# Recent progress in four-dimensional conformal field theory

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Abstract

It is often argued that 2-dimensional conformal field theory (2D CFT) is too special to expect that its methods will work in four space-time dimensions (4D). We shall demonstrate that most objections can, in fact, be overcome. Using, in particular, the principle of global conformal invariance (GCI) in Minkowski space one can extend the notion of a (chiral) vertex algebra to higher dimensions. Although there are no scalar Lie fields in more than two dimensions, harmonic bilocal fields, which naturally arise in operator product expansions in 4D GCI models, give rise to infinite dimensional Lie algebras so that powerful methods of modern representation theory happily apply.

The talk is based on joint work with B. Bakalov, N.M. Nikolov, K.-H. Rehren (and, at an earlier stage, Ya.S. Stanev).

## 1. Can 2D CFT methods work in higher dimensions?

A number of reasons are given why 2-dimensional conformal field theory is, in a way, exceptional so that extending its methods to higher dimensions appears to be hopeless.

**1.** The $2D$ conformal group is infinite dimensional: it is the direct product of the diffeomorphism groups of the left and right (compactified) light rays. (In the euclidean picture it is the group of analytic and antianalytic conformal mappings.) By contrast, for $D > 2$, according to the Liouville theorem, the (quantum mechanical) conformal group in $D$ space-time dimensions is finite (in fact, $\frac{(D+1)(D+2)}{2}$-) dimensional: it is (a covering of) the spin group $\mathrm{Spin}(D, 2)$.

**2.** The representation theory of affine Kac-Moody algebras [Kac] and of the Virasoro algebra [K79] [KR] is playing a crucial role in constructing soluble $2D$ models of (rational) CFT. There are, on the other hand, no local Lie

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fields in higher dimensions: after an inconclusive attempt by Robinson [R64] (criticized in [L67]) this was proven for scalar fields by Baumann [B76].

**3.** The light cone in two dimensions is the direct product of two light rays. This geometric fact is the basis of splitting $2D$ variables into right- and left-movers' *chiral variables*. No such splitting seems to be available in higher dimensions.

**4.** There are chiral algebras in $2D$ CFT whose *local currents* satisfy the axioms of *vertex algebras*[1] and have rational correlation functions. It was believed for a long time that they have no higher dimensional CFT analogue.

**5.** Furthermore, the chiral currents in a $2D$ CFT on a torus have elliptic correlation functions [Zh96], the 1-point function of the stress energy tensor appearing as a modular form (these can be also interpreted as finite temperature correlation functions and a thermal energy mean value on the Riemann sphere). Again, there seemed to be no good reason to expect higher dimensional analogues of these attractive properties.

We shall argue that each of the listed features of $2D$ CFT does have, when properly understood, a higher dimensional counterpart.

1. The presence of a conformal anomaly (a non-zero Virasoro central charge $c$) tells us that the infinite conformal symmetry in $1+1$ dimension is, in fact, broken. What is actually used in $2D$ CFT are the (conformal) *operator product expansions* (OPEs) which can be derived for any $D$ and allow to extend the notion of a primary field (for instance with respect to the stress-energy tensor).

2. For $D = 4$, infinite dimensional Lie algebras are generated by *bifields* $V_{ij}(x_1, x_2)$ which naturally arise in the OPE of a (finite) set of (say, hermitean, scalar) local fields $\phi_i$ of dimension $d\,(> 1)$:

$$(x_{12}^2)^d\, \phi_i(x_1)\, \phi_j(x_2) = N_{ij} + x_{12}^2\, V_{ij}(x_1, x_2) + O((x_{12}^2)^2)\,,$$
$$x_{12} = x_1 - x_2\,, \quad x^2 = \mathbf{x}^2 - (x^0)^2\,, \quad N_{ij} = N_{ji} \in \mathbb{R} \tag{1.1}$$

where $V_{ij}$ are defined as (infinite) sums of OPE contributions of (twist two) conserved local tensor currents (and the real symmetric matrix $(N_{ij})$ is positive definite). We say more on this in what follows (reviewing results of [NST02, 03], [NRT05, 08], [BNRT07, 08]).

3. We shall exhibit a factorization of higher dimensional intervals by using the following parametrization of the conformally compactified space-time ([U63], [T86], [N05], [NT05]):

$$\bar{M} = \{z_\alpha = e^{it} u_\alpha\,, \alpha = 1, \dots, D;\, t, u_\alpha \in \mathbb{R}\,;\; u^2 = \sum_{\alpha=1}^{D} u_\alpha^2 = 1\} = \frac{\mathbb{S}^{D-1} \times \mathbb{S}^1}{\{1, -1\}}\,. \tag{1.2}$$

[1] As a mathematical subject vertex algebras were anticipated by I. Frenkel and V. Kac [FK80] and introduced by R. Borcherds [B86]; for reviews and further references see e.g. [K] [FB-Z] [dSK]

The real interval between two points $z_1 = e^{it_1}\, u_1$, $z_2 = e^{it_2}\, u_2$ is given by:

$$z_{12}^2\,(z_1^2\,z_2^2)^{-1/2} = 2\,(\cos t_{12} - \cos\alpha) = -4\sin t_+ \sin t_-\,, \quad z_{12} = z_1 - z_2 \quad (1.3)$$

$$t_\pm = 1/2\,(t_{12} \pm \alpha)\,, \quad u_1 \cdot u_2 = \cos\alpha\,, \quad t_{12} = t_1 - t_2\,. \quad (1.4)$$

Thus $t_+$ and $t_-$ are the compact picture counterparts of "left" and "right" chiral variables (see [NT05]). The factorization of $2D$ cross ratios into chiral parts again has a higher dimensional analogue [DO01]:

$$s := \frac{x_{12}^2\,x_{34}^2}{x_{13}^2\,x_{24}^2} = u_+\,u_-\,, \quad t := \frac{x_{14}^2\,x_{23}^2}{x_{13}^2\,x_{24}^2} = (1-u_+)\,(1-u_-)\,, \quad x_{ij} = x_i - x_j \quad (1.5)$$

which yields a separation of variables in the d'Alembert equation (cf. Remark 2.1). It would be, in fact, interesting to relate the factorization (1.5) to (1.3).

4. It turns out that the requirement of *global conformal invariance* (GCI) in Minkowski space together with the standard Wightman axioms of local commutativity and energy positivity entails the rationality of correlation functions in any even number of space-time dimensions [NT01]. Indeed, GCI and local commutativity of Bose fields (for space-like separations of the arguments) imply the *Huygens principle* and, in fact, the strong (algebraic) locality condition

$$(x_{12}^2)^n[\phi_i(x_1),\,\phi_j(x_2)] = 0 \quad \text{for } n \text{ sufficiently large,} \quad (1.6)$$

a condition only consistent with the theory of free fields for an even number of space time dimensions only.It is this Huygens locality condition which allows the introduction of higher dimensional vertex algebras [N05] [NT05] [BN06].

5. Local GCI fields have elliptic thermal correlation functions with respect to the (differences of) conformal time variables in any even number of space-time dimensions; the corresponding energy mean values in a Gibbs (KMS) state (see e.g. [H]) are expressed as linear combinations of modular forms [NT05].

The rest of the paper elaborates on the recent developments [BNRT07] [NRT08] [BNRT08] concerned mostly with the relevant infinite dimensional Lie algebras and their representations (issue 2). It is organized as follows. In Sect. 2 we reproduce the general form of the 4-point function of the bifield $V$ and the leading term in its conformal partial wave expansion. The case of a theory of scalar fields of dimension $d = 2$ is singled out, in which the bifields (and the unit operator) close a commutator algebra. In Sect. 3 we classify the arising infinite dimensional Lie algebras $\mathcal{L}$ in terms of the three real division rings $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. In Sect. 4 we formulate the main result of [BNRT07] and [BNRT08] on the Fock space representations of the infinite dimensional Lie algebra $\mathcal{L}(\mathbb{F})$ coupled to the (dual, in the sense of Howe [H85]) compact gauge group $U(N,\mathbb{F})$ where $N$ is the central charge of $\mathcal{L}$.

## 2. Four-point functions and conformal partial wave expansions

The conformal bifields $V(x_1, x_2)$ of dimension $(1,1)$ which arise in the OPE (1.1) (as sums of integrals of conserved tensor currents) satisfy the d'Alembert equation in each argument [NST03]; we shall call them *harmonic bifields*. Their correlation functions depend on the dimension $d$ of the local scalar fields $\phi$. For $d = 1$ one is actually dealing with the theory of a free massless field. We shall, therefore, assume $d > 1$. A basis $\{f_{\nu i}\,,\ \nu = 0, 1, \ldots, d-2,\ i = 1, 2\}$ of invariant amplitudes $F(s,t)$ such that

$$\langle 0 \mid V_1(x_1, x_2)\, V_2(x_3, x_4) \mid 0 \rangle = \frac{1}{\rho_{13}\, \rho_{24}}\ F(s,t)\,,$$

$$\rho_{ij} = x_{ij}^2 + i0x_{ij}^0\,, \quad x^2 = \boldsymbol{x}^2 - (x^0)^2 \tag{2.1}$$

is given by

$$(u_+ - u_-)\, f_{\nu 1}(s,t) = \frac{u_+^{\nu+1}}{(1-u_+)^{\nu+1}} - \frac{u_-^{\nu+1}}{(1-u_-)^{\nu+1}}\,,$$

$$(u_+ - u_-)\, f_{\nu 2}(s,t) = (-1)^\nu (u_+^{\nu+1} - u_-^{\nu+1})\,, \quad \nu = 0, 1, \ldots, d-2\,, \tag{2.2}$$

where $u_\pm$ are the "chiral variables" (1.5);

$$f_{01} = \frac{1}{t}\,, \quad f_{02} = 1\,; \quad f_{11} = \frac{1-s-t}{t^2}\,, \quad f_{12} = t - s - 1\,;$$

$$f_{21} = \frac{(1-t)^2 - s(2-t) + s^2}{t^3}\,, \quad f_{\nu 2}(s,t) = \frac{1}{t}\, f_{\nu 1}\left(\frac{s}{t}\,,\ \frac{1}{t}\right) \tag{2.3}$$

$f_{\nu,i}$, $i = 1, 2$ corresponding to *single pole terms* [NRT08] in the 4-point correlation functions $w_{\nu i}(x_1, \ldots, x_4) = f_{\nu i}(s,t)/\rho_{13}\,\rho_{24}$:

$$w_{01} = \frac{1}{\rho_{14}\,\rho_{23}}\,, \quad w_{02} = \frac{1}{\rho_{13}\,\rho_{24}}\,;$$

$$w_{11} = \frac{\rho_{13}\,\rho_{24} - \rho_{14}\,\rho_{23} - \rho_{12}\,\rho_{34}}{\rho_{14}^2\,\rho_{23}^2}\,, \quad w_{12} = \frac{\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24} - \rho_{12}\,\rho_{34}}{\rho_{13}^2\,\rho_{24}^2}\,;$$

$$w_{21} = \frac{(\rho_{13}\,\rho_{24} - \rho_{14}\,\rho_{23})^2 - \rho_{12}\,\rho_{34}\,(2\,\rho_{13}\,\rho_{24} - \rho_{14}\,\rho_{23}) + \rho_{12}^2\,\rho_{34}^2}{\rho_{14}^3\,\rho_{23}^3}\,,$$

$$w_{22} = \frac{(\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24})^2 - \rho_{12}\,\rho_{34}\,(2\,\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24}) + \rho_{12}^2\,\rho_{34}^2}{\rho_{13}^3\,\rho_{24}^3}\,. \tag{2.4}$$

We have $w_{\nu 2} = P_{34} w_{\nu 1} (= P_{12} w_{\nu 1})$ where $P_{ij}$ stands for the substitution of the arguments $x_i$ and $x_j$. Clearly, for $x_1 = x_2$ (or $s = 0$, $t = 1$) only the amplitudes $f_{0i}$ contribute to the 4-point function (2.1). Indeed, it has been demonstrated in [NRT05] that the lowest angular momentum ($\ell$) contribution to $f_{\nu i}$ corresponds to $\ell = \nu$. The corresponding OPE of the bifield $V$ starts with a local scalar field $\phi$ of dimension $d = 2$ for $\nu = 0$; with a conserved current $j_\mu$ (of $d = 3$) for $\nu = 1$; with a conserved tensor field $T_{\lambda\mu}$ with all properties of the (unique) stress-energy tensor for $\nu = 2$. Indeed, the amplitude $f_{\nu 1}$ admits an expansion in twist two[2] *conformal partial waves* $\beta_\ell(s,t)$ [DMPPT] starting with $\ell = \nu$ (for a derivation see [NRT05], Appendix B).

$$\beta_\nu(s,t) = \frac{G_{\nu+1}(u_+) - G_{\nu+1}(u_-)}{u_+ - u_-},\ G_\mu(u) = u^\mu F(\mu,\mu;2\mu;u). \tag{2.5}$$

Remark 2.1 Eqs. (2.2), (2.5) provide examples of solutions of the d'Alembert equation in any of the arguments $x_i, i = 1,2,3,4$. In fact, the general conformal covariant (of dimension 1 in each argument) such solution has the form of the right hand side of (2.1) with

$$F(s,t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.6}$$

Remark 2.2 We note that albeit each individual conformal partial wave is a transcendental function (like (2.5)) the sum of all such twist two contributions is the rational function $f_{\nu 1}(s,t)$.

It can be deduced from the analysis of 4-point functions that the commutator algebra of a set of harmonic bifields generated by OPE of scalar fields of dimension $d$ can only close on the $V$'s and the unit operator for $d = 2$. In this case the bifields $V$ are proven, in addition, to be *Huygens bilocal* [NRT08].

Remark 2.3 In general, irreducible positive energy representations of the (connected) conformal group are labeled by triples $(d; j_1, j_2)$ including the dimension $d$ and the Lorentz weight $(j_1, j_2)$ $(2j_i \in \mathbb{N})$, [M77]. It turns out that for $d = 3$ there is a spin-tensor bifield of weight $((3/2; 1/2, 0), (3/2; 0, 1/2))$ whose commutator algebra does close; for $d = 4$ there is a conformal tensor bifield of weight $((2; 1, 0), (2; 0, 1))$. These bifields may be termed *lefthanded*: they are analogues of chiral $2D$ currents; a set of bifields invariant under space reflections would also involve their righthanded counterparts (of weights $((3/2; 0, 1/2), (3/2; 1/2, 0))$ and $((2; 0, 1), (2; 1, 0))$, respectively).

[2]The twist of a symmetric traceless tensor field is defined as the difference between its conformal dimension and its rank. All conserved symmetric tensors in $4D$ have twist two.

## 3. Infinite dimensional Lie algebras and real division rings

Our starting point is the following result of [NRT08].

**Proposition 3.1.** *The harmonic bilocal fields $V$ arising in the OPEs of a (finite) set of local hermitean scalar fields of dimension $d = 2$ can be labeled by the elements $M$ of an algebra $\mathcal{M} \subset \mathrm{Mat}(L, \mathbb{R})$ of real matrices closed under transposition, $M \to {}^tM$, in such a way that the following commutation relations (CR) hold:*

$$\begin{aligned}[][V_{M_1}(x_1,x_2), V_{M_2}(x_3,x_4)] = {} & \Delta_{13} V_{{}^tM_1 M_2}(x_2,x_4) + \Delta_{24} V_{M_1\,{}^tM_2}(x_1,x_3) \\ & + \Delta_{23} V_{M_1 M_2}(x_1,x_4) + \Delta_{14} V_{M_2 M_1}(x_3,x_2) \\ & + tr(M_1 M_2)\,\Delta_{12,34} + tr({}^tM_1 M_2)\,\Delta_{12,43}\,; \quad (3.1)\end{aligned}$$

*here $\Delta_{ij}$ is the free field commutator, $\Delta_{ij} := \Delta^+_{ij} - \Delta^+_{ji}$, and $\Delta_{12,ij} = \Delta^+_{1i}\,\Delta^+_{2j} - \Delta^+_{i1}\,\Delta^+_{j2}$ where $\Delta^+_{ij} = \Delta^+(x_i - x_j)$ is the 2-point Wightman function of a free massless scalar field.*

We call the set of bilocal fields closed under the CR (3.1) a *Lie system.* The types of a Lie systems are determined by the corresponding *$t$-algebras* - i.e., real associative matrix algebras $\mathcal{M}$ closed under transposition. We first observe that each such $\mathcal{M}$ can be equipped with a Frobenius inner product

$$< M_1, M_2 >= tr^t M_1 M_2 = \sum_{ij} (M_1)_{ij}(M_2)_{ij}, \qquad (3.2)$$

which is symmetric, positive definite, and has the property $< M_1M_2, M_3 >=< M_1, M_3\,{}^tM_2 >$. This implies that for every right ideal $\mathcal{I} \subset \mathcal{M}$ its orthogonal complement is again a right ideal while its transposed ${}^t\mathcal{I}$ is a left ideal. Therefore, $\mathcal{M}$ is a *semisimple* algebra so that every module over $\mathcal{M}$ is a direct sum of irreducible modules.

Let now $\mathcal{M}$ be irreducible. It then follows from the Schur's lemma (whose real version [L] is less popular than the complex one) that its commutant $\mathcal{M}'$ in $Mat(L, \mathbb{R})$ coincides with one of the three *real division rings* (or not necessarily commutative *fields*): the fields of real and the complex numbers $\mathbb{R}$ and $\mathbb{C}$, and the noncommutative division ring $\mathbb{H}$ of quaternions. In each case the Lie algebra of bilocal fields is a central extension of an infinite dimensional Lie algebra that admits a discrete series of highest weight representations[3]

[3] Finite dimensional simple Lie groups $G$ with this property have been extensively studied by mathematicians (for a review and references - see [EHW]); for an extension to the infinite dimensional case - see [S90]. If $Z$ is the centre of $G$ and $K$ is a closed maximal subgroup of $G$ such that $K/Z$ is compact then $G$ is characterized by the property that $(G, K)$ is a *hermitean symmetric pair.* Such groups give rise to *simple space-time symmetries* in the sense of [MR07] (see also earlier work - in particular by Günaydin - cited there).

It was proven, first in the theory of a single scalar field $\phi$ (of dimension two) [NST02], and eventually for an arbitrary set of such fields [NRT08], that the bilocal fields $V_M$ can be written as linear combinations of normal products of free massless scalar fields $\varphi_i(x)$:

$$V_M(x_1,x_2) = \sum_{i,j=1}^{L} M^{ij} : \varphi_i(x_1)\varphi_j(x_2) : . \tag{3.3}$$

For each of the above types of Lie systems $V_M$ has a canonical form, namely

$$\begin{aligned}
\mathbb{R}: \ V(x_1,x_2) &= \sum_{i=1}^{N} : \varphi_i(x_1)\varphi_i(x_2) :, \\
\mathbb{C}: \ W(x_1,x_2) &= \sum_{j=1}^{N} : \varphi_j^*(x_1)\varphi_j(x_2) :, \\
\mathbb{H}: \ Y(x_1,x_2) &= \sum_{m=1}^{N} : \varphi_m^+(x_1)\varphi_m(x_2) :
\end{aligned} \tag{3.4}$$

where $\varphi_i$ are real, $\varphi_j$ are complex, and $\varphi_m$ are quaternionic valued fields (corresponding to (3.2) with $L = N$, $2N$ and $4N$, respectively). We shall denote the associated infinite dimensional Lie algebra by $\mathcal{L}(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Remark 3.1 We note that the quaternions (realized as $4 \times 4$ real matrices) appear both in the definition of $Y$ - i.e., of the matrix algebra $\mathcal{M}$, and of its commutant $\mathcal{M}'$, the two mutually commuting sets of imaginary quaternionic units $\ell_i$ and $r_j$ corresponding to the splitting of the Lie algebra $so(4)$ of real skew-symmetric $4 \times 4$ matrices into a direct sum of "a left and a right" $so(3)$ Lie subalgebras:

$$\begin{aligned}
&\ell_1 = \sigma_3 \otimes \epsilon\,, \ \ell_2 = \epsilon \otimes \mathbf{1}\,, \ \ell_3 = \ell_1\ell_2 = \sigma_1 \otimes \epsilon\,, \\
&(\ell_j)_{\alpha\beta} = \delta_{\alpha 0}\delta_{j\beta} - \delta_{\alpha j}\delta_{0\beta} - \varepsilon_{0j\alpha\beta}\,, \quad \alpha,\beta = 0,1,2,3\,, \quad j = 1,2,3\,; \\
&r_1 = \epsilon \otimes \sigma_3\,, \ r_2 = \mathbf{1} \otimes \epsilon\,, \ r_3 = r_1 r_2 = \epsilon \otimes \sigma_1
\end{aligned} \tag{3.5}$$

where $\sigma_k$ are the Pauli matrices, $\epsilon = i\sigma_2$ and $\varepsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor normalized by $\varepsilon_{0123} = 1$. We have

$$\begin{aligned}
Y(x_1,x_2) &= V_0(x_1,x_2)\mathbf{1} + V_1(x_1,x_2)\ell_1 + V_2(x_1,x_2)\ell_2 + V_3(x_1,x_2)\ell_3 \\
&= Y(x_2,x_1)^+ \qquad (\ell_i^+ = -\ell_i, [\ell_i, r_j] = 0)\,; \\
V_\kappa(x_1,x_2) &= \sum_{m=1}^{N} : \varphi_m^\alpha(x_1)(\ell_\kappa)_{\alpha\beta}\varphi_m^\beta(x_2) :, \quad \ell_0 = \mathbf{1}.
\end{aligned} \tag{3.6}$$

In order to determine the Lie algebra corresponding to the CR (3.1) in each of the three cases (3.5) we choose a discrete basis and specify the topology of the resulting infinite matrix algebra in such a way that the generators of the conformal Lie algebra (most importantly, the conformal Hamiltonian $H$) belong to it. The basis, say $(X_{mn})$ where $m, n$ are multiindices, corresponds to the expansion [T86] of a free massless scalar field $\varphi$ in creation and annihilation operators of fixed energy states

$$\varphi(z) = \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{(\ell+1)^2} ((z^2)^{-\ell-1} \varphi_{\ell+1,\mu} + \varphi_{-\ell-1,\mu}) h_{\ell\mu}(z), \tag{3.7}$$

where $(h_{\ell\mu}(z), \mu = 1, ..., (\ell+1)^2)$ form a basis of homogeneous harmonic polynomials of degree $\ell$ in the complex 4-vector $z$ (of the parametrization (1.2) of $\bar{M}$). The generators of the conformal Lie algebra $su(2,2)$ are expressed as infinite sums in $X_{mn}$ with a finite number of diagonals (i.e. with bounded $|m-n|$ - cf. Appendix B to [BNRT07]). The requirement $su(2,2) \subset \mathcal{L}$ thus restricts the topology of $\mathcal{L}$ implying that the last (c-number) term in (3.1) gives rise to a non-trivial central extension of $\mathcal{L}$.

The analysis of [BNRT07], [BNRT08] yields the following

**Proposition 3.2** *The Lie algebras $\mathcal{L}(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ are 1-parameter central extensions of appropriate completions of the following inductive limits of matrix algebras:*

$$\begin{aligned} \mathbb{R}: \;\; & sp(\infty, \mathbb{R}) = \lim_{n\to\infty} sp(2n, \mathbb{R}) \\ \mathbb{C}: \;\; & u(\infty, \infty) = \lim_{n\to\infty} u(n,n) \\ \mathbb{H}: \;\; & so^*(4\infty) = \lim_{n\to\infty} so^*(4n). \end{aligned} \tag{3.8}$$

*In the free field realization (3.4) the suitably normalized central charge coincides with the positive integer $N$.*

## 4. Fock space representation of the dual pair $\mathcal{L}(\mathbb{F}) \times U(N, \mathbb{F})$

To summarize the discussion of the last section: there are three infinite dimensional irreducible Lie algebras, $\mathcal{L}(\mathbb{F})$ that are generated in a theory of GCI scalar fields of dimension $d = 2$ and correspond to the three real division rings $\mathbb{F}$ (Proposition 3.2). For an integer central charge $N$ they admit a free field realization of type (3.3) and a Fock space representation with (compact) gauge group $U(N, \mathbb{F})$:

$$U(N,\mathbb{R}) = O(N) \;, \quad U(N,\mathbb{C}) = U(N) \;, \quad U(N,\mathbb{H}) = Sp(2N) \;. \tag{4.1}$$

It is remarkable that this situation is general.

**Theorem 4.1** (i) *In any unitary irreducible positive energy representation (UIPER) of* $\mathcal{L}(\mathbb{F})$ *the central charge* $N$ *is a positive integer.*

(ii) *All UIPERs of* $\mathcal{L}(\mathbb{F})$ *are realized (with multiplicities) in the Fock space* $\mathcal{F}$ *of* $N dim_{\mathbb{R}}\mathbb{F}$ *free hermitean massless scalar fields.*

(iii) *The ground states of equivalent UIPERs in* $\mathcal{F}$ *form irreducible representations of the gauge group* $U(N, \mathbb{F})$ (4.1). *This establishes a one-to-one correspondence between UIPERs of* $\mathcal{L}(\mathbb{F})$ *occurring in the Fock space and the irreducible representations of* $U(N, \mathbb{F})$.

The *proof* of this theorem for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is given in [BNRT07] (the proof of (i) is already contained in [NST02]); the proof for $\mathbb{F} = \mathbb{H}$ is given in [BNRT08].

Remark 4.1 Theorem 4.1 is also valid - and its proof becomes technically simpler - for a 2-dimensional chiral theory (in which the local fields are functions of a single complex variable). For $\mathbb{F} = \mathbb{C}$ the representation theory of the resulting infinite dimensional Lie algebra $u(\infty, \infty)$ is then essentially equivalent to that of the vertex algebra $W_{1+\infty}$ studied in [KR96] (see the introduction to [BNRT07] for a more precise comparison).

Theorem 4.1 provides a link between two parallel developments, one in the study of highest weight modules of reductive Lie groups (and of related dual pairs) [KV78], [J81], [EHW], [S90] (and [H85] [H89]), the other in the work of Haag-Doplicher-Roberts [H], [DR90] on the theory of (global) gauge groups and superselection sectors in the operator algebra approach to local quantum physics (which actually both originated - in the talks of Irving Segal and Rudolf Haag, respectively - at the same Lille 1957 conference on mathematical problems in quantum field theory). Albeit the settings are not equivalent the results match. The observable algebra (in our case, the commutator algebra generated by the set of bilocal fields $V_M$) determines the (compact) gauge group and the structure of the superselection sectors of the theory. (For a more careful comparison between the two approaches - see Sections 1 and 4 of [BNRT07].)

The infinite dimensional Lie algebra $\mathcal{L}(\mathbb{F})$ and the compact gauge group $U(N, \mathbb{F})$ appear as a rather special (limit-) case of a *dual pair* in the sense of Howe [H85], [H89]. It would be interesting to explore whether other (inequivalent) pairs would appear in the study of commutator algebras of (spin)tensor bifields (discussed in Remark 2.2) and of their supersymmetric extension (e.g. a limit as $m, n \to \infty$ of the series of Lie superalgebras $osp(4m^*|2n)$ studied in [GS91]).

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