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RESEARCH

Can the cosmological ^{7}Li problem be solved in the Weyl-type $f(Q,T)$ modified gravity theory?

Jian Ge[1] · Lei Ming[2,3,1] · Shi-Dong Liang[1] · Hong-Hao Zhang[1] · Tiberiu Harko[4,5]

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Abstract

One of the most powerful evidence for the Big Bang theory is the prediction of the primordial abundances of the elements by the Big Bang Nucleosynthesis (BBN) theory. The BBN theory in its standard formulation is a parameter-free theory, with the precise knowledge of the baryon-to-photon ratio of the Universe, obtained from studies of the anisotropies of cosmic microwave background radiation. The theoretical abundances of light elements during primordial nucleosynthesis and those determined from observations are in good agreement throughout a range of nine orders of magnitude. However, there is still a significant difference of the ^{7}Li abundance, overestimated by a factor of ~ 2.5 when calculated theoretically. In the present work we will consider the nucleosynthesis process in the framework of the Weyl-type f(Q,T) theory, a modified gravity theory representing an extension of the f(Q) and f(Q,T) type theories, obtained under the assumption that the scalar non-metricity Q of the space-time is expressed in its standard Weyl form. Hence, the nonmetricity of the spacetime is fully determined by a vector field w^μ. The theory can give a good description of the observational data, and of the evolution of the late-time Universe. We show that in some parameter ranges the Lithium abundance can be explained, and these ranges have a relatively weak dependence on the initial value of the Weyl vector.

Keywords Big Bang Nucleosynthesis · Weyl-type f(Q,T) theory · Lithium abundance · Freeze-out temperature

1 Introduction

The Big Bang theory is based on three fundamental principle, the Hubble expansion of the Universe, the Cosmic Microwave Background Radiation (CMBR), and the Big Bang Nucleosynthesis (BBN), respectively. A very large number of observational data support the Big Bang theory. In particular, the BBN theory predicts the cosmological abundances of the light elements like D, ^{3,4}He and ^{6,7}Li. After the Big Bang the Universe began expanding rapidly, allowing only the formation of the lightest elements. During the BBN, the formation of unstable or radioactive isotopes like tritium or ^{3}H and ^{7,8}Be also took place. Some stable isotopes were later increased by the decay of these unstable isotopes. Once the temperature and the density of the Universe became lower, the formation of elements heavier than

✉ T. Harko
tiberiu.harko@aira.astro.ro

J. Ge
2672537261@qq.com

L. Ming
minglei@scnu.edu.cn

S.-D. Liang
stslsd@mail.sysu.edu.cn

H.-H. Zhang
zhh98@mail.sysu.edu.cn

1 School of Physics, Sun Yat-Sen University, Guangzhou, 510275, People's Republic of China

2 Key Laboratory of Atomic and Subatomic Structure and Quantum Control(Ministry of Education), Guangdong Basic Research Center of Excellence for Structure and Fundamental Interactons of Matter, School of Physics, South China Normal University, Guangzhou, 510006, People's Republic of China

3 Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials, Guangdong-Hong Kong Joint Laboratory of Quantum Matter, South China Normal University, Guangzhou, 510006, People's Republic of China

4 Department of Physics, Babes-Bolyai University, Kogalniceanu Street, Cluj-Napoca, 400084, Romania

5 Astronomical Observatory, 19 Ciresilor Street, Cluj-Napoca, 400487, Romania

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beryllium could not take place, while light elements became the dominant matter component of the very early Universe. The BBN period lasted about seventeen minutes, beginning after three minutes of the formation of the Universe, and ended when the Universe was about twenty minutes old. Before this period the temperature dropped from beyond 10^{16} GeV $= 1.2 \times 10^{29}$ K to 1 MeV $= 1.2 \times 10^{10}$ K, which represents the temperature at which the density of the nucleon component allowed for nuclear reactions to form stable nuclei (Copi et al. 1995; Fields and Olive 2006). Therefore the small nucleon content and the rapid expansion of space allowed for the production of light nuclides only, such as hydrogen, the most abundant element in the Universe, helium, the second most common element and an isotope of lithium. The Big Bang Nucleosynthesis process lasted until the nuclear reactions ceased due to the cosmic expansion that led to the decrease of the density and temperature of the Universe.

Therefore, BBN provides a means to investigate cosmological scenarios that might have influenced the conditions during the primordial nucleosynthesis of elements (Copi et al. 1995; Fields and Olive 2006; Kneller and Steigman 2004; Serpico et al. 2004; Molaro 2008; Steigman 2007; Simha and Steigman 2008; Steigman 2012; Cyburt et al. 2016; Husdal 2016; Fields et al. 2020; Barrow et al. 2021; Anagnostopoulos et al. 2023).

There is a considerable concordance between the primordial abundances of D and ^{3,4}He as obtained from observations and the theoretical results of the BBN theory. However, those of ^{6,7}Li differ by significantly. The estimations of the abundances in the standard BBN theory depend on three extra factors, namely, the number of light neutrino flavors, the neutron lifetime, as well as the baryon-to photon ratio, and on the astrophysical nuclear reaction rates (Singh et al. 2024). The baryon-to-photon ratio of the Universe can be precisely obtained from the data on cosmic microwave anisotropy obtained by the Planck satellite (Ade et al. 2014, 2016), which is $\eta = 6.0914 \pm 0.0438 \times 10^{-10}$. While the observed abundance of isotopes of hydrogen ^{1,2}H and helium ^{3,4}He are consistent with the theoretical predictions, the observed abundance of lithium is smaller than the theoretically predicted value by a factor of 3-4 (Fields 2011).

The standard model of cosmology incorporates the standard model of particle physics, including three families of neutrinos. The abundances of the light nuclei depend on diverse factors such as temperature, nucleon density, expansion rate, neutrino content and neutrino-antineutrino asymmetry.

The theory of general relativity plays a fundamental role in modern physics and cosmology (Dodelson 2003; Kolb and Turner 1990). Einstein and Hilbert did an extensive use of the Riemannian geometry, endowing a space-time with a metric and an affine structure. The geometric and gravitational properties of the space-time can be described by the curvature tensor.

Shortly after Einstein's theory was developed, Weyl did propose in 1918 (Weyl 1919, 1993) a generalization of the Riemannian geometry, with the main goal of proposing first unified theory of gravity and electromagnetism. For a detailed presentation of Weyl's approach see (Weyl 1918). In the Weyl geometry, the existence of the electromagnetic field can be attributed to the presence of the nonmetricity of the spacetime. In the first 50 years of existence, Weyl geometry was generally neglected by physicists. But recently the interest towards the physical and mathematical applications of the Weyl geometry on both microscopic and on macroscopic levels has significantly increased (Scholz 2017), so that presently Weyl geometry and its effects are at the forefront of the scientific research.

Theoretical investigations suggests the gravitational interaction can be described in (at least) three mathematically equivalent formulation, by using three distinct geometric quantities. The curvature formulation was initially introduced by Einstein. However, gravity can be described by using the torsion T only within the $f(\mathsf{T})$ theory of gravity (Hayashi and Shirafuji 1979; Li et al. 2011; Myrzakulov 2011; Cai et al. 2016; Kavya et al. 2024). The nonmetricity Q can also be used to formulate a theory equivalent to the curvature description of general relativity. Initially proposed in (Nester and Yo 1999), and called the symmetric teleparallel gravity formulation, this approach was further extended into the $f(Q)$ gravity theory (Jiménez et al. 2018; Nashed 2025; Jensko 2025; Heisenberg 2024), which is also called the coincident general relativity.

A generalization of the $f(Q)$ theory was proposed in (Xu et al. 2019), called the $f(Q,T)$ gravity theory, where the gravitational Lagrangian L is given by an arbitrary function f of the non-metricity Q and of the trace of the matter-energy-momentum tensor T. A particular case of the $f(Q,T)$ theory was considered in (Xu et al. 2020), under the assumption that the nonmetricity takes the standard form as introduced in Weyl geometry, via the expression of the covariant derivative as $\nabla_\mu g_{\nu\sigma} = \omega_\mu g_{\nu\sigma}$, where ω_μ is the Weyl vector. The cosmological implications of the $f(Q,T)$ and Weyl-type $f(Q,T)$ theories were extensively investigated recently (Nashed and Harko 2024; Bhagat and Mishra 2024; Chalavadi et al. 2024; Gadbail et al. 2024; Kaczmarek et al. 2025; Belchior et al. 2025; Moreira et al. 2025).

The BBN observational data offers a powerful method to constrain modified gravity theories. BBN constraints on the energy-momentum squared gravity have been obtained in (Akarsu et al. 2024) and (Jang et al. 2025), respectively. The nucleosyntesis process was investigated within the framework of $f(T,\tau)$, where T is the matter energy-momentum tensor, and τ is the torsion, in (Mishra et al. 2024). The parameter χ for the $f(R,T) = R + \chi T$ gravity was estimated in (Bhattacharjee and Sahoo 2020) by using constraints coming from big bang nucleosynthesis. The possibility of primordial black holes with masses $M < 10^9$ g undergo Hawking evaporation, around the BBN epoch was considered in (Boccia et al. 2025). The effects of Barrow cosmology on the primordial abundances of light elements, i.e., helium He^4, deuterium D, and lithium Li^7 were analyzed in (Sheykhi and Shahbazi Sooraki 2025). A BBN simulation with a bubble universe scenario around a rotating black hole in Kerr-AdS5 spacetime was performed in (Dohi et al. 2025) to explain the recently updated observations of light elements such as the primordial helium abundance.

It is the main goal of the present work to consider the nucleosynthesis era within the framework of the Weyl-type $f(Q,T)$ modified gravity theory. In particular, we will focus on the possibility of solving the Li^7 problem by investigating the effects of the corrections to the generalized Friedmann equations of the theory on the Big-Bang Nucleosynthesis. Our present approach is based on the choice $f(Q,T) = \alpha Q + \left(\beta/6\kappa^2\right) T$, where α and β are model parameters, and κ^2 is the gravitational coupling constant.

The observational values are combined with the theoretical results on the ^{4}He mass fraction, and with the numerically computed ^{7}Li abundances to obtain constraints on the parameters of the adopted model (Ge et al. 2024). These results show that in the adopted Weyl-type $f(Q,T)$ model there are ranges of the model parameters α and β that allow to fulfill both the ^{4}He and ^{7}Li constraints. Moreover, we study the effects of the initial values Ψ_{ini} of the Weyl vector on the parameter ranges for this model, by giving the fits of the maximum or minimum values of the parameters.

The present paper is organized as follows. In Sect. 2 we briefly present the fundamentals of the Weyl-type $f(Q,T)$ gravity theory, including its geometric aspects, the action, and the field equations. The generalized Friedmann equations are also written down. In Sect. 3 we introduce the basic results of the standard ΛCDM Big Bang Nucleosynthesis, and we also discuss the lithium problem. The application of a specific Weyl-type $f(Q,T)$ model for the description of the Big Bang Nucleosynthesis, and possible solutions of the lithium problem as well as their fits are presented in Sect. 4. We conclude and discuss our results in Sect. 5.

2 Brief review of Weyl geometry, and of Weyl-type f(Q,T) gravity

In the present Section we introduce the basics of the Weyl geometry, together with the action and the field equations of the Weyl-type $f(Q,T)$ gravity. The generalized Friedmann equations are also written down. In the following we will denote the temperature by $\mathcal{T}$, and the trace T^μ_μ of the energy-momentum tensor by T, respectively.

2.1 Geometry, action, and field equations

If we parallelly transport a vector v^μ along an infinitesimal path, then in Riemann geometry the variation of its components is given by $\delta v^\mu = v^\kappa R^\mu_{\kappa\sigma\nu} s^{\sigma\nu}$, where $R^\mu_{\kappa\sigma\nu}$ is the Riemann curvature tensor, and $s^{\sigma\nu}$ is the area encircled by the loop.

During the parallel transport in Riemann geometry the length of a vector does not vary, and we have $\delta(|v|^2) = \delta(g_{\mu\nu} v^\mu v^\nu) = 2v^\kappa v^\nu R_{\nu\kappa\sigma\rho} s^{\sigma\rho} = 0$.

A simultaneous change of both the direction and the length of a vector during the parallel transport is obtained in Weyl geometry, by introducing an intrinsic Weyl vector field w^μ, and a semi-metric connection. This connection in Weyl geometry is given by (Xu et al. 2020)

$$\bar{\Gamma}^\lambda_{\ \mu\nu} \equiv \Gamma^\lambda_{\ \mu\nu} + g_{\mu\nu} w^\lambda - \delta^\lambda_\mu w_\nu - \delta^\lambda_\nu w_\mu, \tag{1}$$

where $\Gamma^\lambda_{\ \mu\nu}$ denotes the standard Levi-Civita connection. In the following we will denote the quantities in Weyl geometry by a bar.

We define the Weyl geometric curvature tensor is a way similarly to the Riemannian case as

$$\bar{R}^\mu_{\ \nu\beta\alpha} \equiv 2\bar{\nabla}_{[\alpha}\bar{\nabla}_{\beta]} A_\nu = \bar{R}_{(\mu\nu)\alpha\beta} + \bar{R}_{[\mu\nu]\alpha\beta}, \tag{2}$$

with the symmetric part

$$\begin{aligned}\bar{R}_{[\mu\nu]\alpha\beta} =& R_{\mu\nu\alpha\beta} + 2\nabla_\alpha w_{[\mu} g_{\nu]\beta} + 2\nabla_\beta w_{[\nu} g_{\mu]\alpha} \\ &+ 2w_\alpha w_{[\mu} g_{\nu]\beta} + 2w_\beta w_{[\nu} g_{\mu]\alpha} \\ &- 2w^2 g_{\alpha[\mu} g_{\nu]\beta},\end{aligned} \tag{3}$$

anti-symmetric part

$$\bar{R}_{(\mu\nu)\alpha\beta} = g_{\mu\nu} F_{\alpha\beta}, \tag{4}$$

and with the field strength tensor of the vector field w^μ defined as

$$F_{\mu\nu} \equiv \nabla_\nu w_\mu - \nabla_\mu w_\nu, \tag{5}$$

where $R_{\mu\nu\alpha\beta}$ is the Riemann curvature tensor.

From the relation $\delta|v| = |v| F_{\sigma\rho} s^{\sigma\rho}$, we can obtain the geometric interpretation of the Weyl vector, as describing the variation of the length of a vector when it is transported along a closed path. If $F_{\sigma\rho} \equiv 0$, the length of the vector is conserved, with this case corresponding to the Weyl Integrable Geometry (WIG).

From the first and the second contractions of the Weyl curvature tensor we obtain the generalized Ricci tensor $\bar{R}^\mu_\nu$, and the Weyl scalar, with the expression given by

$$\bar{R} = R + 6(\nabla_\nu w^\nu - w^2). \tag{6}$$

The metric tensor has a vanishing covariant derivative in Riemann geometry. In Weyl geometry, the covariant derivative of the metric tensor is not vanishing anymore, and we have

$$Q_{\alpha\mu\nu} \equiv \bar{\nabla}_\alpha g_{\mu\nu} = 2 g_{\mu\nu} w_\alpha, \tag{7}$$

where $Q_{\alpha\mu\nu}$ is the non-metricity tensor. The deformation tensor is defined according to

$$L^\lambda_{\ \mu\nu} \equiv -\frac{1}{2} g_{\lambda\gamma}(Q_{\mu\gamma\nu} + Q_{\nu\gamma\mu} - Q_{\gamma\mu\nu}) = -\delta^\lambda_\nu w_\mu - \delta^\lambda_\mu w_\nu + g_{\mu\nu} w^\lambda. \tag{8}$$

The non-metricity scalar is obtained as

$$Q \equiv -g^{\mu\nu}(L^\alpha_{\ \beta\mu} L^\beta_{\ \nu\alpha} - L^\alpha_{\ \beta\alpha} L^\beta_{\ \mu\nu}) = -6w^2. \tag{9}$$

The action of the Weyl-type $f(Q,T)$ modified gravity theory was introduced in (Xu et al. 2020), and it has the form

$$\mathcal{S} = \int d^4x \sqrt{-g}[\kappa^2 f(Q,T) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} m^2 w^2 + \mathcal{L}_m], \tag{10}$$

where $\kappa^2 = 1/16\pi G$, and $\mathcal{L}_m$ denotes the ordinary matter Lagrangian. The flatness condition, which requires $\bar{R} = 0$, is also imposed. This condition is included in the action with the help of a Lagrangian multiplier λ. Hence, we introduce the action of the Weyl-type $f(Q,T)$ gravity via the definition

$$\mathcal{S} = \int d^4x \sqrt{-g}[\kappa^2 f(Q,T) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} m^2 w^2 + \lambda(R + 6\nabla_\alpha w^\alpha - 6w^2) + \mathcal{L}_m]. \tag{11}$$

In the following we denote $f_T = \partial f(Q,T)/\partial T$, and $f_Q = \partial f(Q,T)/\partial Q$. After varying the action with respect to the metric $g^{\mu\nu}$ and making $\delta S = 0$, we obtain the field equations (Xu et al. 2020)

$$\frac{1}{2}(T_{\mu\nu} + S_{\mu\nu}) - \kappa^2 f_T (T_{\mu\nu} + \Theta_{\mu\nu}) + \frac{\kappa^2}{2} g_{\mu\nu} f = -6\kappa^2 f_Q w_\mu \omega_\nu + \lambda(R_{\mu\nu} - 6w_\mu w_\nu + 3g_{\mu\nu}\nabla_\rho w^\rho) + 3g_{\mu\nu} w^\rho \nabla_\rho \lambda - 6w_{(\mu}\nabla_{\nu)}\lambda + g_{\mu\nu}\Box\lambda - \nabla_\mu\nabla_\nu\lambda, \tag{12}$$

In Eq. (12) we have introduced the ordinary matter energy-momentum tensor, defined according to

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} = g_{\mu\nu} - 2\frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}, \tag{13}$$

and the tensor $\Theta_{\mu\nu}$, obtained as

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}_m - 2T_{\mu\nu}. \tag{14}$$

The tensor $S_{\mu\nu}$ represents the energy-momentum tensor of the Proca field

$$S_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma} + F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{2} m^2 g_{\mu\nu} w^2 + m^2 w_\mu w_\nu. \tag{15}$$

The field equations of the Weyl-type $f(Q,T)$ theory can be rewritten in a form similar to the standard Einstein gravitational equations as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2\lambda}(T_{\mu\nu} + S_{\mu\nu}) - \frac{\kappa^2}{\lambda} f_T (T_{\mu\nu} + \Theta_{\mu\nu}) + \frac{\kappa^2}{2\lambda} g_{\mu\nu} f + \frac{6\kappa^2}{\lambda} f_Q w_\mu w_\nu - \frac{3}{\lambda} g_{\mu\nu} w^\rho \nabla_\rho \lambda + \frac{6}{\lambda} w_{(\mu}\nabla_{\nu)}\lambda - \frac{1}{\lambda}(g_{\mu\nu}\Box\lambda - \nabla_\mu\nabla_\nu\lambda) + 6w_\mu w_\nu - 3g_{\mu\nu} w^2. \tag{16}$$

For the trace of the tensor $S_{\mu\nu}$ we have $S = S^\mu_\mu = -m^2\omega^2$.

Now we vary the action with respect to the Weyl vector w^μ. Thus, the field equation of the Weyl vector is obtained as

$$\nabla^\nu F_{\mu\nu} - (m^2 + 12\kappa^2 f_Q + 12\lambda) w_\mu = 6\nabla_\mu \lambda, \tag{17}$$

where we have denoted

$$m^2_{\text{eff}} = m^2 + 12\kappa^2 f_Q + 12\lambda. \tag{18}$$

2.2 The generalized Friedmann equations

For the investigation of the cosmological applications of the Weyl-type $f(Q,T)$ gravity we adopt the flat, homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which is given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j, \tag{19}$$

wherein $a(t)$ is the scale factor. The Hubble function is defined according to $H = \dot{a}/a$, with a dot denoting the derivative with respect to the cosmological time. The isotropy and the homogeneity of the geometry imposes for the Weyl vector the following form

$$w_\mu = (\Psi(t), 0, 0, 0). \tag{20}$$

For this choice we obtain $w^2(t) = -\Psi^2(t)$. The flatness condition (zero total curvature) gives

$$\dot{H} + 2H^2 - \dot{\Psi} - 3H\Psi + \Psi^2 = 0,$$

$$(\dot{\Psi}-\dot{H})-(\Psi-2H)(\Psi-H)=0. \tag{21}$$

If the equation $\Psi-H=0$, is satisfied at a certain moment in time, then we will have $\Psi^{(n)}-H^{(n)}=0$, for any time, and consequently $\Psi-H=0$, will be always true for any time interval. Otherwise, we obtain

$$\frac{d(\Psi-H)}{dt}-(\Psi-2H)=0. \tag{22}$$

By using the FLRW metric and the field equation (12), the energy balance equation and the generalized Friedmann equations can be written in the Weyl-type $f(Q,T)$ gravity as

$$\dot{\rho}+3H(\rho+p)=\frac{-\kappa^2}{1+2\kappa^2 f_T}[2(\rho+p)\dot{f}_T +f_T(\dot{\rho}-\dot{p})], \tag{23}$$

$$\begin{aligned}3H^2=&\frac{1}{2\lambda}[(1+2\kappa^2 f_T)\rho+2\kappa^2 f_T p-\kappa^2 f\\&-\frac{m^2\psi^2}{2}-6\lambda\psi^2+m_{\rm eff}^2 H\psi]\\=&\frac{1}{2\lambda}(\rho+\rho_{\rm eff}),\end{aligned} \tag{24}$$

$$\begin{aligned}2\dot{H}=&\frac{-1}{2\lambda}[(1+2\kappa^2 f_T)(\rho+p)-\frac{m_{\rm eff}^2}{3}(\dot{\psi}+\psi^2\\&-H\psi)-4\kappa^2\dot{f}_Q\psi]\\=&\frac{-1}{2\lambda}(\rho+\rho_{\rm eff}+p+p_{\rm eff}),\end{aligned} \tag{25}$$

respectively, wherein we have denoted

$$\begin{aligned}\rho_{eff}\equiv& m_{eff}^2 H\Psi+2\kappa^2 f_T(\rho+p)-\kappa^2 f\\&-\frac{m^2\Psi^2}{2}-6\lambda\Psi^2,\end{aligned} \tag{26}$$

and

$$\begin{aligned}p_{eff}\equiv&-\frac{m_{eff}^2}{3}(\dot{\Psi}+\Psi^2-H\Psi)-4\kappa^2\dot{f}_Q\Psi\\&-\left(m_{eff}^2 H\Psi-\kappa^2 f-\frac{m^2\Psi^2}{2}-6\lambda\Psi^2\right)\\=&-\frac{m_{eff}^2}{3}(\dot{\Psi}+\Psi^2+2H\Psi)-4\kappa^2\dot{f}_Q\Psi\\&+\kappa^2 f+\frac{m^2\Psi^2}{2}+6\lambda\Psi^2,\end{aligned} \tag{27}$$

respectively.

3 Standard Big Bang Nucleosynthesis and the lithium problem

In the present Section we briefly introduce the fundamentals of the standard BBN theory, formulated within the framework of the ΛCDM model. We also present the lithium problem, as it originates from the observational data.

3.1 Standard BBN theory

The standard BBN theory is essentially based on the general relativistic cosmological models, and on the radiation thermodynamics.

In the ΛCDM paradigm, the two Friedmann equations that describe the cosmological evolution are

$$3H_{GR}^2=\frac{\rho}{2\kappa^2},\ \ 2\dot{H}_{GR}=-\frac{1}{2\kappa^2}(\rho+p), \tag{28}$$

where by H_{GR} we have denoted the value of the standard Hubble function, as obtained in the framework of general relativity. The content of the early Universe consisted mostly of radiation, with the equation of state $p=\rho/3$, and with a temperature dependent energy density given by

$$\rho\approx\rho_r=\frac{\pi^2 g_*}{30}\mathcal{T}^4. \tag{29}$$

Here g_* is the effective number of degrees of freedom given by $g_*=10.75$. For the standard cosmology, the Hubble function in the early Universe is mainly determined by the temperature of the relativistic gas:

$$H_{GR}=\sqrt{\frac{\pi^2 g_*}{180\kappa^2}}\mathcal{T}^2. \tag{30}$$

In the radiation-dominated era, from the energy conservation equation we can easily obtain the expression of the radiation energy density, after we omit the rest mass of the particles, as

$$\rho=\frac{\rho_1 a_1^4}{a^4}, \tag{31}$$

where a_1 and ρ_1 are the scale factor and the energy density of an arbitrary time of this era. Combining the above relation with Eqs. (28) we obtain

$$a=\left(\frac{2\rho_1 a_1^4}{3\kappa^2}t^2\right)^{1/4}, \tag{32}$$

and

$$\frac{1}{t}=\sqrt{\frac{2\rho_r}{3\kappa^2}}=\sqrt{\frac{\pi^2 g_*}{45\kappa^2}}\mathcal{T}^2, \tag{33}$$

respectively.

The conversion rate between protons and neutrons is given by (Dodelson 2003)

$$\begin{aligned}\Gamma =&\lambda_{np} = \frac{255}{\tau_n x^5}(12 + 6x + x^2)\\ =&A(12\mathcal{T}^5 + 6\mathcal{Q}\mathcal{T}^4 + \mathcal{Q}^2\mathcal{T}^3),\end{aligned} \tag{34}$$

where $x \equiv \mathcal{Q}/\mathcal{T}$, and $A \equiv 5.235 \times 10^{-11}$ GeV^{-4}.

When the temperature is large, the total conversion rate is much larger than the Hubble rate, and the protons and the neutrons are in the equilibrium. The equilibrium situation is maintained until a freeze-out temperature is reached, when the two particles decouple from each other. The freeze-out temperature of the neutron-proton conversion satisfies the condition

$$H_{GR}(\mathcal{T}_f) \approx \Gamma(\mathcal{T}_f). \tag{35}$$

At the freeze-out time, the neutron-proton ratio is given as

$$\left(\frac{n}{p}\right)_f \simeq e^{-\frac{\mathcal{Q}}{\mathcal{T}_f}} \tag{36}$$

The neutron mass fraction is defined as

$$X \equiv \frac{n}{n+p}, \tag{37}$$

and it satisfies the differential equation (Dodelson 2003)

$$\frac{dX}{dx} = \frac{x\lambda_{np}}{\sqrt{\frac{4\pi^3 G\mathcal{Q}^4 g_*}{45}}}[e^{-x} - X(1+e^{-x})], \tag{38}$$

with the general solution

$$\begin{aligned}X(x) =&0.5e^{-\mu(x)}\\ &+ \int_{x_i}^{x} \frac{x'\lambda_{np}(x')e^{-x'}}{\sqrt{\frac{4\pi^3 G\mathcal{Q}^4 g_*}{45}}} e^{\mu(x')-\mu(x)}dx'.\end{aligned} \tag{39}$$

Hence we obtain

$$X(\infty) = 0.148. \tag{40}$$

The value of X_f in the standard BBN is also represented as

$$X_f = \frac{e^{-\frac{\mathcal{Q}}{\mathcal{T}_f}}}{1+e^{-\frac{\mathcal{Q}}{\mathcal{T}_f}}} = X(\infty). \tag{41}$$

Thus we have

$$\left(\frac{n}{p}\right)_f = 0.1737, \quad \mathcal{T}_f = 0.7387 \text{ MeV}, \tag{42}$$

With the degrees of freedom in the freeze-out given by $g_* = 10.75$ (Husdal 2016), with the equation (33), we obtain for the freeze-out time $t_f = 1.35$ s.

If we suppose that all neutrons decay into ^{4}He after the BBN, the ^{4}He mass fraction is given by

$$Y_P = 2X_n(t_N) = \chi \frac{2e^{-\frac{\mathcal{Q}}{\mathcal{T}_f}}}{1+e^{-\frac{\mathcal{Q}}{\mathcal{T}_f}}}. \tag{43}$$

From the observational data, we have the ^{4}He mass fraction

$$Y_P = 0.245, \quad |\delta Y_P| < 2 \times 10^{-3}. \tag{44}$$

Combining this result with the freeze-out time, we obtain

$$t_N = 167.34s = 2.54 \times 10^{26} \text{ GeV}^{-1}. \tag{45}$$

We input this value into the equation (33), and by assuming the effective degrees of freedom as $g_* = 3.37$, we obtain the nucleosynthesis temperature of the deuterium as $\mathcal{T}_N = 8.83 \times 10^{-2}$ MeV.

Figure 1 shows the time evolution of the light isotope abundances, and the dependence of the light isotope abundances on the baryon-photon ratio, respectively. As one can see from the Figure, the abundances of the lighter isotopes begin to increase early than those of the heavier isotopes. It is also worth noting that ^{3}H decays into ^{3}He, and ^{7}Be decays into ^{7}Li shortly after the BBN.

3.2 The lithium problem

Lithium is also produced during the BBN period, and it has two stable isotopes, ^{6}Li and ^{7}Li, wherein the latter is absolutely dominant. The numerical fit of the lithium abundance is given as below (Steigman 2012),

$$\frac{\text{Li}}{\text{H}} = 4.82(1 \pm 0.1) \times 10^{-10} \times \left(\frac{10^{10}\eta - 3(Z-1)}{6}\right)^2, \tag{46}$$

where we have denoted $Z \equiv H/H_{GR}$. In the standard BBN case, $Z = 1$, and we obtain

$$\frac{\text{Li}}{\text{H}} = 4.82(1 \pm 0.1) \times 10^{-10} \times \left(\frac{10^{10}\eta}{6}\right)^2. \tag{47}$$

With the use of the observed value of the baryon-photon ratio

$$\eta = 6.1 \times 10^{-10}, \tag{48}$$

we obtain

$$\frac{\text{Li}}{\text{H}} = 4.98(1 \pm 0.1) \times 10^{-10}, \tag{49}$$

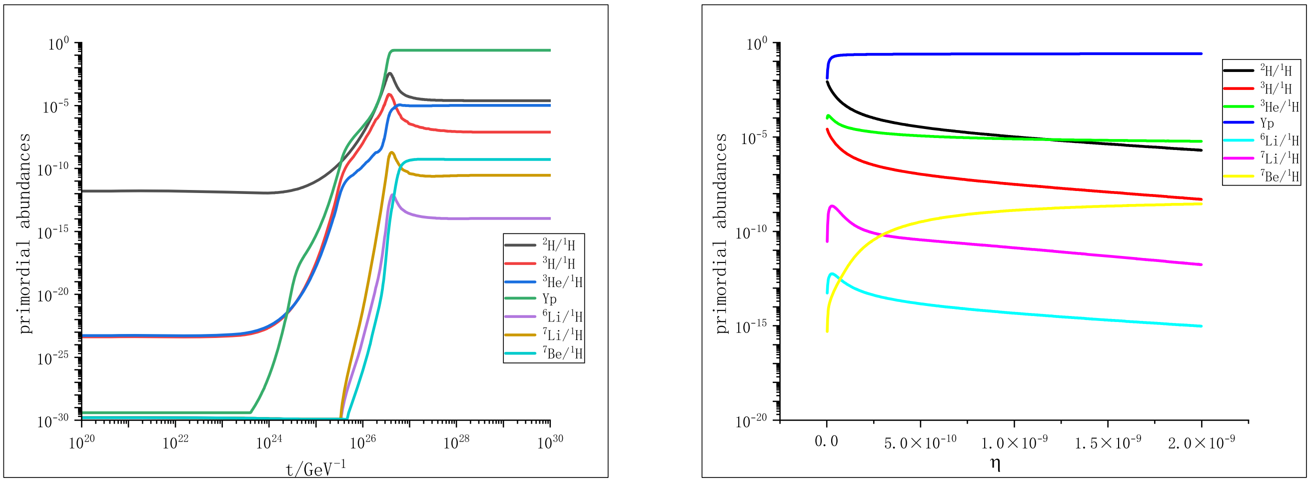

Fig. 1 The evolutions of the light isotope abundances around the BBN epoch, wherein Y_p denotes the ^{4}He mass fraction (left panel), and the dependence of the light isotope abundances on the baryon-photon ratio η (right panel) for the standard BBN model. For the left panel, the baryon-photon ratio is fixed at $\eta = 6.1 \times 10^{-10}$

which is three times larger than the observed value

$$\left.\frac{\text{Li}}{\text{H}}\right|_{obs} = (1.6 \pm 0.3) \times 10^{-10}. \tag{50}$$

3.3 The freeze-out constraint

By taking the time derivative of Eq. (33) we obtain

$$\frac{dt_f}{d\mathcal{T}_f} = -\frac{2t_f}{\mathcal{T}_f}. \tag{51}$$

The time derivative of the right hand side of Eq. (43) gives

$$\frac{dY_P}{d\mathcal{T}_f} = -\frac{2t_f}{\tau_n \mathcal{T}_f} Y_P + \frac{\mathcal{Q}}{\mathcal{T}_f^2(1+x)} Y_P$$
$$= -\frac{2t_f}{\tau_n \mathcal{T}_f} Y_P + (1 - \frac{Y_P}{2\chi}) \ln(\frac{2\chi}{Y_P}$$
$$- 1)\frac{Y_P}{\mathcal{T}_f}. \tag{52}$$

Hence, the deviation of the ^{4}He abundance can be estimated in terms of the deviation of the freeze-out temperature

$$\delta Y_P = Y_P[(1 - \frac{Y_P}{2\chi}) \ln(\frac{2\chi}{Y_P} - 1) - \frac{2t_f}{\tau_n}] \frac{\delta \mathcal{T}_f}{\mathcal{T}_f}. \tag{53}$$

From the observational constraint (44) we obtain the constraint on the freeze-out temperature as

$$\left|\frac{\delta \mathcal{T}_f}{\mathcal{T}_f}\right| < 5.48 \times 10^{-3}. \tag{54}$$

The abundance of ^{4}He strongly depends on the freeze-out temperature. It is important to note that this relation remains unaffected by the change of the cosmological models. Hence, the constraint on the ^{4}He abundance can be transformed into the constraint on the freeze-out temperature $\mathcal{T}_f$.

Thus, it is not necessary to use the value of Y_P. What we need to satisfy is the condition that the range of possible values of the freeze-out temperature is neither too large, nor too small. This is necessary to satisfy the constraint on the ^{4}He abundance, regardless of the specific form of the freeze-out temperature.

The standard BBN model fails to explain the observed abundance of lithium, as given in Eq. (50). However, the new modified gravity model we are examining can account for this abundance, without violating the freeze-out temperature condition (54).

In the following Section, we will explore the lithium problem within the context of a specific model based on the Weyl-type $f(Q,T)$ gravity.

4 Nucleosynthesis in Weyl-type f(Q,T) gravity - a possible solution to the lithium problem

In the present Section we first consider the set of constraints on a specific cosmological model, constructed in the Weyl-type $f(Q,T)$ gravity theory, which can be derived from the BBN observations. Secondly, we consider in detail the possibilities of solving the lithium problem within the framework of this particular cosmological model.

4.1 Weyl-type f(Q,T) cosmology

In our modified Big Bang Nucleosynthesis (BBN) scenario, the matter-radiation ratio and the effective degrees of freedom of particles are the same as in the standard BBN model, for the same temperature and baryon-photon ratio η. The only novel aspects introduced by the Weyl-type $f(Q,T)$ approach are the modified Friedmann equations, and the energy density conservation equation, as one can see from Eqs. (23), (24), and (25), respectively.

Therefore, in our new BBN scenario, the early Universe is primarily composed of radiation, which satisfies the equation of state $p=\rho/3$. Adopting for λ the value $\lambda=\kappa^2$, Eqs. (23), (24), and (25) take the form

$$\dot{\rho}+4H\rho=\frac{-\kappa^2}{1+2\kappa^2 f_T}\left(\frac{8}{3}\rho\dot{f}_T+\frac{2}{3}f_T\dot{\rho}\right), \tag{55}$$

$$H^2=\frac{1}{6\kappa^2}\left[\left(1+\frac{8}{3}\kappa^2 f_T\right)\rho-\kappa^2 f-\frac{m^2\Psi^2}{2}-6\kappa^2\Psi^2+m_{\text{eff}}^2 H\Psi\right], \tag{56}$$

$$\dot{H}=\frac{-1}{4\kappa^2}\left[\left(\frac{4}{3}+\frac{8}{3}\kappa^2 f_T\right)\rho-\frac{m_{eff}^2}{3}(\dot{\Psi}+\Psi^2-H\Psi)-4\kappa^2\dot{f}_Q\Psi\right]. \tag{57}$$

4.1.1 The numerical approach

We can obtain the basic physical quantities $\mathcal{T}$ and Ψ in each step, and then the $\mathcal{T}_f$ and the light isotope abundances, by using the modified version of the BBN code **alterbbn_v2.2** (Arbey 2012). The evolution of these quantities depends on the model parameters, and the initial value of the Weyl field Ψ, denoted as Ψ_{ini}. This value corresponds to the initial temperature $\mathcal{T}=2.7\times10^{10}$ K $=2.33$ MeV. To apply the modified gravity theory, we need to modify the BBN code **alterbbn_v2.2**, according to the equations (21), (56), and (57). Note that these equations are independent of one another.

In our computations of the new BBN models we use the baryon-to-photon ratio obtained from the CMB, $\eta=6.1\times10^{-10}$. Our aim is to identify the parameter ranges that satisfy both the $\mathcal{T}_f$ and the ^{7}Li abundance constraints.

4.2 A specific f(Q,T) model

We assume in the following that the function f is given by

$$f(Q,T)=\alpha Q+\frac{\beta}{6\kappa^2}T, \tag{58}$$

where α and β are constants. In this case $f_Q=\alpha$, and $f_T=\beta/6\kappa^2$. It is worth noting that there is no multiplicative term between Q and T, and f contains only linear term in Q and T, respectively, which means there is no coupling between Q and T.

For the adopted model the modified Friedmann equations (56) and (57) take the form

$$H^2=\frac{1}{108\kappa^2}[(18+8\beta)\rho+18m_{\text{eff}}^2 H\Psi-9m_{\text{eff}}^2\Psi^2], \tag{59}$$

and

$$\dot{H}=\frac{-1}{12\kappa^2}[\frac{4}{3}(3+\beta)\rho+m_{\text{eff}}^2(-\dot{\Psi}-\Psi^2+H\Psi)], \tag{60}$$

respectively, where the effective mass is given by

$$m_{\text{eff}}^2=m^2+12(1+\alpha)\kappa^2. \tag{61}$$

The energy balance equation (55) becomes

$$\dot{\rho}+4H\rho=\frac{-\beta}{9+3\beta}\dot{\rho}. \tag{62}$$

In the following, we have selected five Ψ values: -2×10^{-24} GeV, -1×10^{-24} GeV, 0, 1×10^{-24} GeV and 2×10^{-24} GeV, to compute the BBN constraints.

We set Ψ_{ini} in such a range because we have assumed Ψ_{ini} should be smaller than the Hubble rate of standard general relativity obtained for the temperature value $\mathcal{T}_{ini}=2.33$ MeV, i.e., $H_{GR}(\mathcal{T}_{ini})=\sqrt{\frac{\pi^2 g_*}{180\kappa^2}}\mathcal{T}_{ini}^2=2.4\times10^{-24}$ GeV.

We must also note that α and m are not independent parameters, since Eqs. (59), (60) and (62), do not involve an independent α or an independent m, as both parameters are combined into the effective mass m_{eff}^2. The combined parameter $\alpha+\frac{m^2}{12\kappa^2}$, is thus related to the effective mass through the linear relation $m_{\text{eff}}^2=12\kappa^2\left(\alpha+\frac{m^2}{12\kappa^2}+1\right)$. Hence, the two parameters α and m always appear in the linear combination $\alpha+m^2/12\kappa^2$, and thus we can replace these two parameters with a single new parameter in all our estimations.

Figure 2 shows the time evolution of the light isotope abundances, and the dependence of the light isotope abundances on the baryon-photon ratio, respectively for this new BBN model. As one can see, this Figure is similar to Fig. 1.

4.2.1 The freeze-out temperature constraint

The freeze-out temperature $\mathcal{T}_f$ satisfies the relation (35), and based on it we can numerically determine $\mathcal{T}_f$ for the model. After adopting the above mentioned Ψ_{ini} values, we can obtain the constraints on the model parameters $(\alpha+m^2/12\kappa^2,\beta)$ in Fig. 3, wherein we have solely shown

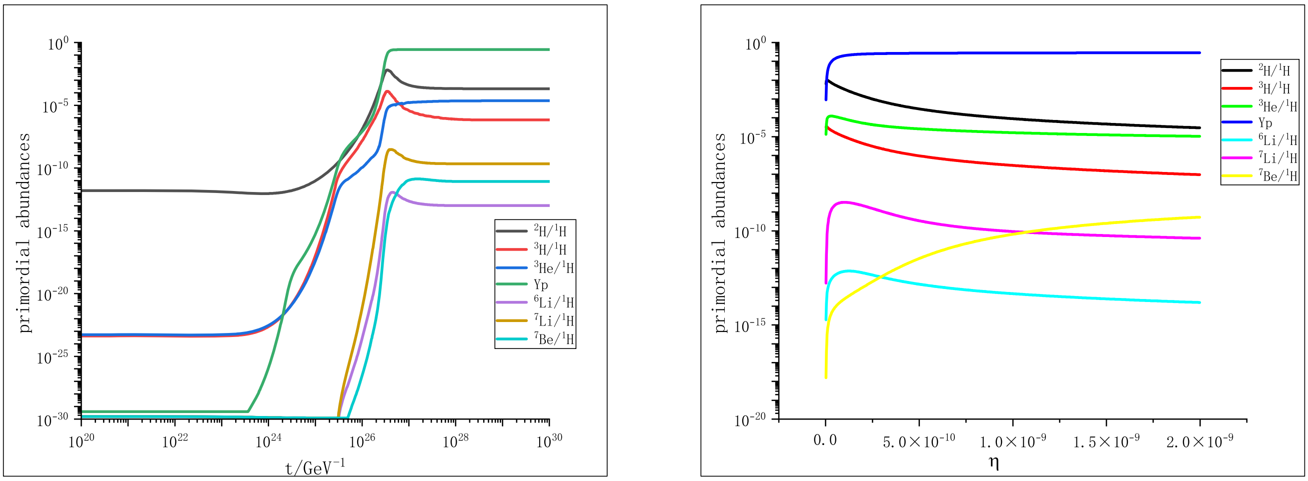

Fig. 2 The evolutions of the light isotope abundances around the BBN epoch, wherein Y_p denotes the ^{4}He mass fraction (left panel), and the dependence of the light isotope abundances on the baryon-photon ratio η (right panel) for the specific Weyl-type f(Q,T) BBN model, with $\alpha = -0.8$, $\beta = 1$ and $\Psi_{ini} = 0$. For the left panel, the baryon-photon ratio is fixed at $\eta = 6.1 \times 10^{-10}$

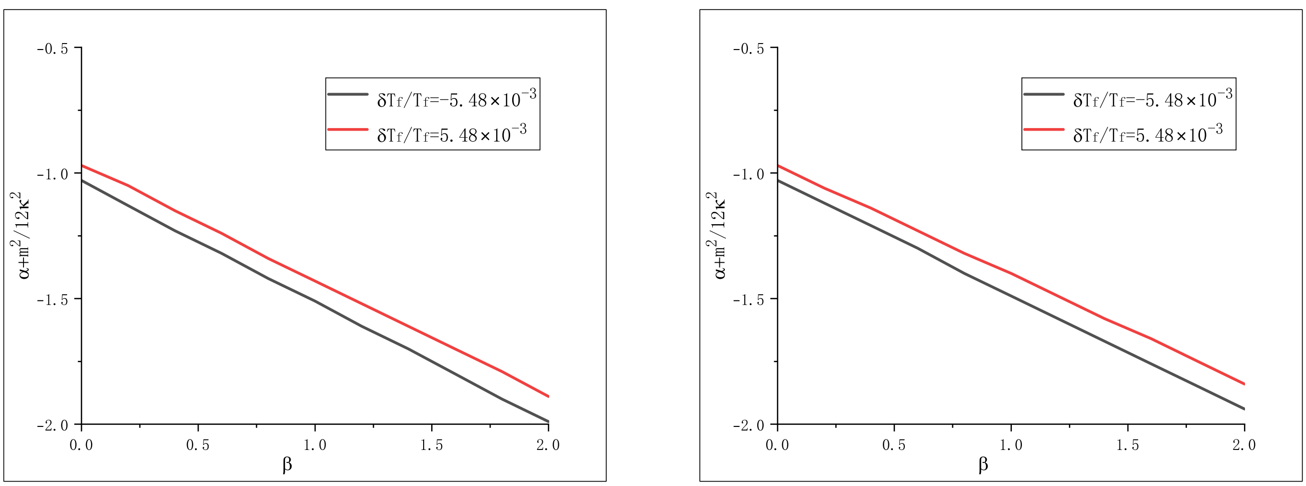

Fig. 3 Numerical freeze-out temperature constraint on the parameters $(\alpha + m^2/12\kappa^2, \beta)$ in the Weyl-type $f(Q,T)$ cosmological model. The two panels correspond to the Ψ_{ini} values -2×10^{-24} GeV and 2×10^{-24} GeV, respectively. The permitted region of the parameters is the area between the two curves.

the plots corresponding to two Ψ_{ini} values, -2×10^{-24} GeV and 2×10^{-24} GeV, respectively. It is worth noting that the two curves in each panel are all nearly linear.

4.2.2 The lithium-hydrogen ratio constraint

The ^{7}Li constraint area is presented in Fig. 4. Here we have adopted the same Ψ_{ini} values as in Fig. 3. The constraint area of ^{7}Li is quite different for the Ψ_{ini} value 2×10^{-24} GeV when $\alpha + m^2/12\kappa^2 \lesssim -2$, while the constraint areas for the other Ψ_{ini} values are similar to that for -2×10^{-24} GeV. This difference may signify a change of the lithium abundance dependence on $(\alpha + m^2/12\kappa^2)$, as shown in Fig. 5, but it has little effects on the common permitted area of the $\mathcal{T}_f$ constraint and the ^{7}Li constraint.

4.2.3 Combining lithium and freeze-out temperature constraints

From Fig. 6 we can see that there is an overlap between the $\mathcal{T}_f$ constraint and the ^{7}Li constraint for the two selected Ψ_{ini} values. The Figure has been obtained as the combination of Figs. 3 and 4.

We can find two areas enclosed by both the $\mathcal{T}_f$ boundary and the ^{7}Li abundance boundary in each subfigure, which are slightly different for these two Ψ_{ini} values.

These two separate areas correspond to different cosmological scenarios. The second area corresponds to the scenarios that show the strongest deviations from the standard

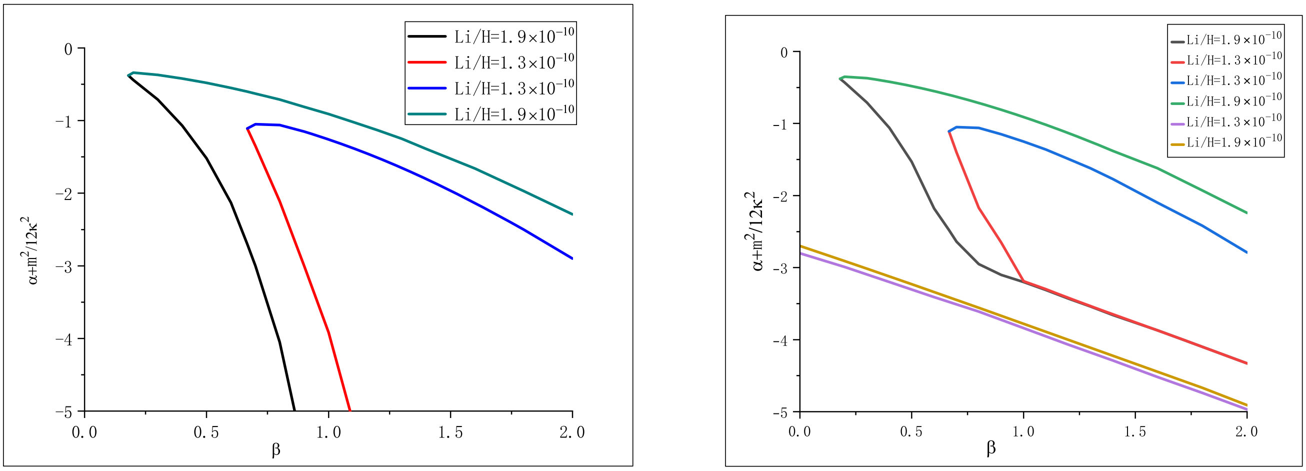

Fig. 4 Numerical Lithium abundance constraints on the parameters $(\alpha + m^2/12\kappa^2, \beta)$ in the Weyl-type $f(Q, T)$ cosmological model. The two panels correspond to these Ψ_{ini} values -2×10^{-24} GeV and 2×10^{-24} GeV, respectively. The permitted region of the parameters is the area between the two curves.

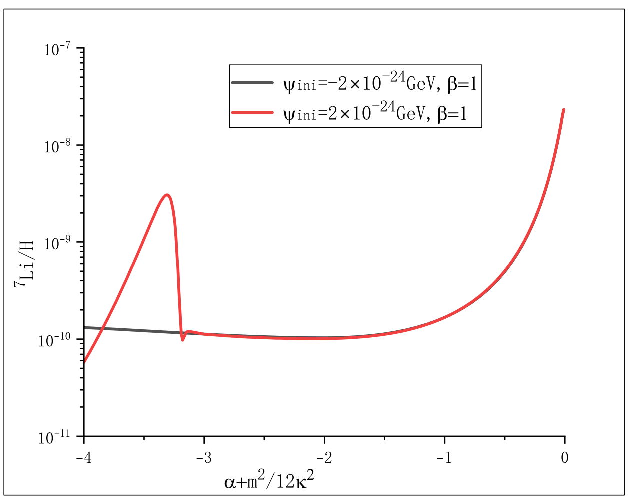

Fig. 5 Lithium abundance dependence on the parameter $(\alpha + m^2/12\kappa^2)$. The two panels correspond to these Ψ_{ini} values -2×10^{-24} GeV and 2×10^{-24} GeV, respectively. β is set at $\beta = 1$

primordial nucleosynthesis scenario, which corresponds to the parameter values of $\alpha + \frac{m^2}{12\kappa^2} = -1$ and $\beta = 0$.

Then we can obtain the constraint ranges of parameters for each Ψ_{ini} value as follows

When $\Psi_{ini} = -2 \times 10^{-24}$ GeV,

- First area

$$-1.3721 < \alpha + \frac{m^2}{12\kappa^2} < -1.159,$$
$$0.4195 < \beta < 0.7027, \tag{63}$$

- Second area

$$-1.867 < \alpha + \frac{m^2}{12\kappa^2} < -1.5252,$$
$$1.202 < \beta < 1.7337. \tag{64}$$

When $\Psi_{ini} = 2 \times 10^{-24}$ GeV,

- First area

$$-1.3461 < \alpha + \frac{m^2}{12\kappa^2} < -1.1491,$$
$$0.4195 < \beta < 0.6949, \tag{65}$$

- Second area

$$-1.8164 < \alpha + \frac{m^2}{12\kappa^2} < -1.4892,$$
$$1.2016 < \beta < 1.7287. \tag{66}$$

We can get the constraint parameter ranges for these Ψ_{ini} values in Figs. 7 and 8.

Although these ranges are relatively close, we can see that the extreme values of β have a quadratic dependence on Ψ_{ini}, and the extreme values of $\alpha + \frac{m^2}{12\kappa^2}$ have a linear dependence on Ψ_{ini}.

The fits of the values corresponding to Figs. 7 and 8, respectively, are

- For the left panel of Fig. 7,

$$\beta_{\min} = -8.4 \times 10^{-4} x^2 + 0.42293,$$
$$\beta_{\max} = 2.06 \times 10^{-3} x^2 - 1.58 \times 10^{-3} x + 0.69047. \tag{67}$$

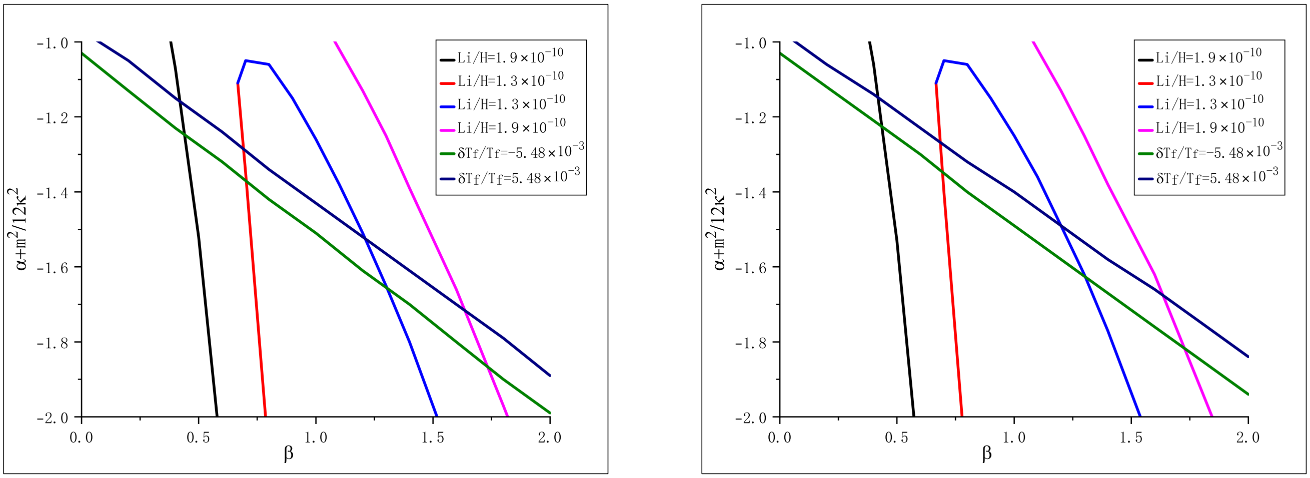

Fig. 6 The combined freeze-out temperature and Lithium abundance constraints on the parameters $\left(\alpha + m^2/12\kappa^2\right)$ and β. The two panels correspond to the Ψ_{ini} values -2×10^{-24} GeV and 2×10^{-24} GeV, respectively. The overlap between the freeze-out temperature constraint area and the Lithium constraint area is represented in the two areas enclosed by both the $\mathcal{T}_f$ boundary and the Lithium abundance boundary

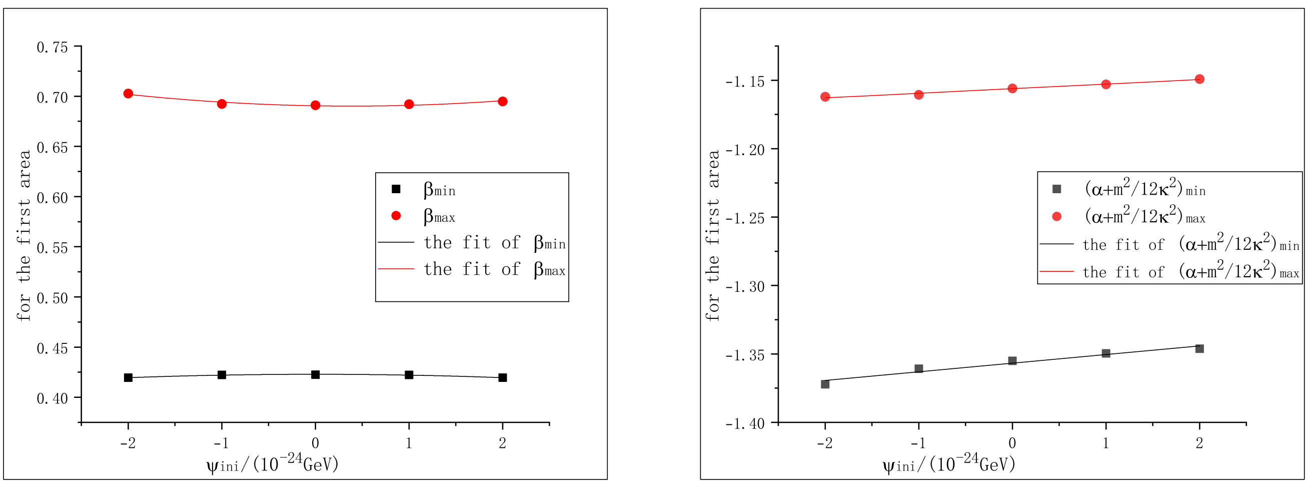

Fig. 7 The maximum and minimum β values (left panel) and the $\alpha + \frac{m^2}{12\kappa^2}$ values (right panel) for the first area

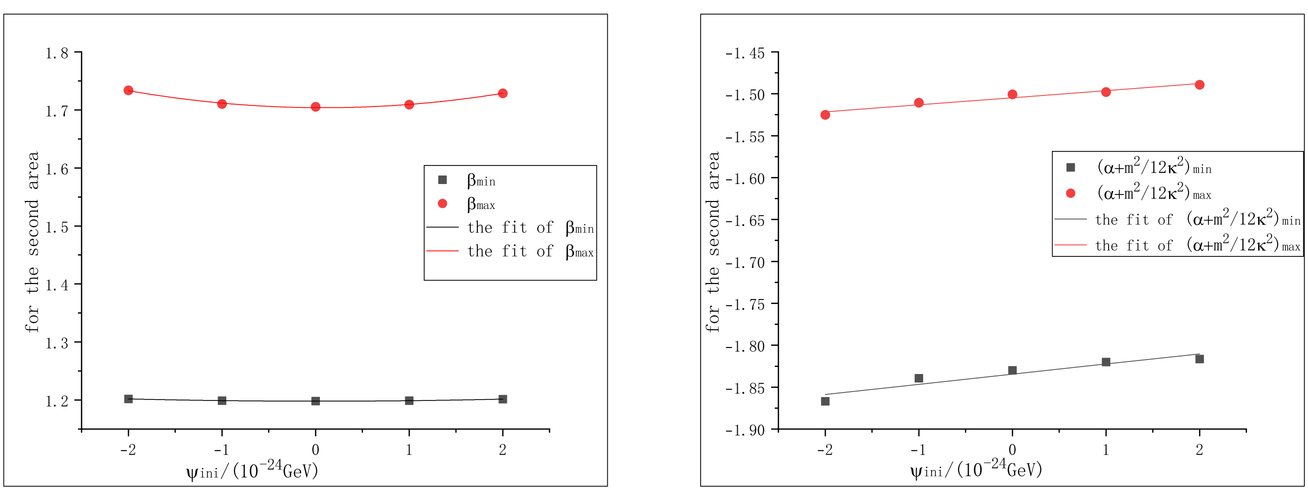

Fig. 8 The maximum and minimum β values (left panel) and the $\alpha + \frac{m^2}{12\kappa^2}$ values (right panel) for the second area

- For the right panel of Fig. 7,

$$\left(\alpha+\frac{m^2}{12\kappa^2}\right)\Big|_{\min} =6.33\times 10^{-3}x-1.35672,$$
$$\left(\alpha+\frac{m^2}{12\kappa^2}\right)\Big|_{\max} =3.35\times 10^{-3}x-1.15616. \quad (68)$$

- For the left panel of Fig. 8,

$$\beta_{\min} =9.3\times 10^{-4}x^2-8\times 10^{-5}x+1.19804,$$
$$\beta_{\max} =6.77\times 10^{-3}x^2-1.14\times 10^{-3}x + 1.70388. \quad (69)$$

- For the right panel of Fig. 8

$$\left(\alpha+\frac{m^2}{12\kappa^2}\right)\Big|_{\min} =1.205\times 10^{-2}x-1.83454,$$
$$\left(\alpha+\frac{m^2}{12\kappa^2}\right)\Big|_{\max} =8.45\times 10^{-3}x-1.5047, \quad (70)$$

with

$$x\equiv\frac{\Psi_{ini}}{10^{-24}\ \mathrm{GeV}}. \quad (71)$$

5 Conclusions and final remarks

In the present work we have investigated the BBN constraints on a specific gravity model belonging to the class of Weyl-type $f(Q,T)$ gravity theories. The model is constructed as a linear combination of the nonmetricity Q, a purely geometric quantity, and the trace T of the matter energy-momentum tensor. The model depends on three free parameters, but two of the parameters can be combined into a single one, and thus there are only two parameters, $\alpha+\frac{m^2}{12\kappa^2}$ and β, which fully determine the cosmological dynamics.

The main goal of our analysis is to confirm that there are ranges of the parameters satisfying the BBN Helium and Lithium constraints for the different Ψ_{ini} values, and to find the dependence of the ranges on Ψ_{ini}.

To numerically integrate cosmological evolution equations we must impose an initial condition for the Weyl vector Ψ. In the present work we have adopted a very small initial value for Ψ, of the order of 10^{-24} GeV. It is important to note that the generalized Friedmann equations do not admit any analytical solutions no matter the value of Ψ_{ini} is.

The numerical solutions of the generalized Friedmann equations leads to a set of BBN constraints on the free parameters of the model. To satisfy the BBN constraints the parameter β can only take positive values, and the sum $\Sigma=\alpha+m^2/12\kappa^2$ should satisfy the condition $\Sigma\lesssim -1$. There are acceptable ranges of the model parameters for which all BBN constraints are satisfied. This indicates that this model has the potential to satisfy all the important BBN constraints that must be passed by any viable physical theory.

Although the ranges for the different Ψ_{ini} values are relatively close to each other, we still obtain the fits of the parameter boundary values of each permitted area, which are quadratic for β and are linear for $\alpha+\frac{m^2}{12\kappa^2}$. This conclusion opens an important perspective on the Weyl-type $f(Q,T)$ gravity, and on its potential for further applications.

The cosmological model we have considered in the present investigations has the Lagrangian density given by $f(Q,T)=\alpha Q+\left(\beta/6\kappa^2\right)T$, with α and β being two model parameters.

The cosmological implications of the Weyl-type $f(Q,T)$ gravity have been considered recently in (Alfedeel et al. 2024) and (Gadbail et al. 2024), respectively. For the model described by the Lagrange density (58), to perform the comparison with the observational data, in (Alfedeel et al. 2024) the values $\alpha=-1.07$ and $\beta=0.5$ have been adopted. These values of the parameters guarantee the positivity and thus the physical acceptability of the matter energy density. Moreover, with these values one can obtain a dark energy equation of state parameter that satisfies the observational constraints of the cosmological evolution in the late Universe. But it is important to point out that the values considered in (Alfedeel et al. 2024) also nearly satisfy the freeze-out constrain. Moreover, the adopted range of values of α and β also satisfies the lithium constraint. However, if β takes values in the range $0.5<\beta<3$, the Strong Energy Condition (SEC) is violated. But in the present Weyl-type $f(Q,T)$ cosmological model, a range of the model parameters α and β satisfying the SEC and the BBN constraints does exist, specifically, $0.423\lesssim\beta<0.5$ and $-1.266\lesssim\alpha\lesssim-1.157$. It is important to note that (Alfedeel et al. 2024) only uses a set of parameter values, which does not necessarily indicate a contradiction between the late-time acceleration and the SEC. Hence the model based on the Lagrangian density (58) satisfies simultaneously both the early and late Universe constraints, a condition which is generally very difficult to be satisfied by the cosmological models.

On the other hand, in (Gadbail et al. 2024) a more general Weyl-type $f(Q,T)$ model was considered, with given by $f(Q,T)=\alpha Q^{m+1}+\left(\beta/6\kappa^2\right)$. For the case $m=0$, the statistical study of the model using the Pantheon+ (Without SHOES Calibrated) dataset with 1701 data points gives the values $\alpha=-1.05$ and $\beta=0.46$, which are also consistent with the constraints obtained in the present study. Hence our study complements the previous studies on the late Universe, and leads to a similar parameter range that satisfies both the early and late cosmological requirements.

In the present study we have only investigated the BBN constraints on the specific model of the Weyl-type $f(Q,T)$

gravity for a fixed value of η obtained from CMB. More investigations are certainly necessary for further considering the dependence of the permitted parameter ranges on η. These investigations will certainly enhance our understanding on both the BBN and on the Weyl-type $f(Q, T)$ gravity theory.

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Author contributions J. G. did the numerical analysis of the field equations and prepared the figures L.M. and S. L wrote the manuscript text and contributed to the development of the model H. Z. contributed to the cosmological interpretation of the results H. T. contributed to the manuscript text and the formulation of the model

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Declarations

Competing interests The authors declare no competing interests.

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