby, Universal large-order asymptotic behavior of the strong-coupling and high-temperature series expansions. Phys. Rev. D **105**, 045004 (2022). https://doi.org/10.1103/PhysRevD.105.045004. arXiv: 1911.03571
21. A.M. Shalaby, High-order parametrization of the hypergeometric-Meijer approximants, Ann. Phys. (N.Y.), 169376 (2023). https://doi.org/10.1016/j.aop.2023.169376. arXiv:2210.04575
22. A.M. Shalaby, Extrapolating the precision of the hypergeometric resummation to strong couplings with application to the $ix^3\mathcal{PT}$-symmetric symmetric $i\phi^3$ field theory. Int. J. Mod. Phys. A **35**, 2050041 (2020). https://doi.org/10.1142/S0217751X20500414
23. C.M. Bender, T.T. Wu, Anharmonic oscillator. Phys. Rev. **184**, 1231 (1969). https://doi.org/10.1103/PhysRev.184.1231
24. A.A. Kilbas, R.K. Saxena, M. Saigo, J.J. Trujillo, The generalized hypergeometric function as the Meijer G-function. Analysis **36**, 1 (2016). https://doi.org/10.1515/anly-2015-5001

25. L. Skála, J. Cízek, J. Zamastil, Strong coupling perturbation expansions for anharmonic oscillators. Numerical results. J. Phys. A: Math. Gen. **32**, 5715 (1999). https://doi.org/10.1088/0305-4470/32/30/314
26. I.A. Ivanov, Reconstruction of the exact ground-state energy of the quartic anharmonic oscillator from the coefficients of its divergent perturbation expansion. Phys. Rev. A **54**, 81 (1996). https://doi.org/10.1103/PhysRevA.54.81
27. A. Sergeev, Rayleigh-Schroedinger perturbation series for various quantum-mechanical problems. https://www.asergeev.com/rsptexp/x6.htm
28. C.M. Bender, T.T. Wu, Large-order behavior of perturbation theory. Phys. Rev. Lett. **27**, 461 (1971). https://doi.org/10.1103/PhysRevLett.27.461
29. F. Vinette, J. Čížek, Upper and lower bounds of the ground state energy of anharmonic oscillators using renormalized inner projection. J. Math. Phys. **32**, 3392 (1991). https://doi.org/10.1063/1.529452
30. O. Schnetz, Maple package HyperlogProcedrues, https://www.math.fau.de/person/oliver-schnetz/ (2023a)
31. O. Schnetz, Numbers and functions in quantum field theory. Phys Rev. D **97**, 085018 (2018). https://doi.org/10.1103/PhysRevD.97.085018. arXiv: 1606.08598
32. O. Schnetz, Theory at seven loops ϕ^4. Phys. Rev. D **107**, 036002 (2023b). https://doi.org/10.1103/PhysRevD.107.036002
33. P. de Gennes, Exponents for the excluded volume problem as derived by the Wilson method. Phys. Lett. A **38**, 339 (1972). https://doi.org/10.1016/0375-9601(72)90149-1
34. N. Clisby, B. Dünweg, High-precision estimate of the hydrodynamic radius for self-avoiding walks. Phys. Rev. E **94**, 052102 (2016). https://doi.org/10.1103/PhysRevE.94.052102
35. H. Shimada, S. Hikami, Fractal dimensions of self-avoiding walks and Ising high-temperature graphs in 3D conformal bootstrap. J. Stat. Phys. **165**, 1006 (2016). https://doi.org/10.1007/s10955-016-1658-x
36. M. Kompaniets, K.J. Wiese, Fractal dimension of critical curves in the $O(n)$-symmetric ϕ^4 model and crossover exponent at 6-loop order: Loop-erased random walks, self-avoiding walks, Ising, XY, and Heisenberg models. Phys. Rev. E **101**, 012104 (2020). https://doi.org/10.1103/PhysRevE.101.012104
37. N. Clisby, Scale-free Monte Carlo method for calculating the critical exponent γ of self-avoiding walks. J. Phys. A: Math. Theor. **50**, 264003 (2017). https://doi.org/10.1088/1751-8121/aa7231
38. M. Hasenbusch, Finite size scaling study of lattice models in the three-dimensional Ising universality class. Phys. Rev. B **82**, 174433 (2010). https://doi.org/10.1103/PhysRevB.82.174433
39. S. El-Showk, M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, A. Vichi, Solving the 3d ising model with the conformal bootstrap II. c-minimization and precise critical exponents. J. Stat. Phys. **157**, 869 (2014). https://doi.org/10.1007/s10955-014-1042-7
40. M. Hasenbusch, Monte Carlo study of an improved clock model in three dimensions. Phys. Rev. B **100**, 224517 (2019). https://doi.org/10.1103/PhysRevB.100.224517
41. A.C. Echeverri, B. von Harling, M. Serone, The effective bootstrap. J. High Energy Phys. **9**, 97 (2016). https://doi.org/10.1007/JHEP09(2016)097
42. F. Kos, D. Poland, D. Simmons-Duffin, A. Vichi, Precision islands in the Ising and O(N) models. J. High Energy Phys. **8**, 36 (2016). https://doi.org/10.1007/JHEP08(2016)036
43. M. Hasenbusch, Eliminating leading corrections to scaling in the three-dimensional $O(N)$-symmetric ϕ^4 model: $N = 3$ and 4. J. Phys. A: Math. Gen. **34**, 8221 (2001). https://doi.org/10.1088/0305-4470/34/40/302
44. M. Hasenbusch, E. Vicari, Anisotropic perturbations in three-dimensional $O(n)$-symmetric vector models. Phys. Rev. B **84**, 125136 (2011). https://doi.org/10.1103/PhysRevB.84.125136