SPACE-CHARGE EFFECTS IN ELECTRON-ELECTRON AND POSITRON-ELECTRON COLLIDING OR CROSSING BEAM RINGS

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I. INTRODUCTION

The interaction between two proton beams in storage rings has been studied, but since Liouville's theorem does not allow the achievement of beam densities higher than the transfer densities, it need not be considered as a strongly limiting factor. In electron-electron or electron-positron storage rings, the strong damping due to radiation losses causes the beam to shrink to a cross section which can be as small as 10^{-4} cm^2. Accordingly, current densities up to 10^3 or 10^4 amp/cm^2 can be achieved by conventional means.

Our results, which apply to the case of electron-electron or electron-positron rings, show that if two beams collide, the less intense beam A will be stable in the presence of beam B up to some limiting effective density of beam B and will then break up into a diffuse halo around beam B. Therefore, the calculations can be separated into an investigation of the point at which beam A becomes unstable and breaks up, and an investigation of the point at which the diffuse halo becomes stable.

The two calculations agree, and therefore give confidence that our description is correct. The calculations for the diffuse halo are made in the approximation that the field due to a ribbon of charge is constant, except close to the ribbon plane, and changes sign as this plane is crossed.

We have also calculated whether arranging for the two beams to intersect, at an angle in the vertical plane, can change the allowed effective space-charge density. The results give substantially the same limits as those obtained for the head-on collision of two beams. While the calculations and effects are complicated and tedious, the simple result is obtained that the effective charge-density achievable under the condition that the two beams still interact is given by:

$$\frac{N_B}{wh} \sim \frac{k}{p}\,\frac{Q\Delta Q}{2Fr_e}\left(\frac{\gamma}{R}\right) \tag{1}$$

$$\frac{N_B}{w\ell\delta} \sim \frac{k}{p}\,\frac{Q\Delta Q}{2Fr_e}\left(\frac{\gamma}{R}\right) \tag{2}$$

for crossing beams, where:

- N_B is the number of charges in the more intense beam, B;
- k is the number of bunches per turn;
- p is the number of interaction regions per turn;
- w, h and ℓ are the radial, vertical and azimuthal dimensions of the beam;
- δ is the angle between the direction of the bunch axis of one of the beams and the median plane of the ring (see Fig. 1); if δ is different for the two beams, in Eq. (2) the higher value of the two δ's must be used.
- Q is the number of betatron wavelengths around the ring, for the vertical betatron mode;
- ΔQ is the distance to the closest proper resonance;
- γ is the relativistic factor $E/(mc^2)$
- F $=\beta Q/R$ is the "beat factor" in the interaction region (β is the amplitude factor);
- R is the mean radius of the machine;
- r_e $=e^2/(mc^2)$ is the classical electron radius.

Since γ/R is proportional to the mean magnetic field along the ring, this result shows that the effective charge density is almost a constant irrespective of machine size or design, when limited to values of the order of 10^{12}

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particles/cm^2. This is compared to 10^{14} to 10^{16} particles/cm^2 which could be quite easily achieved in the absence of the space-charge limitation. Our calculations for the space-charge-limited density agree with the calculations made for the Stanford electron-electron storage ring.[1]

We have used the notation of Green and Courant[2] and the method of solution follows their treatment.

II. HEAD-ON COLLISION; INSTABILITY LIMIT

We consider the motion of an electron or a positron in beam A interacting head-on with a more intense electron beam B. Beam B has k bunches, a total number of electrons N_B, its length is ℓ, width w, height h, and the distribution of charges is assumed to be uniform.

The impulse I given to an electron (or positron) passing through one of the p interaction regions with vertical displacement z is:

$$I = \pm \frac{4\pi N_B e^2}{whkc} z \quad \text{for} \quad |z| \leq h/2. \tag{3}$$

As a result, the region behaves like a thin lens with a transfer matrix T_i:

$$T_i = \begin{vmatrix} 1 & 0 \\ \pm A & 1 \end{vmatrix} \tag{4}$$

with:

$$A = \frac{4\pi N_B r_e}{\gamma whk} \tag{5}$$

The unperturbed transfer matrix from one collision region to the next, under the condition that the number of periods of the magnetic structure to be a multiple of the number of collision regions p, is:

$$T_m = \mathbf{1} \cos\frac{\mu}{p} + \begin{vmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{vmatrix} \sin\frac{\mu}{p} \tag{6}$$

where μ is the phase shift of the betatron oscillations around the ring.

The resulting transfer matrix is the product of (4) and (6); the phase shift between two collision regions becomes:

$$\cos\frac{\mu + \Delta\mu}{p} = \cos\frac{\mu}{p} \pm \frac{A\beta}{2} \sin\frac{\mu}{p} \tag{7}$$

if $\Delta\mu/p$ is small and $(\mu+\Delta\mu)/p$ is not close to a multiple of π, we can rewrite (7):

$$\Delta\mu/p \cong \mp A\beta/2 \tag{8}$$

If the next integral or half-integral resonance is ΔQ removed, the allowed change in phase shift is:

$$\Delta\mu < 2\pi\Delta Q \tag{9}$$

or

$$\mp p(A\beta)/2 < 2\pi\Delta Q \tag{10}$$

and, with (5), remembering that $\beta = FR/Q$,

$$\frac{N}{wh} \lessapprox \mp \frac{2k}{p} \frac{Q\Delta Q}{2Fr_e} \left(\frac{\gamma}{R}\right) \tag{11}$$

The $-$ or $+$ sign in Eq. (11) means that, in the electron-electron case, the closest lower resonance must be taken into account (ΔQ must be negative), while in the positron-electron case, it is the closest higher resonance which must be considered.

If $(\mu+\Delta\mu)/p$ is very close to a multiple of π (which is, for instance, the case when $p=1$), the approximation in (8) is wrong by a factor of 2; we can take it into account by redefining p as a parameter which depends on the number of collision regions per turn, which is always bigger than 2, and is approximately equal to the number of collision regions only when they are more than 2 per turn.

III. HEAD-ON COLLISION; STABLE DIFFUSE ORBITS

Consider the motion of an electron (or positron) in beam A executing an orbit which does not intersect beam B. We shall approximate the impulse per unit length, I, given to the electron (or positron) in traversing the interaction region as:

$$I = \pm \frac{4\pi N_B e^2}{wk\ell c} \frac{z}{|z|} \text{ for } |z| \geq h/2 \tag{12}$$

The resulting equation for the vertical motion is:

$$\frac{d^2z}{ds^2} + k(s)z = \pm \frac{4\pi N_B r_e}{\gamma kw\ell} \frac{z}{|z|} L(s) \tag{13}$$

where s is the distance along the equilibrium orbit, $k(s)$ represents the focusing properties of the machine, $L(s)$ is equal to 1 within the p regions of interaction and to zero outside, and the length of interaction regions is $\ell/2$.

The equation may be transformed, in the standard manner, in terms of the variables:

$$\left.\begin{aligned} \eta &= \beta^{-1/2} z \\ \phi &= \int ds/(Q\beta) \end{aligned}\right\} \qquad (14)$$

into:

$$\frac{d^2\eta}{d\phi^2} + Q^2\eta = \pm Q^2\beta^{3/2}\frac{4\pi r_e}{\gamma k w \ell}\frac{\eta}{|\eta|}L(\phi) \qquad (15)$$

where $L(\phi)$ is equal to 1 within the p interaction regions and to zero outside. Because of the nonlinearity, Eq. (15) may have periodic solutions (closed orbits) with periods $2\pi q$ in ϕ, where q is an integer.

Let us expand, in Fourier series, the perturbation term over q turns:

$$\left.\begin{aligned} & Q^2\,\beta^{3/2}\frac{4\pi r_e N_B}{\gamma k w h \ell}\frac{\eta}{|\eta|}L\left(\frac{\phi}{q}\right) \\ & \cong 2\frac{Q\beta^{1/2}N_B r_e}{\gamma k w q}\sum_{i=1}^{p}\left[\frac{\eta_i}{|\eta_i|}\frac{1}{2} + \frac{\eta_i}{|\eta_i|}\sum_{r=1}^{S}\cos r\frac{\phi+\phi_i}{q}\right] \end{aligned}\right\} \qquad (16)$$

where ϕ_i are the coordinates of the p interaction regions; $\eta_i/|\eta_i|$ gives the sign of the perturbation in the interaction regions.

The expansion (16) is approximate, and is valid under the assumption that

$$(r\ell)/(4Rq) \ll 1 \qquad (17)$$

Accordingly, it is limited to a certain harmonic term of order S; it can be seen *a posteriori* that this approximation is usually quite good.

The behaviour of the periodic solutions of Eq. (15) is dominated by the term of (16) for which

$$r/q \simeq Q \qquad (18)$$

Taking only this term we have:

$$\eta = \pm\frac{2Q\beta^{1/2}N_B r_e}{\gamma k w q}\sum_{i=1}^{p}\frac{\cos(r/q)(\phi-\phi_i)}{Q^2-(r/q)^2}\frac{\eta_i}{|\eta_i|} \qquad (19)$$

The solution, when $\langle\eta\rangle = 0$, is consistent for:

$$\left.\begin{aligned} & Q > r/q \text{ for electron-electron interaction.} \\ & Q < r/q \text{ for positron-electron interaction.} \end{aligned}\right. \qquad (20)$$

We find, again, that in the electron-electron case the relevant resonance is the closest lower one, while in the positron-electron case it is the closest higher one.

The numerical value of the sum in Eq. (19) for the jth interaction region depends on p and on the closed orbit. Let us introduce its maximum value which is given by:

$$\sum_{i=1}^{p}\frac{\eta_i}{|\eta_i|}\cos\frac{r}{q}(\phi_j-\phi_i) \le qp \qquad (21)$$

Substitution of

$$\eta = \beta^{-1/2}z \qquad (22)$$

$$\frac{r}{q} = Q + \Delta Q \qquad (23)$$

in Eq. (19), for the jth interaction region, gives

$$z_j = \mp\frac{\beta N_B r_e p}{\gamma w \Delta Q k} \qquad (24)$$

The condition both for the validity of the potential used and for all the orbits of beam A to lie outside beam B is:

$$z_j \ge h \qquad (25)$$

which finally gives the density limitation:

$$\frac{N_B}{wh} \cong \mp\frac{2k}{p}\frac{Q\Delta Q}{2Fr_e}\left(\frac{\gamma}{R}\right) \qquad (26)$$

The − sign is for electron-electron (ΔQ must be negative for the solution to be consistent) and the + sign is for the positron-electron interactions.

ΔQ should be the distance from the closest resonance of every order; in fact it can be easily seen that for $q4\lessgtr$ the treatment given here begins to fail; ΔQ, for higher values of q, decreases, but at the same time the value of the sum given by (21) decreases. Within the limits of this approximation we think that Eq. (26) can be taken as a rule-of-thumb for resonances up to third or fourth order; the higher order resonances, except in special cases, should not give a worse limit.

IV. CROSSING BEAMS

Let us consider now a crossing-beam ring. In the interaction region, the equilibrium orbits $\bar{Z}_A$ and $\bar{Z}_B$ of the two beams, considered separately, cross at an angle 2α in the vertical plane. This crossing angle is assumed to be given by suitable perturbations in the guide field of harmonic order higher than the Q of the two beams, so that a change of about one unit in Q leaves α substantially unchanged.

Figure 1 represents a possible situation in which the charge density in both A and B beams is high enough to affect the other one. The bunch axes are tilted by angles θ_A and θ_B with respect to the direction of $\bar{Z}_A$ and $\bar{Z}_B$, but the velocities of the particles are always directed along $\bar{Z}_A$ and $\bar{Z}_B$. The vertical displacement $\bar{Z}_A$ is defined with respect to the orbit $\bar{Z}_A$.

A particle of beam A, in traversing the interaction region, receives an impulse I given by:

$$I = \pm \left\{ \frac{2\pi N_B e^2}{wkc} \frac{2x-1}{\ell} + \frac{4\pi N_B e^2}{wkc} \frac{Z_A}{\ell \delta_B} \right\} \text{ for } -x < \frac{Z_A}{\delta_B} < 1 - x \tag{27}$$

and

$$I = \pm \frac{2\pi N_B e^2}{wkc} \frac{Z_A}{|Z_A|} \text{ for } \frac{Z_A}{\delta_B} < -x \text{ or } \frac{Z_A}{\delta_B} > 1 - x \tag{28}$$

where x is the distance of the particle from the leading edge of the bunch, and $\delta_B = \alpha + \theta_B$ is the angle of the axis of beam B with respect to the median plane of the ring.

If the density of beam A is very small, then $\theta_B \simeq 0$ and $\delta_B \simeq \alpha$, since the tilt angle θ_B is due to the effect of beam A. The Q-value of beam A will change (decreasing in the electron-electron case, increasing in the positron-electron case), and the direction of the bunch axis will be tilted by an angle θ_A with respect to the $\bar{Z}_A$ direction, because of the interaction with beam B and as a result of its effective charge density.

The sign of the tilt angle depends on the closest integral resonance: in the electron-electron case θ_A will be negative (see Fig. 1 for the definition of the sign of the angles) if the closest integral resonance is lower than the $\bar{Q}$, and positive if the closest integral resonance is higher than the $\bar{Q}$; the opposite is true for the positron-electron case. $\bar{Q}$ is the number of betatron wavelengths per turn due both to the main-field focusing and to the lens effect of the interaction region.

The density of beam B, which gives rise to the instability limit for crossing beams, comes out to be the same as the one calculated for colliding beams, provided we substitute, for the height of the beam, the equivalent height $\ell\delta_B$:

$$\frac{N_B}{\ell \delta_B w} \leq \mp \frac{2k}{p} \frac{Q \Delta Q}{2F r_e} \left(\frac{\gamma}{R} \right) \tag{29}$$

All the considerations made for Eq. (11) are valid for Eq. (29).

With a method similar to the one used in Section III, we can find the tilt angle θ_A for the case of one interaction region per turn. In the electron-electron case, for a weak-focusing ring, we have:

$$\theta_A = -4F \frac{R}{\gamma} \frac{N_B}{w \ell} \frac{r_e}{k} \left\{ \frac{1}{2\bar{Q}^2} - \frac{1}{1 - \bar{Q}^2} \right\} \tag{30}$$

As an example, with $Q \simeq 0.8$, the lens effect of the interaction region tends to drive $\bar{Q}$ close to 0.5, which is the closest lower half-integral resonance. If we combine Eq. (29) with Eq. (30), at the space-charge limit ($\bar{Q} = 0.5$), we obtain:

$$\theta_A \simeq -\frac{\delta_B}{2} \tag{31}$$

In this case, then, when the density of beam B is high enough to drive beam A into the half-integral resonance, the tilt angle is negative and is given by Eq. (31).

Because the tilt angle depends on $\bar{Q}$, it is not easy to write down a general formula. However, it can be calculated by introducing the impulse given by Eq. (27) into the equations of motion and solving them for each particular case. If the crossing regions are more than one per turn, they can be arranged to make θ small.

The tilt angle θ_A changes the length L of the interaction region along the direction $\bar{Z}_B$, from the initial value L_0 (for $\theta_A = 0$).

$$L_0 = \frac{h}{2\alpha} \tag{32}$$

to

$$L' = \frac{h}{2\alpha} + \frac{\ell Q_A}{2\alpha} \tag{33}$$

If we drop the assumption $N_A \ll N_B$, and consider beams of roughly equal density, then, if the effective densities are close to the limit given by Eq. (29), the equilibrium will be un-

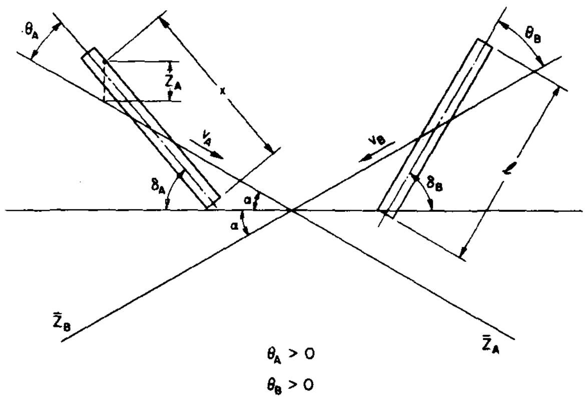

Fig. 1 Interaction region in a crossing beam ring.

stable. It seems, then, that the crossing angle 2α must be controlled to keep it large enough to avoid these unstable situations.

At the beginning of the present section we made the assumption that α remains constant when Q changes of the order of one unit. If the crossing angle is obtained with perturbations of low harmonic order, close to Q, the treatment must be modified, since α will also depend on $\bar{Q}$.

A final remark must be made about the case of crossing beams. Every beam-interaction effect has been calculated with the approximation that the impulse given by beam B to a particle of beam A, outside of beam B, does not depend on the distance from the axis of beam B. This approximation is correct when the distance, in a vertical plane, from the axis is smaller than the radial width of the beam; this is not always the case for a crossing-beam ring. Nevertheless, we have used this approximation, which probably gives a pessimistic value for the space-charge-limited density; however, it permits us more readily to evaluate the effects.

V. RESULTS

The space-charge limit sets in at a charge density given by Eqs. (11), (26), and (29). With the densities given by these relations, there should not be any collision between beams A and B. If we allow for the interaction between the two beams, it seems more correct to take as a space-charge-limited density, a value at least a factor of two lower. Then, we obtain the values given in Section I.

The interaction rates per interaction region, $\dot{n}$, for colliding beams, are given by:

$$\dot{n} = N_A \frac{N_B}{wh} \frac{f}{k} \sigma \quad \text{events/sec} \qquad (34)$$

and, for crossing beams, by

$$\dot{n} = N_A \frac{N_B}{w\ell\delta} \frac{f}{k} \sigma \quad \text{events/sec} \qquad (35)$$

where f is the revolution frequency, and σ is the cross section in cm^2.

If we introduce the space-charge-limited densities for beam B with the numerical values:

$$Q\Delta Q \simeq 0.1 \text{ (weak focusing)}$$
$$\gamma/R \simeq 4 \quad \text{cm}^{-1}$$
$$F \simeq 1$$

we find:

$$\dot{n} \simeq \frac{4 \times 10^{30}}{p} I_A \, \sigma \quad \text{events/sec} \qquad (36)$$

where I_A is the current, in amperes, of the weaker beam A and p the number of the interaction regions per turn.

This result seems to set an effective upper limit for the cross sections that can be investigated with colliding or crossing-beam rings.

In the case of colliding beams, where the cross section of the beams is determined by the radiation effects, one should provide some means to increase this cross section if he wants to use, for I_A, a value bigger than the one which is given by the space-charge-limited density times the unperturbed cross section of the beam (which is of the order of 1 ma for a 750-Mev weak focusing ring).

No problems arise in crossing-beam rings because, in this case, the effective beam cross section can be varied by changing the crossing angle.

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