Quantum field theory at the boundary

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Abstract

This thesis explores various aspects of boundary conformal field theories. They provide a continuum description of lattice systems with boundaries when both the bulk and the boundary are tuned to criticality. A given CFT may admit multiple conformally invariant boundary conditions which describe different boundary critical points of the same bulk system . In other words, there is plenty of room at the boundary. We explore this rich boundary behavior of several well known bulk conformal field theories.

After giving a brief introduction to the subject, we start in Chapter 2 by studying conformal boundary conditions in one of the simplest conformal field theory, namely a system of N free scalars interacting only through the boundary with an $O(N)$ invariant interaction. By a combination of large N and epsilon expansions, we provide evidence for the existence of non-trivial $O(N)$ BCFTs in $1 < d < 4$ where d is the dimension of the boundary. We then point out that these models are closely related to long range $O(N)$ models which describe lattice systems with long range interactions. We continue the study of long range $O(N)$ models in Chapter 3, where we study the spectrum of heavy operators. To be specific, we consider operators with a charge j under $O(N)$ symmetry and study their conformal dimensions in the limit of large j and N with j/N held fixed.

Chapter 4 is devoted to boundary critical behavior in interacting $O(N)$ vector model. To study it, we use the idea that a boundary conformal field theory is Weyl equivalent to a CFT in anti-de Sitter (AdS) space. We recover the known boundary fixed points for $O(N)$ vector models and study these fixed points in an large N expansion in general bulk dimension d as well as in an epsilon expansion near $d = 2, 4$ and 6. Then in Chapter 5, we use similar techniques to identify new boundary fixed points in the Gross-Neveu (GN) model. We verify the conjectured boundary F-theorem and compute several pieces of BCFT data at these boundary fixed points.

This thesis is based on [1, 2] with Simone Giombi and [3, 4] with Simone Giombi and Elizabeth Helfenberger.

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To my parents and my sister, for always being on my side.

Contents

Chapter 1

Introduction

Quantum field theory (QFT) is the language of modern theoretical physics and can be considered one of the greatest developments of 20th century physics. As like all of physics, symmetries are a powerful tool that help us analyze and organize quantum field theory efficiently. These symmetries may act on the spacetime that the theory is defined on, or internally on the degrees of freedom on this spacetime. All conventional quantum field theories at least have a spacetime Poincaré symmetry, which consists of translations in space and time, and rotations of spacetime. Since the seminal work of Wilson [5] we think of a quantum field theory as being defined at a scale. This scale may be the lattice spacing, while thinking about condensed matter systems, or it may be the momentum cutoff when thinking about particle physics. The quantum field theories at different scales are related by renormalization group (RG) transformations.

In the vast landscape of quantum field theories, conformal field theories (CFT) are special points with enhanced symmetry. They are fixed points of the RG flow, and hence are scale invariant. In addition to Poincaré and scaling symmetry, they are also invariant under special conformal transformations. These symmetries combine to form a group $SO(1, d+1)$ in d Euclidean dimensions. These additional symmetries make CFTs ideal for doing calculations and checking our ideas about quantum field theory, but they are not just a calculational tool. They also have a number of practical applications from providing a description of critical systems near a second order phase transition, to describing quantum gravity in Anti-de Sitter (AdS) space via AdS/CFT correspondence.

Let us now briefly explain what we mean by scale invariance in the context of lattice system by a simple example (see for instance [6]). Consider a ferromagnetic spin system with the following

Hamiltonian

$$H = -J \sum_{<ij>} \mathbf{s_i} \cdot \mathbf{s_j} \tag{1.1}$$

where the sum runs over nearest neighbors. Let us further restrict to the case of the Ising model, such that $\mathbf{s_i}$ may only take values ± 1. Then at very low temperatures, the spins are all aligned either up $(+1)$ or down (-1). At very high temperatures, the spins are completely random. At intermediate temperatures, there are domains within which all the spins point in a given direction. The average size of domains is called the correlation length, ξ. At the critical temperature, T_c, the correlation length diverges, and there are domains of all possible sizes. At this temperature, there is an emergent scale invariance in the system. So if we zoom out, say by summing over every alternate spin, the resulting effective coupling of the new system is the same as the original coupling. In figure 1.1 taken from [6], we show a simulation of Ising model at the critical temperature. The black represents spin up, while the white represents spin down. The bottom picture is magnified by a factor of two, but they look essentially the same because the system demonstrates scale invariance at the critical temperature. The essence of scale invariance is that just by looking at the picture, we cannot really tell which one is the magnified picture.

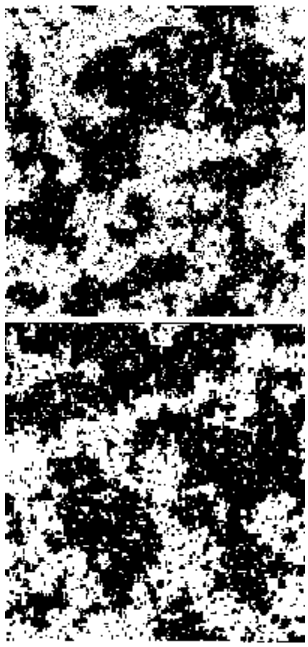


Figure 1.1: Simulation of Ising model at critical temperature. The black represents spin up, while the white represents spin down. The bottom picture is magnified by a factor of two. This figure was taken from [6].

In this thesis, we will only be dealing with the continuum limit where the symmetries of the lattice

become continuous symmetries on spacetime. The translations and the Lorentz transformations part of the conformal transformations acts on the spacetime in the usual way. In the Euclidean signature, the Lorentz transformations are just rotations

$$\text{Translations}: \quad x^\mu \to x^\mu + a^\mu, \qquad \text{Rotations}: \quad x \to R \cdot x. \tag{1.2}$$

The scale transformations and the special conformal transformations (SCT) act in the following way

$$\text{Scale}: \quad x \to \sigma x, \qquad \text{SCT}: \quad x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}. \tag{1.3}$$

The CFT is then defined by the spectrum of operators and OPE (operator product expansion) coefficients between them. For local operators, the OPE takes the following form

$$O_1(x_1)O_2(x_2) \sim \sum_i C_{12}^i(x_{12})O_i(x_2). \tag{1.4}$$

The function $C_{12}^i(x_{12})$ is completely fixed by the conformal symmetry, up to a number, that we call the OPE coefficient. The correlation functions are also constrained by the conformal symmetry (see [7] for a review). The one-point function of all local operators vanishes by translational invariance. Moreover, the local operators are organized into a set of primary operators, and descendants which may be obtained by acting on the primaries with derivatives. For the primary operators, two and three point functions are completely fixed by symmetry (we only write the result for scalar operators, but similar considerations apply for spinning operators)

$$\begin{aligned} \langle O_1(x_1)O_2(x_2)\rangle &= \frac{\delta_{\Delta_1,\Delta_2}}{|x_{12}|^{2\Delta_1}}, \\ \langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle &= \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}. \end{aligned} \tag{1.5}$$

When we have four-points, it is possible to construct two cross-ratios, which are following combinations of positions invariant under conformal transformations

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{1.6}$$

So the four-point function, in addition to the kinematic dependence, is a function of these cross-ratios. We come back to the four-point function and how it may be expanded into conformal blocks, in the next chapter.

So far, we talked about systems which are infinite in spacetime, so the symmetries can be considered exact. This is a good approximation as long as we are far from the edge of the system. But realistic systems are finite, and the situation needs to be modified when we want to describe the system near the boundary. The situation that preserves the most symmetry is when we have conformal symmetry on the boundary. The resulting theory has $SO(1,d)$ symmetry and is called a boundary conformal field theory (BCFT). Conformal field theories with a boundary have been studied for a long time [8, 9, 10, 11, 12] and have a variety of physical applications (for a recent review, see [13]). A renewed interest in the subject has also taken place in light of the progress in conformal bootstrap methods [14, 15, 16, 17, 18]. Recently, boundary conformal field theories have also been proposed to play a role as holographic duals of certain single sided black hole microstates [19, 20]. In this thesis, we add to this growing literature, by finding new examples of interacting BCFTs and developing new techniques to analyze them. But let us start by reviewing some basic facts about BCFT (see [21, 22] for more detailed discussions) which will be useful in the later chapters.

1.1 BCFT basics

For simplicity, let us discuss the case of flat Euclidean space in the presence of a flat boundary (The discussion may be generalized to conformally equivalent cases, for example, a hemisphere.). Let the coordinates be $x = (\mathbf{x}, y)$ where $y \geq 0$ with the boundary located at $y = 0$. So clearly the translational symmetry in the y direction is broken. In a BCFT, we still have the following leftover symmetries

$$\begin{aligned} &\text{Translations}: \quad (\mathbf{x},y) \to (\mathbf{x}+\mathbf{a},y), \qquad \text{Rotations}: \quad (\mathbf{x},y) \to (\mathbf{R}\cdot\mathbf{x},y) \\ &\text{Scale}: \quad (\mathbf{x},y) \to (\sigma\mathbf{x},\sigma y), \qquad \text{SCT}: \quad (\mathbf{x},y) \to \left(\frac{\mathbf{x}+\mathbf{b}x^2}{1+2\mathbf{b}\cdot\mathbf{x}+\mathbf{b}^2x^2}, \frac{y}{1+2\mathbf{b}\cdot\mathbf{x}+\mathbf{b}^2x^2}\right). \end{aligned} \tag{1.7}$$

Notice that if we restrict to the boundary, i.e. $y = 0$, we recover the $d-1$ dimensional conformal group on the boundary. So all correlation functions of operators localized on the boundary are the same as the usual CFT. But we also have operators in the bulk. The OPE between two bulk operators far from the defect also remains unaffected by the presence of the boundary. However, when a bulk operator is close to the boundary, it may be expanded in to a sum of boundary operators,

known as boundary operator expansion

$$O(\mathbf{x}, y) \sim \sum_i \frac{b_O^{\hat{O}_i}}{y^{\Delta - \hat{\Delta}_i}} \hat{O}_i. \tag{1.8}$$

So in addition to the usual CFT data, the observables in a BCFT also include spectrum of boundary operators and the boundary expansion coefficients of bulk operators[1].

Now the bulk local operators may get one-point function, which just depends on the distance of the operator from the boundary

$$\langle O(\mathbf{x}, y) \rangle = \frac{a_O}{y^\Delta}. \tag{1.9}$$

The bulk-defect two-point function is also fixed

$$\langle O(\mathbf{x}_1, y_1) \hat{O}(\mathbf{x}_2) \rangle = \frac{b_{O\hat{O}}}{y_1^{\Delta - \hat{\Delta}} (y_1^2 + \mathbf{x}_{12}^2)^{\hat{\Delta}}}. \tag{1.10}$$

However, with two bulk points, it is possible to form an invariant cross-ratio

$$\xi = \frac{(x_1 - x_2)^2}{4 y_1 y_2}. \tag{1.11}$$

It is easy to check that this is invariant under all the transformations written in (1.7). The bulk two-point function may then be a function of this cross-ratio

$$\langle O_1(\mathbf{x}_1, y_1) O_2(\mathbf{x}_2, y_2) \rangle = \frac{G(\xi)}{(y_1)^{\Delta_1} (y_2)^{\Delta_2}}. \tag{1.12}$$

Notice that the two-point function may be nonzero even when the operators are not identical. Recall that we can expand this correlator in two different ways: we can do the usual OPE between the two bulk operators and expand them in to bulk operators as in (1.4) which gives the bulk channel expansion of the correlator, or we can expand both the operators in to boundary operators as in (1.8) which gives the boundary channel expansion of the correlator. The two expansions must be equal, and this is the essence of boundary bootstrap [14]. We give more details on conformal blocks in both the channels in the next chapter, when we talk about specific examples of the bulk two-point function.

Another feature of BCFT we mention before ending this section is the displacement operator, which is present in the boundary spectrum of every BCFT, and is the operator that may be used to

[1] There are also additional conformal anomalies in the presence of boundaries, and the anomaly coefficients also constitute CFT data. But we will only mention them briefly, and it is an active subject of research to find relations between these anomaly coefficients.

change the shape of the boundary. Since there is no translational invariance perpendicular to the boundary, the stress tensor is not conserved in that direction

$$\partial_i T^{iy} = D(\mathbf{x})\delta(y). \tag{1.13}$$

This equation defines the displacement operator $D(\mathbf{x})$ and also fixes its scaling dimension to be d. Since the displacement is protected, its two-point function always takes the following form in any CFT

$$\langle D(\mathbf{x}_1)D(\mathbf{x}_2)\rangle = \frac{C_D}{|\mathbf{x}_{12}|^{2d}}. \tag{1.14}$$

The coefficient C_D is also a piece of BCFT data.

1.2 Overview of the thesis

The rest of this thesis is organized as follows: In Chapter 2, which is based on [1] with Simone Giombi, we study a system of N scalars free in the bulk, and interacting at the boundary with an $O(N)$ invariant interaction. We find non trivial fixed points for the boundary interaction in boundary dimensions $1 < d < 4$. Due to having free fields in the bulk, these models possess bulk higher-spin currents which are conserved up to terms localized on the boundary. We suggest that this should lead to a set of protected spinning operators on the boundary, and give evidence that their anomalous dimensions vanish. We also discuss the closely related long range $O(N)$ models which provide continuum description of long range spin systems. Long range spin systems have a Hamiltonian that contains all to all interaction decaying as a power of distance between the spins, rather than the usual nearest neighbor interaction we talked about above

$$H = -J\sum_{i,j}\frac{\mathbf{s}_i\cdot\mathbf{s}_j}{|i-j|^{d+s}} \tag{1.15}$$

with s being a continuous parameter. In the continuum limit, the long range models may be described as defects in a higher dimensional local free field theory, with interactions localized on the defect. It is in this sense that they are related to free field theories with boundary localized interaction.

We further study these long range models in Chapter 3 (based on [3] with Simone Giombi and Elizabeth Helfenberger) where we focus on operators having a large charge under $O(N)$ symmetry. It has been observed time and again that the dynamics of quantum field theories simplify in the limit of large quantum numbers. Notable examples include large spin expansion in CFT [23, 24, 25] or

an expansion in large number of fields [26, 27]. This motivates us to study large charge sector in the long range models. In particular, we study large N long range $O(N)$ models and focus on operators with charge j such that $\hat{j} = j/N$ is finite. The usual $1/N$ perturbation theory breaks down in this regime. We identify a new semiclassical saddle point which captures correlation functions involving such operators. We find that the scaling dimensions for general s interpolate between $\Delta_j \sim \frac{(d-s)}{2} j$ at small $\hat{j}$ and $\Delta_j \sim \frac{(d+s)}{2} j$ at large $\hat{j}$, which is a qualitatively different behavior from the one found in the short range version of the $O(N)$ model. We also derive results for the structure constants and 4-point functions with two large charge and one or two finite charge operators.

Chapter 4 of the thesis is based on [2] with Simone Giombi. In this chapter, we move on to studying boundary critical behavior of interacting bulk CFTs. Using the fact that flat space with a boundary is related by a Weyl transformation to anti-de Sitter (AdS) space, we recover the various known boundary critical behaviors of the critical $O(N)$ vector model. We also discuss the free energy of the CFT computed on the AdS space with hyperbolic ball metric, i.e. with a spherical boundary, which we propose should decrease under boundary renormalization group flows. We calculate this quantity for various boundary fixed points in the $O(N)$ model and find results which are consistent with the conjectured F-theorem in a continuous range of dimensions.

In Chapter 5 (based on [4] with Simone Giombi and Elizabeth Helfenberger), we extend these ideas to theories with fermions. After reviewing some aspects of free fermion theories in AdS, we use both large N methods and the epsilon expansion near 2 and 4 dimensions to study the conformal boundary conditions in the Gross-Neveu CFT. At large N and general dimension d, we find three distinct boundary conformal phases. Near four dimensions, where the CFT is described by the Wilson-Fisher fixed point of the Gross-Neveu-Yukawa model, two of these phases correspond respectively to the choice of Neumann or Dirichlet boundary condition on the scalar field, while the third one corresponds to the case where the bulk scalar field acquires a classical expectation value. One may flow between these boundary critical points by suitable relevant boundary deformations. We again compute the AdS free energy on each of them, and verify that its value is consistent with the boundary version of the F-theorem. We also compute some of the BCFT observables in these theories, including bulk two-point functions of scalar and fermions, and four-point functions of boundary fermions.

Chapter 2

$O(N)$ Models with Boundary Interactions and their Long Range Generalizations

In this chapter, we study a special type of boundary conformal field theory (BCFT) which is obtained by taking free fields in a $(d+1)$-dimensional bulk and adding interactions localized on a d-dimensional boundary. Part of the work in this chapter was presented during the poster session for "Boundaries and Defects in Quantum Field Theory" held at Perimeter Institute in August 2019. Free field theories with localized boundary interactions have been considered before in several different contexts including applications to dissipative quantum mechanics, open string theory and edge states in quantum hall effect [28, 29, 30, 31, 32, 33]. More recently, several examples of BCFT with non-interacting bulk fields were considered in [34, 35]. A particularly interesting model, with possible applications to graphene, is obtained by taking a free Maxwell field in four dimensions coupled to fermions localized on a three-dimensional boundary (or "brane") [36, 37, 38, 39, 40, 41, 42, 43, 44, 34, 35, 45].

Here we focus on the case of scalar field theory with $O(N)$ invariant boundary interactions. In particular, we investigate the critical properties of the model defined by N real scalar fields ϕ^I with the standard quartic interaction restricted to the boundary

$$S = \int d^{d+1}x \frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I + \int d^d x \frac{g}{4}(\phi^I \phi^I)^2. \tag{2.1}$$

With (generalized) Neumann boundary conditions $\partial_n \phi \sim g\phi^3$, the quartic interaction is marginal in $d = 2$ and relevant in $d < 2$, and hence one may have a non-trivial IR fixed point. As we show below, working in the framework of the ϵ-expansion one indeed finds a weakly coupled Wilson-Fisher fixed point in $d = 2 - \epsilon$, with real and positive coupling constant (here and below, we shall always assume that relevant quadratic terms have been tuned to criticality). This model was analyzed before in [46, 47] with an additional ϕ^6 coupling in the bulk. Here we will not turn on this bulk coupling. As in the well-known case of the standard critical $O(N)$ models, one may also develop a large N expansion for any d by introducing a Hubbard-Stratonovich field, which in the present case is localized on the d-dimensional boundary. This yields a large N BCFT which appears to be unitary in $1/N$ perturbation theory in the range $1 < d < 4$. We perform explicit calculations of various physical quantities in this BCFT, and show that the large N expansion precisely matches onto the ϵ-expansion in the quartic model in $d = 2 - \epsilon$. On the other hand, in $d = 1 + \epsilon$ we show that it matches onto the UV fixed point of a non-local non-linear $O(N)$ sigma model with the sphere constraint localized on the boundary. The action of this sigma model is given by

$$S = \int d^{d+1}x \frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I + \int d^d x \; \sigma(\phi^I \phi^I - \frac{1}{t^2}) \,, \tag{2.2}$$

where t is the boundary coupling constant for which we compute the beta function to order t^5. The large N expansion can be formally continued above the upper critical dimension $d = 2$, where it remains perturbatively unitary for $d < 4$. In $d = 4 - \epsilon$, we provide strong evidence that the large N expansion matches onto the IR fixed point of a metastable (for sufficiently large N and small ϵ) mixed "$\sigma\phi$" theory

$$S = \int d^{d+1}x \frac{1}{2}(\partial_\mu \phi^I)^2 + \int d^d x \left(\frac{1}{2}(\partial\sigma)^2 + \frac{g_1}{2}\sigma\phi^I\phi^I + \frac{g_2}{4!}\sigma^4 \right). \tag{2.3}$$

The instability arises because at the fixed point the quartic self-interaction of the σ field is negative, as we will show below by explicitly computing the beta functions of the model. Correspondingly, one finds real instanton solutions localized on the boundary, which are expected to produce imaginary parts in the scaling dimensions of boundary operators and other observables, as is well-known for the standard ϕ^4 theory with negative coupling. A summary of the various descriptions of the boundary $O(N)$ BCFTs in $1 < d < 4$ is given in Figure 2.1. The picture we find is a close analogue of the one found for the standard critical $O(N)$ models as a function of d. The large N expansion in those models can be developed for any d and it is perturbatively unitary in $2 < d < 6$. It matches onto the

UV fixed points of the non-linear sigma model near $d = 2$, and onto the Wilson-Fisher fixed point of the ϕ^4 theory near $d = 4$. As one approaches $d = 6$, one finds instead a cubic $O(N)$ symmetric theory [48, 49] that has perturbative fixed points in $d = 6 - \epsilon$; non-perturbatively, these are unstable due to instanton effects, which produce small imaginary parts of physical observables [50].

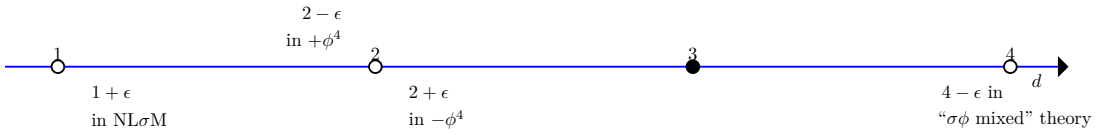


Figure 2.1: $O(N)$ BCFT in $1 < d < 4$

The fact that the BCFTs we study contain fields which are non-interacting in the bulk has interesting consequences. In particular, it implies that the boundary operator spectrum has several operators with protected scaling dimensions, as we elaborate on in Section 2.1. The simplest protected boundary operator is just the one induced by the free bulk field ϕ^I, and has protected dimension $\Delta = (d-1)/2$. While our prime example in this chapter are the scalar $O(N)$ models, similar properties are expected to hold in other similar models with free fields in the bulk.

Recall that a flat boundary in $d+1$ Euclidean dimensions breaks the conformal symmetry from $SO(d+2,1)$ to $SO(d+1,1)$, which is the conformal group on the d dimensional boundary. In particular, translational invariance perpendicular to the boundary is broken, which results in a delta-function localized source for the divergence of stress-tensor

$$\partial_\mu T^{\mu y} = D(\mathbf{x})\delta(y). \tag{2.4}$$

In most of the chapter we assume flat space with a flat boundary, and we will use $\mathbf{x}$ for the d coordinates on the boundary and y for the transverse direction with $x^\mu = (\mathbf{x}, y)$. The above equation is to be understood as an operator equation and it defines the displacement operator denoted by $D(\mathbf{x})$. This relation also fixes the dimension of displacement operator to be same as that of stress tensor, $\Delta = d+1$. Since the stress tensor is conserved in the bulk, the displacement operator remains protected even in the presence of interactions and its scaling dimension is not renormalized. This holds in any BCFT. If the bulk theory is free, as in the models we study in this chapter, then we also have a set of higher spin currents (see e.g. [51] for a review) which in the scalar field theory take the schematic form

$$J^{\mu_1\mu_2\ldots\mu_s} = \sum_{k=0}^{s} c_{sk}\partial_{\{\mu_1\ldots\mu_k}\phi\partial_{\mu_{k+1}\ldots\mu_s\}}\phi. \tag{2.5}$$

If the bulk fields are free, the divergence of these currents vanishes in the bulk. Then, as we explain in section 2.1.1 below, one expects an equation similar to (2.4) with a delta-function localized source, defining a set of spinning operators on the boundary with spin ranging from 0 to $s-2$, which we call higher spin displacement operators.[1] Since the higher-spin currents are conserved in the bulk, we expect that the scaling dimensions of these higher-spin displacement operators should be non-renormalized, despite the presence of interactions at the boundary. We obtain several perturbative checks of this expectation in Section 2.3. It would be nice to further study the consequences of having such protected operators in the spectrum, and also study the analogous operators in other examples of BCFT with free fields in the bulk.

In light of our $O(N)$ BCFT results, it would be interesting to extend the higher-spin versions of AdS/CFT (see [53, 51] for reviews) to the case of AdS/BCFT [54]. Type A Vasiliev theory in AdS_{d+1} space [55, 56, 57] is conjectured to be dual to a d dimensional $O(N)$ model, free or interacting depending on the boundary conditions of a bulk scalar field [58]. Similarly, the $O(N)$ BCFT we study should be dual to Vasiliev theory on hAdS_{d+1}, where we have half of AdS_{d+1} space ending on a AdS_d brane as shown in figure 2.2. In such a setup, boundary conditions of AdS_{d+1} fields on the AdS_d brane should be determined by the boundary conditions of $O(N)$ BCFT, while as usual, the boundary condition on the asymptotic AdS_{d+1} boundary will be determined by whether the $O(N)$ model is free or interacting in the bulk of the BCFT (in this chapter, we turn off interactions in the bulk, but one could more generally allow for a bulk coupling constant in addition to the boundary one, and study the RG flow of both couplings).

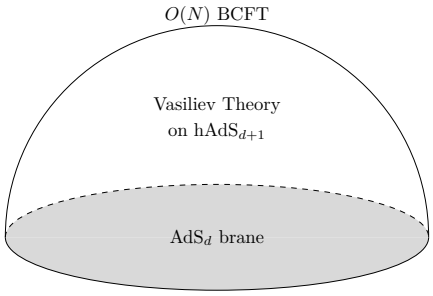


Figure 2.2: AdS/ BCFT setup for $O(N)$ BCFT

From the point of view of perturbative calculations of purely boundary observables in the models we study, one essentially computes boundary Feynman diagrams where the scalar fields has a $1/|\mathbf{p}|$ propagator, which is induced by the free kinetic term in the bulk (recall that we focus on Neumann boundary conditions). This may be thought of as a particular kind of non-local scalar field theory in d

[1] These operators were also considered in the context of replica twist defect in [52] but they are not protected in that case.

dimensions. A natural generalization is to consider more general non-local propagator parametrized by an arbitrary power s, with a propagator $1/|\mathbf{p}|^s$ in momentum space. This corresponds to a non-local kinetic term proportional to

$$\int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}}. \tag{2.6}$$

as can be checked by a Fourier transform to momentum space. Adding $O(N)$ invariant quartic interactions to such a non-local model, one finds fixed points which are expected to describe second order phase transition in a system of N-component unit spins interacting with a long range Hamiltonian

$$H = -J \sum_{i,j} \frac{\mathbf{s}_i \cdot \mathbf{s}_j}{|i-j|^{d+s}}. \tag{2.7}$$

Critical exponents for the long range interactions fall in three categories [59, 60, 61, 62, 63, 64, 65, 66, 67, 68] : 1) For $s < d/2$, critical exponents are the same as the ones for Gaussian fixed point, 2) for $d/2 < s < s_*$ there is a non trivial long range fixed point and critical exponents can be calculated and 3) for $s > s_*$, the critical exponents take the same value as the corresponding short range fixed point. The value of s_* is such that the conformal dimension of ϕ is continuous at the long range to short range crossover. In the long range fixed point, ϕ has no anomalous dimension and its scaling dimension is fixed to be $(d-s)/2$ (an argument for this is that ϕ can be formally thought of as a free field satisfying Laplace equation in a higher dimensional bulk, where $p = 2 - s$ is the co-dimension). On the other hand, at the short range fixed point, ϕ has an anomalous dimension and its scaling dimension is $\Delta_{SR} = (d - 2 + 2\gamma_\phi^{SR})/2$. This fixes $s_* = 2 - 2\gamma_\phi^{SR}$.

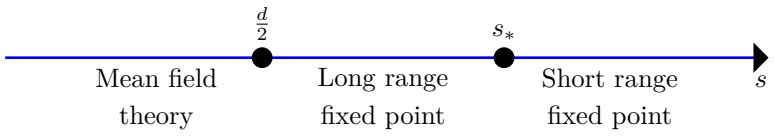


Figure 2.3: Continuum models for various values of s.

The crossover from mean field theory to long-range fixed point is relatively under control and perturbation theory can be developed since the usual ϕ^4 interaction is weakly coupled. An alternative scaling theory was proposed in [67, 68], which is weakly coupled near short range to long range crossover and can be used to do perturbation theory. However, in $d = 1$, there is no short range fixed point, since there is no phase transition in $d = 1$ $O(N)$ model, except at zero temperature. At zero temperature, all correlation functions are constant, and hence the anomalous dimension of

ϕ is commonly assigned an exact value $\gamma_\phi^{SR} = 1/2$ which makes $\Delta_\phi^{SR} = 0$ and $s_* = 1$. In the long range model, there is a phase transition for $0 < s < 1$ as was shown by Dyson in [69] and further studied in [70, 71, 72, 73]. So $s = 1$ is the upper critical value for the long range universality class in $d = 1$, which is what we would have naively expected by extrapolating the crossover region from higher dimensions. Hence for $d = 1$, the picture in figure 2.3 is modified to figure 2.4. Below we will study a non-local non-linear sigma model which becomes weakly coupled in $s = d - \epsilon$ for all d, and is a natural generalization of the boundary model (2.2). Precisely in $d = 1$, it is weakly coupled near the upper critical value of s for the long range model, and is well suited to do perturbation theory in the vicinity of $s = 1$. Unlike the usual local non-linear sigma model, the β function for this model is proportional to $N - 1$ instead of $N - 2$, hence the description is only valid for $N > 1$. This is in agreement with what was found long ago in [70]. Combining results from non-linear sigma model and the quartic model, we give some Padé estimates for critical exponents in the $d = 1$ long range $O(N)$ model. They are in good agreement with the Monte Carlo results of [65] for the values of s given there. It would be interesting to bootstrap this model using techniques similar to the one used for $d = 3$ long range Ising in [74], and compare the results with our estimates.

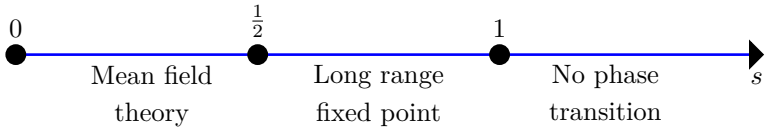


Figure 2.4: Continuum picture for one dimensional $O(N)$ model for various s.

This Chapter is organized as follows: In Section 2.1, we discuss some general aspects of free field theories with interactions localized on the boundary. In Section 2.2, we introduce the boundary $O(N)$ models in $1 < d < 4$ and its various descriptions as a function of dimension, and present various calculations of physical quantities at the fixed points. We explicitly construct a set of spinning operators induced on the boundary by bulk higher spin currents and provide evidence for the vanishing of their anomalous dimension in section 2.3. We end by describing long range generalizations of our models and give some estimates for $d = 1$ long range $O(N)$ model in section 2.4. Appendices contain some other interesting examples of BCFT with free fields in the bulk and some technical details.

2.1 Free fields with boundary interactions: some general remarks

The models we consider in this chapter have an action of the following general form

$$S = \int d^{d+1}x \ \mathcal{L}_{free} + \int d^d x \ \mathcal{L}_{int}. \tag{2.8}$$

To be concrete, let us consider the case of scalar fields, so that $\mathcal{L}_{free} = (\partial_\mu \phi)^2/2$, but most of what we discuss below should have a generalization to the case of other fields. The usual variational principle gives the equation of motion $\partial_\mu \partial^\mu \phi = 0$, and we have to satisfy either Dirichlet or generalized Neumann boundary condition

$$\phi(\mathbf{x}, 0) = 0, \quad \text{or} \quad \partial_y \phi(\mathbf{x}, 0) - \frac{\delta \mathcal{L}_{int}}{\delta \phi} = 0. \tag{2.9}$$

We will be focusing on generalized Neumann in this chapter, which allows for the possibility of interesting critical behavior for the boundary $O(N)$ models in $1 < d < 4$.

In a CFT with a boundary, in addition to the usual bulk OPE, we also have the boundary OPE where we expand the bulk field ϕ into a set of boundary primary operators

$$\phi(\mathbf{x}, y) = \sum_{\hat{O}} \frac{B_\phi^{\hat{O}}}{(2y)^{\Delta - \hat{\Delta}}} D^{\hat{\Delta}}(y^2 \vec{\partial}^2) \hat{O}(\mathbf{x}) \tag{2.10}$$

The differential operator $D^{\hat{\Delta}}(y^2\vec{\partial}^2)$ can be fixed using conformal invariance as we now review [12]. We know by conformal invariance that

$$\langle \phi(\mathbf{x}, y) \hat{O}(\mathbf{x}') \rangle = \frac{B_{\phi\hat{O}}}{(2y)^{\Delta - \hat{\Delta}} ((\mathbf{x} - \mathbf{x}')^2 + y^2)^{\hat{\Delta}}}, \quad \langle \hat{O}(\mathbf{x}) \hat{O}(\mathbf{x}') \rangle = \frac{C_{\hat{O}}}{(\mathbf{x} - \mathbf{x}')^{2\hat{\Delta}}}. \tag{2.11}$$

Using $B_{\phi\hat{O}} = C_{\hat{O}} B_\phi^{\hat{O}}$, this is satisfied if (here and elsewhere the symbol $(x)_m$ refers to the Pochhammer symbol and is defined by $(x)_m = \Gamma(x+m)/\Gamma(x)$)

$$D^{\hat{\Delta}}(y^2\vec{\partial}^2) \frac{1}{(\mathbf{x} - \mathbf{x}')^{2\hat{\Delta}}} = \frac{1}{((\mathbf{x} - \mathbf{x}')^2 + y^2)^{\hat{\Delta}}} = \sum_{m=0}^{\infty} \frac{(\hat{\Delta})_m}{m!} \frac{(-y^2)^m}{(\mathbf{x} - \mathbf{x}')^{2\hat{\Delta} + 2m}} \tag{2.12}$$

which implies

$$D^{\hat{\Delta}}(y^2\vec{\partial}^2) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{(\hat{\Delta} + 1 - \frac{d}{2})_m} \left(-\frac{1}{4} y^2 \vec{\partial}^2 \right)^m \tag{2.13}$$

Applying the bulk equation of motion $\partial_\mu \partial^\mu \phi = 0$ to this OPE, one finds

$$\begin{aligned}\partial_\mu \partial^\mu \phi &= \sum_{\hat{O}} \frac{B_\phi^{\hat{O}}}{(2y)^{\Delta-\hat{\Delta}}} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{(\hat{\Delta}+1-\frac{d}{2})_m} \Big((-\frac{1}{4}y^2)^m (\vec{\partial}^2)^{m+1} \hat{O}(\mathbf{x}) \\ &+ (2m-\Delta+\hat{\Delta})(2m-1-\Delta+\hat{\Delta})(-\frac{1}{4}\vec{\partial}^2)^m (y^2)^{m-1} \hat{O}(\mathbf{x}) \Big) \\ &= \sum_{\hat{O}} \frac{B_\phi^{\hat{O}}}{(2y)^{\Delta-\hat{\Delta}}} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{(\hat{\Delta}+1-\frac{d}{2})_m} \Big(1 - \frac{(2m+2-\Delta+\hat{\Delta})(2m+1-\Delta+\hat{\Delta})}{4(m+1)(m+1+\hat{\Delta}-\frac{d}{2})} \Big) \\ &\times \left(-\frac{1}{4} y^2 \vec{\partial}^2 \right)^m \hat{O}(\mathbf{x}).\end{aligned} \tag{2.14}$$

The only allowed operators will be the ones for which the above coefficient vanishes for all integer m, because different descendants with different m are independent. Plugging in $\Delta = (d-1)/2$, it is easy to see that the coefficient vanishes only for $\hat{\Delta} = (d-1)/2$ and $\hat{\Delta} = (d+1)/2$, so these are the only two operators allowed in the boundary OPE of a free scalar field. In the case where there are no interactions at the boundary, one has either one or the other of these operators, corresponding to Neumann and Dirichlet boundary conditions respectively. For the generalized Neumann boundary conditions in the presence of boundary interactions, as we show below one has both of these operators present in the boundary spectrum. Their dimensions are protected and add to d, satisfying a kind of "shadow relation". Intuitively, the reason for this is clear from the structure of the generalized Neumann boundary condition in (2.9). The operator of dimension $\Delta = (d-1)/2$ is just ϕ restricted to the boundary, while the one of dimension $\Delta = (d+1)/2$ is the operator $\frac{\delta \mathcal{L}_{int}}{\delta \phi}$ (this is a cubic operator in the $O(N)$ models we discuss below), which is related to ϕ by the boundary condition.

We can gain further insight on these protected operators by considering the bulk two-point function. Corresponding to two different OPE limits, there are two different ways to decompose the bulk two point function (see e.g. [16, 14, 21]). We could do the usual OPE in the bulk and then do the boundary OPE of the fields that appear in the bulk OPE, or do the boundary OPE first and then do the usual OPE on the boundary. Correspondingly, a bulk two-point function can be expanded into either a set of boundary conformal blocks or a set of bulk conformal blocks, and the two expansions must be equal. Let us define the following cross-ratios

$$\xi \equiv \frac{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (y_1 - y_2)^2}{4 y_1 y_2}, \quad z \equiv \frac{1}{1+\xi} \tag{2.15}$$

so that $\xi \to \infty,\ z \to 0$ in the boundary OPE limit and $\xi \to 0,\ z \to 1$ in the bulk OPE limit. We

can then express the bulk two-point function of a scalar operator of dimension Δ_O as

$$\begin{aligned}\langle O(x_1)O(x_2)\rangle &= \frac{C_O}{(4y_1y_2)^{\Delta_O}}\mathcal{G}(z)\\ \mathcal{G}(z) &= \frac{z^{\Delta_O}}{(1-z)^{\Delta_O}}\sum_k \lambda_k f_{\rm bulk}(\Delta_k;1-z) = \sum_l \mu_l^2 f_{\rm bdy}(\hat{\Delta}_l;z)\end{aligned} \tag{2.16}$$

where λ_k is the product of the bulk OPE coefficient and one point function of the operator, and $C_O\mu_l^2 = (B_O^{\hat{O}})^2\hat{C}_{\hat{O}}$. The bulk and boundary blocks can be determined to be [12]

$$\begin{aligned}f_{\rm bulk}(\Delta_k;z) &= z^{\frac{\Delta_k}{2}}\ {}_2F_1(\frac{\Delta_k+1-d}{2},\frac{\Delta_k}{2};\Delta_k+\frac{1-d}{2};z)\\ f_{\rm bdy}(\hat{\Delta}_l;z) &= z^{\hat{\Delta}_l}\ {}_2F_1\Big(\hat{\Delta}_l,\hat{\Delta}_l+\frac{1-d}{2};2\hat{\Delta}_l+1-d;z\Big).\end{aligned} \tag{2.17}$$

In the case of a bulk free field ϕ, the equation of motion for the bulk two-point function $\langle\phi(x)\phi(x')\rangle$ has two solutions corresponding to Neumann and Dirichlet boundary conditions

$$\begin{aligned}G_\phi^{N/D}(x,x') &= \frac{\Gamma(\frac{d+1}{2})}{(d-1)2\pi^{\frac{d+1}{2}}}\Big(\frac{1}{((\mathbf{x}-\mathbf{x}')^2+(y-y')^2)^{\frac{d-1}{2}}}\pm\frac{1}{((\mathbf{x}-\mathbf{x}')^2+(y+y')^2)^{\frac{d-1}{2}}}\Big)\\ &= \frac{\Gamma(\frac{d+1}{2})}{(d-1)2\pi^{\frac{d+1}{2}}(4y_1y_2)^{\frac{d-1}{2}}}\Big(\Big(\frac{z}{1-z}\Big)^{\frac{d-1}{2}}\pm z^{\frac{d-1}{2}}\Big).\end{aligned} \tag{2.18}$$

In general, the bulk two-point function can then be a linear combination of these two solutions

$$\langle\phi(x)\phi(x')\rangle = \frac{\Gamma(\frac{d+1}{2})}{(d-1)2\pi^{\frac{d+1}{2}}(4y_1y_2)^{\frac{d-1}{2}}}\Big(\Big(\frac{z}{1-z}\Big)^{\frac{d-1}{2}}+\lambda_{\phi^2}\ z^{\frac{d-1}{2}}\Big), \tag{2.19}$$

where the λ_{ϕ^2} coefficient is related to the bulk one-point function of the ϕ^2 operator. To see this, note that the bulk OPE expansion of the two-point function of ϕ contains, in addition to the identity block, a single block corresponding to the operator ϕ^2 with $\Delta_{\phi^2} = d-1$. The coefficient of the identity is just fixed by the normalization of the field ϕ. Comparing with (2.16)-(2.17), we see that the second term in (2.19) indeed correspond to the ϕ^2 operator. The coefficient λ_{ϕ^2} is equal to ±1 for Neumann or Dirichlet boundary conditions, but is arbitrary for generalized Neumann case. On the boundary, there are two possible blocks corresponding to operators with dimensions $(d-1)/2$ and $(d+1)/2$, as shown above, with OPE coefficients say μ_N^2 and μ_D^2. The blocks simplify for these values of conformal dimensions and the crossing equation relating the bulk and boundary OPE coefficients simply becomes

$$1+\lambda_{\phi^2}\ (1-z)^{\frac{d-1}{2}} = \frac{\mu_N^2}{2}(1+(1-z)^{\frac{d-1}{2}})\ +\ \frac{2\mu_D^2}{d-1}(1-(1-z)^{\frac{d-1}{2}})\,. \tag{2.20}$$

Equating the coefficients gives

$$\frac{\mu_N^2}{2}+\frac{2\mu_D^2}{d-1}=1,\quad \frac{\mu_N^2}{2}-\frac{2\mu_D^2}{d-1}=\lambda_{\phi^2}. \tag{2.21}$$

As we expect, $\lambda_{\phi^2}=1$ corresponds to Neumann and gives $\mu_D^2=0$, while $\lambda_{\phi^2}=-1$ corresponds to Dirichlet and gives $\mu_N^2=0$. The case of generic λ_{ϕ^2} has both operators present in the boundary spectrum and corresponds to the case of interacting theory on the boundary.

2.1.1 Displacement operator and its higher spin cousins

This section uses several results from [11] about curved manifolds with a boundary. We refer the reader to [11, 21] for more detailed derivations. The action for the kind of theories we consider can be written in curved space as

$$S=\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(\frac{g^{\mu\nu}}{2}\partial_\mu\phi^I\partial_\nu\phi^I+\frac{\tau}{2}R\phi^I\phi^I\right)+\int_{\partial\mathcal{M}} d^d\hat{x}\sqrt{\gamma}\left(\mathcal{L}_{\text{int}}+\frac{\rho}{2}K\phi^I\phi^I\right) \tag{2.22}$$

where the boundary (or defect) is located at $x^\mu=X^\mu(\hat{x}^i)$, $K=\gamma^{ij}K_{ij}$ is the trace of the extrinsic curvature, and the boundary metric is defined by

$$\gamma_{ij}=e_i^\mu e_j^\nu g_{\mu\nu},\qquad e_i^\mu=\frac{\partial X^\mu}{\partial\hat{x}^i}. \tag{2.23}$$

By the usual variational principle, we can determine the following equation of motion and the boundary condition

$$\nabla^2\phi^I-\tau R\phi^I=0,\quad (\partial_n\phi^I-\rho K\phi^I-\mathcal{L}'_{\text{int}})|_{\partial\mathcal{M}}=0. \tag{2.24}$$

It can be shown [11] that for Weyl invariance, we need $\rho=2\tau=\frac{d-1}{2d}$. From the variation of the above action with respect to the metric, we can determine the stress energy tensor, which in flat space with a flat defect reduces to

$$\begin{aligned} T^{\text{tot}}_{\mu\nu}&=T_{\mu\nu}+\delta_D(y)\delta^i_\mu\delta^j_\nu\delta_{ij}(-\mathcal{L}_{\text{int}}(\phi^I)+2\tau\mathcal{L}'_{\text{int}}(\phi^I)\phi^I)\\ T_{\mu\nu}&=\partial_\mu\phi^I\partial_\nu\phi^I-\frac{\delta_{\mu\nu}}{2}(\partial_\rho\phi^I)^2-\frac{d-1}{4d}(\partial_\mu\partial_\nu-\delta_{\mu\nu}\partial^2)\phi^I\phi^I. \end{aligned} \tag{2.25}$$

In a similar fashion, we can derive the displacement operator which can be defined by the variation of action with respect to the embedding coordinate $X^\mu(\hat{x}^i)$. Let n^μ be the normal to the defect.

We shift the boundary along the normal as $\delta_t X^\mu(\hat{x}^i) = -n^\mu \delta t$ and we let δt be a function of the boundary coordinates here. Under this variation, the trace of the extrinsic curvature changes as [11]

$$\delta_t K = 3\delta t K_{ij} K^{ij} - \hat{\gamma}^{ij} \hat{\nabla}_i \partial_j \delta t - \hat{\gamma}^{ij} R_{njni} \delta t. \tag{2.26}$$

Using this, one can see that, specializing to flat space with a flat defect, the variation of action is given by

$$\delta_t S = \int d^d\mathbf{x} \left[\delta t \left(\frac{1}{2}(\partial_y \phi^I)^2 + \frac{1}{2}(\partial_i \phi^I)^2 - \frac{\delta \mathcal{L}_{\text{int}}}{\delta \phi^I} \partial_y \phi^I \right) - \frac{\rho}{2} \phi^I \phi^I (\partial_i \partial^i \delta t) \right]. \tag{2.27}$$

The first two terms in the above equation come from the bulk piece of the action. The third term comes from the $\mathcal{L}_{\text{int}}$ piece of the boundary action. Since $\mathcal{L}_{\text{int}}$ is a function of boundary fields, which are just the bulk fields restricted to the boundary, its variation when we move the boundary should be given by $-\partial_y \mathcal{L}_{\text{int}}\ \delta t$, which simplifies to what we wrote above. The variation of K, as written in (2.26), has three pieces, but only one of them survives in the flat space case, yielding the last term in (2.27). After using the boundary condition and integration by parts, we get the displacement operator

$$D(\mathbf{x}) = n^\mu \frac{\delta S}{\delta X^\mu} = \left[\frac{1}{2}(\partial_y \phi^I)^2 - \frac{1}{2d}(\partial_i \phi^I)^2 + \frac{d-1}{2d} \phi^I \partial_i^2 \phi^I \right] \Bigg|_{y \to 0} = T_{yy}|_{y \to 0}. \tag{2.28}$$

Another way to define the same operator is through its appearance in the divergence of stress tensor, as reviewed in the introduction

$$\partial_\mu T^{\mu i} = 0, \quad \partial_\mu T^{\mu y} = D(\mathbf{x}) \delta(y) \tag{2.29}$$

By doing a volume integral over a Gaussian pill box located at the boundary, we can get the following relation

$$T^{yy}|_{y \to 0} = D(\mathbf{x}). \tag{2.30}$$

which agrees with what we get from the other definition above. Since the stress tensor is conserved, the displacement operator must be protected on the boundary. Now, if the bulk theory is free, as in the models we study in this chapter, we will have a tower of exactly conserved higher spin currents. These are then expected to imply a tower of spinning protected operators on the boundary, which

we may view as higher-spin "cousins" of the displacement operator

$$\partial_\mu J^{\mu\mu_1\cdots\mu_s y} = D^{\mu_1\cdots\mu_{s-2}}(\mathbf{x})\delta(y), \quad \Longrightarrow \; J^{y\mu_1\cdots\mu_{s-2}y}|_{y\to 0} = D^{\mu_1\cdots\mu_{s-2}}(\mathbf{x}). \tag{2.31}$$

From the point of view of the theory on the boundary, the operator $D^{\mu_1\cdots\mu_{s-2}}$ contains operators of all spins between 0 and $s-2$, with 0 being the case when all the $\mu's$ are equal to y while $s-2$ being the case when none of the $\mu's$ are equal to y. So we expect to see protected boundary operators of dimension $d+1+s-2$ (same as the dimension of bulk spin s current) and a spin between 0 and $s-2$. In the boundary theory, these will be bilinears in the boundary operator[2] ϕ schematically of the form $\phi\vec{\partial}^{2n}\partial_{\nu_1}\partial_{\nu_2}..\partial_{\nu_l}\phi$ with dimensions $d-1+2n+l$ and spin l. In section 2.3, we will give several pieces of evidence, within perturbation theory, for the fact that these boundary operators are protected.

2.2 $O(N)$ BCFT in $1 < d < 4$

In this section, we describe perturbative fixed points of $O(N)$ invariant field theories with boundary localized interactions in boundary dimensions $1 < d < 4$. We calculate anomalous dimensions of various boundary operators and two point function of the bulk fundamental field at these fixed points and perform appropriate checks wherever different perturbative expansions are expected to match.

2.2.1 ϕ^4 theory in $d = 2-\epsilon$

Let us first consider N scalar fields on $d+1$ dimensional flat space with a d dimensional flat boundary, and a quartic $O(N)$ invariant interaction localized at the boundary:[3]

$$S = \int d^{d+1}x \frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I + \int d^dx \frac{g}{4}(\phi^I\phi^I)^2. \tag{2.32}$$

The coupling becomes marginal in $d=2$, and it is relevant for $d<2$, so we will study this model in $d=2-\epsilon$. To do the calculation in momentum space, we can Fourier transform the free propagator

[2]We use the same letter ϕ for the bulk field $\phi(\mathbf{x}, y)$ and its boundary value $\phi(\mathbf{x})$. It will be clear which one we mean from the context. This will make the expressions less messy by reducing the appearance of "hats".

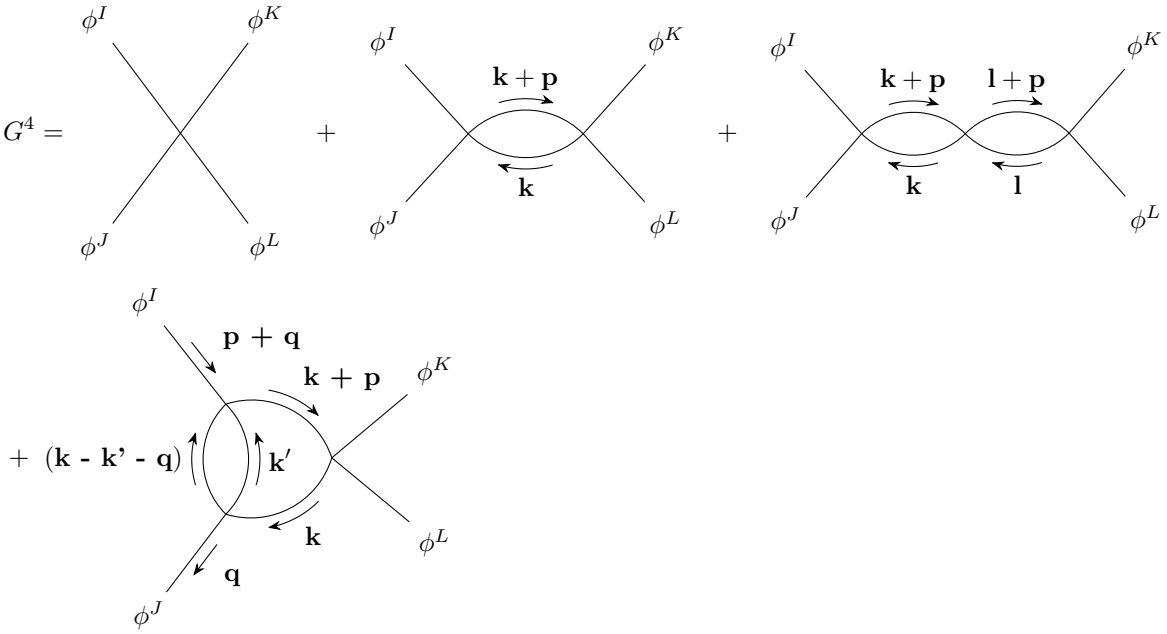
[3]We thank Igor Klebanov for useful suggestions and initial collaboration on the calculations presented in this Section.

along the boundary directions to get

$$\begin{aligned}\langle\phi^I(-\mathbf{p},y)\phi^J(\mathbf{p},y)\rangle = \delta^{IJ}\tilde{G}^0_\phi(p) &= \delta^{IJ}\int_{\partial\mathcal{M}} d^d\mathbf{x} e^{-i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)} G^0_\phi(y_1,\mathbf{x}_1;y_2,\mathbf{x}_2) \\ &= \delta^{IJ}\frac{e^{-p|y_1-y_2|}+e^{-p(y_1+y_2)}}{2p}\end{aligned} \tag{2.33}$$

which becomes $1/p$ on the boundary where $y_1, y_2 \to 0$.

To look for a fixed point, we compute the β function up to two loops by first evaluating the following four point function and then requiring that it satisfies the Callan-Symanzik equation:



$$\begin{aligned}&= 2\delta^{IJ}\delta^{KL}\Bigg[-(g+\delta_g) \;+\; (g+\delta_g)^2(N+8)\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}+\mathbf{p}||\mathbf{k}|} - g^3(N^2+6N+20) \\ &\times\left(\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}+\mathbf{p}||\mathbf{k}|}\right)^2 - 4g^3(5N+22)\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\frac{1}{|\mathbf{k}'||\mathbf{k}-\mathbf{k}'-\mathbf{q}|}\Bigg] \\ &= 2\delta^{IJ}\delta^{KL}\Bigg[-(g+\delta_g) + \frac{(g+\delta_g)^2(N+8)\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\pi\Gamma(d-1)(p^2)^{1-\frac{d}{2}}} \\ &- \frac{g^3(N^2+6N+20)\Gamma(\frac{d-1}{2})^4\Gamma(1-\frac{d}{2})^2}{(4\pi)^d\;\pi^2\Gamma(d-1)^2(p^2)^{2-d}} - \frac{4g^3(5N+22)\Gamma(\frac{d-1}{2})^3\Gamma(1-\frac{d}{2})\Gamma(d-\frac{3}{2})\Gamma(2-d)}{(4\pi)^d\pi^{3/2}\Gamma(d-1)\Gamma(\frac{3-d}{2})\Gamma(\frac{3d}{2}-2)(p^2)^{2-d}}\Bigg].\end{aligned} \tag{2.34}$$

where we used an integral given in appendix 2.6 and evaluated the fourth diagram at $\mathbf{q} = 0$.

Expanding this in $d = 2 - \epsilon$ and demanding that the divergent terms cancel gives

$$\delta_g = \frac{g^2(N+8)}{2\pi\epsilon} - \frac{g^3(5N+22)\log 2}{\pi^2\epsilon} + \frac{g^3(N+8)^2}{4\pi^2\epsilon^2}. \tag{2.35}$$

After canceling the divergent parts, the remaining finite parts need to satisfy Callan-Symanzik equation. Noting that in $2-\epsilon$ dimensions, the bare coupling has a factor of μ^ϵ on dimensional grounds, and then applying following equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)G^4 = 0 \tag{2.36}$$

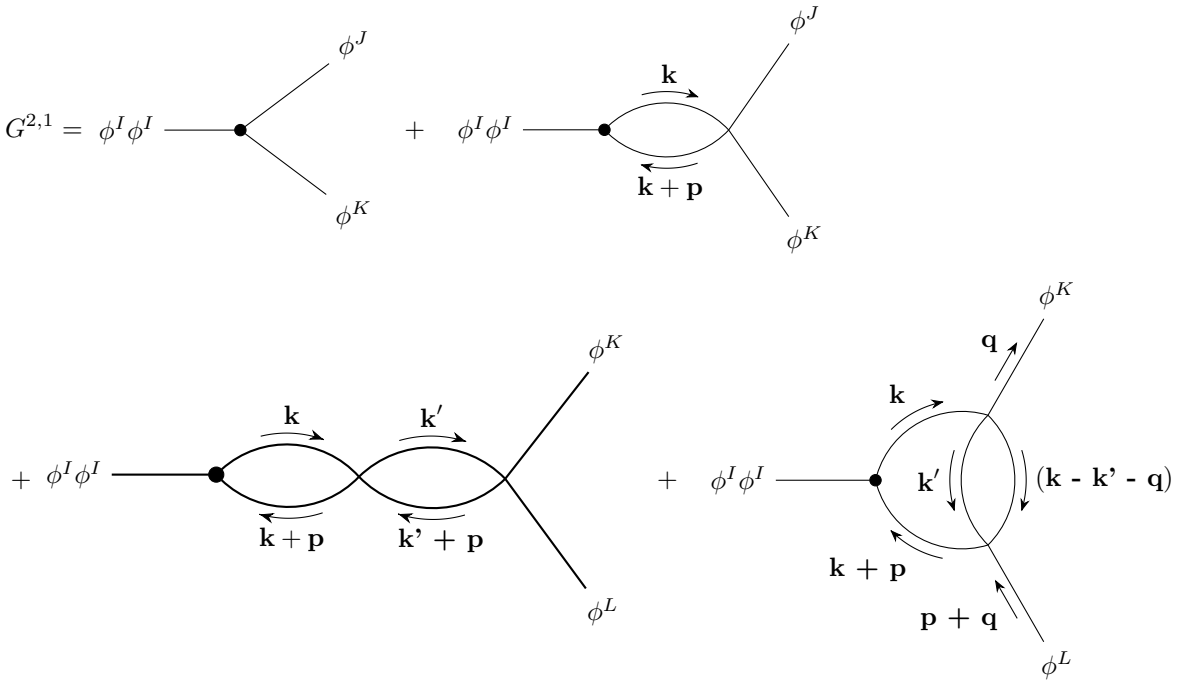
gives us

$$\beta = -\epsilon g + \frac{g^2(N+8)}{2\pi} - \frac{2g^3(5N+22)\log 2}{\pi^2}. \tag{2.37}$$

There is a unitary IR fixed point at

$$g_* = \frac{2\pi\epsilon}{N+8} + \frac{16\pi(5N+22)\epsilon^2\log 2}{(N+8)^3}. \tag{2.38}$$

We can compute the anomalous dimensions of various operators at this fixed point. The simplest operator that gets an anomalous dimension is the $O(N)$ singlet on the boundary, $\phi^I\phi^I$. Its anomalous dimensions up to two loops can be determined from the following contributions to the boundary correlation function $\langle\phi^I\phi^I(\mathbf{x})\phi^J(\mathbf{y})\phi^K(\mathbf{z})\rangle$



$$
\begin{aligned}
&= 2\delta^{JK}\Bigg[1+\delta_{\phi^2} \;-\; (1+\delta_{\phi^2})(g+\delta_g)(N+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|} \\
&+ g^2(N+2)^2\bigg(\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\bigg)^2 \;+\; 6g^2(N+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\frac{1}{|\mathbf{k}'||\mathbf{k}-\mathbf{k}'-\mathbf{q}|}\Bigg] \\
&= 2\delta^{JK}\Bigg[1+\delta_{\phi^2} \;-\; (1+\delta_{\phi^2})\frac{(g+\delta_g)(N+2)\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\pi\Gamma(d-1)(p^2)^{1-\frac{d}{2}}} \\
&+ \frac{g^2(N+2)^2\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})^2}{(4\pi)^d\pi^2\Gamma(d-1)^2(p^2)^{2-d}} \;+\; \frac{6g^2(N+2)\Gamma(\frac{d-1}{2})^3\Gamma(1-\frac{d}{2})\Gamma(d-\frac{3}{2})\Gamma(2-d)}{(4\pi)^d\pi^{3/2}\Gamma(d-1)\Gamma(\frac{3-d}{2})\Gamma(\frac{3d}{2}-2)(p^2)^{2-d}}\Bigg].
\end{aligned}
\tag{2.39}
$$

where we evaluated the last diagram at $\mathbf{q}=\mathbf{0}$ in this case as well. Again, expanding in $d=2-\epsilon$ and requiring that the divergent terms cancel gives

$$
\delta_{\phi^2} = \frac{g(N+2)}{2\pi\epsilon} - \frac{3g^2(N+2)\log 2}{2\pi^2\epsilon} + \frac{g^2(N+2)(N+5)}{4\pi^2\epsilon^2} \tag{2.40}
$$

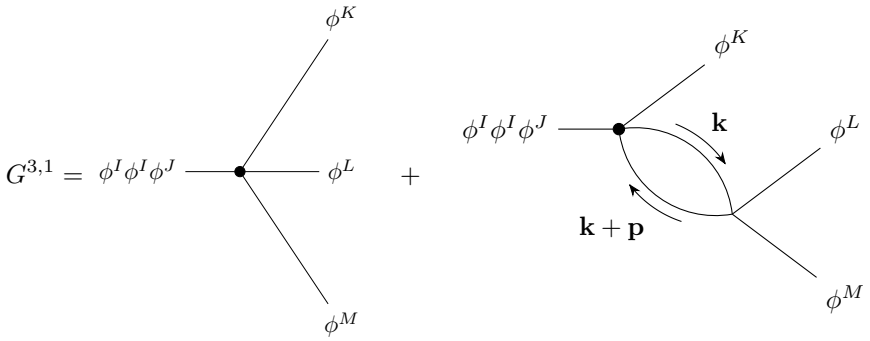
Then applying Callan-Symanzik equation to the correlation function

$$
\bigg(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \hat{\gamma}_{\phi^2}\bigg)G^{2,1} = 0 \tag{2.41}
$$

gives us the anomalous dimension

$$\begin{aligned}\hat{\gamma}_{\phi^2} &= \frac{g_*(N+2)}{2\pi} - \frac{12g_*^2(N+2)\log 2}{4\pi^2} = \frac{N+2}{N+8}\epsilon + \frac{4(N+2)(7N+20)\log 2}{(N+8)^3}\epsilon^2 \\ \hat{\Delta}_{\phi^2} &= d - 1 + \hat{\gamma}_{\phi^2} = 1 - \frac{6\epsilon}{N+8} + \frac{4(N+2)(7N+20)\log 2}{(N+8)^3}\epsilon^2\end{aligned} \tag{2.42}$$

Another interesting operator to look at on the boundary is the $(\phi^I\phi^I)\phi^J$ operator which we dub as ϕ^3 operator. For that we compute the following one loop contributions to the boundary correlator $\langle(\phi^I\phi^I)\phi^J(\mathbf{x})\phi^K(\mathbf{y})\phi^L(\mathbf{z})\phi^M(\mathbf{w})\rangle$



$$\begin{aligned}&= 2(\delta^{KL}\delta^{MJ} + \delta^{KM}\delta^{LJ} + \delta^{LM}\delta^{KJ})\left(1 + \delta_{\phi^3} - g(N+8)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\right) \\ &= 2(\delta^{KL}\delta^{MJ} + \delta^{KM}\delta^{LJ} + \delta^{LM}\delta^{KJ})\left(1 + \delta_{\phi^3} - g(N+8)\frac{\Gamma(\frac{2-d}{2})}{(4\pi)^{\frac{d}{2}}(p^2)^{\frac{2-d}{2}}}\right).\end{aligned} \tag{2.43}$$

To cancel the divergence we impose the condition that the order g term vanish at momentum scale μ which implies

$$\begin{aligned}\delta_{\phi^3} &= \frac{g(N+8)\Gamma(\frac{2-d}{2})}{(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{2-d}{2}}}, \quad \hat{\gamma}_{\phi^3} = -\mu\frac{\partial}{\partial\mu}\delta_{\phi^3} = \epsilon \\ \hat{\Delta}_{\phi^3} &= \frac{3(d-1)}{2} + \epsilon = \frac{3-\epsilon}{2} = \frac{d+1}{2}\end{aligned} \tag{2.44}$$

which agrees with our expectation since the boundary condition fixes $\phi^3 \sim \partial_y\phi$, so it must have dimension $\Delta_\phi + 1$.

We will next compute the bulk two point of ϕ at this fixed point. In the free theory, it is still given by eq. (2.18) but this will receive corrections because of interactions starting at order g^2. The leading perturbative correction is depicted in Figure 2.5. The computation of the corresponding

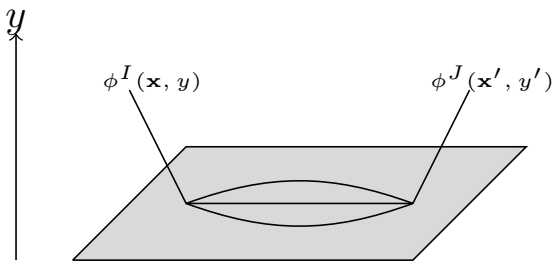
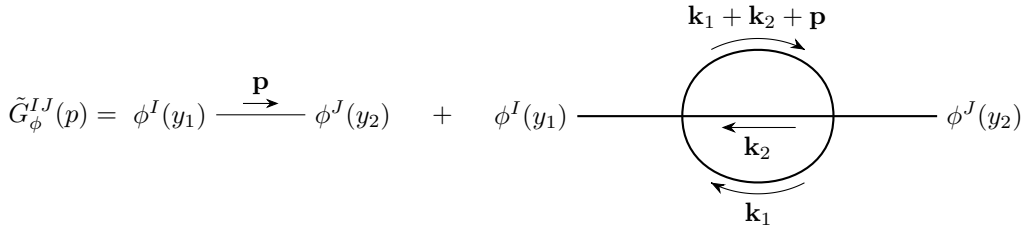


Figure 2.5: Bulk two-point function at leading non-trivial order with ϕ^4 interaction on the boundary

Feynman diagram yields



$$
\begin{aligned}
&= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{2p} + \frac{\delta^{IJ} 2g^2(N+2)e^{-p(y_1+y_2)}}{p^2} \int \frac{d^d\mathbf{k_1}}{(2\pi)^d} \frac{d^d\mathbf{k_2}}{(2\pi)^d} \frac{1}{|k_1||k_2||\mathbf{k_1}+\mathbf{k_2}+\mathbf{p}|} \\
&= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{2p} + \frac{\delta^{IJ} 2g^2(N+2)e^{-p(y_1+y_2)}\Gamma(\frac{2-d}{2})\Gamma(\frac{d-1}{2})^2}{(4\pi)^{\frac{d}{2}}\pi\Gamma(d-1)p^2} \int \frac{d^d\mathbf{k_2}}{(2\pi)^d} \frac{1}{|k_2||\mathbf{k_2}+\mathbf{p}|^{2-d}} \\
&= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{2p} + \frac{\delta^{IJ} 2g^2(N+2)e^{-p(y_1+y_2)}\Gamma(\frac{3-2d}{2})\Gamma(\frac{d-1}{2})^3(p^2)^{d-\frac{5}{2}}}{(4\pi)^d\pi^{\frac{3}{2}}\Gamma(\frac{3d-3}{2})}.
\end{aligned}
\tag{2.45}
$$

This doesn't have a divergence, in accordance with the fact that ϕ^I is a free field and does not get anomalous dimension. We can transform it back to position space and at the fixed point, this gives

$$
G_\phi^{IJ}(x_1, x_2) = \delta^{IJ} G_\phi^0(x_1, x_2) - \frac{\delta^{IJ}\epsilon^2(N+2)}{\pi(N+8)^2\sqrt{(\mathbf{x}_1-\mathbf{x}_2)^2 + (y_1+y_2)^2}} \tag{2.46}
$$

This in particular gives corrections to the one point function of $\phi^I\phi^I$

$$
\langle \phi^I\phi^I(\mathbf{x}, y)\rangle = \frac{N}{2\pi y}\left(\frac{1}{4} - \frac{\epsilon^2(N+2)}{(N+8)^2}\right) \tag{2.47}
$$

2.2.2 Large N description for general d

We can rewrite the quartic model introduced in the previous section in terms of a Hubbard-Stratonovich auxiliary field that lives only at the d-dimensional boundary:

$$S = \int d^{d+1}x \frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I + \int d^d x \left(\frac{\sigma \phi^I \phi^I}{2} - \frac{\sigma^2}{4g} \right). \tag{2.48}$$

The equation of motion of σ sets it equal to $g\phi^I\phi^I$ and plugging this in gives us back the original action. On the boundary, this is analogous to the usual O(N) model except for the fact that the propagator for ϕ is different. We can integrate out ϕ^I on the boundary to get a boundary effective action for σ

$$\begin{aligned} e^{-S^{\text{eff}}_{\text{bdry}}[\sigma]} &= \int D\phi \; e^{-\int d^{d+1}x \frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I - \int d^d x (\frac{\sigma \phi^I \phi^I}{2} - \frac{\sigma^2}{4g})} \\ &= e^{\frac{1}{8}\int d^d x_1 d^d x_2 \; \sigma(x_1)\sigma(x_2)\langle \phi^I\phi^I(x_1)\phi^J\phi^J(x_2)\rangle_0 + \int d^d x \frac{\sigma^2}{4g} + O(\sigma^3)} \end{aligned} \tag{2.49}$$

where

$$\langle \phi^I \phi^I(\mathbf{x}_1)\phi^J\phi^J(\mathbf{x}_2)\rangle_0 = 2N[G_\phi(\mathbf{x}_1 - \mathbf{x}_2)]^2 \tag{2.50}$$

with

$$\begin{aligned} [G_\phi(\mathbf{x_1}-\mathbf{x}_2)]^2 &= \int \frac{d^d\mathbf{k_1}}{(2\pi)^d}\int \frac{d^d\mathbf{k_2}}{(2\pi)^d}\frac{e^{i(\mathbf{k_1}+\mathbf{k_2})\cdot(\mathbf{x_1}-\mathbf{x_2})}}{k_1 k_2} = \int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot(\mathbf{x_1}-\mathbf{x_2})}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q|\mathbf{p}-\mathbf{q}|} \\ &= -\int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot(\mathbf{x_1}-\mathbf{x_2})}\frac{2}{\tilde{C_\sigma}}(p)^{d-2} \end{aligned} \tag{2.51}$$

where

$$\tilde{C_\sigma} = -\frac{2\pi(4\pi)^{\frac{d}{2}}\Gamma(d-1)}{\Gamma(\frac{2-d}{2})\Gamma(\frac{d-1}{2})^2}. \tag{2.52}$$

This gives the quadratic part of the boundary effective action for sigma to be

$$S_2 = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{\sigma(\mathbf{p})\sigma(-\mathbf{p})}{2}\left(\frac{N}{\tilde{C_\sigma}}(p)^{d-2} - \frac{1}{2g}\right). \tag{2.53}$$

From here, it is clear that for $d < 2$, the second term in the quadratic action can be dropped in the IR limit, while for $d > 2$, it can be dropped in the UV limit. This only leaves the induced kinetic term in the quadratic action and leads to the following two point function for σ

$$\langle \sigma(\mathbf{p})\sigma(-\mathbf{p})\rangle = \frac{\tilde{C_\sigma}}{N}(p^2)^{\frac{2-d}{2}} \tag{2.54}$$

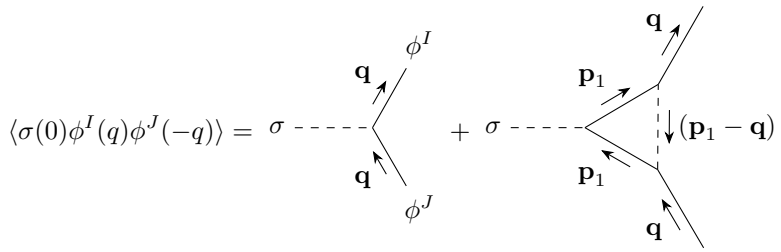
which gives in position space

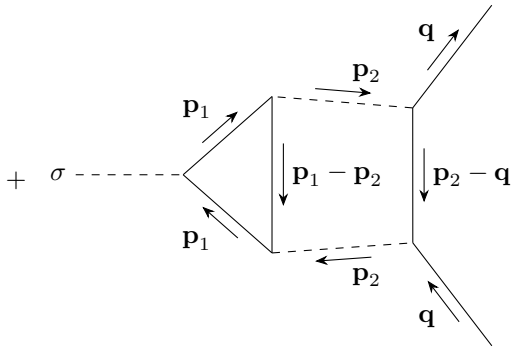
$$\langle\sigma(\mathbf{x}_1)\sigma(\mathbf{x}_2)\rangle = \frac{C_\sigma}{|\mathbf{x}_1-\mathbf{x}_2|^2}, \quad C_\sigma = \tilde{C}_\sigma \frac{4}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}-1)} \tag{2.55}$$

which implies that the conformal dimension of sigma operator to this order is 1. The power law correlation suggests the existence of an IR fixed point in $d < 2$ and a UV fixed point in $d > 2$.

We can also compute the anomalous dimension of σ to order $1/N$. In general, it should be computed using the two loop correction to the σ propagator, but in this case, since ϕ does not get an anomalous dimension, we can use the $1/N$ corrections to the following correlator

$\langle\sigma(0)\phi^I(q)\phi^J(-q)\rangle =$





$$\begin{aligned}
&= \delta^{IJ} + \frac{\tilde{C}_\sigma \delta^{IJ}}{N}\int \frac{d^dp_1}{(2\pi)^d}\frac{1}{|p_1|^2|p_1-q|^{d-2}} + \frac{\tilde{C}_\sigma^2\delta^{IJ}}{N}\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\frac{1}{|p_1|^2|p_1-p_2||p_2-q||p_2|^{2(d-2)}} \\
&= \delta^{IJ}\left(1 - \frac{2\log q\ \tilde{C}_\sigma}{N(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} - \frac{4\log q\ \tilde{C}_\sigma{}^2\ \Gamma(\frac{d-1}{2})\Gamma(\frac{3-d}{2})}{N(4\pi)^d(d-2)\sqrt{\pi}\Gamma(d-\frac{3}{2})}\right) \\
&= \delta^{IJ} + \frac{\delta^{IJ}\log(q^2/\mu^2)}{2N}\left(\frac{2^d\sqrt{\pi}}{\Gamma(\frac{2-d}{2})\Gamma(\frac{d-1}{2})} - \frac{2^{2d-1}\sqrt{\pi}\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})\Gamma(\frac{d-2}{2})}{\Gamma(d-\frac{3}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{2-d}{2})^2}\right).
\end{aligned} \tag{2.56}$$

Applying Callan-Symanzik equation to it gives the anomalous dimension

$$\hat{\Delta}_\sigma = 1 + \hat{\gamma}_\sigma = 1 + \frac{1}{N}\left(\frac{2^{2d-1}\sqrt{\pi}\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})\Gamma(\frac{d-2}{2})}{\Gamma(d-\frac{3}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{2-d}{2})^2} - \frac{2^d\sqrt{\pi}}{\Gamma(\frac{2-d}{2})\Gamma(\frac{d-1}{2})}.\right) \tag{2.57}$$

This can be expanded in $d = 2 - \epsilon$

$$\hat{\Delta}_\sigma = 1 - \frac{6\epsilon}{N} + \frac{28\epsilon^2 \log 2}{N} \tag{2.58}$$

This precisely agrees with the dimension of ϕ^2 operator in the ϵ expansion at large N in eq. (2.42). This can also be expanded in $d = 1 + \epsilon$

$$\hat{\Delta}_\sigma = 1 - \frac{\epsilon^2}{N} \tag{2.59}$$

and we will show that it agrees with the result obtained from non-linear sigma model in eq. (2.88) in the next subsection. Expanding in $d = 4 - \epsilon$

$$\hat{\Delta}_\sigma = 1 - \frac{\epsilon^2}{N} \tag{2.60}$$

which agrees with mixed $\sigma\phi$ theory described below in subsection 2.2.4.

The bulk propagator for ϕ now involves following contributions

$\langle \phi^I(-p, y_1)\phi^J(p, y_2)\rangle =$ $\phi^I(y_1)$ —$\xrightarrow{\mathbf{p}}$— $\phi^J(y_2)$ + $\phi^I(y_1)$ —$\xrightarrow{\mathbf{p}}$ $\overset{\mathbf{p}-\mathbf{q}}{\frown}$ $\xrightarrow{\mathbf{q}}$ $\xrightarrow{\mathbf{p}}$— $\phi^J(y_2)$

$$\begin{aligned}
&= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{|p|} + \frac{\tilde{C}_\sigma \delta^{IJ} e^{-p(y_1+y_2)}}{N|p|^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{|q|((p-q)^2)^{\frac{d-2}{2}}} \\
&= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{|p|} + \frac{\delta^{IJ} 8\pi\ \Gamma(d-1) e^{-p(y_1+y_2)}}{N|p|(d-1)\Gamma(\frac{d-2}{2})\Gamma(\frac{2-d}{2})\Gamma(\frac{d-1}{2})^2}.
\end{aligned} \tag{2.61}$$

We can Fourier transform it back to position space to get

$$G_\phi^{IJ}(x_1, x_2) = \delta^{IJ} G_\phi^0(x_1, x_2) \ + \ \frac{4\delta^{IJ}\ \Gamma(d-1)}{N\pi^{\frac{d-1}{2}}(d-1)\Gamma(\frac{d-2}{2})\Gamma(\frac{2-d}{2})\Gamma(\frac{d-1}{2})} \frac{1}{((y_1+y_2)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2)^{\frac{d-1}{2}}}. \tag{2.62}$$

The $1/N$ correction can be expanded in $d = 2 - \epsilon$ and it matches with what we got in the previous subsection from the ϵ expansion. It can also be expanded in $d = 4 - \epsilon$ and it agrees with what we get from ϵ expansion in subsection 2.2.4.

2.2.3 Non-linear sigma model in $d = 1 + \epsilon$

Next model we will consider is related to the usual $O(N)$ non-linear sigma model, so let us first review the calculation of beta function for the usual case to set the notation. We define the model as

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi^I \partial^\mu \phi^I + \sigma(\phi^I \phi^I - \frac{1}{t^2}) \right) \tag{2.63}$$

where the Lagrange multiplier σ imposes the constraint that $\phi^I \phi^I = \frac{1}{t^2}$. We can choose the following parametrization that solves the constraint

$$\phi^I = \psi^I, \quad I = 1, ..., N-1; \qquad \phi^N = \frac{1}{t}\sqrt{1 - t^2 \psi^I \psi^I} = \frac{1}{t} - \frac{t}{2}\psi^I \psi^I + O(t^3). \tag{2.64}$$

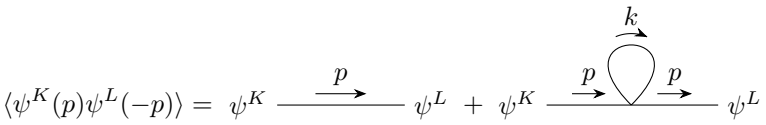
In terms of these variables, the action becomes

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \psi^I \partial^\mu \psi^I + \frac{t^2}{2} \frac{(\psi^I \partial_\mu \psi^I)^2}{1 - t^2 \psi^I \psi^I} \right) = \int d^d x \left(\frac{1}{2} \partial_\mu \psi^I \partial^\mu \psi^I + \frac{t^2}{2} (\psi^I \partial_\mu \psi^I)^2 + O(t^4) \right) \tag{2.65}$$

We can then calculate the β function by requiring that the correlation functions obey Callan-Symanzik equation

$$(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial t} + n\gamma(t)) G^n = 0 \tag{2.66}$$

and the original O(N) symmetry forces the anomalous dimensions for all the ϕ^I to be the same. We can apply this to the two point function

$\langle \psi^K(p) \psi^L(-p) \rangle =$ 

$$\begin{aligned} &= \frac{\delta^{KL}}{p^2} - \frac{t^2\ \delta^{KL}}{(p^2)^2} \int \frac{d^d k}{(2\pi)^d} \frac{p^2 + k^2}{k^2 + m^2} \\ &= \frac{\delta^{KL}}{p^2} - \frac{t^2 \delta^{KL}}{p^2} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} (m^2)^{1 - \frac{d}{2}}} - \frac{t^2 \delta^{KL}}{(p^2)^2} \frac{\frac{d}{2}\ \Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}} (m^2)^{-\frac{d}{2}}} \end{aligned} \tag{2.67}$$

where we have introduced an IR cutoff m^2. The last term vanishes as $m \to 0$ for all $d \geq 0$. The other two terms in $d = 2 + \epsilon$ give

$$\langle \psi^K(p) \psi^L(-p) \rangle = \frac{\delta^{KL}}{p^2} \left(1 - \frac{t^2}{4\pi} \log \frac{\mu^2}{m^2} \right). \tag{2.68}$$

This satisfies Callan-Symanzik equation with

$$\gamma_\phi(t) = \frac{t^2}{4\pi}. \tag{2.69}$$

We next consider the one point function of ϕ^N

$$\begin{aligned}
\langle \phi^N(0)\rangle &= \frac{1}{t} - \frac{t}{2}\langle \psi^a\psi^a(0)\rangle - \frac{t^3}{8}\langle \psi^a\psi^a(0)\psi^b\psi^b(0)\rangle \\
&= \frac{1}{t} - \frac{t(N-1)}{2}G_0(0,0) + \frac{t^3(N-1)}{2}\int d^dx G_0(x,x)(\partial_\mu G_0(0,x))^2 \\
&- \frac{t^3((N-1)^2+2(N-1))}{8}(G_0(0,0))^2 \\
&= \frac{1}{t} - \frac{t(N-1)}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} - \frac{t^3((N-1)^2-2(N-1))}{8}\left(\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\right)^2 \\
&= \frac{1}{t} - \frac{t(N-1)}{8\pi}\log\frac{\mu^2}{m^2} - \frac{t^3(N-1)(N-3)}{8(4\pi)^2}\left(\log\frac{\mu^2}{m^2}\right)^2
\end{aligned} \tag{2.70}$$

where in the last line, we plugged in $d = 2+\epsilon$. We can now apply Callan-Symanzik equation to it and we find

$$\beta(t) = \frac{\epsilon}{2}t - \frac{t^3(N-2)}{4\pi} \tag{2.71}$$

where the first term is present because in $2+\epsilon$ dimensions, t has engineering dimensions $-\epsilon/2$. The sign of β function suggests a UV fixed point in $2+\epsilon$ dimensions at

$$t^2 = t_*^2 = \frac{2\pi\epsilon}{N-2}. \tag{2.72}$$

The anomalous dimensions of the field ϕ at the fixed point $\gamma_\phi = \frac{\epsilon}{2(N-2)}$ agrees with the known results. The anomalous dimensions of the Lagrange multiplier field σ which is the analogue of the field σ in the large N analysis, can be found by the following relation

$$\Delta_\sigma = d + \beta'(t_*) = d + \frac{\epsilon}{2} - \frac{3t_*^2(N-2)}{4\pi} = 2 + O(\epsilon^2). \tag{2.73}$$

We will now consider a variant of the non-linear sigma model where the sphere constraint is only imposed on the d-dimensional boundary:

$$S = \int d^{d+1}x\frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I + \int d^dx\ \sigma(\phi^I\phi^I - \frac{1}{t^2})\,. \tag{2.74}$$

As in the case of the local models, the auxiliary field σ is related to the Hubbard-Stratonovich field introduced in the large N treatment. The fact that $\hat{\Delta}_\sigma = 1 + O(1/N)$, as shown in the previous section, suggests that the lower critical dimension is $d = 1$, and we should look for UV fixed points of the above model in $d = 1 + \epsilon$ boundary dimensions.

As in previous sections, the bulk propagator induces a $1/|\mathbf{p}|$ propagator on the boundary, which in the position space looks like a non-local kinetic term

$$S_{\rm bdry} = -\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int d^d x \; d^d y \; \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+1}} + \int d^d x \; \sigma(\phi^I\phi^I - \frac{1}{t^2}) \tag{2.75}$$

We can now solve the constraint on the boundary in terms of the variables ψ^a as before to get

$$S_{\rm bdry} = -\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int d^d x \; d^d y \; \frac{\psi^a(x)\psi^a(y)}{|x-y|^{d+1}} - \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{t^2}{4} \int d^d x \; d^d y \; \frac{\psi^a\psi^a(x)\psi^b\psi^b(y)}{|x-y|^{d+1}} + ... \tag{2.76}$$

where we dropped a constant unimportant shift, as well as corrections at higher orders in t^2. So, for the purpose of computing boundary correlation functions, this action gives a propagator for the ψ^a field that goes like $1/|p|$, and we can use this to develop perturbation theory with the interaction term from above expression. Let us first try to compute the diagram that would give us the anomalous dimension of the field ψ^a. We will show that it vanishes in accord with the expectation since ϕ^I is a free field in the bulk. The two point function of the field ψ^a goes like

$$\begin{aligned} \langle \psi^a(x)\psi^b(y)\rangle = \delta^{ab} G_0(x,y) &- \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} t^2 \delta^{ab} \int d^d z \; d^d w \frac{G_0(x,w)G_0(y,z)G_0(z,w)}{|z-w|^{d+1}} \\ &- \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} t^2 \delta^{ab} \int d^d z \; d^d w \frac{(N-1)G_0(x,w)G_0(y,w)G_0(z,z)}{|z-w|^{d+1}}. \end{aligned} \tag{2.77}$$

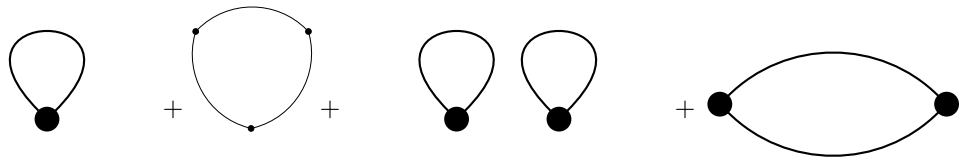
The term in the second line vanishes when we do the integral over z. We can now go to momentum space to get

$$\langle \psi^a(-p)\psi^b(p)\rangle = \frac{\delta^{ab}}{|p|} + \frac{t^2}{|p|^2} \int \frac{d^d q}{(2\pi)^d} \frac{|p-q|}{|q|}. \tag{2.78}$$

The integral can be evaluated in dimensional regularization by adding a small mass and then expanding in mass in $d = 1 + \epsilon$ to get

$$\begin{aligned} \langle \psi^a(-p)\psi^b(p)\rangle &= \frac{\delta^{ab}}{|p|} - \frac{t^2}{|p|^2} \frac{(m^2)^{\frac{d}{2}} \Gamma(-\frac{d}{2})\Gamma(\frac{d+1}{2})\Gamma(-\frac{1}{2}) \; {}_2F_1(-\frac{d}{2}, -\frac{1}{2}, \frac{d}{2}, -\frac{k^2}{m^2})}{2^{d+1}\pi^{\frac{d+2}{2}}\Gamma(\frac{d}{2})} \\ &= \frac{\delta^{ab}}{|p|} - \frac{t^2}{|p|} \frac{(2 + \log\frac{m^2}{4p^2})}{2\pi} + O(m^2). \end{aligned} \tag{2.79}$$

Since there is no $1/\epsilon$ pole, this implies that the field ψ^a does not get an anomalous dimension. We next go on to compute the beta function for the coupling t. For that, we will apply the Callan-Symanzik equation to the one point function of the field $\phi^N(0)$ as before

$\langle \phi^N(0) \rangle =$ 

$$
\begin{aligned}
&= \frac{1}{t} - \frac{t}{2}\langle \psi^a \psi^a(0) \rangle - \frac{t^3}{8}\langle \psi^a \psi^a(0) \psi^b \psi^b(0) \rangle \\
&= \frac{1}{t} - \frac{t(N-1)}{2} G_0(0,0) - \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{4(N-1)t^3}{8} \int d^d z \; d^d w \frac{G_0(0,w) G_0(0,z) G_0(z,w)}{|z-w|^{d+1}} \\
&- \frac{t^3((N-1)^2 + 2(N-1))}{8} G_0(0,0)^2 \\
&= \frac{1}{t} - \frac{t(N-1)}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{|k|} + \frac{(N-1)t^3}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{|k|^2} \int \frac{d^d l}{(2\pi)^d} \frac{|k-l|}{|l|} \\
&- \frac{t^3((N-1)^2 + 2(N-1))}{8} \int \frac{d^d k}{(2\pi)^d} \frac{1}{|k|} \int \frac{d^d l}{(2\pi)^d} \frac{1}{|l|}.
\end{aligned}
\tag{2.80}
$$

The integrals in the second and fourth term are straightforward. However, the integral in the third term is a bit subtle. Let us introduce an IR regulator mass, and perform the integral over l first, which gives in d dimensions

$$
\int \frac{d^d l}{(2\pi)^d} \frac{|k-l|}{\sqrt{l^2 + m^2}} = - \frac{(m^2)^{\frac{d}{2}} \Gamma(\frac{-d}{2}) \Gamma(\frac{d+1}{2}) \Gamma(-\frac{1}{2}) \; {}_2F_1(\frac{-d}{2}, -\frac{1}{2}, \frac{d}{2}, -\frac{k^2}{m^2})}{2^{d+1} \pi^{\frac{d+2}{2}} \Gamma(\frac{d}{2})}.
\tag{2.81}
$$

Fortunately, it is possible to do the integral over k now, and doing that and then taking $d = 1 + \epsilon$, gives, to leading order in ϵ

$$
\int \frac{d^d k}{(2\pi)^d} \frac{1}{|k|^2} \int \frac{d^d l}{(2\pi)^d} \frac{|k-l|}{|l|} = \frac{1}{8\pi^2} \left(\frac{4}{\epsilon^2} + \frac{4(\gamma + \log m^2 - \log 4\pi)}{\epsilon} \right) - \frac{1}{4\pi^2} \left(\frac{-2}{\epsilon} \right) + O(\epsilon^0)
\tag{2.82}
$$

The other two integrals can be evaluated by usual means, and overall it gives

$$
\begin{aligned}
\langle \phi^N(0) \rangle = \frac{1}{t} \; + \; & \frac{t(N-1)}{2\pi} \left(\frac{1}{\epsilon} + \frac{\gamma + \log m^2 - \log 4\pi}{2} \right) + \frac{(N-1)t^3}{4\pi^2 \epsilon} \\
& - \frac{t^3 (N-1)^2}{8\pi^2} \left(\frac{1}{\epsilon^2} + \frac{\gamma + \log m^2 - \log 4\pi}{\epsilon} \right)
\end{aligned}
\tag{2.83}
$$

We can now introduce the counterterms to cancel the divergences by redefining $t \to t_0 = t + \delta_t$ to get

$$\begin{aligned}\langle \phi^N(0) \rangle = \frac{1}{t} \ - \frac{\delta_t}{t^2} + \frac{\delta_t^2}{t^3} \ + \frac{(t+\delta_t)(N-1)}{2\pi}\left(\frac{1}{\epsilon} + \frac{\gamma + \log m^2 - \log 4\pi}{2}\right) + \frac{(N-1)t^3}{4\pi^2\epsilon} \\ - \frac{t^3(N-1)^2}{8\pi^2}\left(\frac{1}{\epsilon^2} + \frac{\gamma + \log m^2 - \log 4\pi}{\epsilon}\right).\end{aligned} \tag{2.84}$$

The counterterm is fixed by the requirement that it should cancel all the divergent terms which gives the original bare coupling in terms of renormalized coupling

$$t_0 = \mu^{-\epsilon/2}\left(t + \frac{(N-1)t^3}{2\pi\epsilon} + \frac{(N-1)t^5}{4\pi^2\epsilon} + \frac{3(N-1)^2t^5}{8\pi^2\epsilon^2}\right). \tag{2.85}$$

This gives the β function

$$\beta(t) = \frac{\epsilon}{2}t - \frac{t^3(N-1)}{2\pi} - \frac{t^5(N-1)}{2\pi^2}. \tag{2.86}$$

Notice that the β function here is proportional to $N-1$ as opposed to $N-2$ in the usual local case. This tells us that the $N = 1$ case has to be treated separately, similar to what happens for $N = 2$ case in the usual $O(N)$ model in two dimensions [75, 76]. This beta function gives a fixed point at

$$t_*^2 = \frac{\epsilon\pi}{(N-1)} - \frac{\epsilon^2\pi}{(N-1)^2} \tag{2.87}$$

This gives the dimension of the field σ

$$\hat{\Delta}_\sigma = d + \beta'(t_*) = 1 - \frac{\epsilon^2}{(N-1)} \tag{2.88}$$

in exact agreement with the prediction of the large N expansion.

2.2.4 Mixed $\sigma\phi$ theory in $d = 4 - \epsilon$

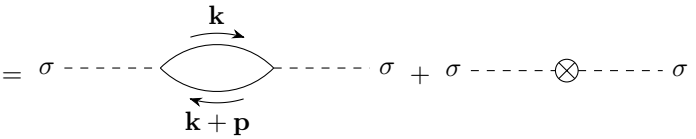
The large N analysis described in subsection 2.2.2 applies for general d, and in particular it can be formally pushed to $d > 2$. In $d = 2+\epsilon$, one finds formal UV fixed points of the quartic model (2.32). The fact that at large N the dimension of σ is near 1 suggests that it becomes a free propagating field in $d = 4$ boundary dimensions. Then, in close analogy with the situation for local $O(N)$ models [48], one expects that a UV completion of the formal UV fixed point of the quartic model in $d > 2$

is provided by the following model

$$S = \int d^{d+1}x \frac{1}{2}(\partial_\mu \phi^I)^2 + \int d^d x \left(\frac{1}{2}(\partial\sigma)^2 + \frac{g_1}{2}\sigma\phi^I\phi^I + \frac{g_2}{4!}\sigma^4 \right). \tag{2.89}$$

where σ propagates only on the boundary. The couplings g_1 and g_2 are classically marginal in $d = 4$, and we can look for perturbative IR fixed points in $d = 4 - \epsilon$.

The leading correction to σ propagator is given by the one-loop diagram

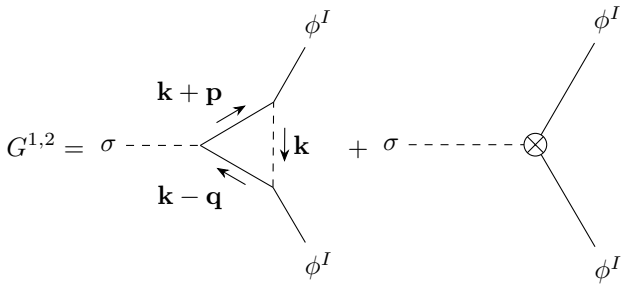
$G^{2,0} =$ 

$$\begin{aligned} &= \frac{N(-g_1)^2}{2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{p}+\mathbf{k}|\ k} - p^2\delta_\sigma \\ &= \frac{Ng_1^2\Gamma(\frac{2-d}{2})(p^2)^{\frac{d-2}{2}}\Gamma(\frac{d-1}{2})^2}{2(4\pi)^{\frac{d}{2}}\Gamma(d-1)\pi} - p^2\delta_\sigma. \end{aligned} \tag{2.90}$$

We then take a derivative with p^2 at $p^2 = \mu^2$ and set the divergent part to 0. This gives

$$\delta_\sigma = -\frac{Ng_1^2\Gamma(\frac{4-d}{2})\Gamma(\frac{d-1}{2})^2}{2(4\pi)^{\frac{d}{2}}\Gamma(d-1)\pi(\mu^2)^{\frac{4-d}{2}}} = -\frac{Ng_1^2}{8\epsilon(4\pi)^2}. \tag{2.91}$$

Next, we can compute the corrections to the vertex g_1

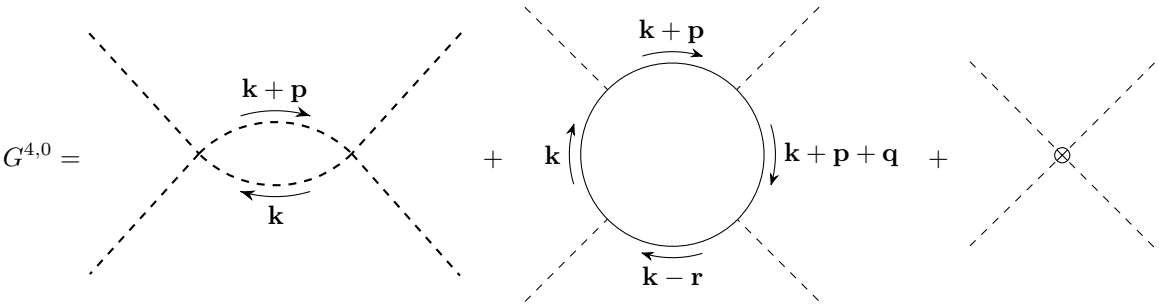
$G^{1,2} =$ 

$$\begin{aligned} &(-g_1)^3 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}-\mathbf{q}||\mathbf{k}+\mathbf{p}|k^2} - \delta_{g_1} \\ &= -g_1^3 \frac{\Gamma(\frac{4-d}{2})}{(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{4-d}{2}}} - \delta_{g_1} \end{aligned} \tag{2.92}$$

which in $d = 4 - \epsilon$ gives

$$\delta_{g_1} = -g_1^3 \frac{\Gamma(\frac{4-d}{2})}{(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{4-d}{2}}} = -\frac{g_1^3}{8\pi^2\epsilon}. \tag{2.93}$$

Similarly, the one loop correction to g_2 is given by the following diagrams (we are evaluating these at all external momenta $= \mu^2$)



$$\begin{aligned} &= \frac{3(-g_2)^2}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{k^2(\mathbf{k}+\mathbf{p})^2} + 3(-g_1)^4 N\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{k|\mathbf{k}-\mathbf{r}||\mathbf{k}+\mathbf{p}||\mathbf{k}+\mathbf{p}+\mathbf{q}|} - \delta_{g_2} \\ &= \frac{3g_2^2\Gamma(\frac{4-d}{2})}{2(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{4-d}{2}}} + \frac{3g_1^4 N\Gamma(\frac{4-d}{2})}{(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{4-d}{2}}} - \delta_{g_2} \end{aligned} \tag{2.94}$$

which implies

$$\delta_{g_2} = \frac{3g_2^2\Gamma(\frac{4-d}{2})}{2(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{4-d}{2}}} + \frac{3g_1^4 N\Gamma(\frac{4-d}{2})}{(4\pi)^{\frac{d}{2}}(\mu^2)^{\frac{4-d}{2}}} = \frac{3g_2^2 + 6g_1^4 N}{16\pi^2\epsilon}. \tag{2.95}$$

Using these counterterms, we can calculate the β function. The Callan-Symanzik equation for a correlation function with m external σ lines and n external ϕ lines is

$$(\mu\frac{\partial}{\partial\mu} + \beta_1\frac{\partial}{\partial g_1} + \beta_2\frac{\partial}{\partial g_2} + m\gamma_\sigma + n\gamma_\phi)G^{m,n} = 0. \tag{2.96}$$

Applying this to $G^{1,2}$ gives

$$\beta_1 = -\frac{\epsilon}{2}g_1 + \mu\frac{\partial}{\partial\mu}(-\delta_{g_1} + \frac{g_1}{2}(2\delta_\phi + \delta_\sigma)) = -\frac{\epsilon}{2}g_1 + \frac{(N-32)g_1^3}{16(4\pi)^2} \tag{2.97}$$

Applying Callan-Symanzik equation to $G^{4,0}$ gives

$$\beta_2 = -\epsilon g_2 + \mu\frac{\partial}{\partial\mu}(-\delta_{g_2} + \frac{g_2}{2}(4\delta_\sigma)) = -\epsilon g_2 + \frac{12g_2^2 + 24g_1^4 N + g_1^2 g_2 N}{4(4\pi)^2}. \tag{2.98}$$

It is possible to find two unitary fixed point at $N > N_{\text{crit}} = 4544$ with coupling constants given by

$$(g_1^*)^2 = \frac{8(4\pi)^2\epsilon}{N-32}, \quad (g_2^*)_\pm = \frac{12288N\pi^2\epsilon}{(N-32)(\pm\sqrt{1024 + N(N-4544)} - (N+32))}. \tag{2.99}$$

Since we find two fixed points here, we should look at their IR stability by looking at the eigenvalues

of the following matrix for the positive and negative sign root

$$M_{ij} = \frac{\partial \beta_i}{\partial g_j}, \quad M = \begin{bmatrix} \frac{-\epsilon}{2} + \frac{3(N-32)}{16(4\pi)^2}(g_1^*)^2 & 0 \\ \\ \frac{48N(g_1^*)^3 + g_1^* g_2^* N}{2(4\pi)^2} & -\epsilon + \frac{24 g_2^* + N(g_1^*)^2}{4(4\pi)^2} \end{bmatrix} \tag{2.100}$$

For IR stability, we want both the eigenvalues of this matrix to be positive, and that only happens when we choose the negative root $(g_2^*)_-$ (sign of g_1^* does not actually affect the eigenvalues). So the fixed point with $(g_2^*)_-$ is the IR stable fixed point and should be the one that matches the large N fixed point near four dimensions. Note that the value of g_2^* is negative for both the fixed points, indicating that this fixed point is non-perturbatively unstable, in the sense that the vacuum is not stable. For sufficiently large N, we may regard it as a metastable BCFT, similarly to the local $O(N)$ models in $4 < d < 6$ [50].

We can also compute the anomalous dimensions at the fixed point. The field ϕ does not get any anomalous dimensions, while the anomalous dimension of the field σ can be computed from δ_σ

$$\hat{\gamma}_\sigma = \frac{\mu}{2}\frac{\partial}{\partial \mu} \log Z_\sigma = \frac{\epsilon N}{2(N-32)} \tag{2.101}$$

which gives

$$\hat{\Delta}_\sigma = 1 + \frac{16\epsilon}{N-32} \tag{2.102}$$

in precise agreement with the large N prediction, expanded near $d = 4$.

The correction to bulk propagator of the field ϕ is given by

$$\langle \phi^I(-p, y_1)\phi^J(p, y_2)\rangle = \ \phi^I(y_1) \xrightarrow{\ \mathbf{p}\ } \phi^J(y_2) \ + \ \phi^I(y_1) \ \overset{\mathbf{k}+\mathbf{p}}{\overset{\curvearrowright}{\underset{\mathbf{k}}{\curvearrowleft}}} \ \phi^J(y_2) \tag{2.103}$$

$$= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{|p|} + \frac{\delta^{IJ} g_1^2 e^{-p(y_1+y_2)}}{|p|^2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{(\mathbf{p}+\mathbf{k})^2\ k}$$

$$= \frac{\delta^{IJ}(e^{-p|y_1-y_2|} + e^{-p(y_1+y_2)})}{|p|} + \frac{\delta^{IJ} g_1^2 e^{-p(y_1+y_2)} \Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)\Gamma(\frac{d-1}{2})(p^2)^{\frac{d-5}{2}}}{(4\pi)^{\frac{d}{2}}\sqrt{\pi}\Gamma(d-\frac{3}{2})}.$$

We can again Fourier transform back to position space to get

$$G_\phi^{IJ}(x_1, x_2) = \delta^{IJ} G_\phi^0(x_1, x_2) - \frac{8\delta^{IJ}\epsilon}{3\pi^2(N-32)((\mathbf{x}_1 - \mathbf{x}_2)^2 + (y_1+y_2)^2)^{\frac{3}{2}}}. \tag{2.104}$$

At large N, this agrees with the result obtained from large N expansion expanded in $d = 4 - \epsilon$.

Boundary instanton

The mixed $\sigma\phi$ theory described in eq. (2.89) can be written on the boundary as

$$S_{\text{bdry}} = \frac{2\Gamma(\frac{d+1}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{1}{2})}\int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|\mathbf{x}-\mathbf{y}|^{d+1}} + \int d^d x \left(\frac{1}{2}(\partial\sigma)^2 + \frac{g_1}{2}\sigma\phi^I\phi^I + \frac{g_2}{4!}\sigma^4\right) \tag{2.105}$$

Since the coupling g_2 is negative at the fixed point, the vacuum $\sigma = \phi^I = 0$ can only be metastable and must tunnel to large absolute values of σ. Indeed for negative g_2, there is a real instanton solution responsible for this tunneling found in [77, 78, 79] in the context of usual ϕ^4 interaction in four dimensions

$$\phi^I = 0, \quad \sigma = \sqrt{\frac{-48}{g_2}}\frac{\lambda}{1+\lambda^2(\mathbf{x}-\mathbf{a})^2}. \tag{2.106}$$

This instanton solution is expected to give non-perturbatively small imaginary parts to critical exponents [80]. This is because the fluctuations of σ about the instanton background include a negative mode which yields an imaginary contribution to the partition function.

We can perform a conformal mapping of the boundary to S^4, which will result in a σ^2 conformal coupling term in the action, and the solution just changes by a Weyl factor

$$\sigma = \sqrt{\frac{-12}{g_2}}\frac{\lambda(1+\mathbf{x}^2)}{1+\lambda^2(\mathbf{x}-\mathbf{a})^2}. \tag{2.107}$$

For $\lambda = 1$ and $\mathbf{a} = 0$, it just becomes a constant VEV on the sphere, and the action evaluated on the solution turns out to be

$$S^{\text{inst}}_{\text{bdry}} = -\frac{16\pi^2}{g_2}. \tag{2.108}$$

This can be evaluated at the fixed point and then we can take the large N limit to compare with the result from large N calculation

$$S^{\text{inst}}_{\text{bdry}} = -\frac{16\pi^2}{(g_2^*)_-} = \frac{(N-32)(\sqrt{1024+N(N-4544)}+(N+32))}{768N\epsilon} + O(\epsilon^0) \overset{N\gg 1}{\approx} \frac{N}{384\epsilon}. \tag{2.109}$$

The same result can be derived in the large N theory by writing eq. (2.48) as an action on the boundary

$$S = \frac{2\Gamma(\frac{d+1}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{1}{2})}\int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|\mathbf{x}-\mathbf{y}|^{d+1}} + \int d^d x \frac{\sigma\phi^I\phi^I}{2}. \tag{2.110}$$

We can conformally map it to a sphere

$$S = \frac{2\Gamma(\frac{d+1}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{1}{2})}\int d^dx d^dy\sqrt{g(x)}\sqrt{g(y)}\frac{\phi^I(x)\phi^I(y)}{s(x,y)^{d+1}} + \int d^dx\sqrt{g(x)}\frac{\sigma\phi^I\phi^I}{2} \tag{2.111}$$

We will again look for the classical solution with a constant σ on the sphere and compute the instanton action by integrating out ϕ^I

$$S^{\text{inst}}_{\text{bdry}}(\sigma) = \frac{N}{2}\log\det\left(\frac{2\Gamma(\frac{d+1}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{1}{2})}\frac{1}{s(x,y)^{d+1}} + \frac{\sigma}{2}\delta(\mathbf{x}-\mathbf{y})\right). \tag{2.112}$$

In general, the chordal distance on the sphere can be decomposed into spherical harmonics as follows [50]

$$\frac{1}{s(x,y)^{2\Delta}} = \sum_{n=0}^{\infty} k_n(\Delta)Y^*_{n,\vec{m}}(x)Y_{n,\vec{m}}(y), \quad k_n(\Delta) = \pi^{d/2}2^{d-2\Delta}\frac{\Gamma(\frac{d}{2}-\Delta)\Gamma(n+\Delta)}{\Gamma(\Delta)\Gamma(d+n-\Delta)} \tag{2.113}$$

These spherical harmonics form a complete set of eigenfunctions with the following eigenvalue equation

$$\int d^dy\sqrt{g(y)}\frac{1}{s(x,y)^{2\Delta}}Y_{n,\vec{m}}(y) = k_n(\Delta)Y_{n,\vec{m}}(x). \tag{2.114}$$

Using this, the required determinant becomes

$$S^{\text{inst}}_{\text{bdry}}(\sigma) = \frac{N}{2}\sum_n D_n\log\left(\frac{2\Gamma(\frac{d+1}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{1}{2})}k_n\left(\frac{d+1}{2}\right) + \frac{\sigma}{2}\right), \quad D_n = \frac{(2n+d-1)\Gamma(n+d-1)}{n!\Gamma(d)} \tag{2.115}$$

where D_n is the degeneracy of the eigenvalue k_n with all the degenerate states labeled by $\vec{m}$ above. The constant value of σ which extremizes this action can be found by solving

$$\frac{\partial S^{\text{inst}}_{\text{bdry}}}{\partial\sigma} = 0 = \frac{N}{4}\sum_n\frac{D_n}{\frac{\Gamma(n+(d+1)/2)}{\Gamma(n+(d-1)/2)}+\frac{\sigma}{2}} = \frac{N\sigma\Gamma(1-d)\Gamma(\frac{d-1+\sigma}{2})}{4\Gamma(\frac{3-d+\sigma}{2})}. \tag{2.116}$$

So apart from the usual vacuum $\sigma = 0$, we also have other saddles

$$\sigma = d-3-2n \tag{2.117}$$

for positive integer n. The saddle point value of σ is effectively the mass of field ϕ^I at large N. We want it to be positive for stability of $\phi^I = 0$ vacuum. Hence for $d < 3$, $\sigma = 0$ is the only allowed saddle, while for $3 < d < 4$, the $n = 0$ saddle in eq. 2.117 is also allowed. So we expect the $n = 0$ instanton configuration to match the classical solution found above in $4-\epsilon$ dimensions. Instanton

action for this configuration is

$$S^{\text{inst}}_{\text{bdry}}(\sigma) - S_{\text{bdry}}(0) = \int_0^{d-3} d\sigma \frac{\partial S^{\text{inst}}_{\text{bdry}}}{\partial \sigma}. \tag{2.118}$$

This clearly vanishes in $d = 3$. We can perform this integral in $d = 4 - \epsilon$ and compare with the result of the ϵ expansion in the previous section. We find

$$S^{\text{inst}}_{\text{bdry}}(\sigma) - S_{\text{bdry}}(0) = \frac{N}{384\epsilon} + O(\epsilon^0) \tag{2.119}$$

which precisely matches the ϵ expansion result (2.109).

2.3 Higher-spin displacement operators

As discussed in section 2.1.1, a spin s conserved current in the bulk induces a tower of protected operators on the boundary with dimension $d+1+s-2$ and spin ranging between 0 and $s-2$. They are bilinears in the boundary operator ϕ and have the schematic form $\sim \phi \vec{\partial}^{2n} \partial_{\nu_1} \partial_{\nu_2} .. \partial_{\nu_l} \phi$ with $n \geq 1$. They appear in the conformal block decomposition of the four point function of the boundary field ϕ. The scalar ones with boundary spin 0 also appear in the boundary channel conformal block decomposition of two point function of the bulk scalar $\phi^I \phi^I$. In the following subsections, we will see that these operators have protected dimensions in perturbation theory using their appearance in both these conformal block decompositions. Then we will go on to calculate the anomalous dimensions of the first few of these operators using Feynman diagrams and verify that they vanish.

2.3.1 ϕ^4 theory in $d = 2 - \epsilon$

Decomposition of boundary four-point function

Let us compute the four-point function of the leading boundary operator ϕ^I in the quartic theory of subsection 2.2.1. In the free theory, the four-point function just comes from the Wick contractions

$$\langle \phi^I(\mathbf{x}_1)\phi^J(\mathbf{x}_2)\phi^K(\mathbf{x}_3)\phi^L(\mathbf{x}_4)\rangle_0 = \hat{C}^2_{\phi\phi}\left(\frac{\delta^{IJ}\delta^{KL}}{(\mathbf{x}^2_{12})^{\hat{\Delta}}(\mathbf{x}^2_{34})^{\hat{\Delta}}} + \frac{\delta^{IK}\delta^{JL}}{(\mathbf{x}^2_{13})^{\hat{\Delta}}(\mathbf{x}^2_{24})^{\hat{\Delta}}} + \frac{\delta^{IL}\delta^{JK}}{(\mathbf{x}^2_{14})^{\hat{\Delta}}(\mathbf{x}^2_{23})^{\hat{\Delta}}}\right). \tag{2.120}$$

In the s-channel, $12 \to 34$, the leading term just comes from the identity operator, while the other two come from the double trace operators of dimensions $2\hat{\Delta} + 2n + l$ [81]

$$\frac{1}{(\mathbf{x}_{13}^2)^{\hat{\Delta}}(\mathbf{x}_{24}^2)^{\hat{\Delta}}} = \frac{(-1)^l}{(\mathbf{x}_{14}^2)^{\hat{\Delta}}(\mathbf{x}_{23}^2)^{\hat{\Delta}}} = \frac{1}{(\mathbf{x}_{12}^2)^{\hat{\Delta}}(\mathbf{x}_{34}^2)^{\hat{\Delta}}} \sum_{l,n} a_{\tau=2\hat{\Delta}+2n,l} u^{\hat{\Delta}+n} g_{\tau=2\hat{\Delta}+2n,l}(u,v) \tag{2.121}$$

where

$$a_{\tau=2\hat{\Delta}+2n,l} = \frac{(-1)^l[(\hat{\Delta} - \frac{d}{2} + 1)_n(\hat{\Delta})_{l+n}]^2}{l!n!(l + \frac{d}{2})_n(2\hat{\Delta} + n - d + 1)_n(2\hat{\Delta} + 2n + l - 1)_l(2\hat{\Delta} + n + l - \frac{d}{2})_n} \tag{2.122}$$

and

$$u = \frac{\mathbf{x}_{12}^2\mathbf{x}_{34}^2}{\mathbf{x}_{13}^2\mathbf{x}_{24}^2}, \quad v = \frac{\mathbf{x}_{14}^2\mathbf{x}_{23}^2}{\mathbf{x}_{13}^2\mathbf{x}_{24}^2}. \tag{2.123}$$

In our case $\hat{\Delta} = \frac{d-1}{2}$ and $g_{\tau,l}(u,v)$ is the d dimensional conformal block for four-point function. At first order in the coupling, we have the following connected contribution to the four-point function

$$\langle \phi^I(\mathbf{x}_1)\phi^J(\mathbf{x}_2)\phi^K(\mathbf{x}_3)\phi^L(\mathbf{x}_4)\rangle_1 = -2g(\delta^{IJ}\delta^{KL}+\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK}) \int d^d\mathbf{x}_0 \frac{\hat{C}_{\phi\phi}^4}{(\mathbf{x}_{10}^2)^{\hat{\Delta}}(\mathbf{x}_{20}^2)^{\hat{\Delta}}(\mathbf{x}_{30}^2)^{\hat{\Delta}}(\mathbf{x}_{40}^2)^{\hat{\Delta}}}. \tag{2.124}$$

To make life simpler, we are going to evaluate this integral in $d = 2$ so that $\hat{\Delta} = 1/2$. In that case, the integral can be computed in terms of the $\bar{D}$ function

$$\langle \phi^I(\mathbf{x}_1)\phi^J(\mathbf{x}_2)\phi^K(\mathbf{x}_3)\phi^L(\mathbf{x}_4)\rangle_1 = -\frac{2g}{\pi}(\delta^{IJ}\delta^{KL} + \delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}) \frac{\hat{C}_{\phi\phi}^4}{(\mathbf{x}_{12}^2\mathbf{x}_{34}^2)^{\frac{1}{2}}} u^{\frac{1}{2}} \bar{D}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(u,v) \tag{2.125}$$

This particular $\bar{D}$ function can be expressed in terms of the H function, which can then be expanded in a power series in u and $1 - v$ [82, 83, 84]

$$\begin{aligned} \bar{D}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(u,v) &= -\pi^2 \log u \; G(\frac{1}{2},\frac{1}{2},1,1;u,1-v) + \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+m)^2\Gamma(\frac{1}{2}+m+n)^2}{(m!)^2\; n!\; \Gamma(1+2m+n)} f_{mn} u^m (1-v)^n, \\ f_{mn} &= 2\psi(1+m) + 2\psi(1+2m+n) - 2\psi(\frac{1}{2}+m) - 2\psi(\frac{1}{2}+m+n). \end{aligned} \tag{2.126}$$

The G function appearing above can also be expanded in to powers

$$G(\alpha,\beta,\gamma,\delta;u,1-v) = \sum_{m,n=0}^{\infty} \frac{(\delta-\alpha)_m(\delta-\beta)_m}{m!(\gamma)_m} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{n!(\delta)_{2m+n}} u^m (1-v)^n \tag{2.127}$$

and in particular,

$$\begin{aligned} G(\frac{1}{2},\frac{1}{2},1,1;u,1-v) &= \sum_{m,n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+m)^2\Gamma(\frac{1}{2}+m+n)^2}{\pi^2(m!)^2\ n!\ \Gamma(1+2m+n)} u^m (1-v)^n \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}+m)^4}{\pi^2(m!)^2\ \Gamma(1+2m)} u^m\ {}_2F_1(\frac{1}{2}+m,\frac{1}{2}+m,1+2m,1-v). \end{aligned} \tag{2.128}$$

The $\log u$ term appearing above in the four-point function directly gives the anomalous dimensions as we now discuss. On general grounds, we can decompose the four point function as follows

$$\begin{aligned} \langle \phi^I(\mathbf{x}_1)\phi^J(\mathbf{x}_2)\phi^K(\mathbf{x}_3)\phi^L(\mathbf{x}_4)\rangle = \delta^{IJ}\delta^{KL}\mathcal{G}_S + \left(\frac{\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK}}{2} - \frac{\delta^{IJ}\delta^{KL}}{N}\right)\mathcal{G}_T \\ + \frac{\delta^{IK}\delta^{JL}-\delta^{IL}\delta^{JK}}{2}\mathcal{G}_A \end{aligned} \tag{2.129}$$

where S, T, A refer to singlet, traceless symmetric and anti-symmetric representations of $O(N)$. For each of these representations, we can have a decomposition into conformal blocks

$$\mathcal{G} = \frac{\hat{C}^2_{\phi\phi}}{(\mathbf{x}_{12}^2\mathbf{x}_{34}^2)^{\frac{1}{2}}}\mathcal{F}(u,v), \quad \mathcal{F}(u,v) = \sum_{\tau,l} a_{\tau,l} u^{\frac{\tau}{2}} g_{\tau,l}(u,v). \tag{2.130}$$

From our discussion above, we have

$$\begin{aligned} \mathcal{F}_S(u,v) &= 1 + \sum_{\substack{l,n \\ l:\text{even}}} a^0_{S\ n,l} u^{\frac{1}{2}+n} g_{\tau^0_n,l} - \frac{2g(N+2)\hat{C}^2_{\phi\phi}}{\pi N} u^{\frac{1}{2}} \bar{D}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(u,v) \\ \mathcal{F}_T(u,v) &= \sum_{\substack{l,n \\ l:\text{even}}} a^0_{T\ n,l} u^{\frac{1}{2}+n} g_{\tau^0_n,l} - \frac{4g\hat{C}^2_{\phi\phi}}{\pi} u^{\frac{1}{2}} \bar{D}_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}(u,v) \\ \mathcal{F}_A(u,v) &= \sum_{\substack{l,n \\ l:\text{odd}}} a^0_{A\ n,l} u^{\frac{1}{2}+n} g_{\tau^0_n,l} \end{aligned} \tag{2.131}$$

where $\tau^0_n = 1+2n$ and

$$a^0_{S\ n,l} = \frac{1}{N} a^0_{T\ n,l} = \frac{1}{N} a^0_{A\ n,l} = \frac{2}{N} \frac{(-1)^l[(\frac{1}{2})_n(\frac{1}{2})_{l+n}]^2}{l!n!(l+1)_n(n)_n(2n+l)_l(n+l)_n}. \tag{2.132}$$

Leading corrections to $\mathcal{F}$ can also be expressed in terms of anomalous dimensions and corrections to OPE coefficients: using $\tau_{n,l} = \tau^0_n + \hat{\gamma}_{n,l}$ and $a_{n,l} = a^0_{n,l} + \delta a_{n,l}$ we have

$$\delta\mathcal{F}(u,v) = u^{\frac{1}{2}} \sum_{n=0}^{\infty} u^n \sum_{l:\text{even}} \left(\frac{1}{2} a^0_{n,l}\hat{\gamma}_{n,l}\log u + \delta a_{n,l} + \frac{1}{2} a^0_{n,l}\hat{\gamma}_{n,l}\partial_n\right) g_{\tau^0_n,l}(u,v). \tag{2.133}$$

It is clear that the operators in the anti-symmetric representation do not get anomalous dimension or corrections to OPE coefficient to leading order in g. For the singlet representation, comparing the terms proportional to $\log u$, we have the following equation which implicitly determines the anomalous dimensions

$$\sum_{\substack{l,n=0\\ l:\text{even}}}^{\infty} u^n \frac{1}{2} a^0_{S\ n,l} \hat{\gamma}^S_{n,l} g_{\tau^0_n,l}(u,v) = \frac{2\pi g(N+2)\hat{C}^2_{\phi\phi}}{N} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}+m)^4}{\pi^2 (m!)^2\ \Gamma(1+2m)} u^m \times {}_2F_1(\frac{1}{2}+m, \frac{1}{2}+m, 1+2m, 1-v). \tag{2.134}$$

A similar equation can be obtained for symmetric traceless case. For small values of u, in two dimensions and for even spins, the conformal block on the LHS has the following expansion [82] to leading order in u

$$g_{\tau^0_n,l}(u,v) = (1-v)^l {}_2F_1(\frac{1}{2}+n+l, \frac{1}{2}+n+l, 1+2n+2l, 1-v) + O(u). \tag{2.135}$$

Also, for $l=0$, we have the following expansion to all orders in u

$$g_{\tau^0_n,l=0} = \sum_{m=0}^{\infty} u^m \frac{\Gamma(\frac{1}{2}+m+n)^4 \Gamma(1+2n)^2}{\Gamma(\frac{1}{2}+n)^4\ m!(m+2n)!(2m+2n)!}\ {}_2F_1(\frac{1}{2}+m+n, \frac{1}{2}+m+n, 1+2m+2n, 1-v). \tag{2.136}$$

We can use these expansions to compare coefficients of different powers of u in eq. (2.134). At zeroth order in u, this implies

$$\sum_{l:\text{even}} \frac{1}{2} a^0_{S\ 0,l} \hat{\gamma}^S_{0,l} x^l F_{\frac{1}{2}+l}(x) = \frac{2g\pi(N+2)\hat{C}^2_{\phi\phi}}{N} F_{\frac{1}{2}}(x) \tag{2.137}$$

where $F_\beta(x)$ is defined by

$$F_\beta(x) \equiv {}_2F_1(\beta,\beta,2\beta,x), \quad x \equiv 1-v \tag{2.138}$$

and it obeys an orthogonality relation

$$\frac{1}{2\pi i} \oint_{x=0} x^{\beta-\beta'-1} F_\beta(x) F_{1-\beta'}(x) = \delta_{\beta,\beta'}. \tag{2.139}$$

Using this and $\hat{C}_{\phi\phi} = 1/2\pi$, we get

$$\hat{\gamma}^S_{0,l} = \delta_{0l} \frac{g(N+2)}{2\pi}. \tag{2.140}$$

For $l = 0$, it agrees with the anomalous dimension of the boundary operator ϕ^2 found in eq. 2.42. It vanishes for all other spins, which is perhaps not so surprising given that in the usual $O(N)$ model, the anomalous dimensions of leading twist bilinear operators (weakly broken higher spin currents) start at $O(\epsilon^2)$ in $4-\epsilon$ dimensions. Similarly for the symmetric traceless case

$$\hat{\gamma}^T_{0,l} = \delta_{0l}\frac{g}{\pi}. \tag{2.141}$$

At next order in u, equation (2.134) implies

$$\sum_{l:\text{even}} \frac{1}{2} a^0_{S\ 1,l}\hat{\gamma}^S_{1,l} x^l F_{\frac{3}{2}+l}(x) + \frac{1}{64} a^0_{S\ 0,0}\hat{\gamma}^S_{0,0} F_{\frac{3}{2}}(x) = \frac{2g\pi(N+2)\hat{C}^2_{\phi\phi}}{32N} F_{\frac{3}{2}}(x) \tag{2.142}$$

which just gives

$$\sum_{l:\text{even}} \frac{1}{2} a^0_{S\ 1,l}\hat{\gamma}^S_{1,l} x^l F_{\frac{3}{2}+l}(x) = 0 \tag{2.143}$$

which implies

$$\hat{\gamma}^S_{1,l} = 0 \tag{2.144}$$

for all values of l. For $l = 0$, this is just the displacement operator. We could use this result to go to next subleading twist and so on, since we know the conformal block for $l = 0$ to all orders in u. In general, it follows that if the anomalous dimensions of operators with all spins vanish from level 1 through level $n-1$, then at level n, we have the following equation

$$\begin{aligned} &\sum_{l:\text{even}} \frac{1}{2} a^0_{S\ n,l}\hat{\gamma}^S_{n,l} x^l F_{n+\frac{1}{2}+l}(x) + \frac{1}{2} a^0_{S\ 0,0}\hat{\gamma}^S_{0,0} \frac{\Gamma(\frac{1}{2}+n)^4}{\pi^2(\ n!)^2(2n)!} F_{n+\frac{1}{2}}(x) \\ &= \frac{2g\pi(N+2)\hat{C}^2_{\phi\phi}}{N} \frac{\Gamma(\frac{1}{2}+n)^4}{\pi^2(\ n!)^2(2n)!} F_{n+\frac{1}{2}}(x) \end{aligned} \tag{2.145}$$

which gives

$$\hat{\gamma}^S_{n,l} = 0. \tag{2.146}$$

In this way we can extend this result to all values of twist. Note that it was important that the leading twist anomalous dimensions vanish for all spins other than $l = 0$. These subleading twist operators with free dimension $d-1+2n+l, n \geq 1$ and spin l are exactly the operators we called higher-spin "cousins" of displacement and we have just shown that their anomalous dimension vanishes to leading order in g. Similar reasoning goes through for the symmetric traceless case.

Decomposition of bulk two point function

Let us now discuss the conformal block decomposition of the bulk two-point function of the $\phi^I\phi^I$ operator. In the case of free theory, using the cross-ratio z defined in section 2.1, we can write

$$\begin{aligned}\langle\phi^I\phi^I(\mathbf{x}_1,y_1)\phi^J\phi^J(\mathbf{x}_2,y_2)\rangle_0 &= N^2(G^0(0,0))^2+2N(G^0_\phi(x_1,x_2))^2\\ &= \frac{N\Gamma(\frac{d-1}{2})^2}{16\pi^{d+1}(4y_1y_2)^{d-1}}\left[N+2\left(\frac{z}{1-z}\right)^{d-1}+2\,z^{d-1}+4\,\frac{z^{d-1}}{(1-z)^{\frac{d-1}{2}}}\right]\\ &= \frac{N\Gamma(\frac{d-1}{2})^2}{16\pi^{d+1}(4y_1y_2)^{d-1}}\mathcal{G}(z).\end{aligned} \tag{2.147}$$

We can determine the coefficients of the blocks using Euclidean inversion formulae [16, 85]. On the boundary, we can define the coefficient function

$$\hat{I}_{\hat{\Delta}}=\frac{1}{\Gamma(\frac{d+1}{2})}\int_0^1 dz\; z^{-(d+1)}\;(1-z)^{\frac{d-1}{2}}\;{}_2F_1\left(\hat{\Delta},d-\hat{\Delta};\frac{d+1}{2};\frac{z-1}{z}\right)\mathcal{G}(z) \tag{2.148}$$

and its residues are related to the coefficients of conformal block expansion as

$$\hat{I}_{\hat{\Delta}}\;\frac{\Gamma(\hat{\Delta})\Gamma(\hat{\Delta}+\frac{1-d}{2})}{2\Gamma(2\hat{\Delta}-d)}\sim\;-\frac{\mu^2_{\hat{O}}}{\hat{\Delta}-\hat{\Delta}_{\hat{O}}}. \tag{2.149}$$

Doing this procedure tells us that we have the identity block on the boundary, with coefficient $\mu_0^2=N$, and a tower of blocks with dimensions $d-1+2n$ and coefficients

$$\begin{aligned}\mu^2_{d-1+2n}=\frac{2}{\Gamma(2n+1)}\Bigg[2\;\delta_{n,0}\;+\;&\frac{\Gamma\left(\frac{3-d}{2}\right)\;{}_2F_1(1-2n,-2n;-d-4n+3;1)}{\Gamma\left(\frac{3-d}{2}-2n\right)}+\Gamma\left(\frac{d+1}{2}\right)\\ &\times\Gamma(-d-4n+3)\,{}_3\tilde{F}_2\left(\frac{3-d}{2}-2n,-2n,1-2n;-d-4n+3,\frac{d+1}{2}-2n;1\right)\Bigg]\end{aligned} \tag{2.150}$$

where regularized Hypergeometric function is defined by

$${}_3\tilde{F}_2(a1,a2,a2;b1,b2;z)=\frac{{}_3F_2(a1,a2,a2;b1,b2;z)}{\Gamma(b1)\Gamma(b2)}. \tag{2.151}$$

Similarly in the bulk, we have the coefficient function

$$I_\Delta=\int_0^1 dy\; y^{\frac{d-5}{2}}(1-y)^{-d+1}{}_2F_1\left(\frac{\Delta}{2},\frac{d+1-\Delta}{2},1,1-\frac{1}{y}\right)\mathcal{G}(1-y) \tag{2.152}$$

and then the bulk data is determined using

$$I_\Delta \ \frac{\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta+1-d}{2})}{2\Gamma(\Delta-\frac{d+1}{2})} \sim \ -\frac{\lambda_O}{\Delta-\Delta_O}. \tag{2.153}$$

Using this, it can be seen that in the bulk channel, the two-point function contains identity, ϕ^2 (with dimension $d-1$), and a tower of primaries $\phi^2\partial^{2n}\phi^2$ with dimensions $2d-2+2n$ with following OPE coefficients

$$\lambda_0 = 2, \quad \lambda_{d-1} = 4,$$

$$\begin{aligned}\lambda_{2d-2+2n} = (-1)^n N\Gamma(1-d)\Bigg(&\frac{\pi(-2d^2-3dn+4d-2n^2+5n-2)\sec(\frac{3\pi d}{2})\Gamma(1-d)}{2\ \Gamma(-d-n+2)^2\ \Gamma(n+1)\Gamma(-\frac{3d}{2}-n+\frac{7}{2})\Gamma(\frac{3d}{2}+2n-\frac{5}{2})}\\ &+\frac{{}_2F_1(1-n,\frac{-d-2n+3}{2};\frac{-3d-4n+7}{2};1)}{\Gamma(-d-n+2)\Gamma(n)}\Bigg)\\ &-\frac{(2\pi(-1)^n\sec(\frac{3\pi d}{2})\,{}_3\tilde{F}_2(-d-n+2,\frac{-d-2n+3}{2},-n;\frac{-3d-4n+7}{2},1-n;1)}{\Gamma(\frac{1}{2}(3d-5)+2n)\ \Gamma(n+1)}.\end{aligned} \tag{2.154}$$

When we add boundary interactions to the theory, the dimensions of the operators in the bulk channel will remain the same since the theory is free in the bulk, but the OPE coefficients λ_O can receive corrections which will depend on the interaction strength.

Note that the operators appearing in the boundary channel are scalars with dimensions $d-1+2n$. We will now show by an explicit perturbative calculation in the interacting theory, that for $n \geq 1$, they don't acquire anomalous dimensions, which is consistent with the fact that they are induced by bulk conserved higher spin currents. At leading order, we have

$$\langle\phi^I\phi^I(\mathbf{x}_1,y_1)\phi^J\phi^J(\mathbf{x}_2,y_2)\rangle_1 = -2gN(N+2)\int d^d\mathbf{x}_0(G^0_\phi(\mathbf{x}_1,y_1;\mathbf{x}_0,0))^2(G^0_\phi(\mathbf{x}_0,0;\mathbf{x}_2,y_2))^2. \tag{2.155}$$

This requires computing the following integral, which can be done, for example, by using Feynman parameters

$$\int d^d\mathbf{x}_0\frac{1}{(\mathbf{x}_{10}^2+y_1^2)^{d-1}(\mathbf{x}_{20}^2+y_2^2)^{d-1}} = \frac{\pi z}{2y_1y_2\sqrt{1-z}}\tanh^{-1}\left(\frac{2\sqrt{1-z}}{2-z}\right) \tag{2.156}$$

where we already set $d=2$ for the integral since we are computing the leading correction in $d=2-\epsilon$.

This gives the two point function as

$$\langle\phi^I\phi^I(\mathbf{x}_1,y_1)\phi^J\phi^J(\mathbf{x}_2,y_2)\rangle = \frac{N\Gamma(\frac{d-1}{2})^2}{16\pi^{d+1}(4y_1y_2)^{d-1}}\left[N+2\left(\frac{z}{1-z}\right)^{d-1}+2\,z^{d-1}+4\,\frac{z^{d-1}}{(1-z)^{\frac{d-1}{2}}}\right] - \frac{gN(N+2)z}{16\pi^3 y_1y_2\sqrt{1-z}}\tanh^{-1}\left(\frac{2\sqrt{1-z}}{2-z}\right). \tag{2.157}$$

We can compute the anomalous dimensions of the operators appearing in boundary channel decomposition by extracting $\log z$ from our two point function. In the boundary channel, $\log z$ comes from the $z^{\hat{\Delta}}$ present in the boundary conformal block. So in the following, we only keep track of the $\epsilon\log z$ term of the leading order perturbation to the free propagator. Then using the decomposition from above, we have at the fixed point

$$\langle\phi^I\phi^I(\mathbf{x}_1,y_1)\phi^J\phi^J(\mathbf{x}_2,y_2)\rangle \ni \frac{N\Gamma(\frac{d-1}{2})^2}{16\pi^{d+1}(4y_1y_2)^{d-1}}\left[N+\sum_{n=0}^{\infty}\mu^2_{d-1+2n}f_{\text{bdry}}(d-1+2n;z)\right] + \frac{N}{64\pi^2y_1y_2}\left(8\log z\frac{N+2}{N+8}\epsilon\right) \tag{2.158}$$

where there will be other order ϵ terms which will contribute to the corrections to OPE coefficients, but we have only kept $\log z$ terms. Noting that the boundary block for $\hat{\Delta}=d-1$ simplifies, this again precisely gives the value of anomalous dimension of the boundary operator ϕ^2 found in (2.42) and tells us that none of the other operators get anomalous dimensions. This is consistent since the operators with $n\geq 1$ correspond to higher spin displacements with boundary spin 0 and are equal to the boundary value of conserved currents with all $2n$ indices being y, $J^{yy\cdots y}$.

Direct Computation

It is possible to compute these anomalous dimensions more directly as well, by explicitly writing down the operator induced by conserved currents on the boundary and computing their anomalous dimensions. For the displacement, the operator is

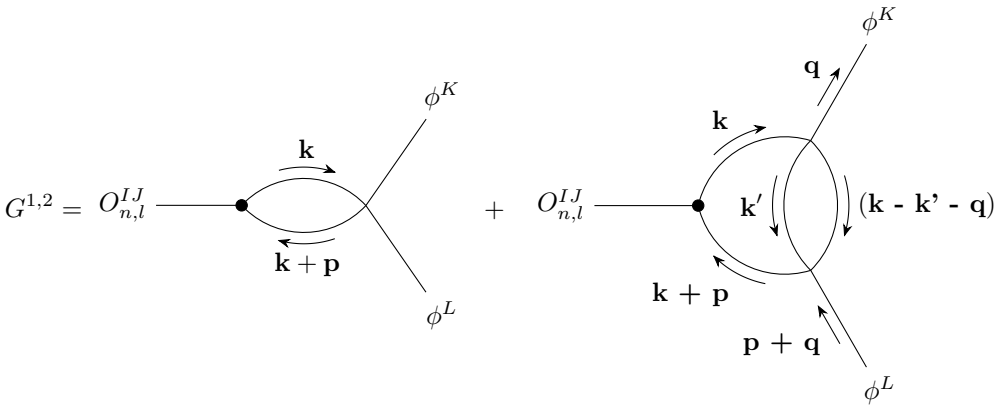
$$\begin{aligned} D=T_{yy} &= \frac{d-1}{4d}(\phi^I(\partial_i^2\phi^I)+\phi^I(\partial_i^2\phi^I))-\frac{1}{2d}\partial_i\phi^I\partial^i\phi^I+\frac{1}{2}\partial_y\phi^I\partial_y\phi^I \\ &= \frac{d-1}{4d}(\phi^I(\partial_i^2\phi^I)+\phi^I(\partial_i^2\phi^I))-\frac{1}{2d}\partial_i\phi^I\partial^i\phi^I+\frac{g^2}{2}((\phi^K\phi^K))\phi^I(\phi^L\phi^L)\phi^I \end{aligned} \tag{2.159}$$

where we used modified Neumann boundary condition $\partial_y\phi^I = g(\phi^J\phi^J)\phi^I$. We will calculate its anomalous dimension to order g^2. To this order, the last term in the above expression will not contribute and it will start contributing at order g^3. This is actually a primary operator in the

boundary theory as it matches up to a coefficient to a "double trace"[4] operator. We will denote by $O^{IJ}_{n,l}$, the operator with dimensions $2\hat{\Delta}+2n+l$ and spin l. For $n=1$ and $l=0$, the "double trace" primary operator takes the form [86]

$$O^{IJ}_{1,0} = \frac{d-1}{2}((\partial_i^2\phi^I)\phi^J + \phi^I(\partial_i^2\phi^J)) - \partial_i\phi^I\partial^i\phi^J. \tag{2.160}$$

We want to show that the anomalous dimension of this operator vanishes by computing its three point function with two other ϕ. To two loop order, following are the non trivial diagrams that will contribute, and we want to show that these do not have any logarithmic divergence.



$$\begin{aligned} &= 2g(\delta^{IJ}\delta^{KL} + \delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|} \tilde{O}_{1,0}(\mathbf{k},\mathbf{p}) \\ &+ g^2(8\delta^{IJ}\delta^{KL} + 2(N+6)(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK})\mathcal{I}_1 \end{aligned} \tag{2.161}$$

where

$$\tilde{O}_{1,0}(\mathbf{k},\mathbf{p}) = \frac{d}{2}(\mathbf{k}^2 + (\mathbf{k}+\mathbf{p})^2) - \frac{\mathbf{p}^2}{2}. \tag{2.162}$$

It is easy to see that the first one loop diagram vanishes identically, which is why we do not need to consider other two loop diagrams which contain this diagram as a subdiagram. Now for the second

[4]The operators we discuss here are bilinears in the fundamental fields ϕ^I and hence should be thought of as single trace operators. However, we will sometimes loosely use the terminology "double trace" to make contact with some of the literature on the subject.

two loop diagram, we have to perform the integral

$$\begin{aligned}\mathcal{I}_1 &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}||\mathbf{k}'||\mathbf{k}-\mathbf{k}'-\mathbf{q}|}\tilde{O}_{1,0}(\mathbf{k},\mathbf{p}) \\ &= \frac{\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\pi\Gamma(d-1)}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}||\mathbf{q}-\mathbf{k}|^{2-d}}\tilde{O}_{1,0}(\mathbf{k},\mathbf{p}) \\ &= -\frac{2^{1-d}\sec\left(\frac{\pi d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(d-\frac{3}{2}\right)\Gamma(\frac{d-1}{2})^2\Gamma(2-\frac{d}{2})}{\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)\Gamma\left(\frac{3d}{2}-1\right)(4\pi)^d\pi\Gamma(d-1)(p^2)^{1-d}}\end{aligned} \tag{2.163}$$

where we computed the integral at $\mathbf{q}=0$, since we are just using this diagram to calculate the anomalous dimension. This is finite in $d=2-\epsilon$ which implies that to this order, the operator $O^{IJ}_{1,0}$ does not get anomalous dimensions.

Let us now talk about the operators induced by the bulk spin 4 current on the boundary. If the bulk is 3 dimensional (which will be sufficient for our perturbative calculation), it can be explicitly constructed using the generating function

$$O^{IJ}(x,\epsilon)=\sum_{s=0}^{\infty}J^{IJ}_{\mu_1\ldots\mu_s}(x)\epsilon^{\mu_1}\cdots\epsilon^{\mu_s}. \tag{2.164}$$

This generating function can be calculated by using the conditions of current conservation and tracelessness and it turns out to be [87]

$$O^{IJ}(x,\epsilon)=\phi^I(x-\epsilon)\sum_{n=0}^{\infty}\frac{(2\epsilon^2\overleftarrow{\partial}_x\cdot\overrightarrow{\partial}_x-4(\epsilon\cdot\overleftarrow{\partial}_x)(\epsilon\cdot\overrightarrow{\partial}_x))^n}{(2n)!}\phi^J(x+\epsilon). \tag{2.165}$$

This can be expanded to fourth order in ϵ, which gives the spin 4 current

$$\begin{aligned}J^{IJ}_{\mu\nu\rho\sigma} = \frac{1}{4!}\Big[&\frac{1}{24}\partial_{(\mu}\partial_\nu\partial_\rho\partial_{\sigma)}\phi^I\phi^J-\frac{7}{6}\partial_{(\mu}\partial_\nu\partial_\rho\phi^I\partial_{\sigma)}\phi^J+\frac{1}{2}\delta_{(\mu\nu}\partial_\rho\partial_{\sigma)}\partial_\alpha\phi^I\partial^\alpha\phi^J+(I\leftrightarrow J)\\ &+\frac{1}{6}\delta_{(\mu\nu}\delta_{\rho\sigma)}\partial_\alpha\partial_\beta\phi^I\partial^\alpha\partial^\beta\phi^J-\frac{5}{3}\delta_{(\mu\nu}\partial_\alpha\partial_\rho\phi^I\partial^\alpha\partial_{\sigma)}\phi^J+\frac{35}{12}\partial_{(\mu}\partial_\nu\phi^I\partial_\rho\partial_{\sigma)}\phi^J\Big]\end{aligned} \tag{2.166}$$

where the symmetrization sign means that we add all the terms related by exchange of indices. Now, we can take all its components to be transverse to the boundary and obtain an operator on the boundary, which with Neumann boundary condition looks like

$$\begin{aligned}J^{IJ}_{yyyy} = \Big[&\frac{1}{24}((\partial_i^2)^2\phi^I)\phi^J-\frac{1}{2}(\partial_i^2\partial_j\phi^I)\partial^j\phi^J+(I\leftrightarrow J)\\ &+\frac{1}{6}(\partial_i\partial_j\phi^I\partial^i\partial^j\phi^J)+\frac{17}{12}(\partial_i^2\phi^I)(\partial_i^2\phi^J)\Big]+O(g^2).\end{aligned} \tag{2.167}$$

From the boundary point of view, this is an operator with dimensions $2\Delta_\phi+4$ and spin 0. Using

recursion relations from [86], we can write down the form of a primary of the same dimension and spin in d dimensions

$$O^{IJ}_{2,0} = \left(\partial_i\partial_j\phi^I\partial^i\partial^j\phi^J + \frac{(d+1)(d+3)+2}{2}(\partial_i^2\phi^I)(\partial_i^2\phi^J)\right) - (d+1)((\partial_i^2\partial_j\phi^I)\partial^j\phi^J + (I\leftrightarrow J)) + \frac{(d+1)(d-1)}{12}(((\partial_i^2)^2\phi^I)\phi^J + (I\leftrightarrow J)). \tag{2.168}$$

The relative coefficients of various terms in this operator indeed match what we get from the operator that the spin 4 current defines on the boundary. So they are the same operator up to a constant. We can now try to compute its anomalous dimensions using the following correlation function, which involves the same set of diagrams as the displacement operator $O^{IJ}_{1,0}$ case but with different factors of external momentum

$$\langle O^{IJ}_{2,0}(-\mathbf{p})\phi^K(-\mathbf{q})\phi^L(\mathbf{p}+\mathbf{q})\rangle = 2g(\delta^{IJ}\delta^{KL}+\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\tilde{O}_{2,0}(\mathbf{k},\mathbf{p}) + g^2(8\delta^{IJ}\delta^{KL}+2(N+6)(\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\mathcal{I}_2 \tag{2.169}$$

where

$$\tilde{O}_{2,0}(\mathbf{k},\mathbf{p}) = \frac{(d+2)(d+4)}{12}(|\mathbf{k}|^4+|\mathbf{k}+\mathbf{p}|^4) + \frac{(d+2)(d+4)}{2}|\mathbf{k}|^2|\mathbf{k}+\mathbf{p}|^2 - \frac{d+2}{2}|\mathbf{p}|^2(|\mathbf{k}|^2+|\mathbf{k}+\mathbf{p}|^2) + \frac{1}{4}|\mathbf{p}|^4. \tag{2.170}$$

The one loop diagram again vanishes identically and the two loop diagram requires the following integral, which we again evaluate at $\mathbf{q}=0$

$$\begin{aligned}\mathcal{I}_2 &= \int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\frac{1}{|\mathbf{k}'||\mathbf{k}-\mathbf{k}'-\mathbf{q}|}\tilde{O}_{2,0}(\mathbf{k},\mathbf{p})\\ &= \frac{\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\pi\Gamma(d-1)}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}||\mathbf{q}-\mathbf{k}|^{2-d}}\tilde{O}_{2,0}(\mathbf{k},\mathbf{p})\\ &= \frac{\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})^2 2^{-d-4}\Gamma\left(d-\frac{3}{2}\right)}{(4\pi)^d\pi\Gamma(d-1)(p^2)^{-d}}\left(\frac{(d^5+8d^4+39d^3-80d^2-4d+48)\Gamma\left(\frac{d-1}{2}\right)}{\pi\Gamma\left(\frac{3d}{2}+1\right)}\right.\\ &\left. - \frac{16(d+2)\sec\left(\frac{\pi d}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)\Gamma\left(\frac{3d}{2}-1\right)}\right)\end{aligned} \tag{2.171}$$

and this is finite in $d=2-\epsilon$. This implies that to this order, the operator $O^{IJ}_{2,0}$ does not get anomalous dimensions.

The next operator we consider is the spin 2 operator on the boundary induced by the spin 4

current in the bulk. It can be obtained by taking two of the components of the current to be in the normal direction and it gives

$$J^{IJ}_{yyij} = \frac{1}{12}\Big[-\frac{1}{2}\partial_k^2\partial_i\partial_j\phi^I\phi^J + \frac{7}{2}\partial_k^2\partial_{(i}\phi^I\partial_{j)}\phi^J + \partial_i\partial_j\partial_k\phi^I\partial^k\phi^J - \frac{35}{6}\partial_k^2\phi^I\partial_i\partial_j\phi^J + (I\leftrightarrow J) \\ - \frac{5}{3}\partial_k\partial_{(i}\phi^I\partial^k\partial_{j)}\phi^J - \frac{8}{3}\delta_{ij}\partial_k^2\phi^I\partial_k^2\phi^J + \frac{2}{3}\delta_{ij}\partial_k\partial_l\phi^I\partial^k\partial^l\phi^J - \delta_{ij}(\partial_k^2\partial_l\phi^I\partial^l\phi^J + I\leftrightarrow J)\Big] \tag{2.172}$$

where $\partial_{(i}\phi^I\partial_{j)}\phi^J = \partial_i\phi^I\partial_j\phi^J + \partial_j\phi^I\partial_i\phi^J$. This is symmetric in i,j indices and we can project it onto a symmetric traceless part

$$J^{IJ}_{yyij(T)} = \Big(\delta_{ik}\delta_{jl} - \frac{\delta_{ij}\delta_{kl}}{d}\Big)V_{kl} = \frac{1}{12}\Big[-\frac{5}{3}\partial_k\partial_{(i}\phi^I\partial^k\partial_{j)}\phi^J \\ + \Big(-\frac{1}{2}\partial_k^2\partial_i\partial_j\phi^I\phi^J + \frac{7}{2}\partial_k^2\partial_{(i}\phi^I\partial_{j)}\phi^J + \partial_i\partial_j\partial_k\phi^I\partial^k\phi^J - \frac{35}{6}\partial_k^2\phi^I\partial_i\partial_j\phi^J + (I\leftrightarrow J)\Big) \\ + \delta_{ij}\Big(\Big(\frac{1}{4}\partial_k^2\partial_l^2\phi^I\phi^J - 4\partial_k^2\partial_l\phi^I\partial^l\phi^J + I\leftrightarrow J\Big) + \frac{35}{6}\partial_k^2\phi^I\partial_l^2\phi^J + \frac{5}{3}\partial_k\partial_l\phi^I\partial^k\partial^l\phi^J\Big)\Big]. \tag{2.173}$$

As is probably familiar by now, we can write the "double trace" primary with spin 2 and dimensions $2\Delta_\phi + 4$ using results from [86]

$$O^{IJ}_{1,2\ ij(T)} = \\ \Big[\frac{1-d}{2}\partial_k^2\partial_i\partial_j\phi^I\phi^J + \frac{d+5}{2}\partial_k^2\partial_{(i}\phi^I\partial_{j)}\phi^J + \partial_i\partial_j\partial_k\phi^I\partial^k\phi^J - \frac{(3+d)(5+d)}{2(1+d)}\partial_k^2\phi^I\partial_i\partial_j\phi^J + (I\leftrightarrow J)\Big] \\ - \frac{d+3}{d+1}\partial_k\partial_{(i}\phi^I\partial^k\partial_{j)}\phi^J + \delta_{ij}\Big[\Big(\frac{d-1}{2d}\partial_k^2\partial_l^2\phi^I\phi^J - \frac{d+6}{d}\partial_k^2\partial_l\phi^I\partial^l\phi^J + I\leftrightarrow J\Big) \\ + \frac{(d+3)(d+5)}{d(1+d)}\partial_k^2\phi^I\partial_l^2\phi^J + \frac{2(d+3)}{d(d+1)}\partial_k\partial_l\phi^I\partial^k\partial^l\phi^J\Big] \tag{2.174}$$

which matches, up to an overall constant, to the operator we need. Repeating the same procedure as other operators

$$\langle O^{IJ}_{1,2ij(T)}(-\mathbf{p})\phi^K(-\mathbf{q})\phi^L(\mathbf{p}+\mathbf{q})\rangle = \\ 2g(\delta^{IJ}\delta^{KL} + \delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK})\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\tilde{O}_{1,2,T}(\mathbf{k},\mathbf{p}) \\ + g^2(8\delta^{IJ}\delta^{KL} + 2(N+6)(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK})\mathcal{I}_3 \tag{2.175}$$

where

$$\begin{aligned}\tilde{O}_{1,2,T}(\mathbf{k},\mathbf{p}) = \Big[& k_i k_j \Big(-\frac{2(d+4)(d+2)}{d+1}(|\mathbf{k}|^2+|\mathbf{k}+\mathbf{p}|^2)+\frac{2(d+2)}{(d+1)}|\mathbf{p}|^2\Big) \\ & + k_{(i}p_{j)}\Big(-\frac{(d+6)(d+2)}{d+1}|\mathbf{k}|^2-\frac{(d+2)^2}{d+1}|\mathbf{k}+\mathbf{p}|^2+\frac{(d+2)}{(d+1)}|\mathbf{p}|^2\Big) \\ & + p_i p_j\Big(-\frac{d}{2}|\mathbf{k}+\mathbf{p}|^2-\frac{d^2+9d+16}{2(d+1)}|\mathbf{k}|^2+\frac{|\mathbf{p}|^2}{2}\Big) \\ & + \delta_{ij}\Big(\frac{(d+2)^2}{d(d+1)}(|\mathbf{k}|^4+|\mathbf{k}+\mathbf{p}|^4)+\frac{2(d+6)(d+2)}{d(d+1)}|\mathbf{k}|^2|\mathbf{k}+\mathbf{p}|^2 \\ & - \frac{d^2+9d+16}{2d(d+1)}(|\mathbf{k}|^2+|\mathbf{k}+\mathbf{p}|^2)|\mathbf{p}|^2+\frac{d+3}{2d(1+d)}|\mathbf{p}|^4\Big)\Big].\end{aligned} \tag{2.176}$$

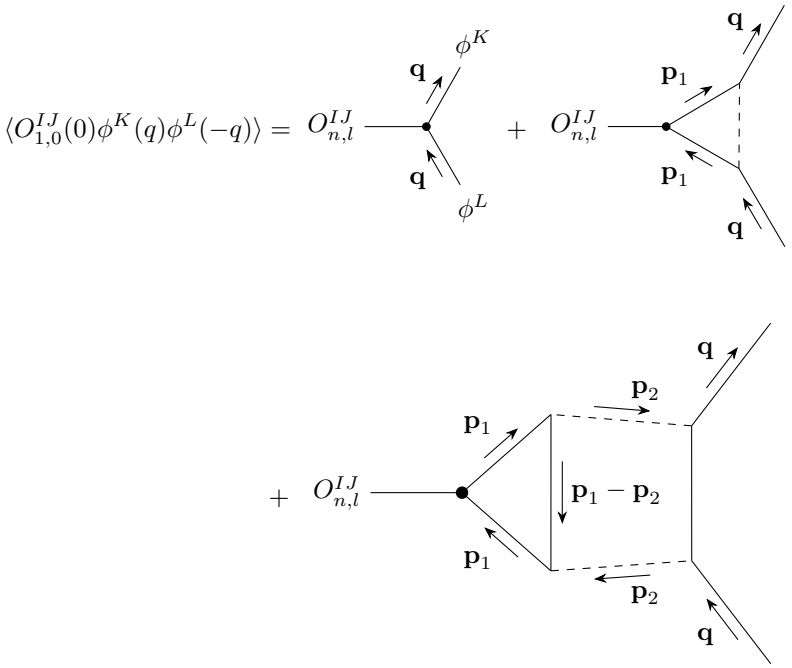
The one loop contribution vanishes, and we can use some integrals from the appendix 2.6 to evaluate the integral appearing in the two loop diagram

$$\begin{aligned}\mathcal{I}_3 &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d}\frac{1}{|\mathbf{k}||\mathbf{k}+\mathbf{p}|}\frac{1}{|\mathbf{k}'||\mathbf{k}-\mathbf{k}'-\mathbf{q}|}\tilde{O}_{1,2,T}(\mathbf{k},\mathbf{p}) \\ &= \frac{\Gamma(\frac{d-1}{2})^2\Gamma(1-\frac{d}{2})}{(4\pi)^d\pi\Gamma(d-1)(p^2)^{-d}}\Big[\frac{p_i p_j}{2^{d+2}p^2\Gamma(\frac{3d}{2})}\Big(\frac{(5d^3+26d^2-40d-16)\Gamma(1-\frac{d}{2})\Gamma(d-\frac{3}{2})\Gamma(\frac{d-1}{2})}{\pi} \\ &- \frac{8(d+2)(d+4)\sec(\frac{\pi d}{2})\Gamma(-\frac{d}{2})\Gamma(d+\frac{1}{2})}{3\Gamma(\frac{3}{2}-\frac{d}{2})}\Big)+\delta_{ij}\frac{(d-2)\sec(\frac{\pi d}{2})\Gamma(4-\frac{d}{2})\Gamma(d-\frac{3}{2})}{2^d d\Gamma(\frac{3}{2}-\frac{d}{2})\Gamma(\frac{3d}{2}+1)}\Big].\end{aligned} \tag{2.177}$$

As anticipated, this is finite in $d = 2-\epsilon$ which implies that to this order, the operator $O^{IJ}_{1,2ij(T)}$ also does not get anomalous dimensions.

2.3.2 Large N expansion

We will now do the calculation of anomalous dimensions of the same operators in the large N model of subsection 2.2.2 using the Feynman diagrams. Starting with the displacement, we have the following contributions



$$
\begin{aligned}
&= (\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK})\bigg(- dq^2 + \frac{\tilde{C}_\sigma}{N}\int \frac{d^dp_1}{(2\pi)^d}\frac{(-dp_1^2)}{|p_1|^2|p_1-q|^{d-2}}\bigg) \\
&+ \frac{2\tilde{C}_\sigma^2\delta^{IJ}\delta^{KL}}{N^2}\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\frac{(-dp_1^2)}{|p_1|^2|p_1-p_2||p_2-q||p_2|^{2(d-2)}} \\
&= q^2\bigg[(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK})\bigg(- d + \frac{2d\ \tilde{C}_\sigma}{N(4\pi)^{\frac{d}{2}}d\Gamma(\frac{d}{2}-1)}\bigg) + \frac{\delta^{IJ}\delta^{KL}\ d{\tilde{C}_\sigma}^2(d-3)\ \Gamma(\frac{d-1}{2})\Gamma(\frac{1-d}{2})}{2N^2(4\pi)^d\sqrt{\pi}d\Gamma(d-\frac{1}{2})}\bigg].
\end{aligned}
\tag{2.178}
$$

There is no $\log q$ term which tells us that there is no anomalous dimension. Both the $1/N$ corrections start at $O(\epsilon^2)$ in $d = 2-\epsilon$ which is consistent with the fact that the $O(g)$ contribution to this correlator vanish in the ϵ expansion. Similar computation can be done for the two operators induced by the spin 4 current on the boundary. For the boundary scalar, we have

$$\langle O^{IJ}_{2,0}(0)\phi^K(q)\phi^L(-q)\rangle = \frac{2(d+2)(d+4)}{3}\Bigg[(\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\Bigg(q^4+\frac{\tilde{C}_\sigma}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{p_1^4}{|p_1|^2|p_1-q|^{d-2}}\Bigg)$$
$$+\frac{2\tilde{C}_\sigma^2\delta^{IJ}\delta^{KL}}{N^2}\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\frac{p_1^4}{|p_1|^2|p_1-p_2||p_2-q||p_2|^{2(d-2)}}\Bigg]$$
$$=\frac{2(d+2)(d+4)q^4}{3}\Bigg[(\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\Bigg(1-\frac{\tilde{C}_\sigma}{N(4\pi)^{\frac{d}{2}}(d+2)\Gamma(\frac{d}{2}-1)}\Bigg)$$
$$-\frac{\delta^{IJ}\delta^{KL}\ 3\tilde{C_\sigma}^{\,2}\ \Gamma(\frac{d-1}{2})^2\Gamma(\frac{-1-d}{2})}{4N^2(4\pi)^d\sqrt{\pi}(d+2)\Gamma(d+\frac{1}{2})\Gamma(\frac{d-5}{2})}\Bigg]. \tag{2.179}$$

This also does not have any $\log q$ terms indicating no anomalous dimensions. The corrections here also start at $O(\epsilon^2)$ in $d=2-\epsilon$. Finally, for the spin two operator, we have

$$\begin{aligned}\langle O^{IJ}_{1,2ij(T)}(0)\phi^K(q)\phi^L(-q)\rangle &= \frac{4(d+2)(d+4)}{d+1}\Bigg[(\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\\ &\times\Bigg(-q_iq_jq^2+\frac{\delta_{ij}q^4}{d}+\frac{\tilde{C}_\sigma}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{1}{|p_1|^2|p_1-q|^{d-2}}(-p_{1i}p_{1j}p_1^2+\frac{\delta_{ij}p_1^4}{d})\Bigg)\\ &+\frac{2\tilde{C}_\sigma^2\delta^{IJ}\delta^{KL}}{N^2}\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\frac{1}{|p_1|^2|p_1-p_2||p_2-q||p_2|^{2(d-2)}}(-p_{1i}p_{1j}p_1^2+\frac{\delta_{ij}p_1^4}{d})\Bigg]\\ &=\frac{4(d+2)(d+4)}{d+1}(-q_iq_jq^2+\frac{\delta_{ij}q^4}{d})\Bigg[(\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\\ &\times\Bigg(1-\frac{2\tilde{C}_\sigma}{N(4\pi)^{\frac{d}{2}}(d+4)\Gamma(\frac{d}{2}-1)}\Bigg)-\frac{15\ \delta^{IJ}\delta^{KL}\tilde{C_\sigma}^{\,2}\ (d-3)\sqrt{\pi}\sec(\frac{d\pi}{2})}{4\ N^2(4\pi)^d(d+4)(d-1)\Gamma(d+\frac{3}{2})}\Bigg]\end{aligned} \tag{2.180}$$

which also does not contain $\log q$ implying that there is no anomalous dimension.

2.4 Long Range $O(N)$ Models

It is natural to generalize the analysis of the previous sections to general non local models in d-dimensional Euclidean space, where the free propagator takes the form $1/|p|^s$ in momentum space, and the kinetic term in position space is

$$\frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}\int d^dxd^dy\frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}},\quad \Delta_\phi=\frac{d-s}{2}. \tag{2.181}$$

For the applications discussed below, d is some fixed dimension (which can be taken to be integer), and s is a free parameter that controls the power of the long range propagator.

2.4.1 Quartic interaction

First we consider the following model with a quartic interaction

$$S = \frac{2^s \Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}} \Gamma(-\frac{s}{2})} \int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \frac{g}{4} \int d^d x (\phi^I \phi^I)^2. \tag{2.182}$$

This coupling becomes marginal when $s = d/2$, so we will study this model perturbatively in $s = \frac{d+\epsilon}{2}$ when g has dimensions equal to ϵ. For $s = 1$ this is equivalent to the boundary model we studied in subsection 2.2.1 and all the diagrams remain the same with modified propagators. So we will not give all the details here and just sketch out the main points.

The computation of the four point function now requires the following integrals

$$\begin{aligned} G^4 &= 2\delta^{IJ}\delta^{KL}\Bigg[-(g+\delta_g) \; + \; (g+\delta_g)^2(N+8) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}+\mathbf{p}|^s |\mathbf{k}|^s} - g^3(N^2+6N+20) \\ &\times \left(\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}+\mathbf{p}|^s |\mathbf{k}|^s} \right)^2 - 4g^3(5N+22) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{d^d\mathbf{k'}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^s |\mathbf{k}+\mathbf{p}|^s} \frac{1}{|\mathbf{k'}|^s |\mathbf{k}-\mathbf{k'}-\mathbf{q}|^s} \Bigg] \\ &= 2\delta^{IJ}\delta^{KL}\Bigg[-(g+\delta_g) + \frac{(g+\delta_g)^2(N+8)\Gamma(\frac{d-s}{2})^2\Gamma(s-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{s}{2})^2\Gamma(d-s)(p^2)^{s-\frac{d}{2}}} \\ &- \frac{g^3(N^2+6N+20)\Gamma(\frac{d-s}{2})^4\Gamma(s-\frac{d}{2})^2}{(4\pi)^d\,\Gamma(\frac{s}{2})^4\Gamma(d-s)^2(p^2)^{2s-d}} - \frac{4g^3(5N+22)\Gamma(\frac{d-s}{2})^3\Gamma(s-\frac{d}{2})\Gamma(d-\frac{3s}{2})\Gamma(2s-d)}{(4\pi)^d\Gamma(\frac{s}{2})^3\Gamma(d-s)\Gamma(\frac{3s-d}{2})\Gamma(\frac{3d}{2}-2s)(p^2)^{2s-d}} \Bigg]. \end{aligned} \tag{2.183}$$

Requiring that the divergent terms cancel when $s = \frac{d+\epsilon}{2}$ fixes δ_g and then applying Callan-Symanzik equation on the finite piece gives the β function

$$\beta(g) = -\epsilon g + \frac{2g^2(N+8)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} + \frac{8g^3(5N+22)}{(4\pi)^d\Gamma(\frac{d}{2})^2}(\gamma + 2\psi(d/4) - \psi(d/2)). \tag{2.184}$$

This gives the fixed point at

$$g = g_* = \frac{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}{2(N+8)}\,\epsilon + \frac{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(5N+22)(-\gamma - 2\psi(d/4) + \psi(d/2))}{(N+8)^3}\epsilon^2. \tag{2.185}$$

The computation of anomalous dimensions of the operator $\phi^I\phi^I$ at this fixed point also closely

follows the boundary case and the result is

$$\begin{aligned}\gamma_{\phi^2} &= \frac{2g_*(N+2)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} + \frac{12(N+2)g_*^2(\gamma+2\psi(d/4)-\psi(d/2))}{(4\pi)^d\Gamma(\frac{d}{2})^2} \\ &= \frac{(N+2)}{(N+8)}\epsilon - \frac{(N+2)(7N+20)(\gamma+2\psi(d/4)-\psi(d/2))}{(N+8)^3}\epsilon^2 \\ \Delta_{\phi^2} &= d-s+\gamma_{\phi^2} = \frac{d}{2} + \frac{(N-4)\epsilon}{2(N+8)} - \frac{(N+2)(7N+20)(\gamma+2\psi(d/4)-\psi(d/2))}{(N+8)^3}\epsilon^2.\end{aligned} \tag{2.186}$$

This agrees with what was found in [59].

2.4.2 Large N description

Similar to subsection 2.2.2 we can develop a complementary approach to study the fixed point studied above in continuous and arbitrary s and d, but in an expansion in $1/N$. For that, we consider the following action with an auxiliary field σ

$$S = \frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}\int d^dx d^dy \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \int d^dx\left(\frac{\sigma\phi^I\phi^I}{2} - \frac{\sigma^2}{4g}\right). \tag{2.187}$$

As usual, we will integrate out the ϕ field to get an effective quadratic action in terms of σ

$$S_2 = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{\sigma(\mathbf{p})\sigma(-\mathbf{p})}{2}\left(\frac{N}{\tilde{C}_\sigma}(p^2)^{\frac{d}{2}-s} - \frac{1}{2g}\right). \tag{2.188}$$

where

$$\tilde{C}_\sigma = -\frac{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{s}{2})^2\Gamma(d-s)}{\Gamma(s-\frac{d}{2})\Gamma(\frac{d-s}{2})^2}. \tag{2.189}$$

From here, it is clear that for $s > \frac{d}{2}$, the second term in the quadratic action can be dropped in the IR limit, while for $s < \frac{d}{2}$, it can be dropped in the UV limit. This only leaves the induced kinetic term in the quadratic action and leads to the following two point function for σ

$$\langle\sigma(x_1)\sigma(x_2)\rangle = \frac{C_\sigma}{N|x_1-x_2|^{2s}}, \quad C_\sigma = \tilde{C}_\sigma\frac{2^{2s}\ \Gamma(s)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}-s)} \tag{2.190}$$

which implies that the conformal dimension of sigma operator, to this order, is s. The computation of its anomalous dimension involves same diagrams and similar integrals as the boundary case and the result is

$$\Delta_\sigma = s + \frac{1}{N}\left(\frac{8\Gamma(\frac{d}{2}-s)\Gamma(\frac{3s-d}{2})\Gamma(\frac{s}{2})^3\Gamma(d-s)^2}{\Gamma(s-\frac{d}{2})^2\Gamma(\frac{d-s}{2})^3\Gamma(s)\Gamma(d-\frac{3s}{2})\Gamma(\frac{d}{2})} - \frac{4\Gamma(\frac{s}{2})^2\Gamma(d-s)}{\Gamma(s-\frac{d}{2})\Gamma(\frac{d-s}{2})^2\Gamma(\frac{d}{2})}.\right) \tag{2.191}$$

This agrees with what was found in [59, 88]. We can expand it in an ϵ expansion with $s = \frac{d+\epsilon}{2}$

$$\Delta_\sigma = s + \frac{1}{N}(-6\epsilon - 7(\gamma + 2\psi(d/4) - \psi(d/2))\epsilon^2 + O(\epsilon^3)). \tag{2.192}$$

It agrees with the ϵ expansion result above in eq. 2.186 when expanded at large N. We can also expand when $s = d - \epsilon$ which gives

$$\Delta_\sigma = s + \frac{1}{N}\bigg(- \epsilon^2 \frac{(\gamma + \psi(\frac{-d}{2}) - \psi(\frac{d}{2}) + \psi(d))}{2} + O(\epsilon^3)\bigg). \tag{2.193}$$

As we show below, this agrees with the result from the non-local non-linear sigma model in eq. 2.198 at large N.

2.4.3 Non-local non-linear sigma model

In line with subsection 2.2.3 we can also study this fixed point by an epsilon expansion at the other end, $s = d - \epsilon$ using a non-local non-linear sigma model (note that the scalar becomes dimensionless at $s = d$). A variant of this model, aiming at a more general target manifold, was considered in [89]. We restrict ourselves to $O(N)$, but it should be possible to generalize our approach to other homogeneous spaces. To do that, we consider the following action

$$S = \frac{2^s \Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})} \int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \int d^d x \sigma(\phi^I\phi^I - \frac{1}{t^2}). \tag{2.194}$$

The constraint can be solved using the same parametrization as the boundary case. The one-point function required for β function computation now involves the following modified integrals

$$\begin{aligned}
\langle \phi^N(0)\rangle &= \frac{1}{t} - \frac{t}{2}\langle \varphi^a\varphi^a(0)\rangle - \frac{t^3}{8}\langle \varphi^a\varphi^a(0)\varphi^b\varphi^b(0)\rangle \\
&= \frac{1}{t} - \frac{t(N-1)}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{|k|^s} + \frac{(N-1)t^3}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{|k|^{2s}}\int \frac{d^dl}{(2\pi)^d}\frac{|k-l|^s}{|l|^s} \\
&- \frac{t^3((N-1)^2 + 2(N-1))}{8}\int \frac{d^dk}{(2\pi)^d}\frac{1}{|k|^s}\int \frac{d^dl}{(2\pi)^d}\frac{1}{|l|^s} \\
&= \frac{1}{t} + \frac{t(N-1)}{2^d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}\bigg(\frac{1}{\epsilon} + \frac{\gamma + \log m^2 + \psi(\frac{d}{2})}{2}\bigg) + \frac{t^3(N-1)(\gamma + \psi(-\frac{d}{2}) - \psi(\frac{d}{2}) + \psi(d))}{2\Gamma(\frac{d}{2})^2(4\pi)^d\epsilon} \\
&- \frac{t^3(N-1)^2}{8}\bigg(\frac{1}{2^{2d-2}\pi^d\Gamma(\frac{d}{2})^2\epsilon^2} + \frac{\gamma + \log m^2 + \psi(\frac{d}{2})}{\epsilon\Gamma(\frac{d}{2})^2 2^{2d-2}\pi^d}\bigg)
\end{aligned} \tag{2.195}$$

where we used techniques similar to boundary case to perform the integrals and expanded in $s = d-\epsilon$. The β function can be extracted from this one-point function

$$\beta(t) = \frac{\epsilon}{2}t - \frac{t^3(N-1)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} - \frac{t^5(N-1)(\gamma+\psi(-\frac{d}{2})-\psi(\frac{d}{2})+\psi(d))}{(4\pi)^d\Gamma(\frac{d}{2})^2}. \tag{2.196}$$

This beta function gives a fixed point at

$$t_*^2 = \frac{\epsilon(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}{2(N-1)} - \frac{\epsilon^2(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(\gamma+\psi(-\frac{d}{2})-\psi(\frac{d}{2})+\psi(d))}{4(N-1)^2} \tag{2.197}$$

and the dimension of the field σ at this fixed point is

$$\Delta_\sigma = d + \beta'(t_*) = s - \frac{\epsilon^2(\gamma+\psi(-\frac{d}{2})-\psi(\frac{d}{2})+\psi(d))}{2(N-1)} \tag{2.198}$$

in agreement with the large N result.

2.4.4 Some Padé estimates for the $d = 1$ long range $O(N)$ model

The quartic model and the non-linear sigma model approximate the fixed point of one dimensional long range $O(N)$ model near the two ends in s, i.e. $s = \frac{d}{2} + \frac{\epsilon}{2}$ and $s = d - \epsilon$ respectively. The large N model interpolates between the two ends, but we can also develop a two-sided Padé approximant to interpolate the intermediate range of s for finite N. By that, we mean that we consider an ansatz Padé$_{m,n} = \frac{\sum_{i=0}^m a_i s^i}{1+\sum_{j=1}^n b_j s^j}$ and equate its series expansion with the available perturbative series expansion. We do this for Δ_σ which is related to the critical exponent ν as $\Delta_\sigma = 1 - 1/\nu$ (this is the dimension of σ in non-linear sigma model and of ϕ^2 in the quartic theory). From the models anlayzed in the previous sections, we have the following series expansions for the anomalous dimension of σ in $d = 1$

$$\begin{aligned}
\Delta_\sigma &= \frac{1}{2} + \frac{(N-4)(s-\frac{1}{2})}{N+8} + \frac{4(N+2)(7N+20)(\pi+4\log 2)}{(N+8)^3}\left(s-\frac{1}{2}\right)^2 + O\left(s-\frac{1}{2}\right)^3,\ s \sim 1/2 \\
\Delta_\sigma &= s - \frac{(1-s)^2}{N-1} + O(1-s)^3, \quad s \sim 1.
\end{aligned} \tag{2.199}$$

We have six possible Padé approximants corresponding to choices of m, n such that $m+n = 5$. Only Padé$_{2,3}$ and Padé$_{3,2}$ are well behaved at all s and N and have a large N behaviour close to our large N result (i.e., they go as $s + 1/N$ at large N). We take their average and plot that to compare it with the large N result in figure 2.6.

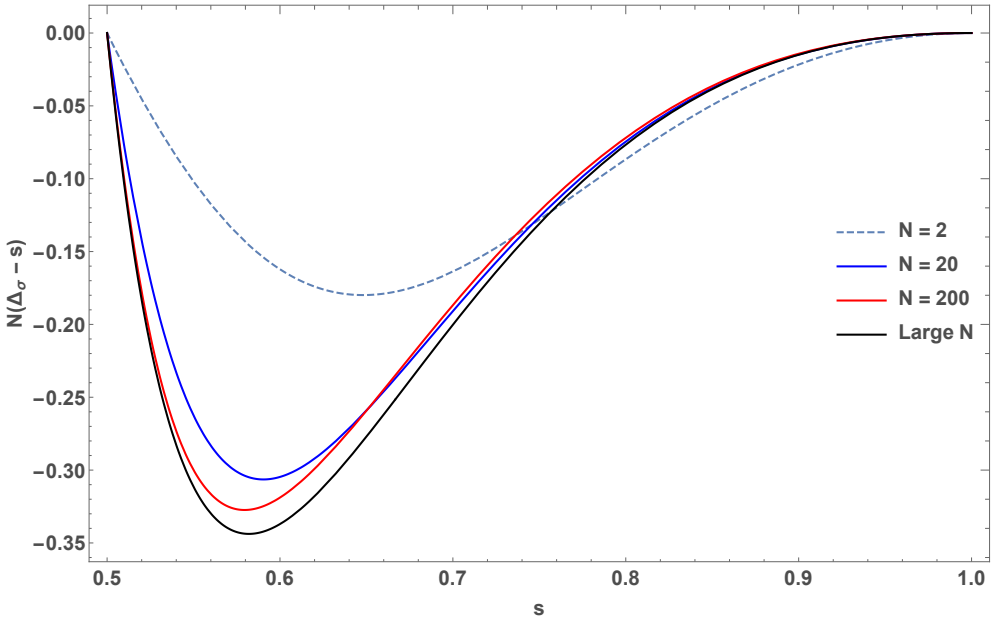


Figure 2.6: Padé result for Δ_σ for $N = 2, 20$ and 200. We plot $N(\Delta_\sigma - s)$ against s because that is easier to compare with the large N result. The Padé result approaches large N result as we go to larger N.

The non-linear sigma model description clearly breaks down for the Ising case $N = 1$, since the β function vanishes and the anomalous dimension diverges in that case. But the dimension of σ near $s = 1$ for the case of long range Ising was found in [70] to be

$$\Delta_\sigma = 1 - \sqrt{2(1-s)}, \quad s \sim 1, \quad N = 1. \tag{2.200}$$

Since there is a square root, we will switch variables to $x = \sqrt{1-s}$ and do a two sided Padé between $0 < x < \frac{1}{\sqrt{2}}$ with the following two constraints

$$\begin{aligned} \Delta_\sigma &= \frac{1}{2} + \frac{\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right) + \frac{3 + 8(\pi + 4\log 2)}{9}\left(x - \frac{1}{\sqrt{2}}\right)^2 + O\left(x - \frac{1}{\sqrt{2}}\right)^3, \; x \sim \frac{1}{\sqrt{2}} \\ \Delta_\sigma &= 1 - \sqrt{2}x + O(x^2), \quad x \sim 0. \end{aligned} \tag{2.201}$$

Again, there are five possibilities and Padé$_{3,1}$, Padé$_{1,3}$ and Padé$_{2,2}$ are all close to each other. We take their average and tabulate the results in table 2.1, where we also include the Padé estimates for higher values of N obtained as described above. For $N = 1$ our estimates are close to the available Monte Carlo results found in [65, 90].

s	0.6	0.65453	0.7	0.8	0.875	0.9
N = 1 Padé	0.488	0.494	0.506	0.553	0.616	0.646
N = 1 Monte Carlo [65]	-	0.494(14)	-	-	0.5876(13)	-
N = 1 Monte Carlo [90]	0.50(2)	-	0.50(4)	0.54(5)	-	0.63(7)
N = 2	0.519	0.565	0.618	0.757	0.858	0. 889
N = 3	0. 534	0.588	0. 643	0.774	0.865	0.894
N = 4	0.544	0.601	0.656	0.781	0.868	0.896
N = 5	0.552	0.610	0.664	0.785	0.870	0.897
N = 10	0.572	0.630	0.681	0.792	0.872	0.898

Table 2.1: The numerical results for $\Delta_\sigma = 1 - 1/\nu$ from our Padé approximants and the available Monte Carlo results for various values of s. As N grows, the results approach the prediction of the large N expansion, which gives $\Delta_\sigma = s + O(1/N)$.

2.5 Appendix: Other Examples of BCFT with free fields in the bulk

In this Apppendix we briefly discuss some other examples of BCFTs with free fields in the bulk and interactions localized on the boundary.

2.5.1 Scalar Yukawa like interaction in $d = 5 - \epsilon$ boundary dimensions

Consider the following model of a free scalar field interacting with N bosons on the boundary with an action

$$S = \int d^{d+1}x \ \frac{1}{2}(\partial\sigma)^2 + \int d^d x \left(\frac{1}{2}(\partial_\mu \phi^I \partial^\mu \phi^I) + \frac{g}{2}\sigma\phi^I\phi^I \right). \tag{2.202}$$

where $I = 1, 2...N$. The interaction becomes marginal in $d = 5$, and it is weakly coupled in $d = 5 - \epsilon$ dimensions. As usual, σ does not get renormalized and has dimensions fixed at classical value. The one loop correction to the propagator of ϕ is

$$\begin{aligned} G^{0,2} &= (-g)^2 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{(\mathbf{p}+\mathbf{k})^2 \ |\mathbf{k}|} - p^2\delta_\phi \\ &= \frac{g^2\Gamma(\frac{d-2}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{3-d}{2})}{(4\pi)^{\frac{d}{2}}\sqrt{\pi}\Gamma(d-\frac{3}{2})(p^2)^{\frac{3-d}{2}}} - p^2\delta_\phi \end{aligned} \tag{2.203}$$

which implies in $d = 5 - \epsilon$

$$Z_\phi = 1 - \frac{g^2}{60\pi^3\epsilon}. \tag{2.204}$$

The one loop correction to the vertex is

$$\begin{aligned} G^{1,2} &= (-g)^3 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{(\mathbf{p}+\mathbf{k})^2(\mathbf{k}-\mathbf{q})^2\ |\mathbf{k}|} - \delta_g \\ &= -\frac{4g^3\Gamma(\frac{5-d}{2})}{3\pi^{\frac{d+1}{2}}2^d(\mu^2)^{\frac{5-d}{2}}} - \delta_g \end{aligned} \tag{2.205}$$

which implies

$$Z_g = g + \delta_g = g - \frac{g^3}{12\pi^3\epsilon}. \tag{2.206}$$

Using the relation $g_0 Z_\sigma^{1/2} Z_\phi = \mu^{\epsilon/2} Z_g$ gives the $\beta-$ function as

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}\Big|_{g_0} = -\mu\frac{\partial_\mu g_0|_g}{\partial_g g_0|_\mu} = -\frac{\epsilon g}{2} - \frac{g^3}{15\pi^3}. \tag{2.207}$$

So there exists a non unitary fixed point at

$$g_*^2 = -\frac{15\pi^3\epsilon}{2}. \tag{2.208}$$

The boundary field ϕ acquires an anomalous dimension

$$\hat{\gamma}_\phi = \mu\frac{\partial}{\partial\mu}\log Z_\phi^{1/2} = \beta(g)\frac{\partial}{\partial g}\log Z_\phi^{1/2} = \frac{g^2}{120\pi^3} \tag{2.209}$$

which at the non unitary fixed point becomes $\hat{\gamma}_\phi|_{g^*} = -\epsilon/16$.

2.5.2 $N+1$ free scalars interacting on $d = 3-\epsilon$ boundary dimensions

Next model we consider is $N+1$ free scalars in the bulk interacting only on the boundary

$$S = \int d^{d+1}x\left(\frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I\right) + \int d^dx\left(\frac{g_1}{2}\sigma\phi^I\phi^I + \frac{g_2}{6}\sigma^3\right). \tag{2.210}$$

where $I = 1, 2...N$. The couplings are marginal in $d = 4$ and the model becomes weakly coupled in $d = 3-\epsilon$. Both σ and ϕ^I are now free bulk fields and they don't get renormalized. The one loop correction to the g_1 vertex is

$$\begin{aligned} G^{1,2} &= ((-g_1)^3 + (-g_1)^2(-g_2)) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}+\mathbf{p}||\mathbf{k}-\mathbf{q}||\mathbf{k}|} - \delta_{g_1} \\ &= -\frac{(g_1^3 + g_1^2 g_2)}{2^{d-1}\pi^{\frac{d+1}{2}}} \frac{\Gamma(\frac{3-d}{2})}{(\mu^2)^{\frac{3-d}{2}}} - \delta_{g_1} \end{aligned} \tag{2.211}$$

which implies

$$Z_{g_1} = g_1 + \delta_{g_1} = g_1 - \frac{(g_1^3 + g_1^2 g_2)}{2\pi^2 \epsilon}. \tag{2.212}$$

The one loop correction to g_2 vertex is similarly

$$\begin{aligned} G^{3,0} &= (N(-g_1)^3 + (-g_2)^3) \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}+\mathbf{p}||\mathbf{k}-\mathbf{q}||\mathbf{k}|} - \delta_{g_2} \\ &= -\frac{(Ng_1^3 + g_2^3)}{2^{d-1}\pi^{\frac{d+1}{2}}} \frac{\Gamma(\frac{3-d}{2})}{(\mu^2)^{\frac{3-d}{2}}} - \delta_{g_2} \end{aligned} \tag{2.213}$$

which implies

$$Z_{g_2} = g_2 + \delta_{g_2} = g_2 - \frac{(Ng_1^3 + g_2^3)}{2\pi^2 \epsilon}. \tag{2.214}$$

The bare couplings are related to the renormalized couplings as

$$\begin{aligned} g_{1_0} Z_\sigma^{1/2} Z_\phi &= \mu^{\epsilon/2}(g_1 + \delta_{g_1}) \\ g_{2_0} Z_\sigma^{3/2} &= \mu^{\epsilon/2}(g_2 + \delta_{g_2}) \end{aligned} \tag{2.215}$$

The β functions can then be computed using following relations

$$\begin{aligned} -\mu\partial_\mu g_{1_0}|_{g_1,g_2} &= \beta(g_1)\frac{\partial g_{1_0}}{\partial g_1}\bigg|_{\mu,g_2} + \beta(g_2)\frac{\partial g_{1_0}}{\partial g_2}\bigg|_{\mu,g_1} \\ -\mu\partial_\mu g_{2_0}|_{g_1,g_2} &= \beta(g_1)\frac{\partial g_{2_0}}{\partial g_1}\bigg|_{\mu,g_2} + \beta(g_2)\frac{\partial g_{2_0}}{\partial g_2}\bigg|_{\mu,g_1}. \end{aligned} \tag{2.216}$$

These give the following β functions

$$\begin{aligned} \beta(g_1) &= -\frac{\epsilon}{2} g_1 - \frac{g_1^3 + g_1^2 g_2}{2\pi^2} \\ \beta(g_2) &= -\frac{\epsilon}{2} g_2 - \frac{Ng_1^3 + g_2^3}{2\pi^2} \end{aligned} \tag{2.217}$$

which give rise to non-unitary fixed points.

2.5.3 Mixed Dimensional QED in $d = 5$ boundary dimensions

Another interesting model to consider is the following higher derivative variant of the mixed dimensional QED discussed in [34]

$$S = \frac{1}{4}\int d^{d+1}x F^{\mu\nu}(-\nabla^2)F_{\mu\nu} - \int d^d x\ \bar{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi. \tag{2.218}$$

The engineering dimension of the gauge field here is $(d+1)/2-2$, hence the coupling is marginal in $d=5$ dimensions. We will analyze this model in $d=5-\epsilon$. The higher derivative term will give a $\frac{\eta_{AB}}{p^4}$ propagator in the bulk. We can Fourier transform back to position space in the direction perpendicular to the boundary and get the propagator on the boundary to be $\frac{\eta_{AB}}{4|p|^3}$. We have the standard propagator for the fermion $-i\frac{\not{p}}{p^2}$. The gauge field is free in the bulk, so it should not receive any anomalous dimensions. So to compute the β function, we need to compute the one loop correction to the fermion propagator and the vertex.

The one loop correction to the fermion propagator is

$$\begin{aligned} G^{0,2} &= (ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^A(-i\not{k})\gamma^B\eta_{AB}}{4|\mathbf{p}-\mathbf{k}|^3k^2} - i\delta_\psi \not{p} \\ &= \frac{-ig^2(d-2)\not{p}\Gamma(\frac{5-d}{2})}{5\sqrt{\pi}(4\pi)^{\frac{d}{2}}} - i\delta_\psi\not{p}. \end{aligned} \tag{2.219}$$

Requiring that the divergent part of the above expression vanish in $d=5-\epsilon$ gives us

$$\delta_\psi = -\frac{3g^2}{80\pi^3\epsilon}. \tag{2.220}$$

The one loop correction to the vertex is

$$G^{1,2} = (ig)^3 \int \frac{d^dp}{(2\pi)^d} \frac{\gamma^C(-i(\not{p}+\not{q}_1))\gamma^A(-i(\not{p}+\not{q}_2))\gamma^B\eta_{BC}}{(p+q_1)^2(p+q_2)^24|\mathbf{p}|^3} + i\delta_g\gamma^A \tag{2.221}$$

We can evaluate the divergent part of the first term in the above expression which must be cancelled by the counterterm which gives

$$\delta_g = -\frac{3g^3}{80\pi^3\epsilon}. \tag{2.222}$$

Using relation $g_0 Z_\gamma^{1/2} Z_\psi = (g+\delta_g)\mu^{\epsilon/2}$ this gives a finite value for g_0. This implies that the beta function actually vanishes in 5 dimensions to this order.

2.6 Appendix: Some useful integrals

In this appendix, we mention some useful integrals which we use throughout the chapter. The first one was performed in [48]

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^{2\alpha}|\mathbf{k}+\mathbf{p}|^{2\beta}} = \frac{1}{(4\pi)^{\frac{d}{2}}|\mathbf{p}|^{2\alpha+2\beta-d}} \frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}-\beta)\Gamma(\alpha+\beta-\frac{d}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}. \tag{2.223}$$

The following two variants of it can be performed by using very similar methods

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_i k_j}{|\mathbf{k}|^{2\alpha}|\mathbf{k}+\mathbf{p}|^{2\beta}} = \frac{1}{(4\pi)^{\frac{d}{2}}|\mathbf{p}|^{2\alpha+2\beta-d-2}} \left(\frac{\delta_{ij}}{2} \frac{\Gamma(\frac{d}{2}+1-\alpha)\Gamma(\frac{d}{2}+1-\beta)\Gamma(\alpha+\beta-\frac{d}{2}-1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2+d-\alpha-\beta)} + \frac{p_i p_j}{|\mathbf{p}|^2} \frac{\Gamma(\frac{d}{2}+2-\alpha)\Gamma(\frac{d}{2}-\beta)\Gamma(\alpha+\beta-\frac{d}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2+d-\alpha-\beta)} \right) \tag{2.224}$$

and

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_i p_j}{|\mathbf{k}|^{2\alpha}|\mathbf{k}+\mathbf{p}|^{2\beta}} = -\frac{p_i p_j}{(4\pi)^{\frac{d}{2}}|\mathbf{p}|^{2\alpha+2\beta-d}} \frac{\Gamma(\frac{d}{2}+1-\alpha)\Gamma(\frac{d}{2}-\beta)\Gamma(\alpha+\beta-\frac{d}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+d-\alpha-\beta)}. \tag{2.225}$$

Chapter 3

Long Range, Large Charge, Large N

As briefly discussed in the previous chapter, long range $O(N)$ models are interesting generalizations of the familiar "short range" $O(N)$ symmetric spin systems. While in the latter the spins only have nearest-neighbor interactions, in the long range models all spins interact with each other with a strength that depends on the distance r as a power law $\sim \frac{1}{r^\alpha}$. The exponent α is usually parameterized as $\alpha = d + s$, where d is the spacetime dimension and s a real parameter. The long range models have second order phase transitions over a range of s, with critical exponents being non-trivial functions of this continuous parameter. Vector models with long range interactions have a long history [59, 60, 61], and various aspects of their physics have also been revisited in several recent works [66, 88, 67, 68, 74, 89, 1, 91, 92, 93]. In this chapter, we focus on the spectrum of operators that carry a large charge under the $O(N)$ symmetry. CFT dynamics simplify significantly when considering operators with a large charge under some global symmetry, as has been observed extensively in the last few years [94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106] (also see [107] for a review and more references).

In the continuum limit, the long range $O(N)$ model may be described by an action containing a non-local kinetic term and a local quartic interaction term [59, 88, 1]

$$S = \frac{C}{2}\int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \frac{g}{4}\int d^d x (\phi^I\phi^I(x))^2, \qquad C = \frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{d/2}\Gamma(-\frac{s}{2})}, \tag{3.1}$$

where we work in the Euclidean flat space R^d. The scaling dimension of the fundamental field

is $\Delta_\phi = \frac{d-s}{2}$ and does not get renormalized due to the non-local nature of the quadratic term (composite operators, on the other hand, can have non-trivial anomalous dimensions). Let us recall from previous chapter following facts about the phase structure of long range models: It is well-known that the model has nontrivial RG fixed points in the range $d/2 < s < s^*$. For $s < d/2$, where the quartic term becomes irrelevant, the low energy behavior of the model is described by the Gaussian (generalized free field) fixed point, while above the upper critical value s_* it is described by the usual short range $O(N)$ symmetric fixed point. The critical value is $s^* = 2 - 2\gamma_\phi^{\rm SR}$, where $\gamma_\phi^{\rm SR}$ is the anomalous dimension of the fundamental field at the short range fixed point (the value of s^* is such that the scaling dimension Δ_ϕ at the long range fixed point becomes equal to that of the short range fixed point). Near the lower limit of the range of s, i.e. for $s = (d+\epsilon)/2$, the model has a perturbative Wilson-Fisher fixed point with $g \sim O(\epsilon)$ [59], while a weakly coupled description near the upper limit s^* was recently proposed in [67, 68].

In this chapter we will focus on the large N limit of the long range theory (3.1). In this limit, it is convenient to introduce a Hubbard-Stratonovich auxiliary field in a way analogous to the standard treatment of the short range $O(N)$ model [59, 88, 1]

$$S = \frac{C}{2}\int d^dx d^dy \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \int d^dx \left(\frac{1}{2}\sigma\phi^I\phi^I(x) - \frac{\sigma^2}{4\lambda}\right). \tag{3.2}$$

The ordinary $1/N$ perturbation theory of the theory for any s may be developed from the above action (where, in the critical limit, one may drop the quadratic term in σ) by expanding around the translational invariant vacuum state, where all one-point functions vanish and the propagator of σ contributes powers of $1/N$ in correlation functions. However, when we consider a correlation function of operators with a large charge j, say $O_j = (\phi^1 + i\phi^2)^j$, with j being of the same order as N, the standard perturbation theory breaks down. This is because j legs in the operators contribute factors of $j \sim N$ to the action. As we will explain in the next section, in the regime where both N and j are large but $\hat{j} = j/N$ is held fixed, there is a new semiclassical saddle where the operator σ acquires a non-trivial classical profile. The two-point function of charge j operators may then be expressed in terms of an effective action at this new saddle point, from which one can extract the scaling dimensions of the operators in this large j, large N limit. The scaling dimensions may be expressed as $\Delta_j = Nh(\hat{j})$ where $h(\hat{j})$ is a non-trivial function of $d, \hat{j}$ and s. We find the following

analytic expansions at small and large $\hat{j}$ for generic s

$$\begin{aligned}\Delta_j &= N\left[\frac{d-s}{2}\hat{j} + O(\hat{j}^2)\right] \\ \Delta_j &= N\left[\frac{d+s}{2}\hat{j} + A(d,s)\hat{j}^{\frac{s}{d+s}} + \dots\right].\end{aligned} \tag{3.3}$$

For small $\hat{j}$, this matches the expectation from the ordinary $1/N$ perturbation theory (the term of order $\hat{j}^2$ can also be explicitly compared to standard diagrammatic expansions). At large $\hat{j}$, the leading behavior of the scaling dimensions is still linear (with a different slope), which is strikingly different from the case of the local $O(N)$ models, where one finds $\Delta_j \sim N\hat{j}^{\frac{d}{d-1}}$ for $\hat{j} \gg 1$ [98, 108].[1] Note that at large N, the upper critical value for the range of s is $s^* = 2 + O(1/N)$. While the behavior (3.3) arises from the dominant saddle point at generic s, we suggest that in the limit $s \to 2$, the $\Delta_j \sim N\hat{j}^{\frac{d}{d-1}}$ behavior of the short range model is recovered due to an interplay between the multiple solutions to the saddle point equation.

An interesting aspect of the long range models (3.1) is their connection to the subject of defect CFT. Indeed, the model (3.1) may be thought of as arising from a free scalar field theory in an auxiliary space of dimension $D = d+2-s$, with the quartic interaction localized on a d dimensional "defect" subspace [109, 66, 1]. The operators in the long range $O(N)$ model map to the operators living on the d-dimensional defect. In the special case $s = 1$, i.e. $D = d+1$, the model is equivalent to a BCFT that is free in the bulk and has boundary localized interactions. In that context, the large charge operators living on the boundary were recently considered in [103]. They used a Weyl transformation to map the problem of calculating scaling dimensions on the half-plane to that of calculating energies on $R \times HS^d$, where HS^d is the hemisphere (with the long range model living on the $R \times S^{d-1}$ boundary). To do the calculation, one then may compute the free energy on $R \times HS^d$ in the presence of a chemical μ for the conserved charge. In this chapter we generalize this calculation to arbitrary s, by mapping the problem to the higher dimensional cylinder $R \times S^{D-1}$ in the presence of a chemical potential μ, with the interaction localized on the subspace $R \times S^{d-1}$. Using this approach, we rederive the scaling dimensions of the large charge operators, obtaining results that precisely match with what we get from the saddle point on R^d, thus providing a useful consistency check. Along the way, we also do perturbative calculations for $s = (d+\epsilon)/2$ in an ϵ expansion valid for any N. We find agreement between large N and ϵ expansions in the overlapping regimes of validity.

[1] The behavior of the scaling dimension $\Delta_j \sim j^{d/(d-1)}$ in the large charge limit holds in a generic strongly coupled CFT [94, 96].

The rest of this chapter is organized as follows: In section 3.1, we set up the calculation of the two-point function of large charge operators in flat space and identify the saddle that provides the dominant contribution in the large j, large N double scaling limit. Solving the saddle point equation requires calculating the Green's function at the large charge saddle point. Having obtained the Green's function, we show that correlation functions of two "heavy" (large charge) and an arbitrary number of light operators can be obtained with little extra effort. We discuss in some detail the calculation of "heavy-heavy-light" three point function and "heavy-heavy-light-light" four-point function. We end the section with a separate discussion of the $d = 1$ long range $O(N)$ model that behaves somewhat differently from its higher dimensional counterparts. Then in section 3.2, we show how our results may be obtained by calculating the energy in a large charge state on a cylinder. The two appendices contain technical details and the standard $1/N$ perturbation theory calculation of the scaling dimensions which is valid at $\hat{j} \ll 1$.

3.1 The large charge, large N saddle point on R^d

In this section, we start by defining the setup and identifying the large charge saddle point in the long range $O(N)$ model at large N. We will closely follow the discussion in [108] where the large charge saddle for the local, short-range $O(N)$ models was discussed. Since there are many similarities in the analysis, we will be brief here, and refer the reader to [108] for details.

We will start with the model defined by (3.2). As mentioned in the introduction, we will study operators that carry a large charge j under $O(N)$ symmetry, and we will work in the double scaling limit such that both j and N are large, with the ratio $\hat{j} = j/N$ fixed and finite. In this note, we focus on operators that transform in the symmetric traceless representation of $O(N)$. Such operators may be written as $\mathcal{O}_j(x) \equiv (u^I \phi^I(x))^j$ with null auxiliary complex vector u^I (a simple representative is the operator $(\phi^1 + i\phi^2)^j$). These are the lowest dimension operators in the given charge sector, and are not expected to undergo mixing. Their two-point function is constrained by conformal symmetry in the usual way:

$$\langle \mathcal{O}_j(x_1)\mathcal{O}_j(x_2)\rangle = (u_1^I u_2^I)^j \frac{C_j}{x_{12}^{2\Delta_j}}. \tag{3.4}$$

This two-point function can also be computed using the path integral (recall that in the critical limit

we drop the σ quadratic term)

$$\begin{aligned}\langle \mathcal{O}_j(x_1)\mathcal{O}_j(x_2)\rangle &= \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\sigma \left(u_1^I\phi^I(x_1)\right)^j \left(u_2^J\phi^J(x_2)\right)^j e^{-\frac{C}{2}\int d^dyd^dz\frac{\phi^K(y)\phi^K(z)}{|y-z|^{d+s}}-\frac{1}{2}\int d^dx\sigma\phi^K\phi^K(x)} \\ &= (u_1^Iu_2^I)^j j!\int \mathcal{D}\sigma[G(x_1,x_2;\sigma)]^j e^{-\frac{N}{2}\log\det\left(\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)\right)} \\ &= (u_1^Iu_2^I)^j j!\int \mathcal{D}\sigma e^{-N\{\frac{1}{2}\log\det[\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)]-\hat{j}\log(G(x_1,x_2;\sigma))\}}\end{aligned} \tag{3.5}$$

where we integrated out the scalars and defined the Green's function in the presence of a non-trivial σ field

$$\delta^{IJ}G(x_1,x_2;\sigma)\equiv\int\mathcal{D}\phi\phi^I(x_1)\phi^J(x_2)e^{-\frac{C}{2}\int d^dyd^dz\frac{\phi^K(y)\phi^K(z)}{|y-z|^{d+s}}-\frac{1}{2}\int d^dx\sigma\phi^K\phi^K(x)}. \tag{3.6}$$

We can perform the path integral over σ in (3.5) using a saddle point approximation by extremizing the effective action

$$\frac{\delta}{\delta\sigma}\left(\frac{1}{2}\log\det\left[\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)\right]-\hat{j}\log(G(x_1,x_2;\sigma))\right)=0. \tag{3.7}$$

This equation will give a profile of $\sigma(x)=\sigma_*(x;x_1,x_2)$ satisfying

$$2\hat{j}G(x_1,x;\sigma_*)G(x_2,x;\sigma_*)=-G(x,x;\sigma_*)G(x_1,x_2;\sigma_*). \tag{3.8}$$

To calculate the Green's function and then to solve the saddle point equation, we start with an ansatz for the σ profile at the saddle point. We observe that this can be viewed as the one-point function of the field $\sigma(x)$ in the presence of the large charge operators, namely a 3-point function:

$$\begin{aligned}\sigma_*(x;x_1,x_2) &= \lim_{N\to\infty}\frac{\int\mathcal{D}\sigma\ \sigma(x)e^{-N\{\frac{1}{2}\log\det[\frac{C}{|y-z|^{d+s}}+\sigma(y)\delta^d(y-z)]-\hat{j}\log(G(x_1,x_2;\sigma))\}}}{\int\mathcal{D}\sigma e^{-N\{\frac{1}{2}\log\det[\frac{C}{|y-z|^{d+s}}+\sigma(y)\delta^d(y-z)]-\hat{j}\log(G(x_1,x_2;\sigma))\}}} \\ &= \lim_{N\to\infty}\frac{\langle\mathcal{O}_j(x_1,u_1)\mathcal{O}_j(x_2,u_2)\sigma(x)\rangle}{\langle\mathcal{O}_j(x_1,u_1)\mathcal{O}_j(x_2,u_2)\rangle}.\end{aligned} \tag{3.9}$$

Conformal symmetry then requires

$$\sigma_*(x;x_1,x_2)=c_\sigma\frac{|x_1-x_2|^s}{|x_1-x|^s|x_2-x|^s} \tag{3.10}$$

since $\Delta_\sigma=s+\mathcal{O}(1/N)$.

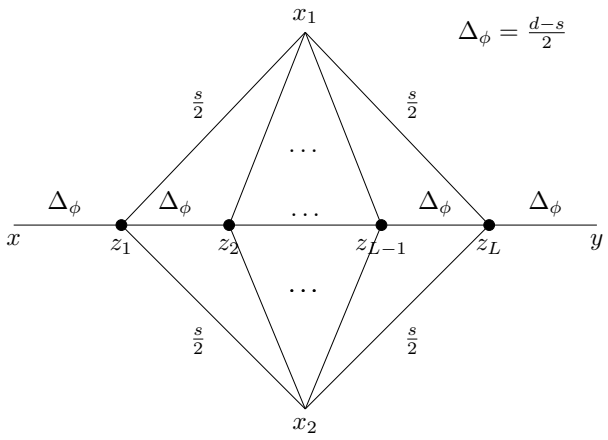


Figure 3.1: The pictorial representation of the integral I_L. The notation is such that the line between the points z_i and z_j contributes $1/|z_{ij}|^{2\alpha}$ to the integral where α is the number written above the line. Every filled dot represents a point that is integrated over.

3.1.1 Green's function

In this subsection, we calculate the Green's function in the presence of a non-trivial σ^*. As usual, it is given by inverting the quadratic term in the action

$$\int d^d x' \left(\frac{C}{|x-x'|^{d+s}} + \sigma_*(x)\delta^d(x-x') \right) G(x', y; \sigma_*) = \delta^d(x-y) \tag{3.11}$$

. We can solve it by expanding the Green's function in powers of σ^*

$$G = G^{(0)} + G^{(1)} + G^{(2)} + ... \tag{3.12}$$

where the superscript indicates the power of σ_*, and the terms in the expansion satisfy following equations

$$\begin{aligned} &\int d^d x' \frac{C}{|x-x'|^{d+s}} G^{(0)}(x', y) = \delta^d(x-y) \\ &\int d^d x' \frac{C}{|x-x'|^{d+s}} G^{(L+1)}(x', y; \sigma^*) = -\sigma_*(x) G^{(L)}(x, y; \sigma^*), \qquad L \geq 1. \end{aligned} \tag{3.13}$$

The leading order result is just the usual two-point function without any large charge operators

$$G^{(0)}(x,y) = \frac{C_\phi}{|x-y|^{d-s}}, \qquad C_\phi = \frac{\Gamma\left(\frac{d-s}{2}\right)}{2^s \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right)}. \tag{3.14}$$

We can then get the result for order L Green's function by iteratively applying the above result

$$
\begin{aligned}
G^L(x,y,\sigma_*) &= (-1)^L \left(\prod_{k=1}^{L}\int d^d z_k \sigma^*(z_k) G^0(z_k,z_{k+1})\right) G^0(x,z_1) \\
&= C_\phi I_L(x,y,x_1,x_2) \\
I_L(x,y,x_1,x_2) &= g_{12}^L \left(\prod_{k=1}^{L}\int \frac{d^d z_k}{|z_k-x_1|^s |z_k-x_2|^s}\right)\left(\prod_{j=0}^{L}\frac{1}{|z_{j+1}-z_j|^{d-s}}\right)
\end{aligned}
\tag{3.15}
$$

where we defined $g_{12} = -C_\phi c_\sigma |x_1 - x_2|^s$ and $z_0 = y, z_{L+1} = x$. The order L result involves doing an integral over L variables. The integral may be visualized as in figure 3.1. Let us now analyze it in more detail. By just shifting all the integration variables, we can see that it is only a function of 3 variables

$$
I_L(x,y,x_1,x_2) = I_L(x-x_1, y-x_1, 0, x_2-x_1). \tag{3.16}
$$

We can then do a change of variables to invert all the variables $z'_k = z_k/z^2$, and the fact that the integral is conformal helps to simplify it as follows

$$
\begin{aligned}
I_L(x-x_1,y-x_1,0,x_2-x_1) &= g_{12}^L \left(\prod_{k=1}^{L}\int \frac{d^d z_k}{|z_k|^s |z_k-(x_2-x_1)|^s}\right)\left(\prod_{j=0}^{L}\frac{1}{|z_{j+1}-z_j|^{d-s}}\right) \\
&= \frac{1}{|x-x_1|^{d-s}|y-x_1|^{d-s}}\left(\frac{g_{12}}{|x_2-x_1|^s}\right)^L \left(\prod_{k=1}^{L}\int \frac{d^d z'_k}{|z'_k-(x_2-x_1)'|^s}\right)\left(\prod_{j=0}^{L}\frac{1}{|z'_{j+1}-z'_j|^{d-s}}\right) \\
&= \frac{1}{|x-x_1|^{d-s}|y-x_1|^{d-s}}\left(\frac{g_{12}}{|x_2-x_1|^s}\right)^L \left(\prod_{k=1}^{L}\int \frac{d^d z'_k}{|z'_k|^s}\right)\left(\prod_{j=0}^{L}\frac{1}{|z'_{j+1}-z'_j|^{d-s}}\right) \\
&= \frac{1}{|x-x_1|^{d-s}|y-x_1|^{d-s}}\left(\frac{g_{12}}{|x_2-x_1|^s}\right)^L \Phi_L(\xi,\eta)
\end{aligned}
\tag{3.17}
$$

where in the third line, we did a shift of variables so that in that equation

$$
z'_0 = \eta = \frac{y-x_1}{|y-x_1|^2} - \frac{x_2-x_1}{|x_2-x_1|^2}, \qquad z'_{L+1} = \xi = \frac{x-x_1}{|x-x_1|^2} - \frac{x_2-x_1}{|x_2-x_1|^2} \tag{3.18}
$$

and in the last line, we defined the integral

$$
\Phi_L(\xi,\eta) = \left(\prod_{k=1}^{L}\int \frac{d^d z'_k}{|z'_k|^s}\right)\left(\prod_{j=0}^{L}\frac{1}{|z'_{j+1}-z'_j|^{d-s}}\right) \tag{3.19}
$$

L = 1

To gain some intuition for the result, let us start by working perturbatively in c_σ which is the same as working order by order in L. For $L = 1$, we have the following integral

$$\Phi_1(\xi,\eta) = \int \frac{d^d z}{|z|^s |z-\xi|^{d-s} |z-\eta|^{d-s}}. \tag{3.20}$$

For the purposes of obtaining the scaling dimensions, we will need several limits of the Green's function : $G(x_1, y, \sigma^*), G(x, x_2, \sigma^*), G(x_1, x_2, \sigma^*)$, and $G(x, x, \sigma^*)$. Let us start with the first one: when $x \to x_1$. We introduce a regulator δ and set $x = x_1 + \delta$ to get

$$\Phi_1(\xi,\eta) = \delta^{d-s} \int \frac{d^d z}{|z|^s |z-\eta|^{d-s}} \tag{3.21}$$

This integral still has a UV divergence, so we introduce a further regulator

$$\begin{aligned} \Phi_1(\xi,\eta) &= \delta^{d-s-2\kappa} \int \frac{d^d z}{|z|^{s+\kappa} |z-\eta|^{d-s+\kappa}} = \frac{\delta^{d-s}\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\kappa} - \frac{2\delta^{d-s}\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \log\left(\delta|\eta|\right) \implies \\ G(x_1, y, \sigma^*) &= \frac{\Gamma\left(\frac{d-s}{2}\right)}{2^s \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right) |x_1 - y|^{d-s}} + \frac{\Gamma\left(\frac{d-s}{2}\right)^2 c_\sigma}{2^{2s-1} \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{d}{2}\right) |x_1 - y|^{d-s}} \log \frac{\delta |x_2 - y|}{|x_1 - x_2||x_1 - y|}. \end{aligned} \tag{3.22}$$

The divergent $1/\kappa$ piece should contribute to wavefunction renormalization, but it will not affect our calculation of scaling dimensions. The $G(x, x_2, \sigma^*)$ should just be related to this one by interchanging $x \leftrightarrow y, x_1 \leftrightarrow x_2$. Finally, to obtain $G(x_1, x_2, \sigma^*)$, we set $y = x_2 + \delta$ in the above to get

$$G(x_1, y, \sigma^*) = \frac{\Gamma\left(\frac{d-s}{2}\right)}{2^s \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right) |x_1 - x_2|^{d-s}} + \frac{\Gamma\left(\frac{d-s}{2}\right)^2 c_\sigma}{2^{2s-1} \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{d}{2}\right) |x_1 - x_2|^{d-s}} \log \frac{\delta^2}{|x_1 - x_2|^2}. \tag{3.23}$$

Next we turn to $G(x, x, \sigma^*)$, for which we need to consider $\xi = \eta$ limit of the integral (3.19) which is easy to obtain

$$\Phi_1(\xi,\xi) = \frac{\pi^{d/2} \Gamma\left(\frac{d-s}{2}\right)^2 \Gamma\left(s - \frac{d}{2}\right)}{\Gamma(d-s) \Gamma\left(\frac{s}{2}\right)^2 |\xi|^{d-s}} \tag{3.24}$$

which implies

$$G^{(1)}(x, x, \sigma_*) = -\frac{C_\phi^2 c_\sigma \pi^{d/2} \Gamma\left(\frac{d-s}{2}\right)^2 \Gamma\left(s - \frac{d}{2}\right) |x_2 - x_1|^{d-s}}{\Gamma(d-s) \Gamma\left(\frac{s}{2}\right)^2 |x - x_1|^{d-s} |x - x_2|^{d-s}}. \tag{3.25}$$

Note that without any insertions, we should set coincident point two-point function to zero, i.e. $G^{(0)}(x, x) = 0$.

General L

Next we turn to the more difficult problem of evaluating the integral in (3.19) for general L. For the local case of $s = 2$, the integral is exactly the kind considered in [110]. But the generalization of that formalism to general s is not straightforward. Fortunately, the integral in (3.19) was already computed exactly in [111] in a very different context of fishnet Feynman integrals. The result can be expressed in terms of Gegenbauer polynomials as follows

$$\begin{aligned}\Phi_L(x,y) &= \frac{\Gamma\left(\frac{d-2}{2}\right)}{(|x||y|)^{\frac{d-s}{2}}\pi^{\frac{-dL}{2}}}\sum_{l=0}^{\infty}\left(l+\frac{d-2}{2}\right)C_l^{\left(\frac{d-2}{2}\right)}\left(\frac{x\cdot y}{|x||y|}\right)\int\frac{du}{2\pi}\left(\frac{x^2}{y^2}\right)^{iu}(Q_l(u))^{L+1}\\ Q_l(u) &= \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{d-s+2l}{4}-iu\right)\Gamma\left(\frac{d-s+2l}{4}+iu\right)}{\Gamma\left(\frac{d-s}{2}\right)\Gamma\left(\frac{d+s+2l}{4}+iu\right)\Gamma\left(\frac{d+s+2l}{4}-iu\right)}\end{aligned} \tag{3.26}$$

In principle, the integral over u can be performed by a sum over residues. The contour can be closed in the upper half-plane and there are infinitely many residues at $u = i\left(\frac{d-s+2l}{4}+n\right)$. But in practice, it becomes hard since the poles are of order $L+1$. To convince the reader that it makes sense, we perform some basic checks of this result starting from $L = 0$ when there is no integral to do. This can be expressed in terms of Gegenbauer polynomials as follows

$$\Phi_0(x,y) = \frac{1}{|x-y|^{d-s}} = \frac{1}{|x|^{d-s}}\sum_{\alpha=0}^{\infty}C_\alpha^{\frac{d-s}{2}}\left(\frac{x\cdot y}{|x||y|}\right)\left|\frac{y}{x}\right|^{\alpha} \tag{3.27}$$

On the other hand, the integral on the right hand side of (3.26) gives the following sum over residues

$$\Phi_0(x,y) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{|x|^{d-s}\Gamma\left(\frac{d-s}{2}\right)}\sum_{l,n=0}^{\infty}\frac{\left(l+\frac{d-2}{2}\right)\Gamma\left(\frac{d-s}{2}+l+n\right)\Gamma\left(1+n-\frac{s}{2}\right)}{n!\Gamma\left(\frac{d}{2}+l+n\right)\Gamma\left(1-\frac{s}{2}\right)}C_l^{\frac{d-2}{2}}\left(\frac{x\cdot y}{|x||y|}\right)\left|\frac{y}{x}\right|^{l+2n} \tag{3.28}$$

This implies the following identity for Gegenbauer polynomials

$$C_\alpha^{\frac{d-s}{2}}(x) = \sum_{n=0}^{\lfloor\frac{\alpha}{2}\rfloor}C_{\alpha-2n}^{\frac{d-2}{2}}(x)\frac{\left(\frac{d-2}{2}+\alpha-2n\right)\Gamma\left(\frac{d-s}{2}+\alpha-n\right)\Gamma\left(1+n-\frac{s}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{n!\Gamma\left(\frac{d}{2}+\alpha-n\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{d-s}{2}\right)}. \tag{3.29}$$

which is known to be true (see for instance Appendix A of [112]).

We can get the full Green's function by summing over L

$$\begin{aligned}G(x,y,\sigma^*) = \sum_{L=0}^{\infty}G^L =& \frac{C_\phi\Gamma\left(\frac{d-2}{2}\right)}{|x-x_1|^{d-s}|y-x_1|^{d-s}\left(|\xi||\eta|\right)^{\frac{d-s}{2}}}\sum_{l=0}^{\infty}\left(l+\frac{d-2}{2}\right)C_l^{\left(\frac{d-2}{2}\right)}\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)\\ &\times\int\frac{du}{2\pi}\left(\frac{\xi^2}{\eta^2}\right)^{iu}\frac{Q_l(u)}{1+C_\phi c_\sigma\pi^{d/2}Q_l(u)}.\end{aligned} \tag{3.30}$$

It is useful to write this Green's function as a function of the conformal cross-ratios defined as

$$X = \frac{|x-x_1|^2|y-x_2|^2}{|x_1-x_2|^2|x-y|^2}, \qquad Y = \frac{|x-x_2|^2|y-x_1|^2}{|x_1-x_2|^2|x-y|^2}. \tag{3.31}$$

The Green's function in terms of these cross-ratios is given by

$$\begin{aligned} G(x,y,\sigma^*) =& \frac{C_\phi \Gamma\left(\frac{d-2}{2}\right)}{|x-y|^{d-s}\left(XY\right)^{\frac{d-s}{4}}} \sum_{l=0}^{\infty} \left(l+\frac{d-2}{2}\right) C_l^{\left(\frac{d-2}{2}\right)}\left(\frac{X+Y-1}{2\sqrt{XY}}\right) \\ & \times \int \frac{du}{2\pi}\left(\frac{Y}{X}\right)^{iu} \frac{Q_l(u)}{1+C_\phi c_\sigma \pi^{d/2} Q_l(u)} \end{aligned} \tag{3.32}$$

The reason that the Green's function can be written in terms of the standard conformal cross-ratios is that, as we clarify below, it is directly related to the 4-point function of two large charge and two charge 1 operators.

One may perform the integral over u by closing the contour in the upper half plane, and the poles are given by the solutions to the following equation

$$\frac{1}{Q_l(u)} + C_\phi c_\sigma \pi^{d/2} = 0, \implies \frac{\Gamma\left(\frac{d+s+2l}{4}-iu\right)\Gamma\left(\frac{d+s+2l}{4}+iu\right)}{\Gamma\left(\frac{d-s+2l}{4}+iu\right)\Gamma\left(\frac{d-s+2l}{4}-iu\right)} = -\frac{c_\sigma}{2^s}. \tag{3.33}$$

We expect the poles to lie on the imaginary axis, and we will parameterize the roots of the above equation by $u = i\mu/2$, where $\mu = \mu(c_\sigma)$ is real.[2]

We can now calculate this Green's function in the various limits needed to extract the scaling dimensions. Let us start by considering the case when either $x \to x_1$ or $y \to x_2$ or both at the same time. In the limit when $x \to x_1$, we have $\xi \to \infty$, the dominant contribution to the integral in (3.30) comes from the pole with the smallest positive imaginary part, i.e. $u = i\mu/2$ with the smallest μ. This also allows to just consider $l = 0$ term in the sum. The same is true when $y \to x_2$, because in that case, $\eta \to 0$. We can find this solution analytically for small and large c_σ as

$$\begin{aligned} \mu(c_\sigma) &= \frac{d-s}{2} + \frac{\Gamma\left(\frac{d-s}{2}\right)c_\sigma}{2^{s-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s}{2}\right)} + \frac{\Gamma\left(\frac{d-s}{2}\right)^2\left(\psi^{(0)}\left(\frac{d-s}{2}\right)-\psi^{(0)}\left(\frac{d}{2}\right)+\psi^{(0)}\left(\frac{s}{2}\right)+\gamma\right)c_\sigma^2}{2^{2s-1}\Gamma\left(\frac{d}{2}\right)^2\Gamma\left(\frac{s}{2}\right)^2} + \dots \\ \mu(c_\sigma) &= \frac{d+s}{2} + \frac{2^{s+1}\Gamma\left(\frac{d+s}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{-s}{2}\right)c_\sigma} + \frac{\Gamma\left(\frac{d+s}{2}\right)^2\left(\psi^{(0)}\left(\frac{d+s}{2}\right)-\psi^{(0)}\left(\frac{d}{2}\right)+\psi^{(0)}\left(-\frac{s}{2}\right)+\gamma\right)}{2^{-1-2s}\Gamma\left(\frac{d}{2}\right)^2\Gamma\left(-\frac{s}{2}\right)^2 c_\sigma^2} + \dots \end{aligned} \tag{3.34}$$

For general values of c_σ, we can find this root numerically. The Green's function in this limit is then

[2]We denote it by μ because it will be equal to the chemical potential when we map the problem to the cylinder.

given by

$$\begin{aligned}
G(x_1, y, \sigma^*) &= \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}|y-x_1|^{d-s}}\mu'(c_\sigma)\left(\frac{\delta|y-x_2|}{|x_2-x_1||y-x_1|}\right)^{\mu(c_\sigma)-\frac{d-s}{2}} \\
G(x, x_2, \sigma^*) &= \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}|x-x_2|^{d-s}}\mu'(c_\sigma)\left(\frac{\delta|x-x_1|}{|x_2-x_1||x-x_2|}\right)^{\mu(c_\sigma)-\frac{d-s}{2}} \\
G(x_1, x_2, \sigma^*) &= \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}|x_2-x_1|^{d-s}}\mu'(c_\sigma)\left(\frac{\delta^2}{|x_2-x_1|^2}\right)^{\mu(c_\sigma)-\frac{d-s}{2}}
\end{aligned} \tag{3.35}$$

At small c_σ, this can be checked to agree with (3.22) and (3.23).

Finally, we consider the coincident point limit, $G(x, x, \sigma^*)$. In this limit, $\xi \to \eta$ and the result simplifies to

$$\begin{aligned}
G^L(x, x, \sigma^*) = &\frac{C_\phi \Gamma\left(\frac{d-2}{2}\right)|x_2-x_1|^{d-s}\left(-C_\phi c_\sigma \pi^{d/2}\right)^L}{\left(|x-x_1||x-x_2|\right)^{d-s}} \\
&\times \sum_{l=0}^{\infty}\left(\frac{2l+d-2}{2}\right)C_l^{\left(\frac{d-2}{2}\right)}(1)\int\frac{du}{2\pi}\left(Q_l(u)\right)^{L+1}.
\end{aligned} \tag{3.36}$$

As usual, this coincident point limit is related to the functional determinant of the quadratic piece in the action, so we discuss it more in the next subsection.

3.1.2 Functional determinant

The functional determinant can be expressed in terms of the Green's function as follows

$$\begin{aligned}
\log\det\left[\frac{C}{|x-y|^{d+s}} + \sigma_*(x)\delta^d(x-y)\right] &= \sum_{L=1}^{\infty}\frac{(-1)^{L-1}}{L}\left(\prod_{i=1}^{L}\int d^d z_i \sigma_*(z_i) G^0(z_i, z_{i+1})\right) \\
&= \sum_{L=1}^{\infty}\frac{1}{L}\int d^d x \sigma_*(x) G^{L-1}(x, x, \sigma_*)
\end{aligned} \tag{3.37}$$

where in the first line, it is to be understood that $z_{L+1} = z_1$. Plugging the result from (3.36), we need to perform an integral over x, which is divergent. We regularize it in the same way as we did in previous subsection

$$\begin{aligned}
\int d^d x \frac{|x_2-x_1|^d}{|x-x_1|^d|x-x_2|^d} &\to \int d^d x \frac{\delta^{-2\kappa}|x_2-x_1|^{d-2\kappa}}{|x-x_1|^{d-2\kappa}|x-x_2|^{d-2\kappa}} \\
&= \frac{2\pi^{d/2}}{\kappa\Gamma\left(\frac{d}{2}\right)} - \frac{4\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\log\left(\frac{\delta}{|x_1-x_2|}\right) + O(\kappa).
\end{aligned} \tag{3.38}$$

Again, the $1/\kappa$ piece will be canceled by an appropriate counterterm and will not be important for us, so we will only keep the log term in the following. Then performing the sum over L, we obtain

the following result for the functional determinant

$$\begin{aligned}\log\det[...] &= -2F(c_\sigma)\log\left(\frac{\delta^2}{|x_{12}|^2}\right)\\ F(c_\sigma) &= \sum_{l=0}^{\infty}\frac{(2l+d-2)\,\Gamma(d-2+l)}{\Gamma(d-1)\,l!}\int\frac{du}{2\pi}\log\left(1+C_\phi c_\sigma\pi^{d/2}Q_l(u)\right)\end{aligned} \tag{3.39}$$

where we used

$$C_l^{\left(\frac{d-2}{2}\right)}(1) = \frac{\Gamma(d-2+l)}{\Gamma(d-2)\,\Gamma(l+1)}. \tag{3.40}$$

The Green's function at coincident points is related to the derivative of the functional determinant. Indeed by summing over L in (3.36), we can see

$$\begin{aligned}G(x,x,\sigma^*) &= \sum_{l=0}^{\infty}\frac{C_\phi\Gamma\left(\frac{d-2}{2}\right)|x_2-x_1|^{d-s}(2l+d-2)\Gamma(d-2+l)}{2\Gamma(d-2)\,l!\,(|x-x_1||x-x_2|)^{d-s}}\int\frac{du}{2\pi}\frac{Q_l(u)}{\left(1+C_\phi c_\sigma\pi^{d/2}Q_l(u)\right)}\\ &= \frac{\Gamma\left(\frac{d}{2}\right)F'(c_\sigma)}{\pi^{\frac{d}{2}}}\left(\frac{|x_2-x_1|}{|x-x_1||x-x_2|}\right)^{d-s}.\end{aligned} \tag{3.41}$$

In the limit of small c_σ, we can expand in powers of c_σ

$$F(c_\sigma) = -\sum_{l=0}^{\infty}\frac{(2l+d-2)\,\Gamma(d-2+l)}{\Gamma(d-1)\,l!}\int\frac{du}{2\pi}\sum_{L=1}^{\infty}\frac{\left(-C_\phi c_\sigma\pi^{d/2}Q_l(u)\right)^L}{L}. \tag{3.42}$$

For the $L=1$ term, we can explicitly perform the integral by summing up residues, and then perform the sum over l to check that it vanishes. So in the limit of small c_σ, $F(c_\sigma)$ actually goes like c_σ^2. This is also expected because the $L=1$ term is proportional to $G^0(x,x)$ which is the short distance limit of the two-point function in flat space, which should be set to zero. To obtain this c_σ^2 it is easiest to go back to (3.37) and look at the $L=2$ term. The Green's function G^1 was written in (3.25) and after performing the integral over x, we get

$$F(c_\sigma) = -\frac{c_\sigma^2\Gamma\left(\frac{d-s}{2}\right)^4\Gamma\left(s-\frac{d}{2}\right)}{2^{2s+1}\Gamma(d-s)\,\Gamma\left(\frac{s}{2}\right)^4\Gamma\left(\frac{d}{2}\right)}. \tag{3.43}$$

As a check, note that in the special case of $s=2$, corresponding to the local (short range) $O(N)$ model, the expression of $Q_l(u)$ simplifies to

$$Q_l(u) = \frac{1}{\Gamma\left(\frac{d-2}{2}\right)\left(u^2+\frac{1}{4}\left(l+\frac{d-2}{2}\right)^2\right)}. \tag{3.44}$$

We can then perform the integral over u by closing the contour in the upper half plane and using the residue at $u = i\left(\frac{d-2+2l}{4}\right)$. This gives for $s = 2$

$$F(c_\sigma) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} c_\sigma^{k+2} (2k+1)!}{2^{2k+2} k! (k+2)! \Gamma(d-1)} \sum_{l=0}^{\infty} \frac{\Gamma(d-2+l)}{\left(l+\frac{d}{2}-1\right)^{2k+2} l!}. \tag{3.45}$$

It can be checked that this agrees with what was obtained in [108].

To find the large $\hat{j}$ limit of the scaling dimensions, we will also need the large c_σ behavior of $F(c_\sigma)$. To gain some intuition, note that for a constant mass, we have

$$\frac{1}{2}\mathrm{Tr}\log\left((-\nabla^2)^{\frac{s}{2}} + m^2\right) = \frac{\mathrm{Vol}(R^d)}{2}\int \frac{d^d p}{(2\pi)^d}\log\left(p^s + m^2\right) = \frac{\mathrm{Vol}(R^d)(m^2)^{\frac{d}{s}}}{2^{d+1}\pi^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}+1\right)\sin\left(\frac{\pi d}{s}\right)}. \tag{3.46}$$

A natural guess in the presence of a position dependent σ^* is that we should replace

$$\begin{aligned} \mathrm{Vol}(R^d)(m^2)^{\frac{d}{s}} &\to \int d^d x (\sigma_*(x))^{\frac{d}{s}} \implies \\ \frac{1}{2}\mathrm{Tr}\log\left((-\nabla^2)^{\frac{s}{2}} + \sigma_*(x)\right) &= \frac{\int d^d x (\sigma_*(x))^{\frac{d}{s}}}{2^{d+1}\pi^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}+1\right)\sin\left(\frac{\pi d}{s}\right)} \\ &= -\frac{(c_\sigma)^{\frac{d}{s}}\pi}{2^{d-1}\ d\ \Gamma\left(\frac{d}{2}\right)^2 \sin\left(\frac{\pi d}{s}\right)} \log\left(\frac{\delta^2}{|x_{12}|^2}\right) \end{aligned} \tag{3.47}$$

We will show in appendix 3.4 that this is indeed the correct behavior using heat kernel methods. So in the limit of large c_σ, we have (3.150)

$$F(c_\sigma) = \frac{(c_\sigma)^{\frac{d}{s}}\pi}{2^{d-1}\ d\ \Gamma\left(\frac{d}{2}\right)^2 \sin\left(\frac{\pi d}{s}\right)}\left(1 + O\left(\frac{1}{c_\sigma^{2/s}}\right)\right). \tag{3.48}$$

For finite c_σ, one can evaluate the functional determinant numerically using (3.39). But for the numerics to converge, it is necessary to regulate it. One simple way to do this is to use the following form

$$\begin{aligned} F(c_\sigma) = &\sum_{l=0}^{\infty} \frac{(2l+d-2)\Gamma(d-2+l)}{\Gamma(d-1)\, l!}\int_0^\infty \frac{du}{\pi}\Big[\log\left(1 + C_\phi c_\sigma \pi^{d/2} Q_l(u)\right) \\ &- C_\phi c_\sigma \pi^{d/2} Q_l(u) + \frac{\left(C_\phi c_\sigma \pi^{d/2} Q_l(u)\right)^2}{2}\Big] - \frac{c_\sigma^2 \Gamma\left(\frac{d-s}{2}\right)^4 \Gamma\left(s-\frac{d}{2}\right)}{2^{2s+1}\Gamma(d-s)\Gamma\left(\frac{s}{2}\right)^4 \Gamma\left(\frac{d}{2}\right)}, \end{aligned} \tag{3.49}$$

where we subtracted out the linear and quadratic pieces in c_σ and then added them back (the linear in c_σ term vanishes while the quadratic term is given by (3.43)). To avoid confusion, we emphasize that the last term in the above formula is not integrated over u or summed over l. This formula can

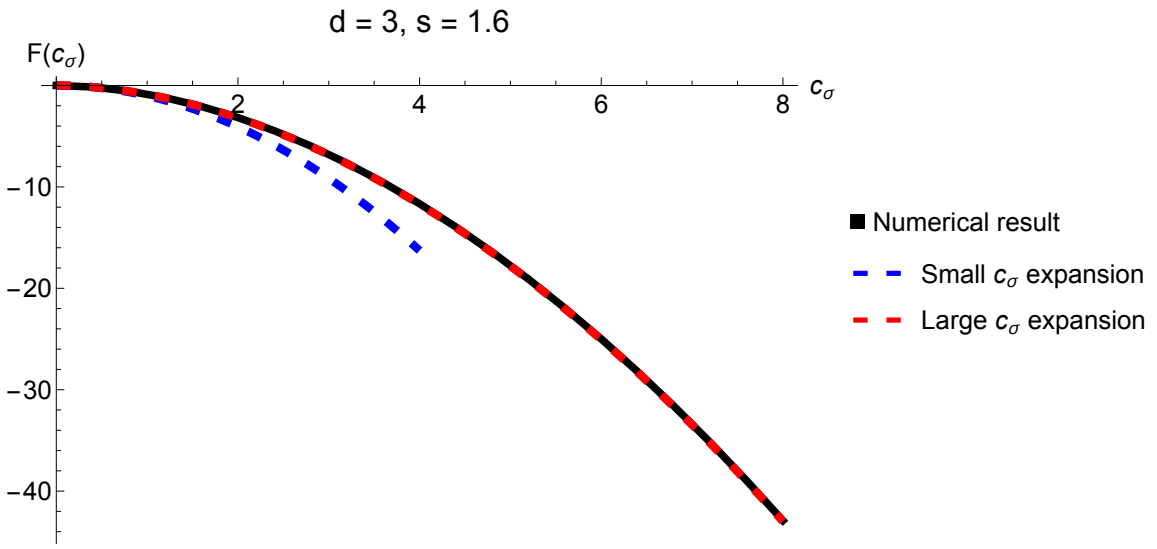


Figure 3.2: The numerical result for $F(c_\sigma)$ in $d = 3, s = 1.6$. We also plot the analytic expansions at small and large c_σ.

be directly used to numerically evaluate $F(c_\sigma)$. We plot the results for $d = 3, s = 1.6$ in (3.2). As is clear, the large c_σ result works very well even down to very small c_σ.

3.1.3 The scaling dimensions

We finally have all the ingredients to calculate the scaling dimensions of the operators $\mathcal{O}_j$, which can be extracted from the two-point function

$$\Delta_j = -\frac{1}{2}|x_{12}|\frac{\partial}{\partial|x_{12}|}\log\langle\mathcal{O}_j(x_1)\mathcal{O}_j(x_2)\rangle. \tag{3.50}$$

At the large N saddle point, using (3.5), this is given by

$$\begin{aligned}\Delta_j &= \frac{N}{2}|x_{12}|\frac{\partial}{\partial|x_{12}|}\left(\frac{1}{2}\log\det\left[\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)\right]-\hat{j}\log(G(x_1,x_2;\sigma))\right)\\ &= N\left(F(c_\sigma)+\hat{j}\mu(c_\sigma)\right)\end{aligned} \tag{3.51}$$

where we used (3.35) and (3.39). The number c_σ is determined by solving the saddle point equation (3.8), which after using (3.35) and (3.41) becomes

$$F'(c_\sigma) = -\hat{j}\mu'(c_\sigma). \tag{3.52}$$

Note that this just corresponds to extremizing $\Delta_j = N(F(c_\sigma)+\hat{j}\mu(c_\sigma))$ with respect to the constant c_σ.

Small $\hat{j}$ expansion

At small c_σ, we can use (3.43) and (3.34) to get the solution to the saddle point equation

$$\begin{aligned} c_\sigma &= \frac{\hat{j}2^{s+1}\Gamma\left(d-s\right)\Gamma\left(\frac{s}{2}\right)^3}{\Gamma\left(\frac{d-s}{2}\right)^3\Gamma\left(s-\frac{d}{2}\right)} \implies \\ \frac{\Delta_j}{N} &= \frac{d-s}{2}\hat{j} + \frac{2\Gamma\left(d-s\right)\Gamma\left(\frac{s}{2}\right)^2}{\Gamma\left(\frac{d-s}{2}\right)^2\Gamma\left(s-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}\hat{j}^2 + O(\hat{j}^3). \end{aligned} \tag{3.53}$$

Note that, recalling that $\hat{j} = j/N$, the quadratic term in $\hat{j}$ above should match the term proportional to j^2 in the anomalous dimension to order $1/N$ computed in the standard large N diagrammatic expansion. We check this explicitly in appendix 3.3.

In the next section, we will also study this model in an ϵ expansion in $s = \frac{d+\epsilon}{2}$ for any N but with ϵj held fixed. To compare with the results in that section, we write here the above result for this value of s to leading order in ϵ

$$\Delta_j = j\left(\frac{d}{4} + \frac{\epsilon j}{N} + O\left((\epsilon j)^2\right)\right). \tag{3.54}$$

Large $\hat{j}$ expansion

At large c_σ, using (3.48) and (3.34) , the saddle point equation gives

$$c_\sigma = \left(\frac{\hat{j}\ 2^{d+s}\ s\ \Gamma\left(\frac{d+s}{2}\right)\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi d}{s}\right)}{\pi\Gamma\left(-\frac{s}{2}\right)}\right)^{\frac{s}{d+s}} \tag{3.55}$$

This gives the dimension of the large charge operator in the limit of large $\hat{j}$

$$\begin{aligned} \frac{\Delta_j}{N} &= \frac{d+s}{2}\hat{j} + A(d,s)\hat{j}^{\frac{d}{d+s}} \\ A(d,s) &= \frac{2\pi(d+s)}{\Gamma\left(\frac{d}{2}\right)^2\sin\left(\frac{\pi d}{s}\right)ds}\left(\frac{\Gamma\left(\frac{d+s}{2}\right)s\sin\left(\frac{\pi d}{s}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(-\frac{s}{2}\right)\pi}\right)^{\frac{d}{d+s}} \end{aligned} \tag{3.56}$$

Notice that the factor in the parenthesis above becomes negative for $s > 2$ and for $s < d/2$ (we are considering only $d > 2$ case here). This implies that outside of the range $d/2 < s < 2$, the factor $A(d,s)$ becomes complex. This is consistent with the fact that the long range real fixed points only exists in the range $d/2 < s < s^*$, with $s^* = 2 + O(1/N)$. As mentioned in the introduction, for

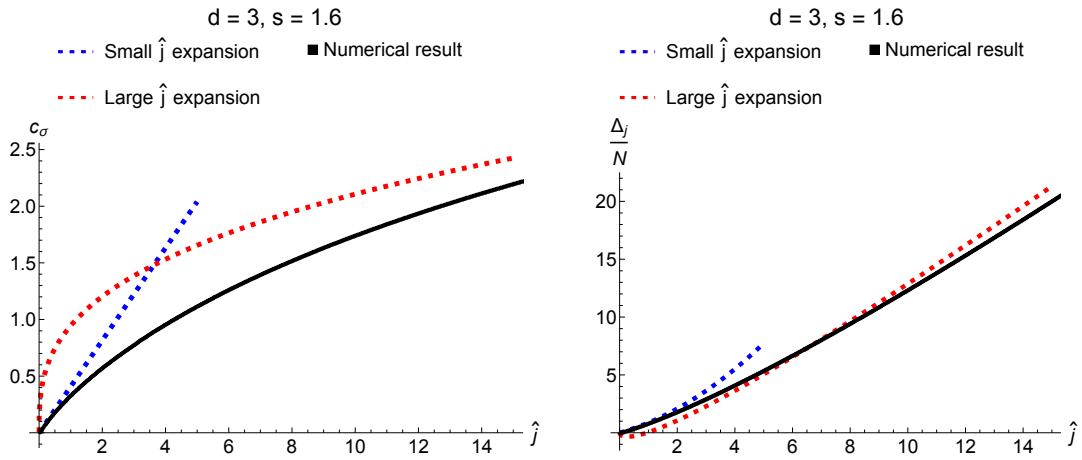


Figure 3.3: The numerical results for the dimension Δ_j and the solution to the saddle point equation for $d = 3, s = 1.6$. In both the plots, black line represent the numerical results, the dashed red line is the analytical result in a large $\hat{j}$ expansion and the dashed blue line is the analytical result in a small $\hat{j}$ expansion.

$s < d/2$ the IR limit of the long range model is described by the Gaussian fixed point, while for $s \geq 2$ the system should cross over to the short range fixed point. We will comment on this more in subsection 3.1.5.

For $s = \frac{d+\epsilon}{2}$, where we can compare to the weakly coupled Wilson-Fisher fixed point, the expansion in ϵj of the above result at leading order in ϵ yields

$$\Delta_j = j\left(\frac{3d}{4} - \frac{3\left(\sin\left(\frac{\pi d}{4}\right)\Gamma\left(\frac{d}{4}+1\right)\Gamma\left(\frac{3d}{4}\right)\right)^{2/3}}{\pi^{2/3}\Gamma\left(\frac{d}{2}\right)^{4/3}}\left(\frac{N}{\epsilon j}\right)^{1/3}\right). \tag{3.57}$$

For general intermediate $\hat{j}$, we can numerically evaluate $F(c_\sigma)$ and $\mu(c_\sigma)$ and then solve the saddle point equation (3.52) numerically to obtain c_σ as a function of $\hat{j}$. We can then plug in this into (3.51) to get the scaling dimensions. We show these numerical results for $d = 3, s = 1.6$ in figure 3.3.

3.1.4 Correlation functions

So far we have focused on the two-point function of large charge operators. But having access to the Green's function at the large charge saddle point, it is easy to obtain also higher point correlation functions involving two heavy (large charge) and an arbitrary number of light (finite charge) operators. In this subsection, we will focus on the "heavy-heavy-light" three point function and "heavy-heavy-light-light" four point function. We will closely follow the approach used in [108].

Let us start with the three point function, and as before we will consider scalar operators in

symmetric traceless representation of $O(N)$, which may be written as $\mathcal{O}_j = (u \cdot \phi)^j$. Their three-point function is fixed by conformal symmetry and $O(N)$ symmetry upto an overall constant

$$\begin{aligned}&\langle \mathcal{O}_{j_1}(x_1,u_1)\mathcal{O}_{j_2}(x_2,u_2)\mathcal{O}_{j_3}(x_3,u_3)\rangle = \\ &C_{j_1 j_2 j_3}\frac{(u_1\cdot u_2)^{(j_1+j_2-j_3)/2}(u_1\cdot u_3)^{(j_1+j_3-j_2)/2}(u_2\cdot u_3)^{(j_2+j_3-j_1)/2}}{|x_{12}|^{\Delta_{j_1}+\Delta_{j_2}-\Delta_{j_3}}|x_{13}|^{\Delta_{j_1}+\Delta_{j_3}-\Delta_{j_2}}|x_{23}|^{\Delta_{j_2}+\Delta_{j_3}-\Delta_{j_1}}}.\end{aligned} \tag{3.58}$$

In the following, we will calculate this overall constant when the two operators are heavy and the third one is light. We will choose a configuration such that j_1 and j_2 are large with j_3 held fixed. To be specific, let us choose $j_1 = j+q, j_2 = j$ where $j \to \infty$ while $\hat{j} = j/N$ and q are held fixed. The correlation function may be explicitly computed using techniques similar to what we used to calculate the two-point function earlier

$$\begin{aligned}&\langle \mathcal{O}_{j_1}(x_1,u_1)\mathcal{O}_{j_2}(x_2,u_2)\mathcal{O}_{j_3}(x_3,u_3)\rangle \\ &= \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\sigma\,(u_1\cdot\phi(x_1))^{j+q}\,(u_2\cdot\phi(x_2))^{j}\,(u_3\cdot\phi(x_3))^{j_3}\,e^{-\frac{C}{2}\int d^dy d^dz \frac{\phi^K(y)\phi^K(z)}{|y-z|^{d+s}} - \frac{1}{2}\int d^dx \sigma\phi^K\phi^K(x)} \\ &= n_{j+q,j,j_3}\int \mathcal{D}\sigma\,(u_1\cdot u_2 G_{12})^{j+(q-j_3)/2}\,(u_1\cdot u_3 G_{13})^{(j_3+q)/2}\,(u_2\cdot u_3 G_{23})^{(j_3-q)/2} \\ &\qquad\qquad \times e^{-\frac{N}{2}\log\det\left(\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)\right)}\end{aligned} \tag{3.59}$$

where in the last line, we just did the Wick contractions which gave rise to the combinatorial factor

$$n_{j+q,j,j_3} = \frac{(j+q)!j!j_3!}{\left(j+\frac{q-j_3}{2}\right)!\left(\frac{q+j_3}{2}\right)!\left(\frac{j_3-q}{2}\right)!}. \tag{3.60}$$

Note that this three point function is only nonzero when $-j_3 \le q \le j_3$ and j_3+q is even. The Green's function $G_{ij} = G(x_i,x_j;\sigma)$ is the two point function in the presence of a non-trivial σ. However, notice that at large j and N, the saddle point will only be affected by the factor of $(G_{12})^j$ in the prefactor above which gives the same exponent as in (3.5). Therefore the large N, large j saddle point is the same as before, and the three-point function is given by simply plugging in the previous saddle point solution into the above equation. Then, using (3.35) and (3.39), we get

$$\begin{aligned}&\langle \mathcal{O}_{j_1}(x_1,u_1)\mathcal{O}_{j_2}(x_2,u_2)\mathcal{O}_{j_3}(x_3,u_3)\rangle = n_{j+q,j,j_3}\left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{2\pi^{d/2}}\right)^{j+\frac{j_3+q}{2}}\delta^{2NF(c_\sigma)+(2j+q)\left(\mu-\left(\frac{d-s}{2}\right)\right)} \\ &\times \frac{(u_1\cdot u_2)^{j+(q-j_3)/2}\,(u_1\cdot u_3)^{(j_3+q)/2}\,(u_2\cdot u_3)^{(j_3-q)/2}}{|x_{12}|^{2NF(c_\sigma)+\mu(2j+q)-j_3\left(\frac{d-s}{2}\right)}|x_{13}|^{\mu q+j_3\left(\frac{d-s}{2}\right)}|x_{23}|^{-\mu q+j_3\left(\frac{d-s}{2}\right)}}.\end{aligned} \tag{3.61}$$

Note that this is scheme dependent through its dependence on δ. To get a scheme independent result, we can choose a normalization in which the coefficient of the two-point function is normalized to one. For that, we need to divide the above result by the square root of the coefficient of the two-point functions. For the large charge operators, the two-point function is normalized as

$$\langle \mathcal{O}_j(x_1,u_1)\mathcal{O}_j(x_2,u_2)\rangle = \left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{2\pi^{d/2}}\right)^j \frac{(u_1\cdot u_2)^j\, j!\delta^{2NF(c_\sigma)+j(2\mu-(d-s))}}{|x_{12}|^{2NF(c_\sigma)+2j\mu}} \tag{3.62}$$

as we found earlier, while for the light operator, we have the usual normalization

$$\langle \mathcal{O}_{j_3}(x_1,u_1)\mathcal{O}_{j_3}(x_2,u_2)\rangle = \frac{(u_1\cdot u_2)^{j_3}\, j_3! C_\phi^{j_3}}{|x_{12}|^{j_3(\frac{d-s}{2})}}. \tag{3.63}$$

Then the normalized coefficient of three-point function is

$$a_{j+q,j,j_3} = \frac{n_{j+q,j,j_3}}{\sqrt{(j+q)!j!j_3!}}\left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{2\pi^{d/2}C_\phi}\right)^{\frac{j_3}{2}} = \frac{\sqrt{j_3!}}{\left(\frac{q+j_3}{2}\right)!\left(\frac{j_3-q}{2}\right)!}\left(\frac{2^{s-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s}{2}\right)\mu'(c_\sigma)N\hat{j}}{\Gamma\left(\frac{d-s}{2}\right)}\right)^{\frac{j_3}{2}} \tag{3.64}$$

where we already took the large N limit. The c_σ in the above expression is the one that solves the saddle point equation (3.52). At large $\hat{j}$, the OPE coefficient has the following scaling

$$a_{j+q,j,j_3} = \frac{\sqrt{j_3!}}{\left(\frac{q+j_3}{2}\right)!\left(\frac{j_3-q}{2}\right)!}\left(-\frac{N\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{d-s}{2}\right)}\left(\frac{\pi}{s\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi d}{s}\right)}\right)^{\frac{2s}{d+s}}\right)^{\frac{j_3}{2}}\left(\frac{\hat{j}\Gamma\left(\frac{d+s}{2}\right)}{\Gamma\left(-\frac{s}{2}\right)}\right)^{\frac{j_3(d-s)}{2(d+s)}} + \dots \tag{3.65}$$

Next, we look at the four-point function of two large charge and two finite charge operators. For simplicity of presentation, let us choose the large charge operators to be $\mathcal{Z}^j$ and $\bar{\mathcal{Z}}^j$ with $\mathcal{Z}^j = (\phi^1 + i\phi^2)^j$. Let us further choose the finite charge operators to have charge one. Then there are two possible four point functions we can consider. The first one is [3]

$$\begin{aligned}\langle \mathcal{Z}^j(x_1)\bar{\mathcal{Z}}^j(x_2)\phi^a(x_3)\phi^b(x_4)\rangle &= \delta^{ab}j!\int \mathcal{D}\sigma\,(2G_{12})^j\, G_{34} e^{-\frac{N}{2}\log\det\left(\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)\right)} \\ &= \frac{\delta^{ab}j!}{|x_{12}|^{2NF(c_\sigma)+2j\mu}}\left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{\pi^{d/2}}\right)^j G(x_3,x_4;\sigma^*)\end{aligned} \tag{3.66}$$

where a, b are not equal to 1 or 2. So this four-point function is just proportional to the Green's

[3] We are now going to not write the factors of renormalization scale δ here anymore, which as we explained in the context of three-point function, may be absorbed in the definition of the operators.

function we found in (3.32). As expected, this has the form of a CFT four-point function

$$\begin{aligned}\langle \mathcal{Z}^j(x_1)\bar{\mathcal{Z}}^j(x_2)\phi^a(x_3)\phi^b(x_4)\rangle &= \frac{\delta^{ab} j!}{|x_{12}|^{2NF(c_\sigma)+2j\mu}|x_{34}|^{d-s}}\left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{\pi^{d/2}}\right)^j \frac{C_\phi \Gamma\left(\frac{d-2}{2}\right)}{(XY)^{\frac{d-s}{4}}} \\ &\times \sum_{l=0}^{\infty}\left(l+\frac{d-2}{2}\right)C_l^{\left(\frac{d-2}{2}\right)}\left(\frac{X+Y-1}{2\sqrt{XY}}\right)\int\frac{du}{2\pi}\left(\frac{Y}{X}\right)^{iu}\frac{Q_l(u)}{1+C_\phi c_\sigma \pi^{d/2}Q_l(u)}\end{aligned} \tag{3.67}$$

with cross-ratios

$$X = \frac{|x_3-x_1|^2|x_4-x_2|^2}{|x_1-x_2|^2|x_3-x_4|^2}, \qquad Y = \frac{|x_3-x_2|^2|x_4-x_1|^2}{|x_1-x_2|^2|x_3-x_4|^2}. \tag{3.68}$$

The other four-point function we can consider in this simple setting is

$$\begin{aligned}&\langle \mathcal{Z}^j(x_1)\bar{\mathcal{Z}}^j(x_2)\mathcal{Z}(x_3)\bar{\mathcal{Z}}(x_4)\rangle = \\ &j!2^{j+1}\int \mathcal{D}\sigma\left(G_{12}^j + jG_{12}^{j-1}G_{14}G_{23}\right)e^{-\frac{N}{2}\log\det\left(\frac{C}{|x-y|^{d+s}}+\sigma(x)\delta^d(x-y)\right)}.\end{aligned} \tag{3.69}$$

Because of the explicit factor of j upfront, the second term dominates. Then using (3.35) and (3.39) as before, we get

$$\langle \mathcal{Z}^j(x_1)\bar{\mathcal{Z}}^j(x_2)\mathcal{Z}(x_3)\bar{\mathcal{Z}}(x_4)\rangle = \left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{\pi^{d/2}}\right)^{j+1}\frac{(j+1)!}{|x_{12}|^{2NF(c_\sigma)+2j\mu}|x_{34}|^{d-s}}X^{\frac{2\mu-(d-s)}{4}}Y^{\frac{-2\mu-(d-s)}{4}}. \tag{3.70}$$

Let us also normalize this four-point function by dividing it by the two-point function coefficients, so that we can extract the OPE coefficients in $13 \to 24$ channel and compare it with what we got by calculating three-point functions

$$\begin{aligned}\langle \mathcal{Z}^j(x_1)\bar{\mathcal{Z}}^j(x_2)\mathcal{Z}(x_3)\bar{\mathcal{Z}}(x_4)\rangle_{\text{norm.}} &= \left(\frac{\Gamma\left(\frac{d}{2}\right)\mu'(c_\sigma)}{2C_\phi\pi^{d/2}}\right)\frac{N\hat{j}}{|x_{12}|^{2\Delta_j}|x_{34}|^{d-s}}X^{\frac{2\mu-(d-s)}{4}}Y^{\frac{-2\mu-(d-s)}{4}} \\ &= \frac{1}{(x_{13}x_{24})^{\Delta_j+\Delta_\phi}}\left(\frac{x_{34}}{x_{12}}\right)^{\Delta_j-\Delta_\phi}\sum_{\Delta,s}a_{\Delta,s}^2 X^{\frac{\Delta-s}{2}}g_{\Delta,s}(X,Y)\end{aligned} \tag{3.71}$$

where we expanded the four-point function into conformal blocks $g_{\Delta,s}$ in the $13 \to 24$ channel (see for instance [113]). Now that we have the four-point function, it is possible to extract all the OPE data from it, but here we will just calculate the OPE coefficient of the leading operator that appears in $13 \to 24$ channel. It should be a scalar operator with charge $j+1$ and should correspond to the leading term in the four-point function in the limit $X \to 0$ and $Y \to 1$. The blocks are normalized

such that $g_{\Delta,s}(X,Y) = 1 + \dots$ in the limit $X \to 0, Y \to 1$. Then the leading operator has the following dimension and OPE coefficient

$$\Delta_{j+1} = \Delta_j + \mu \qquad a^2_{j,1,j+1} = \left(\frac{2^{s-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s}{2}\right)\mu'(c_\sigma)}{\Gamma\left(\frac{d-s}{2}\right)} \right) N\hat{j}. \tag{3.72}$$

This OPE coefficient agrees with what we found using the three-point function calculation (3.65) for $j_3 = q = 1$. The result for the scaling dimension implies that at large j, the derivative of the dimension with respect to j is μ

$$\Delta_{j+1} - \Delta_j = \frac{\partial \Delta_j}{\partial j} = \mu. \tag{3.73}$$

This is in agreement with the structure of the result for the scaling dimension that we found earlier, as can be seen as follows

$$\frac{\partial \Delta_j}{\partial j} = \frac{\partial}{\partial j}\left(NF(c_\sigma) + \mu j\right) = \mu + \left(NF'(c_\sigma) + \mu'(c_\sigma) j\right)\frac{\partial c_\sigma}{\partial j} = \mu \tag{3.74}$$

where we used the saddle point equation (3.52).

3.1.5 Crossover to the short range regime

As we mentioned earlier, at a certain critical s_*, the behavior of the long range model is expected to cross over to that of the short-range $O(N)$ model. For recent discussions of this crossover see [68, 67, 91]. Recall that $s_* = 2 - 2\gamma_\phi^{\rm SR}$ where $\gamma_\phi^{\rm SR}$ is the anomalous dimension of ϕ at the short-range fixed point. However, in the large N expansion, γ_ϕ is of order $1/N$, so in the regime we are working in, the crossover must happen at $s = 2$. In this subsection, we will study how the dimensions of the large charge operators behave near $s = 2$, and how the scaling dimension of the large charge operators may cross over from the long range to the short range behavior.

Let us start by observing that the solution of (3.33), with $u = i\mu/2$ (where $\mu = \mu(c_\sigma)$ will be physically related to the chemical potential on the cylinder), has several branches for any $s < 2$, while for $s = 2$, it has a single solution given by

$$\mu(c_\sigma) = \sqrt{c_\sigma + \left(\frac{d}{2} - 1\right)^2}, \tag{3.75}$$

see figure 3.4. At small c_σ, the values of μ on the various branches go as $\mu(c_\sigma) = \frac{d-s}{2} + 2n + O(c_\sigma)$ with $n = 0, 1, 2, \dots$. One can see that the small c_σ expansion of (3.75) matches what we get by just plugging in $s = 2$ in the small c_σ expansion in (3.34), which gives the value of $\mu(c_\sigma)$ on the first

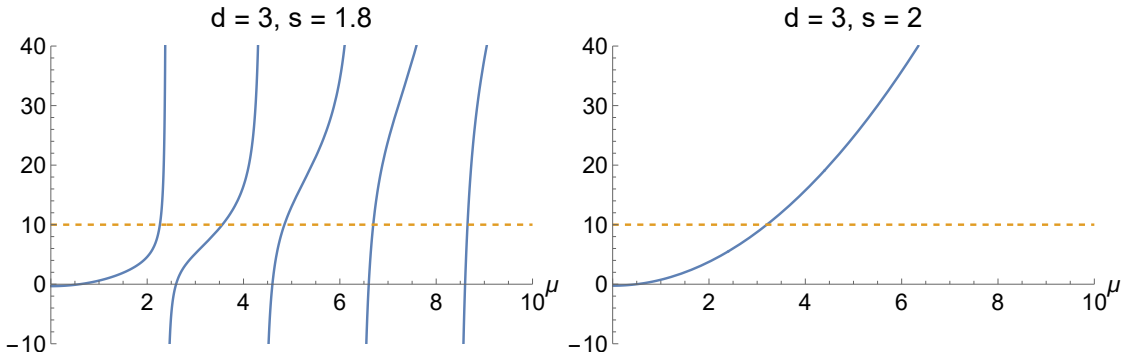


Figure 3.4: The plot of (3.33) for $d = 3$ at $s = 1.8$ and $s = 2$. The horizontal dashed line is the line $c_\sigma = 10$ and for this value of c_σ, the solution μ of (3.33) is given by the point where this dashed line intersects the curve. It is clear that there are several branches of solution for $s < 2$ while there is only one for $s = 2$.

branch (the one with smallest μ). But at large c_σ, the above $s = 2$ result goes like $c_\sigma^{1/2}$ which is very different from the large c_σ expansion in (3.34). In particular, for any $s < 2$ the result (3.34) saturates below $(d + 2)/2$, while the $s = 2$ solution (3.75) crosses that point at $c_\sigma = 2d$ and keeps growing. So if we always stay on the first branch, the function $\mu(c_\sigma)$ can only have a smooth transition from $s < 2$ to the $s = 2$ behavior for $c_\sigma < 2d$, and beyond this value one may expect that the higher roots of (3.33) should play a role. To get further intuition, it is useful to plot the solutions for various branches of μ as s approaches 2 (see figure 3.5). One can see that the $s = 2$ result (3.75) arises essentially by "gluing" portions of different branches as s approaches 2 from below. Note that the function $\mu(c_\sigma)$ on the first branch, as we approach $s = 2$, tends to develop a kink at $c_\sigma = 2d$. This becomes a true kink at $s = 2$, with $\mu(c_\sigma)$ turning to a constant beyond that value. Similar kinks appear on the higher branches.[4]

When we compute the scaling dimension by extremizing $\Delta_j = N(F(c_\sigma)+\hat{j}\mu(c_\sigma))$ with respect to c_σ, we expect the higher branches of μ to lead to additional solutions to the saddle point equation. At small $\hat{j}$, it is easy to see that this leads to a tower of solutions with $\Delta_j^{(n)}/N = (\frac{d-s}{2}+2n)\hat{j}+O(\hat{j}^2)$. For finite $\hat{j}$, one can find these solutions numerically. Given the above discussion, we should see that the $s = 2$ behavior for Δ_j arises by "gluing" the contributions of the saddles obtained from different branches.[5] This is indeed what we find, as shown in figure 3.6. The $s = 2$ case was considered in [98, 108]. In $d = 3$ and at large $\hat{j}$ the result behaves as

$$\frac{\Delta_j}{N} = \frac{2}{3}\hat{j}^{\frac{3}{2}} + \frac{1}{6}\hat{j}^{\frac{1}{2}} + O\left(\frac{1}{\hat{j}^{\frac{1}{2}}}\right). \tag{3.76}$$

[4]The position of the kinks at $s = 2$ is given by $c_\sigma = 2(d + 2n)(n + 1)$, $n = 0, 1, 2, \ldots$.

[5]Note that the functional determinant $F(c_\sigma)$ has a smooth limit as $s \to 2$, which can for instance be seen by setting $s = 2$ in (3.43) and (3.48) and checking that it agrees with the results in [108].

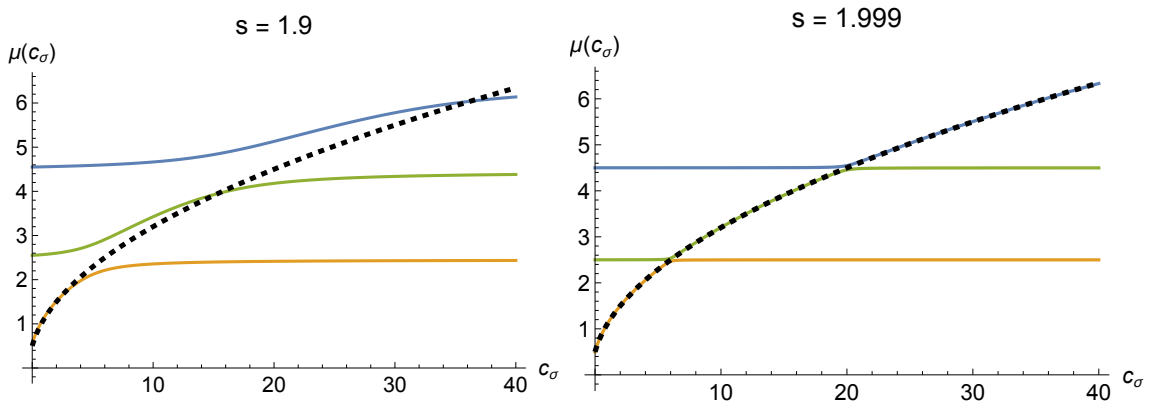


Figure 3.5: The numerical result for various branches of the solution $\mu(c_\sigma)$ for $s = 1.9$ and for $s = 1.999$ in $d = 3$ dimensions. The dashed line is the $s = 2$ result (3.75), so it is clear that $s = 2$ result arises by "gluing" different branches.

In figure 3.6, we just plot the result in (3.76), since, as observed for instance in [108], the large $\hat{j}$ expansion gives a very good approximation to the true numerical value even down to relatively low $\hat{j}$. Note that, as shown in figure 3.6 to the right, the solution coming from the first branch smoothly goes to the $s = 2$ behavior for $\hat{j} < \hat{j}_{\text{crit.}}$. We can get an estimate for this critical value of $\hat{j}$. As we saw above, the $s = 2$ result for $\mu(c_\sigma)$ starts diverging from the $s < 2$ result at $c_\sigma = 6$ in $d = 3$. When $s = 2$ and $d = 3$, c_σ is related to $\hat{j}$ as [98, 108]

$$c_\sigma = \hat{j} - \frac{1}{12} + O\left(\frac{1}{\hat{j}}\right). \tag{3.77}$$

So we expect the curve for the $s = 2$ result to diverge from the $s \lesssim 2$ result at around $\hat{j} \sim 6$. Beyond this value, the $s = 2$ behavior is instead well approximated by the saddle obtained from the second branch of μ, until we reach another critical value of $\hat{j}$ around $\hat{j} \sim 20$, and so on. Note that in the strict $s \to 2$ limit, each branch produces a solution to the saddle point equation only within a certain interval of c_σ (and corresponding $\hat{j}$), outside of which μ becomes a constant (see figure 3.5), which does not allow for solutions to $F'(c_\sigma) + \hat{j}\mu'(c_\sigma) = 0$. Therefore, in the $s \to 2$ limit, the short range behavior $\Delta_j \sim \frac{2}{3}\hat{j}^{\frac{3}{2}}$ is indeed reproduced by "gluing" the saddle point solutions obtained from the different branches.[6] However, for $s < 2$ and infinite N, the dominant behavior always comes from the first branch, which in particular gives scaling dimensions that go as $\Delta_j \sim \frac{d+s}{2} j$ at large $\hat{j}$.

[6]It would be interesting to see if this merging of the branches can be interpreted as some kind of operator mixing. Indeed, it is natural to think of the solutions for Δ_j obtained from the higher branches as the dimensions of operators with the same charge but higher bare dimensions. For instance, on the second branch we have $\Delta_j/N = \left(\frac{d-s}{2} + 2\right)\hat{j} + \ldots$ at small $\hat{j}$, which could be viewed as the dimension of an operator of the schematic form $\sim \left(\partial^2(\phi^1 + i\phi^2)\right)^j$. While at small $\hat{j}$ the scaling dimensions on different branches are well separated, at sufficiently large $\hat{j}$ and $s \to 2$ they can approach each other, and mixing may occur.

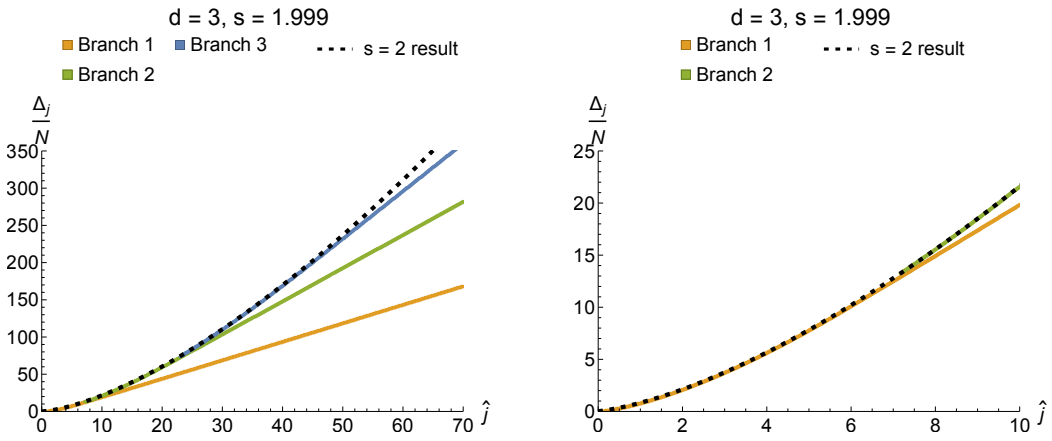


Figure 3.6: The numerical results for the dimension Δ_j obtained from various different branches of $\mu(c_\sigma)$ for $d = 3, s = 1.999$. In the right plot, we just zoom in to the small $\hat{j}$ region.

The picture we described above applies in the infinite N limit we studied in this chapter, where the transtion to the short range regime should happen at precisely $s = 2$. If we include $1/N$ corrections, however, the role of the higher branches should become important slightly below $s = 2$, since the crossover is expected to happen at $s^* = 2-2\gamma_\phi^{SR} = 2-O(1/N) < 2$. It would be interesting to compute the subleading corrections to the scaling dimensions by including the determinant of the fluctuations around the saddle points, and further clarify how the transition to the short range regime works within the large charge sector.

3.1.6 The long range model in $d = 1$

Let us now consider the special case of the $d = 1$ long range $O(N)$ model. Contrary to the usual short range case, there is a non-trivial fixed point for the one-dimensional long range model in the range $0 < s < 1$ [69, 70, 71, 72, 73]. At $s = 1$, we expect a crossover to the short range fixed point, which is the trivial zero temperature fixed point where all correlation functions become constant. So we expect the scaling dimensions to go to zero as s approaches 1. We will show that this is the case for the large charge operators that we have been considering.

Let us start with the Green's function (3.32). In $d = 1$, the two cross-ratios are related to each other and there is only one cross-ratio, which we can take to be χ defined by

$$X = \chi^2, \qquad Y = (1-\chi)^2. \tag{3.78}$$

The makes the argument of the Gegenbauer polynomials equal to 1 and then using (3.40), the result

for the Green's function is

$$
\begin{aligned}
&G(x,y,\sigma^*)\\
&=\sum_{l=0}^{\infty}\frac{C_\phi\Gamma\left(\frac{d-2}{2}\right)(2l+d-2)\Gamma\left(d-2+l\right)}{2|x-y|^{1-s}\left(\chi|1-\chi|\right)^{\frac{1-s}{2}}\Gamma\left(d-2\right)l!}\int\frac{du}{2\pi}\left(\frac{\chi^2}{(1-\chi)^2}\right)^{-iu}\frac{Q_l(u)}{1+C_\phi c_\sigma\pi^{d/2}Q_l(u)} \qquad (3.79)\\
&=\sum_{l=0,1}\frac{\Gamma\left(\frac{1-s}{2}\right)}{2^s\Gamma\left(\frac{s}{2}\right)|x-y|^{1-s}\left(\chi|1-\chi|\right)^{\frac{1-s}{2}}}\int\frac{du}{2\pi}\left(\frac{\chi^2}{(1-\chi)^2}\right)^{-iu}\frac{Q_l(u)}{1+C_\phi c_\sigma\pi^{d/2}Q_l(u)}.
\end{aligned}
$$

In $d=1$, the prefactor of the integral vanishes unless $l=0$ or 1, so the sum collapses to only those two terms. In the next section, we will show that all these calculations may also be done by mapping to a cylinder, $R\times S^{d-1}$. Then the sum over l comes from summing over angular momentum modes on the sphere. However in $d=1$, there is no sphere, so the sum over l must collapse.

Similarly for the functional determinant (3.39), we have

$$F(c_\sigma)=\sum_{l=0,1}\int_{-\infty}^{\infty}\frac{du}{2\pi}\log\left(1+C_\phi c_\sigma\pi^{d/2}Q_l(u)\right). \qquad (3.80)$$

At large c_σ, this goes like (3.146)

$$F(c_\sigma)=\frac{c_\sigma^{1/s}}{\sin\left(\frac{\pi}{s}\right)}\left(1-\frac{1-s^2}{24c_\sigma^{2/s}}+...\right). \qquad (3.81)$$

Since we don't have an infinite sum over l, it is also possible to extract the large c_σ behavior directly from (3.80). To do that, we first differentiate $F(c_\sigma)$ with c_σ and expand at large u. We then rescale the integration variable $u\to c_\sigma^{1/s}\,u$ and then expand at large c_σ. Finally we perform the integral over u and then the integrate back over c_σ. The two results of course agree. At generic values of c_σ, it is possible to evaluate $F(c_\sigma)$ numerically using (3.49), with sum now only running over $l=0$ and $l=1$.

Combining (3.81) with the large c_σ behavior of $\mu(c_\sigma)$ from (3.34), we can solve the saddle point equation at large c_σ

$$c_\sigma=\left(\frac{j2^{s+1}s\Gamma\left(\frac{s+1}{2}\right)}{\sqrt{\pi}\csc\left(\frac{\pi}{s}\right)\Gamma\left(-\frac{s}{2}\right)}\right)^{\frac{s}{s+1}}-\frac{2\sin\left(\frac{\pi s}{2}\right)\Gamma(s+1)\left(\pi s\cot\left(\frac{\pi s}{2}\right)+2s(\psi^{(0)}(s)+\gamma)+2\right)}{\pi(s+1)}. \qquad (3.82)$$

Corrections to the above are of order $O(j^{-\frac{s}{1+s}})$ at large $\hat{j}$. We can then use this result to get the

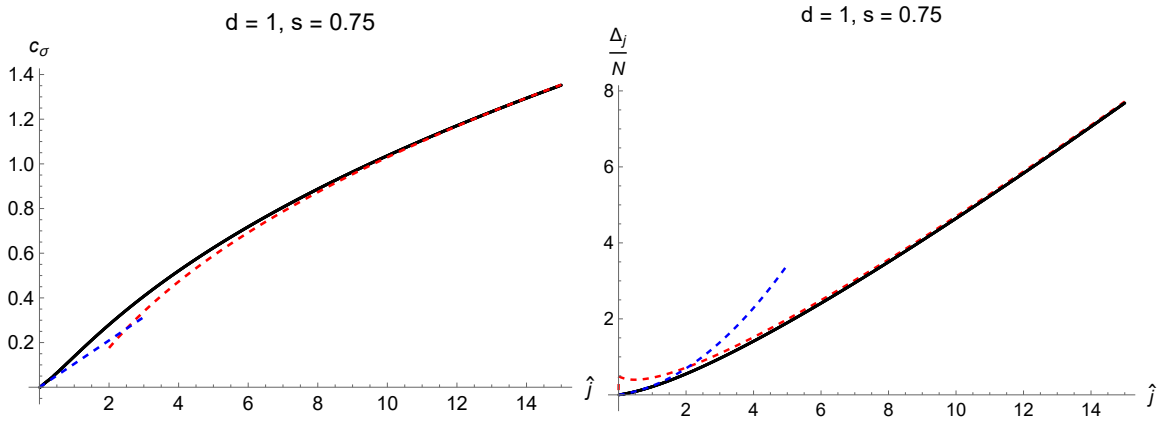


Figure 3.7: The numerical results for the dimension Δ_j and the solution to the saddle point equation for $d = 1, s = 0.75$. In both plots, black line represents the numerical results, the dashed red line is the analytical result in a large $\hat{j}$ expansion and the dashed blue line is the analytical result in a small $\hat{j}$ expansion.

dimensions of the large charge operators in a large $\hat{j}$ expansion

$$\begin{aligned}\frac{\Delta_j}{N} = \frac{1+s}{2}\hat{j} + \frac{2(1+s)}{\sin\left(\frac{\pi}{s}\right)s}\left(\frac{\Gamma\left(\frac{1+s}{2}\right)s\sin\left(\frac{\pi}{s}\right)\hat{j}}{\Gamma\left(-\frac{s}{2}\right)\sqrt{\pi}}\right)^{\frac{1}{1+s}} \\ - \frac{2^{1-s}\Gamma(s)\sin\left(\frac{\pi s}{2}\right)\left(\pi s\cot\left(\frac{\pi s}{2}\right) + 2s(\psi^{(0)}(s)+\gamma)+2\right)}{s\pi\sin\left(\frac{\pi}{s}\right)}\left(\frac{\Gamma\left(\frac{1+s}{2}\right)s\sin\left(\frac{\pi}{s}\right)\hat{j}}{\Gamma\left(-\frac{s}{2}\right)\sqrt{\pi}}\right)^{\frac{1-s}{1+s}}.\end{aligned} \tag{3.83}$$

It is also possible to numerically solve the saddle point equation and hence find the dimension of the large charge operators. We plot the result for the saddle solution and the dimensions for $s = 0.75$ in figure 3.7. Note that the analytical large $\hat{j}$ results in (3.82) and (3.83) work remarkably well.

Now let us discuss the behavior of the model as s approaches 1. Let us start by recalling the small c_σ results for $F(c_\sigma)$ and for $\mu(c_\sigma)$ when s is close to 1

$$\begin{aligned}\mu(c_\sigma) &= \frac{1-s}{2} + \frac{2c_\sigma}{\pi(1-s)} - \frac{4c_\sigma^2}{\pi^2(1-s)^3} + O(c_\sigma^3). \\ F(c_\sigma) &= -\frac{2c_\sigma^2}{\pi^2(1-s)^3}\end{aligned} \tag{3.84}$$

where we only wrote the leading order in $1-s$ result at each order in c_σ. Note that the expansion of $\mu(c_\sigma)$ clearly breaks down unless $c_\sigma \ll (1-s)^2$. Assuming the expansion of $F(c_\sigma)$ has a similar validity, we can solve the saddle point equation using these two expressions and we get

$$c_\sigma = \frac{\hat{j}(1-s)^2\pi}{2}, \qquad \frac{\Delta_j}{N} = \frac{1-s}{2}\hat{j} + \frac{1-s}{2}\hat{j}^2 + O(\hat{j}^3) \tag{3.85}$$

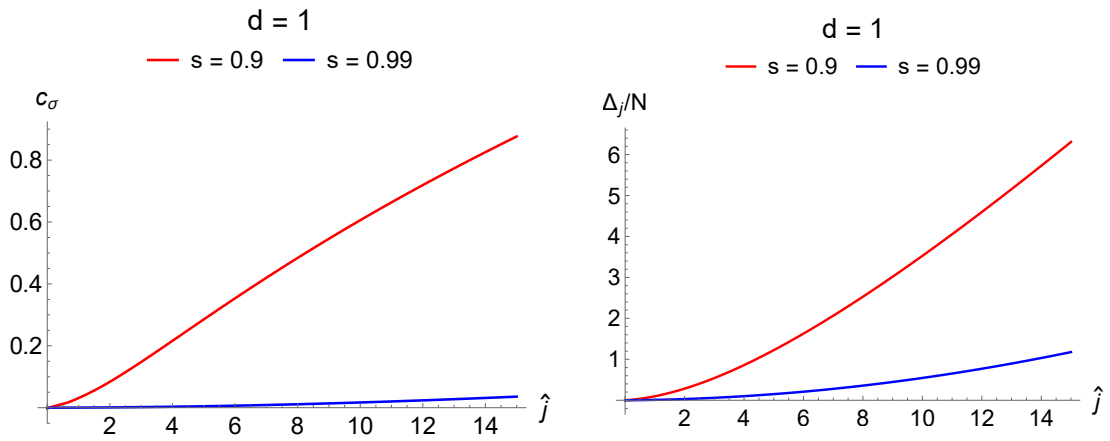


Figure 3.8: The numerical results for the dimension Δ_j and the solution to the saddle point equation in $d = 1$ for $s = 0.9$ (red) and $s = 0.99$ (blue). As one can see, both c_σ and Δ_j appear to approach zero as s gets closer to 1.

So this is only really valid for $\hat{j} \ll 1$. But clearly in this regime, both c_σ and Δ_j go to zero as $s \to 1$. This is consistent with the expectation that the dimensions should go to zero as s approaches 1. Next let us look at the large c_σ expansions

$$\begin{aligned} \mu(c_\sigma) &= 1 - \frac{2}{\pi c_\sigma} + \frac{4}{\pi^2 c_\sigma^2} + O(c_\sigma^3). \\ F(c_\sigma) &= \frac{c_\sigma}{\pi(s-1)}. \end{aligned} \tag{3.86}$$

Clearly, $F'(c_\sigma)$ diverges as $s \to 1$, while $\mu'(c_\sigma)$ is finite. So there is no solution to the saddle point equation with large c_σ close enough to $s = 1$.

In order to clarify what happens for finite $\hat{j}$, let us then look at the numerics as s gets closer to 1. In figure 3.8, we plot the solution to the saddle point equation for c_σ and scaling dimension for $s = 0.9$ and $s = 0.99$, and it seems clear that as s approaches 1, both of these quantities approach zero. To see how they approach 0, we can numerically evaluate Δ_j as a function of s for a fixed $\hat{j}$. We show these results in figure 3.9. The results seem to suggest the dimension goes to zero linearly as $1 - s$ even when $\hat{j}$ is not too small. It would be interesting to clarify this further, perhaps using the non-local non-linear sigma model considered in [1], which has a perturbative fixed point in $d = 1$ and $s = 1 - \epsilon$.

3.2 Scaling dimensions from the cylinder

In this section, we show that the scaling dimensions we calculated above can also be derived by studying the theory on a cylinder, which may be obtained by a Weyl transformation from the flat

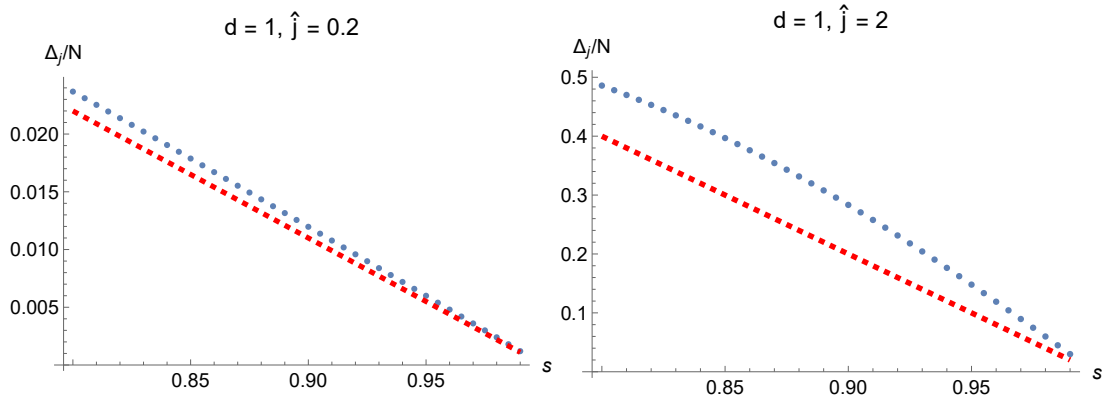


Figure 3.9: The numerical results for the dimension Δ_j as a function of s for $\hat{j} = 0.2$ and for $\hat{j} = 2$. The blue dots are the numerical points while the red dashed line is the analytical result for small $\hat{j}$ (3.85).

space. We will use this approach mainly as a check of the results we obtained above, so we will be brief. Such an approach has been used in several recent works [94, 95, 96, 97, 98, 103]. We will follow and generalize the approach used in [103], which studied large charge operators in a boundary conformal field theory.

3.2.1 ϵ expansion

We start by considering the long range $O(N)$ model (3.1) in the vicinity of the lower critical value of s. For $s = \frac{d+\epsilon}{2}$, the model has a perturbative fixed point where the coupling is given by [59]

$$g_* = \frac{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}{2(N+8)}\epsilon. \tag{3.87}$$

We will work at this fixed point to leading order in ϵ. It is convenient to think of the model as coming from the following model in $D = d + 2 - s$ dimensions, with interactions localized to the d dimensional subspace [66]

$$S = \frac{\Gamma\left(\frac{s}{2}\right)}{(4\pi)^{1-\frac{s}{2}}}\int d^d x\, d^{2-s} w \frac{1}{2}(\partial_\mu \Phi^I)^2 + \frac{g}{4}\int d^d x (\phi^I \phi^I)^2. \tag{3.88}$$

The field Φ^I is the D-dimensional extension of ϕ^I

$$\Phi^I(x, w = 0) = \phi^I(x) \tag{3.89}$$

and we restrict for simplicity to field configurations which depend on the extra coordinates only through $|w|$. We are interested in calculating the dimensions of fixed charge operators with a large charge j while holding ϵj fixed. For that purpose, we perform a Weyl transformation to the cylinder $R\times S^{D-1}$. By the state-operator correspondence, the large charge operators on the plane are mapped to large charge states on the cylinder. We pick the following coordinates on S^{D-1}

$$ds^2 = d\theta^2 + \sin^2\theta d\Omega_{d-1} + \cos^2\theta d\Omega_{1-s} \tag{3.90}$$

where $0 \leq \theta \leq \pi/2$. The limit $\theta \to \pi/2$ then brings us to the d dimensional subspace our original model was defined on. We want to consider fixed large charge states on the cylinder. By ensemble equivalence, this can be done by introducing a fixed chemical potential μ. Without loss of generality, we introduce this chemical potential for the $U(1)$ subgroup that rotates Φ^1 and Φ^2. In practice, the chemical potential may be implemented by having a background gauge field in the time direction [7] (see for instance [98] for a related discussion)

$$S \to S + \frac{\Gamma\left(\frac{s}{2}\right)}{(4\pi)^{1-\frac{s}{2}}}\int_{R\times S^{D-1}}\left[i\mu\left(\dot{\Phi}^1\Phi^2 - \dot{\Phi}^2\Phi^1\right) - \frac{\mu^2}{2}\left((\Phi^1)^2 + (\Phi^2)^2\right)\right] \tag{3.91}$$

where dot represents the time derivative. We expand the field around the following ansatz [8]

$$\Phi^1 + i\Phi^2 = \sqrt{2}f(\theta), \qquad \Phi^3 = \Phi^4 = \cdots = \Phi^N = 0. \tag{3.92}$$

In this background, the classical action is

$$\begin{aligned} S_{\text{cl}} =& \frac{T4^{\frac{s}{2}}\pi^{\frac{d}{2}}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\int_0^{\pi/2} d\theta\,(\sin\theta)^{d-1}(\cos\theta)^{1-s}\left((\partial_\theta f(\theta))^2 + \left(-\mu^2 + \frac{(d-s)^2}{4}\right)f(\theta)^2\right) \\ &+ \frac{2gT\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}f(\theta)^4\Big|_{\theta=\frac{\pi}{2}} \end{aligned} \tag{3.93}$$

where $(d-s)^2/4$ comes from the conformal coupling on the cylinder and T is the length of the cylinder along the Euclidean time direction, which is formally infinite. The variational principle gives the following equation of motion

$$\frac{1}{(\sin\theta)^{d-1}(\cos\theta)^{1-s}}\partial_\theta\left((\sin\theta)^{d-1}(\cos\theta)^{1-s}\partial_\theta f(\theta)\right) + \left(\mu^2 - \frac{(d-s)^2}{4}\right)f(\theta) = 0 \tag{3.94}$$

[7] This is done by modifying the kinetic term in the action so that $\partial_0\Phi^1 \to D_0\Phi^1 = \partial_0\Phi^1 + i\mu\Phi^2$ and $\partial_0\Phi^2 \to D_0\Phi^2 = \partial_0\Phi^2 - i\mu\Phi^1$.

[8] An equivalent way to introduce the chemical potential is to have a time dependent ansatz given by $\Phi^1 + i\Phi^2 = \sqrt{2}f(\theta)e^{-i\mu t}$ where t now is the Lorentzian time.

along with the boundary condition

$$\left[\frac{4^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)(\sin\theta)^{d-1}(\cos\theta)^{1-s}}{\Gamma\left(1-\frac{s}{2}\right)}\partial_\theta f(\theta)+4gf^3(\theta)\right]_{\theta=\frac{\pi}{2}}=0 \tag{3.95}$$

The solution that is regular at $\theta=0$ is given by

$$f(\theta)=v(\cos\theta)^{-\frac{d-s-2\mu}{2}}{}_2F_1\left(\frac{d-s-2\mu}{4},\frac{d+s-2\mu}{4};\frac{d}{2};-\tan^2\theta\right). \tag{3.96}$$

Using the boundary condition fixes (we can set $s=d/2$ for this calculation, to leading order in ϵ)

$$v^2=-\frac{2^{\frac{d}{2}-1}\Gamma\left(\frac{3d}{8}-\frac{\mu}{2}\right)^3\Gamma\left(\frac{3d}{8}+\frac{\mu}{2}\right)^3}{g\Gamma\left(\frac{d}{4}\right)^2\Gamma\left(\frac{d}{2}\right)^2\Gamma\left(\frac{1}{8}(d-4\mu)\right)\Gamma\left(\frac{1}{8}(d+4\mu)\right)}. \tag{3.97}$$

The scaling dimensions of the operators on the plane are then related to the energy on the cylinder and may be calculated by extremizing the following expression (this is essentially a Legendre transform from the free energy at fixed chemical potential to the free energy at fixed charge)

$$\begin{aligned}\Delta_j&=\left[\frac{S_{\text{cl}}}{T}+\mu j\right]_{\mu=\mu^*}=\left[-\frac{2g\pi^{\frac{d}{2}}f^4(\theta=\pi/2)}{\Gamma\left(\frac{d}{2}\right)}+\mu j\right]_{\mu=\mu^*}\\&=\left[-\frac{2^{d-1}\pi^{\frac{d}{2}}\Gamma\left(\frac{3d}{8}-\frac{\mu}{2}\right)^2\Gamma\left(\frac{3d}{8}+\frac{\mu}{2}\right)^2}{g\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{8}(d-4\mu)\right)^2\Gamma\left(\frac{1}{8}(d+4\mu)\right)^2}+j\mu\right]_{\mu=\mu^*}\end{aligned} \tag{3.98}$$

where we used the boundary condition and μ^* is the value of μ that extremizes the above expression

$$\frac{2^{d-1}\pi^{d/2}\Gamma\left(\frac{3d}{8}-\frac{\mu^*}{2}\right)^2\Gamma\left(\frac{3d}{8}+\frac{\mu^*}{2}\right)^2\left[\psi^{(0)}\left(\frac{3d}{8}+\frac{\mu^*}{2}\right)-\psi^{(0)}\left(\frac{1}{8}(d+4\mu^*)\right)-\mu^*\to-\mu^*\right]}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{8}(d-4\mu^*)\right)^2\Gamma\left(\frac{1}{8}(d+4\mu^*)\right)^2}=gj \tag{3.99}$$

It is hard to find this extremal value analytically in general, but we can make progress in the limit of small and large gj. For small gj we get

$$\mu^*=\frac{d}{4}+\frac{gj}{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}+O(gj)^2\implies\Delta_j=j\left(\frac{d}{4}+\frac{j\epsilon}{N+8}+O(j\epsilon)^2\right). \tag{3.100}$$

For large gj, we get instead

$$\begin{aligned}\mu^* &= \frac{3d}{4} - \frac{2^{\frac{d+2}{3}}\pi^{\frac{d-4}{6}}\left(\sin\left(\frac{\pi d}{4}\right)\right)^{\frac{2}{3}}\left(\Gamma\left(\frac{d}{4}+1\right)\Gamma\left(\frac{3d}{4}\right)\right)^{2/3}}{(gj)^{\frac{1}{3}}\Gamma\left(\frac{d}{2}\right)} \implies \\ \Delta_j &= j\left[\frac{3d}{4} - \frac{3\left(\sin\left(\frac{\pi d}{4}\right)\Gamma\left(\frac{d}{4}+1\right)\Gamma\left(\frac{3d}{4}\right)\right)^{2/3}}{\pi^{2/3}\Gamma\left(\frac{d}{2}\right)^{4/3}}\left(\frac{N+8}{\epsilon j}\right)^{\frac{1}{3}}\right]\end{aligned} \tag{3.101}$$

where we plugged in the fixed point value of g. At large N these results agree with what we found before in (3.54) and (3.57).

3.2.2 Large N expansion

We now revisit the large N expansion for the long range $O(N)$ model from the cylinder approach. We start with the following action on $R \times S^{D-1}$

$$S = \frac{\Gamma\left(\frac{s}{2}\right)}{(4\pi)^{1-\frac{s}{2}}}\int_{R\times S^{D-1}}\left(\frac{1}{2}(\partial_\mu \Phi^I)^2 + \frac{(d-s)^2}{8}(\Phi^I\Phi^I)\right) + \frac{1}{2}\int d^dx\sigma\phi^I\phi^I. \tag{3.102}$$

We take the same ansatz as in (3.92) and the classical equation of motion and its solution are the same as in (3.94) and (3.96), but the boundary condition is now given by

$$\left[\frac{4^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)(\sin\theta)^{d-1}(\cos\theta)^{1-s}}{\Gamma\left(1-\frac{s}{2}\right)}\partial_\theta f(\theta) + c_\sigma f(\theta)\right]_{\theta=\frac{\pi}{2}} = 0. \tag{3.103}$$

The c_σ here is the classical value of σ, which is a constant and may be obtained by a Weyl transformation from (3.10). This boundary condition requires

$$\frac{\Gamma\left(\frac{d+s-2\mu}{4}\right)\Gamma\left(\frac{d+s+2\mu}{4}\right)}{\Gamma\left(\frac{d-s-2\mu}{4}\right)\Gamma\left(\frac{d-s+2\mu}{4}\right)} = -\frac{c_\sigma}{2^s}. \tag{3.104}$$

Note that this is precisely the same as (3.33) that we found in the flat space approach. Using the boundary condition, one can see that the action actually vanishes on the classical solution. However, the effective action at large N also involves the fluctuations. We expand the field around the classical background $\Phi^I = \Phi^I_{\text{cl}} + \delta\Phi^I$, and the the action up to quadratic order in fluctuations is given by

$$S_{\text{fluct}} = \frac{\Gamma\left(\frac{s}{2}\right)}{(4\pi)^{1-\frac{s}{2}}}\int_{R\times S^{D-1}}\left(\frac{1}{2}(\partial_\mu \delta\Phi^I)^2 + \frac{(d-s)^2}{8}(\delta\Phi^I\delta\Phi^I)\right) + \frac{1}{2}\int d^dx\ c_\sigma\delta\phi^I\delta\phi^I. \tag{3.105}$$

To calculate the large N free energy, we need to calculate the determinant of the fluctuations.

One way to proceed is to reduce it down to $R \times S^{d-1}$ again

$$S = \frac{2^{s-1}\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})} \int d^d x d^d y \sqrt{g_x}\sqrt{g_y} \frac{\delta\phi^I(x)\delta\phi^I(y)}{(s(x,y))^{d+s}} + \frac{1}{2}\int d^d x \sqrt{g_x}\ c_\sigma \delta\phi^I \delta\phi^I \tag{3.106}$$

where $s(x,y)$ is the Weyl map of the flat space distance to the cylinder. Let us use the coordinates $(\tau, \vec{x})$ on the cylinder, then

$$s(x,y)^2 = 2\left(\cosh\left(\tau_x - \tau_y\right) - \cos\theta\right) \tag{3.107}$$

where θ is the angle between $\vec{x}$ and $\vec{y}$ on the cylinder. The free energy on the cylinder is then given by $\log\det\left(K\right)$ with K defined by

$$K = \frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})} \frac{1}{(s(x,y))^{d+s}} + c_\sigma \frac{\delta^d(x-y)}{\sqrt{g_x}}. \tag{3.108}$$

Let us expand this operator into eigenfunctions of the Laplacian on the cylinder

$$\frac{1}{(s(x,y))^{d+s}} = \sum_{l,m}\int \frac{d\omega}{2\pi} g(l,\omega) e^{i\omega(\tau_x - \tau_y)} Y^*_{lm}(\vec{x}) Y_{lm}(\vec{y}) \tag{3.109}$$

where Y_{lm} are $d-1$ dimensional spherical harmonics and $e^{i\omega\tau}$ is the eigenfunction on the real line. Using orthogonality, we get

$$g(l,\omega) = \frac{e^{i\omega\tau_y}}{Y_{lm}(\vec{y})} \int d^d x \sqrt{g_x} \frac{1}{(s(x,y))^{d+s}} e^{-i\omega\tau_x} Y_{lm}(\vec{x}) \tag{3.110}$$

Note that the eigenvalue should not depend on m because of the symmetries of S^{d-1}, so we can just evaluate it at $m = 0$. Using the same symmetries, we can fix y to be at $\tau_y = 0$ and at the north pole of the sphere S^{d-1}. Also note that $Y_{l0}(\theta)$ is proportional to the Gegenbauer polynomial $C_l^{(d-2)/2}(\cos\theta)$. This results in the following integral

$$g(l,\omega) = \frac{\mathrm{Vol}(S^{d-2})}{C_l^{(d-2)/2}(1)} \int d\tau d\theta \left(\sin\theta\right)^{d-2} \frac{1}{\left(2\left(\cosh\tau - \cos\theta\right)\right)^{\frac{d+s}{2}}} e^{-i\omega\tau} C_l^{(d-2)/2}(\cos\theta) \tag{3.111}$$

We can then use

$$\frac{1}{\left(2\left(\cosh\tau - \cos\theta\right)\right)^{\frac{d+s}{2}}} = \sum_k C_k^{\frac{d+s}{2}}\left(\cos\theta\right) e^{-|\tau|\left(k+\frac{d+s}{2}\right)}. \tag{3.112}$$

to turn the integral into a more useful form

$$g(l,\omega)=\frac{\text{Vol}(S^{d-2})}{C_l^{(d-2)/2}(1)}\sum_{k=0}^{\infty}\int_0^{\infty}d\tau 2\cos\omega\tau e^{-\tau\left(k+\frac{d+s}{2}\right)}\int_{-1}^{1}dz\left(1-z^2\right)^{\frac{d-3}{2}}C_k^{\frac{d+s}{2}}(z)\,C_l^{(d-2)/2}(z). \tag{3.113}$$

The integral over τ may be immediately performed. To perform the integral over z, we first use (3.29) and then use the orthogonality relations to get

$$g(l,\omega)=\frac{4\pi^{d/2}}{\Gamma\left(\frac{d+s}{2}\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{d+s}{2}+l+k\right)\Gamma\left(1+\frac{s}{2}+k\right)}{k!\Gamma\left(\frac{d}{2}+l+k\right)\Gamma\left(1+\frac{s}{2}\right)}\left(\frac{1}{(d+s+2l+4k)+2i\omega}+\text{c.c.}\right) \tag{3.114}$$

The sum over k can be computed in terms of generalized hypergeometric functions

$$\begin{aligned}g(l,\omega)=&\frac{\pi^{d/2}\Gamma(-s)\Gamma\left(\frac{d+s+2l}{2}\right)}{\Gamma\left(\frac{d+2l}{2}\right)\Gamma\left(\frac{d+s}{2}\right)}\Bigg[\frac{\Gamma\left(\frac{d+s+2l}{4}+\frac{i\omega}{2}\right)}{\Gamma\left(\frac{d-3s+2l}{4}+\frac{i\omega}{2}\right)}\\&\times{}_3F_2\left(\frac{d+s+2l}{4}+\frac{i\omega}{2},\frac{d-s+2l}{2}-1,-\frac{s}{2};\frac{d+2l}{2},\frac{d-3s+2l}{4}+\frac{i\omega}{2};1\right)+\text{c.c.}\Bigg].\end{aligned} \tag{3.115}$$

For $s=2$, this gives, as expected

$$\frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}g(l,\omega)\Big|_{s\to 2}=\omega^2+\left(\frac{d}{2}-1+l\right)^2. \tag{3.116}$$

The cylinder free energy may then be computed in terms of these eigenvalues

$$\begin{aligned}\mathcal{F}&=\frac{N}{2}\log\det(K)=\frac{N}{2}\int d^dx\sqrt{g_x}\,\langle x|\log K|x\rangle\\&=\frac{N}{2}\sum_{l,m}\int\frac{d\omega}{2\pi}\int d^dx\sqrt{g_x}\,|Y_{l,m}(\vec{x})|^2\log\left(\frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}g(l,\omega)+c_\sigma\right).\end{aligned} \tag{3.117}$$

The spherical harmonics are normalized such that the integral over the sphere just gives one, while the integral over the real line gives the length of the cylinder. The sum over m gives a factor of the degeneracy

$$\begin{aligned}\mathcal{F}&=\sum_l\frac{NT(2l+d-2)\Gamma(l+d-2)}{2l!\Gamma(d-1)}\int\frac{d\omega}{2\pi}\log\left(\frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}g(l,\omega)+c_\sigma\right)\\&=\sum_l\frac{NT(2l+d-2)\Gamma(l+d-2)}{2l!\Gamma(d-1)}\int\frac{d\omega}{2\pi}\Bigg[\log\left(\frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}g(l,\omega)\right)\\&\qquad+\log\left(1+\frac{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}{2^s\Gamma(\frac{d+s}{2})g(l,\omega)}c_\sigma\right)\Bigg].\end{aligned} \tag{3.118}$$

In the second line we separated out the $c_\sigma=0$ piece of the free energy. This is the vacuum energy,

which should be subtracted while computing the scaling dimensions. After this subtraction, the scaling dimensions may be calculated as in the previous subsection, by extremizing the following expression with respect to μ

$$\begin{aligned}\Delta_j &= \frac{\mathcal{F}}{T} + \mu j \\ &= N\left[\sum_l \frac{(2l+d-2)\Gamma\left(l+d-2\right)}{l!\Gamma\left(d-1\right)} \int \frac{d\omega}{4\pi} \log\left(1 + \frac{\pi^{\frac{d}{2}}\Gamma(-\frac{s}{2})}{2^s\Gamma(\frac{d+s}{2})g(l,\omega)} c_\sigma\right) + \mu\hat{j}\right]\end{aligned} \tag{3.119}$$

For equivalence to the flat space calculation in (3.51), we want the first term to be identified with $F(c_\sigma)$ in (3.39) which requires

$$g(l,\omega) = \frac{\Gamma\left(-\frac{s}{2}\right)\pi^{\frac{d}{2}}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{d+s}{2}\right)\Gamma\left(\frac{d-s}{2}\right)Q_l\left(\frac{\omega}{2}\right)}. \tag{3.120}$$

Comparing (3.26) and (3.115), we need the following hypergeometric identity to hold

$$\begin{aligned}&\frac{{}_3F_2(a,b,c;a-b+1,2b+c;1)}{\Gamma(a-2b-c+1)\Gamma(a-b+1)\Gamma(c+2b)} + \frac{{}_3F_2(a,b,a-2b-c+1;a-b+1,a-c+1;1)}{\Gamma(c)\Gamma(a-b+1)\Gamma(a-c+1)} \\ &= \frac{\sqrt{\pi}2^{1-2b}}{\Gamma\left(b+\frac{1}{2}\right)\Gamma(a-2b+1)\Gamma(b+c)\Gamma(a-b-c+1)}\end{aligned} \tag{3.121}$$

We could not prove this identity or find it in the literature, but we checked that it holds numerically for a wide range of parameters, so we expect it to be true. Hence, as promised, we have shown that the scaling dimensions calculated on R^d and from the cylinder approach match.

3.3 Appendix: Scaling dimensions from standard $1/N$ perturbation theory

In the regime when $j << N$ or $\hat{j} << 1$, we can use ordinary $1/N$ perturbation theory to calculate the scaling dimensions of the operators $\mathcal{O}_j$. We will do that in this appendix, and it will serve as a check of the calculations in section 3.1. We will calculate the correlator of the operator $\mathcal{O}_j$ with j fundamental fields ϕ. Let us look at this correlator in momentum space. Just by dimensional analysis [9]

$$\langle \mathcal{O}_j(0)\phi(\mathbf{k}_1)\dots\phi(\mathbf{k}_1)\rangle = \tilde{G}(k_1,\dots,k_j) \propto \frac{1}{|p|^{jd-\Delta_j-j\Delta_\phi}}. \tag{3.122}$$

[9] We are suppressing $O(N)$ indices in this appendix

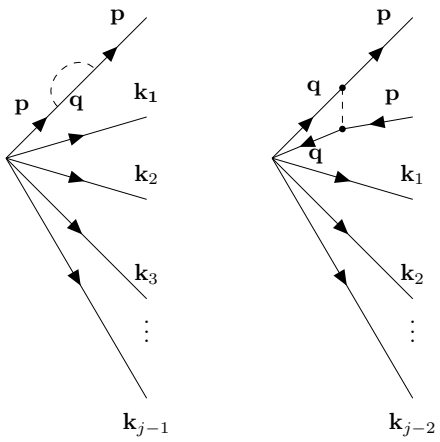


Figure 3.10: The two diagrams that contribute to the $1/N$ correction to the correlator. The dashed line represents the σ propagator.

The momentum conservation requires $\sum_j \mathbf{k}_j = 0$. The last term is just schematic and is meant to count the powers of momentum. It is well known that the field ϕ in the long range model does not receive anomalous dimensions [59, 1]

$$\Delta_\phi = \frac{d-s}{2}, \qquad \Delta_j = j\left(\frac{d-s}{2}\right) + \frac{\gamma_j}{N} + O\left(\frac{1}{N^2}\right). \tag{3.123}$$

Therefore all the logarithmic terms in the correlator must contribute to the anomalous dimensions of $\mathcal{O}_j$

$$\tilde{G}(k_1,\ldots,k_j) \propto \frac{1}{|p|^{js-\gamma_j/N}} = \frac{1}{|p|^{js}}\left(1 + \frac{\gamma_j}{2N}\log(p^2) + \mathcal{O}(\tfrac{1}{N^2})\right). \tag{3.124}$$

There are 2 types of diagrams that contribute to the $1/N$ correction to this $(j+1)$-point function as shown in figure 3.10.

The left diagram does not contribute to the anomalous dimension because it does not give rise to any $\log(p^2)$ terms. This is also the reason why ϕ does not get anomalous dimensions.

$$\begin{aligned} &= \frac{1}{|k_1|^s\cdots|k_{j-2}|^s}\frac{\delta_{IJ}\tilde{C}_\sigma}{N|p|^{2s}}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2)^{s/2}((p-q)^2)^{d/2-s}} \\ &= \frac{1}{|k_1|^s\cdots|k_{j-2}|^s}\frac{\delta_{IJ}\tilde{C}_\sigma}{(4\pi)^{d/2}N|p|^s}\frac{\Gamma(-\frac{s}{2})\Gamma(\frac{d-s}{2})\Gamma(s)}{\Gamma(\frac{d}{2}-s)\Gamma(\frac{s}{2})\Gamma(\frac{d+s}{2})} \end{aligned} \tag{3.125}$$

We used that the σ propagator is given by

$$\langle\sigma(-\mathbf{q})\sigma(\mathbf{q})\rangle = \frac{\tilde{C}_\sigma}{N(q^2)^{\frac{d}{2}-s}}, \qquad \tilde{C}_\sigma = -\frac{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{s}{2})^2\Gamma(d-s)}{\Gamma(s-\frac{d}{2})\Gamma(\frac{d-s}{2})^2}. \tag{3.126}$$

On the other hand, the right diagram does give rise to logarithms. There are $\binom{j}{2} = j(j-1)/2$

diagrams of this form. For simplicity, we picked a configuration such that $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{p}$ where the diagram may be computed by the following simple integral

$$\frac{1}{|k_1|^s \cdots |k_{j-2}|^s} \delta_{IJ} \int \frac{d^d q}{(2\pi)^d} \frac{1}{|p|^{2s}|q|^{2s}} \frac{\tilde{C}_\sigma}{N} \frac{1}{((q-p)^2)^{d/2-s}} \tag{3.127}$$

The integral can be computed using Feynman parameters and introducing a regulator η.

$$\begin{aligned}
&\int \frac{d^d q}{(2\pi)^d} \frac{1}{|p|^{2s}|q|^{2s}} \frac{\tilde{C}_\sigma}{N} \frac{1}{((q-p)^2)^{d/2-s}} \\
&\quad = \frac{\tilde{C}_\sigma}{N|p|^{2s}} \frac{\Gamma(\frac{d}{2})}{\Gamma(s)\Gamma(\frac{d}{2}-s)} \int d\alpha \alpha^{s-1}(1-\alpha)^{d/2-s-1} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2+\alpha(1-\alpha)p^2]^{d/2}} \\
&\quad = \frac{\tilde{C}_\sigma}{N|p|^{2s}} \frac{\Gamma(\frac{d}{2})}{\Gamma(s)\Gamma(\frac{d}{2}-s)} \int d\alpha \alpha^{s-1}(1-\alpha)^{d/2-s-1} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int \frac{dq}{(2\pi)^d} \frac{q^{d-1-\eta}}{[q^2+\alpha(1-\alpha)p^2]^{d/2}} \\
&\quad = \frac{\tilde{C}_\sigma (4\pi)^{-d/2}}{N|p|^{2s}\Gamma(s)\Gamma(\frac{d}{2}-s)\Gamma(\frac{d}{2})} \left[\mathcal{O}(\tfrac{1}{\eta}) - \Gamma(\tfrac{d}{2}-s)\Gamma(s)(\log(p^2)+\dots) + \mathcal{O}(\eta)\right]
\end{aligned} \tag{3.128}$$

Dropping the $1/\eta$ pole and taking the limit $\eta \to 0$, we see that the coefficient of the $\log(p^2)$ piece, after summing all the diagrams of the form 3.127 is

$$\frac{1}{|k_1|^s \cdots |k_{j-2}|^s |p|^{2s}} \frac{-\tilde{C}_\sigma j(j-1)}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \frac{1}{2N} \tag{3.129}$$

Thus the dimension of $\mathcal{O}_j$ is given by

$$\Delta_j = \left(\frac{d-s}{2}\right) j + \frac{2\Gamma(d-s)\Gamma(\frac{s}{2})^2 j(j-1)}{\Gamma(\frac{d-s}{2})^2\Gamma(s-\frac{d}{2})\Gamma(\frac{d}{2})N} + O\left(\frac{1}{N^2}\right). \tag{3.130}$$

At large j this is consistent with what we found in (3.53).

3.4 Appendix: Large c_σ expansion of $F(c_\sigma)$ using heat kernel methods

In this appendix, we show that at large c_σ, the functional determinant $F(c_\sigma)$ behaves as (3.48). We will use heat kernel methods to calculate the functional determinant (see for instance [114, 115, 116] for reviews). We start with the following representation of the functional determinant

$$\frac{1}{2}\mathrm{Tr}\log\left((-\nabla^2)^{\frac{s}{2}} + \sigma_*(x)\right) = -\frac{1}{2}\int d^d x \langle x| \int_0^\infty \frac{dT}{T} e^{-T(P^s + \sigma^*(X))} |x\rangle \tag{3.131}$$

where we use capital letters to denote operators. Recall that

$$\sigma^*(x) = c_\sigma \frac{|x_1 - x_2|^s}{|x_1 - x|^s |x_2 - x|^s} \equiv c_\sigma V(x). \tag{3.132}$$

We then use Trotter formula to write the determinant as a path integral. For that, we divide T into N pieces, then for N very large, we may write

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\log\left((-\nabla^2)^{\frac{s}{2}} + \sigma_*(x)\right) = -\frac{1}{2}\int d^d x_0 \int_0^\infty \frac{dT}{T} \langle x_0| \left(e^{-\frac{T}{N}P^s} e^{-\frac{T}{N}\sigma^*(X)}\right)^N |x_0\rangle \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T} \int_{x(0)=x(T)=x_0} Dx(\tau)Dp(\tau) e^{\int_0^T d\tau \left(ip(\tau)\cdot\dot{x}(\tau) - (p^2)^{s/2}(\tau) - c_\sigma V(x)\right)} \\
&= -\frac{1}{2}\int_0^\infty \frac{dT}{T} \int_{x(0)=x(1)=x_0} Dx(t)Dp(t) e^{\int_0^1 dt \left(ip(t)\cdot\dot{x}(t) - T(p^2)^{s/2}(t) - Tc_\sigma V(x)\right)}.
\end{aligned} \tag{3.133}$$

To get the large c_σ behavior, we rescale $T \to T/c_\sigma$ and at the same time, also rescale $p \to c_\sigma^{1/s} p$ to get

$$\frac{1}{2}\mathrm{Tr}\log\left((-\nabla^2)^{\frac{s}{2}} + \sigma_*(x)\right) = -\frac{1}{2}\int_0^\infty \frac{dT}{T} \int Dx(t)D[c_\sigma^{1/s}p(t)] e^{\int_0^1 dt \left(ic_\sigma^{1/s}p(t)\cdot\dot{x}(t) - T(p^2)^{s/2} - TV(x)\right)}. \tag{3.134}$$

At large c_σ, the path integral will be dominated by constant x configurations. A path integral over x fluctuations will then also force momenta to be constant at large c_σ. So we can expand about the constant x and p configurations to obtain an expansion in $1/c_\sigma$

$$\begin{aligned}
&x(t) = x_0 + \frac{1}{c_\sigma^{1/s}}\chi(t), \qquad p(t) = p_0 + \Pi(t) \implies \\
&\frac{1}{2}\mathrm{Tr}\log(...) = -\frac{1}{2}\int_0^\infty \frac{dT}{T} \int \frac{d^d x_0 d^d p_0 c_\sigma^{d/s}}{(2\pi)^d} e^{-T(p_0^2)^{s/2}} e^{-TV(x_0)} \int D\chi(t)D\Pi(t) e^{-S}
\end{aligned} \tag{3.135}$$

where the action to quadratic order in fluctuations is given by

$$\begin{aligned}
S = \int_0^1 dt \Big[&- i\Pi(t)\cdot\dot{\chi}(t) - \frac{Tp_0^{s-2}s}{2}\left(\Pi^2 + 2p_0\cdot\Pi\right) + \frac{Tp_0^{s-4}s(s-2)}{2}\left(p_0\cdot\Pi\right)^2 \\
&+ \frac{T}{c_\sigma^{1/s}}\chi^\mu\partial_\mu V + \frac{T}{2c_\sigma^{2/s}}\chi^\mu\chi^\nu\partial_\mu\partial_\nu V\Big].
\end{aligned} \tag{3.136}$$

First, note that at leading order in c_σ we can ignore the fluctuations, and then we can do the integral

over p_0 followed by an integral over T and finally over x_0

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\log(...) &= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int \frac{d^dx_0 d^dp_0 c_\sigma^{d/s}}{(2\pi)^d} e^{-T(p_0^2)^{s/2}} e^{-TV(x_0)} \\ &= -\frac{\Gamma\left(\frac{d}{s}\right)\Gamma\left(-\frac{d}{s}\right)c_\sigma^{d/s}}{s(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int d^dx_0 \frac{|x_1-x_2|^d}{|x_0-x_1|^d|x_0-x_2|^d} \\ &= -\frac{(c_\sigma)^{\frac{d}{s}}\pi}{2^{d-1}\ d\ \Gamma\left(\frac{d}{2}\right)^2 \sin\left(\frac{\pi d}{s}\right)}\log\left(\frac{\delta^2}{|x_{12}|^2}\right)\end{aligned} \tag{3.137}$$

Note that the last integral over x_0 is the same as in (3.38). To calculate corrections to it, we expand the fluctuations into Fourier modes

$$\chi^\mu(t) = \sum_{m=1}^\infty \left(\chi_m^\mu \sin(2\pi m t) + \tilde{\chi}_m^\mu \cos(2\pi m t)\right), \quad \Pi^\mu(t) = \sum_{m=1}^\infty \Pi_m^\mu \sin(2\pi m t) + \tilde{\Pi}_m^\mu \cos(2\pi m t). \tag{3.138}$$

The action in terms of these modes is given by

$$\begin{aligned}S = \sum_m \Bigg[&- i\pi m\left(\tilde{\Pi}_m\cdot\chi_m - \Pi_m\cdot\tilde{\chi}_m\right) + \frac{T}{4c_\sigma^{2/s}}\left(\chi_m^\mu\chi_m^\nu + \tilde{\chi}_m^\mu\tilde{\chi}_m^\nu\right)\partial_\mu\partial_\nu V + \\ &+ T\left(\frac{sp_0^{s-2}}{4}\left(\Pi_m\cdot\Pi_m + \tilde{\Pi}_m\cdot\tilde{\Pi}_m\right) + \frac{s(s-2)p_0^{s-4}}{4}\left((p_0\cdot\Pi_m)^2 + (p_0\cdot\tilde{\Pi}_m)^2\right)\right)\Bigg].\end{aligned} \tag{3.139}$$

The last term in the above action mixes the Π modes in different directions, and is therefore slightly tedious to deal with. Let us start with the case when $d = 1$, so there is only one direction and no mixing. Then the action simplifies to

$$S = \sum_m \left[- i\pi m\left(\tilde{\Pi}_m\chi_m - \Pi_m\tilde{\chi}_m\right) + \frac{T}{4c_\sigma^{2/s}}\left(\chi_m^2 + \tilde{\chi}_m^2\right)\partial^2 V + \frac{Ts(s-1)p_0^{s-2}}{4}\left(\Pi_m^2 + \tilde{\Pi}_m^2\right)\right]. \tag{3.140}$$

Then the path integral over Π and χ may be easily performed

$$\begin{aligned}\int D\chi(t)D\Pi(t)e^{-S} &= \prod_m \int d\chi_m d\tilde{\chi}_m \left(\frac{\pi m^2}{Ts(s-1)p_0^{s-2}}\right) e^{-\frac{\pi^2m^2}{Ts(s-1)p_0^{s-2}}\left(\chi_m^2+\tilde{\chi}_m^2\right) - \frac{T}{4c_\sigma^{2/s}}\left(\chi_m^2+\tilde{\chi}_m^2\right)\partial^2 V} \\ &= 1 - \frac{T^2\partial^2 V s(s-1)p_0^{s-2}}{24c_\sigma^{2/s}}.\end{aligned} \tag{3.141}$$

We chose the path integral measure such that the path integral is normalized to one when the

potential vanishes. The functional determinant is then given by

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\log(...) &= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int \frac{dx_0 dp_0 c_\sigma^{1/s}}{2\pi} e^{-T(p_0^2)^{s/2}} e^{-TV(x_0)}\left(1-\frac{T^2\partial^2 V s(s-1)p_0^{s-2}}{24c_\sigma^{2/s}}\right)\\ &= -\int_0^\infty \frac{dT}{T^{\frac{1}{s}+1}}\int \frac{dx_0 c_\sigma^{1/s}\Gamma\left(\frac{1}{s}\right)}{2\pi s} e^{-TV(x_0)}\left(1-\frac{T^{\frac{2}{s}+1}\partial^2 V s(s-1)\Gamma\left(1-\frac{1}{s}\right)}{24c_\sigma^{2/s}\Gamma\left(\frac{1}{s}\right)}\right) \qquad (3.142)\\ &= -\int \frac{dx_0 c_\sigma^{1/s} V^{1/s}\Gamma\left(\frac{1}{s}\right)\Gamma\left(-\frac{1}{s}\right)}{2\pi s}\left(1-\frac{(1-s)\partial^2 V}{24\ s\ c_\sigma^{2/s}V^{\frac{2}{s}+1}}\right).\end{aligned}$$

Recall that

$$V(x_0)=\frac{|x_1-x_2|^s}{|x_0-x_1|^s|x_0-x_2|^s} \implies \frac{\partial^2 V}{V^{\frac{2}{s}+1}} = s(s+2-d)+2s(2s+2-d)\frac{(x_0-x_1)\cdot(x_0-x_2)}{|x_1-x_2|^2}. \qquad (3.143)$$

But the second term above, when multiplied by $V^{d/s}$, is proportional to a total derivative. This can be seen from the following

$$\frac{(x_0-x_1)\cdot(x_0-x_2)}{|x_0-x_1|^d|x_0-x_2|^d} = -\frac{1}{2(d-2)}\partial_i\left(\frac{(x_0-x_1)^i}{|x_0-x_1|^d|x_0-x_2|^{d-2}}+\frac{(x_0-x_2)^i}{|x_0-x_1|^{d-2}|x_0-x_2|^d}\right) \qquad (3.144)$$

so it does not contribute to the integral. Therefore $\partial^2 V$ term only changes the x_0 integral by a constant factor. The integral over x_0 may then be easily performed by using (3.38)

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\log(...) &= -\int \frac{dx_0 c_\sigma^{1/s} V^{1/s}\Gamma\left(\frac{1}{s}\right)\Gamma\left(-\frac{1}{s}\right)}{2\pi s}\left(1-\frac{(1-s^2)}{24c_\sigma^{2/s}}\right)\\ &= -\frac{c_\sigma^{1/s}}{\sin\left(\frac{\pi}{s}\right)}\left(1-\frac{(1-s^2)}{24c_\sigma^{2/s}}\right)\log\left(\frac{\delta^2}{|x_{12}|^2}\right).\end{aligned} \qquad (3.145)$$

This implies that in $d=1$, we get

$$F(c_\sigma)=\frac{c_\sigma^{1/s}}{\sin\left(\frac{\pi}{s}\right)}\left(1-\frac{(1-s^2)}{24c_\sigma^{2/s}}\right) \qquad (3.146)$$

Another case when there is no mixing in (3.139) is $s=2$ for any d when the last term in (3.139) vanishes. In that case also, the path integral over χ and Π may be done

$$\begin{aligned}\int D\chi(t)D\Pi(t)e^{-S} &= \prod_{m,\mu}\int d\chi_m d\tilde{\chi}_m\left(\frac{\pi m^2}{2T}\right)e^{-\frac{\pi^2 m^2}{2T}(\chi_m^2+\tilde{\chi}_m^2)-\frac{T}{4c_\sigma}(\chi_m^2+\tilde{\chi}_m^2)\partial^2 V}\\ &= 1-\frac{T^2\partial^2 V}{12c_\sigma}.\end{aligned} \qquad (3.147)$$

We can then calculate the functional determinant by integrating over p_0, T and x_0 as before

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\log(...) &= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int \frac{d^d x_0 d^d p_0 c_\sigma^{d/2}}{(2\pi)^d} e^{-Tp_0^2} e^{-TV(x_0)}\left(1-\frac{T^2\partial^2 V}{12 c_\sigma}\right) \\ &= -\frac{\Gamma\left(-\frac{d}{2}\right)(c_\sigma)^{d/2}}{2}\int \frac{d^d x_0 V^{d/2}}{(4\pi)^{d/2}}\left[1+\frac{d(d-2)(d-4)}{24\ c_\sigma}+O\left(\frac{1}{c_\sigma^2}\right)\right] \\ &= \frac{\Gamma\left(-\frac{d}{2}\right)(c_\sigma)^{d/2}}{2^d\Gamma\left(\frac{d}{2}\right)}\left[1+\frac{d(d-2)(d-4)}{24\ c_\sigma}+O\left(\frac{1}{c_\sigma^2}\right)\right]\log\left(\frac{\delta^2}{|x_{12}|^2}\right)\end{aligned} \tag{3.148}$$

which then implies

$$F(c_\sigma) = -\frac{\Gamma\left(-\frac{d}{2}\right)(c_\sigma)^{d/2}}{2^d\Gamma\left(\frac{d}{2}\right)}\left[1+\frac{d(d-2)(d-4)}{24\ c_\sigma}+O\left(\frac{1}{c_\sigma^2}\right)\right]. \tag{3.149}$$

This agrees with what was found in [108]. We will not do the general calculation for general d and s, but from the structure of (3.139), we expect the corrections to the leading large c_σ behavior to be of order $1/c_\sigma^{2/s}$, so that

$$F(c_\sigma) = \frac{(c_\sigma)^{\frac{d}{s}}\pi}{2^{d-1}\ d\ \Gamma\left(\frac{d}{2}\right)^2 \sin\left(\frac{\pi d}{s}\right)}\left(1+O\left(\frac{1}{c_\sigma^{2/s}}\right)\right). \tag{3.150}$$

Chapter 4

CFT in AdS and boundary RG flows

A boundary conformal field theory (BCFT) defined on the flat half-space may be related via a Weyl transformation to the same conformal field theory defined in anti-de Sitter space. Indeed, the flat metric on the half-space with coordinates $(y, \mathbf{x})$, $y > 0$ can be written as

$$ds^2 = dy^2 + d\mathbf{x}^2 = y^2 ds^2_{AdS_d}, \quad \mathbf{x} = (x_1, ..., x_{d-1}), \tag{4.1}$$

where $ds^2_{AdS_d}$ is the standard Poincaré metric

$$ds^2_{AdS_d} = \frac{1}{y^2}\left(dy^2 + d\mathbf{x}^2\right). \tag{4.2}$$

The correlation functions of the BCFT can be then translated to correlation functions in AdS by performing the required Weyl rescaling of the operators. For instance, for a scalar operator of dimension Δ, we have $\mathcal{O}_{AdS} = y^{\Delta}\mathcal{O}_{\text{half-space}}$. This implies, in particular, that the BCFT one-point functions in AdS are simply constant. In this chapter, we use the AdS approach to study various properties of boundary conformal field theories. We will see that several aspects of a BCFT appear naturally when the CFT is defined on AdS, and the technical machinery developed in the AdS/CFT literature can be used to extract new results about the BCFT data. The connection between CFT on AdS and the BCFT problem has been noted before several times in the literature, see e.g. [117, 118, 119, 120], and [121, 122, 123] for earlier related work. The more general idea of studing quantum field theory in AdS background appeared a long time ago in [124].

In a BCFT, it is possible to add relevant perturbations localized on the boundary, which may then drive non-trivial boundary RG flows connecting boundary critical points. Under such boundary flows, the bulk theory stays conformal and the bulk OPE data remains unaffected, but the boundary data changes. There has been considerable progress on studying quantities that are argued to be monotonic under boundary RG flows (and more general flows in defect CFT) [125, 126, 127, 128, 54, 129, 130, 131, 132, 133, 134, 135]. For $d = 3$ Euclidean BCFT, a proof was given in [132] that the coefficient of the Euler density term in the boundary trace anomaly decreases under a boundary RG flow. Such anomaly coefficient may be extracted from the logarithmic term in either the 3d hemisphere [132] or round ball [129] free energy. In $d = 4$, the free energy on a hemisphere [130] (suitably normalized by the round 4-sphere free energy) was proposed and checked in free and perturbative examples to decrease under boundary RG flows. The boundary trace anomaly is also related to the entanglement entropy in the presence of a boundary which has been discussed in [136, 137, 138, 139, 140, 141].

It is natural to expect that in general d the free energy of the BCFT defined on a space with spherical boundary may be used to define a suitable quantity that decreases under boundary RG flows. When the CFT is placed in AdS space, such a free energy can be defined by using the hyperbolic ball coordinates of AdS, so that the boundary is a sphere and the problem is conformally related to the BCFT on the round ball. Extending the idea of the generalized F-theorem proposed in [142] for the case of CFT with no boundaries, it is then a plausible conjecture that the quantity

$$\tilde{F} = -\sin\left(\frac{\pi(d-1)}{2}\right) F_{\mathrm{AdS}_d}\,, \tag{4.3}$$

where F_{AdS_d} is the free energy of the CFT on the hyperbolic space with sphere boundary, decreases under boundary RG flows in general d.[1] A similar conjecture was presented in [134], but our main point here is the suggestion of using the AdS background to compute the free energy of the BCFT. In odd d, i.e. even-dimensional boundary, there is no bulk conformal anomaly, but the free energy F_{AdS_d} has a logarithmic divergence coming from the regularized volume of hyperbolic space [144, 145]. The coefficient of the logarithmic divergence is related to one of the boundary conformal anomaly coefficients (the one that does not vanish for round sphere boundary). When working in dimensional regularization, this logarithmic divergence appears as a pole, which is cancelled by the sine factor in (4.3). Thus, in odd d, the quantity $\tilde{F}$ captures the boundary anomaly coefficient. In even d, the regularized volume of hyperbolic space is finite, but the free energy F_{AdS_d} has UV

[1] A proposal to use AdS space to define a candidate c-function for *bulk* RG flows was made in [143].

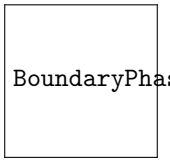


Figure 4.1: Phase diagram of the $O(N)$ model as we vary the boundary interaction strength c [150], defined such that $c = 0$ corresponds to tuning to the special transition. The blue line describes the boundary critical temperature when it is above the bulk critical temperature, and immediately below this line one has a phase where there is ordering only on the boundary and not in the bulk.

logarithmic divergences related to the bulk conformal anomaly.[2] Since this is fixed by short distance physics in the bulk, it is not expected to change under boundary RG flows. Hence, the difference of $\tilde{F}$ between UV and IR should be a finite quantity in even d, and it still makes sense to ask for positivity of $\tilde{F}^{UV} - \tilde{F}^{IR}$. One can verify this explicitly in the case of free fields, as we show below, but should be true more generally.

After some warm-up calculations in free field theory in section 4.1, we will compute the quantity (4.3) in interacting BCFT and verify that the boundary F-theorem for $\tilde{F}$ holds for boundary RG flows. Our primary example in this chapter will be the critical $O(N)$ model. In the case of the free $O(N)$ model, there are two possible boundary conditions, Neumann or Dirichlet, for each of the fundamental fields. One may flow from Neumann to Dirichlet by adding a boundary mass term, and it is easy to verify that $\tilde{F}^{\text{Neumann}} > \tilde{F}^{\text{Dirichlet}}$ for all d.[3] But when we add interactions in the bulk, there is a much richer phase structure on the boundary. The theory of boundary fixed points in the $O(N)$ model has been studied in great detail in the literature before [149, 150, 151]. It can be studied perturbatively near 4 dimensions by means of an ϵ expansion [10, 11, 12], in general d by means of a large N expansion [152, 153, 12], or using bootstrap techniques [14, 15, 18, 17]. The $O(N)$ model defined in AdS has also received some attention on its own right [154, 118]. We briefly review some of the main features here and explain some of the phenomena from a large N perspective.

In terms of a lattice system of spins interacting with a ferromagnetic nearest-neighbor interaction, the boundary has a lower coordination number than bulk. So we expect ordering and magnetization on the boundary to be driven by the bulk, and hence the boundary should undergo the phase transition at the same temperature as the bulk. This is what typically happens and it is referred to as the "ordinary transition". However, in the presence of sufficiently strong interactions at the

[2]The conformal anomaly in even d in the presence of a boundary includes, in addition to the bulk terms, various boundary terms, see [146] where the case of $d = 4$ was worked out in generality. But for the case of round sphere boundary, the only surviving boundary term should be the topological term completing the bulk Euler density. The combination of bulk Euler density and corresponding boundary term is proportional to the a-anomaly coefficient, which is fixed by short-distance bulk physics.

[3]Calculations of the free energy for free conformal fields in hyperbolic space as well as round ball, and their relation to conformal anomalies, were carried out previously in [147, 148].

boundary, the boundary can undergo a phase transition at a temperature higher than the bulk. As we lower the temperature and reach the bulk critical temperature (with the boundary already ordered), we have the so-called "extraordinary" transition. In this case, the $O(N)$ symmetry is broken and the fundamental field has a non-zero one-point function. Finally, there is a critical value of the boundary interaction strength at which the bulk and boundary critical temperatures become equal. This corresponds to the so-called "special transition". We reproduce the well-known phase diagram in figure 4.1 showing different phases of the system at different values of the surface interaction strength c.

In order to describe these phases and the RG flows between them more explicitly, let us turn to the field theory description. Near four dimensions, the critical behavior of the $O(N)$ model can be described by the Wilson-Fisher fixed point of the scalar field theory with quartic interactions. The bulk action on the flat half-space is[4]

$$S = \int d^{d-1}\mathbf{x}dy \left(\frac{1}{2}(\partial_\mu \phi^I)^2 + \frac{\lambda}{4}(\phi^I \phi^I)^2 \right) . \tag{4.4}$$

This model has a perturbative IR fixed point in $d = 4 - \epsilon$ at a certain critical coupling $\lambda = \lambda_*$. This is determined by bulk physics, and it is fixed by the renormalization of the theory in the usual flat space without boundary. As it is well-known, the large N expansion of the critical theory may be developed by performing a Hubbard-Stratonovich transformation, which yields the action in terms of the auxiliary field $\sigma(x)$

$$S = \int d^{d-1}\mathbf{x}dy \left(\frac{1}{2}(\partial_\mu \phi^I)^2 + \frac{1}{2}\sigma \phi^I \phi^I \right) . \tag{4.5}$$

In this action we omitted the $\sigma^2/4\lambda$ term, which can be dropped in the critical limit (see e.g. [48, 51] for reviews). Note that the σ operator, with bulk dimension $2 + O(1/N)$, plays the role of ϕ^2 at the interacting fixed point. Let us assume that we are at the bulk critical point, and further tune the boundary interactions so that we reach the special transition point. In the field theory description, this corresponds to tuning the boundary mass term $\hat{\phi}^2 = \phi^2(\mathbf{x}, 0)$, which is relevant at the special transition (in addition, the operator $\hat{\phi}^I = \phi^I(\mathbf{x}, 0)$ is also relevant. Operators with a hat will denote operators in the boundary spectrum throughout this chapter). Then, we can flow out of the special transition by adding the relevant boundary interaction $c\hat{\phi}^2$. For $c > 0$, this drives the system to the ordinary transition, where the $O(N)$ symmetry is unbroken. The case of $c < 0$ corresponds

[4]We will always assume that the bulk is critical, so any other mass terms have been tuned to zero.

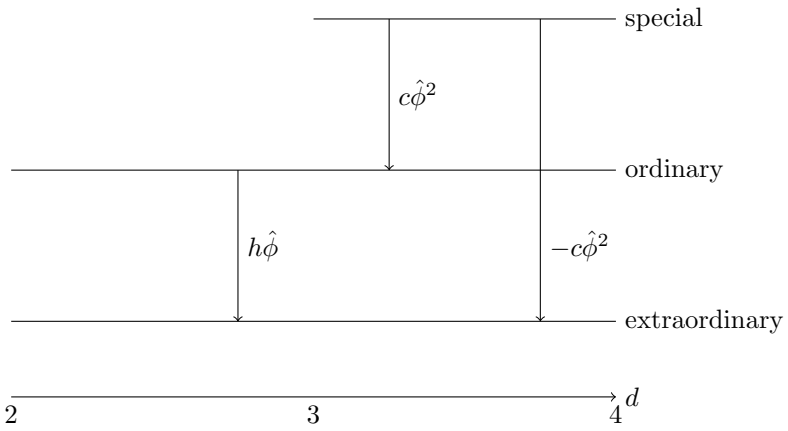


Figure 4.2: Surface RG flow in the large N $O(N)$ model between $2 < d < 4$.

instead to flowing to the extraordinary transition, which favours a non-zero vev for ϕ. As we will see later, in the large N description of the special transition, there is a boundary operator induced by σ with dimension 2 at large N: this plays the role of the boundary mass term driving the flow. One can see that this operator is relevant on the boundary only for $d > 3$, which is consistent with the fact that $d = 3$ is the lower critical dimension for the special transition for $N > 1$ [150] (as we will review below, from the large N approach one finds that the dimension of the leading boundary operator induced by ϕ^I goes to zero as $d \to 3$). Note that another possible relevant interaction we can add on the boundary is $h\hat{\phi}^I$, which is like adding a surface magnetic field. This also has the effect of ordering the boundary and drives the system to the extraordinary transition.[5] In the case of the ordinary transition, we will see below that the leading boundary operator induced by ϕ is relevant and has dimension $d - 2$ at large N. This operator can be used to drive a flow from ordinary to extraordinary transition, and such flow exists also in the range $2 < d < 3$. To summarize, assuming we are at bulk criticality, there are three distinct boundary critical behaviors: the special transition, which has two relevant boundary operators; the ordinary transition, with a single relevant boundary operator; and the extraordinary transition, which has no relevant boundary operators. We show all the three fixed points in figure 4.2. We will compute the free energy for all these three boundary critical points, both at large N and using ϵ-expansion, and find that $\tilde{F}$ is highest for the special transition, followed by ordinary and then by extraordinary transition, in agreement with the conjectured boundary F-theorem.

Near 4 dimensions, all the fixed points described above should match with the possible boundary

[5]To be precise, the fixed point that is reached by $h\hat{\phi}^I$ flow is referred to as the "normal transition" in some of the literature, but normal and extraordinary transition belong to the same universality class [150] and we will not distinguish between them.

conditions of the model in eq. (4.4). Indeed, it is well-known that the special transition corresponds to perturbing the free theory with Neumann boundary conditions for all the N fields, and the ordinary transition to perturbing the free theory with Dirichlet boundary conditions. The flow from special to ordinary transition described above corresponds in the free field limit to the familiar fact that we can flow from Neumann to Dirichlet boundary conditions by adding a boundary mass term. On the other hand, the extraordinary transition involves giving a non-zero one-point function to one of the ϕ^I (and having Dirichlet boundary condition for the remaining $N-1$ fields). We will see below that this phase has a natural realization in the AdS description: it simply corresponds to the non-trivial minimum of the scalar potential of the theory in AdS, which arises due to the negative conformal coupling term to the AdS curvature, see eq. (4.65).

As was recently noted in [155], and as we also observe in subsection 4.3.2, the description in terms of simple Neumann and Dirichlet boundary conditions is really only appropriate in the vicinity of $d = 4$ where the CFT is nearly free. In general, the boundary critical behaviors of the model may have realizations in terms of different boundary conditions in different perturbative descriptions of the same underlying BCFT, and we will see some explicit examples of this in the chapter (see figure 4.6).

Recall that near 2 dimension, the critical properties of the $O(N)$ model can also be described by the non-linear sigma model (NLσM). In the flat half-space, the action is

$$S = \int d^{d-1}\mathbf{x}dy \left(\frac{1}{2}(\partial_\mu \phi^I)^2 + \sigma\left(\phi^I\phi^I - \frac{1}{t^2}\right)\right). \tag{4.6}$$

In the presence of the boundary, after solving the constraint, we can assign either Neumann or Dirichlet boundary conditions to the $N-1$ unconstrained fields. We will check below, using our calculations of the free energy and anomalous dimensions in section 4.3, that the Neumann case matches onto the ordinary transition while the Dirichlet case matches onto the extraordinary transition in the large N theory. Note that the Neumann/Dirichlet boundary conditions near $d = 2$ are not correlated with the boundary conditions in the Wilson-Fisher description near $d = 4$.

The large N $O(N)$ model can be formally continued above $d = 4$, and near 6 dimensions, it is described by the $6-\epsilon$ expansion in a cubic theory with $N+1$ fields [48, 49]. When mapped to the AdS background, the action reads

$$S = \int d^{d-1}\mathbf{x}dy \left[\frac{1}{2}(\partial_\mu \phi^I)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma\phi^I\phi^I + \frac{g_2}{6}\sigma^3\right]. \tag{4.7}$$

For large enough N, the model has a perturbatively unitary fixed point with real couplings [48, 49].[6] We will see that the continuation of the special transition above $d = 4$ matches onto the critical point of the cubic model with Dirichlet boundary conditions, while the continuation of the ordinary transition matches onto a phase where σ acquires a vev. The case of Neumann boundary conditions in the $d = 6 - \epsilon$ theory matches instead with an additional phase of the large N theory which appears above $d = 5$, where the leading boundary operator in the fundamental of $O(N)$ has dimension $d - 4 + O(1/N)$. The extraordinary transition may also be formally continued to $d > 4$, though it becomes non-unitary as we explain below; it matches onto a phase of the cubic model where both ϕ^I and σ get a non-zero one-point function.

The rest of this chapter is organized as follows: in section 4.1, we compute the AdS free energy in free theories and in conformal perturbation theory, and spell out the connection to trace anomalies in $d = 3$. In section 4.2 we study the $O(N)$ model in AdS, focusing on the large N expansion, and describe the different boundary critical behaviors of the model. We calculate the corresponding values of the AdS free energy and verify consistency with the conjectured F-theorem. We also make explicit comparisons between the large N and the various ϵ expansions near even d. In section 4.3, we give more details on the BCFT spectrum in these models. We suggest that using the equations of motion obeyed by the bulk fields gives a convenient way to extract the anomalous dimensions of boundary operators. This is essentially an application to the case of BCFT of the idea described in [156]. For the bulk two-point function, it is particularly convenient to do this calculation in the AdS setup, because the correlation function is then just a function of the chordal distance. Moreover, the equation of motion operator takes a simple form and the boundary conformal blocks are the eigenfunctions of this operator. Using this idea, we reproduce in a straightforward manner the ϵ expansion results for Wilson-Fisher fixed point previously obtained in [18, 17]. At large N, we combine this idea with the BCFT crossing equation to get the $1/N$ correction to the anomalous dimension of the leading boundary operator and OPE coefficients of subleading boundary operators in the case of the ordinary transition.[7] This can be thought of as a version of analytic bootstrap for BCFT. We then go on to calculate some examples of boundary four-point functions using Witten diagrams in AdS, and obtain the boundary data appearing in the conformal block decomposition of the four-point function. In section 4.4 we make some concluding remarks and comment on possible future directions. Part of the work in this and the next chapter was presented at a student talk at TASI 2021, and during a poster session at Strings 2021.

[6] Non-perturbatively, instantons generates exponentially suppressed imaginary parts for any N [50]. In this chapter we will focus on perturbation theory and use the cubic model (4.84) as a useful check of the large N results.

[7] The anomalous dimension was originally obtained some time ago by using different methods [153]

4.1 AdS free energy and boundary RG flows: simple examples

In preparation to the calculations in the interacting $O(N)$ model, in this section we compute the AdS free energy in simple free field theory examples, and check consistency with the conjectured boundary F-theorem in terms of the quantity defined in (4.3). We also briefly discuss the case of weakly relevant boundary flows, and elaborate on the relation of the free energy to the trace anomaly coefficients, focusing on the $d = 3$ case.

As explained in the introduction, to calculate the free energy we consider the case in which the boundary of AdS is a round sphere, in other words we will be computing the free energy on a hyperbolic ball. The metric may be obtained, for instance, from the Poincaré metric (4.2), by the following stereographic projection (throughout this chapter, the index i runs from 1 to $d-1$, while the index μ runs from 1 to d)

$$x_i = \frac{2u\Omega_{i+1}}{1+u^2-2u\Omega_1}, \qquad y = \frac{1-u^2}{1+u^2-2u\Omega_1} \tag{4.8}$$

where $(\Omega_1, ...\Omega_d)$ are the coordinates on the $d-1$ sphere with $|\Omega_i|^2 = 1$. This gives the hyperbolic ball metric

$$ds^2 = \frac{4}{(1-u^2)^2}\left(du^2 + u^2 d\Omega_{d-1}^2\right) . \tag{4.9}$$

Then conjecture is that under a RG flow driven by a relevant boundary operator, $\tilde{F}$ computed on AdS_d with sphere boundary decreases. From now on, whenever we write F or $\tilde{F}$ it should be understood to be computed on AdS.

4.1.1 Neumann to Dirichlet flow in free field theory

The simplest example that we can study is the case of a conformally coupled scalar on AdS, which can be obtained via a Weyl transformation from a free massless scalar on half-space. The action in AdS reads

$$S = \int d^d x\sqrt{g}\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{d(d-2)}{8}\phi^2\right), \tag{4.10}$$

where the "mass" term comes from the conformal coupling to the AdS curvature (the Ricci scalar is $\mathcal{R} = -d(d-1)/R^2$, and we have set the radius to one for convenience). Using the usual AdS/CFT mass-dimension relation, we can get the conformal dimension of the boundary operator induced by

ϕ

$$\hat{\Delta}(\hat{\Delta}-(d-1))=-\frac{d(d-2)}{4} \implies \hat{\Delta}_+=\frac{d}{2}, \quad \hat{\Delta}_-=\frac{d}{2}-1. \tag{4.11}$$

The case of $\hat{\Delta}=d/2-1$ corresponds to the Neumann boundary condition while $d/2$ to Dirichlet boundary condition.

A relevant boundary mass term, $c\hat{\phi}(\mathbf{x})^2=c\phi(\mathbf{x},0)^2$ triggers a RG flow from Neumann (UV) to Dirichlet (IR) boundary conditions. The free energy can be computed by calculating determinants on AdS, since the action is quadratic (for later use, we consider slightly more general massive case)

$$\begin{aligned} F(m^2) &= \frac{1}{2}\mathrm{tr}\log\left(-\nabla^2+m^2-\frac{d(d-2)}{4}\right) \\ &= \frac{\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int_{-\infty}^{\infty}d\nu\frac{\Gamma(i\nu+\frac{d-1}{2})\Gamma(-i\nu+\frac{d-1}{2})}{\Gamma(i\nu)\Gamma(-i\nu)}\log\left(\nu^2+m^2+\frac{1}{4}\right) \end{aligned} \tag{4.12}$$

where we used the fact that the eigenvalues of Laplacian on AdS_d are $\nu^2+\frac{(d-1)^2}{4}$ with spectral density [157, 158]

$$\mu(\nu)=\frac{\mathrm{Vol}(H^d)}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\frac{\Gamma(i\nu+\frac{d-1}{2})\Gamma(-i\nu+\frac{d-1}{2})}{\Gamma(i\nu)\Gamma(-i\nu)}. \tag{4.13}$$

We can also write the free energy in terms of the boundary dimension $\hat{\Delta}$

$$\begin{aligned} F(\hat{\Delta}) &= \frac{\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int_{-\infty}^{\infty}d\nu\frac{\Gamma(i\nu+\frac{d-1}{2})\Gamma(-i\nu+\frac{d-1}{2})}{\Gamma(i\nu)\Gamma(-i\nu)}\log\left(\nu^2+\left(\hat{\Delta}-\frac{d-1}{2}\right)^2\right) \\ &= -\frac{\partial}{\partial\alpha}\left[\frac{\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int_{-\infty}^{\infty}d\nu\frac{\Gamma(i\nu+\frac{d-1}{2})\Gamma(-i\nu+\frac{d-1}{2})}{\Gamma(i\nu)\Gamma(-i\nu)}\frac{1}{\left(\nu^2+\left(\hat{\Delta}-\frac{d-1}{2}\right)^2\right)^{\alpha}}\right]\Bigg|_{\alpha\to 0} \end{aligned} \tag{4.14}$$

where the second line is equivalent to using the standard spectral zeta function regularization. This integral can be explicitly computed in three dimensions and gives the result

$$F(\hat{\Delta})=-\frac{\mathrm{Vol}(H^3)}{12\pi}(\hat{\Delta}-1)^3. \tag{4.15}$$

As defined in the introduction in eq. (4.3), the quantity that is conjectured to decrease under a boundary RG flow is $\tilde{F}$. Using the regularized volume of the Hyperbolic space [144, 145] $\mathrm{Vol}(H^d)=\pi^{\frac{d-1}{2}}\Gamma(\frac{1-d}{2})$, we get

$$\tilde{F}^N=\tilde{F}\left(\hat{\Delta}=\frac{1}{2}\right)=\frac{\pi}{96}, \qquad \tilde{F}^D=\tilde{F}\left(\hat{\Delta}=\frac{3}{2}\right)=-\frac{\pi}{96}. \tag{4.16}$$

So $\tilde{F}^N>\tilde{F}^D$ in agreement with the expected boundary F-theorem. The quantity $\tilde{F}$ is related to

one of the boundary anomaly coefficient, as we review in section 4.1.3 below in the $d = 3$ case, and hence the inequality for the AdS free energy is in accordance with what was proved in [132]. We can also evaluate the integral in $d = 5$, where it gives

$$\begin{aligned} F(\hat{\Delta}) &= \frac{\text{Vol}(H^5)}{360\pi^2}(\hat{\Delta}-2)^3(7+3\ \hat{\Delta}(\hat{\Delta}-4)) \\ \implies \tilde{F}^N &= \tilde{F}\left(\hat{\Delta}=\frac{3}{2}\right) = \frac{17\pi}{23040}, \qquad \tilde{F}^D = \tilde{F}\left(\hat{\Delta}=\frac{5}{2}\right) = -\frac{17\pi}{23040}. \end{aligned} \tag{4.17}$$

For general dimensions, this integral is not easy to perform, but there is a shortcut if we are just interested in the free energy difference between the two boundary conditions. We can think of the flow from Neumann to Dirichlet boundary conditions as analogous to a "double trace" flow in the $d-1$ dimensional CFT on the boundary driven by operator $c\hat{\phi}^2$. Under such a flow, the dimension of the boundary operator flows from $\hat{\Delta}$ to $d-1-\hat{\Delta}$. So we can use the general result for the free energy change under a double trace flow driven by the square of a primary scalar operator of dimension $\hat{\Delta}$ [144, 142]

$$F_{d-1-\hat{\Delta}} - F_{\hat{\Delta}} = -\frac{1}{\sin(\frac{\pi(d-1)}{2})\Gamma(d)}\int_0^{\hat{\Delta}-\frac{d-1}{2}} du\ u\ \sin\pi u\ \Gamma\left(\frac{d-1}{2}+u\right)\Gamma\left(\frac{d-1}{2}-u\right). \tag{4.18}$$

This implies that the free energy change under the flow from Neumann to Dirichlet is

$$\begin{aligned} \delta F = F^N - F^D &= F\left(\hat{\Delta}=\frac{d}{2}-1\right) - F\left(\hat{\Delta}=\frac{d}{2}\right) \\ &= -\frac{1}{\sin(\frac{\pi(d-1)}{2})\Gamma(d)}\int_0^{\frac{1}{2}} du\ u\ \sin\pi u\ \Gamma\left(\frac{d-1}{2}+u\right)\Gamma\left(\frac{d-1}{2}-u\right) \end{aligned} \tag{4.19}$$

It is easy to check that this result agrees with our calculation in $d = 3, 5$ done above. It also agrees with a calculation of the free energy difference between Neumann and Dirichlet performed on a hemisphere in [119]. In $d = 4$, (4.19) gives

$$\delta\tilde{F} = \frac{\zeta(3)}{8\pi^2}\,, \tag{4.20}$$

while in $d = 6$, we find

$$\delta\tilde{F} = \frac{\pi^2\zeta(3)+3\zeta(5)}{96\pi^6}. \tag{4.21}$$

These results agree with the hemisphere free energy calculation done in $d = 4$ in [130] and later

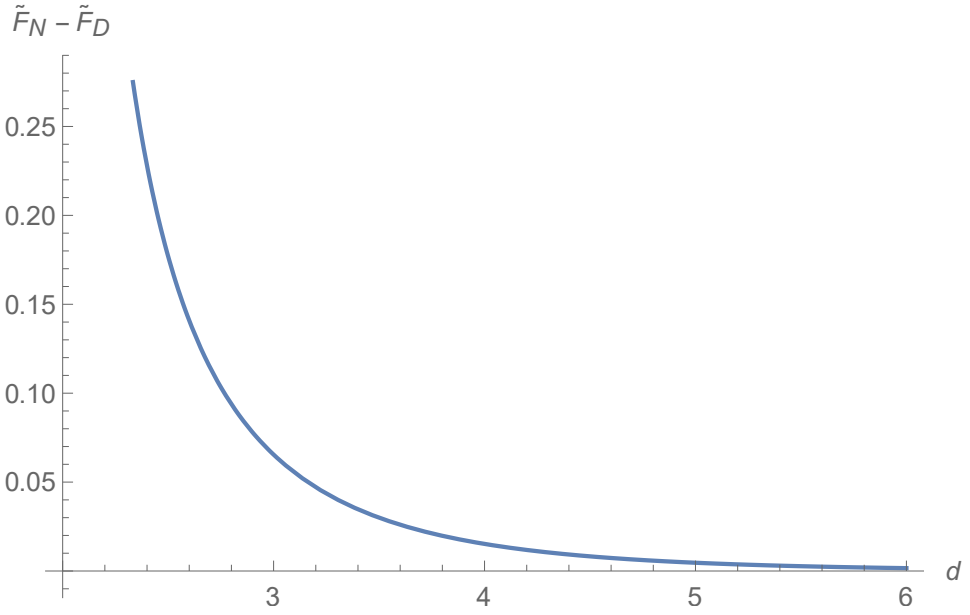


Figure 4.3: Free energy difference between the Neumann and Dirichlet boundary fixed points of the free bulk theory.

generalized to even dimensions in [159, 160, 161]. For general d, in terms of $\tilde{F}$, we may write

$$\delta\tilde{F} = \tilde{F}^N - \tilde{F}^D = \frac{1}{\Gamma(d)} \int_0^{\frac{1}{2}} du\ u\ \sin \pi u\ \Gamma\left(\frac{d-1}{2} + u\right)\Gamma\left(\frac{d-1}{2} - u\right). \tag{4.22}$$

This can be checked to be positive numerically in all $d > 2$,[8] in accordance with $\tilde{F}^{UV} > \tilde{F}^{IR}$ for continuous d. We plot this quantity in fig. 4.3. Note that in even d the free energies F^N and F^D have UV divergences related to the bulk conformal anomaly, but they cancel when taking the difference leaving a well-defined, finite quantity.[9]

4.1.2 Weakly relevant boundary flows

Another simple situation where we can discuss the boundary F-theorem using the AdS free energy is in conformal perturbation theory. Since the boundary is a sphere, for small perturbations localized on the boundary, the statement is similar to the statement of generalized F-theorem for the sphere free energy in field theories without a boundary [142, 164] (even though in the BCFT case the

[8] In the limit $d \to 2$, one can see that (4.22) diverges logarithmically. This is due to the fact that for Neumann boundary condition the free scalar develops a zero mode in $d = 2$. The zero mode should be separated out and treated carefully, but we will not discuss this case in detail in the chapter.

[9] In the standard spectral zeta function approach, the bulk UV divergence is captured by $\zeta(0)$ [162] (equivalently in the heat kernel approach, it is captured by the Seeley coefficient b_d). This can be computed from the second line in (4.14) by evaluating the integral by analytic continuation in α, and setting $\alpha = 0$ at the end (without taking the derivative in front). The result is easily seen to be independent on the boundary condition, as expected since it is related to short-distance physics in the bulk. For instance, in $d = 4$ one finds $\zeta(0) = -1/90$ for both $\hat{\Delta} = 1$ and $\hat{\Delta} = 2$, see e.g. [163].

boundary theory is not a local CFT by itself, this is not important for the calculation of F in conformal perturbation theory). It was shown in [164] that when a CFT is perturbed by a weakly relevant operator, the universal part of the free energy decreases. The same argument basically goes through when we put the CFT in AdS and perturb the boundary by a weakly relevant boundary operator, with just the replacement $d \to d-1$. We give a very brief review of the argument here and refer the reader to [164] for more details.

Consider a CFT in AdS_d perturbed by a weakly relevant operator localized on the boundary

$$S = S_{CFT_0} \; + \; g_b \int_{S^{d-1}} O \tag{4.23}$$

where g_b is the bare coupling constant and O has bare dimension $\hat{\Delta} = d - 1 - \epsilon$ with small ϵ. The correlation functions of O can be written in terms of the chordal distance $s(\mathbf{x}, \mathbf{y})$ on the boundary sphere

$$\begin{aligned} &\langle O(\mathbf{x})O(\mathbf{y})\rangle_0 = \frac{\mathcal{C}_2}{s(\mathbf{x},\mathbf{y})^{2\hat{\Delta}}}, \qquad \langle O(\mathbf{x})O(\mathbf{y})O(\mathbf{z})\rangle_0 = \frac{\mathcal{C}_3}{s(\mathbf{x},\mathbf{y})^{\Delta} s(\mathbf{y},\mathbf{z})^{\Delta} s(\mathbf{x},\mathbf{z})^{\hat{\Delta}}}, \\ &s(\mathbf{x},\mathbf{y}) = \frac{2|\mathbf{x}-\mathbf{y}|}{(1+\mathbf{x}^2)^{1/2}(1+\mathbf{y}^2)^{1/2}}. \end{aligned} \tag{4.24}$$

It is possible to find an IR fixed point using conformal perturbation theory at the renormalized coupling

$$g = g_* = \frac{\Gamma\left(\frac{d-1}{2}\right)\mathcal{C}_3}{\pi^{\frac{d-1}{2}}\mathcal{C}_2}\epsilon + O(\epsilon^2). \tag{4.25}$$

The change in the AdS free energy under the flow can be calculated as [164]

$$\delta F = F^{UV} - F^{IR} = \frac{g_b^2}{2}\int\int_{S^{d-1}}\langle OO\rangle - \frac{g_b^3}{6}\int\int\int_{S^{d-1}}\langle OOO\rangle. \tag{4.26}$$

These integrals over the sphere can be evaluated explicitly. They are divergent as $\epsilon \to 0$, and the divergences get cancelled when we plug in the bare coupling in terms of the renormalized coupling. All in all, the answer to leading order in ϵ turns out to be [164]

$$\delta\tilde{F} = \tilde{F}^{UV} - \tilde{F}^{IR} = \frac{\pi\Gamma\left(\frac{d-1}{2}\right)^2\mathcal{C}_2^3}{3\Gamma(d)\mathcal{C}_3^2}\epsilon^3 \tag{4.27}$$

which is always positive since in an unitary theory $\mathcal{C}_2 > 0$ and $\mathcal{C}_3$ is real. Similar observations were made in [130] about the hemisphere free energy.

4.1.3 Relation to trace anomaly coefficients in $d = 3$

The boundary free energy is related to the conformal anomaly that appears in the trace of energy momentum tensor when we put the theory on a curved space with a boundary [165, 132, 166, 137, 167, 168, 169]. We confine to $d = 3$ for this discussion, where there is no conformal anomaly without the presence of a boundary. In the presence of a boundary, there is a conformal anomaly localized at the boundary, and the trace of the energy momentum tensor takes the form [165, 132, 167]

$$\langle T^{\mu}{}_{\mu}\rangle^{d=3} = \frac{\delta(x_{\perp})}{4\pi}\left(a_{3d}\hat{\mathcal{R}} + b\mathrm{tr}\hat{K}^2\right) \tag{4.28}$$

where $\hat{\mathcal{R}}$ is the boundary Ricci scalar and $\hat{K}_{ij}$ is the traceless part of the extrinsic curvature K_{ij} associated to the boundary

$$\hat{K}_{ij} = K_{ij} - \frac{1}{2}\gamma_{ij}K \implies \mathrm{tr}\hat{K}^2 = \mathrm{tr}K^2 - \frac{1}{2}K^2 \tag{4.29}$$

with γ_{ij} being the boundary metric. The coefficient b is related to the displacement operator two-point function, and we come back to it in appendix 4.7. The other coefficient a_{3d} is proportional to the logarithmic divergence in the AdS free energy as we now discuss. The change in the free energy under a Weyl transformation, $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$ is given by

$$\delta^W F = -\frac{1}{2}\int d^dx\sqrt{g}\delta g_{\mu\nu}\langle T^{\mu}{}_{\nu}\rangle = -\sigma\int d^dx\sqrt{g}\langle T^{\mu}{}_{\mu}\rangle \tag{4.30}$$

In our case of Euclidean hyperbolic ball in three dimensions (we restore the AdS radius just for this discussion), the boundary is just a two-sphere of radius R, so the Ricci scalar is $2/R^2$ and the extrinsic curvature is $K_{ij} = \frac{1}{2}\gamma_{ij}K$ so that the traceless part vanishes. So we have the change in free energy under the Weyl transformation in terms of the anomly coefficient

$$\delta^W F = -2\sigma a_{3d}. \tag{4.31}$$

Under the above Weyl transformation, $R \to e^{\sigma}R$. The regularized volume of AdS_3 can be computed by imposing a radial cutoff, and it is equal to $-2\pi\log(R/\epsilon)$ [170, 171] where ϵ is a UV cutoff and R the radius of the boundary sphere. This gives the free energy and its change under the Weyl transformation using eq. (4.15) as

$$F(\hat{\Delta}) = \frac{\log(R/\epsilon)}{6}(\hat{\Delta}-1)^3, \qquad \delta^W F(\hat{\Delta}) = \frac{\sigma}{6}(\hat{\Delta}-1)^3 \tag{4.32}$$

which implies that

$$a_{3d} = -\frac{1}{12}(\hat{\Delta} - 1)^3 \,. \tag{4.33}$$

This tells us that in the free theory of a single scalar, $a_{3d}^N = 1/96$ and $a_{3d}^D = -1/96$ for Neumann and Dirichlet boundary conditions respectively, in agreement with the known results [132, 167].

4.2 Large N $O(N)$ model in AdS: boundary critical points and free energy

In this section, we study the critical $O(N)$ model in AdS, focusing on the large N expansion. As explained in the introduction, this is related by a Weyl transformation to studying the $O(N)$ model on the flat half-space (or on a ball with spherical boundary). Mapping the action (4.5) to AdS by a Weyl transformation, we obtain the action for the critical $O(N)$ model in hyperbolic space as

$$S = \int d^d x \sqrt{g} \left(\frac{1}{2}(\partial \phi^I)^2 - \frac{d(d-2)}{8} \phi^I \phi^I + \frac{1}{2} \sigma \phi^I \phi^I \right) . \tag{4.34}$$

The various boundary critical points of the model can be then recovered by solving the saddle point equations arising by integrating out the scalar fields.

Before moving on to the interacting theory, let us first discuss in a bit more detail the case of free scalar BCFT, viewed from the AdS approach. The free scalar in half-space is related to a conformally coupled scalar in AdS. As expected from the Weyl transformation, it is easy to see explicitly that the two-point function of a massless scalar on the half-space with Neumann or Dirichlet boundary conditions is the same as the bulk-to-bulk propagator in AdS up to an overall conformal factor, namely

$$\begin{aligned} \langle \phi(x_1)\phi(x_2) \rangle_{N/D}^{\text{flat}} &= \frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{\frac{d}{2}} (4 y_1 y_2)^{\frac{d}{2}-1}} \left(\frac{1}{\xi^{\frac{d}{2}-1}} \pm \frac{1}{(\xi+1)^{\frac{d}{2}-1}} \right), \quad \xi = \frac{\mathbf{x}_{12}^2 + y_{12}^2}{4 y_1 y_2} \\ &= \frac{1}{(y_1 y_2)^{\frac{d}{2}-1}} G^{bb}_{\hat{\Delta}_{N/D}}, \quad \hat{\Delta}_N = \frac{d}{2} - 1 \,, \quad \hat{\Delta}_D = \frac{d}{2} \end{aligned} \tag{4.35}$$

where $G^{bb}_{\hat{\Delta}}$ is the well-known bulk-to-bulk propagator in AdS given by

$$G^{bb}_{\hat{\Delta}} = \frac{\Gamma(\hat{\Delta})}{2\pi^{\frac{d-1}{2}} \Gamma\left(\hat{\Delta} + \frac{3-d}{2}\right) (4\xi)^{\hat{\Delta}}} {}_2F_1\left(\hat{\Delta}, \hat{\Delta} - \frac{d}{2} + 1, 2\hat{\Delta} - d + 2, -\frac{1}{\xi} \right). \tag{4.36}$$

and the values of the conformal dimensions $\hat{\Delta}_{N/D}$ corresponding to Neumann and Dirichlet boundary

conditions can be obtained by the AdS/CFT mass/dimension relation in eq. (4.11).

In any BCFT the bulk two-point function can be expanded in conformal blocks in two different channels: 1) Bulk channel, which corresponds to taking the two operators close to each other in the bulk, i.e. $\xi \to 0$, or 2) Boundary channel, which corresponds to taking both the operators close to the boundary and then using their boundary operator expansion (BOE), i.e. $\xi \to \infty$ (see for instance [21])

$$\begin{aligned} \langle O(x)O(x')\rangle &= \frac{A}{(4yy')^{\Delta_O}}\xi^{-\Delta_O}G(\xi) \\ G &= 1+\sum_k \lambda_k f_{\text{bulk}}(\Delta_k;\xi) = \xi^{\Delta_O}(a_O^2+\sum_l \mu_l^2 f_{\text{bdry}}(\hat{\Delta}_l;\xi)). \end{aligned} \tag{4.37}$$

The bulk and boundary blocks have following expressions [12]

$$\begin{aligned} f_{\text{bulk}}(\Delta_k;\xi) &= \xi^{\frac{\Delta_k}{2}}\ {}_2F_1\left(\frac{\Delta_k}{2},\frac{\Delta_k}{2};\Delta_k+1-\frac{d}{2};-\xi\right) \\ f_{\text{bdry}}(\hat{\Delta}_l;\xi) &= \xi^{-\hat{\Delta}_l}\ {}_2F_1\left(\hat{\Delta}_l,\hat{\Delta}_l+1-\frac{d}{2};2\hat{\Delta}_l+2-d;-\frac{1}{\xi}\right). \end{aligned} \tag{4.38}$$

To express the two-point function (4.37) in AdS, we simply strip off the conformal factor of $(yy')^{\Delta_O}$, and everything else stays the same. Note that the boundary conformal block is proportional to the AdS bulk-to-bulk propagator, which is consistent with the fact that a free bulk field induces a single operator on the boundary of dimension $d/2-1$ or $d/2$.

When we add interactions in the bulk and tune to criticality, we can have phases with more interesting boundary conditions. We can have phases that preserves the $O(N)$ symmetry, and also phases that spontaneously breaks it to $O(N-1)$. These phases correspond to different boundary critical behaviors of the model, and we will present their large N analysis below.

4.2.1 $O(N)$ invariant boundary fixed points

Assuming that the $O(N)$ symmetry is preserved, we can start from the action in eq. (4.34) and integrate out the N fundamental fields ϕ^I to get an effective action for σ

$$Z = \exp[-F] = \int [d\sigma] \exp\left[-\frac{N}{2}\text{tr}\log\left(-\nabla^2+\sigma-\frac{d(d-2)}{4}\right)\right]. \tag{4.39}$$

At large N, we can use a saddle point approximation to do the integral over σ and look for a field configuration with a constant value of $\sigma = \sigma_*$.[10] Therefore, at leading order in large N, σ_* just acts

[10] In the flat half-space picture, one would instead find that at the saddle point $\sigma = \sigma_*/y^2$, with σ_* the same constant found in the AdS calculation.

like a bulk "mass" for the ϕ^I fields (note, however, that we are still describing a BCFT, i.e. the bulk theory remains critical: the non-zero expectation value for σ is simply a reflection of the non-zero one-point function of the operator $\phi^2 \sim \sigma$). As in section 4.1, the free energy can then be written as

$$\begin{aligned} F(\sigma) &= \frac{N}{2}\mathrm{tr}\log\left(-\nabla^2 + \sigma - \frac{d(d-2)}{4}\right) \\ &= \frac{N\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int_{-\infty}^{\infty} d\nu \frac{\Gamma(i\nu+\frac{d-1}{2})\Gamma(-i\nu+\frac{d-1}{2})}{\Gamma(i\nu)\Gamma(-i\nu)}\log\left(\nu^2+\sigma+\frac{1}{4}\right). \end{aligned} \tag{4.40}$$

The constant σ_* can then be fixed by demanding that it extremizes the free energy, which happens when the following derivative of free energy with σ vanishes

$$\begin{aligned} &\left.\frac{\partial F(\sigma)}{\partial\sigma}\right|_{\sigma=\sigma_*} = \frac{N\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int_{-\infty}^{\infty} d\nu \frac{\Gamma\left(i\nu+\frac{d-1}{2}\right)\Gamma\left(-i\nu+\frac{d-1}{2}\right)}{\Gamma(i\nu)\Gamma(-i\nu)}\frac{1}{\nu^2+\sigma_*+\frac{1}{4}} \\ &= \frac{N\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\Bigg[-\sin\left(\pi\sqrt{\sigma_*+\frac{1}{4}}\right)\Gamma\left(\frac{d-1-\sqrt{4\sigma_*+1}}{2}\right)\Gamma\left(\frac{d-1+\sqrt{4\sigma_*+1}}{2}\right) \\ &-\sum_{n=0}^{\infty}\frac{4(-1)^n(d+2n-1)\cos\left(\frac{\pi}{2}(d+2n)\right)\Gamma(d+n-1)}{\Gamma(n+1)\left(4dn+(d-2)d+4n^2-4(n+\sigma_*)\right)}\Bigg] \end{aligned} \tag{4.41}$$

To go from the first line to second line, we performed the ν-integral by closing the contour in the complex ν plane and summing over residues. The arc at infinity can be dropped for $d < 2$, but in dimensional regularization we may continue the final result to $d > 2$. Note that one of the Gamma functions also introduces poles at $\nu = i(d-1+2n)/2$, which lie on the upper half plane for $d > 1$: these poles give the sum over n above. The sum can be performed by analytic continuation in d, and we get the final result

$$\begin{aligned} \left.\frac{\partial F(\sigma)}{\partial\sigma}\right|_{\sigma=\sigma_*} &= \frac{N\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\frac{\Gamma\left(\frac{d-1}{2}+\sqrt{\sigma_*+\frac{1}{4}}\right)\Gamma\left(\frac{d-1}{2}-\sqrt{\sigma_*+\frac{1}{4}}\right)\sin\left(\pi\left(\frac{d-1}{2}-\sqrt{\sigma_*+\frac{1}{4}}\right)\right)}{\sin\left(\frac{\pi d}{2}\right)} \\ &= \frac{N\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}}\frac{\Gamma(\hat{\Delta})\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(-d+\hat{\Delta}+2)} \end{aligned} \tag{4.42}$$

where we used again the familiar AdS/CFT relation

$$\hat{\Delta}(\hat{\Delta}-(d-1)) = \sigma_* - \frac{d(d-2)}{4}. \tag{4.43}$$

We also had to use $\hat{\Delta} > (d-1)/2$ to get to the last line in eq. (4.42) which is where the above spectral representation is valid, but the final result can be analytically continued in $\hat{\Delta}$.

Another way to arrive at the same result, that does not involve the spectral representation, is to note that $\partial F/\partial\sigma$ is just the integral over AdS of the one point function $\langle\phi^I\phi^I(x)\rangle/2$, and the integral only produces the volume factor since the one-point functions are constant on AdS. At leading order in large N, $\sigma=\sigma_*$ acts as a constant mass term, so ϕ is a free massive field in AdS and its propagator must be the usual bulk-bulk propagator, eq. (4.36). The required one-point function $\langle\phi^I\phi^I(x)\rangle/2$ then is equal to the coincident point limit of the two-point function, and can be obtained from its $\xi\to 0$ limit

$$\begin{aligned}\langle\phi^I(x_1)\phi^J(x_2)\rangle &= \delta^{IJ}G_{bb}^{\hat{\Delta}} \\ &= \frac{\delta^{IJ}}{\xi^{\frac{d}{2}-1}}\left(\frac{\Gamma\left(\frac{d}{2}-1\right)}{(4\pi)^{d/2}}+O(\xi)\right)+\left(\frac{\Gamma(\hat{\Delta})\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(-d+\hat{\Delta}+2)}+O(\xi)\right).\end{aligned} \tag{4.44}$$

One can see that the constant piece of the above expression, which is the coincident limit of the two point function, is the same as the derivative of free energy in eq. (4.42) up to a factor of $N\text{Vol}(H^d)/2$.

The saddle point requirement of the vanishing of the free energy derivative in eq. (4.42) or equivalently vanishing of the constant piece in eq. (4.44) can also be now motivated in another way: in a BCFT, the bulk OPE data should be unaffected by the boundary, and hence the operator spectrum encoded in the $\phi\phi$ bulk OPE should be the same as the one for the critical $O(N)$ model in flat space with no boundary. In particular, the bulk spectrum should be such that the operator ϕ^2 of dimension $d-2$ in the free theory is replaced by the operator σ of dimension 2. The $\xi\to 0$ expansion in eq. (4.44) is the same as doing the bulk OPE, and we recognize that the second term would correspond to the contribution of an operator of dimension $d-2$, which we must then set to zero. This yields the same condition as the above saddle point analysis, and fixes the dimension of the leading boundary operator $\hat{\Delta}$ and hence the value of σ^* at leading order in large N. We will also use the same argument in subsection 4.3.2 to find the $1/N$ corrections to the dimension of the leading boundary operator.

Setting eq. (4.42) equal to zero requires $\hat{\Delta}=d-n$, for integer n with $n\geq 2$. We will restrict to the case of unitary theories, and so we will only consider solutions satisfying the boundary unitarity bound. The existence of these saddles was also noted in [120].

In dimensions $3\leq d\leq 5$, there are two unitary solutions with dimension $\hat{\Delta}$ of the form $d-n$:

$$\hat{\Delta} = d-2 \implies \sigma_* = \frac{(d-2)(d-4)}{4} \tag{4.45}$$

$$\hat{\Delta} = d-3 \implies \sigma_* = \frac{(d-4)(d-6)}{4} \tag{4.46}$$

The $\hat{\Delta} = d - 2$ solution describes the ordinary transition, while the $\hat{\Delta} = d - 3$ solution describes the special transition. In $d = 4 - \epsilon$, these match onto the possible boundary critical behaviors of the weakly coupled Wilson-Fisher fixed point in the ϕ^4 theory (4.65) (see figure 4.6): the $\hat{\Delta} = d - 2$ and $\hat{\Delta} = d - 3$ solutions correspond to the Wilson-Fisher BCFTs obtained by perturbing the free theory respectively with Dirichlet or Neumann boundary conditions on the N fundamental fields (note that the description in terms of Dirichlet or Neumann boundary conditions is really only appropriate in the vicinity of $d = 4$, where we perturb a free scalar field theory). These results agree with what was found in [12] in the flat space setup. As we review in section 4.3 below, computing the σ two-point function around these saddle points, one finds that in the case of the special transition, σ induces at the boundary an operator of dimension 2 and an operator of dimension d, while for the ordinary transition it induces only an operator of dimension d (the latter is related to the displacement operator). Therefore, at the special transition ($\hat{\Delta} = d - 3$) there is a single $O(N)$ invariant relevant operator at the boundary, which can be used to trigger a flow from the special to the ordinary transition ($\hat{\Delta} = d - 2$). We may think of such operator as $\hat{\sigma}$, defined by the boundary limit of the bulk field σ. Since in the Hubbard-Stratonovich description σ plays the role of ϕ^2, the deformation by $\hat{\sigma}$ can be viewed as the large N counterpart of adding a boundary mass term. Hence, we expect that the AdS free energy at the two saddle points (4.45)-(4.46) should satisfy $\tilde{F}^{\hat{\Delta}=d-3} > \tilde{F}^{\hat{\Delta}=d-2}$. We will verify this explicitly below.

While the solution $\hat{\Delta} = d - 3$ for the special transition does not extend to $d < 3$,[11] the solution $\hat{\Delta} = d - 2$ corresponding to the ordinary transition smoothly continues to $d < 3$. As we will discuss below, this solution matches near $d = 2$ with the critical point of the $O(N)$ non-linear sigma model in $d = 2 + \epsilon$, for the case of Neumann boundary conditions on the unconstrained $N - 1$ fields [172]. We will compute the anomalous dimension of the leading boundary operator in the non-linear sigma model in eq. (4.139) below, which is seen to be precisely consistent with the large N result.

As we go above $d = 5$, for $5 < d < 7$, we have now three possible solutions consistent with unitarity bounds. Besides (4.45) and (4.46), there is an additional one

$$\hat{\Delta} = d - 4 \implies \sigma_* = \frac{(d-6)(d-8)}{4}. \tag{4.47}$$

In $d = 6 - \epsilon$, all these solutions match onto possible phases of the cubic theory of eq. (4.84), as we verify below, see figure 4.6 (since we are interested in unitary theories, we will restrict our attention to the case of $d \leq 6$ in this chapter). In particular, the new phase with $\hat{\Delta} = d - 4 + O(1/N)$

[11] The lower critical dimension for special transition is $d = 3$ for $N > 1$. In the $N = 1$ case, however, the lower critical dimension is $d = 2$ [150]. In this chapter we focus on the large N theory.

corresponds in $d = 6-\epsilon$ to the fixed point of the cubic theory with Neumann boundary conditions on the N fields ϕ^I, while the $\hat{\Delta} = d-3+O(1/N)$ phase corresponds to Dirichlet boundary conditions. Therefore, at least in the perturbative description near $d = 6$, we expect that we can flow from the $\hat{\Delta} = d-4$ to the $\hat{\Delta} = d-3$ phase by adding a boundary mass term of dimension ~ 4 (in the large N description, this should correspond to an operator of dimension $4+O(1/N)$ contained in the bulk-boundary operator expansion of σ). Finally, the phase with $\hat{\Delta} = d-2$, which is the smooth continuation of the ordinary transition, corresponds in the cubic model description to a $O(N)$ invariant saddle point with non-zero expectation value for the σ field. We expect that this phase can be reached by perturbing either the $\hat{\Delta} = d-4$ or $\hat{\Delta} = d-3$ phases by the dimension 2 operator $\sim h\hat{\sigma}$. To summarize, if the boundary F-theorem holds, we then expect that the free energies should satisfy $\tilde{F}^{\hat{\Delta}=d-4} > \tilde{F}^{\hat{\Delta}=d-3} > \tilde{F}^{\hat{\Delta}=d-2}$. We will verify this shortly.

Having identified the various boundary fixed points with $O(N)$ symmetry, we can go on and compute the corresponding values of the AdS free energy. In $d = 3$, recall from section 4.1 that the free energy can be computed exactly for any value of $\hat{\Delta}$ and we can directly use eq. (4.15), or, in terms of $\tilde{F}$:

$$\tilde{F}(\hat{\Delta}) = -\frac{\pi}{12}\left(\hat{\Delta}-1\right)^3, \tag{4.48}$$

where we used that $-\sin(\pi(d-1)/2)\mathrm{Vol}(H^d)|_{d\to 3} = \pi^2$. Special and ordinary transitions correspond to $\hat{\Delta} = d-3$ and $\hat{\Delta} = d-2$ respectively, and hence

$$\tilde{F}^S = \frac{N\pi}{12}, \qquad \tilde{F}^O = 0. \tag{4.49}$$

So clearly $\tilde{F}^S > \tilde{F}^O$. We can also immediately get the anomaly coefficient a_{3d} by using eq. (4.33)

$$a_{3d}^S = \frac{N}{12}, \qquad a_{3d}^O = 0. \tag{4.50}$$

For $d = 5$ as well, we can use eq. (4.17) to calculate the free energy for all three symmetry preserving phases

$$\tilde{F}(\hat{\Delta}=1) = \frac{\pi}{360}, \qquad \tilde{F}(\hat{\Delta}=2) = 0, \qquad \tilde{F}(\hat{\Delta}=3) = -\frac{\pi}{360}. \tag{4.51}$$

To make progress for other values of d, we can use the derivative of the free energy in eq. (4.42) and express the free energy as a function of $\hat{\Delta}$ in terms of a reference value, say for a conformally

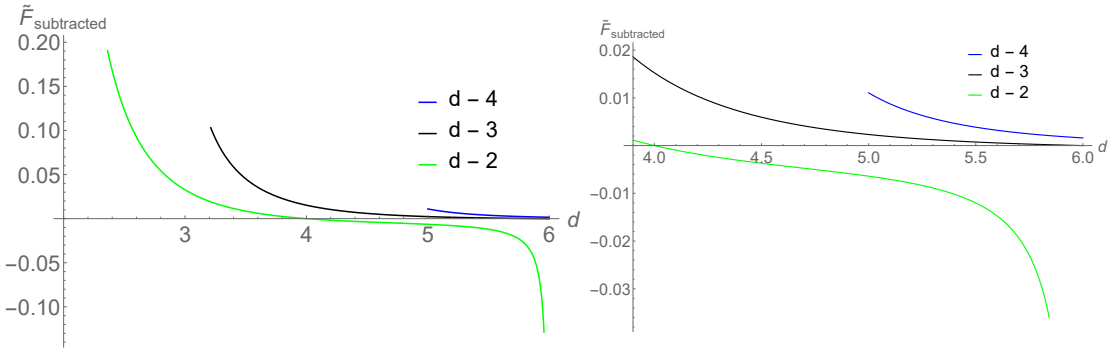


Figure 4.4: Large N free energy between $2 < d < 6$ for the different boundary fixed points. We are plotting $\tilde{F}_{\text{subtracted}} = (\tilde{F} - N\tilde{F}(d/2))/N$ on $y-$ axis. On right, we zoom in to the region between 4 and 6 dimensions for clarity.

coupled scalar with Dirichlet boundary, $F(\hat{\Delta} = d/2)$, and we find

$$F(\hat{\Delta}) = N\ F(d/2) + \int_{\frac{d}{2}}^{\hat{\Delta}} \frac{\partial F(\hat{\Delta})}{\partial \hat{\Delta}}\, d\hat{\Delta}, \qquad \frac{\partial F(\hat{\Delta})}{\partial \hat{\Delta}} = (2\hat{\Delta} - d + 1)\frac{\partial F(\sigma(\hat{\Delta}))}{\partial \sigma} \tag{4.52}$$

The integral can be evaluated numerically, and we plot the result for all the large N phases we discussed above in figure 4.4. The values of the free energy are indeed consistent for all d in $2 < d < 6$ with the RG flows discussed above and the boundary F-theorem in terms of $\tilde{F}$ in AdS.

In subsection 4.2.3, we will compute these free energies in an ϵ expansion near even dimensions and verify that they are consistent with the results from the large N expansion. For comparison, we present here some of our large N result near $d = 4$ and 6 by performing the integral in eq. (4.52) near these even dimensions. In $d = 4 - \epsilon$, we have two possible phases of the large N theory which preserve $O(N)$ symmetry

$$\begin{aligned} F(\hat{\Delta} = d - 3) &= NF\left(\frac{d}{2} - 1\right) + N\text{Vol}(H^d)\left[\frac{\epsilon}{128\pi^2} + \frac{\epsilon^2(\gamma + 1 + \log 4\pi)}{256\pi^2}\right] \\ F(\hat{\Delta} = d - 2) &= NF\left(\frac{d}{2}\right) + N\text{Vol}(H^d)\left[\frac{\epsilon}{128\pi^2} + \frac{\epsilon^2(\gamma - 1 + \log 4\pi)}{256\pi^2}\right] \end{aligned} \tag{4.53}$$

where we express the $\hat{\Delta} = d - 3$ free energy in terms of $\hat{\Delta} = d/2 - 1$ because that is what we will directly obtain from ϵ expansion. This is because, as we said before, this phase is obtained by perturbing free theory in 4 dimensions with Neumann boundary conditions. In $d = 6 - \epsilon$, we have

three possible phases

$$\begin{aligned} F(\hat{\Delta} = d-4) &= NF\left(\frac{d}{2}-1\right) \; + \; \frac{N\text{Vol}(H^d)\epsilon}{512\pi^3} \\ F(\hat{\Delta} = d-3) &= NF\left(\frac{d}{2}\right) \; + \; \frac{N\text{Vol}(H^d)\epsilon}{512\pi^3} \\ F(\hat{\Delta} = d-2) &= NF\left(\frac{d}{2}\right) \; - \; \frac{N\text{Vol}(H^d)}{96\pi^3\epsilon} \end{aligned} \tag{4.54}$$

Lastly, in $d = 2$, we have one phase with $\hat{\Delta} = d - 2$

$$F(\hat{\Delta} = d-2) = NF\left(\frac{d}{2}-1\right) \; + \; \frac{N\text{Vol}(H^d)(1-2\log 2)}{8\pi}. \tag{4.55}$$

Note that there are higher order corrections to all of these formulas which go like higher powers in ϵ.

4.2.2 $O(N)$ symmetry breaking phase: extraordinary transition

There is also a phase of this model when the $O(N)$ symmetry is spontaneously broken to $O(N-1)$ and the fundamental field ϕ^I also gets a one-point function, in addition to σ. This is what is referred to in the literature as the "extraordinary transition". As explained in the introduction, it can be obtained by either perturbing the special transition by a boundary mass term with negative coefficient, or by adding a "boundary magnetic field" term $\sim h\phi^N$ (picking a particular direction in the scalar field space). See e.g. [14] for discussion of this phase in the ϵ expansion, and [173] for a large N treatment in the flat space setup. In this section we discuss the large N description of this phase, using the AdS setup. In section 4.2.3 below we then discuss how this phase arises in the various ϵ expansions near the even dimensions $d = 2, 4$ and 6, and match the results with large N wherever applicable.

We again start with eq. (4.34), but unlike the previous section we do not assume $O(N)$ symmetry and only integrate out $N-1$ fields

$$\begin{aligned} Z &= \exp[-F^E] = \int d[\phi^N] d[\sigma] \exp\left[-S\right] \\ S &= \int d^d x\sqrt{g}\left(\frac{1}{2}(\partial\phi^N)^2 + \frac{1}{2}(\phi^N)^2\left(\sigma - \frac{d(d-2)}{4}\right)\right) + \frac{N-1}{2}\text{tr}\log\left(-\nabla^2 + \sigma - \frac{d(d-2)}{4}\right) \end{aligned} \tag{4.56}$$

At large N, we look for a saddle point with constant one point functions $\sigma = \sigma_*$ and $\phi^N = \phi^N_*$ and

require that the derivatives vanish

$$\begin{aligned}\left.\frac{\partial F^E}{\partial\sigma}\right|_{\sigma_*,\phi_*^N} &= \frac{\mathrm{Vol}(H^d)(\phi_*^N)^2}{2}+\frac{(N-1)\mathrm{Vol}(H^d)}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\\ &\times\frac{\Gamma\left(\frac{d-1}{2}+\sqrt{\sigma_*+\frac{1}{4}}\right)\Gamma\left(\frac{d-1}{2}-\sqrt{\sigma_*+\frac{1}{4}}\right)\sin\left(\pi\left(\frac{d-1}{2}-\sqrt{\sigma_*+\frac{1}{4}}\right)\right)}{\sin\left(\frac{\pi d}{2}\right)}\\ \left.\frac{\partial F^E}{\partial\phi^N}\right|_{\sigma_*,\phi_*^N} &= \mathrm{Vol}(H^d)\phi_*^N\left(\sigma_*-\frac{d(d-2)}{4}\right)\end{aligned} \tag{4.57}$$

Assuming $\phi_*^N\neq 0$ (otherwise we fall back to the $O(N)$ invariant phases discussed earlier), the second equation gives us $\sigma_*=d(d-2)/4$, which when plugged into the first equation yields

$$(\phi_*^N)^2=-\frac{(N-1)\Gamma(d-1)\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}. \tag{4.58}$$

We can expand this result near even dimensions $d=2,4$ and 6

$$(\phi_*^N)^2\Big|_{d=2+\epsilon}=\frac{N-1}{2\pi\epsilon},\quad (\phi_*^N)^2\Big|_{d=4-\epsilon}=\frac{N-1}{4\pi^2\epsilon},\quad (\phi_*^N)^2\Big|_{d=6-\epsilon}=-\frac{3(N-1)}{8\pi^3\epsilon} \tag{4.59}$$

which, as we will see below, match the various ϵ expansions. Note that the result (4.58) is negative for $4<d<6$, indicating that this phase is non-unitary in that range of dimensions. We will still discuss below, the $d=6-\epsilon$ description of this phase as a useful cross-check of our results.

At large N, the $N-1$ transverse fields are just free fields in AdS with their dimensions given by

$$\hat{\Delta}^T(\hat{\Delta}^T-d+1)=\sigma_*-\frac{d(d-2)}{4}=0\implies\hat{\Delta}^T=d-1+O(1/N). \tag{4.60}$$

The fact that the fields are massless is related to the fact that these are Goldstone modes for the spontaneously broken $O(N)$ symmetry. Therefore, we expect that the relation $\hat{\Delta}^T=d-1$ may hold to all orders in $1/N$ (this was also observed in [173]).

At leading order at large N, the free energy only receives contribution from the determinants of these transverse fields, and knowing their dimension, we can compute its value using (4.15). In $d=3$ we get

$$F^E=-\frac{N\mathrm{Vol}(H^3)}{12\pi},\qquad \tilde{F}^E=-\frac{N\pi}{12},\qquad a_{3d}^E=-\frac{N}{12}. \tag{4.61}$$

So clearly $\tilde{F}^S>\tilde{F}^O>\tilde{F}^E$ consistent with the expected F-theorem for $\tilde{F}$. For other values of d, we

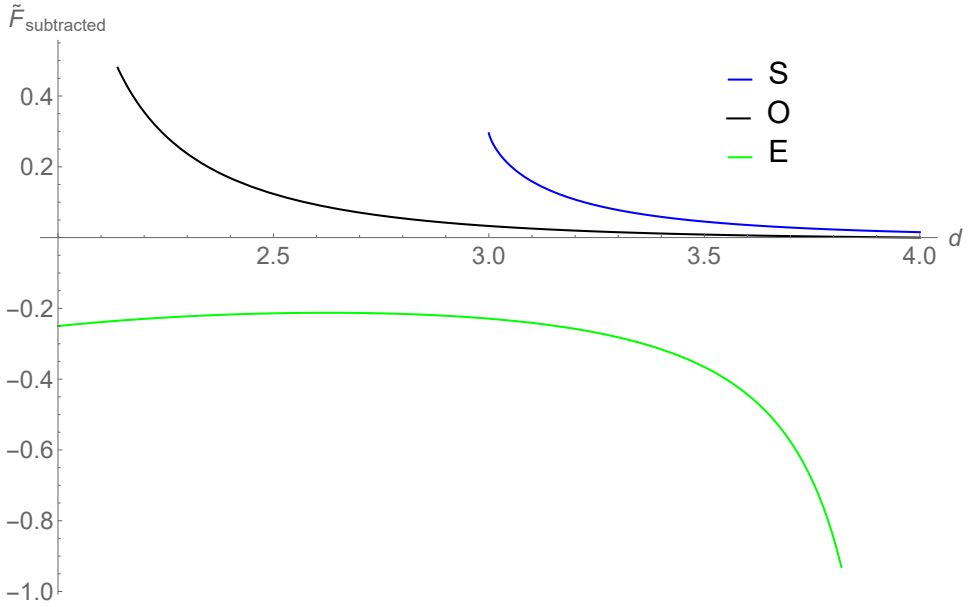


Figure 4.5: Large N free energy between $2 < d < 4$ for special(S), ordinary(O) and extraordinary(E) transitions. We are plotting $\tilde{F}_{\text{subtracted}} = (\tilde{F} - N\tilde{F}(d/2))/N$ on $y-$ axis.

can use eq. (4.52) to get

$$F(\hat{\Delta} = d-1) = N\ F(d/2) + \int_{\frac{d}{2}}^{d-1} \frac{\partial F(\hat{\Delta})}{\partial \hat{\Delta}}\, d\hat{\Delta})\,. \tag{4.62}$$

We plot the free energies for three phases (special, ordinary and extraordinary) in figure 4.5, again showing agreement with the boundary F-theorem in the continuous range $2 < d < 4$. We do not plot extraordinary above $d = 4$ because this phase becomes non-unitary for $d > 4$, as mentioned above. Hence, we do not expect the conjectured F theorem to hold for this phase in $4 < d < 6$.

In preparation for the ϵ expansion calculation, we report the results for the large N free energy for the extraordinary phase near dimensions $2, 4$ and 6 to leading order in ϵ

$$\begin{aligned}
F(\hat{\Delta} = d-1)|_{d=2+\epsilon} &= NF\left(\frac{d}{2}\right) - \frac{N\text{Vol}(H^d)}{8\pi} \\
F(\hat{\Delta} = d-1)|_{d=4-\epsilon} &= NF\left(\frac{d}{2}\right) - \frac{N\text{Vol}(H^d)}{8\pi^2\epsilon} \\
F(\hat{\Delta} = d-1)|_{d=6-\epsilon} &= NF\left(\frac{d}{2}\right) + \frac{9N\text{Vol}(H^d)}{32\pi^3\epsilon}.
\end{aligned} \tag{4.63}$$

Note that the free energy in extraordinary transition in $d = 6-\epsilon$ is actually higher than all the other phases in $6 - \epsilon$ dimensions computed in eq. (4.54). This should be related to the non-unitarity of the extraordinary transition above 4 dimensions. We can also check this explicitly in $d = 5$, where

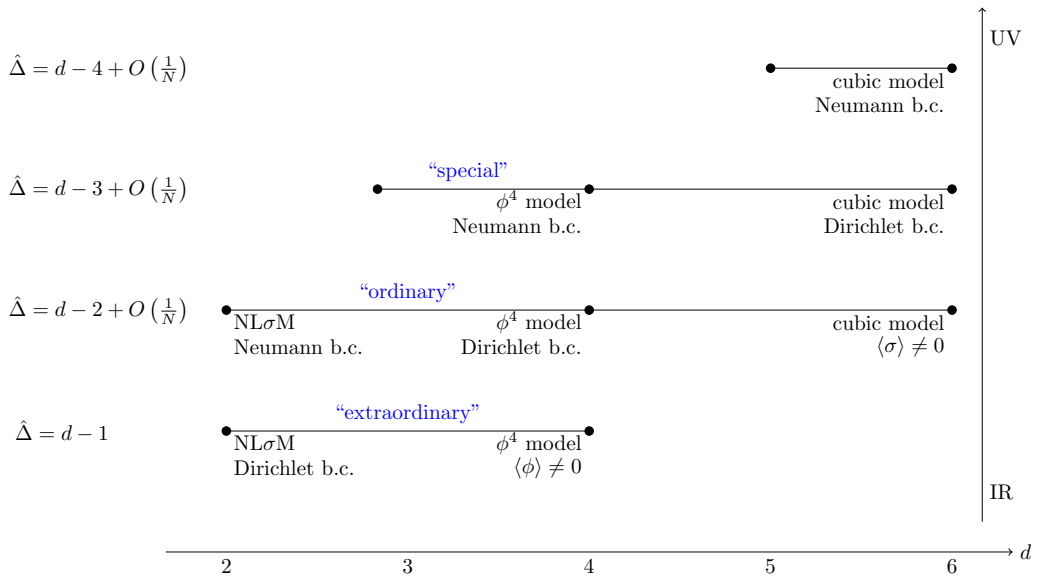


Figure 4.6: All the large N unitary boundary fixed points and their weakly coupled descriptions. The values of $\hat{\Delta}$ on the left side are the large N scaling dimensions of the leading boundary operator induced by ϕ^I.

using (4.17) we get

$$\tilde{F}(\hat{\Delta} = 4) = \frac{7\pi}{90} \tag{4.64}$$

which is again higher, rather than lower, than all the other three symmetry preserving phases in $d = 5$ in eq. (4.51). We view this as an indication that the validity of the boundary F-theorem is tied with unitarity of the BCFT.

4.2.3 ϵ expansion near even dimensions

The various phases discussed in the previous two subsections all have weakly coupled descriptions in the ϵ expansion near even dimensions for arbitrary N. We discuss those descriptions in this subsection, and in particular match the free energy results that we computed above by large N methods. A summary of all the phases and their perturbative descriptions is given in figure 4.6.

$d = 4 - \epsilon$ dimensions

In $d = 4-\epsilon$, the critical behavior of $O(N)$ model is described by the Wilson-Fisher fixed point of the quartic $O(N)$ model written in eq. (4.4). Mapping the action to AdS by including the appropriate conformal coupling term, we have

$$S = \int d^d x \sqrt{g} \left(\frac{1}{2} (\partial_\mu \phi^I)^2 - \frac{d(d-2)}{8} \phi^I \phi^I + \frac{\lambda}{4} (\phi^I \phi^I)^2 \right) . \tag{4.65}$$

Ordinary and special transition can be described by the fixed points obtained by perturbing the free theory with Neumann or Dirichlet boundary condition, respectively. We can compute their free energy in perturbation theory as

$$\begin{aligned} F = NF_{\text{free}} \ &+ \frac{\lambda}{4}\int d^dx\sqrt{g_x}\langle(\phi^I\phi^I(x))^2\rangle_0 + \frac{\delta_\lambda}{4}\int d^dx\sqrt{g_x}\langle(\phi^I\phi^I(x))^2\rangle_0 \\ &- \frac{\lambda^2}{32}\int d^dx\int d^dx'\sqrt{g_x}\sqrt{g'_x}\langle(\phi^I\phi^I(x))^2(\phi^J\phi^J(x'))^2\rangle_0 \end{aligned} \tag{4.66}$$

where the expectation values are taken in the free theory and we have introduced counterterms to deal with the divergences that arise (note that the counterterm δ_λ is fixed by the flat space divergences and is unaffected by the presence of the boundary). To do the integrals on the second line, it will be convenient to work with the ball coordinates in AdS introduced in eq. (4.8). In terms of these coordinates, we can put one of the points at the center of the ball ($u' = 0$) and then it can be checked that the cross ratio becomes

$$\xi = \frac{(\mathbf{x}-\mathbf{x}')^2 + (y-y')^2}{4yy'} \to \frac{u^2}{1-u^2}\,. \tag{4.67}$$

The two-point function becomes (here and elsewhere in this section, the top sign refers to Neumann and the bottom sign to Dirichlet boundary condition)

$$\langle\phi^I(x)\phi^J(x')\rangle_{N/D} = \delta^{IJ}G(x,x') = \frac{\delta^{IJ}\Gamma\left(\frac{d}{2}-1\right)(1-u^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\left(u^{2-d}\pm 1\right). \tag{4.68}$$

Plugging these in, we get

$$\begin{aligned} F = NF_{\text{free}} \ &+ \frac{(\lambda+\delta_\lambda)\text{Vol}(H^d)}{4}N(N+2)\frac{\Gamma\left(\frac{d}{2}-1\right)^2}{(4\pi)^d} - \frac{\lambda^2\text{Vol}(H^d)}{4}\frac{\Gamma(\frac{d}{2}-1)^4}{(4\pi)^{2d}}\frac{2\pi^{\frac{d}{2}}2^d}{\Gamma\left(\frac{d}{2}\right)}\times \\ &\int_0^1 \frac{u^{d-1}du}{(1-u^2)^d}\left[N(N^2+4(N+1))(1-u^2)^{d-2}(u^{2-d}\pm 1)^2 + N(N+2)(1-u^2)^{2d-4}(u^{2-d}\pm 1)^4\right] \end{aligned} \tag{4.69}$$

where the integral over one of the points gave the volume of AdS and the integral over spherical coordinates of the second point gave the volume of the sphere, leaving behind a single integral over u in the last term. The integrals above diverge in the case of Neumann boundary condition. The divergence occurs as the insertion approaches the boundary, $u \to 1$. However we can regularize it

by introducing a regulator δ and doing instead the following integral

$$F = NF_{\text{free}} + \frac{(\lambda+\delta_\lambda)\text{Vol}(H^d)}{4}N(N+2)\frac{\Gamma\left(\frac{d}{2}-1\right)^2}{(4\pi)^d} - \frac{\lambda^2\text{Vol}(H^d)\Gamma(\frac{d}{2}-1)^3}{\pi^{\frac{3d}{2}}(d-2)2^{3d}}\int_0^1 \frac{u^{d-1}du}{(1-u^2)^d}$$
$$\times\left[N(N^2+4(N+1))(1-u^2)^{d-2+\delta}(u^{2-d-\delta}\pm 1)^2 + N(N+2)(1-u^2)^{2d-4+2\delta}(u^{2-d-\delta}\pm 1)^4\right] \tag{4.70}$$

and then taking limit $\delta \to 0$. We can then plug in $d = 4-\epsilon$ and $\delta_\lambda = \frac{N+8}{8\pi^2}\frac{\lambda^2}{\epsilon}$ (see for instance [174]), and with Neumann boundary conditions, we get

$$\begin{aligned} F^N &= NF\left(\frac{d}{2}-1\right) + \text{Vol}(H^d)\left[\frac{N(N+2)}{4}\left(\lambda+\frac{N+8}{8\pi^2}\frac{\lambda^2}{\epsilon}\right)\left(\frac{1}{256\pi^4}+\frac{\gamma+\log 4\pi}{256\pi^4}\epsilon\right) - \right. \\ &\left. \lambda^2\left(\frac{N(N+2)(N+8)}{8192\pi^6\epsilon}+\frac{N(N+2)(9(N+8)(\gamma+\log 4\pi)-3N+25)}{49152\pi^6}\right)\right] \\ &= NF\left(\frac{d}{2}-1\right) + \text{Vol}(H^d)\left[\lambda\frac{N(N+2)}{1024\pi^4}(1+\epsilon(\gamma+\log 4\pi))\right. \\ &\left. -\frac{\lambda^2 N(N+2)}{49152\pi^6}(3(N+8)(\gamma+\log 4\pi)-3N+25)\right] \end{aligned} \tag{4.71}$$

where we used the fact that in the free theory with Neumann boundary condition the dimension of the $\hat{\phi}$ boundary operator is $\hat{\Delta} = d/2-1$. Plugging in the critical point value of the coupling $\lambda = \lambda_* = \frac{8\pi^2}{N+8}\epsilon + \frac{24(3N+14)\pi^2}{(N+8)^3}\epsilon^2$, we find

$$\begin{aligned} F^N = NF\left(\frac{d}{2}-1\right) + \text{Vol}(H^d)\left[\frac{\epsilon}{128\pi^2}\frac{N(N+2)}{(N+8)} + \frac{3\epsilon^2}{128\pi^2}\frac{N(N+2)(3N+14)}{(N+8)^3}\right. \\ \left. + \frac{\epsilon^2 N(N+2)}{(N+8)^2 768\pi^2}(3(N+8)(\gamma+\log 4\pi)-25+3N)\right] \end{aligned} \tag{4.72}$$

which to leading order in N becomes

$$F^N = NF\left(\frac{d}{2}-1\right) + N\text{Vol}(H^d)\left[\frac{\epsilon}{128\pi^2}+\frac{\epsilon^2}{256\pi^2}(\gamma+1+\log 4\pi)\right]. \tag{4.73}$$

This agrees with $d = 4-\epsilon$ value of the large N expansion found above in eq. (4.53). With Dirichlet boundary conditions, the integral gives

$$\begin{aligned} F^D = NF\left(\frac{d}{2}\right) + \text{Vol}(H^d)\left[\lambda\frac{N(N+2)}{1024\pi^4}(1+\epsilon(\gamma+\log 4\pi))\right. \\ \left. -\frac{\lambda^2 N(N+2)}{49152\pi^6}(3N+3(N+8)(\gamma+\log 4\pi)+13))\right] \end{aligned} \tag{4.74}$$

Again, plugging in the critical point value, we find

$$F^D = NF\left(\frac{d}{2}\right) + \mathrm{Vol}(H^d)\Big[\frac{\epsilon}{128\pi^2}\frac{N(N+2)}{(N+8)} + \frac{3\epsilon^2}{128\pi^2}\frac{N(N+2)(3N+14)}{(N+8)^3} + \frac{\epsilon^2 N(N+2)}{(N+8)^2 768\pi^2}(-3N+3(N+8)(\gamma+\log 4\pi)-13))\Big] \tag{4.75}$$

which at large N becomes

$$F^D = NF\left(\frac{d}{2}\right) + N\mathrm{Vol}(H^d)\Big[\frac{\epsilon}{128\pi^2} + \frac{\epsilon^2}{256\pi^2}(\gamma+\log 4\pi - 1)\Big]. \tag{4.76}$$

This again agrees with the $4-\epsilon$ value of the large N expansion result, see eq. (4.53).

Let us now discuss the phase of this model where $O(N)$ symmetry is broken and which describes the extraordinary transition in $d = 4-\epsilon$ dimensions. In the AdS approach, this is simply obtained by minimizing the potential on the hyperbolic space for the action in eq. (4.65), $V(\phi) = -\frac{d(d-2)}{8}\phi^I\phi^I + \frac{\lambda}{4}(\phi^I\phi^I)^2$. Extremizing the potential one finds[12]

$$\phi^K\phi^K = \frac{d(d-2)}{4\lambda} = \frac{N+8}{4\pi^2\epsilon} \implies \phi^a = 0, \quad a = 1,...N-1; \ \phi^N = \sqrt{\frac{d(d-2)}{4\lambda}} = \sqrt{\frac{N+8}{4\pi^2\epsilon}} \tag{4.77}$$

where we used the value of the coupling at the fixed point $\lambda = \lambda_* = (8\pi^2\epsilon)/(N+8)$. The value of the one-point function above is precisely consistent with the result of the large N expansion in eq. (4.59). We can expand around this minimum in terms of ϕ^a and $\phi^N = \sqrt{\frac{d(d-2)}{4\lambda}} + \chi$

$$\begin{aligned} S &= \int d^dx\sqrt{g}\left(\frac{1}{2}(\partial_\mu\phi^I)^2 - \frac{d(d-2)}{8}\phi^I\phi^I + \frac{\lambda_0}{4}(\phi^I\phi^I)^2\right) \\ &= -\frac{1}{\lambda_0}\int d^dx\sqrt{g} \ + \ \int d^dx\sqrt{g}\Big(\frac{1}{2}(\partial_\mu\phi^a)^2 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{d(d-2)}{4}\chi^2 \\ &+ \frac{\sqrt{d(d-2)\lambda_0}}{2}(\chi^3 + (\phi^a\phi^a)\chi) + \frac{\lambda_0}{4}\left((\phi^a\phi^a)^2 + \chi^4 + 2\chi^2(\phi^a\phi^a)\right)\Big) \end{aligned} \tag{4.78}$$

where we use λ_0 to emphasize that we are using bare coupling here. So we are left with $N-1$ massless fields with boundary dimension 3 and a single massive field χ. The mass terms of χ also tell us the dimensions of the boundary operator corresponding to χ

$$\hat{\Delta}(\hat{\Delta}-d+1) = \frac{d(d-2)}{4}, \implies \hat{\Delta}_\pm = \frac{d-1\pm\sqrt{3d^2-6d+1}}{2} \xrightarrow{d\to 4} 4, -1 \tag{4.79}$$

We obviously choose the + boundary condition for χ for boundary unitarity, which gives a dimension

[12] In the flat space approach, this phase corresponds to the classical solution of the equation of motion $\nabla^2\phi^N = \lambda(\phi^N)^3$ given in $d=4$ by $\phi^N = \sqrt{2/\lambda}\frac{1}{y}$, see e.g. [14].

4 operator at the boundary. To leading order in ϵ, the free energy can then be written as

$$F^E = (N-1)F_{\phi^a}(\hat{\Delta}=3)+F_\chi(\hat{\Delta}=4)+S_{\text{clas.}} = (N-1)F_{\phi^a}(\hat{\Delta}=3)+F_\chi(\hat{\Delta}=4)-\frac{\text{Vol}(H^d)}{\lambda_0}. \quad (4.80)$$

Using eq. (4.52) in $4-\epsilon$ dimensions, we get

$$F_{\phi^a}(\hat{\Delta}=3) = F\left(\frac{d}{2}\right) - \frac{\text{Vol}(H^d)}{8\pi^2\epsilon}; \qquad F_\chi(\hat{\Delta}=4) = F\left(\frac{d}{2}\right) - \frac{9\text{Vol}(H^d)}{8\pi^2\epsilon}; \quad (4.81)$$

where both of these results are true up to terms that are finite as $\epsilon \to 0$. These $1/\epsilon$ poles in the free energy must be cancelled by the ones present in the bare coupling coming from the classical action. Recall that in this model, the coupling gets renormalized as (see for instance, [164])

$$\frac{1}{\lambda_0} = \mu^{-\epsilon}\left(\frac{1}{\lambda} - \frac{N+8}{8\pi^2\epsilon} + O(\lambda)\right). \quad (4.82)$$

The pole in ϵ here clearly cancels the ones coming from one-loop determinants for χ and ϕ^a, so that the free energy becomes

$$F^E = NF\left(\frac{d}{2}\right) - \frac{\text{Vol}(H^d)}{\lambda_*} = NF\left(\frac{d}{2}\right) - \frac{\text{Vol}(H^d)(N+8)}{8\pi^2\epsilon} \quad (4.83)$$

where we finally plugged in the value of the renormalized coupling at fixed point. At large N, this precisely matches the result of large N computation in eq. (4.63) near 4 dimensions.

$d = 6-\epsilon$ dimensions

Near 6 dimensions, the large N $O(N)$ model can be described by the fixed point of the cubic $O(N)$ scalar theory written in (4.7) in $d = 6-\epsilon$ dimensions. In AdS, we work with the action

$$S = \int d^dx\sqrt{g}\left[\frac{1}{2}(\partial_\mu\phi^I)^2 - \frac{d(d-2)}{8}(\phi^I\phi^I+\sigma^2) + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^I\phi^I + \frac{g_2}{6}\sigma^3\right]. \quad (4.84)$$

At large enough N, one finds a fixed point at [48]

$$(g_1)_* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}}, \quad (g_2)_* = 6(g_1)_*. \quad (4.85)$$

The $\hat{\Delta} = d-4$ and $d-3$ phases can be described by imposing Neumann or Dirichlet boundary conditions on N ϕ^I fields, respectively. We will only do the leading order in N calculation here, and this is sufficient for our purposes of comparing with the large N. To leading order, the free energy

is given by

$$F = NF_{\text{free}} \; - \frac{g_1^2 N^2}{8} \int d^d x d^d x' \sqrt{g_x}\sqrt{g_{x'}} G(x,x) G(x',x') G(x,x'). \tag{4.86}$$

Plugging in the correlators gives

$$F = NF_{\text{free}} \; - \; \frac{(g_1^2 N^2)\text{Vol}(H^d)\Gamma(\frac{d}{2}-1)^2}{2^{2d+1}\pi^d (d-2)} \int_0^1 du \; \frac{u^{d-1}}{(1-u^2)^d} (1-u^2)^{\frac{d+\delta}{2}-1}(u^{2-d-\delta} \pm 1) \tag{4.87}$$

where we already introduced the regulator δ as before, since this integral also diverges in Neumann case. By $\pm$ above, we really mean Neumann or Dirichlet boundary condition on the σ correlator, since ϕ^I only contributes through one-point function at this order. We should not be able to tell which boundary condition we are using for σ just from the leading large N calculation, and indeed both the signs give the same answer. In the case of Neumann condition on ϕ, we get

$$F^N = NF\left(\frac{d}{2}-1\right) \; + \; \frac{g_1^2 N^2 \text{Vol}(H^d)}{6(2)^{15}\pi^6} = NF\left(\frac{d}{2}-1\right) \; + \; \frac{\epsilon N \text{Vol}(H^6)}{512\pi^3} \tag{4.88}$$

while in the Dirichlet case

$$F^D = NF\left(\frac{d}{2}\right) \; + \; \frac{g_1^2 N^2 \text{Vol}(H^d)}{6(2)^{15}\pi^6} = NF\left(\frac{d}{2}\right) \; + \; \frac{\epsilon N \text{Vol}(H^6)}{512\pi^3} \tag{4.89}$$

This agrees with what we found in eq. (4.54).

There is another phase of this model that preserves $O(N)$ symmetry, and which turns out to be counterpart of the large N phase with $\hat{\Delta} = d-2$ (i.e., this is the smooth continuation of the "ordinary" transition above $d = 4$). It corresponds to the following extremum of the potential in AdS where the field σ gets a one-point function[13]

$$\sigma = \frac{d(d-2)}{2g_2}; \qquad \phi^I = 0. \tag{4.90}$$

We can expand the action around this solution as

$$\begin{aligned} S &= \int d^d x \sqrt{g} \left(\frac{1}{2}(\partial_\mu \phi^I)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{d(d-2)}{8}(\sigma^2 + \phi^I \phi^I) + \frac{g_1}{2}\sigma \phi^I \phi^I + \frac{g_2}{6}\sigma^3 \right) \\ S &= -\frac{d^3(d-2)^3}{96\ (g_{2,0})^2} \int d^d x \sqrt{g} + \int d^d x \sqrt{g} \Big(\frac{1}{2}(\partial_\mu \phi^I)^2 + \frac{1}{2}(\partial_\mu \delta\sigma)^2 + \frac{d(d-2)}{8}(\delta\sigma)^2 \\ &\quad - \left(1 - \frac{2g_{1,0}}{g_{2,0}}\right) \frac{d(d-2)}{8}\phi^I \phi^I + \frac{g_{1,0}}{2}\delta\sigma \phi^I \phi^I + \frac{g_{2,0}}{6}(\delta\sigma)^3 \Big). \end{aligned} \tag{4.91}$$

[13] In $d = 6$ flat space with flat boundary, this corresponds to the solution of $\nabla^2 \sigma = \frac{g_2}{2}\sigma^2$ given by $\sigma = \frac{12}{g_2 y^2}$.

where $\delta\sigma$ is the fluctuation of σ, and we emphasize that the coupling constants are bare. Given the mass of the fluctuations, we can read off the boundary dimensions

$$\begin{aligned}\hat{\Delta}^{\sigma}(\hat{\Delta}^{\sigma}-d+1)&=\frac{d(d-2)}{4},\implies \hat{\Delta}^{\sigma}_{\pm}=6,-1;\\ \hat{\Delta}^{\phi^I}(\hat{\Delta}^{\phi^I}-d+1)&=-\frac{d(d-2)}{4}\left(1-\frac{2g_{1,0}}{g_{2,0}}\right),\implies \hat{\Delta}^{\phi^I}_{\pm}=4,1\end{aligned} \tag{4.92}$$

We obviously choose the $\hat{\Delta}_+$ boundary condition for both σ and ϕ^I. The dimension for ϕ^I is indeed consistent with the phase of the large N theory with leading boundary dimension being $d-2$. The σ dimension is consistent with the fact that for this phase, as we will review in section 4.3 below, the leading boundary operator induced by σ has dimension $\hat{\Delta}=d$.

The free energy at leading order in ϵ then becomes

$$F^O=NF_{\phi^I}(\hat{\Delta}=4)\,+\,F_{\delta\sigma}(\hat{\Delta}=6)\,+\,S_{\text{clas.}}=NF_{\phi^I}(\hat{\Delta}=4)\,+\,F_{\delta\sigma}(\hat{\Delta}=6)-\frac{144\ \text{Vol}(H^d)}{(g_{2,0})^2}. \tag{4.93}$$

At large N, we only need to take care of the N ϕ^I fields, and using eq. (4.52)

$$F_{\phi^I}(\hat{\Delta}=4)=F\left(\frac{d}{2}\right)\ -\frac{\text{Vol}(H^d)}{96\pi^3\epsilon} \tag{4.94}$$

This $1/\epsilon$ pole gets cancelled when we plug in the bare coupling in terms of the renormalized coupling [49]

$$\begin{aligned}g_{1,0}&=\mu^{\frac{\epsilon}{2}}\left(g_1+\frac{Ng_1^3+g_1g_2^2-8g_1^3-12g_1^2g_2}{12(4\pi)^3\epsilon}+...\right)\\ g_{2,0}&=\mu^{\frac{\epsilon}{2}}\left(g_2+\frac{Ng_1^2g_2-3g_2^3-4Ng_1^3}{4(4\pi)^3\epsilon}+...\right).\end{aligned} \tag{4.95}$$

The free energy then becomes a finite function of the renormalized coupling, and plugging in the fixed point value we find

$$F^O=NF\left(\frac{d}{2}\right)-\frac{144\ \text{Vol}(H^d)}{(g_2)_*^2}=NF\left(\frac{d}{2}\right)\ -\ \frac{N\text{Vol}(H^d)}{96\pi^3\epsilon} \tag{4.96}$$

This agrees with the large N result in eq. (4.54).

As discussed above, there is also a (non-unitary) $O(N)$ symmetry breaking vacuum of the cubic theory in eq. (4.84) which describes the extraordinary transition in $d=6-\epsilon$. It corresponds to an

extremum of the potential in AdS at the following complex values

$$\sigma = \frac{d(d-2)}{4g_1}; \quad \phi^N = \pm\frac{d(d-2)\sqrt{2g_1 - g_2}}{4g_1^{3/2}} = \pm 12i\sqrt{\frac{N}{6\epsilon(4\pi)^3}}; \quad \phi^a = 0, \quad a = 1, .., N-1. \quad (4.97)$$

The one-point function of ϕ^N agrees with the large N result above (4.59), and its being complex indicates that the theory is non-unitary. We can expand around this classical solution in terms of fluctuations

$$S = \frac{36(g_{2,0} - 3g_{1,0})}{g_{1,0}^3}\int d^dx\sqrt{g} + \int d^dx\sqrt{g}\bigg(\frac{1}{2}(\partial_\mu\phi^a)^2 + \frac{1}{2}(\partial_\mu\delta\sigma)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \left(1 - \frac{g_{2,0}}{g_{1,0}}\right)\frac{d(d-2)}{8}(\delta\sigma)^2 + \frac{d(d-2)\sqrt{2g_{1,0} - g_{2,0}}}{4\sqrt{g_{1,0}}}\delta\sigma\chi + \frac{g_{1,0}}{2}\delta\sigma(\phi^a\phi^a + \chi^2) + \frac{g_{2,0}}{6}(\delta\sigma)^3\bigg). \quad (4.98)$$

where $\delta\sigma, \chi$ and ϕ^a are fluctuations of σ, ϕ^N and the $N-1$ transverse fields respectively. The transverse fields are massless Goldstone bosons and the leading boundary operator in their boundary operator expansion has dimensions 5. So the free energy is

$$F^E = (N-1)F_{\phi^a}(\hat{\Delta} = 5) \; + \; F_{\delta\sigma} \; + \; F_\chi \; + \frac{36(g_{2,0} - 3g_{1,0})\mathrm{Vol}(H^d)}{g_{1,0}^3}. \quad (4.99)$$

At large N, we only really need to take into account the contribution of $N-1$ massless fields, and using eq. (4.52) in $d = 6 - \epsilon$, we get

$$F_{\phi^a}(\hat{\Delta} = 5) = F\left(\frac{d}{2}\right) \; + \; \frac{9\mathrm{Vol}(H^d)}{32\pi^3\epsilon} + .. \quad (4.100)$$

up to terms that are finite as $\epsilon \to 0$. This $1/\epsilon$ pole is cancelled by the one coming from classical action when we plug in the bare coupling in terms of renormalized coupling, thus rendering the free energy finite in terms of renormalized coupling. We can then plug in the fixed point value of the coupling to get at large N

$$F^E = NF\left(\frac{d}{2}\right) \; + \frac{36((g_2)_* - 3(g_1)_*)\mathrm{Vol}(H^d)}{g_{1,0}^3} = NF\left(\frac{d}{2}\right) \; + \; \frac{9N\mathrm{Vol}(H^d)}{32\pi^3\epsilon}. \quad (4.101)$$

This again agrees with the result we found using a large N expansion (4.63).

$d = 2 + \epsilon$ dimensions

A similar analysis can be done for the non-linear sigma model in $2+\epsilon$ dimensions. Mapping the flat space action (4.6) to AdS, we have

$$S = \int d^d x \sqrt{g} \left(\frac{1}{2} \partial_\mu \phi^I \partial^\mu \phi^I - \frac{d(d-2)}{8} \phi^I \phi^I + \sigma \left(\phi^I \phi^I - \frac{1}{t^2} \right) \right). \tag{4.102}$$

For the free energy calculation in this section, we will restrict to the simpler case of Dirichlet boundary conditions, which we expect to be related to the extraordinary transition. We can solve the constraint explicitly in terms of the following unconstrained variables

$$\phi^a = \psi^a, \quad a = 1, ..., N-1; \quad \phi^N = \frac{1}{t} \sqrt{1 - t^2 \psi^a \psi^a}\,. \tag{4.103}$$

and the ψ^a are quantized with Dirichlet boundary conditions. Recall that in $d = 2 + \epsilon$, this model has a fixed point at $t^2 = (t_*)^2 = (2\pi\epsilon)/(N-2)$ [175]. Using this fixed point value, we find the one-point function of ϕ^N to leading order in ϵ

$$\phi^N = \frac{1}{t_*} = \sqrt{\frac{N-2}{2\pi\epsilon}}\,, \tag{4.104}$$

which is seen to match the large N expansion result for the extraordinary transition, given in (4.59). Using the parametrization (4.103) we can write down the action in terms of the unconstrained variables

$$S = \int d^d x \sqrt{g} \left(\frac{1}{2} \partial_\mu \psi^a \partial^\mu \psi^a - \frac{d(d-2)}{8 t_0^2} + \frac{t_0^2}{2} (\psi^a \partial_\mu \psi^a)^2 + \dots \right) \tag{4.105}$$

where we omitted additional terms of higher order in t. So we are left with $N-1$ massless fields ψ^a, and the extraordinary transition corresponds to choosing $\hat{\Delta}_+ = d-1$ boundary condition for these fields in AdS. We will check this further by computing the anomalous dimensions of these transverse fields in eq. (4.139) and comparing it to the large N result for the extraordinary transition. To leading order, the free energy will be just the classical action plus the fluctuations of $N-1$ massless scalars

$$F^E = (N-1) F(\hat{\Delta} = d-1) \; - \; \frac{\epsilon \mathrm{Vol}(H^d)}{4 t_0^2} \tag{4.106}$$

The bare coupling in this case is (see for instance [1])

$$t_0 = \mu^{-\frac{\epsilon}{2}} \left(t + \frac{t^3 (N-1)}{4\pi\epsilon} + ... \right) \tag{4.107}$$

and using eq. (4.52) in $d = 2 + \epsilon$ gives

$$F(\hat{\Delta} = d - 1) = F\left(\frac{d}{2}\right) \; - \; \frac{\text{Vol}(H^d)}{8\pi} + .. \tag{4.108}$$

up to terms that vanish as $\epsilon \to 0$. Combining these results and using the fixed point value $t_*^2 = \frac{2\pi\epsilon}{N-2}$, we get a precise match with the large N result in eq. (4.63)

$$F^E = (N-1)F\left(\frac{d}{2}\right) \; - \; \frac{(N-2)\text{Vol}(H^d)}{8\pi}. \tag{4.109}$$

4.3 Bulk correlators and extracting BCFT data

In this section, we study in more detail the BCFT data of the models discussed in this chapter. We start with the discussion of the bulk two-point functions of both ϕ and σ at large N. We then move on to the Wilson-Fisher model in $d = 4 - \epsilon$ dimensions, where we compute the bulk two-point function of the fundamental field ϕ, to second order in ϵ, for both Neumann and Dirichlet boundary conditions on the fundamental field. This can be used to calculate the anomalous dimensions and OPE coefficients of various boundary operators appearing in boundary operator expansion of ϕ. Instead of directly computing loop diagrams, we make essential use of the fact that ϕ satisfies an equation of motion in the bulk. We use similar ideas and the BCFT crossing equation to compute $1/N$ corrections to some of the BCFT data in the large N description. We find that at order $1/N$, for the ordinary transition, the boundary operator expansion of ϕ contains a tower of operators with dimensions $2d - 2 + 2k$. From the boundary point of view, these can be schematically written as $\hat{\phi}(\vec{\partial}^2)^k\hat{\sigma}$ with $\hat{\phi}$ and $\hat{\sigma}$ being the leading boundary operators induced by ϕ and σ. We find the boundary operator expansion coefficients for this tower, and use this result to "bootstrap" the $1/N$ correction to the scaling dimension of $\hat{\phi}$. For the special transition, we find that two such towers appear, with dimensions $d - 1 + 2p$ and $2d - 3 + 2q$. This is in accordance with the fact that, as we will see, for special transtion, σ induces two boundary operators with dimensions 2 and d.

We also give a formula for the anomalous dimensions of higher-spin displacement operators which are induced on the boundary by higher-spin currents in the bulk. We will find the anomalous dimensions by using the fact that the current conservation is weakly broken in the bulk, similarly to the analysis in [176, 177]. Lastly, we calculate the boundary four-point function in the Wilson-Fisher model and in the non-linear sigma model, which in AdS language is given by a relatively simple contact Witten diagram. We extract the anomalous dimensions of boundary operators from

these four-point functions using techniques familiar from AdS/CFT literature.

4.3.1 Bulk two-point functions

Let us start by analyzing the ordinary ($\hat{\Delta} = d - 2 + O(1/N)$) and special ($\hat{\Delta} = d - 3 + O(1/N)$) transition in a little more detail. Knowing the dimension of the boundary operator, we can immediately get the two point function of the bulk fundamental field ϕ to leading order in large N. For ordinary transition (O), it is

$$\langle \phi^I \phi^J \rangle = \delta^{IJ} G_\phi(\xi) = \delta^{IJ} G_{bb}^{\hat{\Delta}=d-2} = \frac{\Gamma\left(\frac{d}{2}-1\right)}{(4\pi)^{\frac{d}{2}}} \frac{1}{(\xi(1+\xi))^{\frac{d}{2}-1}} \tag{4.110}$$

while for special transition(S), we have

$$\langle \phi^I \phi^J \rangle = \delta^{IJ} G_\phi(\xi) = \delta^{IJ} G_{bb}^{\hat{\Delta}=d-3}(\xi) = \frac{\Gamma\left(\frac{d}{2}-1\right)}{(4\pi)^{\frac{d}{2}}} \frac{1+2\xi}{(\xi(1+\xi))^{\frac{d}{2}-1}} \tag{4.111}$$

In the boundary channel, clearly there is only a single operator in both cases (this is because, as mentioned above, the bulk-to-bulk propagator in AdS is proportional to a single boundary conformal block). In the bulk channel, there is a tower of operators with dimensions $2n+2$ with the following OPE coefficients

$$\begin{aligned}
(\lambda)_n^O &= -\frac{(-1)^n \Gamma\left(\frac{d}{2}\right)\ {}_2F_1\left(-n-1,-n;\frac{1}{2}(d-4n-2);1\right)}{\Gamma(n+2)\Gamma\left(\frac{d}{2}-n-1\right)} \\
(\lambda)_n^S &= \frac{(d^2-4d(n+2)+8(1+n)^2+4)\Gamma(1-\frac{d}{2}+n)\Gamma(2-\frac{d}{2}+n)^2}{4\Gamma(2-\frac{d}{2})^2\Gamma(n+2)\Gamma(2-\frac{d}{2}+2n)}.
\end{aligned} \tag{4.112}$$

We show how to derive these formulae in the appendix 4.5.

To do a large N perturbation theory, we still need the σ propagator. To calculate that, we decompose $\sigma = \sigma_* + \delta\sigma(x)$ into a constant background and fluctuations around it. We can plug this into eq. (4.39) and read off the quadratic piece of the effective action for σ

$$S_2 = -\frac{N}{4} \int d^dx d^dy \sqrt{g_x}\sqrt{g_y} \delta\sigma(x) B(x,y) \delta\sigma(y), \quad B(x,y) = \left[\frac{1}{-\nabla^2 + \sigma_* - \frac{d(d-2)}{4}}\right]^2. \tag{4.113}$$

This tells us that the connected propagator of σ must satisfy

$$\int d^dx \sqrt{g_x}\ \left(G_\phi(\xi_{x_1,x})\right)^2 G_\sigma^0(\xi_{x,x_2}) = -\frac{2}{N} \frac{\delta^d(x_1-x_2)}{\sqrt{g_{x_1}}}. \tag{4.114}$$

So we need to invert the square of the ϕ propagator in order to get the σ propagator. The general

method to invert functions on half-space was described in [12]. The procedure can be straightforwardly adapted to AdS and we show how to do that in Appendix 4.6. Here we just report the results. The full two point function of σ takes the form

$$\langle\sigma(x)\sigma(x')\rangle = (\sigma_*)^2 + G_\sigma(\xi). \tag{4.115}$$

For ordinary transition with $\sigma_* = \frac{(d-2)(d-4)}{4}$, we get

$$G_\sigma(\xi) = B\frac{\Gamma(d)\Gamma(d-2)}{\Gamma(2d-4)}\xi^{-d}\ \ {}_2F_1(d-2,d;2d-4;-\frac{1}{\xi}) \tag{4.116}$$

where

$$B = \frac{(4-d)\Gamma(d-2)}{N\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^3}. \tag{4.117}$$

In the limit when we push one of the points to the boundary, $y' \to 0$, the leading term is

$$G_\sigma(\xi) = B\frac{\Gamma(d)\Gamma(d-2)}{\Gamma(2d-4)}\left(\frac{4yy'}{(\vec{x}-\vec{x}')^2+y^2}\right)^d. \tag{4.118}$$

This tells us that the dimension of leading boundary operator induced by σ is $\hat{\Delta} = d$, and we identify it as the being proportional to the displacement operator (more comments on the displacement operators are collected in Appendix 4.7). For the special transition with $\sigma_* = \frac{(d-4)(d-6)}{4}$, we get the more complicated expression

$$\begin{aligned} G_\sigma(\xi) = B\Bigg[&\frac{1}{3}\frac{(6-d)\Gamma(d)\Gamma(d-2)}{(d-2)\Gamma(2d-5)}\frac{\xi+\frac{1}{2}}{(\xi(\xi+1))^{\frac{d+1}{2}}}\ \ {}_3F_2(\frac{d}{2}+\frac{1}{2},\frac{d}{2}-\frac{3}{2},\frac{3}{2};d-\frac{5}{2},\frac{5}{2};-\frac{1}{4\xi(1+\xi)}) \\ &+\frac{\pi\Gamma(\frac{d}{2}-1)^2}{\Gamma(d-3)\Gamma(\frac{d}{2}-\frac{3}{2})\Gamma(\frac{7}{2}-\frac{d}{2})}\frac{8}{(1+2\xi)^2}\ \ {}_2F_1(\frac{3}{2},1;\frac{7}{2}-\frac{d}{2};\frac{1}{(1+2\xi)^2})\Bigg]. \end{aligned} \tag{4.119}$$

Again, if we push one of the points to the boundary, $y' \to 0$, the leading terms are

$$\begin{aligned} G_\sigma(\xi) = B\Bigg[&\frac{1}{3}\frac{(6-d)\Gamma(d)\Gamma(d-2)}{(d-2)\Gamma(2d-5)}\left(\frac{4yy'}{(\vec{x}-\vec{x}')^2+y^2}\right)^d \\ &+\frac{8\pi\Gamma(\frac{d}{2}-1)^2}{\Gamma(d-3)\Gamma(\frac{d}{2}-\frac{3}{2})\Gamma(\frac{7}{2}-\frac{d}{2})}\left(\frac{4yy'}{(\vec{x}-\vec{x}')^2+y^2}\right)^2\Bigg]. \end{aligned} \tag{4.120}$$

So there are two operators of dimensions d and 2 induced by σ on the boundary. The dimension d operator is proportional to the displacement. The dimension 2 operator is the boundary mass operator which drives the transition from special to ordinary transition.

Let us now turn to the extraordinary transition, where the $O(N)$ symmetry is broken to $O(N-1)$ and both σ and ϕ^N acquire one-point functions. The correlator of the transverse $N-1$ fields to leading order at large N is easily obtained by simply plugging into the bulk-to-bulk propagator the boundary dimension $\hat{\Delta} = d-1$

$$\langle \phi^a(x_1)\phi^b(x_2)\rangle = \delta^{ab} G^T(\xi); \quad G^T(\xi) = \frac{\Gamma\left(\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(d-1)} \xi^{-d+1} {}_2F_1\left(d-1, \frac{d}{2}; d; -\frac{1}{\xi}\right) \tag{4.121}$$

To compute the correlators for the fields ϕ^N and σ, we can expand them around their background values, $\phi^N = \phi^N_* + \chi$ and $\sigma = \sigma_* + \delta\sigma$ and then the action up to the quadratic terms for the fluctuations becomes

$$\begin{aligned} S_2 &= \int d^dx\sqrt{g}\left(\frac{1}{2}(\partial\chi)^2 + \phi^N_*\chi\delta\sigma\right) - \frac{N-1}{4}\int d^dx d^dy \sqrt{g_x}\sqrt{g_y}\delta\sigma(x)B(x,y)\delta\sigma(y) \\ B(x,y) &= \left[\frac{1}{-\nabla^2 + \sigma_* - \frac{d(d-2)}{4}}\right]^2 = \frac{1}{(\nabla^2)^2} \end{aligned} \tag{4.122}$$

where we already used the large N values of σ_* and ϕ^N_*. To get the σ correlator G_σ, we integrate out χ exactly, which gives the effective quadratic term for σ, and tells us that the correlator must satisfy

$$\begin{aligned} \int d^dx\sqrt{g_x}\ \Pi(\xi_{x_1,x})G_\sigma(\xi_{xx_2}) &= \frac{\delta^d(x_1-x_2)}{\sqrt{g_{x_1}}} \\ \Pi(\xi_{x_1,x}) &= -\frac{N-1}{2}(G^T(\xi))^2 + (\phi^N_*)^2 G^T(\xi) \end{aligned} \tag{4.123}$$

Then, following [173], to invert $\Pi(\xi)$, it is convenient to first apply the differential operator corresponding to the massless equation of motion to it, which annihilates one of the terms and simplifies the other term in $\Pi(\xi)$

$$\begin{aligned} \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)\Pi(\xi_{x_1,x}) &= \left(\xi(\xi+1)\partial_\xi^2 + d(\xi+\frac{1}{2})\partial_\xi\right)\Pi(\xi_{x_1,x}) \\ &= -\frac{(N-1)\Gamma\left(\frac{d}{2}\right)^2}{(4\pi)^d\left(\xi(\xi+1)\right)^{d-1}} = \tilde{\Pi}(\xi_{x_1,x}). \end{aligned} \tag{4.124}$$

When we apply this to the equation of motion of σ in eq. (4.123), we find

$$\int d^dx\ y^{-d}\ \tilde{\Pi}(\xi_{x_1,x})G_\sigma(\xi_{xx_2}) = y_1^{d+2}\left(\vec{\nabla}^2_{x_1} + \frac{\partial^2}{\partial y_1^2} + \frac{d+2}{y_1}\frac{\partial}{\partial y_1} + \frac{d}{y_1^2}\right)\delta^d(x_1-x_2) \tag{4.125}$$

Now that we have a sufficiently simpler function to invert, we turn to the method reviewed in

Appendix 4.6, being careful about the modifications wherever necessary because of the differential operator present on the right hand side. Integrating the above equation over the boundary coordinates $\mathbf{x}_1$, we get

$$\begin{aligned}\int_0^\infty \frac{dy}{y}\ \tilde{\pi}(\rho_{y_1,y})g_\sigma(\rho_{yy_2}) &= \frac{y_1^{\frac{d+5}{2}}y_2^{\frac{1-d}{2}}}{4^{d-1}}\left(\frac{\partial^2}{\partial y_1^2}+\frac{d+2}{y_1}\frac{\partial}{\partial y_1}+\frac{d}{y_1^2}\right)\delta(y_1-y_2)\\ &= \frac{y_1^3}{4^{d-1}}\left(\frac{\partial^2}{\partial y_1^2}+\frac{3}{y_1}\frac{\partial}{\partial y_1}+\frac{(-d^2+2d+3)}{4y_1^2}\right)\delta(y_1-y_2)\end{aligned} \tag{4.126}$$

Then making a change of variables to $y = e^{2\theta}$ and doing a Fourier transform gives

$$\begin{aligned}\int_{-\infty}^\infty d\theta\ \tilde{\pi}(\sinh^2(\theta_1-\theta))g_\sigma(\sinh^2(\theta-\theta_2)) &= \frac{1}{4^{d+1}}\left(\frac{\partial^2}{\partial\theta_1^2}-(d-1)^2\right)\delta(\theta_1-\theta_2)\\ \implies \hat{\tilde{\pi}}(k)\hat{g}_\sigma(k) &= \frac{-k^2-(d-1)^2}{4^{d+1}}\end{aligned} \tag{4.127}$$

Then, following the Appendix 4.6, the transform $\hat{\tilde{\pi}}(k)$ corresponding to $\tilde{\Pi}(\xi)$ in eq. (4.124) can be worked out, and it gives

$$\begin{aligned}\hat{\tilde{\pi}}(k) &= -\frac{(N-1)\pi^{\frac{3-d}{2}}\csc\left(\frac{\pi d}{2}\right)}{4^d\,\Gamma\left(\frac{d-1}{2}\right)}\frac{\Gamma\left(\frac{3d-3-ik}{4}\right)\Gamma\left(\frac{3d-3+ik}{4}\right)}{\Gamma\left(\frac{d+1-ik}{4}\right)\Gamma\left(\frac{d+1+ik}{4}\right)}\\ \hat{g}_\sigma(k) &= \frac{4\pi^{\frac{d-3}{2}}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{(N-1)}\left[\frac{\Gamma\left(\frac{d+5-ik}{4}\right)\Gamma\left(\frac{d+5+ik}{4}\right)}{\Gamma\left(\frac{3d-3-ik}{4}\right)\Gamma\left(\frac{3d-3+ik}{4}\right)}-\frac{d}{4}\frac{\Gamma\left(\frac{d+1-ik}{4}\right)\Gamma\left(\frac{d+1+ik}{4}\right)}{\Gamma\left(\frac{3d-3-ik}{4}\right)\Gamma\left(\frac{3d-3+ik}{4}\right)}\right].\end{aligned} \tag{4.128}$$

This yields

$$\begin{aligned}G_\sigma(\xi) = \frac{2^{-d-1}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma(d+2)}{\pi(N-1)\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(d+\frac{1}{2}\right)}\Bigg[&\frac{1}{\xi^{d+2}}\ {}_2F_1\left(d+2,d;2d;-\frac{1}{\xi}\right)\\ &-\frac{4(2d-1)}{(d+1)(d-4)}\frac{1}{\xi^d}\ {}_2F_1\left(d,d-1;2d-2;-\frac{1}{\xi}\right)\Bigg].\end{aligned} \tag{4.129}$$

This is the result for σ correlator. We will not try to compute the correlator of χ here. This σ correlator immediately shows that the leading operator induced by σ on the boundary has dimensions d, which we identify to be proportional to the displacement operator. There are no relevant operators at the boundary, consistently with the fact that this phase has the lowest free energy in $2 < d < 4$.

Let us also report the results for extraordinary transition in $d = 4-\epsilon$ for comparison. Near $d = 4$, the dimension d displacement operator can be seen in the boundary operator expansion of the fluctuation χ. From our discussion below eq. (4.78), it is clear that the two-point functions of

the fields ϕ^a and χ are just the bulk-to-bulk propagators with dimensions 3 and 4 respectively

$$\begin{aligned}\langle\phi^a(x_1)\phi^b(x_2)\rangle &= \frac{\delta^{ab}}{16\pi^2}\left(\frac{1}{\xi}+\frac{1}{\xi+1}+2\log\frac{\xi}{1+\xi}\right)\\ \langle\chi(x_1)\chi(x_2)\rangle &= \frac{1}{16\pi^2}\left(\frac{1}{\xi}-\frac{1}{\xi+1}+12+6(1+2\xi)\log\frac{\xi}{1+\xi}\right).\end{aligned} \tag{4.130}$$

To identify the contribution of the boundary operators, we can take one of the points to the boundary which corresponds to taking $\xi\to\infty$

$$\langle\chi(x_1)\chi(x_2)\rangle = \frac{1}{160\pi^2}\frac{1}{\xi^4}+O\left(\frac{1}{\xi^5}\right);\quad \langle\phi^a(x_1)\phi^b(x_2)\rangle = \frac{\delta^{ab}}{48\pi^2}\frac{1}{\xi^3}+O\left(\frac{1}{\xi^4}\right). \tag{4.131}$$

In this setting, the order ϵ corrections to these two-point functions require computing loop diagrams in AdS. We do not do that here. But in a flat space setting, these corrections were recently reported in [178, 179].

4.3.2 Using bulk equations of motion

To warm-up, we will first apply the equation of motion on the bulk-boundary two-point function, and then on the more complicated case of bulk two-point functions.

Let us start with the case of the Wilson-Fisher fixed points in $d=4-\epsilon$, focusing on the $O(N)$ invariant phases corresponding to perturbing Neumann and Dirichlet boundary conditions. Consider the bulk-boundary two point function of the operators ϕ^I and $\hat{\phi}^I$, which is fixed by the conformal symmetry to take the following form in flat half-space

$$\langle\phi^I(x_1)\hat{\phi}^J(\mathbf{x}_2)\rangle = \frac{B_{\phi\hat{\phi}}\delta^{IJ}}{(2y_1)^{\Delta_\phi-\hat{\Delta}_\phi}(\mathbf{x}_{12}^2+y_1^2)^{\hat{\Delta}_\phi}}. \tag{4.132}$$

Applying the Laplacian operator on the bulk point x_1 gives

$$\begin{aligned}\nabla^2\langle\phi^I(x_1)\hat{\phi}^J(\mathbf{x}_2)\rangle = \frac{B_{\phi\hat{\phi}}\delta^{IJ}}{(2y_1)^{\Delta_\phi-\hat{\Delta}_\phi}(\mathbf{x}_{12}^2+y_1^2)^{\hat{\Delta}_\phi+1}}\Big(&\hat{\Delta}_\phi(3-2d+\hat{\Delta}_\phi)+\Delta_\phi(1+2\hat{\Delta}_\phi+\Delta_\phi)\\ &+(\hat{\Delta}_\phi-\Delta_\phi)(\hat{\Delta}_\phi-\Delta_\phi-1)\frac{\mathbf{x}_{12}^2}{y_1^2}\Big).\end{aligned} \tag{4.133}$$

In the free theory, we have $\nabla^2\phi^I(x_1)=0$ and setting the right hand side above to 0 gives two possibilities: 1) $\hat{\Delta}_\phi=\Delta_\phi=d/2-1$ corresponding to Neumann boundary condition and, 2) $\hat{\Delta}_\phi=\Delta_\phi=d/2$ corresponding to Dirichlet boundary condition. In the interacting theory with $\lambda(\phi^I\phi^I)^2/4$

interaction, $\nabla^2\phi^I(x_1) = \lambda\phi^I\phi^K\phi^K(x_1)$. Plugging this into the above equation gives

$$\begin{aligned}\frac{\lambda\langle\phi^I\phi^K\phi^K(x_1)\hat{\phi}^J(\mathbf{x}_2)\rangle}{\langle\phi^I(x_1)\hat{\phi}^J(\mathbf{x}_2)\rangle} &= \frac{\hat{\Delta}_\phi(3-2d+\hat{\Delta}_\phi)+\Delta_\phi(1+2\hat{\Delta}_\phi+\Delta_\phi)}{\mathbf{x}_{12}^2+y_1^2} \\ &+ \frac{(\hat{\Delta}_\phi-\Delta_\phi)(\hat{\Delta}_\phi-\Delta_\phi-1)\ \mathbf{x}_{12}^2}{y_1^2(\mathbf{x}_{12}^2+y_1^2)}\end{aligned} \tag{4.134}$$

This equation is exact at the fixed point where the coupling becomes $\lambda_* = \frac{8\pi^2}{N+8}\epsilon$. So we can compute the correlators on the left hand side in the free theory in four dimension, and the factor of λ in front will ensure that the BCFT data on the right is correct to order ϵ. Recall the free theory correlators

$$\langle\phi^I\phi^K\phi^K(x_1)\hat{\phi}^J(\mathbf{x}_2)\rangle^{N/D} = (N+2)\frac{A_{\phi^2}^{N/D}}{(2y_1)^{\Delta_{\phi^2}}}\langle\phi^I(x_1)\hat{\phi}^J(\mathbf{x}_2)\rangle, \quad \Delta_{\phi^2}=2, \quad A_{\phi^2}^{N/D} = \pm\frac{1}{4\pi^2}. \tag{4.135}$$

Using the fact that at order ϵ, the bulk dimension $\Delta_\phi = d/2-1$ does not get corrected (it is well-known that the anomalous dimension of ϕ at the Wilson-Fisher fixed point starts at order ϵ^2), the above equation gives us

$$\hat{\Delta}_\phi^N = \frac{d}{2}-1+\hat{\gamma}_\phi^N, \quad \hat{\Delta}_\phi^D = \frac{d}{2}+\hat{\gamma}_\phi^D, \quad \hat{\gamma}_\phi^N = \hat{\gamma}_\phi^D = -\frac{\epsilon}{2}\frac{N+2}{N+8}. \tag{4.136}$$

This agrees with the results in [12]. Notice that our approach using the equation of motion did not require us to do any loop calculations or regularization.

We can also do a similar analysis in the non-linear sigma model in $d = 2+\epsilon$. The equation of motion for $N-1$ unconstrained fields in that case is $\nabla^2\varphi^a(x_1) = -t^2\varphi^a\partial_\mu\varphi^b\partial^\mu\varphi^b(x_1)+O(g^4)$. This then gives us

$$\begin{aligned}-\frac{t^2\langle\varphi^a(\partial_\mu\varphi^b)^2(x_1)\hat{\varphi}^c(\mathbf{x}_2)\rangle}{\langle\varphi^a(x_1)\hat{\varphi}^c(\mathbf{x}_2)\rangle} &= \frac{\hat{\Delta}_\phi(3-2d+\hat{\Delta}_\phi)+\Delta_\phi(1+2\hat{\Delta}_\phi+\Delta_\phi)}{\mathbf{x}_{12}^2+y_1^2} \\ &+ \frac{(\hat{\Delta}_\phi-\Delta_\phi)(\hat{\Delta}_\phi-\Delta_\phi-1)\ \mathbf{x}_{12}^2}{y_1^2(\mathbf{x}_{12}^2+y_1^2)}.\end{aligned} \tag{4.137}$$

The relevant correlation function can again be computed in the free theory to be

$$\langle\varphi^a(\partial_\mu\varphi^b)^2(x_1)\hat{\varphi}^c(\mathbf{x}_2)\rangle = \pm\frac{\langle\varphi^a(x_1)\hat{\varphi}^c(\mathbf{x}_2)\rangle}{S_d(2y_1)^d}\left(2(N-1)(d-1)+4\Delta_\phi+4\hat{\Delta}_\phi\frac{y_1^2-\mathbf{x}_{12}^2}{y_1^2+\mathbf{x}_{12}^2}\right). \tag{4.138}$$

Using the bulk results to order ϵ, $\Delta_\phi = \frac{d}{2} - 1 + \frac{\epsilon}{2(N-2)}$ and $t_*^2 = \frac{2\pi\epsilon}{N-2}$, we get

$$\begin{aligned}\hat{\Delta}_\phi^N &= \frac{d}{2} - 1 + \frac{N}{2(N-2)}\epsilon + O(\epsilon^2) = \epsilon\frac{N-1}{N-2} + O(\epsilon^2), \\ \hat{\Delta}_\phi^D &= \frac{d}{2} + \frac{\epsilon}{2} + O(\epsilon^2) = 1 + \epsilon + O(\epsilon^2)\end{aligned} \tag{4.139}$$

The Neumann case agrees with what was found in [172] and is consistent with the large N result for the ordinary transition found in section 4.2.1, i.e. $\hat{\Delta} = d - 2 + O(1/N)$. The Dirichlet case is consistent with the large N result for extraordinary transition found in section 4.2.2, i.e. $\hat{\Delta} = d - 1$.

It is possible to extend this idea of applying the equation of motion to the bulk two-point function. This method can be applied to either the AdS or flat half-space approach, but it is slightly more convenient to work in AdS, since then the two-point function is simply a function of the single cross-ratio ξ

$$\langle \phi^I(x_1)\phi^J(x_2)\rangle = \delta^{IJ} G_\phi(\xi). \tag{4.140}$$

Hence, the free equation of motion $(\nabla^2 - m^2)\phi^I = 0$ when applied to the two point function just gives

$$\left(\nabla^2 + \frac{d(d-2)}{4}\right) G_\phi(\xi) = \left(\xi(\xi+1)\partial_\xi^2 + d(\xi + \frac{1}{2})\partial_\xi + \frac{d(d-2)}{4}\right) G_\phi(\xi) = D^{(2)} G_\phi(\xi) = 0 \tag{4.141}$$

This differential equation has two solutions, which of course correspond to the Neumann or Dirichlet boundary conditions in the free theory [14]

$$G_\phi(\xi) = b_1 \left(\frac{1}{(\xi)^{\frac{d}{2}-1}} + \frac{1}{(\xi+1)^{\frac{d}{2}-1}}\right) + b_2 \left(\frac{1}{(\xi)^{\frac{d}{2}-1}} - \frac{1}{(\xi+1)^{\frac{d}{2}-1}}\right). \tag{4.142}$$

One of the constants above can be fixed by the normalization of the field ϕ and the other one can be fixed by demanding that the boundary spectrum contains a single operator of dimension $d/2 - 1$ or $d/2$ for Neumann or Dirichlet boundary conditions respectively. The canonical normalization corresponds to $b_1 = \Gamma\left(\frac{d}{2}\right) / \left(2^{d-1}(d-2)\pi^{\frac{d}{2}}\right), b_2 = 0$ for Neumann boundary condition and vice versa for Dirichlet boundary condition. However, in the rest of this subsection, we will find it convenient to work in a normalization in which $b_1 = 1$ for Neumann and $b_2 = 1$ for Dirichlet boundary conditions.

In the interacting theory in $d = 4 - \epsilon$ dimensions, the equation of motion gets modified to

[14] To be precise, the differential equation for the two-point function must have a delta function on the right, which is indeed reproduced by the solution we mention.

$(\nabla^2 - m^2)\phi^I(x) = \lambda\phi^I\phi^K\phi^K(x)$, which tells us that the two-point function must satisfy

$$D^{(2)}G_\phi^{N/D}(\xi) = \lambda_*(N+2)A_{\phi^2}^{N/D}G_\phi^{N/D}(\xi) + O(\lambda_*^2) \tag{4.143}$$

and in this normalization, $\lambda_* = \epsilon/(2(N+8))$, and $A_{\phi^2}^{N/D} = \pm 1$. We can also apply the equation of motion operator on the other ϕ in the two-point function, which gives the following fourth order differential equation to $O(\lambda^2)$

$$\begin{aligned}&\Big[\xi(1+\xi)\left(\xi(1+\xi)\partial_\xi^4 + (d+2)(1+2\xi)\partial_\xi^3\right) + \frac{(d(d+2)+(8+6d(d+2))\xi(1+\xi))}{4}\partial_\xi^2 \\ &+ \frac{d^3(1+2\xi)}{4}\partial_\xi + \frac{d^2(d-2)^2}{16}\Big]G_\phi(\xi) = D^{(4)}G_\phi(\xi) = \lambda_*^2(N+2)\left[(A_{\phi^2}^{N/D})^2(N+2)G_\phi + 2G_\phi^3\right].\end{aligned} \tag{4.144}$$

These differential equations can be used to extract some of the BCFT data. To see how to do that, let us recall that the two point function can be expanded into boundary conformal blocks as

$$G_\phi(\xi) = \sum_l \mu_l^2 f_{\text{bdry}}(\hat{\Delta}_l;\xi). \tag{4.145}$$

It is easy to see that applying the quadratic and quartic differential operator on the blocks returns the block itself with a coefficient

$$\begin{aligned}D^{(2)}f_{\text{bdry}}(\hat{\Delta}_l;\xi) &= \frac{(d-2\hat{\Delta}_l)(d-2-2\hat{\Delta}_l)}{4}f_{\text{bdry}}(\hat{\Delta}_l;\xi), \\ D^{(4)}f_{\text{bdry}}(\hat{\Delta}_l;\xi) &= \frac{(d-2\hat{\Delta}_l)^2(d-2-2\hat{\Delta}_l)^2}{16}f_{\text{bdry}}(\hat{\Delta}_l;\xi).\end{aligned} \tag{4.146}$$

Plugging this decomposition in to eq. (4.143) gives us again the anomalous dimension of the leading boundary operator $\hat{\gamma}_\phi^{N/D}$, eq. (4.136), which we already found above. At next order, plugging this decomposition in to eq. (4.144) gives us a relation for the boundary operator expansion coefficients for blocks other than the leading block. In Neumann case, it gives

$$\sum_l \frac{(d-2\hat{\Delta}_l)^2(d-2-2\hat{\Delta}_l)^2}{16}(\mu_l^N)^2 f_{\text{bdry}}(\hat{\Delta}_l;\xi) = \frac{(N+2)\epsilon^2}{2(N+8)^2}\frac{(1+2\xi)^3}{\xi^3(1+\xi)^3} \tag{4.147}$$

where the sum does not include the leading operator of dimension $d/2-1 = 1+O(\epsilon)$. This equation tells us that the first subleading operator has dimension 3 and appears with a boundary operator expansion coefficient $(\mu_3^N)^2 = \frac{N+2}{(N+8)^2}\epsilon^2$. In the Dirichlet case, we get the following equation instead

$$\sum_l \frac{(d-2\hat{\Delta}_l)^2(d-2-2\hat{\Delta}_l)^2}{16}(\mu_l^D)^2 f_{\text{bdry}}(\hat{\Delta}_l;\xi) = \frac{(N+2)\epsilon^2}{2(N+8)^2}\frac{1}{\xi^3(1+\xi)^3}. \tag{4.148}$$

Again, the sum does not run over the leading operator of dimension $d/2 = 2 + O(\epsilon)$. This tells us that the first subleading operator has dimension 6 in this case and appears with a coefficient $(\mu_6^D)^2 = \frac{N+2}{800(N+8)^2}\epsilon^2$. We can go on and recursively find all other boundary operator expansion coefficients to order ϵ^2. These results agree with what was found in [18].

In fact, in this case, we can do better and fix the full two-point function by solving the differential equations explicitly. Let us start with the second order equation in eq. (4.143). We work perturbatively in ϵ and write the two point function and the differential operator as

$$\begin{aligned} G_\phi^{N/D}(\xi) &= G_0^{N/D}(\xi) + \epsilon G_1^{N/D}(\xi) + \epsilon^2 G_2^{N/D}(\xi) + O(\epsilon^3) \\ D^{(2)} &= D_0^{(2)} + \epsilon D_1^{(2)} + O(\epsilon^2). \end{aligned} \tag{4.149}$$

$G_0^{N/D}(\xi)$ is just the solution of the homogeneous differential equation and is given by the free theory two-point function found above in eq. (4.142). At next order, $G_1^{N/D}(\xi)$ must satisfy the differential equation (4.143) to order ϵ

$$D_0^{(2)} G_1^{N/D}(\xi) = \pm\frac{(N+2)}{2(N+8)} G_0^{N/D}(\xi) - D_1^{(2)} G_0^{N/D}(\xi). \tag{4.150}$$

Plugging in the free solution, this can be solved to give

$$G_1^{N/D}(\xi) = \frac{c_1}{\xi} + \frac{c_2}{1+\xi} + \frac{\log\xi}{2\xi} \pm \frac{\log(1+\xi)}{2(1+\xi)} + \frac{N+2}{2(N+8)}\left(\frac{\log(1+\xi)}{\xi} \pm \frac{\log\xi}{1+\xi}\right). \tag{4.151}$$

We work in the normalization such that in the bulk OPE limit, $\xi \to 0$, the leading term in the two-point function goes like $1/\xi^{\Delta_\phi}$ which fixes $c_1 = 0$. As we just saw, to order ϵ, we still just have a single boundary block of dimension $d/2-1+\hat{\gamma}_\phi^N$ or $d/2+\hat{\gamma}_\phi^D$. Consistency with this requires setting $c_2 = 0$ for both Neumann and Dirichlet cases. This solution agrees with what was found using a one-loop calculation in [11, 12]. For the fourth order equation in eq. (4.144), we can again expand $D^{(4)} = D_0^{(4)} + \epsilon D_1^{(4)} + \epsilon^2 D_2^{(4)}$ and get a differential equation for the second order correction to the two-point function

$$D_0^{(4)} G_2^{N/D} = \frac{(N+2)}{4(N+8)^2}\left[(N+2)G_0^{N/D}(\xi) + 2(G_0^{N/D})^3\right] - D_1^{(4)} G_1^{N/D} - D_2^{(4)} G_0^{N/D}. \tag{4.152}$$

This can again be solved to give

$$\begin{aligned} G_2^{N/D}(\xi) = &\frac{d_1}{\xi} + \frac{d_2}{1+\xi} + d_3\frac{\log\xi}{1+\xi} + d_4\frac{\log(1+\xi)}{\xi} \\ &- \frac{N+2}{4(N+8)^2}\left(\frac{\log\xi}{\xi} \pm \frac{\log(1+\xi)}{1+\xi}\right) + \frac{\log^2(\xi)}{8\xi} \pm \frac{\log^2(1+\xi)}{8(1+\xi)} \\ &+ \frac{(N+2)^2}{8(N+8)^2}\left(\frac{\log^2(1+\xi)}{\xi} \pm \frac{\log^2(\xi)}{1+\xi}\right) + \frac{N+2}{4(N+8)}\left(\frac{1}{\xi} \pm \frac{1}{1+\xi}\right)\log(\xi)\log(1+\xi). \end{aligned} \tag{4.153}$$

Fixing the normalization yields $d_1 = 0$, and consistency with the expansion in the boundary channel spectrum that we just found above fixes $d_2 = 0$ and $d_4^{N/D} = \pm d_3^{N/D}$. The last constant left can be fixed by the bulk OPE behaviour. We know that as $\xi \to 0$, the correlator must go like

$$\begin{aligned} &G_\phi(\xi) = \xi^{-\Delta_\phi} + \lambda_{\phi^2}\xi^{\frac{1}{2}(\Delta_{\phi^2}-2\Delta_\phi)} \ + \text{ higher orders in } \xi \\ &= \xi^{-\Delta_\phi} + \lambda_{\phi^2}^{(0)} + \epsilon\left[\lambda_{\phi^2}^{(1)} + \left(\frac{\gamma_{\phi^2}^{(1)}}{2} - \gamma_\phi^{(1)}\right)\lambda_{\phi^2}^{(0)}\log\xi\right] + \\ &\epsilon^2\left[\lambda_{\phi^2}^{(2)} + \left[\lambda_{\phi^2}^{(0)}\left(\frac{\gamma_{\phi^2}^{(2)}}{2} - \gamma_\phi^{(2)}\right) + \lambda_{\phi^2}^{(1)}\left(\frac{\gamma_{\phi^2}^{(1)}}{2} - \gamma_\phi^{(1)}\right)\right]\log\xi + \lambda_{\phi^2}^{(0)}\left(\frac{\gamma_{\phi^2}^{(1)}}{2} - \gamma_\phi^{(1)}\right)^2\frac{\log^2(\xi)}{2}\right] \end{aligned} \tag{4.154}$$

where $\gamma_{\phi^2}^{(2)}$ is the second order correction to the anomalous dimension of bulk operator ϕ^2 and similarly for all the other notation. Comparing to the correlator we found above at order ϵ tells us $\lambda_{\phi^2}^{(1)} = (N+2)/(2(N+8))$. Then comparing the $\log\xi$ terms at order ϵ^2 and small ξ gives us the following relation

$$\lambda_{\phi^2}^{(0)}\left(\frac{\gamma_{\phi^2}^{(2)}}{2} - \gamma_\phi^{(2)}\right) + \lambda_{\phi^2}^{(1)}\left(\frac{\gamma_{\phi^2}^{(1)}}{2} - \gamma_\phi^{(1)}\right) = d_3 + \frac{N+2}{4(N+8)}. \tag{4.155}$$

Using the following bulk data (see for instance [174])

$$\gamma_{\phi^2}^{(2)} = \frac{(N+2)(13N+44)}{2(N+8)^3}, \quad \gamma_\phi^{(2)} = \frac{N+2}{4(N+8)^2}, \quad \gamma_{\phi^2}^{(1)} = \frac{(N+2)}{(N+8)}, \quad \gamma_\phi^{(1)} = 0, \quad \lambda_{\phi^2}^{(0)} = \pm 1 \tag{4.156}$$

fixes the coefficient

$$d_3^N = \frac{3(N+2)(N-2)}{2(N+8)^3}, \quad d_3^D = -\frac{3(N+2)(3N+14)}{2(N+8)^3}. \tag{4.157}$$

This determines the two-point function completely to order ϵ^2 and agrees exactly with what was found in [18] using analytic bootstrap methods.

To do a similar analysis in the large N theory, we first decompose the field σ into its constant one-

point function and fluctuation, $\sigma = \sigma_* + \delta\sigma$. The fluctuation $\delta\sigma$ then only has connected correlators and the equation of motion is $(\nabla^2 - m^2 - \sigma_*)\phi^I(x) = \delta\sigma\phi^I(x)$ which to leading order in large N gives

$$(D^{(2)} - \sigma_*)G_\phi^{N/D}(\xi) = 0. \tag{4.158}$$

This is the equation of motion of a massive particle and its solution must be the usual bulk-to-bulk propagator, as we have already seen in the previous section. As we expect, expanding this into boundary conformal blocks just tells us that the only allowed blocks at leading order in large N are the ones whose dimensions satisfy the AdS/CFT relation which we used several times above

$$\frac{(d-2\hat{\Delta})(d-2-2\hat{\Delta})}{4} = \sigma_*, \implies \hat{\Delta} = \frac{d-1}{2} \pm \sqrt{\sigma_* + \frac{1}{4}}. \tag{4.159}$$

If we apply the equation of motion operator on the other ϕ, we obtain the following quartic differential equation to order $1/N$

$$\left(D^{(4)} - 2\sigma_* D^{(2)} + (\sigma_*)^2\right) G_\phi(\xi) = G_\phi(\xi)G_\sigma(\xi) \tag{4.160}$$

where $G_\sigma(\xi)$ is the correlator of $\delta\sigma$ found in subsection 4.2.1 and it appears at order $1/N$ in large N perturbation theory. We can then plug in the boundary channel conformal block decomposition for the correlator (eq. (4.145)) into this differential equation. It is easy to see that the leading boundary operator, the one that satisfies eq. (4.159), only starts contributing at order $1/N^2$ on the left hand side. So at order $1/N$, we only have subleading operators contributing, which yield

$$\sum_l \left(\frac{(d-2\hat{\Delta}_l)(d-2-2\hat{\Delta}_l)}{4} - \sigma_*\right)^2 (\mu_l)^2 f_{\text{bdry}}(\hat{\Delta}_l;\xi) = G_\phi(\xi)G_\sigma(\xi) \tag{4.161}$$

where the sum runs over all operators other than the leading ones of dimensions $d-2$ or $d-3$ for ordinary or special transition respectively. For ordinary transition, plugging in the correlators on the right hand side and comparing powers of $1/\xi$ tells us that the operators appearing must have dimensions $2d-2+2k$ with coefficients

$$(\mu_k^O)^2 = \frac{2^{-d-4k+2}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{3(d-1)}{2}+k\right)\Gamma\left(\frac{d}{2}+k\right)\Gamma(d+2k)}{N\pi d(d+2k)(2d+2k-3)\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}\right)\Gamma(k+1)\Gamma\left(d+k-\frac{1}{2}\right)\Gamma\left(\frac{3(d-1)}{2}+2k\right)}. \tag{4.162}$$

Near four dimensions, this agrees with the large N limit of what was found using ϵ expansion in [18].

Given these OPE coefficients, we can write down the bulk scalar two point function to order $1/N$ as

$$G_\phi(\xi) = (\mu^O_{d-2})^2 f_{\rm bdry}(d-2+\gamma,\xi) + \sum_{k=0}^{\infty} \mu^O_k f_{\rm bdry}(2d-2+2k,\xi) \tag{4.163}$$

and in our convention $(\mu^O_{d-2})^2 = 1 + O(1/N)$. By crossing symmetry, this expansion must reproduce the bulk OPE expansion in the limit $\xi \to 0$. Now recall from subsection 4.2.1 that the boundary conformal block has the following expansion in the bulk OPE limit, $\xi \to 0$

$$\begin{aligned} f_{\rm bdry}(\hat{\Delta},\xi) = \frac{1}{\xi^{\frac{d}{2}-1}} &\left(\frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma(-d+2\hat{\Delta}+2)}{\Gamma(\hat{\Delta})\Gamma\left(-\frac{d}{2}+\hat{\Delta}+1\right)} + O(\xi) \right) \\ &+ \left(\frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma(-d+2\hat{\Delta}+2)}{\Gamma(-d+\hat{\Delta}+2)\Gamma\left(-\frac{d}{2}+\hat{\Delta}+1\right)} + O(\xi) \right). \end{aligned} \tag{4.164}$$

The constant term in the second line above corresponds to the presence of ϕ^2 in the bulk OPE, which is supposed to be absent at the large N fixed point. This term vanishes exactly when $\hat{\Delta} = d-2$. At order $1/N$, we can allow for an anomalous dimension $\hat{\Delta} = d-2+\hat{\gamma}^O/N$, and its contribution to this constant term must be cancelled by the subleading operators of dimensions $2d-2+2k$. Plugging in the dimensions, for consistency of bulk and boundary expansions, we must demand

$$\frac{\Gamma(1-\frac{d}{2})\Gamma(d-2)}{\Gamma(\frac{d}{2}-1)}\,\hat{\gamma}^O + \sum_{k=0}^{\infty} (\mu^O_k)^2 \frac{\Gamma(1-\frac{d}{2})\Gamma(3d-2+4k)}{\Gamma(d+2k)\Gamma\left(\frac{3d}{2}-1+2k\right)} = 0 \tag{4.165}$$

which finally gives the result for the $1/N$ correction to the scaling dimension of the leading boundary operator $\hat{\phi}$ as

$$\begin{aligned} \hat{\gamma}^O &= -\frac{\Gamma(\frac{d}{2}-1)}{\Gamma(d-2)} \sum_{k=0}^{\infty} (\mu^O_k)^2 \frac{\Gamma(3d-2+4k)}{\Gamma(d+2k)\Gamma\left(\frac{3d}{2}-1+2k\right)} \\ &= \frac{2^{d-5}(d-4)\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{3d}{2}-\frac{1}{2}\right)}{\pi^2\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(d+\frac{1}{2}\right)} \left[\frac{\pi^{\frac{3}{2}} 2^d \Gamma\left(d-\frac{3}{2}\right)\, {}_3F_2\left(1,1-\frac{d}{2},\frac{d}{2}+1;2-\frac{d}{2},d+\frac{1}{2};1\right)}{\Gamma(d)\Gamma(d-2)\sin(\pi d)} \right. \\ &\left. + \frac{4(d-2)}{\Gamma(d)} \left(\frac{\pi^2}{\sin(\pi d)}\, {}_3F_2\left(1-\frac{d}{2},\frac{d-1}{2},\frac{d+2}{2};2-\frac{d}{2},d+\frac{1}{2};1\right) + \frac{4\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)}{d^2(2d-3)} \right) \right] \end{aligned} \tag{4.166}$$

where we performed the sum after plugging in the explicit OPE coefficients of the subleading operators we found above [15]. The hypergeometric functions appearing in this result are well defined for $d \geq 1$. While we have not been able to find relevant hypergeomtric identities to simplify this

[15] For reader interested in reproducing this result, note that to perform the summation, we had to separate out the $k=0$ piece and then add it back at the end. Also, the result that Mathematica gives has to be analytically continued using the formula given in eq. 2.12 of [180] in order to obtain an expression that is well defined for positive d.

formula analytically, we have verified numerically that for all $d \geq 1$ the result precisely agrees with the simpler formula given in [153], which reads

$$\hat{\gamma}^O = \frac{(4-d)\Gamma(2d-3)}{d\Gamma(d-2)\Gamma(d-1)}\,. \tag{4.167}$$

In $d=3$, this gives $\hat{\Delta} = 1+2/(3N)+O(1/N^2)$. We can also verify that in $d=4-\epsilon$ and $d=2+\epsilon$ the large N prediction agrees with (4.136) and (4.139) for Dirichlet and Neumann conditions respectively. Indeed using (4.167) we find

$$\begin{aligned} \hat{\Delta} &= 2-\epsilon+\frac{3\epsilon}{N}+\dots\,, \qquad d=4-\epsilon \\ \hat{\Delta} &= \epsilon+\frac{\epsilon}{N}+\dots\,, \qquad d=2+\epsilon\,. \end{aligned} \tag{4.168}$$

For the case of the special transition, eq. (4.161) tells us that there are two towers of operators that appear at the subleading order: the ones with dimension $d-1+2p$ and the ones with dimension $2d-3+2q$. The coefficients for these can be found recursively using eq. (4.161). Carrying out the calculation explicitly is more involved because of the two towers involved, so we leave it for future work. But we expect a similar reasoning as explained above for the ordinary transition to also give anomalous dimension of leading boundary operator at order $1/N$ for the special transition case. The result for this case was also reported previously in [153] using different methods. The explicit result reads

$$\hat{\gamma}^S = \frac{2(4-d)}{\Gamma(d-3)}\left(\frac{(6-d)\Gamma(2d-6)}{d\Gamma(d-3)}+\frac{1}{\Gamma(5-d)}\right) \tag{4.169}$$

Note that in $d=3$, the anomalous dimension vanishes, consistently with the expectation that this should be lower critical dimension for the special transition (presumably $\hat{\Delta}=0$ to all orders in the $1/N$ expansion in $d=3$). In $d=4-\epsilon$, (4.169) gives

$$\hat{\Delta} = 1-\epsilon+\frac{3\epsilon}{N}+\dots \tag{4.170}$$

in agreement with the ϵ-expansion result in (4.136).

4.3.3 Using weakly broken higher spin symmetry

We can generalize the equation of motion idea, in a manner similar to [176, 177], to find the anomalous dimensions of the higher spin displacement operators, which are the operators that appear in the boundary operator expansion of the higher spin currents. The bulk higher spin currents are

conserved in the free theory, but are weakly broken in the interacting theory, and hence the corresponding "higher-spin displacement" operators acquire anomalous dimensions (except of course the spin-2 case, which corresponds to the bulk stress-tensor and displacement operator at the boundary). As usual, it is convenient to package the currents in the index-free notation (see for instance, [51] for a review)

$$\mathcal{J}_s(x,z) = J_{\mu_1\ldots\mu_s}(x) z^{\mu_1}\ldots z^{\mu_s}, \quad z^2 = 0. \tag{4.171}$$

We can free the indices by acting with the Todorov differential operator

$$D_z^\mu = \left(\frac{d}{2} - 1 + z^\nu \frac{\partial}{\partial z^\nu}\right)\frac{\partial}{\partial z_\mu} - \frac{1}{2} z^\mu \frac{\partial}{\partial z^\nu}\frac{\partial}{\partial z_\nu}. \tag{4.172}$$

Similar tensors can be constructed for the boundary operators and we use the notation $\hat{\mathcal{J}}_l^s$ for the spin l operator appearing in the BOE of a spin s operator. The two-point function of a spin s operator in the bulk and a spin l operator on the boundary is fixed by the conformal symmetry. We will focus on the correlator $\langle \mathcal{J}_s(x_1,z_1)\hat{\mathcal{J}}_l^s(\mathbf{x}_2,\mathbf{z}_2)\rangle$. The tensor structures appearing in this correlator were found in [21] for the case of general defect, and only two of those structures survive in the case of co-dimension one

$$Q^0_{b\partial} = \mathbf{z}_1\cdot\mathbf{z}_2 - 2\frac{x_{12}\cdot z_1 (\mathbf{x}_{12}\cdot\mathbf{z}_2)}{x_{12}^2}, \quad Q^2_{b\partial} = z_1^y - \frac{2y_1 x_{12}\cdot z_1}{x_{12}^2}. \tag{4.173}$$

In terms of these structures, the bulk-boundary two point function is

$$\langle \mathcal{J}_s(x_1,z_1)\hat{\mathcal{J}}_l^s(\mathbf{x}_2,\mathbf{z}_2)\rangle = b_{s,l}\frac{(Q^0_{b\partial})^l (Q^2_{b\partial})^{s-l}}{(x_{12}^2)^{\hat{\Delta}_l^s}(y_1)^{\Delta_s-\hat{\Delta}_l^s}}. \tag{4.174}$$

Now consider a CFT with an interaction strength g[16] and suppose we are in the regime where g is small. The dimension of the spin s current then is

$$\Delta_s = d - 2 + s + \gamma_s(g). \tag{4.175}$$

The current conservation is weakly broken and its divergence defines a spin $s-1$ descendant

$$\partial_\mu D_z^\mu \mathcal{J}_s(x,z) = g\ \mathcal{K}_{s-1}(x,z). \tag{4.176}$$

[16]Not to be confused with the determinant of the metric. We hope there is no cause for confusion, because this subsection is entirely in flat space and the metric does not appear.

Applying this to the correlator gives

$$\partial_{1\mu} D^{\mu}_{z_1} \langle \mathcal{J}_s(x_1, z_1) \hat{\mathcal{J}}^s_l(\mathbf{x}_2, \mathbf{z}_2) \rangle = g \, \langle \mathcal{K}_{s-1}(x_1, z_1) \hat{\mathcal{J}}^s_l(\mathbf{x}_2, \mathbf{z}_2) \rangle. \tag{4.177}$$

We can also apply this differential operator on the right hand side of eq. (4.174). As a first step, we need

$$\begin{aligned} D^{\mu}_{z_1}((Q^0_{b\partial})^l (Q^2_{b\partial})^{s-l}) &= l \left(\frac{d}{2} + s - 2\right) (Q^0_{b\partial})^{l-1} (Q^2_{b\partial})^{s-l} \left(z_2^i \delta_i^{\mu} - \frac{2 x_{12}^{\mu} (\mathbf{x}_{12} \cdot \mathbf{z}_2)}{x_{12}^2} \right) \\ &+ (s-l)(Q^0_{b\partial})^l (Q^2_{b\partial})^{s-l-2} \left[\left(\frac{d}{2} + s - 2\right) (Q^2_{b\partial}) \left(\delta_y^{\mu} - \frac{2 y_1 x_{12}^{\mu}}{x_{12}^2} \right) - \frac{(s-l-1)}{2} z_1^{\mu} \right] \end{aligned} \tag{4.178}$$

Using this and after a bit of algebra, we get the following result for the ratio of descendant correlator to the current correlator

$$\begin{aligned} \frac{g \, \langle \mathcal{K}_{s-1}(x_1, z_1) \hat{\mathcal{J}}^s_l(\mathbf{x}_2, \mathbf{z}_2) \rangle}{\langle \mathcal{J}_s(x_1, z_1) \hat{\mathcal{J}}^s_l(\mathbf{x}_2, \mathbf{z}_2) \rangle} &= \frac{(\hat{\Delta}^s_l - \Delta_s)(s-l)(d+s+l-3)}{2 y_1 Q^2_{b\partial}} \\ &+ \frac{\gamma_s(g)}{x_{12}^2} \left[\frac{(s-l)(s-l-1) x_{12} \cdot z_1}{(Q^2_{b\partial})^2} + (d + 2s - 4) \left(\frac{y_1 (s-l)}{Q^2_{b\partial}} + \frac{l (\mathbf{x}_{12} \cdot \mathbf{z}_2)}{Q^0_{b\partial}} \right) \right]. \end{aligned} \tag{4.179}$$

The left hand side of the above equation vanishes if there are no interactions in the bulk ($g = 0$). The bulk anomalous dimension vanishes in that case, $\gamma_s(g) = 0$. The equation above then tells us that in the case of no bulk interactions the dimension of the higher spin displacement operator is fixed to be equal to the dimension of current, $\hat{\Delta}^s_l = \Delta_s = d - 2 + s$. It was shown to be true for stress-tensor in [12], but here we get it for higher spin currents as well. This proves the observation made in [1] about the higher spin displacements being protected in the presence of interactions localized on the boundary. In the presence of bulk interactions, we parametrize $\hat{\Delta}^s_l = d - 2 + s + \hat{\gamma}^s_l(g)$ and the anomalous dimensions can be obtained from

$$\begin{aligned} \frac{g \, \langle \mathcal{K}_{s-1}(x_1, z_1) \hat{\mathcal{J}}^s_l(\mathbf{x}_2, \mathbf{z}_2) \rangle}{\langle \mathcal{J}_s(x_1, z_1) \hat{\mathcal{J}}^s_l(\mathbf{x}_2, \mathbf{z}_2) \rangle} &= \frac{(\hat{\gamma}^s_l(g) - \gamma_s(g))(s-l)(d+s+l-3)}{2 y_1 Q^2_{b\partial}} \\ &+ \frac{\gamma_s(g)}{x_{12}^2} \left[\frac{(s-l)(s-l-1) x_{12} \cdot z_1}{(Q^2_{b\partial})^2} + (d + 2s - 4) \left(\frac{y_1 (s-l)}{Q^2_{b\partial}} + \frac{l (\mathbf{x}_{12} \cdot \mathbf{z}_2)}{Q^0_{b\partial}} \right) \right]. \end{aligned} \tag{4.180}$$

We can compute the correlators on the left hand side in the free theory, and this will give us the anomalous dimensions to leading order in g. We will demonstrate it explicitly in Wilson-Fisher model in $4 - \epsilon$ dimensions. We know that in this model, the anomalous dimension of the bulk current, $\gamma_s(g)$ starts at order g^2 (see e.g. [177] and references therein), so we can drop the second

line at leading order. This gives us the anomalous dimensions of the boundary operators

$$\hat{\gamma}_l^s = \frac{2g}{(s-l)(d+s+l-3)b_{s,l}} \frac{\langle \mathcal{K}_{s-1}(x_1,z_1)\hat{\mathcal{J}}_l^s(\mathbf{x}_2,\mathbf{z}_2)\rangle (x_{12}^2)^{\hat{\Delta}_l^s} y_1}{(Q_{b\partial}^0)^l (Q_{b\partial}^2)^{s-l-1}}. \tag{4.181}$$

Let us look at some explicit examples now. Consider the spin 2 current in the $O(N)$ model near 4 dimensions, which has a boundary scalar in its BOE. We will find that there is a non-zero anomalous dimension only for the symmetric traceless sector of $O(N)$, which is what we expect because the trace piece is just the stress-tensor. We have the following operators for Dirichlet[17] and Neumann case

$$\begin{aligned}
\mathcal{J}_2^{IJ}(x_1,z_1) &= \frac{\sqrt{\pi}\Gamma(d-1) z_1^{\mu_1} z_1^{\mu_2}}{2^{d-3}\Gamma(\frac{d-3}{2})\Gamma(\frac{d}{2}-1)} \left[\phi^I \partial_{\mu_1}\partial_{\mu_2}\phi^J + I \leftrightarrow J - \frac{2d}{d-2}\partial_{\mu_1}\phi^I \partial_{\mu_2}\phi^J \right] \\
\mathcal{K}_{s-1}^{IJ}(x_1,z_1) &= z_1^{\mu_1} \left(-6\phi^a\phi^a \partial_{\mu_1}(\phi^I\phi^J) + 6\phi^I\phi^J\partial_{\mu_1}(\phi^a\phi^a) \right) \\
(\hat{\mathcal{J}}_0^{2KL}(\mathbf{x}_2))_D &= \hat{\phi}^K\hat{\phi}^L, \quad (\hat{\mathcal{J}}_0^{2KL}(\mathbf{x}_2))_N = \frac{d-2}{2}\left(\hat{\phi}^K \partial_{\mathbf{i}}{}^2 \hat{\phi}^L + K \leftrightarrow L \right) - \partial_i\hat{\phi}^K\partial^i\hat{\phi}^L.
\end{aligned} \tag{4.182}$$

The form of the current is fixed by conservation and condition of being symmetric traceless. The descendant $\mathcal{K}$ can then be figured out using its definition in eq. (4.176). The boundary operator $\hat{\mathcal{J}}$ is the boundary limit of the bulk current with both of its Lorentz indices equal to y since we are considering a boundary scalar. Computing the correlator then is just a matter of free-field Wick contractions and it gives

$$(\hat{\gamma}_{l=0}^{s=2})^{T,D} = -\frac{N\epsilon}{N+8}, \qquad (\hat{\gamma}_{l=0}^{s=2})^{T,N} = -\frac{N\epsilon}{N+8} \tag{4.183}$$

where T stands for symmetric traceless. Note that this operator is a composite primary operator on the boundary and also appears in the OPE of $\hat{\phi}\hat{\phi}$ on the boundary. We will check below that the result found here agrees with what we get from a boundary four-point computation in (4.200). In principle, this method can be used to calculate anomalous dimensions of all the higher spin displacements, although the algebra gets tedious very soon.

4.3.4 Boundary four-point functions

In this subsection, we compute the four-point function of the leading boundary operator $\hat{\phi}^I$ induced by the bulk scalar ϕ^I. This will help us compute the anomalous dimensions of the boundary composite operators appearing in the OPE $\hat{\phi}^I\hat{\phi}^J$. This four-point function can be decomposed into singlet,

[17] Note that we are using the notation where $\hat{\phi}$ is the leading boundary operator, so in the Dirichlet case, $\hat{\phi}(\mathbf{x}) = \partial_y\phi(\mathbf{x},0)$

symmetric traceless and anti-symmetric representations of $O(N)$

$$\langle\hat{\phi}^I(\mathbf{x}_1)\hat{\phi}^J(\mathbf{x}_2)\hat{\phi}^K(\mathbf{x}_3)\hat{\phi}^L(\mathbf{x}_4)\rangle = \delta^{IJ}\delta^{KL}\mathcal{G}_S + \left(\frac{\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK}}{2} - \frac{\delta^{IJ}\delta^{KL}}{N}\right)\mathcal{G}_T + \frac{\delta^{IK}\delta^{JL}-\delta^{IL}\delta^{JK}}{2}\mathcal{G}_A. \tag{4.184}$$

Each of these structures can be decomposed in terms of the conformal blocks

$$\mathcal{G} = \frac{\hat{C}^2_{\phi\phi}}{(\mathbf{x}^2_{12}\mathbf{x}^2_{34})^{\hat{\Delta}_\phi}}\mathcal{F}(u,v), \quad \mathcal{F}(u,v) = \sum_{\hat{\Delta},l} a_{\hat{\Delta},l}G_{\hat{\Delta},l}(u,v) \tag{4.185}$$

where

$$u = \frac{\mathbf{x}^2_{12}\mathbf{x}^2_{34}}{\mathbf{x}^2_{13}\mathbf{x}^2_{24}}, \quad v = \frac{\mathbf{x}^2_{14}\mathbf{x}^2_{23}}{\mathbf{x}^2_{13}\mathbf{x}^2_{24}}. \tag{4.186}$$

In the free theory, the four-point function takes the simple form

$$\langle\hat{\phi}^I(\mathbf{x}_1)\hat{\phi}^J(\mathbf{x}_2)\hat{\phi}^K(\mathbf{x}_3)\hat{\phi}^L(\mathbf{x}_4)\rangle_0 = \hat{C}^2_{\phi\phi}\left(\frac{\delta^{IJ}\delta^{KL}}{(\mathbf{x}^2_{12})^{\hat{\Delta}_\phi}(\mathbf{x}^2_{34})^{\hat{\Delta}_\phi}} + \frac{\delta^{IK}\delta^{JL}}{(\mathbf{x}^2_{13})^{\hat{\Delta}_\phi}(\mathbf{x}^2_{24})^{\hat{\Delta}_\phi}} + \frac{\delta^{IL}\delta^{JK}}{(\mathbf{x}^2_{14})^{\hat{\Delta}_\phi}(\mathbf{x}^2_{23})^{\hat{\Delta}_\phi}}\right). \tag{4.187}$$

We will be looking at its decomposition in the s-channel $12 \to 34$ where the first term is the identity operator and the other two come from the composite operators schematically given by $\hat{\phi}^I(\partial)^{2n}\partial^l\hat{\phi}^J$ with dimension $2\hat{\Delta}_\phi + 2n + l$ and spin l [81]

$$\frac{1}{(\mathbf{x}^2_{13})^{\hat{\Delta}_\phi}(\mathbf{x}^2_{24})^{\hat{\Delta}_\phi}} = \frac{(-1)^l}{(\mathbf{x}^2_{14})^{\hat{\Delta}_\phi}(\mathbf{x}^2_{23})^{\hat{\Delta}_\phi}} = \frac{1}{(\mathbf{x}^2_{12})^{\hat{\Delta}_\phi}(\mathbf{x}^2_{34})^{\hat{\Delta}_\phi)}}\sum_{n,l} a^0_{n,l}G_{n,l}(u,v) \tag{4.188}$$

where

$$a^0_{n,l} = a^0_{2\hat{\Delta}_\phi+2n+l,l} = \frac{(-1)^l[(\hat{\Delta}_\phi - \frac{d}{2} + \frac{3}{2})_n(\hat{\Delta}_\phi)_{l+n}]^2}{l!n!(l+\frac{d-1}{2})_n(2\hat{\Delta}_\phi + n - d + 2)_n(2\hat{\Delta}_\phi + 2n + l - 1)_l(2\hat{\Delta}_\phi + n + l + \frac{1-d}{2})_n} \tag{4.189}$$

and $G_{n,l}(u,v) = G_{2\hat{\Delta}_\phi+2n+l,l}(u,v)$ is the four-point conformal block. In the interacting theory, we expect these dimensions and OPE coefficients to receive corrections. To first order in the expansion parameter, we have

$$\delta\mathcal{F}(u,v) = \sum_{n,l}\left(a^1_{n,l} + \frac{1}{2}a^0_{n,l}\hat{\gamma}_1(n,l)\partial_n\right)G_{n,l} \tag{4.190}$$

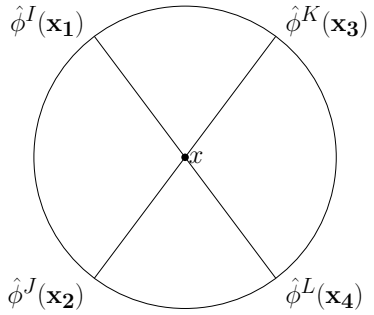
We will calculate these corrections in an ϵ expansion in $d = 4-\epsilon$ in the Wilson-Fisher model, for both Neumann and Dirichlet boundary conditions with $\hat{\Delta}_\phi = d/2-1$ and $d/2$ respectively. We will also do an ϵ expansion calculation in non-linear sigma model in $d = 2+\epsilon$, for Dirichlet boundary

conditions (we leave the case of Neumann boundary conditions to future work).

ϵ expansion in $d = 4 - \epsilon$

The first order correction to this four-point function in ϕ^4 theory can be computed using the following contact Witten diagram in AdS

$$\langle \hat{\phi}^I(\mathbf{x}_1)\hat{\phi}^J(\mathbf{x}_2)\hat{\phi}^K(\mathbf{x}_3)\hat{\phi}^L(\mathbf{x}_4)\rangle_1 \;=\; \tag{4.191}$$



This contact Witten diagram evaluates to the D function that appears frequently in AdS/CFT literature

$$\begin{aligned}\langle \hat{\phi}^I(\mathbf{x}_1)\hat{\phi}^J(\mathbf{x}_2)\hat{\phi}^K(\mathbf{x}_3)\hat{\phi}^L(\mathbf{x}_4)\rangle_1 &= -2\;\hat{C}^4_{\phi\phi}\;\lambda(\delta^{IJ}\delta^{KL}+\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})\int d^dx\sqrt{g}\prod_{i=1}^{4}K_{\hat{\Delta}_\phi}(\mathbf{x}_i,x)\\ &= -2\;\hat{C}^4_{\phi\phi}\;\lambda(\delta^{IJ}\delta^{KL}+\delta^{IK}\delta^{JL}+\delta^{IL}\delta^{JK})D_{\hat{\Delta}_\phi\hat{\Delta}_\phi\hat{\Delta}_\phi\hat{\Delta}_\phi}(\mathbf{x}_i).\end{aligned} \tag{4.192}$$

Here we are using a normalization where bulk-boundary propagator takes the form

$$K_{\hat{\Delta}_\phi}(\mathbf{x}_i,x) = \left(\frac{y}{y^2+(\mathbf{x}_i-\mathbf{x})^2}\right)^{\hat{\Delta}_\phi}. \tag{4.193}$$

The conformal block decomposition of this D function was worked out in [181, 182] [18]

$$\begin{aligned}&D_{\hat{\Delta}_\phi\hat{\Delta}_\phi\hat{\Delta}_\phi\hat{\Delta}_\phi}(\mathbf{x}_i) =\\ &\frac{1}{(\mathbf{x}_{12}^2)^{\hat{\Delta}_\phi}(\mathbf{x}_{34}^2)^{\hat{\Delta}_\phi}}\Bigg[\sum_{\substack{m,n=0\\ m\neq n}}^{\infty}\frac{2(-1)^{m+n}(\hat{\Delta}_\phi)^2_m(\hat{\Delta}_\phi)^2_n\;\;G_{n,0}(u,v)}{\mathcal{N}_{\hat{\Delta}_n}m!n!\,(m_n^2-m_m^2)\left(2\hat{\Delta}_\phi+m+\frac{1-d}{2}\right)_m\left(2\hat{\Delta}_\phi+n+\frac{1-d}{2}\right)_n}\\ &+\sum_{n=0}^{\infty}\frac{4\;(\hat{\Delta}_\phi)^4_n\;\Gamma(\hat{\Delta}_n)^2}{(n!)^2\left(2\hat{\Delta}_\phi+n+\frac{1-d}{2}\right)_n^2\Gamma\left(\frac{\hat{\Delta}_n}{2}\right)^4\;\partial_n m_n^2}\partial_n\left(\frac{\Gamma\left(\frac{\hat{\Delta}_n}{2}\right)^4}{4\;\Gamma(\hat{\Delta}_n)^2\mathcal{N}_{\hat{\Delta}_n}}G_{n,0}(u,v)\right)\Bigg]\end{aligned} \tag{4.194}$$

[18] We thank Christian Jepsen for pointing out a normalization typo in eq. 4.8 of [181] which introduces an additional factor of $\mathcal{N}_{\hat{\Delta}_n}$ in our eq. (4.194) compared to eq. 4.24 of [181].

where $m_n^2 = \hat{\Delta}_n(\hat{\Delta}_n - d + 1)$ with $\hat{\Delta}_n = 2\hat{\Delta}_\phi + 2n$ and

$$\mathcal{N}_\Delta = -\frac{\Gamma(\Delta)}{\pi^{\frac{d-1}{2}}(2\Delta - d + 1)\Gamma(\Delta + \frac{1-d}{2})}. \tag{4.195}$$

Comparing this with eq. (4.190) tells us that only the spin 0 operators get anomalous dimension to this order, i.e. $\gamma_1(n, l > 0) = 0$. Moreover, the spin 0 anomalous dimensions in the singlet sector are given by

$$\frac{N}{2} a^0_{S\ n,0} \hat{\gamma}_1^S(n, 0) = -\frac{2\hat{C}^2_{\phi\phi}\ \lambda(N+2)(\hat{\Delta}_\phi)^4_n}{(n!)^2(2\hat{\Delta}_\phi + n + \frac{1-d}{2})^2_n(\partial_n m_n^2)\mathcal{N}_{\hat{\Delta}_n}} \tag{4.196}$$

where $a^0_{S\ n,0} = a^0_{n,l}(1 + (-1)^l)/N$. For Dirichlet boundary condition, we plug in $\hat{C}_{\phi\phi} = \Gamma\left(\frac{d}{2}\right)/\pi^{\frac{d}{2}}$ and $\hat{\Delta}_\phi = d/2$ to get

$$\hat{\gamma}_1^{S,D}(n, 0) = \frac{\epsilon(N+2)}{(N+8)} \tag{4.197}$$

where we used the fixed point value of the coupling $\lambda = \lambda^*$. This gives the dimension of the boundary operators up to 1-loop order

$$\hat{\Delta}^{S,D}_{n,0} = 2\hat{\Delta}_\phi + 2n + \hat{\gamma}_1^{S,D}(n, 0) = d + 2n + 2\hat{\gamma}_\phi^D + \hat{\gamma}_1^{S,D}(n, 0) = d + 2n + O(\epsilon^2) \tag{4.198}$$

where we used the result for $\hat{\gamma}_\phi^D$ from eq. (4.136). The $n = 0$ operator is proportional to the displacement operator, which is supposed to be protected to all orders. Similarly, for the Neumann boundary condition, using $\hat{C}_{\phi\phi} = \Gamma\left(\frac{d}{2} - 1\right)/(2\pi^{\frac{d}{2}})$ and $\hat{\Delta}_\phi = d/2 - 1$ along with the result for $\hat{\gamma}_\phi^N$ from eq. (4.136) gives

$$\begin{aligned} \hat{\gamma}_1^{S,N}(n, 0) &= \frac{\epsilon(N+2)}{(N+8)} \implies \hat{\Delta}^{S,N}_{n,0} = d - 2 + 2n, \quad \forall\ n > 0 \\ \hat{\gamma}_1^{S,N}(0, 0) &= \frac{2\epsilon(N+2)}{(N+8)} \implies \hat{\Delta}^{S,N}_{0,0} = 2 - \frac{6\epsilon}{N+8}. \end{aligned} \tag{4.199}$$

Note that the $n = 0$ case has to be treated separately here because first setting $\hat{\Delta} = d/2 - 1$ and then taking $n \to 0$ gives the wrong result for the OPE coefficient $a^0_{0,0}$. Instead one should directly take $n \to 0$ which gives $a^0_{0,0} = 1$ for all values of $\hat{\Delta}$. Recall that from the boundary operator expansion of σ at large N, we saw leading operators of dimension 2 and d to be present in the boundary spectrum, and in (4.199) we just see the $4 - \epsilon$ description of the same operators. We can obtain similar results

in the symmetric traceless sector, for which we get

$$\begin{aligned}
&\hat{\gamma}_1^{T,D}(n,0) = \frac{2\epsilon}{(N+8)}, \quad \hat{\gamma}_1^{T,N}(n>0,0) = \frac{2\epsilon}{(N+8)}, \quad \hat{\gamma}_1^{T,N}(0,0) = \frac{4\epsilon}{(N+8)} \\
\implies &\hat{\Delta}_{n,0}^{T,D} = d + 2n - \frac{N\epsilon}{N+8}, \quad \hat{\Delta}_{n>0,0}^{T,N} = d - 2 + 2n - \frac{N\epsilon}{N+8}, \quad \hat{\Delta}_{0,0}^{T,N} = d - 2 + \frac{(2-N)\epsilon}{N+8}.
\end{aligned} \tag{4.200}$$

These composite operators also appear in the boundary operator expansion of the bulk higher spin currents. In the previous subsection, we gave another method to calculate the anomalous dimension of such operators and the result above agrees with the example we considered there in eq. (4.183). These results for anomalous dimensions and ϵ^2 corrections to some of these results were also reported in [154] by explicitly performing the loop integrals in AdS.

ϵ expansion in $d = 2 + \epsilon$

We can compute a similar four-point function in the non-linear sigma model by evaluating contact Witten diagrams in $\text{AdS}_{2+\epsilon}$. We will work to the leading order in ϵ, hence the calculation really reduces to Witten diagrams in AdS_2.[19] We will restrict ourselves to the simpler case of Dirichlet boundary conditions. The calculation is technically similar to the ones relevant for the defect CFT on BPS Wilson line operators [187]. The case of Neumann boundary conditions is expected to be more subtle, similarly to what was discussed in [188], due to the presence of zero modes responsible for restoration of the $O(N)$ symmetry, and we will not discuss it in detail here.

The sigma-model interactions now involves derivatives, so the expression is a little longer to write down

$$\begin{aligned}
&\langle\hat{\varphi}^a(\mathbf{x}_1)\hat{\varphi}^b(\mathbf{x}_2)\hat{\varphi}^c(\mathbf{x}_3)\hat{\varphi}^d(\mathbf{x}_4)\rangle_1 = -t^2\hat{C}^4_{\phi\phi}\int d^dx\sqrt{g}g^{\mu\nu}\times \\
&\left[\delta^{ab}\delta^{cd}\left(K^1_{\hat{\Delta}_\phi}\partial_\mu K^2_{\hat{\Delta}_\phi} + 1\leftrightarrow 2\right)\left(K^3_{\hat{\Delta}_\phi}\partial_\nu K^4_{\hat{\Delta}_\phi} + 3\leftrightarrow 4\right) + \{b,2\}\leftrightarrow\{c,3\} + \{b,2\}\leftrightarrow\{d,4\}\right]
\end{aligned} \tag{4.201}$$

where we introduced the notation $K^i_{\hat{\Delta}_\phi} = K_{\hat{\Delta}_\phi}(\mathbf{x}_i, x)$. To deal with the derivatives, the following identity is useful, which can be derived just by using the explicit expression of the bulk-to-boundary propagator

$$g^{\mu\nu}\partial_\mu K^1_{\Delta_1}\partial_\nu K^2_{\Delta_2} = \Delta_1\Delta_2\left(K^1_{\Delta_1}K^2_{\Delta_2} - 2\,\mathbf{x}_{12}^2\,K^1_{\Delta_1+1}K^1_{\Delta_2+2}\right). \tag{4.202}$$

[19] For other recent calculations of Witten diagrams for CFTs in AdS_2, see [183, 184, 185, 186].

So the integral now produces a linear combination of D functions

$$\begin{aligned}
&\langle\hat{\varphi}^a(\mathbf{x}_1)\hat{\varphi}^b(\mathbf{x}_2)\hat{\varphi}^c(\mathbf{x}_3)\hat{\varphi}^d(\mathbf{x}_4)\rangle_1 = -t^2\hat{C}^4_{\phi\phi}\hat{\Delta}^2_\phi\times\\
&\Bigg[\delta^{ab}\delta^{cd}\Bigg(4D_{\hat{\Delta}_\phi,\hat{\Delta}_\phi,\hat{\Delta}_\phi,\hat{\Delta}_\phi} - 2\mathbf{x}^2_{24}D_{\hat{\Delta}_\phi,\hat{\Delta}_\phi+1,\hat{\Delta}_\phi,\hat{\Delta}_\phi+1} - 2\mathbf{x}^2_{14}D_{\hat{\Delta}_\phi+1,\hat{\Delta}_\phi,\hat{\Delta}_\phi,\hat{\Delta}_\phi+1}\\
&- 2\mathbf{x}^2_{23}D_{\hat{\Delta}_\phi,\hat{\Delta}_\phi+1,\hat{\Delta}_\phi+1,\hat{\Delta}_\phi} - 2\mathbf{x}^2_{13}D_{\hat{\Delta}_\phi+1,\hat{\Delta}_\phi,\hat{\Delta}_\phi+1,\hat{\Delta}_\phi}\Bigg) + \{b,2\}\leftrightarrow\{c,3\} + \{b,2\}\leftrightarrow\{d,4\}\Bigg]
\end{aligned} \tag{4.203}$$

where the permutation also exchanges the subscripts of the D function. For example, under $\{b,2\}\leftrightarrow\{c,3\}$ we have $\mathbf{x}^2_{24}D_{\hat{\Delta}_\phi,\hat{\Delta}_\phi+1,\hat{\Delta}_\phi,\hat{\Delta}_\phi+1}\to\mathbf{x}^2_{34}D_{\hat{\Delta}_\phi,\hat{\Delta}_\phi,\hat{\Delta}_\phi+1,\hat{\Delta}_\phi+1}$. These D functions are well known and all we need here is the particular case with $\hat{\Delta}_\phi = 1$. Explicit expressions for this particular case can functions can be explicitly found in, for instance [187, 188]. Moreover, since the boundary theory is essentially one-dimensional,[20] we only have one cross-ratio which we call χ with $u=\chi^2$ and $v=(1-\chi)^2$. In one dimension, the conformal block is just given by [113]

$$G_{n,0} = \chi^{2\hat{\Delta}_\phi+2n}{}_2F_1(2\hat{\Delta}_\phi+2n, 2\hat{\Delta}_\phi+2n, 4\hat{\Delta}_\phi+4n, \chi) \tag{4.204}$$

and the derivative with respect to n gives

$$\partial_n G_{n,0} = 2\chi^{2(\hat{\Delta}_\phi+n)}\ \log(\chi)\ {}_2F_1(2\hat{\Delta}_\phi+2n, 2\hat{\Delta}_\phi+2n, 4\hat{\Delta}_\phi+4n, \chi) + \text{ other terms without a}\log. \tag{4.205}$$

We only focus on the $\log\chi$ term, because that is sufficient for us to extract the anomalous dimensions. Using the explicit expressions for the D-functions, we can collect the log terms and comparing it to the log terms appearing in the boundary operator expansion (4.190), we can read off the anomalous dimensions. For the singlet sector, this gives the following equation

$$\begin{aligned}
&\sum_n (N-1)a^0_{S,\ n,0}\hat{\gamma}^{S,D}_1(n,0)\ \chi^{2n}\ {}_2F_1(2+2n,2+2n,4+4n,\chi) =\\
&-\frac{2\epsilon}{N-2}\Bigg(-\frac{2(N+1)}{1-\chi} - \frac{N(2\chi-3)}{2(1-\chi)^2} + \frac{N(3-\chi)}{2(1-\chi)} - \frac{\chi^2}{(1-\chi)^2}\Bigg)
\end{aligned} \tag{4.206}$$

where we used the coupling at the fixed point $t^2_* = 2\pi\epsilon/(N-2)$ and the $d=2$ Dirichlet value of

[20] More precisely, the boundary theory is $1+\epsilon$ dimensional, but to the order we are working we can set $d=2$ everywhere, and hence the boundary is for our purposes one-dimensional.

$\hat{C}_{\phi\phi} = 1/\pi$. Using the following orthogonality property of $F_\beta(\chi) = {}_2F_1(\beta, \beta, 2\beta, \chi)$

$$\frac{1}{2\pi i}\oint_{\chi=0} \chi^{\beta-\beta'-1} F_\beta(\chi) F_{1-\beta'}(\chi) = \delta_{\beta\beta'} \tag{4.207}$$

it is easy to show that the anomalous dimensions are given by

$$\hat{\gamma}_1^{S,D}(m,0) = -\frac{2\epsilon}{(N-1)a^0_{S,\ m,0}} \frac{1}{2\pi i}\oint_{\chi=0} \chi^{-2m-1}\left(\frac{\chi(\chi-2)+2}{2(\chi-1)^2}\right) F_{-2m-1}(\chi). \tag{4.208}$$

So we just need to calculate the coefficient of χ^{2m} in a product of a Hypergeometric function and a polynomial in order to know the anomalous dimension for any m. The OPE coefficient $a^0_{S,\ m,0}$ is given by

$$(N-1)a^0_{S,\ m,0} = \frac{2\ (\frac{3}{2})_m^2 \Gamma(1+m)}{(\frac{1}{2})_m (2+m)_m (\frac{3}{2}+m)_m}. \tag{4.209}$$

Using the boundary anomalous dimension of $\hat{\phi}$ from eq. (4.139), this gives the following values of dimensions of the composite operators

$$\hat{\gamma}_1^{S,D}(n,0) = -\epsilon, \quad \hat{\Delta}^{S,D}_{n,0} = 2\hat{\Delta}^D_\phi + 2n + \hat{\gamma}_1^{S,D}(n,0) = 2 + \epsilon + O(\epsilon^2) = d + 2n + O(\epsilon^2). \tag{4.210}$$

Again, the operator with $n = 0$ is proportional to the displacement operator, while the others appear in the boundary operator expansion of the bulk higher spin currents. So far, we dealt with the case of Dirichlet boundary conditions on the unconstrained fields, when the dimension of the leading boundary operator is $\hat{\Delta}_\phi = 1 + \epsilon = d - 1$. This describes extraordinary transition in $d = 2 + \epsilon$. As we mentioned above, we do not discuss the four-point function in the case of Neumann boundary conditions, which would have $\hat{\Delta}_\phi = O(\epsilon)$ and hence the propagators would be logarithmic. As we discussed in subsection 4.2.2, that case describes the ordinary transition in $d = 2 + \epsilon$ dimensions. We leave a more detailed study of that case for future work.

4.4 Conclusion

In this chapter, we have explored the idea of placing a CFT in AdS as a way of studying the BCFT problem. Focusing on the concrete example of the large N critical $O(N)$ model, we have explained how to obtain the various boundary critical behaviors of the model in the AdS picture. We have also computed the large N free energies of the model on the hyperbolic ball, and verified consistency of a conjectured F-theorem for the behavior of the quantity $\tilde{F}$ in (4.3) under RG flows

triggered by boundary relevant operators. Then, we showed how to use the AdS setup to extract some of the BCFT data in the theory. In particular we suggested that using the bulk equations of motion in a way similar to [156] one can reconstruct in a convenient way the bulk two-point function and the anomalous dimensions of boundary operators encoded in it. We have also presented some calculations of boundary 4-point functions, where one can use the well-known techniques developed for the calculation of Witten diagrams in AdS/CFT.

It would be interesting to apply the methods used in this chapter to other examples of interacting BCFT, in particular explore the idea of using the bulk equations of motion to extract BCFT data as described in subsection 4.3.2. For instance, one may consider theories with fermions, like the large N Gross-Neveu model and the related Gross-Neveu-Yukawa model (some results for the Gross-Neveu BCFT at large N from the AdS approach were recently obtained in [118]). It would be also quite interesting to study bosonic and fermionic vector models coupled to Chern-Simons gauge theory in $d = 3$ [189, 190] by placing them in AdS. One may then compute their free energy and other BCFT data, and perhaps provide further evidence for the 3d boson/fermion duality.

Another technically interesting direction to pursue would be to develop more methods for the $1/N$ perturbation theory in the $O(N)$ model BCFT. Analytic functionals have been developed in [16] to do perturbative expansions around a mean field solution in BCFT. In a mean field theory, there is an elementary field with dimension Δ_φ and all its higher point correlators factorize into products of two-point functions. In the presence of a boundary, there are two possibilities for the boundary spectrum: Neumann with boundary dimensions being $\Delta_\phi + 2n$ or Dirichlet with boundary dimensions being $\Delta_\phi + 2n + 1$. However, as we saw in subsection 4.2.1, in the large N solution the elementary bulk field has dimensions $d/2 - 1$, while the leading boundary operator for ordinary or special transition has dimension $d-2$ or $d-3$. So the large N theory is quite different from a mean field solution, and the analytic functionals developed so far cannot be used for the $1/N$ perturbation theory. It would be interesting to see if a suitable set of functionals may be developed for the large N expansion. Improving our knowledge of the $O(N)$ model BCFT at large N may also help in better understanding the holographic description of the model, which should presumably be related to Vasiliev higher-spin theory in AdS_{d+1} in the presence of an AdS_d "boundary brane".[21]

[21] To avoid possible confusion, let us stress here that this AdS_d "brane" is not the same as the AdS_d on which we placed the CFT in this chapter.

4.5 Appendix: Bulk OPE coefficients at large N

In this appendix, we show how to obtain the bulk OPE coefficients for the two-point function of ϕ in the case of special and ordinary transtions. As we saw in the main text, the two-point function for any scalar operator in the flat half-space can be written as (the AdS expression is the same with the conformal factors stripped off)

$$\langle O(x)O(x')\rangle = \frac{A}{(4yy')^{\Delta_O}}\xi^{-\Delta_O}G(\xi) = \frac{A}{(4yy')^{\Delta_O}}\mathcal{G}(z), \quad z = \frac{1}{1+\xi}. \tag{4.211}$$

We introduced a new variable z because that is more convenient to work with for us in this appendix. In terms of z, we have

$$\begin{aligned}
\mathcal{G}(z) &= \frac{z^{\Delta_O}}{(1-z)^{\Delta_O}}\sum_k \lambda_k f_{\text{bulk}}\left(\Delta_k; 1-z\right) = \sum_l \mu_l^2 f_{\text{bdy}}(\hat{\Delta}_l; z) \\
f_{\text{bulk}}(\Delta_k; z) &= z^{\frac{\Delta_k}{2}}\ {}_2F_1\left(\frac{\Delta_k - d}{2}+1, \frac{\Delta_k}{2}; \Delta_k + 1 - \frac{d}{2}; z\right) \\
f_{\text{bdy}}(\hat{\Delta}_l; z) &= z^{\hat{\Delta}_l}\ {}_2F_1\left(\hat{\Delta}_l, \hat{\Delta}_l + 1 - \frac{d}{2}; 2\hat{\Delta}_l + 2 - d; z\right).
\end{aligned} \tag{4.212}$$

The bulk and boundary block expansions are obtained respectively in the limit $\xi \to 0$ ($z \to 1$) and $\xi \to \infty$ ($z \to 0$) limit of the two point function. For the special transition, we have

$$G(\xi) = \frac{1+2\xi}{(1+\xi)^{\frac{d}{2}-1}} \implies \mathcal{G}(z) = \frac{z^{d-3}(2-z)}{(1-z)^{\frac{d}{2}-1}}. \tag{4.213}$$

By expanding this for small ξ, it is easy to see that the operators appearing in bulk have dimensions $\Delta_n = 2n+2$. To obtain the OPE coefficients, we can use the Euclidean inversion formula [16] for BCFT, which gives the bulk coefficient function as

$$I_\Delta = \int_0^1 dy\ y^{-2}(1-y)^{1-\frac{d}{2}}\ {}_2F_1\left(\frac{\Delta}{2}, \frac{d-\Delta}{2}, 1, 1-\frac{1}{y}\right)\mathcal{G}(1-y). \tag{4.214}$$

To actually do this integral, we need to transform the parameters of this hypergeometric function, so that it becomes a combination of hypergeometric functions at y instead of $1-1/y$, and the integral can be then performed after that. The bulk coefficients can be determined from the residues of this coefficient function. The coefficient function near behaves near the poles as

$$I_\Delta \frac{\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta-d+2}{2})}{2\Gamma(\Delta-\frac{d}{2})} \sim \frac{(\lambda)_n}{\Delta - \Delta_n}, \quad \text{as} \quad \Delta \to \Delta_n = 2n+2. \tag{4.215}$$

Using this, we can find the bulk coefficients

$$(\lambda)_n^S = \frac{(d^2-4d(n+2)+8(1+n)^2+4)\Gamma(1-\frac{d}{2}+n)\Gamma(2-\frac{d}{2}+n)^2}{4\Gamma(2-\frac{d}{2})^2\Gamma(n+2)\Gamma(2-\frac{d}{2}+2n)}. \tag{4.216}$$

This agrees with the result that was found in [14]. For the ordinary transition, we have

$$G(\xi) = \frac{1}{(1+\xi)^{\frac{d}{2}-1}} \implies \mathcal{G}(z) = \frac{z^{d-2}}{(1-z)^{\frac{d}{2}-1}}. \tag{4.217}$$

The Euclidean inversion formula can again be used to get a coefficient function which gives the bulk coefficients

$$(\lambda)_n^O = -\frac{(-1)^n\Gamma\left(\frac{d}{2}\right)\ {}_2F_1\left(-n-1,-n;\frac{1}{2}(d-4n-2);1\right)}{\Gamma(n+2)\Gamma\left(\frac{d}{2}-n-1\right)}. \tag{4.218}$$

This formula, to our knowledge, has not appeared before.

4.6 Appendix: Details on σ propagator

In this appendix, we give some details on deriving the σ propagator in position space in the large N theory. In order to do that, we first show how to invert functions of chordal distance in AdS space, following a similar derivation on half-space in [12]. We start with the following equation

$$\int d^dx\sqrt{g_x}\ G(\xi_{x_1,x})H(\xi_{x,x_2}) = \int d^dx\ y^{-d}\ G(\xi_{x_1,x})H(\xi_{x,x_2}) = \frac{\delta^d(x_1-x_2)}{\sqrt{g_{x_1}}} \tag{4.219}$$

and the problem is to find H given G. First we note that we can integrate over the boundary directions using the following formulae

$$\int d^{d-1}\mathbf{x}\ G(\xi_{x,x'}) = (4yy')^{\frac{d-1}{2}}g(\rho_{y,y'}), \quad \rho_{y,y'} = \frac{(y-y')^2}{4yy'}, g(\rho) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\int_0^\infty duu^{\frac{d-3}{2}}G(u+\rho) \tag{4.220}$$

and the above transform can be inverted as

$$G(\xi) = \frac{\pi^{-\frac{d-1}{2}}}{\Gamma(-\frac{d-1}{2})}\int_0^\infty d\rho\ u^{\frac{-d-1}{2}}g(\rho+\xi). \tag{4.221}$$

Using this, we can integrate both sides over the boundary coordinates $\mathbf{x}_1$

$$\int_0^\infty \frac{dy}{y}g(\rho_{y_1,y})h(\rho_{y,y_2}) = \frac{y_1\delta(y_1-y_2)}{4^{d-1}}. \tag{4.222}$$

We can make a change of variable to $y = e^{2\theta}$ to simplify the integral to

$$\int d\theta \; g(\sinh^2(\theta_1 - \theta))h(\sinh^2(\theta - \theta_2)) = \frac{\delta(\theta_1 - \theta_2)}{4^d}. \tag{4.223}$$

This can then be solved by a Fourier transform

$$\hat{g}(k) = \int d\theta e^{ik\theta} g(\sinh^2 \theta) \tag{4.224}$$

which gives

$$\hat{g}(k)\hat{h}(k) = \frac{1}{4^d}. \tag{4.225}$$

So we need to be able to go from $\hat{g}(k)$ to $G(\xi)$. It was shown in [12] that $\hat{g}(k)$ of the form

$$\hat{g}_{a,b}(k) = \frac{\Gamma\left(a - \frac{ik}{4}\right)\Gamma\left(a + \frac{ik}{4}\right)}{\Gamma\left(b - \frac{ik}{4}\right)\Gamma\left(b + \frac{ik}{4}\right)} \tag{4.226}$$

corresponds to

$$G_{a,b}(\xi) = \frac{\Gamma\left(2a + \frac{d-1}{2}\right)}{4^{2a-1}\pi^{\frac{d-1}{2}}\Gamma(b-a)\Gamma(b+a)} \frac{1}{\xi^{2a+\frac{d-1}{2}}} {}_2F_1(2a + \frac{d-1}{2}, a + b - \frac{1}{2}; 2a + 2b - 1; -\frac{1}{\xi}). \tag{4.227}$$

For the ordinary transition in section 4.2.1, we have

$$G(\xi) = -\frac{N}{2}G_\phi(\xi)^2 = -\frac{N\Gamma\left(\frac{d}{2} - 1\right)^2}{2(4\pi)^d} \frac{1}{(\xi(1+\xi))^{d-2}} = -\frac{N\Gamma\left(\frac{d}{2} - 1\right)^2 \Gamma\left(-\frac{d}{2} + 2\right)}{2^{d+5}\Gamma(d-2)\pi^{d/2}} G_{\frac{3d-7}{4}, \frac{d+1}{4}} \tag{4.228}$$

which immediately gives us

$$G_\sigma(\xi) = H(\xi) = -\frac{2^{d+5}\Gamma(d-2)\pi^{d/2}}{N4^d\Gamma\left(\frac{d}{2} - 1\right)^2 \Gamma\left(-\frac{d}{2} + 2\right)} G_{\frac{d+1}{4}, \frac{3d-7}{4}}. \tag{4.229}$$

For application to the extraordinary transition in section 4.2.2 we need to use

$$G(\xi) = \tilde{\Pi}(\xi) = -\frac{(N-1)\Gamma\left(\frac{d}{2}\right)^2}{(4\pi)^d} \frac{1}{(\xi(1+\xi))^{d-1}} = -\frac{(N-1)\pi^{\frac{3-d}{2}} \csc\left(\frac{\pi d}{2}\right)}{4^d \; \Gamma\left(\frac{d-1}{2}\right)} G_{\frac{3d-3}{4}, \frac{d+1}{4}}. \tag{4.230}$$

For the special transition, getting the σ propagator needs a little more work. We do not do it here and refer the reader to [12] for details.

4.7 Appendix: Displacement operator and the b-anomaly coefficient

In this appendix we collect some comments on the displacement operator, which appears in every CFT with a boundary or a defect. In the case of a $d-$ dimensional CFT with a boundary this operator is a scalar of conformal dimension d appearing in the spectrum of the $d-1$ dimensional boundary theory. In the flat half-space space setup, the displacement operator D can be defined by

$$\partial^{\mu} T_{\mu y}(\mathbf{x}, y) = D(\mathbf{x})\delta(y) \tag{4.231}$$

and it can be thought of as related to the broken translational symmetry perpendicular to the boundary. This equation can be integrated in a Gaussian pill box located at the boundary which implies $D = T_{yy}|_{y\to 0}$. The coefficient of the two-point function of the displacement operator is a piece of BCFT data, and in $d = 3$ it is related to the trace anomaly coefficient b in (4.28). We will first discuss the calculation of this two-point function in the theory of a single free scalar field, and then move to the interacting case. The improved stress tensor in flat half-space can be written as

$$T_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\delta_{\mu\nu}\partial\phi\cdot\partial\phi - \frac{(d-2)}{4(d-1)}\big(\partial_{\mu}\partial_{\nu} - \delta_{\mu\nu}\partial^2\big)\phi^2. \tag{4.232}$$

We can use this to explicitly write the displacement operator. With Neumann boundary condition, we have $\partial_y\phi(\mathbf{x}, y = 0) = 0$, which gives after using equations of motion

$$D^N = T_{yy}|_{y\to 0} = -\frac{d-2}{2(d-1)}\phi\partial_y^2\phi - \frac{1}{2(d-1)}\partial_i\phi\partial^i\phi\,. \tag{4.233}$$

In the Dirichlet case, $\phi(\mathbf{x}, y = 0) = 0$, which gives

$$D^D = \frac{1}{2}(\partial_y\phi)^2. \tag{4.234}$$

Computing the correlator then is just a matter of Wick contractions and taking derivatives, and yields the result

$$\langle D^D(\mathbf{x})D^D(\mathbf{x}')\rangle = \langle D^N(\mathbf{x})D^N(\mathbf{x}')\rangle = \frac{2}{S_d^2}\frac{1}{|\mathbf{x}-\mathbf{x}'|^{2d}} \tag{4.235}$$

where $S_d = 2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ is the volume of a $d-1$ dimensional sphere. This equation gives the coefficient of the two-point function of the displacement operator for a free scalar, $\hat{C}_{DD} = 2/S_d^2$ for

both Neumann and Dirichlet cases. These results agree with what was found in [12].

In the large N interacting theory, one way to define the displacement operator is through its appearance in the BOE of the σ operator. In hyperbolic space, this means that if we look at the propagator of σ near the boundary, the contribution of the displacement operator should look like

$$\langle\sigma(x)\sigma(x')\rangle_{y,y'\to 0} \sim (B_\sigma{}^D)^2 (yy')^d \langle D(\mathbf{x})D(\mathbf{x}')\rangle. \tag{4.236}$$

Using the boundary limit of the bulk correlator of σ from eq. (4.118), (4.120) and eq. (4.129) for special, ordinary and extraordinary transition gives

$$\begin{aligned}
(B_\sigma{}^D)^2 \hat{C}_{DD}|_{\text{Special}} &= \frac{4^d}{3N}\frac{(4-d)\Gamma(d-2)^2}{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^3}\frac{(6-d)\Gamma(d)}{(d-2)\Gamma(2d-5)} \\
(B_\sigma{}^D)^2 \hat{C}_{DD}|_{\text{Ordinary}} &= \frac{4^d}{N}\frac{(4-d)\Gamma(d-2)^2}{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^3}\frac{\Gamma(d)}{\Gamma(2d-4)} \\
(B_\sigma{}^D)^2 \hat{C}_{DD}|_{\text{Extraordinary}} &= -\frac{2^{d+1}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma(d+1)}{\pi(N-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(d-\frac{1}{2}\right)}.
\end{aligned} \tag{4.237}$$

We can combine these equations with the constraint coming from the Ward identity relating the two point function of a bulk operator O with the displacement to its one-point function [12, 21] [22]

$$B_O{}^D \hat{C}_{DD} = -\Delta_O \frac{2^d A_O}{S_d} \tag{4.238}$$

where A_O is the one-point function coefficient and Δ_O is the conformal dimension of O . Using this relation for the σ operator with dimension 2 gives us

$$\begin{aligned}
\hat{C}_{DD}|_{\text{Special}} &= \frac{24N}{S_d^2}\frac{\Gamma\left(\frac{d}{2}\right)^3\Gamma\left(4-\frac{d}{2}\right)\Gamma\left(2d-5\right)}{\Gamma\left(d\right)\Gamma\left(d-1\right)^2} \\
\hat{C}_{DD}|_{\text{Ordinary}} &= \frac{2N}{S_d^2}\frac{\Gamma\left(\frac{d}{2}\right)^3\Gamma\left(3-\frac{d}{2}\right)\Gamma\left(2d-3\right)}{\Gamma\left(d\right)\Gamma\left(d-1\right)^2} \\
\hat{C}_{DD}|_{\text{Extraordinary}} &= -\frac{(N-1)\pi^{3/2}(d-2)d\,\Gamma\left(d-\frac{1}{2}\right)}{2\,S_d^2\,\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}.
\end{aligned} \tag{4.239}$$

The Neumann and Dirichlet results agree with [12]. Note that in $3 < d < 4$ we have $\hat{C}_{DD}|_{\text{Special}} > \hat{C}_{DD}|_{\text{Ordinary}}$, but the value for the extraordinary transition (which is supposed to be in the IR of both special and ordinary) is not consistent with a possible "$\hat{C}_{DD}$-theorem" for boundary RG flows.

As a check of the large N result, for extraordinary transition, we can also easily calculate this

[22] Note that the conventions for the OPE coefficients in [12] and [21] are different, which makes their Ward identities look different by factors of 2 etc., but the physical content is the same. We use the conventions of [12] suitably adapted to hyperbolic space.

quantity in $d = 4 - \epsilon$. In that case, ϕ^N or its fluctuation χ is the operator that contains the displacement in its BOE. From the BOE of χ in eq. (4.131), we get

$$(B_{\phi^N D})^2 \hat{C}_{DD} = \frac{4^4}{160\pi^2} \tag{4.240}$$

which when combined with the Ward identity (4.238) for $O = \phi^N$ gives

$$\hat{C}_{DD}|_{\text{Extraordinary}} = \frac{40(N+8)}{S_d^2 \epsilon} \tag{4.241}$$

in agreement with the large N result.

As we discussed in subsection (4.1.3), in $d = 3$ boundary CFT, the trace anomaly contains two terms as in eq. (4.28). The coefficient of one of the terms called as a_{3d} is related to the logarithmic term of the free energy with spherical boundary, and it was discussed at length in the main text. The other anomaly coefficient b is related to the two- point function coefficient of the displacement operator as $b = \pi^2 \hat{C}_{DD}/8$ [167]. Plugging in $d = 3$ in the results from above (4.239), we get

$$b_{3d}^{\text{Special}} = \frac{9N\pi^2}{1024}, \quad b_{3d}^{\text{Ordinary}} = \frac{N\pi^2}{1024}, \quad b_{3d}^{\text{Extraordinary}} = \frac{9N\pi^2}{1024}. \tag{4.242}$$

There was a conjectured bound in [167] implying $a_{3d}/b \geq -2/3$. It does not seem to hold true in the case of the extraordinary transition. The bound was based on a conjectured relation between a_{3d} and coefficients appearing in the stress tensor two-point function, which was also found to not hold true in case of a ϕ^6 theory with a ϕ^4 boundary interaction [120].

Chapter 5

Fermions in AdS and Gross-Neveu BCFT

In this chapter, we study theories with fermions in the presence of a boundary. Previous works on fermionic theories with conformal boundary conditions include for instance [11, 34, 35, 118]. To be more specific, we consider the Gross-Neveu (GN) model in dimensions $2 < d < 4$, which is a theory of N Dirac fermions with an $U(N)$ invariant interaction [191]

$$S = -\int d^d x \sqrt{g} \left(\bar{\Psi}_i \gamma \cdot \nabla \Psi^i + \frac{g}{2} (\bar{\Psi}_i \Psi^i)^2 \right). \tag{5.1}$$

The coupling g is dimensionless in two dimensions, and the model has a perturbative UV fixed point in $d = 2+\epsilon$. At large N, the critical point of the model may be described by introducing an auxiliary Hubbard-Stratonvich field and dropping the quadratic term $\sim \sigma^2$ which becomes irrelevant in the critical limit (see for instance [27, 51] for reviews). This yields the following action that can be used to develop the $1/N$ expansion of the large N CFT [1]

$$S = -\int d^d x \sqrt{g} \left(\bar{\Psi}_i \gamma \cdot \nabla \Psi^i + \sigma \bar{\Psi}_i \Psi^i \right). \tag{5.2}$$

This leads to a unitary conformal field theory in the dimension range $2 < d < 4$ and the $1/N$ perturbation theory for this model is well studied. The main goal of this chapter is to study the behavior of this theory in the presence of a boundary.

We again use the AdS description of a BCFT to do calculations. This enables us to directly

[1] Throughout this chapter, we are always going to assume that the bulk theory is at its critical point.

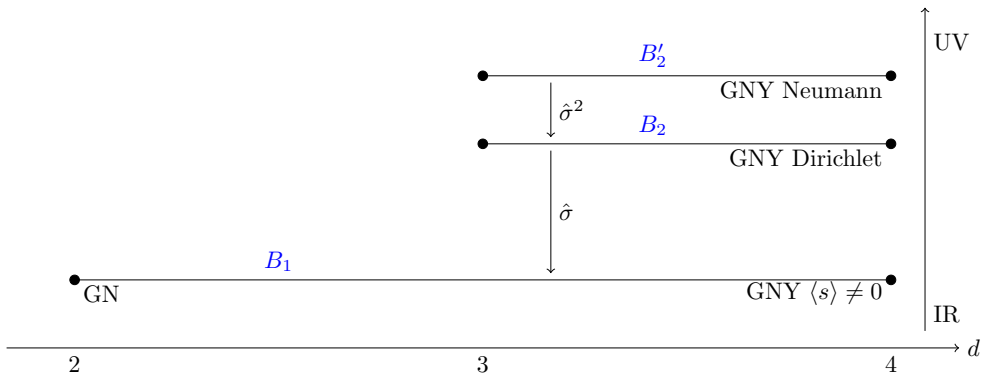


Figure 5.1: Various boundary phases for the large N model and their description in terms of Gross-Neveu and Gross-Neveu Yukawa model near $d = 2$ and 4 respectively. The figure is not to scale, and the top two phases can only be distinguished at subleading order in $1/N$.

apply the results from the extensive AdS/CFT literature about fermions in AdS [192, 193, 194, 195, 196, 197, 198, 122, 199, 200, 201] to the problem of BCFT. At leading order at large N, we obtain the following results that summarize the boundary critical behavior of the Gross-Neveu CFT: If we impose that the boundary spectrum satisfies unitarity bounds, then in dimensions $2 < d < 3$, there is a single boundary conformal phase characterized by the leading fermion operator with scaling dimension $\hat{\Delta}_{(1/2)} = \hat{\Delta} = d - 3/2 + O(1/N)$ in its boundary spectrum. We call this phase B_1. However as we go above three dimensions, in $3 \leq d < 4$, in addition to the above, there is another possible unitary phase, which has $\hat{\Delta}_{(1/2)} = \hat{\Delta} = d - 5/2 + O(1/N)$. At subleading order in $1/N$, we find that this actually splits into two distinct phases, which we call B_2 and B_2'. They have different bulk two-point functions for the fluctuations of the σ field around the saddle point (which is the same for B_2 and B_2' cases, corresponding to the same large N boundary fermion dimension), in particular yielding a different scaling dimension $\hat{\Delta}_{(0)}$ for the leading boundary scalar induced by the σ field. Let us note that, in all of the boundary conformal phases, we find that the bulk-boundary OPE of the bulk field σ includes a scalar operator of dimension d, which corresponds to the displacement operator (the presence of such operator is required by conformal symmetry in any BCFT). To summarize, near four dimensions, there are a total of three boundary critical points of the model. See Figure 5.1 where we summarize various phases and the RG flows between them.

As shown in [202], there is another description of the GN model in terms of the IR fixed point of the Gross-Neveu-Yukawa (GNY) model which has N Dirac fermions and a single scalar

$$S = \int d^d x \left(\frac{(\partial_\mu s)^2}{2} - \left(\bar{\Psi}_i \gamma \cdot \nabla \Psi^i + g_1 s \bar{\Psi}_i \Psi^i \right) + \frac{g_2}{24} s^4 \right). \tag{5.3}$$

This model is weakly coupled near $d = 4$ and one may develop a perturbation theory in ϵ in $d = 4 - \epsilon$,

where one finds an IR fixed point (see for instance [203] for a more extensive review and several results on the CFT data at this fixed point). The field s is essentially identified with σ in (5.2), up to rescaling by the coupling constant g_1. To be consistent with the results in the large N description, we expect to find three boundary phases in the GNY description as well. Indeed as we will show in section 5.3, the phases B_2 and B_2' correspond to doing perturbation theory around Dirichlet or Neumann boundary condition for the scalar s respectively, while B_1 corresponds to having a classical vev for the field s. In this sense, near four dimensions, B_2 and B_2' are analogous to the so-called ordinary and special transition in $O(N)$ scalar BCFT, while B_1 is analogous to extraordinary transition (see e.g. [204, 14, 2]). For easy reference, in Table 5.1 we report the dimensions of the boundary operators induced by the bulk fundamental fields Ψ and σ, along with the dimensions of the same operators in the ϵ expansion description, which can be seen to be precisely consistent with each other.

	Large N	GNY $d = 4 - \epsilon$	GN $d = 2 + \epsilon$
B_1	$\hat{\Delta}_{(1/2)} = d - \frac{3}{2}$	$\hat{\Delta}_{(1/2)} = \frac{3}{2} + \sqrt{\frac{36}{-2N+3+\sqrt{4N^2+132N+9}}} + O(\epsilon)$	$\hat{\Delta}_{(1/2)} = \frac{1}{2} + \frac{4N-3}{4(N-1)}\epsilon$
	$\hat{\Delta}_{(0)} = d$	$\hat{\Delta}_{(0)} = 4 + O(\epsilon)$	$\hat{\Delta}_{(0)} = 2 + \epsilon$
B_2	$\hat{\Delta}_{(1/2)} = d - \frac{5}{2}$,	$\hat{\Delta}_{(1/2)} = \frac{3}{2} - \frac{(8N+7)}{4(3+2N)}\epsilon$	- ,
	$\hat{\Delta}_{(0)} = 2$	$\hat{\Delta}_{(0)} = 2 - \frac{\sqrt{4N^2+132N+9}-2N+21}{12(2N+3)}\epsilon$	-
B_2'	$\hat{\Delta}_{(1/2)} = d - \frac{5}{2}$,	$\hat{\Delta}_{(1/2)} = \frac{3}{2} - \frac{(8N+9)}{4(3+2N)}\epsilon$	-
	$\hat{\Delta}_{(0)} = d - 3$	$\hat{\Delta}_{(0)} = 1 - \frac{\sqrt{4N^2+132N+9}+22N+21}{12(2N+3)}\epsilon$	-

Table 5.1: The dimensions of the leading boundary operators induced by the bulk fundamental fields Ψ and σ at large N in the three boundary phases we find. We also show the corresponding results from ϵ near two and four dimensions. The phase in the first row exists for $2 < d < 4$, while the two phases in the bottom two rows only exist between $3 < d < 4$.

In the free scalar BCFT, one may flow from Neumann to Dirichlet boundary conditions by turning on a boundary mass term. In the GNY model, we still expect this to be true near four dimensions, and one should be able to flow from B_2' to B_2 by turning on $\hat{s}^2$, where $\hat{s}$ is the leading operator in the boundary operator expansion of s. In the large N theory, the role of s is played by the σ field, hence in the large N theory, the flow from B_2' to B_2 must be driven by $\hat{\sigma}^2$ operator. Continuing the analogy with $O(N)$ scalar BCFT, to flow from ordinary to extraordinary transition there, one can turn on the analog of a "boundary magnetic field". In the GNY model description this corresponds to turning on the $\hat{s}$ operator on the boundary, and in the large N description to

turning on $\hat{\sigma}$. So we should be able to flow from the B_2 to B_1 phase by turning on the $\hat{\sigma}$ operator at the boundary. We will see in section 5.2 that there is no relevant scalar in the boundary spectrum for the B_1 phase, so we expect B_1 to be the most stable in the RG sense, followed by B_2 and B_2'.

Following a similar proposal for bulk CFT in [142] and for defect CFT in [134], it was proposed in [2] that the rescaled free energy on AdS with a sphere boundary

$$\tilde{F} = -\sin\left(\frac{\pi(d-1)}{2}\right) F_{AdS_d} \tag{5.4}$$

should decrease under RG flows localized on the boundary: $\tilde{F}_{\rm UV} > \tilde{F}_{\rm IR}$.[2] We check in section 5.2 by computing the AdS free energy for the various boundary conformal phases that it indeed does satisfy such inequality under the boundary RG flow.

In the case of $d = 3$, a very interesting extension that we leave to future work would be to gauge the $U(N)$ global symmetry and couple the fermions to the Chern-Simons gauge theory. One may consider adding Chern-Simons interactions either in the model of N free fermions, or at the critical point of the Gross-Neveu model. Similarly, one may consider gauging the scalar CFTs with a Chern-Simons term (either in the critical model, or starting with the free scalar theory without quartic interaction). Then, one may study how the bose-fermi dualities [189, 190, 205, 206, 207] (see also [51] for a review) are realized in the presence of a boundary. An interesting observation, which was also pointed out in [118], is that in $d = 3$, the large N dimensions of the leading boundary fermion in the two phases B_1 and B_2 of the GN model are $3/2$ and $1/2$ respectively, see Table 5.1. These coincide with the dimensions of the leading boundary scalar in a free massless boson theory with Dirichlet or Neumann boundary conditions, respectively. On the other hand, for the free massless fermion, there is just one phase, with leading boundary fermion of dimension 1, which happens to be the same as the dimension of the leading boundary fundamental scalar in the so-called ordinary transition in the large N scalar BCFT (see [2]). The fact that the dimensions of the boundary fundamental fermionic and bosonic operators match this way should be related to the bose-fermi duality, and suggests that the Chern-Simons interactions may not affect those boundary scaling dimensions to leading order at large N. It would be interesting to clarify this, as well as compute other observables in the Chern-Simons scalar and fermion theories, like the AdS_3 free energy (which encodes the boundary conformal anomaly), and boundary four-point functions of the fundamental fields.

The rest of this chapter is organized as follows: We start in section 5.1 by studying a single free

[2]Here we are only making a statement about the difference of $\tilde{F}$ between the UV and IR boundary fixed points, and not about the value of $\tilde{F}$ along the flow.

massive fermion in AdS. We calculate the bulk and boundary two-point function of the fermion, and study possible boundary conditions and the AdS free energy for these boundary conditions. Then in section 5.2, we study large N $U(N)$ Gross-Neveu model and discuss the various phases we described above. In section 5.3, we describe these phases in the GN model in $d = 2 + \epsilon$ and in the GNY model in $d = 4 - \epsilon$. Finally, in section 5.4, we calculate the bulk two-point functions to leading order in ϵ in both GN and GNY models and compare the results with those of the large N expansion. In particular, following an approach proposed in [2], we use bulk equations of motion to derive a differential equation that the two-point function must satisfy, and then solve it to extract the bulk two-point function. This work was presented at the virtual conference called "Quantum field theory at the boundary" held at Mainz institue of theoretical physics in September 2021.

5.1 Free massive fermion on hyperbolic space

Let's start by reviewing some facts about free fermions on hyperbolic space to set the notation. This is mostly a review and the material is well discussed [192, 193, 194, 195, 196, 197, 198, 122, 199, 118, 208]. We start with the following Euclidean action

$$S = -\int d^d x \sqrt{g} \bar{\Psi} (\gamma \cdot \nabla + m) \Psi. \tag{5.5}$$

As discussed in [192, 209, 210], one needs to add a boundary term to this action to have a well defined variational principle. But we will not need it so we do not write it down explicitly. For the most part, we will use Poincaré coordinates $(x^0, x^i) = (z, \mathbf{x})$, $i = 1, ..., d-1$ with the metric

$$ds^2 = \frac{dz^2 + d\mathbf{x}^2}{z^2}. \tag{5.6}$$

In these coordinates, the vielbein $e_a^\mu = z\delta_a^\mu$, so that $\gamma \cdot \nabla = z\gamma^a \nabla_a$ with γ^a being the flat space gamma matrices. The spin connection and the Dirac operator take the following form

$$\omega_\mu^{ab} = \frac{\delta_0^a \delta_\mu^b - \delta_0^b \delta_\mu^a}{z}, \qquad \gamma \cdot \nabla \Psi = e_a^\mu \gamma^a \left(\partial_\mu + \frac{\omega_\mu^{bc} [\gamma_b, \gamma_c]}{8} \right) \Psi = \left(z\gamma^a \partial_a - \frac{(d-1)}{2} \gamma_0 \right) \Psi. \tag{5.7}$$

Also note that we are using 0 for the radial (z) direction, so $\bar{\Psi} = \Psi^\dagger \gamma^i$ with i being set equal to the Euclidean time direction.

At the boundary of the hyperbolic space, one can impose on the fermion two possible boundary conditions $\gamma_0 \Psi(z \to 0, \mathbf{x}) = \pm \Psi(z \to 0, \mathbf{x})$ or equivalently $(\bar{\Psi}\gamma_0 = \mp \bar{\Psi})$ which we will refer to as

+ and − boundary condition respectively. Let us first review the calculation of the fermion two-point function $\langle\Psi(x_1)\bar{\Psi}(x_2)\rangle = G_\Psi(x_1,x_2)$, which can be found by solving the following differential equation

$$(\gamma\cdot\nabla + m)G_\Psi(x_1,x_2) = -\delta^d(x_1-x_2). \tag{5.8}$$

To solve it, we start with the following ansatz [195, 198]

$$G_\Psi(x_1,x_2) = \left(-\frac{\gamma_0\gamma_a(\bar{x}_1-x_2)^a}{\sqrt{z_1z_2}}\frac{\alpha(\zeta)}{\sqrt{\zeta+4}} + \frac{\gamma_a(x_1-x_2)^a}{\sqrt{z_1z_2}}\frac{\beta(\zeta)}{\sqrt{\zeta}}\right) \tag{5.9}$$

where the cross-ratio ζ is defined by

$$\zeta = \frac{\mathbf{x}_{12}^2 + z_{12}^2}{z_1z_2} \tag{5.10}$$

and $\bar{x} = (-z,\mathbf{x})$ is the image point with respect to the boundary. We then act on the ansatz with the Dirac operator which gives

$$\begin{aligned}\gamma\cdot\nabla_1 G_\Psi(x_1,x_2) &= \left(z_1\gamma^a\partial_{1a} - \frac{(d-1)}{2}\gamma_0\right)G_\Psi(x_1,x_2)\\ &= \frac{\gamma_a x_{12}^a\sqrt{\zeta+4}}{\sqrt{z_1z_2}}\left(\alpha'(\zeta)+\frac{(d-1)}{2}\frac{\alpha(\zeta)}{\zeta+4}\right) - \frac{\gamma_0\gamma_a\bar{x}_{12}^a\sqrt{\zeta}}{\sqrt{z_1z_2}}\left(\beta'(\zeta)+\frac{(d-1)}{2}\frac{\beta(\zeta)}{\zeta}\right).\end{aligned} \tag{5.11}$$

Hence, the massive Dirac equation on this ansatz gives following set of coupled equations

$$\begin{aligned}\alpha'(\zeta)+\frac{d-1}{2}\frac{\alpha(\zeta)}{4+\zeta} &= -\frac{m\beta(\zeta)}{\sqrt{\zeta(4+\zeta)}}\\ \beta'(\zeta)+\frac{d-1}{2}\frac{\beta(\zeta)}{\zeta} &= -\frac{m\alpha(\zeta)}{\sqrt{\zeta(4+\zeta)}}.\end{aligned} \tag{5.12}$$

We can solve it by substituting for $\beta(\zeta)$ from the first equation into the second one, which gives a second order equation for $\alpha(\zeta)$. This has two solutions, which gives two choices of propagator corresponding to two possible boundary fall-offs. The first one has a leading boundary fermion of dimension $(d-1)/2+|m|$ in its boundary spectrum

$$\begin{aligned}G_\Psi(x_1,x_2) =& \frac{-\Gamma\left(\frac{d}{2}+|m|\right)}{\Gamma\left(\frac{1}{2}+|m|\right)2\pi^{\frac{d-1}{2}}}\left[\frac{\gamma_a x_{12}^a}{\sqrt{z_1z_2}}\frac{(4+\zeta)^{1-\frac{d}{2}}}{\zeta^{|m|+1}}{}_2F_1\left(1+|m|-\frac{d}{2},1+|m|,1+2|m|,-\frac{4}{\zeta}\right)\right.\\ &\left.-\operatorname{sgn}(m)\frac{\gamma_0\gamma_a(\bar{x}_{12})^a}{\sqrt{z_1z_2}}\frac{1}{\zeta^{|m|}(4+\zeta)^{\frac{d}{2}}}{}_2F_1\left(1+|m|-\frac{d}{2},|m|,1+2|m|,-\frac{4}{\zeta}\right)\right].\end{aligned} \tag{5.13}$$

This is allowed for all values of mass and is known in the literature as standard quantization. This satisfies the boundary condition $\gamma_0 G_\Psi(x_1,x_2)|_{z_1\to 0} = -\operatorname{sgn}(m)G_\Psi(x_1,x_2)|_{z_1\to 0}$. We can set z_1 or

$z_2 = 0$ to get the bulk-boundary two-point function of the fermion. Defining the boundary spinors of dimension $\hat{\Delta} = \frac{d-1}{2} + |m|$ as $\hat{\Psi}(\mathbf{x}) = z^{-\hat{\Delta}}\Psi(\mathbf{x}, z \to 0)$, we have

$$\begin{aligned}
\langle \hat{\Psi}(\mathbf{x}_1)\bar{\Psi}(x_2)\rangle &= -\left(\frac{1-\mathrm{sgn}(m)\gamma_0}{2}\right)\frac{(\gamma_a x_{12}^a)\Gamma\left(\hat{\Delta}+\frac{1}{2}\right)}{\sqrt{z_2}\pi^{\frac{d-1}{2}}\Gamma\left(\hat{\Delta}-\frac{d}{2}+1\right)}\left(\frac{z_2}{z_2^2+\mathbf{x}_{12}^2}\right)^{\hat{\Delta}+\frac{1}{2}} \\
\langle \Psi(x_1)\hat{\bar{\Psi}}(\mathbf{x}_2)\rangle &= -\frac{(\gamma_a x_{12}^a)\Gamma\left(\hat{\Delta}+\frac{1}{2}\right)}{\sqrt{z_1}\pi^{\frac{d-1}{2}}\Gamma\left(\hat{\Delta}-\frac{d}{2}+1\right)}\left(\frac{z_1}{z_1^2+\mathbf{x}_{12}^2}\right)^{\hat{\Delta}+\frac{1}{2}}\left(\frac{1+\mathrm{sgn}(m)\gamma_0}{2}\right).
\end{aligned} \tag{5.14}$$

The other possible boundary fall-off is when the leading boundary spinor has dimension $(d-1)/2-|m|$. This is only unitary for $|m| < 1/2$ and is known in the literature as alternative quantization. The corresponding two-point function is

$$\begin{aligned}
G_\Psi(x_1,x_2) =& \frac{-\Gamma\left(\frac{d}{2}-|m|\right)}{\Gamma\left(\frac{1}{2}-|m|\right)2\pi^{\frac{d-1}{2}}}\left[\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}}\frac{(4+\zeta)^{1-\frac{d}{2}}}{\zeta^{-|m|+1}}{}_2F_1\left(1-|m|-\frac{d}{2},1-|m|,1-2|m|,-\frac{4}{\zeta}\right)\right. \\
&\left. +\,\mathrm{sgn}(m)\frac{\gamma_0\gamma_a(\bar{x}_{12})^a}{\sqrt{z_1 z_2}}\frac{1}{\zeta^{-|m|}(4+\zeta)^{\frac{d}{2}}}{}_2F_1\left(1-|m|-\frac{d}{2},-|m|,1-2|m|,-\frac{4}{\zeta}\right)\right].
\end{aligned} \tag{5.15}$$

This satisfies the boundary condition $\gamma_0 G_\Psi(x_1,x_2)|_{z_1\to 0} = \mathrm{sgn}(m) G_\Psi(x_1,x_2)|_{z_1\to 0}$. In the massless limit, $m = 0$, the two cases become degenerate with the propagator given by [3]

$$G_\Psi(x_1,x_2) = -\frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}}\left[\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}}\frac{1}{\zeta^{\frac{d}{2}}} \pm \frac{\gamma_0\gamma_a(\bar{x}_{12})^a}{\sqrt{z_1 z_2}}\frac{1}{(4+\zeta)^{\frac{d}{2}}}\right] \tag{5.16}$$

which satisfies the boundary condition $\gamma_0 G_\Psi(x_1,x_2)|_{z_1\to 0} = \pm G_\Psi(x_1,x_2)|_{z_1\to 0}$.

5.1.1 Boundary correlation functions

In this subsection, we explain how to obtain correlation functions in the boundary theory from the bulk. As a first step, we need to take the boundary limit of (5.14)

$$\langle \hat{\Psi}(\mathbf{x}_1)\hat{\bar{\Psi}}(\mathbf{x}_2)\rangle = -\frac{\Gamma\left(\hat{\Delta}+\frac{1}{2}\right)}{2\pi^{\frac{d-1}{2}}\Gamma\left(\hat{\Delta}-\frac{d}{2}+1\right)}\frac{(1\pm\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12}}{(\mathbf{x}_{12}^2)^{\hat{\Delta}+\frac{1}{2}}}. \tag{5.17}$$

Note that the fermion operators on the boundary have half as many components as the ones in the bulk, because the boundary condition sets the other half to 0. So we need to project the above

[3] This is related, by a Weyl transformation, to the result in flat space written in [11], if we pick $U = \pm\gamma_0$ and $\bar{U} = \mp\gamma_0$.

two-point function onto the boundary fermion representation. When the bulk is even, the boundary fermions are Dirac fermions, while when the bulk is odd, the boundary fermions are Weyl. Let us start with the case when the bulk is even dimensional. For concreteness, let us choose the following representation of Dirac matrices

$$\gamma_0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \Gamma_i \\ \Gamma_i & 0 \end{pmatrix} \tag{5.18}$$

where Γ_i are the Dirac matrices in $d-1$ dimensions and $\mathbb{I}$ is the $c_{d-1} \times c_{d-1}$ dimensional identity. We defined $c_d = 2^{\lfloor \frac{d}{2} \rfloor}$ as the number of components of a Dirac spinor in d dimensions. We now only restrict to + boundary condition on the fermion and $m > 0$ (the other cases are identical), in which case, we can choose the following bulk polarization spinors

$$S = \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \bar{S} = \begin{pmatrix} \bar{v} & 0 \end{pmatrix} \tag{5.19}$$

where v is the boundary polarization spinor. We can then define the boundary fermion operator by $\bar{v}\psi(\mathbf{x}) = \bar{S}\hat{\Psi}(\mathbf{x})$. Contracting the two-point function with these polarization spinors, we get

$$\langle \bar{v}_1 \psi(\mathbf{x}_1) \bar{\psi}(\mathbf{x}_2) v_2 \rangle = \langle \bar{S}_1 \hat{\Psi}(\mathbf{x}_1) \hat{\bar{\Psi}}(\mathbf{x}_2) S_2 \rangle = -\frac{\Gamma\left(\hat{\Delta} + \frac{1}{2}\right)}{\pi^{\frac{d-1}{2}} \Gamma\left(\hat{\Delta} - \frac{d}{2} + 1\right)} \frac{\bar{v}_1 \mathbf{\Gamma} \cdot \mathbf{x}_{12} v_2}{(\mathbf{x}_{12}^2)^{\hat{\Delta} + \frac{1}{2}}}. \tag{5.20}$$

We can then differentiate with respect to boundary polarization spinors to get the correlation functions on the boundary

$$\langle \psi(\mathbf{x}_1) \bar{\psi}(\mathbf{x}_2) \rangle = -\frac{\Gamma\left(\hat{\Delta} + \frac{1}{2}\right)}{\pi^{\frac{d-1}{2}} \Gamma\left(\hat{\Delta} - \frac{d}{2} + 1\right)} \frac{\mathbf{\Gamma} \cdot \mathbf{x}_{12}}{(\mathbf{x}_{12}^2)^{\hat{\Delta} + \frac{1}{2}}}. \tag{5.21}$$

In the free theory, the higher point functions can then be just constructed by Wick contractions. However, when the bulk has additional interactions, as we will show in section 5.3, we should start with fermions in the bulk representation, and then project onto the boundary fermion representation.

We now comment on what happens when the bulk is odd dimensional. In this case, the Dirac matrices on the boundary have the same dimension as the bulk and are just given by the bulk gamma matrices γ_i with γ_0 being the chirality matrix. The boundary fermion operator is a Weyl fermion and we can take it to be just $\hat{\Psi}(\mathbf{x})$ satisfying $\gamma_0 \hat{\Psi}(\mathbf{x}) = \pm \hat{\Psi}(\mathbf{x})$. The two-point function is given by (5.17). An immediate consequence of the fact that the fermion is Weyl is that for a single

Dirac fermion in the bulk, the leading boundary scalar $\hat{\bar{\Psi}}\hat{\Psi}(\mathbf{x})$ vanishes. So the leading boundary scalar should have dimension $d-2m$ instead of $d-1-2m$ and should include a derivative.

5.1.2 Free energy

Next we calculate the free energy on hyperbolic space. To do that, we compactify the boundary of the hyperbolic space to a sphere. The free energy is then given by the following trace

$$F = -\mathrm{tr}\log\left(\gamma\cdot\nabla + m\right). \tag{5.22}$$

We need to know the spectrum of Dirac operator on hyperbolic space [211, 158]. The eigenvalues of $\gamma\cdot\nabla$ are $\pm i\lambda$ with the degeneracy given by

$$\mu(\lambda) = \frac{\mathrm{Vol}(H^d)c_d}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left|\frac{\Gamma\left(\frac{d}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}+i\lambda\right)}\right|^2. \tag{5.23}$$

The free energy does not depend on the sign of m, so we will just take $m>0$ for this calculation. Using the above results, the free energy is given by the following spectral integral

$$\begin{aligned} F &= -\frac{\mathrm{Vol}(H^d)c_d}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_0^\infty d\lambda \frac{|\Gamma\left(\frac{d}{2}+i\lambda\right)|^2\log\left(\lambda^2+m^2\right)}{|\Gamma\left(\frac{1}{2}+i\lambda\right)|^2} \\ &= \frac{\mathrm{Vol}(H^d)c_d}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\frac{\partial}{\partial\alpha}\left[\int_0^\infty d\lambda \frac{|\Gamma\left(\frac{d}{2}+i\lambda\right)|^2}{|\Gamma\left(\frac{1}{2}+i\lambda\right)|^2\left(\lambda^2+m^2\right)^\alpha}\right]\Bigg|_{\alpha\to 0} \end{aligned} \tag{5.24}$$

The above integral is hard to do analytically for arbitrary d, but can be performed if we plug in $d=3$

$$F(\hat{\Delta}) = \frac{\mathrm{Vol}(H^3)(\hat{\Delta}-1)\left(4(\hat{\Delta}-1)^2-3\right)}{24\pi}. \tag{5.25}$$

where we wrote the answer in terms of $\hat{\Delta} = (d-1)/2+m$. Even though we used the $+$ sign, the final result can be analytically continued for both $\hat{\Delta}=(d-1)/2\mp m$. For the free massless fermion, $\hat{\Delta}=1$, so $F=0$. For $d\neq 3$, the integral can be performed if we first take a derivative with m

$$\begin{aligned} \frac{\partial F}{\partial m} &= -\mathrm{tr}\left(\frac{1}{\gamma\cdot\nabla+m}\right) = -\frac{\mathrm{Vol}(H^d)mc_d}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_{-\infty}^\infty d\lambda \frac{|\Gamma\left(\frac{d}{2}+i\lambda\right)|^2}{|\Gamma\left(\frac{1}{2}+i\lambda\right)|^2\left(\lambda^2+m^2\right)} \\ &= -\frac{\mathrm{Vol}(H^d)c_d\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+m\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(1-\frac{d}{2}+m\right)}. \end{aligned} \tag{5.26}$$

We did the above integral by closing the contour in the upper half λ plane and summing over the residues at $\lambda = im$ and at $\lambda = i(d/2+n)$ for $n\geq 0$. The arc at infinity can only be dropped for

$d < 2$, but the final result can be analytically continued to $d > 2$. This trace can also be obtained by taking the short distance limit and tracing over the two-point function in (5.13).

As a side remark, we note that for the mass range $0 < |m| < 1/2$ where both the boundary conditions are allowed, they are related by a RG flow on the boundary triggered by a fermionic bilinear [212, 209, 210, 199]. The fermion bilinear in alternate quantization is relevant with scaling dimension $d - 1 - 2m$ and may be used to flow to the standard quantization [210] [4]. There is a general formula for the free energy change under a flow by the square of a spin 1/2 single-trace operator in a CFT that obeys large N factorization [142, 209, 199] [5]

$$F_{d-1-\hat{\Delta}} - F_{\hat{\Delta}} = -\frac{c_d}{\sin\left(\frac{\pi(d-1)}{2}\right)\Gamma(d)} \int_0^{\hat{\Delta}-\frac{d-1}{2}} du \cos(\pi u)\Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right). \tag{5.27}$$

In the AdS/CFT context, this corresponds to the difference in free energy between the same bulk theory with the two possible boundary conditions for the bulk fermion dual to the boundary single-trace operator. Even though in our case the flow between the two boundary conditions in the free fermion theory is not a double-trace flow in the usual sense, mathematically the problem is equivalent and we can still calculate the free energy difference between the two boundary conditions using the above formula. In $d = 3$, it gives

$$F_{2-\hat{\Delta}} - F_{\hat{\Delta}} = -\frac{\mathrm{Vol}(H^3)(\hat{\Delta}-1)(1+4\hat{\Delta}(\hat{\Delta}-2))}{12\pi} \tag{5.28}$$

where we used the fact that the regularized volume of hyperbolic space is given by $\mathrm{Vol}(H^d) = \pi^{\frac{d-1}{2}}\Gamma\left(\frac{1-d}{2}\right)$. This agrees with what we get by using the explicit result for $d = 3$ free energy (5.25).

As was discussed in [2] and also in previous chapter, the free energy on hyperbolic space is also related to the trace anomaly coefficients. In $d = 3$, on manifolds with a boundary, the trace anomaly is given by [165, 132, 167]

$$\langle T^{\mu}{}_{\mu}\rangle^{d=3} = \frac{\delta(x_\perp)}{4\pi}\left(a_{3d}\hat{\mathcal{R}} + b\mathrm{tr}\hat{K}^2\right). \tag{5.29}$$

In the above equation, $\hat{\mathcal{R}}$ is the boundary Ricci scalar and $\hat{K}_{ij}$ is the traceless part of the extrinsic curvature associated to the boundary. Following the logic in [2], it can be shown that for free massive

[4] This flow is not possible for a single bulk fermion in odd dimensions. Because in that case, the boundary fermion is Weyl, and hence the leading bilinear scalar vanishes.

[5] Note that when the bulk is odd dimensional, our formula differs from that of [142] by a factor of 2. This is because the result in [142] is given for a Dirac fermion, whereas in our case, when bulk is odd, the boundary condition forces the boundary spinor to be a Weyl fermion, which has half the number of components as that of a Dirac fermion.

fermions, the coefficient a_{3d} is given by

$$a_{3d} = \frac{(\hat{\Delta}-1)\left(4(\hat{\Delta}-1)^2-3\right)}{24}. \tag{5.30}$$

This vanishes for massless fermions, in agreement with the results in [137]. In what follows, we will also calculate this anomaly coefficient for large N interacting fixed points. It should also be possible to extract this coefficient from the fermion free energy on a round ball, which was calculated for free fermions in [213, 148].

5.2 Large N Gross-Neveu model

In this section, we study the Gross-Neveu model for N interacting Dirac fermions in AdS [118] and do perturbation theory in $1/N$. Starting with the action (5.2), we can integrate out the fermions to get an effective action in terms of σ

$$Z = e^{-F} = \int [d\sigma][d\Psi^i][d\bar{\Psi}_i] e^{\int d^d x \sqrt{g_x}\left(\bar{\Psi}_i \gamma\cdot\nabla\Psi^i + \sigma\bar{\Psi}_i\Psi^i\right)} = \int [d\sigma] \exp\left(N \mathrm{tr}\log(\gamma\cdot\nabla+\sigma(x))\right). \tag{5.31}$$

At leading order in large N, the path integral over σ may be performed by a saddle point approximation, assuming a constant saddle at $\sigma(x) = \sigma^*$. This constant can be found by solving

$$\frac{\partial F}{\partial \sigma^*} = -N \mathrm{tr}\left[\frac{1}{\gamma\cdot\nabla+\sigma^*}\right] = 0. \tag{5.32}$$

It is clear that at this order, σ^* acts like a mass for the fermions. The trace above can be obtained from the two-point function (5.13) or (5.15) depending upon the boundary fall-off for the fermion. For the boundary condition $\gamma_0 G_\Psi(x_1,x_2)|_{z_1\to 0} = -\mathrm{sgn}(\sigma^*) G_\Psi(x_1,x_2)|_{z_1\to 0}$, using (5.13)

$$\frac{\partial F}{\partial \sigma^*} = -N \mathrm{tr}\left[\frac{1}{\gamma\cdot\nabla+\sigma^*}\right] = -\frac{N \mathrm{sgn}(\sigma^*) c_d \mathrm{Vol}(H^d)}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+|\sigma^*|\right)}{\Gamma\left(1-\frac{d}{2}+|\sigma^*|\right)}. \tag{5.33}$$

This gives the following large N saddle

$$|\sigma^*| = \frac{d}{2} - 1 - n \implies \hat{\Delta} = d - \frac{3}{2} - n \tag{5.34}$$

for a non-negative integer n. For $2 < d < 4$, there is only one possible solution

$$|\sigma^*| = \frac{d}{2} - 1 \implies \hat{\Delta} = d - \frac{3}{2}. \tag{5.35}$$

The unitarity bound at the boundary requires $\hat{\Delta} \geq (d-2)/2$, which is satisfied for all $2 < d < 4$. This is the phase we called B_1 in the introduction.

For the other boundary condition $\gamma_0 G_\Psi(x_1, x_2)|_{z_1 \to 0} = \text{sgn}(\sigma^*) G_\Psi(x_1, x_2)|_{z_1 \to 0}$, we have, using (5.15)

$$\frac{\partial F}{\partial \sigma^*} = -N \text{tr}\left[\frac{1}{\gamma \cdot \nabla + \sigma^*}\right] = \frac{N \text{sgn}(\sigma^*) c_d \text{Vol}(H^d)}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(1 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - |\sigma^*|\right)}{\Gamma\left(1 - \frac{d}{2} - |\sigma^*|\right)}. \tag{5.36}$$

This gives the following large N saddle

$$|\sigma|^* = -\frac{d}{2} + 1 + n \implies \hat{\Delta} = d - \frac{3}{2} - n \tag{5.37}$$

for a positive integer n. There is no unitary saddle for $d < 3$, while in $3 \leq d \leq 4$, $n = 1$ gives a unitary saddle

$$|\sigma^*| = 2 - \frac{d}{2} \implies \hat{\Delta} = d - \frac{5}{2}. \tag{5.38}$$

This is the saddle for both B_2 and B_2' phases, and as we show below, the two phases can only be distinguished by the σ fluctuations around this saddle which are subleading in $1/N$.

Let us also write explicitly, the fermion two-point function for the two cases. Plugging in $|\sigma^*| = \frac{d}{2} - 1$ into (5.13), we get

$$G_\Psi^{|\sigma^*| = \frac{d}{2} - 1}(x_1, x_2) = -\frac{2^{d-3} \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}} \left(\zeta(4+\zeta)\right)^{\frac{d}{2}} \sqrt{z_1 z_2}} \left(\gamma \cdot (x_1 - x_2)(4+\zeta) - \text{sgn}(\sigma^*) \gamma_0 \, \gamma \cdot (\bar{x}_1 - x_2)\zeta\right). \tag{5.39}$$

As we show below, in $d = 2 + \epsilon$, this saddle should match with the calculation in an ϵ expansion in Gross-Neveu model. The negative value of σ^*, i.e. $\sigma^* = 1 - \frac{d}{2}$ matches the ϵ expansion calculation if we do perturbation theory around free theory with a + boundary condition on the fermion, and similarly for the other sign. This is consistent with the boundary condition obeyed by the propagator we write here i.e. a negative σ^* gives a + boundary condition for the fermion and vice versa. In $d = 4 - \epsilon$, this saddle matches to a phase in Gross-Neveu-Yukawa model where the scalar gets a classical vev.

For the other saddle, we plug in $|\sigma^*| = 2 - \frac{d}{2}$ into (5.15), and we get

$$G_{\Psi}^{|\sigma^*|=2-\frac{d}{2}}(x_1,x_2) = -\frac{2^{d-5}\Gamma\left(\frac{d}{2}-1\right)}{\pi^{\frac{d}{2}}\left(\zeta(4+\zeta)\right)^{\frac{d}{2}}\sqrt{z_1 z_2}}\Big[\gamma\cdot(x_1-x_2)(4+\zeta)\left(d(2+\zeta)-3\zeta-4\right)+ \\ \operatorname{sgn}(\sigma^*)\gamma_0\ \gamma\cdot(\bar{x}_1-x_2)\zeta\left(d(2+\zeta)-3\zeta-8\right)\Big]. \tag{5.40}$$

In $d = 4-\epsilon$, this saddle matches with the ϵ expansion calculation in GNY model where the scalar does not get a classical vev. The positive value of $\sigma^* = 2 - \frac{d}{2}$ matches the perturbation theory around the free theory with + boundary condition. This again, is consistent with the propagator we write here. Note that the two signs of σ^* give two essentially equivalent theories. They only differ by the signs of one-point functions of parity-odd operators.

5.2.1 σ fluctuations

In this subsection, we consider fluctuations about the constant σ saddles that we found above. So we expand the effective action in (5.31) about the constant σ background $\sigma(x) = \sigma^* + \delta\sigma(x)$

$$\begin{aligned} S_{\text{eff}}(\sigma) &= -N\text{tr}\log\left(\gamma\cdot\nabla+\sigma^*+\delta\sigma(x)\right) = -N\text{tr}\log\left(\gamma\cdot\nabla+\sigma^*\right)+\frac{N}{2}\text{tr}\left(\frac{\delta\sigma}{\gamma\cdot\nabla+\sigma^*}\right)^2 \\ &= -N\text{tr}\log\left(\gamma\cdot\nabla+\sigma^*\right)+\frac{N}{2}\int d^dx d^dy\sqrt{g_x}\sqrt{g_y}\ \text{Tr}\left[G_\Psi(x,y)G_\Psi(y,x)\right]\delta\sigma(x)\delta\sigma(y) \end{aligned} \tag{5.41}$$

where Tr is the trace over the fermionic indices while tr includes trace over both spacetime and fermionic indices. The σ propagator can then be read off from the inverse of the quadratic piece

$$\int d^dx_3\sqrt{g}\ \text{Tr}\left[G_\Psi(x_1,x_3)G_\Psi(x_3,x_1)\right]G_\sigma(x_3,x_2) = \frac{1}{N}\frac{\delta^d(x_1-x_2)}{\sqrt{g_{x_1}}}. \tag{5.42}$$

For $\sigma^* = d/2-1$, we need to invert

$$\text{Tr}\left[G_\Psi(x_1,x_3)G_\Psi(x_3,x_1)\right] = -\frac{4^d\Gamma\left(\frac{d}{2}\right)^2 c_d}{16\pi^d\left(\zeta(4+\zeta)\right)^{d-1}} \tag{5.43}$$

while for $\sigma^* = d/2-2$, we need to find the inverse of

$$\text{Tr}\left[G_\Psi(x_1,x_3)G_\Psi(x_3,x_1)\right] = -\frac{4^d\Gamma\left(\frac{d}{2}-1\right)^2 c_d}{64\pi^d\left(\zeta(4+\zeta)\right)^{d-1}}\left((d-2)^2+\frac{(d-1)(d-3)}{4}\zeta(4+\zeta)\right). \tag{5.44}$$

We give the details of this inversion in the appendix 5.5 and just report the result here. For $\sigma^* = d/2 - 1$ i.e. B_1 phase, we find (5.178) [6]

$$G_\sigma(\zeta) = -\frac{2^{4d-5}(d-2)\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(d)}{Nc_d\pi\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\Gamma(2d-2)\zeta^d}\ {}_2F_1\left(d, d-1, 2d-2, -\frac{4}{\zeta}\right). \tag{5.45}$$

The two-point function of a scalar operator O in a BCFT can be expanded into bulk and boundary channel conformal blocks as [12, 21]

$$G_O(\zeta) = \frac{A}{\zeta^{\Delta_O}}\left(1 + \sum_k \lambda_k f_{\text{bulk}}\left(\Delta_k;\zeta\right)\right) = A\left(a_O^2 + \sum_l \mu_l^2 f_{\text{bdry}}(\hat{\Delta}_l, \xi)\right) \tag{5.46}$$

where A is the normalization of the operator. The blocks are known to be

$$\begin{aligned} f_{\text{bulk}}(\Delta_k;\zeta) &= \left(\frac{\zeta}{4}\right)^{\frac{\Delta_k}{2}}\ {}_2F_1\left(\frac{\Delta_k}{2}, \frac{\Delta_k}{2}; \Delta_k + 1 - \frac{d}{2}; -\frac{\zeta}{4}\right) \\ f_{\text{bdry}}(\hat{\Delta}_l;\zeta) &= \left(\frac{4}{\zeta}\right)^{\hat{\Delta}_l}\ {}_2F_1\left(\hat{\Delta}_l, \hat{\Delta}_l + 1 - \frac{d}{2}; 2\hat{\Delta}_l + 2 - d; -\frac{4}{\zeta}\right). \end{aligned} \tag{5.47}$$

Expanding the two-point function (5.45) in powers of $1/\zeta$ tells us that the boundary spectrum consists of operators of dimension $d + 2n$ with OPE coefficients given by

$$\mu_{d+2n}^2 = \frac{\sqrt{\pi}2^{-2d-4n+1}\Gamma\left(-\frac{d}{2}+n+2\right)\Gamma\left(\frac{d+1}{2}+n\right)\Gamma(d+2n)}{\Gamma\left(2-\frac{d}{2}\right)\Gamma(n+1)\Gamma\left(d+n-\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}+2n\right)}. \tag{5.48}$$

The $n = 0$ operator with $\hat{\Delta}_{(0)} = d$ corresponds to the displacement operator. In $d = 3$, the OPE coefficients simplify to [7]

$$\mu_{3+2n}^2 = \frac{(n+1)^2}{4^{1+2n}(4n(2+n)+3)}. \tag{5.49}$$

Note that there is no relevant scalar in the boundary theory in this phase. Hence this phase is the most stable one in the RG sense and must be at the end of the boundary RG flow, consistent with what we wrote in the introduction. In the bulk channel, the operators that appear are even powers of σ, i.e. σ^{2k} with dimensions $2k$. The two-point function in the bulk OPE limit goes like

$$\langle\sigma(x_1)\sigma(x_2)\rangle = (\sigma^*)^2 - \frac{2^d\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\pi^{3/2}c_d N\Gamma\left(\frac{d}{2}-1\right)\zeta} - \frac{\Gamma(d+1)}{c_d N\Gamma\left(\frac{d}{2}-1\right)^2\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)}\log\zeta + \ldots \tag{5.50}$$

where the subleading terms are suppressed in the $\zeta \to 0$ limit. The $\log\zeta$ terms appear because

[6] Note that this is only the connected piece of the σ two-point function, so that the complete two-point function is $(\sigma^*)^2 + G_\sigma$

[7] It does not quite agree with the result in [118]. We suspect this may be due to a different definition of the coefficients, or possibly a typo.

the σ^2 operator already appears at leading order in N. So at order $1/N$, we expect the anomalous dimension of σ^2 to appear, which gives rise to the logarithm. From the structure of the OPE, the coefficient of the log should be related to the anomalous dimension as follows

$$(\sigma^*)^2\left(\frac{\gamma_{\sigma^2}}{2}-\gamma_\sigma\right)=-\frac{\Gamma(d+1)}{c_d N\Gamma\left(\frac{d}{2}-1\right)^2\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)}. \tag{5.51}$$

The bulk anomalous dimensions of σ^2 and σ operators for the large N Gross-Neveu model are known [203, 214, 215, 216] and they satisfy the above relation, providing a non-trivial check of our results.

For $|\sigma^*|=d/2-2$, as we explain in appendix 5.5, we have two choices for the σ propagator. The first one has the following correlator (5.195)

$$\begin{aligned}G^D_\sigma(\zeta)=-\mathcal{B}\Bigg[&\frac{2^{2-d}\cos\left(\frac{\pi d}{2}\right)}{(d-5)\Gamma\left(\frac{d-1}{2}\right)}\frac{1}{(4+\zeta)^2}{}_2F_1\left(2,3-\frac{d}{2},6-d,\frac{4}{4+\zeta}\right)\\&+\frac{\Gamma(d)}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)}\frac{\zeta+2}{(\zeta(4+\zeta))^{\frac{d+1}{2}}}{}_3F_2\left(\frac{d+1}{2},\frac{d-1}{2},\frac{3}{2};d-\frac{3}{2},\frac{5}{2};-\frac{4}{\zeta(4+\zeta)}\right)\Bigg]\end{aligned} \tag{5.52}$$

where $\mathcal{B}$ is a dimension dependent constant defined in (5.181). The boundary spectrum in this phase consists of a leading scalar of dimension 2 and then a tower of operators of dimension $d+2n$ with the following OPE coefficients (the $n=0$ member of this tower should be, as above, the displacement operator)

$$\begin{aligned}\hat{\Delta}_l=\{2,d,d+2,d+4,d+6,...\},\qquad \mu_2^2=-\frac{\sqrt{\pi}2^{1-d}\cos\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{(d-5)\Gamma\left(\frac{d-1}{2}\right)}\\ \mu^2_{d+2n}=\frac{\sqrt{\pi}(-1)^{n+1}2^{3-2(d+2n)}(d+2n-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d+1}{2}+n\right)\Gamma(d+2n-2)}{(2n+3)\Gamma(n+1)\Gamma\left(\frac{d}{2}-n-2\right)\Gamma\left(d+n-\frac{3}{2}\right)\Gamma\left(\frac{d+1}{2}+2n\right)}.\end{aligned} \tag{5.53}$$

This is the phase we called B_2 in the introduction and we use superscript D to indicate that it matches on to GNY model with Dirichlet boundary condition on the scalar s. The dimension 2 scalar operator we find here is relevant for $d>3$ and may be turned on to flow to the B_1 phase.

The σ propagator for the B_2' phase is (5.196)

$$\begin{aligned}G^N_\sigma(\zeta)=-\mathcal{B}\Bigg[&-\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{d}{2}-1\right)}{8\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\frac{1}{(4+\zeta)^{d-3}}{}_2F_1\left(d-3,\frac{d}{2}-2,d-4,\frac{4}{4+\zeta}\right)\\&+\frac{\Gamma(d)}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)}\frac{\zeta+2}{(\zeta(4+\zeta))^{\frac{d+1}{2}}}{}_3F_2\left(\frac{d+1}{2},\frac{d-1}{2},\frac{3}{2};d-\frac{3}{2},\frac{5}{2};-\frac{4}{\zeta(4+\zeta)}\right)\Bigg].\end{aligned} \tag{5.54}$$

The boundary spectrum and the bulk-boundary OPE coefficients are the same as the B_2 phase, apart from the leading boundary scalar, which had dimensions $d-3$ instead of 2 and the OPE

coefficient

$$\mu_{d-3}^2 = \frac{\pi 4^{3-d}\Gamma\left(\frac{d}{2}-1\right)^2}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}. \tag{5.55}$$

The relevant operator $\hat{\sigma}^2$ of dimensions $2d-6$ drives the flow from the B_2' to B_2 phase. The bulk spectrum in B_2 and B_2' phases is of course the same as in B_1 phase. The two-point function in the bulk OPE limit still contains a $\log\zeta$ whose coefficient is related to the bulk anomalous dimension of the σ^2 operator, as we saw in (5.51) for the B_1 phase. We now expand these propagators in $d = 4-\epsilon$

$$\begin{aligned} G_\sigma^N &= \frac{\epsilon}{N}\left(\frac{1}{\zeta}+\frac{1}{4+\zeta}\right) - \frac{\epsilon^2}{N}\left(\frac{1}{\zeta}-\frac{1}{4+\zeta}\right) + \frac{\epsilon^2}{N}\left(\frac{1}{\zeta}+\frac{1}{4+\zeta}\right)\log\left(1+\frac{\zeta}{4}\right) + O(\epsilon^3) \\ G_\sigma^D &= \frac{\epsilon-\epsilon^2}{N}\left(\frac{1}{\zeta}-\frac{1}{4+\zeta}\right) + O(\epsilon^3). \end{aligned} \tag{5.56}$$

As we will see in section 5.4, these exactly match the correlator of s in GNY model, once we normalize the operators in the same way. Note that in $d = 3$ the dimension of the leading boundary scalar induced by σ becomes zero at large N. This may indicate that the B_2' boundary conformal phase may not survive in $d = 3$, though it is present in the range $3 < d < 4$. It would be interesting to clarify this.

5.2.2 Free Energy

In this subsection, we calculate the AdS free energy at the large N boundary fixed points we discussed. At leading order, the one-point function of σ acts as a mass for fermions, so we can use the results from section 5.1. For $d = 3$, we can just use the general formula (5.25). For the two phases, we get

$$F(3/2) = -\frac{N\mathrm{Vol}(H^3)}{24\pi}, \qquad F(1/2) = \frac{N\mathrm{Vol}(H^3)}{24\pi}. \tag{5.57}$$

The value of the trace anomaly coefficient for these phases is (5.30)

$$a_{3d}(3/2) = -\frac{N}{24}, \qquad a_{3d}(1/2) = \frac{N}{24}. \tag{5.58}$$

For other values of d, the free energy can be calculated in terms of some reference value, say the

free energy of free massless fermions. For the B_1 phase, using (5.33), we have

$$\begin{aligned} F_{\sigma^*=d/2-1} &= F[\sigma^*=0] \ + \ \int_0^{d/2-1} d\sigma \frac{\partial F}{\partial \sigma} \\ &= NF_{\text{Free}} \ - \ \frac{Nc_d \text{Vol}(H^d)\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^{d/2-1} d\sigma \frac{\Gamma\left(\frac{d}{2}+\sigma\right)}{\Gamma\left(1-\frac{d}{2}+\sigma\right)} \end{aligned} \tag{5.59}$$

where F_{Free} is the free energy of a single free massless fermion on H^d. It is easy to see that the free energy itself does not depend on the sign of σ^*, so we restrict ourselves to positive σ^* in this section. In $d = 2+\epsilon$, this has the following expression

$$F_{\sigma^*=d/2-1}\big|_{d=2+\epsilon} = NF_{\text{Free}} - \frac{N\text{Vol}(H^2)}{8\pi}\epsilon + O(\epsilon^2) \tag{5.60}$$

while in $d = 4-\epsilon$, this gives

$$F_{\sigma^*=d/2-1}\big|_{d=4-\epsilon} = NF_{\text{Free}} \ - \ \frac{N\text{Vol}(H^4)}{8\pi^2\epsilon} + O(1). \tag{5.61}$$

For $\sigma^* = 2-d/2$, using (5.36) we have

$$\begin{aligned} F_{\sigma^*=2-d/2} &= F[\sigma^*=0] \ + \ \int_0^{2-d/2} d\sigma \frac{\partial F}{\partial \sigma} \\ &= NF_{\text{Free}} \ + \ \frac{Nc_d \text{Vol}(H^d)\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^{2-d/2} d\sigma \frac{\Gamma\left(\frac{d}{2}-\sigma\right)}{\Gamma\left(1-\frac{d}{2}-\sigma\right)} \end{aligned} \tag{5.62}$$

In $d = 4-\epsilon$, this is

$$F_{\sigma^*=2-d/2}\big|_{d=4-\epsilon} = NF_{\text{Free}} \ + \ \frac{N\text{Vol}(H^4)\epsilon}{16\pi^2}. \tag{5.63}$$

In the next section, we will match these with the calculation in ϵ expansion. As we mentioned in the introduction, we expect a RG flow from B_2 to B_1 phase, so we expect $\tilde{F}$ defined in (5.4) to be lower for the B_1 phase. It can be seen numerically that $\tilde{F}$ for $\sigma^* = d/2-1$ is lower than that for $\sigma^* = 2-d/2$. We plot the difference in $\tilde{F}$ between these phases in Figure 5.2.

There is also an RG flow from B_2' to B_2 phase, so we also expect $\tilde{F}$ for B_2 phase to be lower than that for B_2' phase. To calculate the free energy difference, we can think of the flow between the two as a double trace flow on the boundary triggered by a $\hat{\sigma}^2$ operator. The free energy change under an RG flow driven by the square of a scalar operator in a large N CFT is given by [144, 142, 2]

$$F_{d-1-\hat{\Delta}} - F_{\hat{\Delta}} = -\frac{1}{\sin\left(\frac{\pi(d-1)}{2}\right)\Gamma(d)} \int_0^{\hat{\Delta}-\frac{d-1}{2}} duu \sin \pi u \Gamma\left(\frac{d-1}{2}+u\right)\Gamma\left(\frac{d-1}{2}-u\right). \tag{5.64}$$

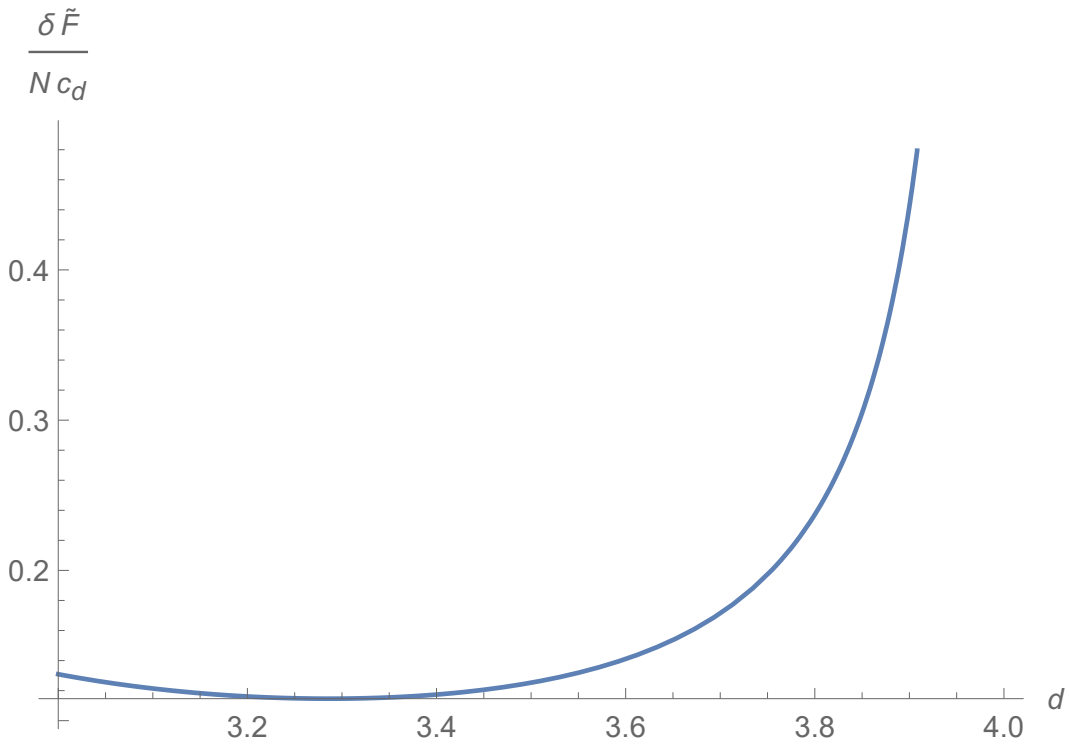


Figure 5.2: The free energy difference at large N between the two phases, $\delta\tilde{F} = \tilde{F}_{\sigma^*=2-d/2} - \tilde{F}_{\sigma^*=d/2-1}$ where both the phases exist, i.e. between $3 < d < 4$.

Applying it for $\hat{\Delta} = 2$, we get

$$F^N - F^D = F_{d-3} - F_2 = -\frac{1}{\sin\left(\frac{\pi(d-1)}{2}\right)\Gamma(d)} \int_0^{\frac{5-d}{2}} duu \sin \pi u \Gamma\left(\frac{d-1}{2}+u\right)\Gamma\left(\frac{d-1}{2}-u\right). \tag{5.65}$$

We can use this to calculate the difference in $\tilde{F} = -\sin\left(\frac{\pi(d-1)}{2}\right) F$ and check that $\tilde{F}_{d-3} - \tilde{F}_2$ is positive between $3 < d < 4$. In $4-\epsilon$, we get

$$\begin{aligned} F^N - F^D &= -\frac{1}{\sin\left(\frac{\pi(d-1)}{2}\right)\Gamma(d)} \int_0^{\frac{1}{2}} duu \sin \pi u \Gamma\left(\frac{d-1}{2}+u\right)\Gamma\left(\frac{d-1}{2}-u\right) + \frac{\epsilon}{24} + O(\epsilon^2) \\ &= \frac{\zeta(3)}{8\pi^2} + A\epsilon + O(\epsilon^2) \end{aligned} \tag{5.66}$$

and numerically, $A = 0.06122$. We will check this against an ϵ expansion calculation in GNY model in next section.

5.3 ϵ expansion

In this section, we study alternative descriptions of the above fixed points near $d = 2$ and $d = 4$. These results are valid for all N.

5.3.1 Gross-Neveu model in $d = 2 + \epsilon$

We start with Gross-Neveu model described in (5.1) and study it near two dimensions, where there is only one boundary phase B_1. There is a fixed point at

$$g^* = \frac{\pi}{N-1}\epsilon. \tag{5.67}$$

The σ operator in the large N theory is related, by equation of motion, to $\bar{\Psi}_i\Psi^i$ as

$$\sigma^* = g^*\langle\bar{\Psi}_i\Psi^i\rangle. \tag{5.68}$$

Taking the short distance limit of (5.16), we get

$$\langle\bar{\Psi}\Psi\rangle = \mp\frac{c_d\Gamma\left(\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \implies \sigma^* = \mp\frac{N}{2(N-1)}\epsilon. \tag{5.69}$$

At large N, this agrees with (5.35). The two possible signs of σ^* correspond to two different boundary conditions we can impose on the fermion, and define two equivalent theories. The free energy to leading order in ϵ in $d = 2 + \epsilon$ is given by

$$\begin{aligned} F &= NF_{\text{Free}} - \frac{g^*}{2}\text{Vol}(H^d)N\left(N-\frac{1}{2}\right)\langle\bar{\Psi}\Psi\rangle^2 \\ &= NF_{\text{Free}} - \frac{\text{Vol}(H^2)N(2N-1)}{16\pi(N-1)}\epsilon \end{aligned} \tag{5.70}$$

At large N, this matches the large N result in (5.60).

Let us now look at the spectrum of the boundary theory. One way to do this is to calculate the boundary correlation functions in the ϵ expansion. We will just do the calculation for the $+$ boundary condition on the fermion, but it goes exactly the same way for the other case. Let's start with the two-point function. In the free theory, it is given by (5.21) with $\hat{\Delta} = (d-1)/2$

$$\langle\psi^i(\mathbf{x}_1)\bar{\psi}_j(\mathbf{x}_2)\rangle_0 = -\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}}\frac{\delta^i_j\boldsymbol{\Gamma}\cdot\mathbf{x}_{12}}{(\mathbf{x}_{12}^2)^{\frac{d}{2}}}. \tag{5.71}$$

In the interacting Gross-Neveu model, the two point function receives corrections which can be calculated using the bulk tadpole Witten diagram. We will need bulk-boundary propagator (5.14), so we will calculate the interaction piece first when the fermions are in the bulk spinor representation, and then project onto the boundary representation

$$\begin{aligned}\langle\hat{\Psi}^i(\mathbf{x}_1)\hat{\bar{\Psi}}_j(\mathbf{x}_2)\rangle_1 = \quad & \\ &= -g\delta^i_j\left(N-\frac{1}{c_d}\right)\langle\bar{\Psi}\Psi\rangle\int d^dx\sqrt{g_x}\langle\hat{\Psi}(\mathbf{x}_1)\bar{\Psi}(x)\rangle\langle\hat{\bar{\Psi}}(\mathbf{x}_2)\Psi(x)\rangle.\end{aligned} \tag{5.72}$$

Using (5.14), we note that the product of two bulk-boundary propagators for fermions can be simplified as

$$\begin{aligned}\langle\hat{\Psi}(\mathbf{x}_1)\bar{\Psi}(x)\rangle\langle\hat{\bar{\Psi}}(\mathbf{x}_2)\Psi(x)\rangle &= \frac{\Gamma\left(\frac{d}{2}\right)^2}{4\pi^d}\frac{z^{d-1}(1+\gamma_0)\gamma_a(x_1-x)^a\gamma_b(x_2-x)^b(1-\gamma_0)}{((z^2+(\mathbf{x}_1-\mathbf{x})^2)(z^2+(\mathbf{x}_2-\mathbf{x})^2))^{\frac{d}{2}}} \\ &= \frac{\Gamma\left(\frac{d}{2}\right)^2}{2\pi^d}\frac{z^d(1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12}}{((z^2+(\mathbf{x}_1-\mathbf{x})^2)(z^2+(\mathbf{x}_2-\mathbf{x})^2))^{\frac{d}{2}}}\end{aligned} \tag{5.73}$$

Plugging in all the factors near $d=2$ gives, to leading order in ϵ

$$\begin{aligned}\langle\hat{\Psi}^i(\mathbf{x}_1)\hat{\bar{\Psi}}_j(\mathbf{x}_2)\rangle_1 &= \frac{\delta^i_j\epsilon(2N-1)(1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12}}{8\pi^2(N-1)}\int d^dx\frac{1}{((z^2+(\mathbf{x}_1-\mathbf{x})^2)(z^2+(\mathbf{x}_2-\mathbf{x})^2))^{\frac{d}{2}}} \\ &= \frac{\delta^i_j\epsilon(2N-1)(1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12}}{16\pi(N-1)}\int d\alpha dz\frac{1}{(z^2+\alpha(1-\alpha)\mathbf{x}_{12}^2)^{\frac{3}{2}}}.\end{aligned} \tag{5.74}$$

The integral above has a divergence at $z=0$, and it corresponds to an anomalous dimension for the boundary operator $\hat{\Psi}$. To calculate this anomalous dimension, we only need the logarithmic piece of the above integral, which can be extracted by regulating it as follows

$$\begin{aligned}\langle\hat{\Psi}^i(\mathbf{x}_1)\hat{\bar{\Psi}}_j(\mathbf{x}_2)\rangle_1 &= \frac{\delta^i_j\epsilon(2N-1)(1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12}}{16\pi(N-1)}\int d\alpha dz\frac{z^\eta}{(z^2+\alpha(1-\alpha)\mathbf{x}_{12}^2)^{\frac{3}{2}}} \\ &= \frac{\delta^i_j\epsilon(2N-1)(1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12}}{8\pi(N-1)\mathbf{x}_{12}^2}\left(\frac{2}{\eta}+\log\left(\mathbf{x}_{12}^2\right)-2\log 2+O(\eta)\right).\end{aligned} \tag{5.75}$$

Projecting it onto the boundary gives

$$\langle\psi^i(\mathbf{x}_1)\bar{\psi}_j(\mathbf{x}_2)\rangle_1 = \frac{\delta^i_j\epsilon(2N-1)\boldsymbol{\Gamma}\cdot\mathbf{x}_{12}}{4\pi(N-1)\mathbf{x}_{12}^2}\left(\frac{2}{\eta}+\log\left(\mathbf{x}_{12}^2\right)-2\log 2+O(\eta)\right). \tag{5.76}$$

The log piece gives us the anomalous dimension of the leading boundary fermion

$$\hat{\gamma} = \frac{(2N-1)}{4(N-1)}\epsilon, \implies \hat{\Delta} = \frac{d-1}{2} + \hat{\gamma} = \frac{1}{2} + \frac{4N-3}{4(N-1)}\epsilon. \tag{5.77}$$

This is consistent with the large N result of $d - 3/2$ (5.35).

Next, let us calculate the four-point function on the boundary. This should give anomalous dimensions of the scalar operators on the boundary, which are bilinears of the leading fermionic operator. In the free theory, the four-point function is given by Wick contractions of (5.21)

$$\begin{aligned}&\langle \bar{\psi}_{i,a}(\mathbf{x}_1)\psi^{j,b}(\mathbf{x}_2)\bar{\psi}_{k,c}(\mathbf{x}_3)\psi^{l,d}(\mathbf{x}_4)\rangle_0 = \\ &\frac{\Gamma\left(\frac{d}{2}\right)^2}{\pi^d}\left(\frac{\delta_i^j\delta_k^l(\mathbf{\Gamma}\cdot\mathbf{x}_{12})^b{}_a(\mathbf{\Gamma}\cdot\mathbf{x}_{12})^d{}_c}{(\mathbf{x}_{12}^2)^{\frac{d}{2}}(\mathbf{x}_{34}^2)^{\frac{d}{2}}} + \frac{\delta_i^l\delta_k^j(\mathbf{\Gamma}\cdot\mathbf{x}_{14})^d{}_a(\mathbf{\Gamma}\cdot\mathbf{x}_{23})^b{}_c}{(\mathbf{x}_{14}^2)^{\frac{d}{2}}(\mathbf{x}_{23}^2)^{\frac{d}{2}}}\right)\end{aligned} \tag{5.78}$$

where indices a, b, c, d are boundary spinor indices. We now restrict to two bulk dimensions, so that boundary is one-dimensional and the boundary gamma matrix is just 1

$$\begin{aligned}\langle \bar{\psi}_i(\mathbf{x}_1)\psi^j(\mathbf{x}_2)\bar{\psi}_k(\mathbf{x}_3)\psi^l(\mathbf{x}_4)\rangle_0 &= \delta_i^j\delta_k^l\mathcal{S}_{\text{sing}} + \left(\delta_i^l\delta_k^j - \frac{\delta_i^j\delta_k^l}{N}\right)\mathcal{S}_{\text{adj}} \\ &= \frac{\Gamma\left(\frac{d}{2}\right)^2}{\pi^d}\left(\frac{\delta_i^j\delta_k^l\mathbf{x}_{12}\mathbf{x}_{34}}{(\mathbf{x}_{12}^2)^{\frac{d}{2}}(\mathbf{x}_{34}^2)^{\frac{d}{2}}} + \frac{\delta_i^l\delta_k^j\mathbf{x}_{14}\mathbf{x}_{23}}{(\mathbf{x}_{14}^2)^{\frac{d}{2}}(\mathbf{x}_{23}^2)^{\frac{d}{2}}}\right)\end{aligned} \tag{5.79}$$

where we defined the $U(N)$ singlet and adjoint parts of the four-point function. For convenience, we now restrict to the configuration $\mathbf{x}_1 > \mathbf{x}_2 > \mathbf{x}_3 > \mathbf{x}_4$. The first term in the correlator above represents the contribution of the identity operator, while the second term contains contributions of operators appearing in the OPE of $\bar{\psi}(x_1)\psi(x_2)$ and can be decomposed into conformal blocks using [118]

$$\frac{1}{(\mathbf{x}_{14}\mathbf{x}_{14})^{2\hat{\Delta}}} = \frac{1}{(\mathbf{x}_{12}\mathbf{x}_{34})^{2\hat{\Delta}}}\sum_{n=0}^{\infty} c_n^2\mathcal{K}_{\hat{\Delta}_n}(\chi), \qquad \hat{\Delta}_n = 2\hat{\Delta} + n, \qquad \chi = \frac{\mathbf{x}_{12}\mathbf{x}_{34}}{\mathbf{x}_{13}\mathbf{x}_{24}}. \tag{5.80}$$

The intermediate scalar operators have the schematic form $\bar{\psi}\left(\partial\!\!\!/\right)^n\psi$ and the OPE coefficients and conformal blocks turn out to be [118]

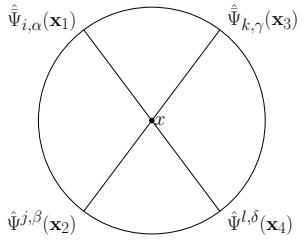
$$c_n^2 = \frac{4^{1-n}\hat{\Delta}\ (4\hat{\Delta})_{n-1}(2\hat{\Delta}+1)_{n-1}}{n!\left(2\hat{\Delta}+\frac{1}{2}\right)_{n-1}}, \qquad \mathcal{K}_{\hat{\Delta}_n}(\chi) = \chi^{\hat{\Delta}_n}{}_2F_1\left(\hat{\Delta}_n, \hat{\Delta}_n, 2\hat{\Delta}_n, \chi\right). \tag{5.81}$$

In the free theory, the dimensions of composite operators are just $\hat{\Delta}_n = (d-1) + n$, but in the

interacting theory, they get corrected to $\hat{\Delta}_n = 2\hat{\Delta} + n + \hat{\gamma}_n$.

The interaction term corrects this four-point function, which can be calculated using the bulk contact Witten diagram

$$\langle \hat{\bar{\Psi}}_{i,\alpha}(\mathbf{x}_1)\hat{\Psi}^{j,\beta}(\mathbf{x}_2)\hat{\bar{\Psi}}_{k,\gamma}(\mathbf{x}_3)\hat{\Psi}^{l,\delta}(\mathbf{x}_4)\rangle_1 = \quad (5.82)$$



$$= \frac{g}{2}\int d^dx\sqrt{g_x}\langle \hat{\bar{\Psi}}_{i,\alpha}(\mathbf{x}_1)\hat{\Psi}^{j,\beta}(\mathbf{x}_2)\hat{\bar{\Psi}}_{k,\gamma}(\mathbf{x}_3)\hat{\Psi}^{l,\delta}(\mathbf{x}_4)(\bar{\Psi}_m\Psi(x)^m)^2\rangle.$$

Here, the greek indices $\alpha, ...$ are bulk spinor indices. Plugging in the bulk-boundary propagators and using (5.73) gives the following integral for the contact interaction

$$\langle \hat{\bar{\Psi}}_{i,\alpha}(\mathbf{x}_1)\hat{\Psi}^{j,\beta}(\mathbf{x}_2)\hat{\bar{\Psi}}_{k,\gamma}(\mathbf{x}_3)\hat{\Psi}^{l,\delta}(\mathbf{x}_4)\rangle_1 = \frac{g\Gamma\left(\frac{d}{2}\right)^4}{4\pi^{2d}}\int d^dx z^{-d}\prod_{m=1}^{4}\left(\frac{z}{z^2+(\mathbf{x}_m-\mathbf{x})^2}\right)^{\frac{d}{2}}\times \quad (5.83)$$
$$\left[\delta_i^j\delta_k^l((1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{12})^\beta{}_\alpha((1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{34})^\delta{}_\gamma + \delta_i^l\delta_k^j((1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{14})^\delta{}_\alpha((1+\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{x}_{23})^\beta{}_\gamma\right].$$

We can contract this with bulk polarization spinors (5.19) and then differentiate with respect to boundary polarization spinors exactly as we did for the two-point function, to get the four-point function in the boundary spinor representation. The integral can be evaluated in terms of well known D-functions [217, 82] defined as the following AdS integral

$$D_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \int\frac{d^dx}{z^d}\prod_{i=1}^{4}\left(\frac{z}{z^2+(\mathbf{x}-\mathbf{x}_i)^2}\right)^{\Delta_i}. \quad (5.84)$$

In $d = 2$, the explicit expression for this D function can be worked out in terms of elementary functions (see for example [187, 188]). This gives the correction to the four-point function

$$\begin{aligned}\langle\bar{\psi}_i(\mathbf{x}_1)\psi^j(\mathbf{x}_2)\bar{\psi}_k(\mathbf{x}_3)\psi^l(\mathbf{x}_4)\rangle_1 &= \frac{g\Gamma\left(\frac{d}{2}\right)^4}{\pi^{2d}}\left(\delta_i^j\delta_k^l\mathbf{x}_{12}\mathbf{x}_{34}+\delta_i^l\delta_k^j\mathbf{x}_{14}\mathbf{x}_{23}\right)D_{\frac{d}{2},\frac{d}{2},\frac{d}{2},\frac{d}{2}}\\ &= -\frac{\epsilon\left(\chi\ \delta_i^j\delta_k^l+(1-\chi)\ \delta_i^l\delta_k^j\right)\chi}{2(N-1)\pi^2\mathbf{x}_{12}\mathbf{x}_{34}}\left(\frac{\log\chi}{1-\chi}+\frac{\log(1-\chi)}{\chi}\right).\end{aligned} \quad (5.85)$$

The $\log\chi$ piece above gives the anomalous dimensions of the composite operators. This is because in the conformal block decompostion, the $\log\chi$ comes from the derivative of the conformal block

with respect to the dimension

$$\partial_n \mathcal{K}_{\hat{\Delta}_n}(\chi) = \log(\chi)\chi^{\hat{\Delta}_n}{}_2F_1\left(\hat{\Delta}_n, \hat{\Delta}_n, 2\hat{\Delta}_n, \chi\right). \tag{5.86}$$

Hence the conformal block expansion contains

$$\sum_{n=0}^{\infty} c_n^2 \hat{\gamma}_n \log(\chi)\chi^{1+n}{}_2F_1\left(1+n, 1+n, 2+2n, \chi\right). \tag{5.87}$$

Comparing the log terms, we get the following anomalous dimensions in the $U(N)$ adjoint sector

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^2 \hat{\gamma}_n^{\text{adj}} \chi^{1+n}{}_2F_1\left(1+n, 1+n, 2+2n, \chi\right) &= -\frac{\epsilon\chi}{2(N-1)} \\ \hat{\gamma}_0^{\text{adj}} = -\frac{\epsilon}{2(N-1)} \implies \hat{\Delta}_0^{\text{adj}} &= 2\hat{\Delta} + \hat{\gamma}_0^{\text{adj}} = 1 + 2\epsilon \end{aligned} \tag{5.88}$$

where we used the corrected dimension of the boundary fermion operator from (5.77). The anomalous dimensions of all the other higher operators vanish. At large N, this matches $2\hat{\Delta}$ resulting from the large N calculation (5.35). This is what we expect because in the large N theory, the correction to $2\hat{\Delta}$ comes from the connected σ exchange diagram, which should be suppressed at large N in the adjoint sector. Similarly, in the singlet sector, we get the following equation to determine anomalous dimension

$$\sum_{n=0}^{\infty} c_n^2 \hat{\gamma}_n^{\text{sing}} \chi^{1+n}{}_2F_1\left(1+n, 1+n, 2+2n, \chi\right) = -\frac{\epsilon\chi}{2(N-1)}\left(1 + \frac{N\chi}{1-\chi}\right). \tag{5.89}$$

Expanding both sides in powers of χ gives anomalous dimensions of all the fermion bilinears in the OPE $\bar{\psi}$ and ψ. We just write the dimensions of the first two operators

$$\begin{aligned} \hat{\gamma}_0^{\text{sing}} = -\frac{\epsilon}{2(N-1)} &\implies \hat{\Delta}_0^{\text{sing}} = 2\hat{\Delta} + \hat{\gamma}_0^{\text{sing}} = 1 + 2\epsilon. \\ \hat{\gamma}_1^{\text{sing}} = -\frac{\epsilon(2N-1)}{2(N-1)} &\implies \hat{\Delta}_1^{\text{sing}} = 2\hat{\Delta} + 1 + \hat{\gamma}_1^{\text{sing}} = 2 + \epsilon \end{aligned} \tag{5.90}$$

where again, we used the corrected dimension of the boundary fermion operator from (5.77). The $n = 0$ operator is the leading singlet scalar operator on the boundary. The $n = 1$ operator is proportional to the displacement operator and has dimension $d = 2 + \epsilon$. We expect this dimension to stay protected to all orders in the perturbation theory. Also, this $n = 1$ singlet operator is the one that in the large N theory corresponds to $\hat{\sigma}$ and its dimension was referred to as $\hat{\Delta}_{(0)}$ in Table 5.1.

5.3.2 Gross-Neveu-Yukawa model in $d = 4 - \epsilon$

In $d = 4 - \epsilon$, the large N theory should match the Gross-Neveu-Yukawa model, which in hyperbolic space, may be described by the following action

$$S = \int d^d x \sqrt{g(x)} \left(\frac{(\partial_\mu s)^2}{2} - \frac{d(d-2)}{8} s^2 - \left(\bar{\Psi}_i \gamma \cdot \nabla \Psi^i + g_1 s \bar{\Psi}_i \Psi^i \right) + \frac{g_2}{24} s^4 \right) \tag{5.91}$$

where $i = 1, ..., N$, so we have N Dirac fermions. There is a fixed point at the following values of the couplings

$$\begin{aligned} (g_1^*)^2 &= \frac{(4\pi)^2}{4N+6}\epsilon \xrightarrow{\text{Large N}} \frac{4\pi^2}{N}\epsilon \\ g_2^* &= \frac{(4\pi)^2 \left(-2N+3+\sqrt{4N^2+132N+9}\right)}{3(4N+6)}\epsilon \xrightarrow{\text{Large N}} \frac{3(4\pi)^2}{N}\epsilon. \end{aligned} \tag{5.92}$$

The operator s in this description is proportional to the σ operator in the large N description. In the B_1 phase, s gets a vev in the classical theory. It appears naturally in the hyperbolic space as the minimum of the potential which occurs at

$$(s^*)^2 = \frac{3d(d-2)}{2g_2^*} \implies |\sigma^*| = |g_1^* s^*| = \sqrt{\frac{36}{-2N+3+\sqrt{4N^2+132N+9}}} + O(\epsilon) \tag{5.93}$$

in agreement with the large N result (5.35). We can expand the classical action around this vev $s = s^* + t$, to obtain an action for the fluctuations

$$\begin{aligned} S = &-\frac{3d^2(d-2)^2}{32 g_2} \int d^d x \sqrt{g(x)} + \int d^d x \sqrt{g(x)} \Big(\frac{(\partial_\mu t)^2}{2} + \frac{d(d-2)}{4} t^2 \\ &- \left(\bar{\Psi}_i \left(\gamma \cdot \nabla + g_1 s^* \right) \Psi^i + g_1 t \bar{\Psi}_i \Psi^i \right) + \frac{g_2}{6} s^* t^3 + \frac{g_2}{24} t^4 \Big). \end{aligned} \tag{5.94}$$

Note that the fermion becomes massive now, with a mass given by $g_1 s^*$. According to our discussion in 5.1, it leads to two possible boundary spinors with dimensions given by

$$\hat{\Delta} = \frac{d-1}{2} \mp |g_1^* s^*|. \tag{5.95}$$

It is easy to see from (5.93) that $|g_1^* s^*| > 1$, so only the plus sign above is consistent with the boundary unitarity bound which requires boundary fermions to have dimensions greater than or equal to $(d-2)/2$. At large N, it gives a $\hat{\Delta}$ which is consistent with (5.35). The fermion satisfies the boundary condition $\gamma_0 \Psi(z \to 0, \mathbf{x}) = -\text{sgn}(g_1^* s^*) \Psi(z \to 0, \mathbf{x})$, which is also in agreement with

the large N result (5.39). The bulk mass of the scalar also gets shifted by the vev, so the dimension of the leading boundary scalar is now given by

$$\hat{\Delta}_s(\hat{\Delta}_s-(d-1))=\frac{d(d-2)}{2}\underset{d\to 4}{\to} 4,-1. \tag{5.96}$$

We pick the unitary value $\hat{\Delta}_s=4$. In the large N theory, this matches with the dimension of the leading operator that appears in boundary operator expansion of σ. This operator is proportional to the displacement operator and its dimension was referred to as $\hat{\Delta}_{(0)}$ in Table 5.1. The free energy in this phase is given by

$$F=S_{\text{clas.}}+F_t+NF_\Psi=-\frac{3d^2(d-2)^2\text{Vol}(H^d)}{32g_{2,0}}+F_t+NF_\Psi. \tag{5.97}$$

We emphasize that we are using bare coupling here. This is because in $d=4-\epsilon$, the bare coupling gets renormalized as follows (see for instance [203])

$$\frac{1}{g_{2,0}}=\mu^{-\epsilon}\left(\frac{1}{g_2}-\frac{3}{16\pi^2\epsilon}-\frac{Ng_1^2}{2\pi^2\epsilon g_2}+\frac{3Ng_1^4}{\pi^2\epsilon g_2^2}.\right) \tag{5.98}$$

The $1/\epsilon$ pole above has to be canceled by the other terms in the free energy. The field t is a scalar with leading boundary operator of dimension 4 in its spectrum. The free energy on hyperbolic space of such a scalar in $d=4-\epsilon$ is [2]

$$F_t=F_{\text{scalar}}\left(\frac{d}{2}\right)-\frac{9\text{Vol}(H^4)}{8\pi^2\epsilon}. \tag{5.99}$$

The fermion Ψ is a massive fermion with mass $g_1^*s^*$ and its free energy is given by

$$\begin{aligned}F_\Psi=F_{\text{free}}+\int_0^{g_1^*s^*}dm\frac{\partial F}{\partial m}&=F_{\text{free}}-\frac{Nc_d\text{Vol}(H^d)\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^{g_1s^*}dm\frac{\Gamma\left(\frac{d}{2}+m\right)}{\Gamma\left(1-\frac{d}{2}+m\right)}\\&=F_{\text{free}}-\frac{3Ng_1^2}{\pi^2\epsilon g_2}+\frac{18Ng_1^4}{\pi^2\epsilon g_2^2}.\end{aligned} \tag{5.100}$$

Adding all the pieces together, we see that the $1/\epsilon$ pieces cancel and we get a finite free energy as a function of coupling

$$\begin{aligned}F&=NF_{\text{free}}+F_{\text{scalar}}\left(\frac{d}{2}\right)-\frac{6\text{Vol}(H^4)}{g_2}\\&=NF_{\text{free}}+F_{\text{scalar}}\left(\frac{d}{2}\right)-\frac{9(2N+3)\text{Vol}(H^4)}{4\pi^2\left(-2N+3+\sqrt{4N^2+132N+9}\right)\epsilon}.\end{aligned} \tag{5.101}$$

This is consistent with the large N result in (5.61).

In the phase where s does not get a vev in the classical theory, we can choose to start from either Dirichlet or Neumann boundary condition on s which corresponds to B_2 and B_2' phases respectively. The one-point function of s to leading order in the coupling may be calculated as

$$\langle s(x)\rangle = g_1 \int d^d x_1 \langle s(x)s(x_1)\rangle \langle \bar{\Psi}_i \Psi^i\rangle (x_1). \tag{5.102}$$

The two-point function of s in this phase is the usual one for a free scalar in hyperbolic space

$$\langle s(x_1)s(x_2)\rangle^N = \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}\left(\frac{1}{\zeta^{\frac{d}{2}-1}} + \frac{1}{(4+\zeta)^{\frac{d}{2}-1}}\right) \tag{5.103}$$

when we impose Neumann boundary condition on s. When we impose Dirichlet, the only change is that there is a $-$ sign between the two terms. In both cases we need to do an integral of the following form in $d = 4-\epsilon$

$$\begin{aligned}\langle s(x)\rangle^N &= \mp \frac{g_1 N c_d \Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{2^{d+2}\pi^d}\int d^{d-1}\mathbf{x}_1 dz_1 \Bigg[\frac{z^{\frac{d}{2}-1} z_1^{-\frac{d}{2}-1}}{((z_1-z)^2+(\mathbf{x}_1-\mathbf{x})^2)^{\frac{d}{2}-1}} \\ &\quad + \frac{z^{\frac{d}{2}-1} z_1^{-\frac{d}{2}-1}}{((z_1+z)^2+(\mathbf{x}_1-\mathbf{x})^2)^{\frac{d}{2}-1}}\Bigg] \\ &= \pm \frac{g_1 N c_d \Gamma\left(\frac{d}{2}\right)}{2^{d+1}\pi^{\frac{d}{2}}}\int_0^\infty dz_1 z^{\frac{d}{2}-1} z_1^{-\frac{d}{2}-1}\left(|z-z_1| + (z+z_1)\right).\end{aligned} \tag{5.104}$$

Again, for the Dirichlet case, there is a minus sign between the two terms. It is then easy to see that the integral of the second term is a pure divergence. It has a divergence at the boundary, $z_1 \to 0$, which must be cancelled by a boundary counterterm. But there is no finite part, so we can just set this integral to 0. This is also expected, because this means that the one-point function is the same for Neumann and Dirichlet cases, and in the large N theory, there is no distinction between Neumann and Dirichlet at the level of the one-point function of σ. The integral of the first term then gives

$$\langle s(x)\rangle = \pm\frac{g_1 N}{8\pi^2} \implies \sigma^* = g_1^* \langle s(x)\rangle = \pm\frac{\epsilon N}{2N+3} \tag{5.105}$$

which is in agreement with the large N result (5.38). The free energy also depends on whether we choose Neumann or Dirichlet boundary conditions on the scalar. For Neumann, i.e. in the B_2' phase, it is given by

$$
\begin{aligned}
F^N &= NF_{\text{Free}} + F_{\text{scalar}}\left(\frac{d}{2}-1\right) + \frac{g_2\text{Vol}(H^d)}{8}\langle s^2\rangle^2 - \frac{g_1^2}{2}\int d^dx_1 d^dx_2 \langle s\bar{\Psi}_i\Psi^i(x_1)s\bar{\Psi}_j\Psi^j(x_2)\rangle \\
&= NF_{\text{Free}} + F_{\text{scalar}} + \frac{g_2\text{Vol}(H^4)}{8(4\pi)^4} - \frac{g_1^2N^2\text{Vol}(H^4)}{128\pi^6}\int d^dx\sqrt{g_x}\left(\frac{1}{\zeta^{\frac{d}{2}-1}}+\frac{1}{(4+\zeta)^{\frac{d}{2}-1}}\right) \\
&- \frac{g_1^2\text{Vol}(H^4)N}{8\pi^6}\int d^dx\sqrt{g_x}\left(\frac{1}{\zeta^{d-1}}-\frac{1}{(4+\zeta)^{d-1}}\right)\left(\frac{1}{\zeta^{\frac{d}{2}-1}}+\frac{1}{(4+\zeta)^{\frac{d}{2}-1}}\right)
\end{aligned}
\tag{5.106}
$$

where we already fixed one of the points at the center of hyperbolic space, and the integral over that point resulted in a factor of the volume of the hyperbolic space. The integral in the first line is the same as what we did in (5.102), so we can just use the same result. To do the integral in the second line, it is convenient to use ball coordinates on the hyperbolic space. In these coordinates, the metric is given by

$$
ds^2 = \frac{4}{(1-u^2)^2}\left(du^2 + u^2 d\Omega_{d-1}^2\right). \tag{5.107}
$$

We then fix one of the points at the center of the ball. The chordal distance in terms of u variables between the center to an arbitrary point is given by

$$
\zeta = \frac{4u^2}{1-u^2}. \tag{5.108}
$$

The integral in $d=4$ then gives

$$
J = -\frac{g_1^2\text{Vol}(H^4)N}{64\pi^4}\int_0^1 du\; u^{d-1}(1-u^2)^{\frac{d}{2}-2}(u^{2-2d}-1)(u^{2-d}+1) = \frac{7g_1^2N\text{Vol}(H^4)}{512\pi^4}. \tag{5.109}
$$

Putting all the pieces together, the free energy to leading order in ϵ is given by

$$
\begin{aligned}
F^N = NF_{\text{Free}} \;+\; F_{\text{scalar}}\left(\frac{d}{2}-1\right) \;+\; \frac{N^2\text{Vol}(H^4)\epsilon}{8\pi^2(2N+3)} + \frac{7N\text{Vol}(H^4)\epsilon}{(2N+3)64\pi^2} \\
+ \;\frac{\text{Vol}(H^4)\left(-2N+3+\sqrt{4N^2+132N+9}\right)\epsilon}{48(2N+3)(4\pi)^2}.
\end{aligned}
\tag{5.110}
$$

In the B_2 phase, when we choose Dirichlet boundary condition on s, we get the following result for the free energy

$$
\begin{aligned}
F^D &= NF_{\text{Free}} + F_{\text{scalar}}\left(\frac{d}{2}\right) + \frac{g_2 \text{Vol}(H^4)}{8(4\pi)^4} - \frac{g_1^2 N^2 \text{Vol}(H^4)}{128\pi^6}\int d^d x\sqrt{g_x}\left(\frac{1}{\zeta^{\frac{d}{2}-1}} - \frac{1}{(4+\zeta)^{\frac{d}{2}-1}}\right) \\
&- \frac{g_1^2 \text{Vol}(H^4) N}{8\pi^6}\int d^d x\sqrt{g_x}\left(\frac{1}{\zeta^{d-1}} - \frac{1}{(4+\zeta)^{d-1}}\right)\left(\frac{1}{\zeta^{\frac{d}{2}-1}} - \frac{1}{(4+\zeta)^{\frac{d}{2}-1}}\right) \\
&= NF_{\text{Free}} \; + \; F_{\text{scalar}}\left(\frac{d}{2}\right) \; + \; \frac{N^2 \text{Vol}(H^4)\epsilon}{8\pi^2(2N+3)} + \frac{3N\text{Vol}(H^4)\epsilon}{(2N+3)64\pi^2} \\
&+ \frac{\text{Vol}(H^4)\left(-2N+3+\sqrt{4N^2+132N+9}\right)\epsilon}{48(2N+3)(4\pi)^2}.
\end{aligned}
\tag{5.111}
$$

In both B_2 and B_2' phases, the order N piece in the free energy at large N agrees with the result of the large N calculation (5.63). However, they differ at order 1 with the difference given by

$$
\begin{aligned}
F^N - F^D &= F_{\text{scalar}}\left(\frac{d}{2}-1\right) - F_{\text{scalar}}\left(\frac{d}{2}\right) + \frac{N\text{Vol}(H^4)\epsilon}{(2N+3)16\pi^2} \\
&= -\frac{1}{\sin\left(\frac{\pi(d-1)}{2}\right)\Gamma(d)}\int_0^{\frac{1}{2}} du u \sin\pi u \Gamma\left(\frac{d-1}{2}+u\right)\Gamma\left(\frac{d-1}{2}-u\right) + \frac{N\epsilon}{(2N+3)12}
\end{aligned}
\tag{5.112}
$$

where we used the general formula (5.64) to calculate the difference in the free energy of a free scalar with dimension $d/2-1$ and $d/2$. This difference is also in agreement with the large N result in (5.66).

5.4 Using equations of motion in the bulk

In this section, we turn to bulk correlation functions. Since we have a Lagrangian description of our models, the bulk fields satisfy equations of motion, which in turn implies that the correlation functions involving bulk fields must satisfy a differential equation. This differential equation can be solved in some situations to yield the correlation function. Such an approach was originally used to calculate anomalous dimensions in a CFT in [156] and was later extended to calculate two-point functions in a BCFT [2] and in a CFT on real projective space [218]. This is an alternative approach to calculating Feynman diagrams in half-space or in AdS. We start with the correlation functions involving the scalar s in GNY model, where the calculation is very similar to [2, 218]. We then move on to the correlation functions involving the fermion and fix the fermion two-point function in both the GN model and the GNY model, to leading order in ϵ.

5.4.1 Scalar

Let us look at the s correlator in the GNY model (5.91) in $d = 4 - \epsilon$ in the phase where s does not get a vev (classically). As a warm up, we start with the bulk-boundary propagator on H^d which must take the following form

$$\langle s(x_1)\hat{s}(\mathbf{x}_2)\rangle = B_{s\hat{s}}\left(\frac{z_1}{z_1^2+\mathbf{x}_{12}^2}\right)^{\hat{\Delta}_s}. \tag{5.113}$$

Applying the H^d equation of motion at x_1 gives

$$\left(\nabla_{x_1}^2 + \frac{d(d-2)}{4}\right)\langle s(x_1)\hat{s}(\mathbf{x}_2)\rangle = \frac{(2\hat{\Delta}_s - d)(2\hat{\Delta}_s - d + 2)}{4} B_{s\hat{s}}\left(\frac{z_1}{z_1^2+\mathbf{x}_{12}^2}\right)^{\hat{\Delta}_s}. \tag{5.114}$$

In the free theory, the right hand side above must be set to 0 which gives the usual boundary dimensions for Neumann and Dirichlet boundary conditions. In the interacting GNY model, the equation of motion, to leading order in ϵ gives

$$\begin{aligned}\left(\nabla_{x_1}^2 + \frac{d(d-2)}{4}\right)\langle s(x_1)\hat{s}(\mathbf{x}_2)\rangle = \frac{g_2}{2}\langle s^2(x_1)\rangle\langle s(x_1)\hat{s}(\mathbf{x}_2)\rangle - g_1^2\langle\bar{\Psi}_i\Psi^i\rangle^2\int d^dx\sqrt{g_x}\langle s(x)\hat{s}(\mathbf{x}_2)\rangle \\ - g_1^2\int d^dx\sqrt{g_x}\langle\bar{\Psi}_{i\alpha}(x_1)\Psi^{j\beta}(x)\rangle\langle\Psi^{i\alpha}(x_1)\bar{\Psi}_{j\beta}(x)\rangle\langle s(x)\hat{s}(\mathbf{x}_2)\rangle.\end{aligned} \tag{5.115}$$

Comparing (5.114) and (5.115) should give us the anomalous dimension of the leading boundary scalar $\hat{s}$ to leading order in ϵ. So let us try to evaluate the right hand side of (5.115). The first term is straightforward. As for the second term, it is easy to see that the integral should be set to 0 (it is pure power law divergence at $z \to 0$ but this can be absorbed in a boundary counterterm). The last term is non-trivial and the integral involved is as follows

$$\begin{aligned}I &= -\frac{g_1^2\Gamma\left(\frac{d}{2}\right)^2 N c_d B_{s\hat{s}}}{4\pi^d}\int d^dx\sqrt{g_x}\left(\frac{1}{\zeta_{x_1x}^{d-1}} - \frac{1}{(4+\zeta_{x_1x})^{d-1}}\right)\left(\frac{z}{z^2+(\mathbf{x}-\mathbf{x}_2)^2}\right)^{\hat{\Delta}_s} \\ &= -\frac{g_1^2\Gamma\left(\frac{d}{2}\right)^2 N c_d z_1^{d-1} B_{s\hat{s}}}{4\pi^d}\int dz d^{d-1}\mathbf{x}\Bigg[\frac{z^{\hat{\Delta}_s-1}}{((z-z_1)^2+(\mathbf{x}-\mathbf{x}_1)^2)^{d-1}(z^2+(\mathbf{x}-\mathbf{x}_2)^2)^{\hat{\Delta}_s}} \\ &- \frac{z^{\hat{\Delta}_s-1}}{((z+z_1)^2+(\mathbf{x}-\mathbf{x}_1)^2)^{d-1}(z^2+(\mathbf{x}-\mathbf{x}_2)^2)^{\hat{\Delta}_s}}\Bigg]\end{aligned} \tag{5.116}$$

The integral over $\mathbf{x}$ can be performed using Feynman parameters, and then the integral over the Feynman parameter can be performed leaving us with the following integral over z

$$
\begin{aligned}
I &= -\frac{g_1^2\Gamma\left(\frac{d}{2}\right)^2 N c_d \Gamma\left(\frac{d-1}{2}\right) B_{s\hat{s}}}{4\pi^{\frac{d+1}{2}}\Gamma(d-1)} \frac{z_1^{d-1}}{(z_1^2+\mathbf{x}_{12}^2)^{\hat{\Delta}_s}} \int dz \left(\frac{z^{\hat{\Delta}_s-1}}{((z-z_1)^2)^{\frac{d-1}{2}}} - \frac{z^{\hat{\Delta}_s-1}}{((z+z_1)^2)^{\frac{d-1}{2}}} \right) \\
&= -\frac{g_1^2\Gamma\left(\frac{d}{2}\right)^2 N c_d \Gamma\left(\frac{d-1}{2}\right) B_{s\hat{s}}}{4\pi^{\frac{d+1}{2}}\Gamma(d-1)} \left(\frac{z_1}{z_1^2+\mathbf{x}_{12}^2}\right)^{\hat{\Delta}_s} \Bigg[-\frac{\Gamma\left(d-1-\hat{\Delta}_s\right)\Gamma\left(\hat{\Delta}_s\right)}{\Gamma(d-1)} \\
&+ \Gamma(2-d)\left(\frac{\Gamma(d-\hat{\Delta}_s-1)}{\Gamma(1-\hat{\Delta}_s)} + \frac{\Gamma(\hat{\Delta}_s)}{\Gamma(-d+\hat{\Delta}_s+2)}\right)\Bigg]
\end{aligned}
\tag{5.117}
$$

where the first integral only converges for $d<2, \hat{\Delta}_s>0$ and $d-\hat{\Delta}_s>1$, but the final answer can be analytically continued in d and $\hat{\Delta}_s$. The two terms in the second line add up to 0 for both Neumann and Dirichlet cases, i.e. for both $\hat{\Delta}_s = d/2-1$ and $d/2$, so the resulting integral for these two cases becomes

$$
I = \frac{g_1^2\Gamma\left(\frac{d}{2}\right)^2 N c_d \Gamma\left(\frac{d-1}{2}\right)\Gamma\left(d-1-\hat{\Delta}_s\right)\Gamma\left(\hat{\Delta}_s\right)}{4\pi^{\frac{d+1}{2}}\Gamma(d-1)^2} B_{s\hat{s}} \left(\frac{z_1}{z_1^2+\mathbf{x}_{12}^2}\right)^{\hat{\Delta}_s}. \tag{5.118}
$$

Using this, and the fact that in the free theory in $d=4$, $\langle s^2(x)\rangle = \pm 1/(4\pi)^2$ we can calculate the dimensions of the leading boundary scalar in GNY model

$$
\begin{aligned}
\hat{\gamma}_s^N &= -\frac{g_2}{2(4\pi)^2} - \frac{g_1^2 N c_d}{8\pi^2} = -\frac{\sqrt{4N^2+132N+9}+10N+3}{12(2N+3)}\epsilon \\
\hat{\Delta}_s^N &= \frac{d}{2} - 1 + \hat{\gamma}_s^N = 1 - \frac{\sqrt{4N^2+132N+9}+22N+21}{12(2N+3)}\epsilon
\end{aligned}
\tag{5.119}
$$

in B_2' phase and

$$
\begin{aligned}
\hat{\gamma}_s^D &= -\frac{g_2}{2(4\pi)^2} + \frac{g_1^2 N c_d}{8\pi^2} = -\frac{\sqrt{4N^2+132N+9}-14N+3}{12(2N+3)}\epsilon \\
\hat{\Delta}_s^D &= \frac{d}{2} + \hat{\gamma}_s^D = 2 - \frac{\sqrt{4N^2+132N+9}-2N+21}{12(2N+3)}\epsilon
\end{aligned}
\tag{5.120}
$$

in the B_2 phase. At leading order in large N, they are equal $1-\epsilon$ and 2 respectively, in agreement with the large N values of $d-3$ and 2.

Next, we look at the two-point function of s, in which case, we can apply the equation of motion at both points. In this case, to leading order in the perturbation theory, we get the following differential equation for the two-point function

$$
\begin{aligned}
&\left(\nabla_{x_2}^2 + \frac{d(d-2)}{4}\right)\left(\nabla_{x_1}^2 + \frac{d(d-2)}{4}\right)\langle s(x_1)s(x_2)\rangle = \\
&g_1^2\langle\bar{\Psi}_i\Psi^i\rangle^2 + g_1^2\langle\bar{\Psi}_{ia}(x_1)\Psi^{jb}(x_2)\rangle\langle\Psi^{ia}(x_1)\bar{\Psi}_{jb}(x_2)\rangle.
\end{aligned}
\tag{5.121}
$$

Writing the propagator as a function of ζ , $G_s(\zeta)$, we get following differential equation for the propagator, keeping only terms to order ϵ on the RHS

$$\Big[\zeta(4+\zeta)\left(\zeta(4+\zeta)\partial_\zeta^4+(d+2)(4+2\zeta)\partial_\zeta^3\right)+\frac{(8d(d+2)+(4+3d(d+2))\zeta(4+\zeta))}{2}\partial_\zeta^2+ \\ \frac{d^3(4+2\zeta)}{4}\partial_\xi+\frac{d^2(d-2)^2}{16}\Big]G_s(\zeta)=D^{(4)}G_s(\zeta)=\frac{g_1^2Nc_d\Gamma\left(\frac{d}{2}\right)^2}{4\pi^d}\left[\frac{Nc_d}{4^{d-1}}+\frac{1}{\zeta^{d-1}}-\frac{1}{(4+\zeta)^{d-1}}\right]. \tag{5.122}$$

Recall from (5.46) the conformal block decomposition for the two-point function. It turns out that the boundary channel block is an eigenfunction of the equation of motion operator

$$D^{(4)}f_{\text{bdry}}(\hat{\Delta}_l;\zeta)=\frac{\left(d-2\hat{\Delta}_l\right)^2\left(d-2-2\hat{\Delta}_l\right)^2}{16}f_{\text{bdry}}(\hat{\Delta}_l;\zeta). \tag{5.123}$$

This allows us to plug in the block decomposition into (5.122) and extract information about the bulk-boundary OPE coefficients at order ϵ. For instance, the one-point function of s is fixed to be

$$A\frac{(d-2)^2d^2}{16}a_s^2=\frac{g_1^2N^2c_d^2\Gamma\left(\frac{d}{2}\right)^2}{(4\pi)^d}. \tag{5.124}$$

The boundary expansion coefficients of all the subleading boundary operators obey the following constraint

$$A\sum_l\frac{\left(d-2\hat{\Delta}_l\right)^2\left(d-2-2\hat{\Delta}_l\right)^2}{16}\mu_l^2f_{\text{bdry}}(\hat{\Delta}_l;\zeta)=\frac{g_1^2Nc_d\Gamma\left(\frac{d}{2}\right)^2}{4\pi^d}\left(\frac{1}{\zeta^3}-\frac{1}{(4+\zeta)^3}\right) \tag{5.125}$$

where the sum does not include the leading boundary operator of dimension $d/2-1$ or $d/2$. In the usual normalization, $A=1/4\pi^2$ in four dimensions. Expanding both sides in powers of ζ tells us that the boundary spectrum contains a tower of operators of dimension $4+2n$ with the OPE coefficients

$$\mu_{4+2n}^2=\frac{N\Gamma(2n+2)\sqrt{\pi}\epsilon}{(2N+3)2^{4n+5}\Gamma\left(2n+\frac{5}{2}\right)}. \tag{5.126}$$

This is consistent with what we found in the large N expansion (5.53). The $n=0$ operator in the tower is proportional to the displacement operator in B_2 and B_2' phases in this description.

We can also directly solve the equation (5.122) perturbatively in ϵ by expanding the differential

operator and the correlator in powers of ϵ

$$\begin{aligned} D^{(4)} &= D_0^{(4)} + \epsilon D_1^{(4)} + O(\epsilon^2) \\ G_s(\zeta) &= G_0(\zeta) + \epsilon G_1(\zeta) + O(\epsilon^2). \end{aligned} \tag{5.127}$$

Let us now work in a more convenient normalization where the free theory correlator is [8]

$$G_0^{N/D}(\zeta) = \frac{1}{\zeta} \pm \frac{1}{4+\zeta}. \tag{5.128}$$

We then get the following differential equation for $G_1(\zeta)$

$$D_0^{(4)} G_1(\zeta) = \frac{32N}{2N+3}\left(\frac{N}{16} + \frac{1}{\zeta^3} - \frac{1}{(4+\zeta)^3}\right) - D_1^{(4)} G_0(\zeta). \tag{5.129}$$

The equation can be solved to give

$$\begin{aligned} G_1^N(\zeta) &= \frac{c_1}{\zeta} + \frac{c_2}{4+\zeta} + \frac{c_3 \log(\zeta/4)}{4+\zeta} + \frac{c_4 \log(1+\zeta/4)}{\zeta} + \frac{N^2}{2(2N+3)} \\ &\quad + \frac{3}{2(2N+3)}\left(\frac{\log\zeta}{\zeta} + \frac{\log(4+\zeta)}{4+\zeta}\right) + \frac{2N}{(2N+3)}\frac{\log(1+\zeta/4)}{4+\zeta} \\ G_1^D(\zeta) &= \frac{c_1}{\zeta} + \frac{c_2}{4+\zeta} + \frac{c_3 \log(\zeta/4)}{4+\zeta} + \frac{c_4 \log(1+\zeta/4)}{\zeta} + \frac{N^2}{2(2N+3)} \\ &\quad + \frac{3}{2(2N+3)}\left(\frac{\log\zeta}{\zeta} - \frac{\log(4+\zeta)}{4+\zeta}\right). \end{aligned} \tag{5.130}$$

We have four undetermined coefficients. One of these is fixed by fixing the normalization of the field s. We are working in a normalization such that the correlator falls off as $1/\zeta$ as $\zeta \to 0$, which sets $c_1 = 0$ for both Neumann and Dirichlet cases. For further analysis, we have to consider Dirichlet and Neumann cases separately. For the Dirichlet case, the leading boundary operator has dimension 2, so the large ζ expansion of the two-point function should not have any $1/\zeta$ or $(\log\zeta)/\zeta$ terms. This implies that $c_2 = 0$ and $c_4 = -c_3$. So we are left with one undetermined coefficient. This can be fixed by looking at the bulk OPE limit ($\zeta \to 0$), where the correlator should behave like

$$\begin{aligned} G_s(\zeta) &= \zeta^{-\Delta_s} + \lambda_{s^2}\zeta^{\frac{1}{2}(\Delta_{s^2}-2\Delta_s)} + \text{higher orders in } \zeta \\ &= \zeta^{-1} + \lambda_{s^2}^{(0)} + \left(\left(\frac{1}{2} - \gamma_s^{(1)}\right)\frac{\log\zeta}{\zeta} + \lambda_{s^2}^{(1)} + \left(\frac{\gamma_{s^2}^{(1)}}{2} - \gamma_s^{(1)}\right)\lambda_{s^2}^0 \log\zeta\right)\epsilon. \end{aligned} \tag{5.131}$$

[8] Note that when we change the normalization of fields, the coupling constant also needs to change accordingly. So in this normalization, coupling constant g_1 changes to $g_1(2\pi)$ to leading order in ϵ.

Free theory result fixed $\lambda^0_{s^2} = -1/4$. Then we get the following result by comparing $\log\zeta$ terms

$$\frac{\gamma^{(1)}_{s^2}}{2} - \gamma^{(1)}_s = -c_3 \tag{5.132}$$

Using the following bulk data from [203], we can calculate c_3

$$\Delta_s = 1 - \frac{3}{2(2N+3)}\epsilon, \qquad \Delta_{s^2} = 2 + \frac{\sqrt{4N^2+132+9} - 2N - 15}{6(2N+3)}\epsilon \tag{5.133}$$

and we get

$$c_3 = \frac{-\sqrt{4N^2+132N+9} + 2N - 3}{12(2N+3)} \tag{5.134}$$

This fixes the s two-point function in the B_2 phase to be

$$G^D_1(\zeta) = c_3\left(\frac{\log(\zeta/4)}{4+\zeta} - \frac{\log(1+\zeta/4)}{\zeta}\right) + \frac{N^2}{2(2N+3)} + \frac{3}{2(2N+3)}\left(\frac{\log\zeta}{\zeta} - \frac{\log(4+\zeta)}{4+\zeta}\right). \tag{5.135}$$

At large N, this agrees with what we found in (5.56). This determines all the BCFT data to order ϵ. In particular, the dimension and boundary expansion coefficient of the leading boundary scalar is

$$\hat{\Delta}^D_s = 2 - \frac{\sqrt{4N^2+132N+9} - 2N + 21}{12(2N+3)}\epsilon, \quad \mu_2^2 = \frac{1}{4} + \left(\frac{c_3(2\log 2 - 1)}{4} - \frac{3}{8(2N+3)}\right)\epsilon \tag{5.136}$$

which agrees with what we found above (5.120) and is also consistent with the large N result (5.53).

Next, we consider the B_2' phase where the leading boundary operator has dimension 1 and the next subleading operator has dimension 4 in the free theory. This implies that $1/\zeta^2$ and $(\log\zeta)/\zeta^2$ terms must be descendants of the leading operator (similarly for $1/\zeta^3$). This puts constraints on the coefficients

$$c_2 = \frac{4N}{3+2N}, \qquad c_4 = c_3 + \frac{2N}{3+2N}. \tag{5.137}$$

We then compare with the bulk channel expansion (5.131). The free theory result implies $\lambda^0_{s^2} = -1/4$ and comparing the coefficient of $\log\zeta$ gives

$$\frac{\gamma^{(1)}_{s^2}}{2} - \gamma^{(1)}_s = c_3 \implies c_3 = \frac{\sqrt{4N^2+132N+9} - 2N + 3}{12(2N+3)}. \tag{5.138}$$

So the full $O(\epsilon)$ correlator in the B_2' phase is the following

$$\begin{aligned} G_1^N(\zeta) = &\frac{4N}{(3+2N)}\frac{1}{4+\zeta} + c_3\left(\frac{\log(\zeta/4)}{4+\zeta} + \frac{\log(1+\zeta/4)}{\zeta}\right) + \frac{N^2}{2(2N+3)} \\ &+ \frac{3}{2(2N+3)}\left(\frac{\log\zeta}{\zeta} + \frac{\log(4+\zeta)}{4+\zeta}\right) + \frac{2N}{(2N+3)}\left(\frac{1}{\zeta} + \frac{1}{4+\zeta}\right)\log(1+\zeta/4). \end{aligned} \tag{5.139}$$

This also agrees with the large N result (5.56). The dimension and boundary expansion coefficient of the leading boundary scalar in this case is

$$\hat{\Delta}_s^N = 1 - \frac{\sqrt{4N^2+132N+9}+22N+21}{12(2N+3)}\epsilon, \qquad \mu_1^2 = \frac{1}{2} + \frac{\epsilon N}{2N+3} \tag{5.140}$$

again in agreement with (5.119) and with the large N result (5.55).

5.4.2 Fermion

In this subsection, we apply the same logic to fermion correlators. As we wrote in (5.14), the bulk-boundary propagator of a fermion can be written as

$$\langle\Psi(x_1)\hat{\bar{\Psi}}(\mathbf{x}_2)\rangle = B_{\Psi\hat{\Psi}}\frac{\gamma_a x_{12}^a\left(1\mp\gamma_0\right)z_1^{\hat{\Delta}}}{(z_1^2+\mathbf{x}_{12}^2)^{\hat{\Delta}+1/2}} \tag{5.141}$$

where the fermion satisfies the boundary condition $\gamma_0\Psi(z_1\to 0,\mathbf{x}_1) = \pm\Psi(z_1\to 0,\mathbf{x}_1)$. Acting with the Dirac operator on the right hand side above gives

$$\gamma\cdot\nabla_1\langle\Psi(x_1)\hat{\bar{\Psi}}(\mathbf{x}_2)\rangle = \left(z_1\gamma^a\partial_{1a} - \frac{(d-1)}{2}\gamma_0\right)\langle\Psi(x_1)\hat{\bar{\Psi}}(\mathbf{x}_2)\rangle = \pm\left(\hat{\Delta}-\frac{d-1}{2}\right)\langle\Psi(x_1)\hat{\bar{\Psi}}(\mathbf{x}_2)\rangle \tag{5.142}$$

For the free massive fermion, the equation of motion sets $\gamma\cdot\nabla\Psi = -m\Psi$ which gives the dimension of the leading boundary spinor $\hat{\Delta} = (d-1)/2 \mp m$. In the Gross-Neveu model, the equation of motion sets

$$\gamma\cdot\nabla_1\langle\Psi^i(x_1)\hat{\bar{\Psi}}_j(\mathbf{x}_2)\rangle = -g\left(N-\frac{1}{2}\right)\langle\bar{\Psi}\Psi\rangle\langle\Psi^i(x_1)\hat{\bar{\Psi}}_j(\mathbf{x}_2)\rangle \tag{5.143}$$

Since there is an explicit factor of g on the right hand side above, we can plug in the one-point function and the correlator for the free theory, and on comparing it with (5.142), this should give us the anomalous dimension for the leading boundary spinor. Using the one-point function from (5.69), we get, in $d = 2+\epsilon$,

$$\pm\left(\hat{\Delta}-\frac{d-1}{2}\right) = \pm g^*\left(N-\frac{1}{2}\right)\frac{c_d\Gamma\left(\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \implies \hat{\Delta} = \frac{1}{2} + \frac{4N-3}{4(N-1)}\epsilon \tag{5.144}$$

in agreement with the result we got by direct calculation (5.77).

We now turn to the bulk two-point function of the fermion. We start with the ansatz in (5.9) and act on it with the Dirac operator which gives (5.11). In the free theory, the equation of motion sets this derivative to zero away from the coincident limit, which gives us two first order differential equations

$$\gamma \cdot \nabla_1 G_\Psi(x_1, x_2) = 0 \implies \alpha'(\zeta) + \frac{(d-1)}{2}\frac{\alpha(\zeta)}{\zeta+4} = 0, \qquad \beta'(\zeta) + \frac{(d-1)}{2}\frac{\beta(\zeta)}{\zeta} = 0. \tag{5.145}$$

These equations can be solved to give

$$\beta(\zeta) = \frac{c_1}{\zeta^{\frac{d-1}{2}}}, \qquad \alpha(\zeta) = \frac{c_2}{(\zeta+4)^{\frac{d-1}{2}}}. \tag{5.146}$$

One of these constants can be fixed by fixing the overall normalization of the two-point function. For convenience, we now work with the convention such that as $\zeta \to 0$, the two-point function goes like $-(\gamma_a x_{12}^a)/\zeta^{\frac{d}{2}}$, which sets $c_1 = -1$. We then recall that the boundary condition requires

$$\gamma_0 G_\Psi(x_1, x_2)|_{z_1 \to 0} = \pm G_\Psi(x_1, x_2)|_{z_1 \to 0} \implies c_2 = \pm 1 \tag{5.147}$$

In the Gross-Neveu model, the equation of motion requires

$$\gamma \cdot \nabla_1 G_\Psi(x_1, x_2) = -g\left(N - \frac{1}{2}\right)\langle \bar{\Psi}\Psi \rangle G_\Psi(x_1, x_2). \tag{5.148}$$

We can then solve this equation perturbatively in $d = 2 + \epsilon$ by expanding

$$\alpha(\zeta) = \alpha_0(\zeta) + \epsilon\alpha_1(\zeta), \quad \alpha_0(\zeta) = \pm\frac{1}{\sqrt{\zeta+4}}; \qquad \beta(\zeta) = \beta_0(\zeta) + \epsilon\beta_1(\zeta), \quad \beta_0(\zeta) = -\frac{1}{\sqrt{\zeta}}. \tag{5.149}$$

Plugging this into (5.148) and comparing the coefficients of $\gamma_a x_{12}^a$ and $\gamma_0\gamma_a \bar{x}_{12}^a$ gives the following equations at order ϵ

$$\begin{aligned} \alpha_1'(\zeta) + \frac{\alpha_1(\zeta)}{2(\zeta+4)} &= \mp\frac{\zeta+2}{\zeta(\zeta+4)^{3/2}} \mp \frac{1}{4(N-1)\zeta\sqrt{\zeta+4}}; \\ \beta_1'(\zeta) + \frac{\beta_1(\zeta)}{2\zeta} &= \frac{\zeta+2}{\zeta^{3/2}(\zeta+4)} + \frac{1}{4(N-1)\sqrt{\zeta}(\zeta+4)}. \end{aligned} \tag{5.150}$$

The solutions are

$$\begin{aligned}\beta_1(\zeta) &= \frac{d_1}{\sqrt{\zeta}} + \frac{1}{2\sqrt{\zeta}}\log\left(\frac{\zeta(\zeta+4)}{16}\right) + \frac{\log(\zeta/4+1)}{4(N-1)\sqrt{\zeta}};\\ \alpha_1(\zeta) &= \frac{d_2}{\sqrt{\zeta+4}} \mp \frac{1}{2\sqrt{\zeta+4}}\log\left(\frac{\zeta(\zeta+4)}{16}\right) \mp \frac{\log(\zeta/4)}{4(N-1)\sqrt{\zeta+4}}.\end{aligned} \tag{5.151}$$

Fixing the normalization sets $d_1 = 0$. And then requiring that the boundary condition (5.147) is satisfied as $z_1 \to 0$ fixes $d_2 = 0$. So the full correlator, to order ϵ in GN model is

$$\begin{aligned}G_\Psi(x_1,x_2) = &\frac{\gamma_0\gamma_a(\bar{x}_1 - x_2)^a}{\sqrt{z_1 z_2}}\left(\mp\frac{1}{\zeta+4} \pm \frac{\epsilon}{2(\zeta+4)}\log\left(\frac{\zeta(\zeta+4)}{16}\right) \pm \epsilon\frac{\log(\zeta/4)}{4(N-1)(\zeta+4)}\right)\\ &+ \frac{\gamma_a(x_1-x_2)^a}{\sqrt{z_1 z_2}}\left(-\frac{1}{\zeta} + \frac{\epsilon}{2\zeta}\log\left(\frac{\zeta(\zeta+4)}{16}\right) + \epsilon\frac{\log(\zeta/4+1)}{4(N-1)\zeta}\right).\end{aligned} \tag{5.152}$$

At large N, this agrees with the fermion two-point function we found at large N in B_1 phase (5.39). As a check, looking at the coefficient of $\log\zeta$ in $\zeta \to \infty$ limit, we recover the dimension of the leading boundary fermion

$$\hat{\Delta} = \frac{1}{2} + \frac{4N-3}{4(N-1)}\epsilon. \tag{5.153}$$

In the GNY model, it is more convenient to apply the Dirac operator on both of the fermions in the two-point function. Acting on the ansatz (5.9) with two Dirac operators we get

$$\begin{aligned}&\left(z_1\gamma^a\partial_{1a} - \frac{(d-1)}{2}\gamma_0\right)G_\Psi(x_1,x_2)\left(z_2\gamma^a\overleftarrow{\partial}_{2a} - \frac{(d-1)}{2}\gamma_0\right) =\\ &\frac{\gamma_0\gamma_a\bar{x}_{12}^a}{\sqrt{z_1 z_2}\sqrt{\zeta+4}}\left(\zeta(\zeta+4)\alpha''(\zeta) + d(\zeta+2)\alpha'(\zeta) + \frac{(d-1)}{4}\left(d - \frac{\zeta}{\zeta+4}\right)\alpha(\zeta)\right)\\ &-\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}\sqrt{\zeta}}\left(\zeta(\zeta+4)\beta''(\zeta) + d(\zeta+2)\beta'(\zeta) + \frac{(d-1)}{4}\left(d - \frac{(\zeta+4)}{\zeta}\right)\beta(\zeta)\right)\end{aligned} \tag{5.154}$$

The GNY equation of motion sets this to

$$(\gamma\cdot\nabla_1)G_\Psi(x_1,x_2)(\gamma\cdot\overleftarrow{\nabla}_2) = -g_1^2\langle s(x_1)s(x_2)\rangle G_\Psi(x_1,x_2). \tag{5.155}$$

In addition to the choice of boundary condition for the fermion, we now have an additional choice for the boundary condition on the scalar. If we choose Neumann boundary condition on the scalar,

then we get the following differential equations for α and β

$$\begin{aligned}\zeta(\zeta+4)\alpha''(\zeta)+d(\zeta+2)\alpha'(\zeta)+\frac{d-1}{4}\left(d-\frac{\zeta}{\zeta+4}\right)\alpha(\zeta)&=\frac{\pm 2\epsilon}{2N+3}\left(\frac{1}{\zeta}+\frac{1}{4+\zeta}\right)\frac{1}{(\zeta+4)^{\frac{3}{2}}}\\ \zeta(\zeta+4)\beta''(\zeta)+d(\zeta+2)\beta'(\zeta)+\frac{d-1}{4}\left(d-\frac{\zeta+4}{\zeta}\right)\beta(\zeta)&=\frac{-2\epsilon}{2N+3}\left(\frac{1}{\zeta}+\frac{1}{4+\zeta}\right)\frac{1}{\zeta^{\frac{3}{2}}}.\end{aligned} \tag{5.156}$$

There is a similar equation for when we choose Dirichlet boundary condition on the scalar, apart from the fact that the propagator on the right is $1/\zeta - 1/(4+\zeta)$. As we did before, we may expand the differential operator and the correlator in powers of ϵ

$$\alpha(\zeta)=\alpha_0(\zeta)+\epsilon\alpha_1(\zeta),\quad \alpha_0(\zeta)=\pm\frac{1}{(\zeta+4)^{\frac{3}{2}}};\qquad \beta(\zeta)=\beta_0(\zeta)+\epsilon\beta_1(\zeta),\quad \beta_0(\zeta)=-\frac{1}{\zeta^{\frac{3}{2}}}. \tag{5.157}$$

Plugging these in to the differential operators above, we can solve the differential equation to get order ϵ correction to the correlator

$$\begin{aligned}\beta_1^N(\zeta)&=\frac{d_1}{\zeta^{\frac{3}{2}}}+\frac{d_2}{\zeta^{\frac{3}{2}}}\left[\log(1+\zeta/4)-\frac{\zeta}{4+\zeta}\right]+\frac{1}{2(3+2N)\zeta^{\frac{3}{2}}}\left(\frac{\zeta}{4+\zeta}-\frac{(4N+5)\log(\zeta/4)}{2}\right)\\ \alpha_1^N(\zeta)&=\frac{d_3}{(4+\zeta)^{\frac{3}{2}}}+\frac{d_4}{(4+\zeta)^{\frac{3}{2}}}\left[\log(\zeta/4)-\frac{\zeta+4}{\zeta}\right]\\ &\quad\pm\frac{1}{2(3+2N)}\left(\frac{1}{\zeta\sqrt{4+\zeta}}+\frac{(4N+7)}{2}\frac{\log(1+\zeta/4)}{(4+\zeta)^{\frac{3}{2}}}\right)\end{aligned} \tag{5.158}$$

for Neumann boundary condition and

$$\begin{aligned}\beta_1^D(\zeta)&=\frac{d_1}{\zeta^{\frac{3}{2}}}+\frac{d_2}{\zeta^{\frac{3}{2}}}\left[\log(1+\zeta/4)-\frac{\zeta}{4+\zeta}\right]-\frac{1}{2(3+2N)\zeta^{\frac{3}{2}}}\left(\frac{\zeta}{4+\zeta}+\frac{(4N+5)\log(\zeta/4)}{2}\right)\\ &\quad\pm\frac{1}{2(3+2N)}\left(\frac{1}{\zeta\sqrt{4+\zeta}}+\frac{(4N+5)}{2}\frac{\log(1+\zeta/4)}{(4+\zeta)^{\frac{3}{2}}}\right)\end{aligned} \tag{5.159}$$

for Dirichlet boundary condition. Now, let's fix the undetermined coefficients. Fixing the normalization fixes $d_1 = 0$. Then, we recall that the contribution of the $\bar{\Psi}\Psi$ operator to the two-point function in the limit $x_1 \to x_2$ i.e. $\zeta \to 0$ should look like

$$G_\Psi(x_1,x_2)\sim\lambda^0_{\bar{\Psi}\Psi}+\epsilon\lambda^1_{\bar{\Psi}\Psi}+\lambda^0_{\bar{\Psi}\Psi}\left(\frac{\gamma_{\bar{\Psi}\Psi}}{2}-\gamma_\Psi\right)\log\zeta \tag{5.160}$$

It is easy to see that the constant and $\log\zeta$ terms can only appear in $\alpha(\zeta)$ and comparing their coefficient fixes

$$d_4=\pm\left(\frac{\gamma_{\bar{\Psi}\Psi}}{2}-\gamma_\Psi\right)=\pm\frac{(2N+1)}{2(2N+3)} \tag{5.161}$$

where we used the bulk data from [203]. The other two constants can be determined by imposing the boundary condition (5.147) in the limit $\zeta \to \infty$ and comparing the coefficients of $1/\zeta^2$ and $\log\zeta/\zeta^2$ in this limit

$$\begin{aligned} d_2^N &= \mp d_4 - \frac{1}{2(3+2N)}, \qquad d_2^D = \mp d_4 \\ d_3^N &= \mp \frac{3}{2(3+2N)}, \qquad d_3^D = 0. \end{aligned} \tag{5.162}$$

This gives the following two-point function for the fermion in GNY model to leading order in ϵ

$$\begin{aligned} G_\Psi^N(x_1,x_2) &= -\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}}\frac{1}{\zeta^2} \mp \frac{\gamma_0\gamma_a \bar{x}_{12}^a}{\sqrt{z_1 z_2}}\frac{1}{(\zeta+4)^2} \\ &+ \frac{\epsilon}{3+2N}\left[\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}}\left(\frac{(3+2N)}{2\zeta(4+\zeta)} - (N+1)\frac{\log(1+\zeta/4)}{\zeta^2} - \frac{(4N+5)}{4}\frac{\log(\zeta/4)}{\zeta^2}\right)\right. \\ &\left. \pm \frac{\gamma_0\gamma_a \bar{x}_{12}^a}{\sqrt{z_1 z_2}}\left(\frac{3}{2(4+\zeta)^2} + \frac{N}{\zeta(4+\zeta)} - \frac{(2N+1)}{2}\frac{\log(\zeta/4)}{(4+\zeta)^2} - \frac{(4N+7)}{4}\frac{\log(1+\zeta/4)}{(4+\zeta)^2}\right)\right] \end{aligned} \tag{5.163}$$

for the B_2' phase and

$$\begin{aligned} G_\Psi^D(x_1,x_2) &= -\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}}\frac{1}{\zeta^2} \mp \frac{\gamma_0\gamma_a \bar{x}_{12}^a}{\sqrt{z_1 z_2}}\frac{1}{(\zeta+4)^2} \\ &+ \frac{\epsilon}{3+2N}\left[\frac{\gamma_a x_{12}^a}{\sqrt{z_1 z_2}}\left(\frac{N}{\zeta(4+\zeta)} - \frac{(2N+1)}{2}\frac{\log(1+\zeta/4)}{\zeta^2} - \frac{(4N+5)}{4}\frac{\log(\zeta/4)}{\zeta^2}\right)\right. \\ &\left. \pm \frac{\gamma_0\gamma_a \bar{x}_{12}^a}{\sqrt{z_1 z_2}}\left(\frac{N}{\zeta(4+\zeta)} - \frac{(2N+1)}{2}\frac{\log(\zeta/4)}{(4+\zeta)^2} - \frac{(4N+5)}{4}\frac{\log(1+\zeta/4)}{(4+\zeta)^2}\right)\right] \end{aligned} \tag{5.164}$$

for the B_2 phase. At large N, both of them go to the large N result (5.40). BCFT data can be extracted from the two-point function. For instance, looking at it in the limit of large ζ gives us following dimensions of the leading boundary operator in B_2' and B_2 phase

$$\hat{\Delta}^N = \frac{3}{2} - \frac{(8N+9)}{4(3+2N)}\epsilon, \qquad \hat{\Delta}^D = \frac{3}{2} - \frac{(8N+7)}{4(3+2N)}\epsilon. \tag{5.165}$$

These are also in agreement with the large N result of $d-5/2$. A curious observation is that for $N=1/4$, the anomalous dimensions of boson and fermion agree, such that to leading order in ϵ, the following relations hold

$$\hat{\Delta}^N = \frac{3}{2} - \frac{11\epsilon}{14} = \hat{\Delta}_s^N + \frac{1}{2}, \qquad \hat{\Delta}^D = \frac{3}{2} - \frac{9\epsilon}{14} = \hat{\Delta}_s^D - \frac{1}{2} \tag{5.166}$$

as can be checked by recalling (5.119) and (5.120). This may be related to the observation in [203] that in $d = 4-\epsilon$, for $N=1/4$, the GNY model respects $\mathcal{N}=1$ emergent supersymmetry, to order

ϵ^2. It will be interesting to check if the boundary preserves this supersymmetry.

It is also possible to apply the equation of motion to the fermion two-point function in the large N theory. In [2], this was used to get 1/N correction to the boundary anomalous dimension for $O(N)$ BCFT. However, this requires deriving bulk and boundary channel conformal blocks for the fermion two-point function, which we did not pursue here. We hope to come back to this question in a future work. Knowing the conformal block expansion for fermion two-point function will also be useful to extract BCFT data from the results we obtained in this section in the ϵ expansion.

5.5 Appendix: σ propagator

To obtain the σ propagator at large N one should solve the following inversion problem (5.42)

$$\int d^d x \sqrt{g} H(\zeta_{x_1,x}) G_\sigma(\zeta_{x,x_2}) = \frac{1}{N} \frac{\delta^d(x_1 - x_2)}{\sqrt{g_{x_1}}}, \qquad \zeta_{x_1,x_2} = \frac{(z_1 - z_2)^2 + \mathbf{x}_{12}^2}{z_1 z_2}. \tag{5.167}$$

In our case, $H(\zeta_{x_1,x}) = \mathrm{Tr}\left[G_\Psi(x_1, x) G_\Psi(x, x_1)\right]$. Such a problem was discussed on half space in [12] and the problem is essentially identical on hyperbolic space as discussed in [2] and the previous chapter. All the details can be found in those two papers, so we will be brief. As a first step, we can integrate over the boundary coordinates as follows

$$\int d^{d-1}\mathbf{x}_1 H(\zeta_{x_1 x_2}) = \frac{\pi^{\frac{d-1}{2}} (z_1 z_2)^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty du u^{\frac{d-3}{2}} H\left(\rho_{z_1 z_2} + u\right) = (z_1 z_2)^{\frac{d-1}{2}} h(\rho_{z_1 z_2}) \tag{5.168}$$

where $\rho_{z_1 z_2} = (z_1 - z_2)^2 / z_1 z_2$. This transform can be inverted as

$$H(\zeta) = \frac{1}{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{-d+1}{2}\right)} \int_0^\infty d\rho \rho^{\frac{-d-1}{2}} h(\rho + \zeta) \tag{5.169}$$

Applying this to (5.167) and changing variables to $z = e^{2\theta}$ gives

$$\int d\theta \; h\left(4\sinh^2(\theta_1 - \theta)\right) g_\sigma\left(4\sinh^2(\theta - \theta_2)\right) = \frac{\delta(\theta_1 - \theta_2)}{4N}. \tag{5.170}$$

This can be Fourier transformed as

$$\tilde{h}(k) = \int d\theta e^{ik\theta} h\left(4\sinh^2\theta\right) \implies \tilde{h}(k)\tilde{g}_\sigma(k) = \frac{1}{4N}. \tag{5.171}$$

Then, following [12], consider the function

$$\tilde{g}_{a,b}(k) = \frac{\Gamma\left(a - \frac{i}{4}k\right)\Gamma\left(a + \frac{i}{4}k\right)}{\Gamma\left(b - \frac{i}{4}k\right)\Gamma\left(b + \frac{i}{4}k\right)} \tag{5.172}$$

The inverse Fourier transform of the above function gives

$$\begin{aligned} g_{a,b}\left(4\sinh^2\theta\right) &= \frac{1}{2\pi}\int dk e^{-ik\theta}\tilde{g}_{a,b}(k) \\ &= \frac{4\Gamma(2a)}{\Gamma(b-a)\Gamma(b+a)}\frac{1}{(4\cosh^2\theta)^{2a}}{}_2F_1\left(2a, a+b-\frac{1}{2}; 2a+2b-1; \frac{1}{\cosh^2\theta}\right). \end{aligned} \tag{5.173}$$

We can then transform it into a function of ζ by writing the hypergeometric as a sum and using

$$\frac{1}{\Gamma(\lambda)}\int_0^\infty du u^{\lambda-1}\frac{\Gamma(p+\lambda)}{(4+\rho+u)^{p+\lambda}} = \frac{\Gamma(p)}{(4+\rho)^p}. \tag{5.174}$$

This gives

$$G_{a,b}(\zeta) = \frac{4\Gamma\left(2a+\frac{d-1}{2}\right)}{\Gamma(b-a)\Gamma(b+a)\pi^{\frac{d-1}{2}}(\zeta)^{2a+\frac{d-1}{2}}}{}_2F_1\left(2a+\frac{d-1}{2}, a+b-\frac{1}{2}; 2a+2b-1; -\frac{4}{\zeta}\right). \tag{5.175}$$

For $\sigma^* = d/2 - 1$, we have

$$H(\zeta) = -\frac{4^d\Gamma\left(\frac{d}{2}\right)^2 c_d}{16\pi^d\left(\zeta(4+\zeta)\right)^{d-1}} = \frac{d\Gamma\left(\frac{d}{2}\right)^2 c_d\Gamma\left(-\frac{d}{2}\right)}{16(\pi)^{\frac{d}{2}}\Gamma(d-1)}G_{\frac{3(d-1)}{4},\frac{d+1}{4}}(\zeta). \tag{5.176}$$

This gives

$$\tilde{g}_\sigma(k) = \frac{4(\pi)^{\frac{d}{2}}\Gamma(d-1)}{dN\Gamma\left(\frac{d}{2}\right)^2 c_d\Gamma\left(-\frac{d}{2}\right)}\tilde{g}_{\frac{d+1}{4},\frac{3(d-1)}{4}}(k) \tag{5.177}$$

which gives the σ propagator

$$\begin{aligned} G_\sigma(\zeta) &= \frac{4(\pi)^{\frac{d}{2}}\Gamma(d-1)}{dN\Gamma\left(\frac{d}{2}\right)^2 c_d\Gamma\left(-\frac{d}{2}\right)}G_{\frac{d+1}{4},\frac{3(d-1)}{4}}(\zeta) \\ &= -\frac{2^{4d-5}(d-2)\Gamma\left(\frac{d-1}{2}\right)^2\Gamma(d)}{Nc_d\pi\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\Gamma(2d-2)\zeta^d}\ {}_2F_1\left(d, d-1, 2d-2, -\frac{4}{\zeta}\right). \end{aligned} \tag{5.178}$$

For $\sigma^* = d/2 - 2$, we have

$$\begin{aligned} H(\zeta) &= -\frac{4^d\Gamma\left(\frac{d}{2}-1\right)^2 c_d}{64\pi^d}\left(\frac{(d-2)^2}{\left(\zeta(4+\zeta)\right)^{d-1}} + \frac{(d-1)(d-3)}{4}\frac{1}{\left(\zeta(4+\zeta)\right)^{d-2}}\right) \\ &= -\frac{\Gamma\left(\frac{d}{2}-1\right)^2 c_d\Gamma\left(2-\frac{d}{2}\right)}{32\pi^{\frac{d}{2}}\Gamma(d-2)}\left(-2G_{\frac{3(d-1)}{4},\frac{d+1}{4}}(\zeta) + (d-1)(d-3)G_{\frac{3d-7}{4},\frac{d+1}{4}}(\zeta)\right). \end{aligned} \tag{5.179}$$

This gives

$$\begin{aligned}\tilde{h}(k) &= \frac{\Gamma\left(\frac{d}{2}-1\right)^2 c_d \Gamma\left(2-\frac{d}{2}\right)}{64(4\pi^{\frac{d}{2}})\Gamma(d-2)}\left(k^2+(d-5)^2\right)\tilde{g}_{\frac{3d-7}{4},\frac{d+1}{4}}(k) \\ \implies \left(k^2+(d-5)^2\right)\tilde{g}_\sigma(k) &= \mathcal{B}\pi^{\frac{d-1}{2}}\tilde{g}_{\frac{d+1}{4},\frac{3d-7}{4}}(k)\end{aligned} \tag{5.180}$$

where

$$\mathcal{B} = \frac{64\pi^{\frac{1}{2}}\Gamma(d-2)}{N\Gamma\left(\frac{d}{2}-1\right)^2 c_d \Gamma\left(2-\frac{d}{2}\right)}. \tag{5.181}$$

This gives the following differential equation for the σ propagator in terms of ρ

$$\begin{aligned}&\left(-\frac{d^2}{d\theta^2}+(d-5)^2\right)g_\sigma(4\sinh^2\theta) = \mathcal{B}\pi^{\frac{d-1}{2}}g_{\frac{d+1}{4},\frac{3d-7}{4}}(4\sinh^2\theta) \implies \\ &\left(\rho(4+\rho)\frac{d^2}{d\rho^2}+(\rho+2)\frac{d}{d\rho}-\left(\frac{d-5}{2}\right)^2\right)g_\sigma(\rho) = -\frac{\mathcal{B}\pi^{\frac{d-1}{2}}}{4}g_{\frac{d+1}{4},\frac{3d-7}{4}}(\rho)\end{aligned} \tag{5.182}$$

We can then use (5.169) to get a differential equation in terms of ζ

$$\begin{aligned}&\left(\zeta(4+\zeta)\frac{d^2}{d\zeta^2}+d(\zeta+2)\frac{d}{d\zeta}+2(d-3)\right)G_\sigma(\zeta) \\ &= \frac{2^{d+3}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma(d)}{Nc_d\pi\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(d-\frac{3}{2}\right)\zeta^d}\ {}_2F_1\left(d,d-2,2d-4,-\frac{4}{\zeta}\right).\end{aligned} \tag{5.183}$$

The differential equation has a solution of the form

$$\begin{aligned}&G_\sigma(\zeta) = G_\sigma^P(\zeta) + \\ &\frac{c_1}{(4+\zeta)^2}{}_2F_1\left(2,3-\frac{d}{2},6-d;\frac{4}{4+\zeta}\right)+\frac{c_2}{(4+\zeta)^{d-3}}{}_2F_1\left(d-3,\frac{d}{2}-2,d-4;\frac{4}{4+\zeta}\right)\end{aligned} \tag{5.184}$$

where $G_\sigma^P(\zeta)$ is the particular solution and the second line is the solution to the homogeneous equation. To calculate the particular solution, we recall from (5.180)

$$\tilde{g}_\sigma^P(k) = \frac{\mathcal{B}\pi^{\frac{d-1}{2}}}{(k^2+(d-5)^2)}\frac{\Gamma\left(\frac{d+1}{4}-\frac{ik}{4}\right)\Gamma\left(\frac{d+1}{4}+\frac{ik}{4}\right)}{\Gamma\left(\frac{3d-7}{4}-\frac{ik}{4}\right)\Gamma\left(\frac{3d-7}{4}+\frac{ik}{4}\right)} \tag{5.185}$$

We then need to perform a Fourier transform of this

$$g_\sigma^P(\rho = 4\sinh^2\theta) = \frac{1}{2\pi}\int dk e^{-ik\theta}\tilde{g}_\sigma(k). \tag{5.186}$$

We can do the integral by a contour integration in the upper half $k-$ plane for $\theta < 0$ while in the lower half $k-$ plane for $\theta > 0$. There are poles at $\pm i(5-d)$ and $\pm i(d+1+4n)$. The arc at infinity

can be dropped for $d > 3$, which is the region we are interested in

$$g_\sigma^P(\rho = 4\sinh^2\theta) = -\mathcal{B}\pi^{\frac{d-1}{2}}\Bigg[\frac{\pi 2^{3-d}e^{-((5-d)|\theta|)}}{(d-5)^2\Gamma\left(\frac{d-5}{2}\right)\Gamma\left(\frac{d-1}{2}\right)} + \sum_{n=0}^{\infty}\frac{(-1)^n e^{-(d+4n+1)|\theta|}\Gamma\left(\frac{d+1}{2}+n\right)}{(2n+3)n!(d+2n-2)\Gamma\left(\frac{d}{2}-n-2\right)\Gamma\left(d+n-\frac{3}{2}\right)}\Bigg]. \tag{5.187}$$

Recall that

$$e^{-2a|\theta|} = \left(\frac{\sqrt{\rho}+\sqrt{4+\rho}}{2}\right)^{-2a} = \frac{1}{(4+\rho)^a}{}_2F_1\left(a, a+\frac{1}{2}, 2a+1, \frac{4}{4+\rho}\right) \tag{5.188}$$

and then using (5.174), we can do the integral over ρ

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{-d+1}{2}\right)}\int_0^\infty d\rho\rho^{\frac{-d-1}{2}}\frac{1}{(4+\rho+\zeta)^a}{}_2F_1\left(a, a+\frac{1}{2}, 2a+1, \frac{4}{4+\rho+\zeta}\right)\\
&= \frac{\Gamma\left(a+\frac{d-1}{2}\right)}{\pi^{\frac{d-1}{2}}\Gamma(a)}\frac{1}{(4+\zeta)^{a+\frac{d-1}{2}}}{}_2F_1\left(a+\frac{d-1}{2}, a+\frac{1}{2}, 2a+1, \frac{4}{4+\zeta}\right)\\
&= \frac{\Gamma\left(a+\frac{d-1}{2}\right)}{\pi^{\frac{d-1}{2}}\Gamma(a)}\frac{\zeta+2}{(\zeta(4+\zeta)^{\frac{2a+d+1}{4}}}{}_2F_1\left(\frac{2a+d+1}{4}, \frac{2a-d+5}{4}, a+1, -\frac{4}{\zeta(4+\zeta)}\right).
\end{aligned} \tag{5.189}$$

This gives

$$\begin{aligned}
G_\sigma^P(\zeta) = -\mathcal{B}\Bigg[&\frac{2^{2-d}\cos\left(\frac{\pi d}{2}\right)}{(d-5)\Gamma\left(\frac{d-1}{2}\right)}\frac{1}{(4+\zeta)^2}{}_2F_1\left(2, 3-\frac{d}{2}, 6-d, \frac{4}{(4+\zeta)}\right)\\
&+\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(\frac{d+1}{2}+n\right)\Gamma(d+2n)(\zeta+2)\,{}_2F_1\left(\frac{d+1}{2}+n, n+\frac{3}{2}; \frac{d+4n+3}{2}; -\frac{4}{\zeta(\zeta+4)}\right)}{(2n+3)n!(d+2n-2)\Gamma\left(\frac{d}{2}-n-2\right)\Gamma\left(d+n-\frac{3}{2}\right)\Gamma\left(\frac{d+4n+1}{2}\right)(\zeta(\zeta+4))^{\frac{d+1}{2}+n}}\Bigg].
\end{aligned} \tag{5.190}$$

The first term is also a solution to the homogeneous equation, so we do not need to include it in the particular solution. So we focus on the sum in the second line. By expanding the hypergeometric, it can be rewritten as

$$G_\sigma^P(\zeta) = -\mathcal{B}\frac{\zeta+2}{(\zeta(4+\zeta))^{\frac{d+1}{2}}}\sum_{N=0}^{\infty}\frac{h_N}{N!}\left(-\frac{4}{\zeta(4+\zeta)}\right)^N \tag{5.191}$$

where

$$h_N = \sum_{n=0}^{N} \frac{N! 4^{-n} \Gamma\left(\frac{d+1}{2}+n\right) \Gamma(d+2n) \left(\frac{d+1}{2}+n\right)_{N-n} (n+\frac{3}{2})_{N-n} \left(\frac{d+4n+1}{2}\right) (\frac{d+3}{2}+n+N)_{n-N}}{(2n+3)n!(N-n)!(d+2n-2)\Gamma\left(\frac{d}{2}-n-2\right)\Gamma\left(d+n-\frac{3}{2}\right)\Gamma}$$
$$= \frac{\Gamma(d)\left(\frac{d+1}{2}\right)_N \left(\frac{3}{2}\right)_N}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)\left(\frac{d+3}{2}\right)_N} {}_5F_4 \left[\begin{matrix} \frac{d+1}{2}, \frac{d+5}{4}, \frac{d}{2}-1, 3-\frac{d}{2}, -N; \\ \\ \frac{d+1}{4}, \frac{5}{2}, d-\frac{3}{2}, \frac{d+3}{2}+N; \end{matrix} \; 1 \right]. \tag{5.192}$$

Using a special case of Dougall's theorem [219], we get

$$h_N = \frac{\Gamma(d)\left(\frac{d+1}{2}\right)_N \left(\frac{3}{2}\right)_N \left(\frac{d-1}{2}\right)_N}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)\left(d-\frac{3}{2}\right)_N \left(\frac{5}{2}\right)_N}. \tag{5.193}$$

This finally determines the particular solution

$$G_\sigma^P(\zeta) = \\ \frac{-\mathcal{B}\Gamma(d)}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)} \frac{\zeta+2}{(\zeta(4+\zeta))^{\frac{d+1}{2}}} {}_3F_2\left(\frac{d+1}{2}, \frac{d-1}{2}, \frac{3}{2}; d-\frac{3}{2}, \frac{5}{2}; -\frac{4}{\zeta(4+\zeta)}\right). \tag{5.194}$$

This equation, along with (5.184) gives us a general solution for the σ correlator in this phase. To fix the constants, we note that at the boundary of hyperbolic space, $\zeta \to \infty$, there are two possible decays: ζ^{-2} or ζ^{3-d}, and they correspond to having a scalar of dimension 2 or $d-3$ in the boundary spectrum, respectively. For the former case, we set $c_2 = 0$. To fix c_1, we look at the bulk limit of the correlator, $\zeta \to 0$. In this limit, we expect the leading term to come from identity operator in the bulk channel, and hence should fall off as ζ^{-1} since the σ operator in the bulk has dimension 1 at large N. This fixes c_1 and hence the correlator

$$G_\sigma^D(\zeta) = -\mathcal{B}\Bigg[\frac{2^{2-d}\cos\left(\frac{\pi d}{2}\right)}{(d-5)\Gamma\left(\frac{d-1}{2}\right)} \frac{1}{(4+\zeta)^2} {}_2F_1\left(2, 3-\frac{d}{2}, 6-d, \frac{4}{(4+\zeta)}\right) \\ + \frac{\Gamma(d)}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)} \frac{\zeta+2}{(\zeta(4+\zeta))^{\frac{d+1}{2}}} {}_3F_2\left(\frac{d+1}{2}, \frac{d-1}{2}, \frac{3}{2}; d-\frac{3}{2}, \frac{5}{2}; -\frac{4}{\zeta(4+\zeta)}\right)\Bigg]. \tag{5.195}$$

If we instead demand that the propagator falls of as ζ^{3-d} at the boundary, we set $c_1 = 0$. The same

argument as above fixes c_2 and the correlator turns out to be

$$G_\sigma^N(\zeta) = -\mathcal{B}\Bigg[-\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{d}{2}-1\right)}{8\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\frac{1}{(4+\zeta)^{d-3}}{}_2F_1\left(d-3,\frac{d}{2}-2,d-4,\frac{4}{(4+\zeta)}\right) \\ +\frac{\Gamma(d)}{3\Gamma\left(d-\frac{3}{2}\right)(d-2)\Gamma\left(\frac{d}{2}-2\right)}\frac{\zeta+2}{(\zeta(4+\zeta))^{\frac{d+1}{2}}}{}_3F_2\left(\frac{d+1}{2},\frac{d-1}{2},\frac{3}{2};d-\frac{3}{2},\frac{5}{2};-\frac{4}{\zeta(4+\zeta)}\right)\Bigg]. \tag{5.196}$$

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