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Fuzzy and discrete black hole models*

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Abstract

Using quantum Riemannian geometry, we solve for a $\text{Ricci} = 0$ static spherically-symmetric solution in 4D, with the S^2 at each t, r a noncommutative fuzzy sphere, finding a dimension jump with solutions having the time and radial form of a classical 5D Tangherlini black hole. Thus, even a small amount of angular noncommutativity leads to radically different radial behaviour, modifying the Laplacian and the weak gravity limit. We likewise provide a version of a 3D black hole with the S^1 at each t, r now a discrete circle \mathbb{Z}_n , with the time and radial form of the inside of a classical 4D Schwarzschild black hole far from the horizon. We study the Laplacian and the classical limit $\mathbb{Z}_n \rightarrow S^1$. We also study the 3D FLRW model on $\mathbb{R} \times S^2$ with S^2 an expanding fuzzy sphere and find that the Friedmann equation for the expansion is the classical 4D one for a closed $\mathbb{R} \times S^3$ Universe.

Keywords: noncommutative geometry, quantum gravity, black hole, FLRW cosmology, discrete gravity, modified gravity, fuzzy sphere

(Some figures may appear in colour only in the online journal)

1. Introduction

The idea that not only quantum phase spaces but spacetime coordinates themselves could be noncommutative or ‘quantum’ due to quantum gravity effects has been around since the first days of quantum theory. An often cited early work was [44], although not proposing a closed spacetime algebra as such. In modern times, such a *quantum spacetime hypothesis* was proposed in [32] on the grounds that the division into position and momentum should be

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arbitrary and hence if these do not commute then so should position and momentum separately noncommute. Several flat quantum spacetimes were studied in the 1990s [20, 28, 38] based on quantum symmetry [12, 30], but only recently has there emerged a constructive formalism of quantum Riemannian geometry [12] to more easily develop curved models [3, 11, 29, 36, 37, 39]. This formalism complements Connes' well-known 'spectral triple' approach to noncommutative geometry [18] based on an axiomatically defined 'Dirac operator' in that the different layers of Riemannian geometry are built up starting with a bimodule of differential forms Ω^1 for the coordinate algebra A , a quantum metric $g \in \Omega^1 \otimes_A \Omega^1$ and a compatible 'quantum Levi-Civita' connection $\nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$, without necessarily having a Dirac operator at all. This bottom-up approach fits well with examples coming from quantum groups and in some cases leads to spectral triples, see [12, chapter 8.5].

In the present work, we take curved quantum spacetime model building to the next level with black hole and FLRW cosmological models. Here [3], introduced an expanding FLRW model based on $\mathbb{R} \times S^1$ with S^1 replaced by a discrete group \mathbb{Z}_n with noncommutative differentials, while the coordinate algebra itself remains commutative. In section 3, we similarly look at the 3D $\mathbb{R} \times S^2$ model but replace S^2 by a noncommutative fuzzy sphere. This has coordinate algebra $\mathbb{C}_\lambda[S^2]$ defined as the usual angular momentum algebra viewed as a quantum space [9, 23, 28] but with a fixed value of the quadratic Casimir (i.e., a coadjoint orbit quantisation) and with differential structure as recently introduced in [12]. Here, $\mathbb{C}_\lambda[S^2]$ are infinite-dimensional algebras but for a discrete series of values of the deformation parameter, they can be reduced to the matrix algebra 'fuzzy spheres' introduced by Madore [31] as 'finite geometries' and studied in several further works, see [7, 25, 26, 47] to name some. A feature of our approach is that we adopt a single functorial framework that includes discrete geometries, noncommutative geometries such as fuzzy spheres, and indeed classical Riemannian geometry, as opposed to ad hoc methods for different settings.

We then proceed to our main results, noncommutative black hole models. Previously, 4D black holes were studied in a semidirect 'almost commutative' quantisation [35] but with the quantum geometry only implicitly defined through the wave operator constructed as a noncommutative extension to the classical differential calculus. Also previously, a 4D FLRW model was constructed in a deformation setting at the Poisson–Riemannian geometry level [24] and with the expected dimension, but at the price of nonassociativity of the exterior algebra due to curvature of the Poisson connection. Hence, the models in the present paper are the first fully noncommutative FLRW and black hole ones that we are aware of within usual (associative) quantum Riemannian geometry. Section 4 does the 4D black hole with S^2 in polars replaced by a fuzzy one, while section 5 looks for a 3D black hole model with the S^1 in polar coordinates replaced by \mathbb{Z}_n . The latter is not flat but Ricci flat (which cannot happen classically in 3D) and has a naked singularity rather than a horizon.

A common feature that we find is what we call 'dimension jump'. It has long been known that quantum exterior algebras often have extra dimensions that could not be predicted classically. Thus, in [3, 36] the calculus on \mathbb{Z} and \mathbb{Z}_n respectively was actually 2D and this made possible curvature effects not expected for a classical line or circle. The limit of the geometry on \mathbb{Z}_n as $n \rightarrow \infty$ was likewise shown in [3] to be a classical circle but with an extra one-form θ' normal to the circle when viewed in the plane. We then found that the Friedmann equations for the discrete $\mathbb{R} \times \mathbb{Z}_n$ model were the same as for a classical flat $\mathbb{R} \times \mathbb{R}^2$ FLRW model (an expanding plane), i.e. a dimension jump. The same will apply now for the fuzzy sphere with, in the classical limit $\lambda_p \rightarrow 0$, an extra 'normal' direction θ' for the sphere embedded in \mathbb{R}^3 . This time the dimension jump means that the radial-time sector matches to the closed (positively curved) 4D FLRW model. For the black hole model, the dimension jump means we land on radial and time behaviour matching the 5D Tangherlini black hole [45] when we use

the fuzzy sphere. Here the $\beta(r) = 1 - r_H/r$ factor in the familiar Schwarzschild metric case for horizon radius r_H is now a factor $\beta(r) = 1 - r_H^2/r^2$. This gets asymptotically flat faster than the Schwarzschild case and the effective gravity in the Newtonian limit is an inverse cubic force law. Finally, for the $\mathbb{R}^2 \times \mathbb{Z}_n$ black hole, we have $\beta = -r_H/r$ which is as for a usual black hole but without the constant term. This therefore approximates the metric *inside* a Schwarzschild black hole of infinite mass (so that the missing 1 is negligible). We also cover the case $\beta(r) = r_H/r$ of interest in its own right. These models have no horizon but naked singularities. We describe the $\mathbb{Z}_n \rightarrow S^1$ limit where S^1 retains a 2D noncommutative differential structure, and the classical projection to the usual calculus on S^1 where the metric is no longer Ricci flat, i.e. this is a purely quantum-geometric solution of the vacuum Einstein equations.

Our noncommutative models are theoretical and we are not aware of an immediate application, but they do indicate an unusual phenomenon which has a purely quantum origin in an extra ‘normal direction’ θ' required for an associative differential calculus in our examples. We also begin to explore some of the physics in our noncommutative backgrounds. In principle, one could use a new framework [10, 13] of quantum geodesics to do this, but the full formulation of that is rather involved and we propose instead a direct approach starting with the Klein–Gordon equation in the noncommutative background. In section 4.1, we introduce the notion of a Schroedinger-like equation for an effective quantum theory relative to an exact solution in the same manner as usual quantum mechanics for a free particle can be obtained as a nonrelativistic limit of the Klein–Gordon equations for solutions of the form $e^{-imt}\psi$ with ψ slowly varying. The novel feature will be to replace e^{-imt} by an exact reference solution of the Klein Gordon equation, and we explain first how this looks for a classical Schwarzschild black hole. This appears to be rather different from well-known methods of quantum field theory on a curved background [14, 42, 43] but fits with the general idea of [10, 13] that a quantum geodesic flow is actually a Schroedinger-like evolution.

The paper starts with some preliminaries in section 2, where we introduce the key points of the formalism for quantum Riemannian geometry from [12] by way of the quantum metric and connection for the fuzzy sphere from [29], and investigate the classical limit of the latter. We use dot or ∂_t for time derivatives and prime or ∂_r for radial derivatives (while ∂_i, ∂_\pm will be noncommutative angular derivatives). We sum over repeated indices. \otimes_s will denote the symmetric tensor product, where we add the two sides flipped, and we work in units where $\hbar = c = 1$. Numerical computations were done with Mathematica. The paper concludes with some remarks about further directions.

2. Recap of the fuzzy sphere and its classical limit

The fuzzy sphere $A = \mathbb{C}_\lambda[S^2]$ in the sense of [12, 23, 28, 29] just means the angular momentum enveloping algebra $U(su_2)$ with an additional relation giving a fixed value of the quadratic Casimir. This is the standard coadjoint quantisation of the unit sphere with its Kirillov–Kostant bracket known since the 1970s, and in our conventions takes the form

$$[x_i, x_j] = 2i\lambda_p \epsilon_{ijk} x_k, \quad \sum_i x_i^2 = 1 - \lambda_p^2, \quad (2.1)$$

where λ_p is a real dimensionless parameter. These conventions are chosen so that the standard spin j representation ρ_j descends to a representation of the the fuzzy sphere if $\lambda_p = 1/(2j+1)$. In this case, ρ_j is surjective but has a large kernel and quotienting out by this gives the ‘reduced fuzzy sphere’ $c_\lambda[S^2] \cong M_{2j+1}(\mathbb{C})$ via ρ_j , recovering the original usage of the term ‘fuzzy sphere’ in [31] as a matrix algebra with additional structure. We are not restricted to

these discrete values, however. Also note that physical coordinate generators with units of length would be

$$X_i := Lx_i, \quad [X_i, X_j] = 2\ell\lambda_p L\epsilon_{ijk}X_k, \quad \sum_i X_i^2 = L^2(1 - \lambda_p^2) \quad (2.2)$$

as a deformation of a sphere of radius L . For a quantum gravity effect, we might expect $\lambda_p \sim l_{\text{Planck}}/L$ in terms of the Planck length.

Next, there is a rotationally invariant differential calculus [12, example 1.46] in the sense of an exterior algebra (Ω, d) given by central basic one-forms $s^i \in \Omega^1$ and exterior derivative

$$dx_i = \epsilon_{ijk}x_js^k, \quad ds^i = -\frac{1}{2}\epsilon_{ijk}s^j \wedge s^k, \quad (2.3)$$

with associated partial derivatives defined by $df(x) = (\partial_i f)s^i$ in this basis (they act in the same way as orbital angular momentum). The s^i are preferable as they graded commute with everything, but they can be recovered in terms of the dx_i by

$$s^i = \frac{1}{1 - \lambda_p^2}(x_i\theta' + \epsilon_{ijk}dx_jx_k); \quad \theta' = x_is^i = \frac{x_i dx_i}{2\ell\lambda_p}. \quad (2.4)$$

There is a similar formula in the context of reduced fuzzy spheres $M_{2j+1}(\mathbb{C})$ in [31]. There is also a $*$ -operation with $x_i^* = x_i$ and $s^{i*} = s^i$. (Then $*$ commutes with d and $\theta'^* = \theta'$.) There is also a top form

$$s^1 \wedge s^2 \wedge s^3 = \frac{1}{2\ell\lambda_p(1 - \lambda_p^2)}\theta'^3 \quad (2.5)$$

in contrast to $\frac{1}{2}\epsilon_{ijk}x_is^j \wedge s^k = \frac{\theta'^2}{2\ell\lambda_p}$ for a deformation of the two-sphere volume form.

A metric on the fuzzy sphere from the point of view of quantum Riemannian geometry means $g \in \Omega^1 \otimes_A \Omega^1$ subject to certain conditions, and is shown in [12] to be necessarily of the form

$$g = g_{ij}s^i \otimes s^j \quad (2.6)$$

for a real symmetric matrix g_{ij} . Here g , in order to have a bimodule inverse, needs to be central and this forces the g_{ij} to be constants. By bimodule inverse, we mean a bimodule map

$$(\cdot, \cdot) : \Omega^1 \otimes_A \Omega^1 \rightarrow A \quad (2.7)$$

inverting g in the sense $((\omega, \cdot) \otimes \text{id})(g) = \omega = (\text{id} \otimes (\cdot, \omega))(g)$ for all $\omega \in \Omega^1$. A ‘bimodule map’ means commuting with the product by elements of A from either side, i.e. tensorial from either side, and we also require tensoriality in the middle in asking that (\cdot, \cdot) is well-defined on the tensor product over A . This then requires g to be central [11], see [12, lemma 1.16] for a short proof. In our case, $(s^i, s^j) = g^{ij}$ is just the inverse matrix to g_{ij} . The rotationally invariant ‘round metric’ is $g_{ij} = \delta_{ij}$ or $g = s^i \otimes s^i$ (sum over i understood). Finally, quantum symmetry in the sense $\wedge(g) = 0$ requires the matrix to be symmetric and reality in the sense $\text{flip}(* \otimes *) (g) = g$ then requires g_{ij} to be real-valued.

We next need a quantum Levi-Civita connection $\nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$ in the sense of torsion free and metric compatible. Here, if $X : \Omega^1 \rightarrow A$ is a right module map or ‘right vector field’

then $\nabla_X := (X \otimes \text{id})\nabla$ is well defined as a kind of ‘covariant derivative’ on Ω^1 . The associated left Leibniz rule is

$$\nabla(a\omega) = da \otimes \omega + a(\nabla\omega) \quad (2.8)$$

for all $a \in A, \omega \in \Omega^1$. We define the torsion tensor and Riemann curvature tensor for any connection as maps [12]

$$T_\nabla : \Omega^1 \rightarrow \Omega^2, \quad T_\nabla = \wedge \nabla - d, \quad R_\nabla : \Omega^1 \rightarrow \Omega^2 \otimes_A \Omega^1, \quad R_\nabla = (d \otimes \text{id} - \text{id} \wedge \nabla)\nabla. \quad (2.9)$$

However, as we can also multiply one-forms by algebra elements from the right, we need another Leibniz rule [22]

$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes da) \quad (2.10)$$

for some bimodule map $\sigma : \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$. The generalised braiding map σ is not additional data as it is determined by the above if it exists. Connections where it exists are called ‘bimodule connections’ and we will focus on these. We then define the metric compatibility tensor by

$$\nabla g := (\nabla \otimes \text{id} + (\sigma \otimes \text{id})(\text{id} \otimes \nabla))g. \quad (2.11)$$

The connection is quantum Levi-Civita if $T_\nabla = \nabla g = 0$. We also require a reality condition

$$\sigma \circ \text{flip}(* \otimes *) \circ \nabla = \nabla \circ *. \quad (2.12)$$

Finally, for physics we need a Ricci tensor and the working definition [12] is to suppose a bimodule map $i : \Omega^2 \rightarrow \Omega^1 \otimes_A \Omega^1$ and define Ricci by a trace of $(i \otimes \text{id})R_\nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$. This can be done explicitly via the metric and its inverse to make the trace between the input and the *first* output factor,

$$\text{Ricci} = ((,) \otimes \text{id})(\text{id} \otimes i \otimes \text{id})(\text{id} \otimes R_\nabla)g. \quad (2.13)$$

These natural definitions mean, however, that Ricci as it comes out from quantum Riemannian geometry is $-\frac{1}{2}$ of the usual Ricci in the classical case. The Ricci scalar $S = (,)\text{Ricci}$ is also $-\frac{1}{2}$ of the usual one.

All of this can be solved for the fuzzy sphere under the assumption that the connection coefficients are constant in the s^i basis, giving [29]

$$\nabla s^i = -\frac{1}{2}\Gamma_{jk}^i s^j \otimes s^k, \quad \Gamma_{jk}^i = g^{il}(2\epsilon_{lkm}g_{mj} + \text{Tr}(g)\epsilon_{ljk}). \quad (2.14)$$

Moreover, as classically, we can just take the map i to be the antisymmetric lift, so

$$i(s^i \wedge s^j) = \frac{1}{2}(s^i \otimes s^j - s^j \otimes s^i). \quad (2.15)$$

The resulting Ricci curvature on the fuzzy sphere is in [29] but in the round metric case one has

$$\nabla s^i = -\frac{1}{2}\epsilon_{ijk}s^j \otimes s^k, \quad \text{Ricci} = -\frac{1}{4}g, \quad S = -\frac{3}{4}. \quad (2.16)$$

This round metric connection projects in the reduced fuzzy sphere $M_{2j+1}(\mathbb{C})$ case to something similar to the covariant derivative in [31]. The Ricci curvatures are not the values you might have expected for a unit sphere even allowing for our conventions. Nor does the Einstein tensor (at least, if defined in the usual way) vanish as would be the case for a classical two-manifold.

To understand this last point better, which is the modest new result of this preliminary section, we look more carefully at the classical limit $\lambda_p \rightarrow 0$. By the Leibniz rule and the values above, we have for the round metric

$$\begin{aligned}\nabla(\theta') &= x_i \nabla s^i + dx_i \otimes s^i = -\frac{1}{2} \epsilon_{ijk} x_i s^j \otimes s^k + \epsilon_{ijk} x_j s^k \otimes s^i = \frac{1}{2} \epsilon_{ijk} x_i s^j \otimes s^k \\ &= \frac{1}{2(1-\lambda_p^2)^2} \epsilon_{ijk} (x_j \theta' + \epsilon_{jmn} (dx_m) x_n) \otimes (x_k \theta' + \epsilon_{kab} (dx_a) x_b) \\ &= \frac{1}{2} \epsilon_{ijk} dx^i \otimes (dx^j) x^k + O(\lambda_p).\end{aligned}$$

This means that we cannot just set $\theta' = 0$ in the classical limit for the given quantum geometry. Meanwhile, we write the commutation relations of the calculus as

$$\begin{aligned}[\theta', x_i] &= 2i\lambda_p dx_i, & x_i dx_i &= 2i\lambda_p \theta', \\ [x_i, dx_j] &= 2i\lambda_p (\delta_{ij} \theta' - \frac{x_j}{1-\lambda_p^2} (x_i \theta' + \epsilon_{imn} (dx_m) x_n))\end{aligned}$$

from which we see that the calculus is commutative and $x_i dx_i = 0$ in the classical limit $\lambda_p \rightarrow 0$ as expected for the unit sphere, but θ' itself does not need to vanish, and we have seen that it cannot if we want to have a limit for ∇ . Rather, we consider the classical limit as the classical sphere plus a single remnant θ' which graded-commutes with everything and (in the classical limit) does not arise from functions and differentials on the sphere. Indeed, this limit is not a strict differential calculus but a generalised one for this reason, but there is no such problem in the quantum case, where

$$\theta' = \frac{1}{2i\lambda_p} x_i dx_i \quad (2.17)$$

shows its origin as ‘normal’ to the sphere as embedded in \mathbb{R}^3 . We now note that the round metric has the limit

$$\begin{aligned}g &= s^i \otimes s^i = (x_i \theta' + \epsilon_{imn} (dx_m) x_n) \otimes (x_i \theta' + \epsilon_{iab} (dx_a) x_b) \\ &= (1 - \lambda_p^2) \theta' \otimes \theta' + \epsilon_{imn} x_i \theta' \otimes_s (dx_m) x_n + (\delta_{ma} \delta_{nb} - \delta_{mb} \delta_{na}) (dx_m) x_n \otimes (dx_a) x_b \\ &= \theta' \otimes \theta' + dx_i \otimes dx_i + O(\lambda_p)\end{aligned}$$

since the calculus is commutative to $O(\lambda_p)$. Thus, we see that the rotationally invariant ‘round’ metric actually has an extra direction required by the calculus. We can recover the completely classical S^2 by the limit $\lambda_p \rightarrow 0$ and projecting $\theta' = 0$, but traces taken for the Ricci curvature before we do this will remember the extra ‘normal direction’ and not map onto the classical values.

3. Expanding fuzzy sphere FLRW model

Here we work with the coordinate algebra $A = C^\infty(\mathbb{R}) \otimes \mathbb{C}_\lambda[S^2]$, where the \mathbb{R} has a classical time variable t with classical dt graded commuting with t, dt and with the generators x_i, s^i of the exterior algebra of the fuzzy sphere. We first consider a general metric of the form

$$g = \beta dt \otimes dt + n_i(dt \otimes s^i + s^i \otimes dt) + g_{ij}s^i \otimes s^j, \quad (3.1)$$

where g_{ij} is a symmetric 3×3 matrix of coefficients and β, n_i further coefficients, *a priori* all valued in A . Centrality of the metric, however, then forces the $n_i = 0$ and the remaining coefficients to be in the centre of $\mathbb{C}_\lambda[S^2]$, which is trivial. Hence g_{ij}, β are functions only of the time t . The reality condition for quantum metrics forces them to be real-valued.

Next, a general QLC for the calculus has the form

$$\nabla s^i = -\frac{1}{2}\Gamma_{jk}^i s^j \otimes s^k + \gamma_j^i s^j \otimes_s dt + \tau^i dt \otimes dt \quad (3.2)$$

$$\nabla dt = \mu_{jk}s^j \otimes s^k + \eta_j s^j \otimes_s dt + \Gamma dt \otimes dt \quad (3.3)$$

again with $\Gamma_{jk}^i, \gamma_j^i, \tau^i, \Gamma, \eta_j, \mu_{jk} \in A$. However, given that the spatial metric g_{ij} are functions only of t , it is natural to assume this also for the spatial Christoffel symbols Γ_{jk}^i just as is done for the fuzzy sphere alone in [29]. In this case, compatibility of ∇ with the relations of commutativity of dt, s^i with t, x_j and the natural assumption that the associated braiding σ has the classical ‘flip’ form when one of the arguments is dt require that $\gamma_j^i, \tau^i, \Gamma, \eta_j, \mu_{jk}$ are also function of time alone in order to have a bimodule connection.

The nontrivial conditions for ∇ to be torsion-free are

$$\frac{1}{2}(-\Gamma_{jk}^i + \epsilon_{jk}^i)s^j \wedge s^k = 0, \quad \mu_{jk} = \mu_{kj}, \quad (3.4)$$

since $d(dt) = 0$. The condition $\nabla g = 0$ for metric compatibility then produces

$$\begin{aligned} dt \otimes s^i \otimes s^j : \dot{g}_{ij} + g_{il}\gamma_j^l + g_{lj}\gamma_i^l &= 0, \\ dt \otimes dt \otimes dt : \dot{\beta} + 2\beta\Gamma &= 0, \\ s^l \otimes s^m \otimes s^j : -\frac{g_{ij}}{2}\Gamma_{lm}^i - \frac{g_{in}}{2}\Gamma_{pj}^n \sigma^{ip}_{lm} &= 0, \\ s^l \otimes dt \otimes s^j : g_{ij}\gamma_l^i + \beta\mu_{lj} &= 0, \\ dt \otimes dt \otimes s^j : g_{ij}\tau^i + \beta\eta_j &= 0, \\ s^n \otimes s^p \otimes dt : g_{ij}\gamma_l^j \sigma^{il}_{np} + \beta\mu_{np} &= 0, \\ dt \otimes s^i \otimes dt : g_{ij}\tau^j + \beta\eta_i &= 0, \\ s^m \otimes dt \otimes dt : 2\beta\eta_m &= 0. \end{aligned}$$

It is clear from third of these and the first of (3.4) that Γ_{jk}^i is indeed the Christoffel symbol for the fuzzy sphere QLC as solved uniquely in the $*$ -preserving case with constant coefficients in [29]. Also, the last equation implies that $\eta_m = 0$ and, using this together with the fifth or seventh equation, we get $\tau^i = 0$. The second equation makes $\Gamma = -\dot{\beta}/(2\beta)$. Using $g_{ik}\gamma_j^k = \gamma_{ij}$ and the symmetry of g_{ij} in the first equation gives the value of γ_j^i , then this together with the

4th equation gives μ_{ij} , resulting in

$$\gamma^i_j = -\frac{1}{2}\dot{g}_{jk}g^{ik}, \quad \mu_{ij} = \frac{\dot{g}_{ij}}{2\beta}. \quad (3.5)$$

Note that μ_{ij} is proportional to the time derivative of the metric, which implies that it is also symmetric if the metric is, solving the second half of (3.4). Because γ^i_j, μ_{ij} just depend on real functions, they are also real-valued functions. This leads to a reasonably canonical QLC.

Theorem 3.1. *Up to a reparametrisation of t , a generic quantum metric on the algebra $C^\infty(\mathbb{R}) \otimes \mathbb{C}_\lambda[S^2]$ can be taken in the form*

$$g = -dt \otimes dt + g_{ij}s^i \otimes s^j,$$

where g_{ij} is a time-dependent real 3×3 symmetric matrix. Moreover, this admits a canonical $*$ -preserving QLC

$$\nabla dt = -\frac{1}{2}\dot{g}_{ij}s^i \otimes s^j, \quad \nabla s^i = -\frac{1}{2}\Gamma^i_{jk}s^j \otimes s^k - \frac{1}{2}g^{ki}\dot{g}_{jk}s^j \otimes_s dt,$$

where $\Gamma_{ijk} = 2\epsilon_{ikm}g_{mj} + \text{Tr}(g)\epsilon_{ijk}$ as for the fuzzy sphere in [29]. The associated Ricci scalar and Laplacian are

$$2S = -g^{ij}\ddot{g}_{ij} - \text{Tr}(g) + \frac{1}{2}(\text{Tr}(g))^2 - \delta_{ij} - \frac{1}{4}(g^{ml}g^{ij}\dot{g}_{ml}\dot{g}_{ij} + g^{kl}g^{mn}\dot{g}_{nk}\dot{g}_{lm}),$$

$$\Delta = g^{ij}\partial_i\partial_j - \frac{1}{2}g^{ij}\dot{g}_{ij}\partial_t - \partial_t^2.$$

Proof. The analysis for the metric (3.1) was done above and we were forced by the requirement for the metric to be central (in order to be invertible) to $n_i = 0$ and only a time dependence $\beta(t), g_{ij}(t)$. We add the $*$ -reality of the metric in the form $\text{flip}(* \otimes *)g = g$ to find β and g_{ij} real. Quantum symmetry also requires the latter to be symmetric, while the expected signature requires $\beta(t) < 0$.

Now substituting the values obtained so far in the analysis of the general form of (3.2) and (3.3), we have the connection

$$\nabla dt = \frac{1}{2\beta}\dot{g}_{ij}s^i \otimes s^j - \frac{1}{2}\frac{\dot{\beta}}{\beta}dt \otimes dt; \quad \nabla s^i = -\frac{1}{2}\Gamma^i_{jk}s^j \otimes s^k - \frac{1}{2}g^{ki}\dot{g}_{jk}s^j \otimes_s dt \quad (3.6)$$

for some unknown $\Gamma^i_{jk}(t)$, where this does not depend on fuzzy sphere variables under our assumptions. The requirement of being $*$ -preserving yields

$$\dot{g}_{jk}(s^j \otimes s^k - \sigma(s^k \otimes s^j)) = 0, \quad \Gamma^i_{jk}s^j \otimes s^k - \bar{\Gamma}^i_{kj}\sigma(s^k \otimes s^j) = 0. \quad (3.7)$$

with the second of these the same as for the fuzzy sphere in [29] at each fixed time. Here we used $dt^* = dt$. Thus all the equations for Γ^i_{jk} are the same as in that paper and hence there is a unique solution (2.14) for it in terms of $g_{ij}(t)$, as stated, under our assumptions. In this case, we know from [29] that $\sigma = \text{flip}$ on $s^j \otimes s^k$ and hence the first of (3.7) is empty, as is the 6th of the metric compatibility equations in our previous analysis. The rest of the $*$ -preserving conditions require $\Gamma, \eta_i, \gamma^i_j$ to be real-valued functions, which already holds as we have solved for them.

The curvature for the connection (3.6) is

$$\begin{aligned}
R_{\nabla} dt &= \left(\frac{1}{2\beta} \ddot{g}_{ij} - \frac{\dot{\beta}}{4\beta^2} \dot{g}_{ij} - \frac{1}{4\beta} g^{ml} g_{il} g_{jm} \right) dt \wedge s^i \otimes s^j \\
&\quad + \frac{1}{4\beta} (-\dot{g}_{lk} \epsilon^l_{ij} + \dot{g}_{il} \Gamma^l_{jk}) s^i \wedge s^j \otimes s^k + \frac{1}{4\beta} g^{lm} \dot{g}_{jl} \dot{g}_{im} s^i \wedge s^j \otimes dt \\
R_{\nabla} s^i &= \left(\frac{1}{4} \Gamma^i_{jl} g^{ml} \dot{g}_{km} - \frac{1}{4} \Gamma^l_{jk} g^{mi} \dot{g}_{lm} \right) dt \wedge s^j \otimes s^k \\
&\quad + \left(\frac{1}{4} g^{li} \dot{g}_{ml} \epsilon^m_{jk} - \frac{1}{4} \Gamma^i_{jl} g^{ml} \dot{g}_{km} \right) s^j \wedge s^k \otimes dt \\
&\quad + \left(\frac{1}{4} \Gamma^i_{ml} \epsilon^m_{jk} - \frac{1}{4} \Gamma^i_{jm} \Gamma^m_{kl} + \frac{1}{4\beta} g^{mi} \dot{g}_{jm} \dot{g}_{kl} \right) s^j \wedge s^k \otimes s^l \\
&\quad + \left(-\frac{1}{2} (\dot{g}^{ki} \dot{g}_{jk} + g^{ki} \ddot{g}_{jk}) + \frac{\dot{\beta}}{4\beta} g^{ki} \dot{g}_{jk} - \frac{1}{4} g^{ki} g^{ml} \dot{g}_{lk} \dot{g}_{jm} \right) dt \wedge s^j \otimes dt,
\end{aligned}$$

leading to the Ricci tensor,

$$\begin{aligned}
2\text{Ricci} &= \left(\frac{1}{2\beta} \left(\ddot{g}_{ij} - \frac{\dot{\beta}}{2\beta} \dot{g}_{ij} - \frac{1}{2} g^{kl} \dot{g}_{il} \dot{g}_{jk} \right) + \frac{1}{2} \Gamma^l_{mj} \epsilon^m_{li} - \frac{1}{4} \Gamma^l_{lm} \Gamma^m_{ij} \right. \\
&\quad \left. + \frac{1}{4\beta} g^{ml} \dot{g}_{ml} \dot{g}_{ij} + \frac{1}{4} \Gamma^l_{im} \Gamma^m_{lj} - \frac{1}{4\beta} g^{ml} \dot{g}_{im} \dot{g}_{lj} \right) s^i \otimes s^j \\
&\quad - \left(-\frac{1}{2} (\dot{g}^{kl} \dot{g}_{lk} + g^{ij} \ddot{g}_{ij}) + \frac{\dot{\beta}}{4\beta} g^{ij} \dot{g}_{ij} - \frac{1}{4} g^{kl} g^{mn} \dot{g}_{nk} \dot{g}_{ml} \right) dt \otimes dt \\
&\quad + \left(\frac{1}{2} g^{nl} \dot{g}_{mn} \epsilon^m_{il} - \frac{1}{4} \Gamma^l_{lm} g^{nm} \dot{g}_{in} + \frac{1}{4} \Gamma^l_{im} g^{nm} \dot{g}_{ln} \right) s^i \otimes dt \\
&\quad + \left(-\frac{1}{4} \Gamma^l_{lm} g^{nm} \dot{g}_{in} + \frac{1}{4} g^{nl} \dot{g}_{mn} \Gamma^m_{li} \right) dt \otimes s^i.
\end{aligned}$$

Next, as we require $\beta(t) < 0$, we can generically absorb this in a redefinition of t (by changing to a new variable t' solving $\frac{dt'}{dt} = \sqrt{-\beta(t)}$). Therefore, up to such a redefinition, we can assume that $\beta = -1$ as stated. In this case, and with Γ^i_{jk} as explained from (2.14), the Ricci tensor simplifies to

$$\begin{aligned}
2\text{Ricci} &= \left(-\frac{\ddot{g}_{ij}}{2} + \frac{1}{2} g^{kl} \dot{g}_{il} \dot{g}_{jk} - \frac{1}{4} g^{ml} \dot{g}_{ml} \dot{g}_{ij} - g_{ij} - \delta_{ij} + \frac{1}{2} \text{Tr}(g) g_{ij} \right) s^i \otimes s^j \\
&\quad - \left(-\frac{1}{2} (\dot{g}^{kl} \dot{g}_{lk} + g^{ij} \ddot{g}_{ij}) - \frac{1}{4} g^{kl} g^{mn} \dot{g}_{nk} \dot{g}_{ml} \right) dt \otimes dt \\
&\quad + \left(\frac{1}{2} g^{nl} \dot{g}_{mn} \epsilon^m_{il} - \frac{1}{2} \epsilon^{kl}_m g^{nm} g_{kl} \dot{g}_{in} + \frac{1}{4} (2\epsilon^{kl}_m g_{ik} + \text{Tr}(g) \epsilon^l_{im}) g^{nm} \dot{g}_{ln} \right) s^i \otimes dt \\
&\quad + \left(-\frac{1}{2} \epsilon^{kl}_m g_{kl} g^{nm} \dot{g}_{in} - \frac{1}{4} g^{nl} \dot{g}_{mn} (2\epsilon^{km}_i g_{kl} + \text{Tr}(g) \epsilon^m_{li}) \right) dt \otimes s^i.
\end{aligned}$$

Making the contraction with the inverse metric, we recover the required Ricci scalar. The Laplacian for a function $f = f(t, x^i)$ follows as

$$\begin{aligned}\Delta f &= (,) \nabla(df) = (,) \nabla(\partial_i f s^i + \dot{f} dt) \\ &= g^{ij} \partial_i \partial_j f - \partial_t^2 f - \frac{1}{2} g^{ij} \dot{g}_{ij} \partial_t f - \frac{1}{2} g^{jk} \Gamma_{jk}^i \partial_i f,\end{aligned}$$

where the last term vanishes when we take into account the explicit form of Γ_{jk}^i , recovering the required Laplacian. \square

The QLC here is unique under the reasonable assumption as in [29] that the Γ_{jk}^i are constant on the fuzzy sphere, given that the g_{ij} have to be. The theorem applies somewhat generally, but now we take the expanding round metric $g_{ij} = R^2(t) \delta_{ij}$ for the spatial part, so the metric, non-zero inverse metric entries, QLC, curvature and Laplacian are

$$g = -dt \otimes dt + R^2(t) s^i \otimes s^i, \quad (dt, dt) = -1, \quad (s^i, s^j) = \frac{\delta^{ij}}{R^2}, \quad (3.8)$$

$$\nabla dt = -R \dot{R} s^i \otimes s^i, \quad \nabla s^i = -\frac{1}{2} \epsilon^i_{jk} s^j \otimes s^k - \frac{\dot{R}}{R} s^i \otimes_s dt, \quad (3.9)$$

$$R \nabla dt = -R \ddot{R} dt \wedge s^i \otimes s^i, \quad (3.10)$$

$$R \nabla s^i = \left(\frac{1}{4} \epsilon^{pi}_n \epsilon_{pkm} - \dot{R}^2 \delta^i_m \delta_{nk} \right) s^m \wedge s^n \otimes s^k + \frac{\ddot{R}}{R} dt \wedge s^i \otimes dt, \quad (3.11)$$

$$\text{Ricci} = - \left(\dot{R}^2 + \frac{1}{2} R \ddot{R} + \frac{1}{4} \right) s^i \otimes s^i + \frac{3}{2} \frac{\ddot{R}}{R} dt \otimes dt, \quad (3.12)$$

$$S = -3 \left(\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{1}{4R^2} \right), \quad (3.13)$$

$$\Delta = \frac{1}{R^2} \sum_i \partial_i^2 - 3 \frac{\dot{R}}{R} \partial_t - \partial_t^2. \quad (3.14)$$

Also of interest is the Einstein tensor and, in the absence of a general theory, we assume as in [3] the ‘naive definition’ $\text{Eins} = \text{Ricci} - \frac{S}{2} g$, which works out as.

$$\text{Eins} = \left(\ddot{R} + \frac{1}{2} \dot{R}^2 + \frac{1}{8} \right) s^i \otimes s^i - \frac{3}{2} \left(\frac{1}{4R^2} + \frac{\dot{R}^2}{R^2} \right) dt \otimes dt \quad (3.15)$$

and is justified by checking that

$$\nabla \cdot \text{Eins} = 0. \quad (3.16)$$

Here, if we have any tensor for the form $T = f dt \otimes dt + p R^2 s^i \otimes s^i$, then the divergence is

$$\nabla \cdot T = ((,) \otimes \text{id}) \nabla T = - \left(\dot{f} + 3(f + p) \frac{\dot{R}}{R} \right) dt + \partial_i p s^i \quad (3.17)$$

and we use this now for the particular form of the Einstein tensor to establish (3.16). We also assume this form of T for the energy–momentum tensor of dust with pressure p and density f ,

in which case the continuity equation $\nabla \cdot T = 0$ for p a function only of t is

$$\dot{f} + 3(f + p)\frac{\dot{R}}{R} = 0 \quad (3.18)$$

as usual, and Einstein's equation $\text{Eins} + 4\pi GT = 0$ in our curvature conventions is

$$4\pi Gf = \frac{3}{2} \left(\frac{\dot{R}^2}{R^2} + \frac{1}{4R^2} \right), \quad 4\pi Gp = -\frac{\ddot{R}}{R} - \frac{1}{2} \frac{\dot{R}^2}{R^2} - \frac{1}{8R^2} = -\frac{\ddot{R}}{R} - \frac{4\pi G}{3} f. \quad (3.19)$$

These are identical to the classical FLRW equations, see e.g. [16, chapter 8], for a 4D closed Universe with curvature constant $\kappa = 1/(4R_0^2)$ in the classical FLRW metric

$$-dt \otimes dt + R(t)^2 \left(\frac{1}{r^2(1 - \kappa r^2)} dr \otimes dr + g_{S^2} \right),$$

where g_{S^2} is the metric on a unit sphere, R_0 is a normalisation constant with dimension of length, and we have adapted $R(t)$ to include a factor r in order to match our conventions.

4. Black hole with the fuzzy sphere

We assume a similar framework as in the previous section, but now with a 4D metric of a static form in polar coordinates. Thus, we add a radial variable r with differential dr and consider the Schwarzschild-like metric

$$g = -\beta(r)dt \otimes dt + H(r)dr \otimes dr + r^2 g_{ij} s^i \otimes s^j. \quad (4.1)$$

The algebra of functions here is $A = C^\infty(\mathbb{R} \times \mathbb{R}_{>0}) \otimes \mathbb{C}_\lambda[S^2]$ with classical variables and differentials t, r, dr, dt for the $\mathbb{R} \times \mathbb{R}_{>0}$ part (so these graded commute among themselves and with the functions and forms on the fuzzy sphere). The coefficients g_{ij} define the metric on the fuzzy sphere, and centrality and reality of the metric dictates that these are constant real values. Thus, g_{ij} is a real symmetric invertible 3×3 matrix (it should also be positive definite for the expected signature) and $\beta(r), H(r)$ are real-valued functions.

We start with the general form of connection on the tensor product calculus,

$$\nabla s^i = -\frac{1}{2} \Gamma_{jk}^i s^j \otimes s^k + \alpha^i dt \otimes dt + \gamma^i dr \otimes dr + \Delta^i dr \otimes_s dt + \eta^i_j dt \otimes_s s^j + \tau^i_j dr \otimes_s s^j,$$

$$\nabla dt = a_{ij} s^i \otimes s^j + b dt \otimes dt + c dr \otimes dr + d dr \otimes_s dt + e_j dr \otimes_s s^j + f_j dt \otimes_s s^j,$$

$$\nabla dr = h_{ij} s^i \otimes s^j + \theta dt \otimes dt + R dr \otimes dr + \phi dr \otimes_s dt + \nu_j dt \otimes_s s^j + \psi_j dr \otimes_s s^j.$$

Assuming that $\sigma(dt \otimes), \sigma(\otimes dt), \sigma(dr \otimes), \sigma(\otimes dr)$ are the flip on the one-forms dr, dt, s^i and the natural restrictions needed for a bimodule connection, one finds that all the coefficients are functions of t and r alone (constant on the fuzzy sphere).

The torsion freeness conditions for $\nabla dt, \nabla dr$ and ∇s^i are

$$a_{ij} = a_{ji}, \quad h_{ij} = h_{ji}, \quad \Gamma_{jk}^i - \Gamma_{kj}^i + 2\epsilon_{jk}^i = 0, \quad (4.2)$$

respectively, and the conditions needed for the compatibility with the metric are

$$dr \otimes dt \otimes dt : \partial_r \beta + 2\beta d = 0,$$

$$dr^{\otimes 3} : \partial_r H + 2HR = 0,$$

$$\begin{aligned}
dr \otimes s^l \otimes s^j : 2rg_{lj} + r^2 g_{ij} \tau^l_i + r^2 g_{lm} \tau^m_j &= 0, \\
s^m \otimes s^n \otimes dt : -\beta a_{mn} + r^2 g_{ij} \eta^j_l \sigma^{il}_{mn} &= 0, \\
dt^{\otimes 3} : -2\beta b &= 0, \\
dr \otimes dr \otimes dt / dr \otimes dt \otimes dr : -\beta c + H\phi &= 0, \\
dt \otimes dr \otimes dr : 2H\phi &= 0, \\
s^i \otimes dt \otimes dr / s^i \otimes dr \otimes dt : -\beta e_i + H\nu_i &= 0, \\
dr \otimes s^j \otimes dt / dr \otimes dt \otimes s^j : -\beta e_j + r^2 g_{ij} \Delta^i &= 0, \\
s^i \otimes dt \otimes dt : -2\beta f_j &= 0, \\
dt \otimes dt \otimes s^j / dt \otimes s^j \otimes dt : -\beta f_j + r^2 g_{ij} \alpha^i &= 0, \\
s^i \otimes dt \otimes s^j : -\beta a_{ij} + r^2 g_{lj} \eta^l_i &= 0, \\
dt \otimes dr \otimes dt / dt \otimes dt \otimes dr : -\beta d + H\theta &= 0, \\
s^m \otimes s^n \otimes dr : Hh_{mn} + r^2 g_{ij} \tau^j_l \sigma^{il}_{mn} &= 0, \\
dt \otimes dr \otimes s^i / dt \otimes s^i \otimes dr : H\nu_i + r^2 g_{ij} \Delta^j &= 0, \\
dr \otimes dr \otimes s^i / dr \otimes s^i \otimes dr : H\psi_i + r^2 g_{ij} \gamma^j &= 0, \\
s^i \otimes dr \otimes dr : 2H\psi_i &= 0, \\
s^i \otimes dr \otimes s^j : Hh_{ij} + r^2 g_{lj} \tau^l_i &= 0, \\
s^p \otimes s^q \otimes s^m : g_{lm} \Gamma^l_{pq} + g_{ij} \Gamma^j_{lm} \sigma^{il}_{pq} &= 0, \\
dt \otimes s^i \otimes s^j : g_{lj} \eta^l_i + g_{il} \eta^l_j &= 0.
\end{aligned}$$

We immediately note that $b = \phi = f_j = \psi_i = 0$ for the 5th, 7th, 10th, and 17th equations respectively. In this case, we have that $\alpha^i = \phi = \gamma^i = 0$ by the 11th, 6th and 16th equations respectively. Also, solving simultaneously the 8th, 9th, 15th equations, we obtain $\Delta^i = e_i = \nu_i = 0$. The values of d and R are deduced from the 1st and 2nd equations respectively, while θ comes from 13th and 1st equations, with result

$$d = -\frac{\partial_r \beta}{2\beta}, \quad R = -\frac{\partial_r H}{2H}, \quad \theta = -\frac{\partial_r \beta}{2H}. \quad (4.3)$$

The 3rd equations together with the symmetry of g_{ij} lead to $\tau^i_j = -\frac{1}{r} \delta^i_j$. Now, we can solve the 18th equation as

$$h_{ij} = \frac{r}{H} g_{ij}. \quad (4.4)$$

The 20th equation gives the condition $\eta_{ij} - \eta_{ji} = 0$, where we used $\eta^k_j g_{ki} = \eta_{ij}$. But the 12th equation produces $a_{ij} = \frac{r^2}{\beta} \eta_{ij}$ so that a_{ij} is anti-symmetric, which together with the torsion freeness conditions imply that $a_{ij} = \eta^i_j = 0$.

Theorem 4.1. *The static Schwarzschild-like metric with spatial part a fuzzy sphere,*

$$g = -\beta(r)dt \otimes dt + H(r)dr \otimes dr + r^2 g_{ij} s^i \otimes s^j,$$

where g_{ij} is a constant real symmetric matrix, has a canonical $*$ -preserving QLC

$$\begin{aligned}\nabla dt &= -\frac{1}{2\beta} \partial_r \beta dr \otimes_s dt, \\ \nabla dr &= -\frac{1}{2H} \partial_r H dr \otimes dr + \frac{r}{H} g_{ij} s^i \otimes s^j - \frac{1}{2H} \partial_r \beta dt \otimes dt, \\ \nabla s^i &= -\frac{1}{2} \Gamma^i_{jk} s^j \otimes s^k - \frac{1}{r} dr \otimes_s s^i,\end{aligned}$$

where $\Gamma_{ijk} = 2\epsilon_{ikm}g_{mj} + \text{Tr}(g)\epsilon_{ijk}$ is the fuzzy sphere QLC from [29]. The corresponding Ricci scalar and Laplacian are

$$\begin{aligned}S &= \frac{1}{2H\beta} \partial_r^2 \beta - \frac{1}{4H\beta^2} (\partial_r \beta)^2 - \frac{1}{4H^2\beta} \partial_r \beta \partial_r H + \frac{3}{2r^2} \\ &\quad + \frac{1}{4rH} (3 + \text{Tr}(g)) \left(\frac{\partial_r \beta}{\beta} - \frac{\partial_r H}{H} \right) + \frac{\text{Tr}(g)}{r^2 H} \left(1 - \frac{H}{2} \right) + \frac{(\text{Tr}(g))^2}{4r^2}, \\ \Delta &= -\frac{1}{\beta} \partial_t^2 + \frac{1}{H} \partial_r^2 + \left(\frac{3}{rH} - \frac{\partial_r H}{2H^2} + \frac{\partial_r \beta}{2H\beta} \right) \partial_r + \frac{g^{ij}}{r^2} \partial_i \partial_j.\end{aligned}$$

Proof. Most of the analysis was done above. The torsion-freeness and metric compatibility conditions for the Christoffel symbol Γ of the fuzzy sphere part are the same as in [29], namely the second half of the $*$ -preserving conditions

$$h_{ij} s^i \otimes s^j - \bar{h}_{ji} \sigma(s^j \otimes s^i) = 0, \quad \Gamma^i_{jk} s^j \otimes s^k - \bar{\Gamma}^i_{kj} \sigma(s^k \otimes s^j) = 0$$

coming from ∇dr and ∇s^i respectively, with $(s^i)^* = s^i$, $dr^* = dr$ and $dt^* = dt$. There is therefore a unique solution for Γ under the assumption that it consists of constants on the fuzzy sphere [29], and we use this solution. This has σ the flip on the s^i and hence Γ real. In this case, the other condition for $*$ -preserving requires h_{ij} to be Hermitian, which already holds because h_{ij} is real and symmetric for (4.4). The 4th and 14th metric compatibility equations also then hold. The connection stated is then obtained by substituting into the general form of the connection. This completes the analysis for the canonical QLC.

The curvature for this connection comes out as

$$\begin{aligned}R_{\nabla} s^i &= \left(\frac{1}{4} \Gamma^i_{jk} \epsilon^j_{mn} - \frac{1}{4} \Gamma^i_{ml} \Gamma^l_{nk} + \frac{1}{H} g_{nk} \delta^i_m \right) s^m \wedge s^n \otimes s^k \\ &\quad - \frac{1}{2rH} \partial_r H s^i \wedge dr \otimes dr + \frac{1}{2r} (\epsilon^i_{jk} - \Gamma^i_{jk}) s^j \wedge s^k \otimes dr - \frac{1}{2rH} \partial_r \beta s^i \wedge dt \otimes dt, \\ R_{\nabla} dt &= \left(\frac{\partial_r^2 \beta}{2\beta} - \left(\frac{\partial_r \beta}{2\beta} \right)^2 - \frac{1}{4\beta H} \partial_r \beta \partial_r H \right) dt \wedge dr \otimes dr + \frac{r}{2H\beta} \partial_r \beta g_{ij} dt \wedge s^i \otimes s^j, \\ R_{\nabla} dr &= -\frac{r}{2H^2} g_{ij} \partial_r H dr \wedge s^i \otimes s^j + \frac{r}{2H} (g_{ml} \Gamma^l_{nj} - g_{ij} \epsilon^l_{mn}) s^m \wedge s^n \otimes s^j \\ &\quad + \left(\frac{1}{4H^2} \partial_r \beta \partial_r H - \frac{1}{2H} \partial_r^2 \beta + \frac{1}{4\beta H} (\partial_r \beta)^2 \right) dr \wedge dt \otimes dt + \frac{g_{ij}}{H} s^i \wedge s^j \otimes dr.\end{aligned}$$

Taking the antisymmetric lift of products of the basic one-forms and tracing gives the associated Ricci tensor

$$\begin{aligned} 4\text{Ricci} = & \left(\frac{\partial_r^2 \beta}{\beta} - \frac{3}{rH} \partial_r H - \frac{1}{2\beta H} \partial_r \beta \partial_r H - \frac{1}{2} \left(\frac{\partial_r \beta}{\beta} \right)^2 \right) dr \otimes dr \\ & + \frac{1}{H} \left(\frac{1}{2H} \partial_r \beta \partial_r H - \partial_r^2 \beta + \frac{1}{2\beta} (\partial_r \beta)^2 - \frac{3}{r} \partial_r \beta \right) dt \otimes dt \\ & + \left(\frac{r}{H\beta} g_{ij} \partial_r \beta - \frac{r}{H^2} g_{ij} \partial_r H + 4 \frac{g_{ij}}{H} - 2g_{ij} - 2\delta_{ij} + \text{Tr}(g)g_{ij} \right) s^i \otimes s^j. \end{aligned}$$

This gives the Ricci scalar as stated. The Laplacian is also immediate from ∇ and the inverse metric. \square

The QLC here is unique under the reasonable assumption as in [29] that the Γ_{jk}^i are constant on the fuzzy sphere, given that the g_{ij} have to be. To do some physics, we focus on the static rotationally invariant case where $g_{ij} = k\delta_{ij}$, for a positive constant k . In this case, it follows from the above that Ricci = 0 if and only if

$$H(r) = \frac{1}{\beta(r)}, \quad \beta(r) = \frac{1}{2} \left(\frac{1}{k} + 1 \right) - \frac{3}{4}k + \frac{c_1}{r^2},$$

where c_1 is an arbitrary constant. The values

$$k = \frac{1}{3}(\sqrt{7} - 1), \quad c_1 = -r_H^2 \quad (4.5)$$

give the form of β for the Tangherlini black hole metric of mass M , namely

$$\beta(r) = 1 - \frac{r_H^2}{r^2}, \quad r_H^2 = \frac{8}{3}G_5 M, \quad (4.6)$$

but note that the latter only makes sense in 5D spacetime due to an extra length dimension in the Newton constant G_5 . We are thinking of our model as 4D so we will *not* take this value but just work with r_H as a free parameter. A different value of k can be absorbed in a different normalisation of the t, r variables while r_H is more physical.

The quantum geometric structures in this ‘fuzzy black hole’ Ricci flat case are

$$g = - \left(1 - \frac{r_H^2}{r^2} \right) dt \otimes dt + \left(1 - \frac{r_H^2}{r^2} \right)^{-1} dr \otimes dr + r^2 k s^i \otimes s^i, \quad (4.7)$$

$$(dt, dt) = -\frac{r^2}{r^2 - r_H^2}, \quad (dr, dr) = 1 - \frac{r_H^2}{r^2}, \quad (s^i, s^j) = \frac{\delta^{ij}}{kr^2}, \quad (4.8)$$

$$\nabla dt = -\frac{r_H^2}{r(r^2 - r_H^2)} dr \otimes_s dt, \quad (4.9)$$

$$\nabla dr = \frac{r_H^2}{r(r^2 - r_H^2)} dr \otimes dr - \frac{r_H^2}{r^3} \left(1 - \frac{r_H^2}{r^2} \right) dt \otimes dt + rk \left(1 - \frac{r_H^2}{r^2} \right) s^i \otimes s^i, \quad (4.10)$$

$$\nabla s^i = -\frac{1}{2} \epsilon^i_{jk} s^j \otimes s^k - \frac{1}{r} dr \otimes_s s^i, \quad (4.11)$$

$$R_{\nabla} dt = -\frac{3r_H^2}{r^2(r^2 - r_H^2)} dt \wedge dr \otimes dr + \left(\frac{r_H}{r} \right)^2 k dt \wedge s^i \otimes s^i, \quad (4.12)$$

$$R_{\nabla} dr = \left(\frac{r_H}{r}\right)^2 k dr \wedge s^i \otimes s^i + 3r_H^2 \frac{r^2 - r_H^2}{r^6} dr \wedge dt \otimes dt, \quad (4.13)$$

$$\begin{aligned} R_{\nabla} s^i &= \left(-\frac{1}{4} + k \left(1 - \frac{r_H^2}{r^2}\right)\right) s^i \wedge s^j \otimes s^j + \left(\frac{r_H}{r}\right)^2 \frac{1}{r^2 - r_H^2} s^i \wedge dr \otimes dr \\ &\quad + \frac{r_H^2}{r^6} (r_H^2 - r^2) s^i \wedge dt \otimes dt, \end{aligned} \quad (4.14)$$

$$\Delta = -\left(1 - \frac{r_H^2}{r^2}\right)^{-1} \partial_t^2 + \left(\frac{3}{r} - \frac{r_H^2}{r^3}\right) \partial_r + \left(1 - \frac{r_H^2}{r^2}\right) \partial_r^2 + \frac{1}{kr^2} \sum_i \partial_i^2. \quad (4.15)$$

For comparison, the classical Tangherlini 5D black hole metric has the form

$$g = -\left(1 - \frac{r_H^2}{r^2}\right) dt \otimes dt + \left(1 - \frac{r_H^2}{r^2}\right)^{-1} dr \otimes dr + r^2 g_{S^3}$$

with the Laplacian

$$\Delta = -\left(1 - \frac{r_H^2}{r^2}\right)^{-1} \partial_t^2 + \left(\frac{3}{r} - \frac{r_H^2}{r^3}\right) \partial_r + \left(1 - \frac{r_H^2}{r^2}\right) \partial_r^2 + \frac{1}{r^2} \Delta_{S^3},$$

where g_{S^3} denotes the metric element on a unit S^3 . We see that this has just the same form as our metric and Laplacian except that our unit fuzzy sphere Laplacian $\sum_i \partial_i^2$ is replaced by the classical unit S^3 Laplacian

$$\Delta_{S^3} = \frac{1}{\sin^2 \psi} \partial_\psi (\sin^2 \psi \partial_\psi) + \frac{1}{\sin^2 \psi \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \psi \sin^2 \theta} \partial_\phi^2$$

in standard angular coordinates.

We will also be interested in the spatial geometry of the fuzzy black hole as a time slice with respect to the t coordinate. This is easily achieved in our formalism.

Proposition 4.2. *Defining the spatial geometry of the fuzzy black hole as a slice of the 4D geometry by setting $dt = 0$, gives*

$$\begin{aligned} g &= \beta^{-1} dr \otimes dr + kr^2 s^i \otimes s^i, \\ \nabla dr &= \frac{r_H^2}{r^3 \beta} dr \otimes dr + kr \beta s^i \otimes s^i, \\ \nabla s^i &= -\frac{1}{2} \epsilon^i_{jk} s^j \otimes s^k - \frac{1}{r} dr \otimes_s s^i, \\ R_{\nabla} dr &= k \left(\frac{r_H}{r}\right)^2 dr \wedge s^i \otimes s^i, \\ R_{\nabla} s^i &= \left(k\beta - \frac{1}{4}\right) s^i \wedge s^j \otimes s^j + \frac{r_H^2}{r^4 \beta} s^i \wedge dr \otimes dr, \\ \text{Ricci} &= \frac{3r_H^2}{2r^4 \beta} dr \otimes dr + \left(k \left(1 - \frac{r_H^2}{2r^2}\right) - \frac{1}{4}\right) s^i \otimes s^i, \\ S &= \frac{3}{r^2} \left(1 - \frac{1}{4k}\right), \end{aligned}$$

using the antisymmetric lift as usual and $\beta = 1 - \frac{r_H^2}{r^2}$. The spatial Einstein tensor $\text{Eins} = \text{Ricci} - \frac{S}{2}g$ comes out as

$$\text{Eins} = \frac{3}{2r^2\beta} \left(\frac{1}{4k} - \beta \right) dr \otimes dr + \frac{1}{2} \left(\frac{1}{4} - k \left(1 + \frac{r_H^2}{r^2} \right) \right) s^i \otimes s^i$$

and is conserved in the sense $\nabla \cdot \text{Eins} = 0$.

Proof. That setting $dt = 0$ gives a QLC for the reduced metric and its curvature follows on general grounds but can be checked explicitly. The computation of Ricci is a trace of R_∇ as usual: we apply this to the second factors of g and then apply $(dr, dr) = \beta^{-1}$, $(s^i, s^j) = \frac{\delta_{ij}}{kr^2}$ (and other cases zero) to the first two tensor factors. The Ricci scalar S and Einstein tensor then follow. For its divergence, we first compute ∇Eins by acting with ∇ on each tensor factor but keeping its left-most output to the far left using the trivial flip σ ,

$$\begin{aligned} \nabla \text{Eins} &= \frac{1}{2} \left(\frac{1}{4} - k \left(1 + \frac{r_H^2}{r^2} \right) \right) \left(-\frac{1}{r} s^i \otimes s^i \otimes dr \right) + d \left(\frac{3}{2r^2\beta} \left(\frac{1}{4k} - \beta \right) \right) \otimes dr \otimes dr \\ &\quad + \frac{3}{2r^2\beta} \left(\frac{1}{4k} - \beta \right) \left(\frac{2r_H^2}{r^3\beta} dr \otimes dr \otimes dr + kr\beta s^i \otimes s^i \otimes dr \right) + \dots, \end{aligned}$$

where \dots refers to terms that involve $dr \otimes s^i$ or $s^i \otimes dr$ in the first two tensor factors. The terms in $\nabla(s^i \otimes s^i)$ with s 's in all tensor factors cancel. We then define $\nabla \cdot \text{Eins}$ by applying $(,)$ to the first two tensor factors to give

$$\begin{aligned} \nabla \cdot \text{Eins} &= \left(-\frac{3}{2r^3k} \left(\frac{1}{4} - k \left(1 + \frac{r_H^2}{r^2} \right) \right) + \beta \left(\frac{1}{4} - k \left(1 + \frac{r_H^2}{r^2} \right) \right) \right)' \\ &\quad + \frac{3}{2r^2} \left(\frac{1}{4k} - \beta \right) \frac{2r_H^2}{r^3\beta} + \frac{9}{2r^3} \left(\frac{1}{4k} - \beta \right) dr \end{aligned}$$

from the displayed terms taken in order. We then check that the function in brackets vanishes. \square

4.1. Motion in the fuzzy black hole background

In terms of physical implications, since the radial form for the fuzzy black hole is the same as that of the Tangherlini solution, we can apply the usual logic that $g_{00} = -(1 + 2\Phi)$ to first approximation contains the gravitational potential Φ per unit mass governing geodesic motion for a mass m in the weak field limit. Therefore in our case, this should be

$$\Phi = -\frac{r_H^2}{2r^2} \quad (4.16)$$

but, because we are thinking of this as a 4D model, we do not set r_H to be the same as for a Tangherlini 5D black hole. Rather, we think of r_H as the physical parameter and equate it for purpose of comparison with $r_H = 2GM$ so that the horizon occurs at the same r as for a Schwarzschild black hole of mass M . The weak field force law is no longer Newtonian gravity, having an inverse cubic form according to $\Phi = -2G^2M^2/r^2$. This is rather different from modified gravity schemes such as MOND for the modelling of dark matter [41], but could still be of interest.

To properly justify the above, we should study geodesics, which is possible but not easy on quantum spacetimes. Here, we instead reach the same conclusion from the point of view of quantum mechanics as the non-relativistic limit of the Klein–Gordon equation

$$\Delta\phi = m^2\phi.$$

As proof of concept, we first do this for a classical Schwarzschild black hole, where $\beta(r) = 1 - \frac{r_H}{r}$. The Laplacian is

$$\Delta_{\text{Sch}} = -\frac{1}{\beta} \frac{\partial^2}{\partial t^2} + \Delta_r + \frac{1}{r^2} \Delta_{S^2}; \quad \Delta_r := \frac{1}{r^2} \frac{\partial}{\partial r} \left(\beta r^2 \frac{\partial}{\partial r} \right)$$

and Δ_{S^2} is normalised to have eigenvalues $\lambda_l = -l(l+1)$ on the spherical harmonics of degree $l \in \mathbb{N} \cup \{0\}$ for the orbital angular momentum. We focus on waves of fixed l and look for solutions of the form

$$\phi = e^{-imt} \psi_l(t, r)$$

with ψ_l slowly varying in t . Accordingly neglecting its double time derivative, the Klein–Gordon equation becomes the Schroedinger-like equation

$$i\dot{\psi}_l = -\frac{\beta}{2m} \left(\Delta_r + \frac{\lambda_l}{r^2} \right) \psi_l + (\beta - 1) \frac{m}{2} \psi_l, \quad (4.17)$$

where $\beta(\Delta_r + \frac{\lambda_l}{r^2})$ is a modification by β of the \mathbb{R}^3 Laplacian on ψ_l in polars, which we think of as the square of a modified momentum (the difference is anyhow suppressed at large r), and $(\beta - 1)m/2 = -GMm/r$ is the expected Newtonian potential for Schroedinger's equation in the presence of a point source of mass M .

Note that e^{-imt} is not itself a solution of the Klein–Gordon equation. Repeating the above but with reference to an actual solution would be analogous to finding the forces experienced by a particle in geodesic motion, where one only sees tidal forces. Doing this in the Schwarzschild case, we first solve (numerically) for spherical $l = 0$ solutions of the form

$$\phi = e^{-i\omega t} \phi_\omega(r); \quad \left(\frac{\omega^2}{\beta} - m^2 \right) \phi_\omega + \Delta_r \phi_\omega = 0 \quad (4.18)$$

with initial conditions specified at large r . We then look for a Schroedinger-like equation relative to a fixed ϕ_ω , by solving the Klein–Gordon equation for solutions of the form

$$\phi = e^{-i\omega t} \phi_\omega(r) \psi_l(t, r) \quad (4.19)$$

with ψ_l of orbital angular momentum labelled by l and slowly varying in t . This time we obtain

$$i\dot{\psi}_l = -\frac{\beta}{2\omega} \left(\Delta_r + \frac{\lambda_l}{r^2} + 2\beta \frac{\phi'_\omega}{\phi_\omega} \frac{\partial}{\partial r} \right) \psi_l \quad (4.20)$$

with a new velocity-dependent correction but without the gravitational point source potential, as expected.

The natural choice for a reference field is to focus on the case $\omega = m$. For large r , we can neglect β' relative to $2/r$ in Δ_r and in this case one has an exact solution for ϕ_m in terms of Bessel I , K functions. We choose conditions which match to Bessel I , say, at large r , and we

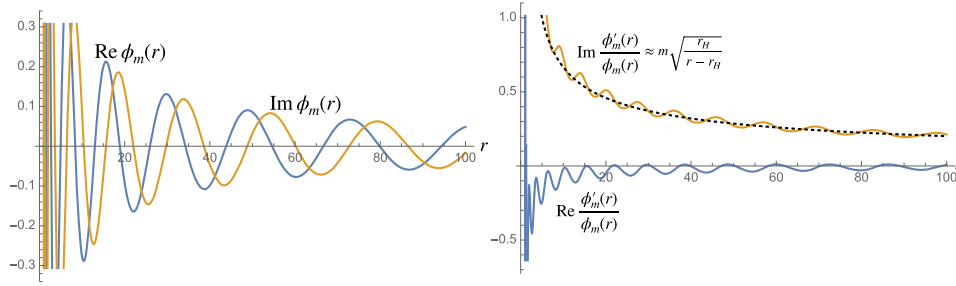


Figure 1. Radial solution $\phi_m(r)$ for the Klein–Gordon equation around a black hole shown for $m = 2/r_H$, $r_H = 1$ and asymptotic form of ϕ'_m/ϕ_m shown dashed.

assume $m > 1/r_H$ so that the test particle Compton wavelength is less than r_H . Then ϕ'_m/ϕ_m is barely oscillatory for larger m and decays gradually as $r \rightarrow \infty$ according to

$$\frac{\phi'_m}{\phi_m} \approx im\sqrt{\frac{r_H}{r-r_H}}, \quad r \gg r_H. \quad (4.21)$$

The exact numerical solution as illustrated in figure 1 is similar to this, although more oscillatory. We see that in this ‘comoving frame’ from a Klein–Gordon point of view, we do not experience the main force of gravity but we do see a novel radial velocity term in the effective Schroedinger-like equation approximated as

$$i\psi_l \approx -\frac{\beta}{2m} \left(\Delta_r + \frac{\lambda_l}{r^2} \right) \psi_l - i\beta^{\frac{3}{2}} \sqrt{\frac{2GM}{r}} \psi'_l \quad (4.22)$$

far from the horizon. Nearer the horizon, one needs to use the actual ϕ'_m/ϕ_m to avoid an instability coming in from the horizon. A numerical solution for ψ_l at $l = 0$ using the actual values is in figure 2, showing an initial Gaussian centred at $r = 5r_H$ evolving much as in regular quantum mechanics but, unlike the latter, decaying over time. Some of the noise in the picture comes from the numerical approximation.

The above is for a regular Schwarzschild black hole, but one can make a similar analysis for the different radial equation for our fuzzy black hole and thereby justify (4.16), provided we know something about the operators $\sum_i \partial_i^2$.

Proposition 4.3. $\sum_i \partial_i^2$ on the fuzzy sphere has eigenvalues $\lambda_l = -l(l+1)$ as for the classical Δ_{S^2} , with eigenspaces

$$H_l = \{x^{i_1}x^{i_2} \dots x^{i_l} f_{i_1 \dots i_l} \mid f \text{ totally symmetric and traceless}\}.$$

Proof. Here, as vector spaces, $\mathbb{C}[x^1, \dots, x^n] = \mathbb{C}[su_2^*] \cong U(su_2)$ by the Duflo map (as for any Lie algebra). This sends a commutative monomial in the x^i to an average of all orderings of its factors (it is an isomorphism because, although there are nontrivial commutation relations in the enveloping algebra, these are strong enough to reorder at the expense of lower degree.) This map is covariant for the coadjoint and adjoint actions, in our case, of su_2 , and therefore descends to a vector space isomorphism between polynomial functions on the classical sphere in cartesian coordinates on one side, and the fuzzy sphere $\mathbb{C}_\lambda[S^2]$ on the other side. Moreover,

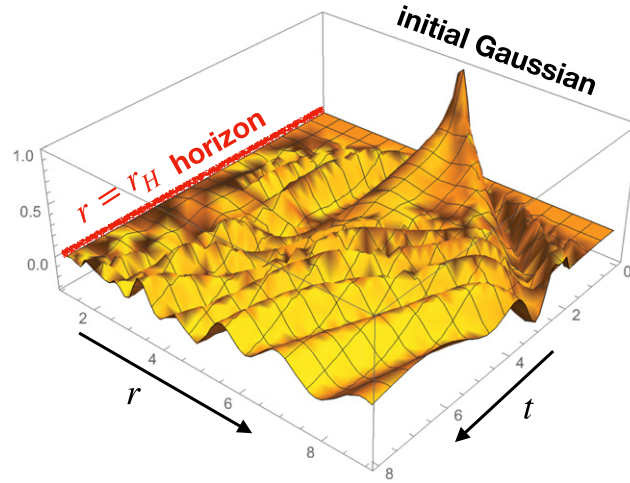


Figure 2. Schroedinger-like evolution relative to the ϕ_m in figure 1. We see an initial Gaussian centred at $r = 5r_H$ evolving much as in quantum mechanics but decaying over time, with the essentially zero initial values at $r = 1.1r_H, 10r_H$ held fixed.

$\partial_k x_i = \epsilon_{ijk} x^j$ for our differential calculus on the latter acts as orbital angular momentum. Hence $\sum_i \partial_i^2$ acts as the quadratic Casimir and can be computed on the classical sphere, where it decomposes the polynomial functions into the spherical harmonics of each degree l . These then correspond to the H_l as stated. One can check this directly on the fuzzy sphere on low degrees by hand, to fix the normalisation. For example, on degree $l = 1$, we have $\sum_k \partial_k^2 x^i = \partial_k \epsilon_{ijk} x^j = \epsilon_{jmk} \epsilon_{ijk} x^m = -2x^i$. \square

An expansion of $M_{2j+1}(\mathbb{C})$ into subspaces similar to H_l also applies in the reduced fuzzy sphere case with $l = 0, \dots, 2j$, see [31]. It means in our case that we can solve the Laplacian and look at the non-relativistic limits by the same methods as we illustrated for the Schwarzschild black hole. The only difference is that our functions have values $\psi_l(t, r) \in \mathbb{C}_\lambda[S^2]$, but the differential equations themselves in t, r are purely classical according to

$$\Delta_{\text{fuz}} = -\frac{1}{\beta} \frac{\partial^2}{\partial t^2} + \Delta_r + \frac{1}{kr^2} \lambda_l; \quad \Delta_r := \frac{1}{r^3} \frac{\partial}{\partial r} \left(\beta r^3 \frac{\partial}{\partial r} \right)$$

with $\beta = 1 - r_H^2/r^2$. Taking e^{-imt} as reference gives the same form as (4.17) but with $\frac{\lambda_l}{kr^2}$ in a modified effective spatial Laplacian. Then $(\beta - 1)m/2 = -2G^2 M^2 m/r^2$ for the gravitational potential energy, in agreement with (4.16).

Next, for the ‘comoving’ version, the $l = 0$ solutions of the Klein–Gordon equation are given by solving (4.18) as before and, relative to this, slowly-varying ψ_l defined by (4.19) obey the Schroedinger-like equation (4.20) but now with $\frac{\lambda_l}{kr^2}$ in place of $\frac{\lambda_l}{r^2}$. Focussing on the $\omega = m$ case, the main difference now is that ϕ_m decays more rapidly and in first approximation,

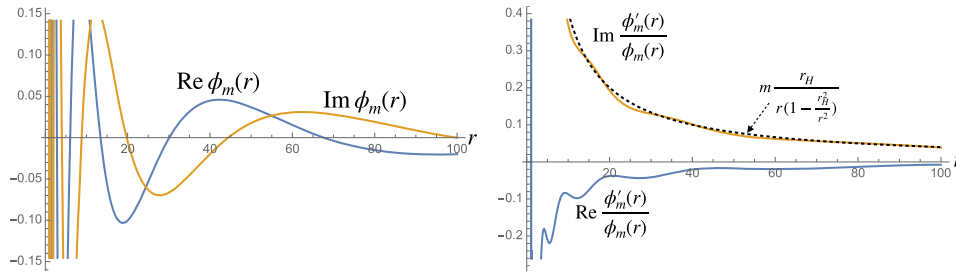


Figure 3. Radial solution $\phi_m(r)$ for the Klein–Gordon equation around a fuzzy black hole shown for $m = 4/r_H$, $r_H = 1$, and function $\phi'_m(r)/\phi_m(r)$.

if we leave out the β' term in Δ_r , is now solved by

$$\phi_m(r) \propto \frac{(r^2 - r_H^2)^{\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - m^2 r_H^2}}}{r^2}.$$

We focus on the $+$ case of the square root, which leads for $m \gg 1/r_H$ to a fair approximation

$$\frac{\phi'_m}{\phi_m} \approx im \frac{r_H}{r(1 - \frac{r_H^2}{r^2})}, \quad r \gg r_H.$$

The exact numerical solution is similar but with a degree of oscillation as illustrated in figure 3. As a result, the long range Schroedinger-like equation is

$$i\psi_l \approx -\frac{\beta}{2m} \left(\Delta_r + \frac{\lambda_l}{kr^2} \right) \psi_l - i\beta \frac{2GM}{r} \psi'_l$$

if we use the Schwarzschild value of r_H , showing a coupling to the velocity term of the same size as the usual gravitational potential per unit mass. As before, near the horizon we need the actual ϕ'_m/ϕ_m values for stability of the solutions. An initial Gaussian breaks up and decays over time, looking much as before.

Finally, although we have used the Schwarzschild value of r_H for purposes of comparison, since the geometry is asymptotically flat, we could naively try to define an actual ADM mass by copying its physical formulation in terms of the Einstein tensor of the spatial geometry [5, 17, 40], which in spherical polars amounts to the limit $r \rightarrow \infty$ of

$$\begin{aligned} M(r) &= \frac{2}{G(n-1)(n-2)\Omega_{n-1}} \int_{S_r^{n-1}} \text{Eins}(r\partial_r, \sqrt{\beta}\partial_r) d^{n-1}\Omega \\ &= \frac{2}{G(n-1)(n-2)} r^{n-1} \text{Eins}(r\partial_r, \sqrt{\beta}\partial_r) \end{aligned}$$

for a spatial geometry of dimension n . Here there is a factor -2 compared to the usual definition because our Ricci and hence Einstein tensor conventions reduce in the classical case to $-\frac{1}{2}$ of the usual ones. Ω_{n-1} is the volume of a unit sphere of dimension $n-1$ and we integrate with measure $d^{n-1}\Omega$ over the sphere S_r^{n-1} at radius r . The conformal Killing vector field in the general formula in [40] is just $r\partial_r$ in our case and the unit outward normal vector field is $\sqrt{\beta}\partial_r$ given the form of the spatial metric. As everything is rotationally invariant, the integration

merely gives a factor $r^{n-1}\Omega_{n-1}$. For a usual Schwarzschild black hole with $n = 3$, the Einstein tensor of the spatial geometry in our conventions can be extracted from [12, cor. 9.9] to find

$$\text{Eins}_{\text{Sch}}(r\partial_r, \sqrt{\beta}\partial_r) = \frac{1}{2r\sqrt{\beta}}(1 - \beta), \quad M(r) = \frac{r_H}{2G\sqrt{\beta}} \rightarrow \frac{r_H}{2G}$$

as expected for the Schwarzschild $\beta(r) = 1 - \frac{r_H^2}{r^2}$. If we instead use the fuzzy quantum black hole spatial geometry in proposition 4.2, the radial sector is completely classical, so it still makes sense to read off $\text{Eins}(\partial_r, \partial_r)$ as the coefficient of $dr \otimes dr$, resulting in our case in

$$\text{Eins}_{\text{fuz}}(r\partial_r, \sqrt{\beta}\partial_r) = \frac{3}{2r\sqrt{\beta}} \left(\frac{1}{4k} - \beta \right),$$

which, since $\beta(r) = 1 - \frac{r_H^2}{r^2}$ and $k \neq \frac{1}{4}$, results in $M(r) \rightarrow \infty$. If we took $k = \frac{1}{4}$ then we would not have a Ricci flat metric in the spacetime quantum geometry and we would get $M(r) \rightarrow 0$, which is not reasonable either. These problems are a consequence of the dimension jump in the quantum model, evidently requiring a more sophisticated approach to ADM mass. By contrast, if we were to set $n = 4$ and $k = \frac{1}{4}$ then we would obtain $M(r) \rightarrow \frac{r_H^2}{2G}$, which is rather close to the value (4.6) for a classical 5D black hole.

There is a similar story if we try to compute the Komar mass in 4D from its classical formula as a surface integral around a two-sphere at infinity. This can be written, cf [46], as the limit $r \rightarrow \infty$ of

$$M_{\text{Komar}}(r) = -\frac{1}{8\pi G} \int_{S^2(r)} i_{\partial_t} i_{\partial_r} (\partial_r \beta \, dt \wedge dr) = \frac{1}{2G} r^2 \partial_r \beta$$

where, for our form of metric, the timelike Killing vector converted to a one-form via the metric is $\xi = -\beta dt$ with $\nabla \wedge \xi = -\nabla \wedge (\beta dt) = -d(\beta dt) = \partial_r \beta dt \wedge dr$ since ∇ has zero torsion. The unit timelike reference field is $\frac{1}{\sqrt{\beta}} \partial_t$ and (and as above) the outward unit normal to the sphere $S^2(r)$ of radius r is $\sqrt{\beta} \partial_r$ (the $\sqrt{\beta}$ factors cancel in their product). For a Schwarzschild black hole, this gives $\frac{r_H}{2G}$ as expected, but for our solution $\beta(r) = 1 - \frac{r_H^2}{r^2}$, we obtain $M_{\text{Komar}}(r) \rightarrow 0$. We would have obtained a reasonable answer if we were integrating over a three-sphere in a four-dimensional spatial geometry, but that is not the case for our fuzzy black hole geometry.

5. Black hole with the discrete circle

We now consider the same ideas as in the preceding section, but for a 3D spacetime metric with S^1 in polar coordinates replaced by the discrete group \mathbb{Z}_n . The 2D FLRW model with S^1 replaced by \mathbb{Z}_n was done in [3] and we use the same notations. Briefly, $i \in \mathbb{Z}_n$ now labels the vertices of a polygon as an integer mod n . The ‘step up’ and ‘step down’ partial derivatives are $\partial_{\pm} = R_{\pm} - \text{id}$, where $(R_{\pm} f)(i) = f(i \pm 1)$ and e^{\pm} are the associated invariant one-forms with $e^{+*} = -e^{-}$. The calculus on \mathbb{Z}_n is not commutative as $e^{\pm} f = R_{\pm}(f)e^{\pm}$ for a function f , the e^{\pm} anticommute with each other and the exterior derivative is $df = (\partial_{\pm} f)e^{\pm}$ (sum over \pm) and $de^{\pm} = 0$. The classical limit $n \rightarrow \infty$ can be seen as a circle with a noncommutative 2D calculus, which is the classical calculus on S^1 extended by a one-form Θ_0 . The latter has no classical analogue but can be viewed as normal to the circle when embedded in a plane [3], but without an actual normal variable. The natural invariant metric on the polygon is $-e^+ \otimes_s e^-$.

Now the spacetime coordinate algebra is $A = C^\infty(\mathbb{R} \times \mathbb{R}_{>0}) \otimes \mathbb{C}(\mathbb{Z}_n)$ with t, r for the time and radial classical variables, and we consider a static Schwarzschild-like metric of the form

$$g = -\beta(r)dt \otimes dt + H(r)dr \otimes dr - \alpha_{ab}(r, i)e^a \otimes e^b. \quad (5.1)$$

Invertibility of the metric requires centrality, which dictates $\alpha_{ab}(r, i) = \alpha_a(r, i)\delta_{ab-1}$ for some real-valued functions α_a . We also require edge-symmetry $\alpha_a = R_a(\alpha_{a-1})$ so that the length of each edge $\bullet_i - \bullet_{i+1}$ for the \mathbb{Z}_n at radius r is the same in either direction, namely given by some real function $a(r, i)$ according to

$$\alpha_+(r, i) = a(r, i), \quad \alpha_-(r, i) = R_-a(r, i). \quad (5.2)$$

We limit attention to this form of metric.

We take analogous conditions on the tensor product calculus as in the previous section, in the sense that the functions of the time t , radius r as well as dt, dr are classical and graded-commute with everything. In view of this, and in line with [3] and with the fuzzy case above, we make the simplifying assumption that the connection braiding σ among the differentials dr, dt and between them and e^\pm is just the flip map. In this case, the most general form of a potential bimodule connection turns out to be

$$\begin{aligned} \nabla e^a &= -\Gamma_{bc}^a e^b \otimes e^c + \nu_{ab}^a dt \otimes_s e^b + \gamma_{ab}^a dr \otimes_s e^b, \\ \nabla dt &= \xi_{ab} e^a \otimes e^b + b dt \otimes dt + c dr \otimes dr + h dr \otimes_s dt, \\ \nabla dr &= A_{ab} e^a \otimes e^b + B dt \otimes dt + C dr \otimes dr + D dr \otimes_s dt, \end{aligned}$$

where the coefficients are elements of the algebra A and of the form

$$\nu_{ab}^a = \nu_a \delta_{a,b-1}, \quad \gamma_{ab}^a = \gamma_a \delta_{a,b-1}, \quad A_{ab} = A_a \delta_{a,b-1}, \quad \xi_{ab} = \xi_a \delta_{a,b-1}. \quad (5.3)$$

We now analyse when such a bimodule connection is a QLC. The requirement to be torsion free comes down to

$$A_{ab} = A_{ba}, \quad \xi_{ab} = \xi_{ba}, \quad \Gamma_{bc}^a = \Gamma_{cb}^a, \quad \wedge(\text{id} + \sigma)(e^a \otimes e^b) = 0, \quad (5.4)$$

while to be metric compatible comes down to the 13 equations:

$$\begin{aligned} dr \otimes dt \otimes dt : \partial_r \beta + 2\beta h &= 0, \\ dr^{\otimes 3} : \partial_r H + 2HC &= 0, \\ dt^{\otimes 3} : 2\beta b &= 0, \\ dr \otimes dt \otimes dr/dr \otimes dr \otimes dt : -\beta c + HD &= 0, \\ dt \otimes dt \otimes dr/dt \otimes dr \otimes dt : -\beta h + HB &= 0, \\ dt \otimes dr \otimes dr : 2HD &= 0, \\ dr \otimes e^a \otimes e^b : -\partial_r \alpha_{ab} - \alpha_{cb} \gamma_a^c - \alpha_{ac} R_a(\gamma_b^c) &= 0, \\ e^a \otimes e^b \otimes dt : -\beta \xi_{ab} - \alpha_{cd} R_c(\nu_f^d) \sigma^{cf}_{ab} &= 0, \\ e^a \otimes dt \otimes e^b : -\beta \xi_{ab} - \alpha_{cb} \nu_a^c &= 0, \\ e^a \otimes e^b \otimes dr : HA_{ab} - \alpha_{cd} R_c(\gamma_f^d) \sigma^{cf}_{ab} &= 0, \end{aligned}$$

$$\begin{aligned}
e^a \otimes dr \otimes e^b : HA_{ab} - \alpha_{cb} \gamma^c_a &= 0, \\
e^a \otimes e^b \otimes e^c : -\partial_a \alpha_{bc} - \alpha_{dc} \Gamma^d_{ab} - \alpha_{df} R_d(\Gamma^f_{gc}) \sigma^{dg}_{ab} &= 0, \\
dt \otimes e^a \otimes e^b : -\alpha_{cb} \nu^c_a - \alpha_{ac} R_a(\nu^c_b) &= 0.
\end{aligned}$$

The 1st and 2nd equations gives h , C respectively, and these together with the 5th equation give B , resulting in

$$h = -\frac{1}{2\beta} \partial_r \beta, \quad C = -\frac{1}{2H} \partial_r H, \quad B = -\frac{1}{2H} \partial_r \beta. \quad (5.5)$$

The 3rd, 6th and 4th equations imply that $c = b = D = 0$. Next, the 9th and 11th equations tell us that

$$\nu_a = -\frac{\beta \xi_a}{\alpha_a}, \quad \gamma_a = \frac{HA_a}{\alpha_a}, \quad (5.6)$$

while, given the edge-symmetry, the 13th and 7th equations reduce to

$$\nu_a + R_a(\nu_{a-1}) = 0, \quad \gamma_a + R_a(\gamma_{a-1}) = -\frac{\partial_r \alpha_a}{\alpha_a}. \quad (5.7)$$

Given that the 12th equation for metric compatibility and the torsion-freeness condition are the same as for the polygon in [3], we are led to take Γ^a_{bc} at each radius r the same as for the QLC $\nabla^{\mathbb{Z}_n}$ on the polygon found there. This has

$$\begin{aligned}
\nabla^{\mathbb{Z}_n} e^+ &= (1 - \rho) e^+ \otimes e^+, & \nabla^{\mathbb{Z}_n} e^- &= (1 - R^2 \rho^{-1}) e^- \otimes e^-, \\
\rho(r, i) &= \frac{a(r, i+1)}{a(r, i)}
\end{aligned}$$

and its braiding obeys $\sigma(e^\pm \otimes e^\mp) = e^\mp \otimes e^\pm$, in which case the 8th and 10th metric compatibility equations become

$$R_a(\nu_{a-1}) = -\frac{\beta}{\alpha_a} \xi_{a-1}, \quad R_a(\gamma_{a-1}) = \frac{HA_{a-1}}{\alpha_a}. \quad (5.8)$$

Using the first of (5.6) and (5.8) in (5.7) leads us to $\xi_a = -\xi_{a-1}$, which together with the second half of the torsion-freeness conditions (5.4) requires $\xi_a = 0$, and as consequence $\nu_a = 0$. Similarly, inserting the second half of (5.6) and (5.8) in (5.7) produces

$$-A_a - A_{a-1} = \frac{\partial_r \alpha_a}{H}. \quad (5.9)$$

In summary, for a QLC, it only remains to solve for A_a, γ_a subject to such residual equations, with the other coefficients zero or determined. It also remains to impose reality in the form of $\nabla*$ -preserving.

Proposition 5.1. *Assuming a static edge-symmetric central metric (5.1) and σ the flip on generators involving dr, dt leads to a $*$ -preserving QLC if and only if $\partial_- \partial_r \alpha_a = 0$ (which needs the underlying $a(r, i)$ to be the sum of a function of r and a function of i). The $*$ -preserving QLC with real coefficients is then unique and given by*

$$\begin{aligned}\nabla dt &= -\frac{1}{2\beta}\partial_r\beta\,dr\otimes_s dt, \\ \nabla dr &= -\frac{1}{2H}\partial_r H\,dr\otimes dr - \frac{\partial_r\alpha_+}{2H}e^+\otimes e^- - \frac{\partial_r\alpha_-}{2H}e^-\otimes e^+ - \frac{1}{2H}\partial_r\beta\,dt\otimes dt, \\ \nabla e^\pm &= \nabla^{\mathbb{Z}_n}e^\pm - \frac{1}{r}dr\otimes_s e^\pm.\end{aligned}$$

Proof. The $*$ -preserving conditions for ∇ include conditions on Γ which coincide at each r with those for a QLC on \mathbb{Z}_n as in [3], for which the solution is unique, so we are forced to this choice for Γ . The remaining $*$ -preserving conditions require B, C, h to be real-valued, which already holds because they are functions of the metric coefficients, together with the conditions

$$\sum_a (\bar{A}_{a-1}\sigma(e^{a-1}\otimes e^a) - A_a e^a\otimes e^{a-1}) = 0, \quad \bar{\gamma}_a = R_a(\gamma_{a-1}), \quad (5.10)$$

$$\sum_a (\bar{\xi}_{a-1}\sigma(e^{a-1}\otimes e^a) - \xi_a e^a\otimes e^{a-1}) = 0, \quad \bar{\nu}_a = R_a(\nu_{a-1}). \quad (5.11)$$

The conditions (5.11) are trivially fulfilled, while the second half of (5.10) implies $\bar{A}_a = A_{a-1}$, which together with the form of the braiding map σ solves the first half of (5.10). In this case, (5.9) takes the form

$$-A_a - \bar{A}_a = \frac{\partial_r\alpha_a}{H}. \quad (5.12)$$

The second halves of (5.6) and (5.8), together with the edge-symmetric condition, tell us that $A_a = R_{a-1}(A_a)$ and hence that A_a is independent of the discrete variable, i.e., just a function of r . In this case, we must have

$$A_\pm = -\frac{\partial_r\alpha_\pm}{2H} \pm y(r), \quad \gamma_\pm = -\frac{\partial_r\alpha_\pm}{2\alpha_\pm} \pm \imath \frac{Hy(r)}{\alpha_\pm}$$

for some real-valued function $y(r)$. It is natural at this point to set $y(r) = 0$ so as to keep coefficients real, and we do this now (this was also done at the parallel point in [3]). Another consequence of A_\pm being constant on the polygon is $\partial_\pm A_\pm = 0$, which lead us to $\partial_\pm\partial_r\alpha_a = 0$. This corresponds to restricting the underlying metric function $a(r, i)$ in (5.2) as stated. \square

This is a general result, but we now focus attention on the \mathbb{Z}_n -invariant metric where $a(r, i)$ is independent of i and moreover of the expected radial form.

Theorem 5.2. *The static \mathbb{Z}_n -invariant Schwarzschild-like metric*

$$g = -\beta(r)dt\otimes dt + H(r)dr\otimes dr - r^2e^+\otimes_s e^-$$

has a canonical $$ -preserving QLC,*

$$\begin{aligned}\nabla dt &= -\frac{1}{2\beta}\partial_r\beta\,dr\otimes_s dt, \\ \nabla dr &= -\frac{1}{2H}\partial_r H\,dr\otimes dr - \frac{r}{H}e^+\otimes_s e^- - \frac{1}{2H}\partial_r\beta\,dt\otimes dt, \\ \nabla e^\pm &= -\frac{1}{r}dr\otimes_s e^\pm\end{aligned}$$

with the corresponding Ricci scalar and Laplacian

$$S = \frac{1}{2H\beta} \partial_r^2 \beta - \frac{1}{4H\beta^2} (\partial_r \beta)^2 - \frac{1}{4H^2\beta} \partial_r H \partial_r \beta - \frac{1}{rH^2} \partial_r H + \frac{1}{rH\beta} \partial_r \beta + \frac{1}{r^2 H},$$

$$\Delta = \frac{2}{r^2} (\partial_+ + \partial_-) - \frac{1}{\beta} \partial_t^2 + \frac{1}{H} \partial_r^2 + \left(\frac{2}{rH} - \frac{1}{2H^2} \partial_r H + \frac{1}{2H\beta} \partial_r \beta \right) \partial_r.$$

This is Ricci flat if and only if

$$H(r) = \frac{1}{\beta(r)}, \quad \beta(r) = \frac{r_H}{r} \quad (5.13)$$

for some constant r_H of length dimension.

Proof. Taking $\alpha_{\pm} = r^2$ in the preceding proposition immediately gives the canonical QLC stated. Its associated curvature comes out as

$$R_{\nabla} dt = \frac{1}{2\beta} \left(\partial_r^2 \beta - \frac{1}{2\beta} (\partial_r \beta)^2 - \frac{1}{2H} \partial_r H \partial_r \beta \right) dt \wedge dr \otimes dr$$

$$- \frac{r}{2H\beta} \partial_r \beta dt \wedge e^+ \otimes_s e^-,$$

$$R_{\nabla} e^{\pm} = -\frac{\partial_r H}{2rH} e^{\pm} \wedge dr \otimes dr - \frac{1}{2rH} \partial_r \beta e^{\pm} \wedge dt \otimes dt - \frac{1}{H} e^{\pm} \wedge e^{\mp} \otimes e^{\pm},$$

$$R_{\nabla} dr = \frac{1}{2H} \left(\frac{1}{2H} \partial_r \beta \partial_r H - \partial_r^2 \beta + \frac{1}{2\beta} (\partial_r \beta)^2 \right) dr \wedge dt \otimes dt$$

$$- \frac{1}{2} r \partial_r \beta dr \wedge e^+ \otimes_s e^-.$$

Taking the antisymmetric lift of products of basic one-forms and tracing gives the associated Ricci tensor

$$2\text{Ricci} = \left(\frac{1}{2\beta} \partial_r^2 \beta - \left(\frac{\partial_r \beta}{2\beta} \right)^2 - \frac{1}{4H\beta} \partial_r H \partial_r \beta - \frac{1}{rH} \partial_r H \right) dr \otimes dr$$

$$+ \left(-\frac{r}{2H\beta} \partial_r \beta + \frac{r}{2H^2} \partial_r H - \frac{1}{H} \right) e^+ \otimes_s e^-$$

$$+ \left(\frac{1}{4H^2} \partial_r \beta \partial_r H - \frac{1}{2H} \partial_r^2 \beta + \frac{1}{4H\beta} (\partial_r \beta)^2 - \frac{\partial_r \beta}{rH} \right) dt \otimes dt.$$

The Ricci scalar and Laplacian follow on application of the inverse metric. We then solve for $\text{Ricci} = 0$. The calculations are straightforward and are omitted. \square

The quantum geometric structures in the discrete black hole-like Ricci-flat case are

$$g = -\frac{r_H}{r} dt \otimes dt + \frac{r}{r_H} dr \otimes dr - r^2 e^+ \otimes_s e^-, \quad (5.14)$$

$$(dt, dt) = -\frac{r}{r_H}, \quad (dr, dr) = \frac{r_H}{r}, \quad (e^{\pm}, e^{\mp}) = -\frac{1}{r^2}, \quad (5.15)$$

$$\nabla dt = \frac{1}{2r} dr \otimes_s dt, \quad (5.16)$$

$$\nabla dr = -\frac{1}{2r} dr \otimes dr - r_H e^+ \otimes_s e^- + \frac{r_H^2}{2r^3} dt \otimes dt, \quad (5.17)$$

$$\nabla e^\pm = -\frac{1}{r} dr \otimes_s e^\pm, \quad (5.18)$$

$$R_\nabla dt = \frac{1}{r^2} dt \wedge dr \otimes dr + \frac{r_H}{2r} dt \wedge e^+ \otimes_s e^-, \quad (5.19)$$

$$R_\nabla dr = -\frac{r_H^2}{r^4} dr \wedge dt \otimes dt + \frac{r_H}{2r} dr \wedge e^+ \otimes_s e^-, \quad (5.20)$$

$$R_\nabla e^\pm = -\frac{1}{2r^2} e^\pm \wedge dr \otimes dr + \frac{r_H^2}{2r^4} e^\pm \wedge dt \otimes dt \mp \frac{r_H}{r} e^+ \wedge e^- \otimes e^\pm, \quad (5.21)$$

$$\Delta = -\frac{r}{r_H} \partial_t^2 + \frac{r_H}{r} \partial_r^2 + \frac{r_H}{r^2} \partial_r + \frac{2}{r^2} (\partial_+ + \partial_-). \quad (5.22)$$

To keep the signature, we can take $r_H > 0$ and we will analyse this case first. However, to approximately match the inside of a black hole, we will then also analyse the case $r_H = -2GM < 0$ with the physical roles of r, t interchanged.

We also note that $\beta = H = 1$ leads to

$$g = -dt \otimes dt + dr \otimes dr - r^2 e_+ \otimes_s e_-, \quad \text{Ricci} = -\frac{1}{2} e^+ \otimes_s e^-, \quad S = \frac{1}{r^2},$$

$$\Delta = -\partial_t^2 + \partial_r^2 + \frac{2}{r} \partial_r + \frac{2}{r^2} (\partial_+ + \partial_-),$$

which is more like the spacetime Laplacian in 3 spatial dimensions, again showing the dimension jump and the constant curvature at each fixed radius and time. Here, S^1 behaves more like S^2 in polar coordinates, just with $2(\partial_+ + \partial_-)$ in the role of the angular Laplacian.

5.1. Klein–Gordon equation on the discrete black hole-like model for $\beta(r) > 0$

Here, we analyse the case of the length scale $r_H > 0$ in the Laplacian (5.22) found for the discrete black hole-like model above in ‘polar coordinates’ form. The eigenvalues of the angular Laplacian $\partial_+ + \partial_-$ are labelled by $l \in \mathbb{Z}_n$ and given by [3]

$$\lambda_l = q^l + q^{-l} - 2 = 2 \left(\cos \left(\frac{2\pi l}{n} \right) - 1 \right) = -4 \sin^2 \left(\frac{\pi l}{n} \right); \quad q = e^{\frac{2\pi i}{n}}$$

with eigenfunctions q^{il} . If we followed the format of section 4.1, we might first consider ‘quantum mechanics-like’ solutions of Klein–Gordon equations $\Delta\phi = m^2\phi$ of the form

$$\phi = e^{-imt} \psi_l(t, r)$$

of orbital angular momentum l and slowly varying in t . This is not particularly justified from the form of the metric but leads to

$$i\dot{\psi} = -\frac{r_H}{2mr} \left(\Delta_r + \frac{2\lambda_l}{r^2} \right) \psi_l + \left(1 - \frac{r_H}{r} \right) \frac{m}{2} \psi_l; \quad \Delta_r = \frac{r_H}{r^2} \partial_r (r \partial_r).$$

The mass term from the Klein–Gordon equation has not cancelled due to the r_H/r factor in the $dt \otimes dt$ term in the metric, except in the vicinity of $r \approx r_H$.

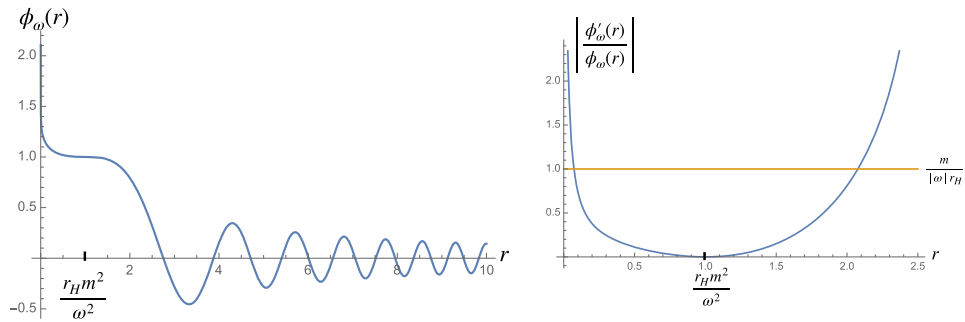


Figure 4. Solution of Klein–Gordon equation for $l = 0$ and $\omega = m = r_H = 1$, with Cauchy boundary condition at $r_0 = r_H m^2 / \omega^2$.

Here, it makes more sense to look in the ‘comoving’ case where we start with an $l = 0$ reference solution of the Klein–Gordon equation of the form

$$\phi = e^{-i\omega t} \phi_\omega; \quad \phi_\omega'' + \frac{1}{r} \phi_\omega' + \left(\frac{r^2}{r_H^2} \omega^2 - \frac{r}{r_H} m^2 \right) \phi_\omega = 0.$$

A generic solution of this for $\omega = m = r_H = 1$ is shown in figure 4, which illustrates that we can have an extended region where ϕ_ω is approximately constant, here with boundary condition

$$\phi_\omega'(r_0) = 0, \quad \phi_\omega(r_0) = 1; \quad r_0 := r_H \frac{m^2}{\omega^2}.$$

This results in

$$\left| \frac{\phi_\omega'(r)}{\phi_\omega(r)} \right| < \frac{m}{|\omega| r_H}, \quad r \approx r_0$$

for a reasonable range around the central value, as illustrated in the second half of the figure. An obvious choice would be $\omega = m$ and hence $r_0 = r_H$, but we can choose other ω to have other central values r_0 .

Next, we use this as reference and look for solutions of the Klein–Gordon equations of the form $\phi = e^{-i\omega t} \phi_\omega(r) \psi_l(t, r)$ with ψ_l in the λ_l eigenspace and slowly varying in t . Discarding $\ddot{\psi}_l$ terms, we have

$$v\dot{\psi}_l = -\frac{r_H}{2\omega r} \left(\Delta_r + \frac{2\phi_\omega'}{\phi_\omega} \frac{r_H}{r} \partial_r + \frac{2\lambda_l}{r^2} \right) \psi_l$$

and hence in any regime where the $\phi_\omega' / \phi_\omega$ term can be neglected, we have approximately

$$v\dot{\psi}_l \approx -\frac{r_H}{2\omega r} \left(\Delta_r + \frac{2\lambda_l}{r^2} \right) \psi_l$$

as an effective Schroedinger-like equation. We still have an expected scale factor out front, but now the unwanted mass terms are absent, i.e. this looks more like free motion as expected.

We can go further and replace r by a new variable

$$\rho(r) = \frac{r^2}{2r_H}, \quad \frac{\partial}{\partial r} = \frac{r}{r_H} \frac{\partial}{\partial \rho}, \quad \frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r} \left(\frac{r}{r_H} \frac{\partial}{\partial \rho} \right) = \frac{r^2}{r_H^2} \frac{\partial^2}{\partial \rho^2} + \frac{1}{r_H} \frac{\partial}{\partial \rho},$$

in which case

$$\psi_l \approx -\frac{1}{2\omega} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{2\lambda_l}{(2\rho)^{\frac{3}{2}} r_H^{\frac{1}{2}}} \right) \psi_l.$$

This absorbs the $\beta^2 = r_H^2/r^2$ factor in front of the radial double derivative so as to look more like flat space quantum mechanics, but has an unusual radial power for the angular contribution. Here ω plays the role of the effective mass and determines the central value

$$\rho_0 = \frac{r_H}{2} \left(\frac{m}{\omega} \right)^4$$

around which we wish our approximation to hold.

5.2. Continuum limit of the discrete black hole-like model

Here, we send $n \rightarrow \infty$ in such a way that the \mathbb{Z}_n geometry becomes S^1 with its usual constant metric. The algebraic way to do this was explained in [3] as a switch from functions on \mathbb{Z}_n to the algebraic circle $\mathbb{C}[s, s^{-1}]$, where classically $s = e^{i\theta}$ for an angle coordinate θ . The limiting calculus is not, however, the classical one on S^1 , being 2D not 1D. Rather, it is the $q \rightarrow 1$ limit of the 2D q -deformed calculus with generators f^\pm and the commutation relations and exterior derivative [3]

$$f^- s = -s f^+, \quad f^+ s = s(f^- + (q + q^{-1})f^+), \quad ds = s f^+, \quad ds^{-1} = s^{-1} f^-.$$

The calculus is inner with

$$\Theta = \frac{q}{(q-1)^2} (f^+ + f^-)$$

and has a quantum metric

$$g_{S^1} = \frac{1}{2} f^+ \otimes_s f^- + \frac{q}{(q-1)^2} (f^+ + f^-) \otimes (f^+ + f^-).$$

One can check that this is central, i.e. commutes with s and obeys the reality property flip $(* \otimes *) (g_{S^1}) = g_{S^1}$ for a quantum metric if q is real or modulus 1. If we impose $q = e^{\frac{2\pi i}{n}}$ and $s^n = 1$ then this is the constant metric $\frac{1}{2}(q - q^{-1})^2 e^+ \otimes_s e^-$ on \mathbb{Z}_n under the correspondence [3]

$$e^\pm = \frac{q f^\pm + f^\mp}{(q - q^{-1})(q - 1)}. \quad (5.23)$$

In this case $(q - q^{-1})^2$ is negative, which is the reason for the $-$ sign that was needed in the discrete model. But we do not impose these restrictions and thereby work on the circle. One still has a flat $*$ -preserving QLC with

$$\nabla f^\pm = 0, \quad f^{+*} = -f^-$$

and σ the flip on the basic one-forms. We now work on $A = C^\infty(\mathbb{R} \times \mathbb{R}_{>0}) \otimes \mathbb{C}[s, s^{-1}]$ with t, r, dt, dr classical and graded-commuting with the s, f^\pm . We take the metric

$$g = -\frac{r_H}{r} dt \otimes dt + \frac{r}{r_H} dr \otimes dr + r^2 g_{S^1}$$

and we look for QLCs with σ assumed to be the flip on the basic one-forms.

Proposition 5.3. *The metric g has a canonical Ricci flat $*$ -preserving QLC and associated geometry*

$$\begin{aligned}\nabla dt &= \frac{1}{2r} dr \otimes_s dt, \\ \nabla dr &= -\frac{1}{2r} dr \otimes dr + \frac{r_H^2}{2r^3} dt \otimes dt + r_H g_{S^1}, \quad \nabla f^\pm = -\frac{1}{r} dr \otimes_s f^\pm, \\ R_\nabla dt &= \frac{1}{r^2} dt \wedge dr \otimes dr - \frac{r_H}{2r} dt \wedge g_{S^1}, \\ R_\nabla dr &= -\frac{r_H^2}{r^4} dr \wedge dt \otimes dt - \frac{r_H}{2r} dr \wedge g_{S^1}, \\ R_\nabla f^\pm &= -\frac{1}{2r^2} f^\pm \wedge dr \otimes dr + \frac{r_H^2}{2r^4} f^\pm \wedge dt \otimes dt + \frac{r_H}{r} f^\pm \wedge g_{S^1}, \\ \Delta &= -\frac{r}{r_H} \partial_t^2 + \frac{r_H}{r} \partial_r^2 + \frac{r_H}{r^2} \partial_r + \frac{1}{r^2} \Delta_{S^1}, \\ \Delta_{S^1} &= -\frac{4(1 + (q-1)s\partial_q)}{(q+1)^2} (s\partial_q)^2,\end{aligned}$$

where ∂_q is the standard q -derivative so that Δ_{S^1} on modes s^l has eigenvalue

$$\lambda_l = -\frac{4q^l[l]_q^2}{(q+1)^2}, \quad [l]_q := \frac{1-q^l}{1-q}.$$

Proof. First, we can redo the discrete black hole-like model with $a(r, i) = ar^2$ for any constant factor a for the angular term $g_{\mathbb{Z}_n} = -ae^+ \otimes_s e^-$ in the metric. This same factor enters in the connection ∇dr as $g_{\mathbb{Z}_n}$ there. The same happens for R_∇ in the term where $e^+ \otimes_s e^-$ entered. We then replace $g_{\mathbb{Z}_n}$ by g_{S^1} to get the connection as stated, noting that f^\pm are a linear combination of e^\pm , so expressions linear in these have the same form. This version is constructed so as to be isomorphic to the discrete black hole-like model when $q = e^{\frac{2\pi i}{n}}$ and $s^n = 1$ are imposed, but these properties do not enter into the computations for a QLC, so this also holds for generic q , and likewise for Ricci flatness and for being $*$ -preserving when $|q| = 1$. One can do a direct check of these features and see that ∇ is $*$ -preserving also when q is real, as a consequence of g_{S^1} being real in the required sense.

For Ricci, the antisymmetric lift $i(f^+ \wedge f^-) = \frac{1}{2}(f^+ \otimes f^- - f^- \otimes f^+)$ of

$$f^+ \wedge f^- = \left(\frac{q-1}{q+1}\right) (q - q^{-1})^2 e^+ \wedge e^-$$

is equivalent to that of $e^+ \wedge e^-$ when we use the correspondence (5.23). We also use the inverse metric which on the f^\pm comes out as

$$(f^\pm, f^\pm) = -\frac{4q}{r^2(q+1)^2}, \quad (f^\pm, f^\mp) = 2\frac{q^2+1}{r^2(q+1)^2}.$$

For the Laplacian, we use $ds^l = -\frac{q[l]_q s^l}{q+1} (q[-1 - l]_q f^+ + [1 - l]_q f^-)$ from [3] and $(,)$ to compute $\Delta s^l = (,) \nabla ds^l = -\frac{4q^{2+l}}{r^2(q+1)^2} [l]_q^2 s^l$, which we write as stated since $s\partial_q s^l = [l]_q s^l$ for the

standard q -derivative $\partial_q f(s) = (f(qs) - f(s))/((q-1)s)$. The other values of Δ on functions of r, t are unchanged from the discrete case. In the classical case with $s = e^{t\theta}$, we have $s \frac{\partial}{\partial s} = -t \frac{\partial}{\partial \theta}$ as the limit of $s \partial_q$. \square

It remains to say a few words about the actual classical limit of the geometry. As explained in [3], this is a joint process $q \rightarrow 1$ and $f^+ = -f^-$, with the latter taking precedence so that $g_{S^1} \rightarrow -f^+ \otimes f^+ = d\theta \otimes d\theta$ as classically in our normalisation of g_{S^1} . In this way, one arrives at the classical 1 + 2-dimensional curved metric

$$g_{\text{class}} = -\frac{r_H}{r} dt \otimes dt + \frac{r}{r_H} dr \otimes dr + r^2 d\theta \otimes d\theta,$$

which is not, however, Ricci flat. One finds in our conventions (which are $-1/2$ of the usual ones)

$$\text{Ricci} = -\frac{1}{2} \left(\frac{r_H^2}{2r^4} dt \otimes dt - \frac{1}{2r^2} dr \otimes dr + \frac{r_H}{r} d\theta \otimes d\theta \right), \quad S = 0,$$

$$\Delta = -\frac{r}{r_H} \partial_t^2 + \frac{r_H}{r} \partial_r^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

The Laplacian agrees with the limit of the q -deformed geometry but Ricci does not. This is due to the 4D cotangent bundle in the quantum model, since the trace gives a different result from the trace in the quotient, where we impose $f^+ = -f^-$. Moreover, the dropped terms in the metric that are singular as $q \rightarrow 1$ contribute in the calculation of $\text{Ricci} = 0$ in the quantum model.

5.3. Discrete black hole model for $\beta(r) < 0$

Here we briefly analyse the case where $r_H < 0$ in our previous presentation of the discrete black hole-like model. More precisely, we still define $r_H = 2GM > 0$ but replace r_H by $-r_H$ and we also replace t by r and r by t in all the formulae (5.14)–(5.22) so as to match the signature. Thus, the quantum metric and resulting quantum geometry are now

$$\begin{aligned} g &= -\frac{t}{r_H} dt \otimes dt + \frac{r_H}{t} dr \otimes dr - t^2 e^+ \otimes_s e^-, \\ \nabla dr &= \frac{1}{2t} dr \otimes_s dt, \quad \nabla dt = -\frac{1}{2t} dt \otimes dt + \frac{r_H^2}{2t^3} dr \otimes dr + r_H e^+ \otimes_s e^-, \\ \nabla e^\pm &= -\frac{1}{t} dt \otimes_s e^\pm, \quad \Delta = -\frac{r_H}{t} \partial_t^2 + \frac{t}{r_H} \partial_r^2 - \frac{r_H}{t^2} \partial_t + \frac{2}{t^2} (\partial_+ + \partial_-) \end{aligned}$$

with a curvature singularity now at $t = 0$. We next make a change of variable

$$t = \left(\frac{3\tau}{2} \right)^{\frac{2}{3}} r_H^{\frac{1}{3}} = \eta(\tau)^2 r_H, \quad \eta(\tau) = \left(\frac{3\tau}{2r_H} \right)^{\frac{1}{3}}$$

in order to have a constant term in the ‘time’ coefficient of the metric, so that the quantum geometric structures become

$$\begin{aligned}
g &= -d\tau \otimes d\tau + \eta^{-2} dr \otimes dr - \eta^4 r_H^2 e^+ \otimes_s e^-, \\
\nabla e^\pm &= -\frac{2}{3\tau} d\tau \otimes_s e^\pm, \quad \nabla d\tau = \frac{1}{3\eta^2 \tau} dr \otimes dr + \eta r_H e^+ \otimes_s e^-, \\
\nabla dr &= \frac{1}{3\tau} dr \otimes_s d\tau, \quad \Delta = -\partial_\tau^2 - \frac{1}{\tau} \partial_\tau + \eta^2 \partial_r^2 + \frac{2}{\eta^4 r_H^2} (\partial_+ + \partial_-).
\end{aligned}$$

We now do the parallel analysis to section 5.1. Using the above Laplacian for the Klein–Gordon equation, we first look for solutions of the form $\phi = e^{-im\tau} \psi_l(\tau, r)$, where ψ_l is slowly varying in τ and with eigenvalue λ_l for the angular sector. Ignoring ψ_l , we have

$$\dot{\psi}_l = -\frac{\eta^2}{2m + \frac{1}{\tau}} \left(\partial_r^2 + \frac{8\lambda_l}{9\tau^2} \right) \psi_l - \frac{im}{\tau(2m + \frac{1}{\tau})} \psi_l$$

where dot denotes ∂_τ . If we assume that we are very far from the $\tau = 0$ singularity in the sense

$$\tau \gg \frac{1}{m}$$

(i.e. at macroscopic times much larger than the Compton wavelength in time units), we have

$$\dot{\psi}_l \approx -\frac{\eta^2}{2m} \left(\partial_r^2 + \frac{8\lambda_l}{9\tau^2} \right) \psi_l. \quad (5.24)$$

This looks a bit like quantum mechanics, not in the presence of a point source potential but rather with an overall time-dependent expansion factor and a time-dependent contribution of the angular momentum. Note that $e^{-im\tau}$ does not itself obey the Klein Gordon equation.

Next, we look for the ‘comoving’ behaviour, noting that reference solutions of the Klein–Gordon equation of mass m and $l = 0$ are in fact given by Hankel functions, of which we focus on the first type,

$$\phi_m(\tau) = H_0^{(1)}(-m\tau).$$

Here, the real and imaginary parts (Bessel J , Y functions respectively) oscillate, $\phi_m(0)$ has an infinite (imaginary) value and $|\phi_m|^2$ gradually decreases with time. This therefore plays the role of an exact plane wave. Relative to this, we look for solutions of the form

$$\phi(\tau, r) = \phi_m(\tau) \psi_l(\tau, r)$$

with ψ_l slowly varying in τ , leading to a Schroedinger-like equation

$$\dot{\psi}_l = -\frac{\eta^2 h(m\tau)}{2m} \left(\partial_r^2 + \frac{8\lambda_l}{9\tau^2} \right) \psi_l,$$

where

$$h(s) = -i \frac{H_0^{(1)}(-s)}{H_1^{(1)}(-s) + \frac{1}{2s} H_0^{(1)}(-s)} \approx 1$$

for large s , as shown on the left in figure 5. Here, one can see that $h(m\tau)$ approaches 1 very rapidly as $\tau \gg 1/m$. In other words, the behaviour near the $\tau = 0$ singularity is different but for larger τ the effective Schroedinger-like equation is now much more sharply approximated by (5.24) than before.

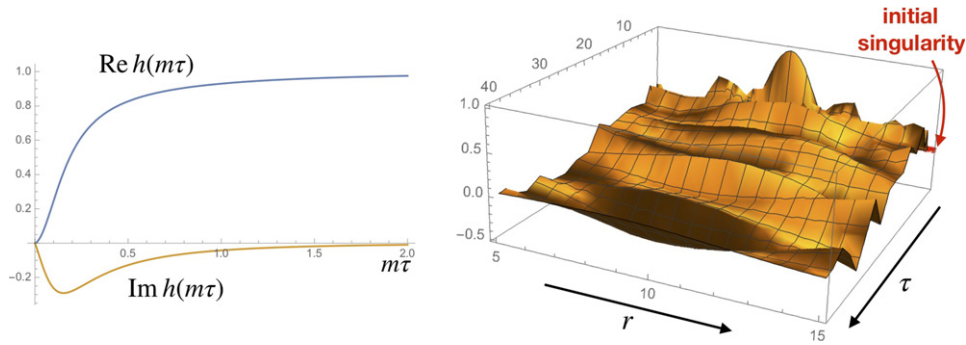


Figure 5. Function $h(m\tau)$ in definition of Schroedinger-like equation for discrete black hole metric and evolution of a Gaussian centred at $r = 10r_H$ at $\tau = 1/m$, for $r_H = m = 1$ and $l = 0$. The essentially zero initial values at $r = 0, 20r_H$ are held fixed.

The numerical solution for the real part of these equation is shown on the right in figure 5, where we used the exact function $h(m\tau)$ and set the initial Gaussian at $m\tau = 1$. The evolution becomes noticeably constant in r compared to regular quantum mechanics. Some of the noise in the picture comes from the numerical approximation.

6. Concluding remarks

We have solved for the quantum Levi-Civita connection and hence found the quantum geometry for quantum metrics with each sphere at radius r, t replaced by a fuzzy sphere $\mathbb{C}_\lambda[S^2]$. We did this for both FLRW-type metrics (3.1) and static black-hole like metrics (4.1) in polar coordinates and general metric g_{ij} on the fuzzy sphere. We also completed the discrete case with static metric (5.1), where each sphere is replaced by \mathbb{Z}_n as a discrete circle or its non-commutative S^1 limit, the FLRW-like case having already been treated in [3]. After the general analysis, we specialised to the ‘round’ metric $g_{ij} = k\delta_{ij}$ in the fuzzy case and the regular polygon metric $-e^+ \otimes_s e^-$ in the discrete case, respectively, and solved the Friedmann equations for the cosmological model and the Ricci = 0 equation for the black-hole-like models.

The four models between them show a remarkably consistent ‘dimension jump’ phenomenon where the radial-time sector behaves as for a classical model of one dimension higher. The origin of this from a mathematical point of view is what has been called a ‘quantum anomaly for differential structures’ [33], where quantisation of an algebra while preserving symmetries typically has an obstruction requiring either a breakdown of associativity or, which is our approach here, an extra cotangent dimension. This then affects both the Ricci tensor and Laplace operator. This is not surprising, but it is remarkable that the result appears so simply as a classical dimension jump. The consequence from a physical point of view is striking: if each sphere at r, t is better modelled as fuzzy due to quantum gravity effects, which is plausible enough if one wanted to preserve rotational symmetry but allow for some noncommutativity of spatial coordinates, then Ricci flat solutions, in particular, have a very different long range behaviour in 4D, being now of the form of a 5D black hole with the black hole appearing as a source of an inverse cubic gravitational force. In the discrete circle case, as well as in its non-commutative circle limit, the fact that the circle has zero constant curvature in contrast to S^2 also resulted in dropping the 1 in the usual Schwarzschild factor $\beta = 1 - 2GM/r$, which meant that we only approximated a black hole far inside the horizon. We are not proposing the model

as 4D physics, since the angular sector remains a circle not a sphere, but it could be of interest in $2 + 1$ gravity. It also illustrates that it is possible to have a nonflat Ricci = 0 quantum geometry in 3D, ultimately because of the hidden extra cotangent direction. The geometric meaning of the extra dimension was discussed in [3] as a kind of normal to the circle but without actually extending the circle to an ambient plane. (This is not unlike the notion of a ribbon or framed knot in TQFTs.) Finally, we have considered θ' as part of the background quantum geometry, but in the light of experience with Connes spectral triple approach (and notably, the appearance of a Higgs field there [19]), it could be interesting to promote this extra one-form to a dynamical field. According to our discussion, such a field might be associated to some kind of ‘normal bundle’. Other ideas for a physical role of θ' have ranged from a spontaneously induced external time as in [33–35] to a direction associated to a renormalisation group flow [23]. In the former context, a functional coefficient in front of θ' was indeed interpreted in [34, 35] as a gravitational potential.

In summary, we offer new models with different radial-time behaviour from those expected. We do not know if such effects could be relevant to real-world cosmology, but the idea of modified gravity [41] is of interest and it is possible that our new theoretical mechanism could likewise be of interest. We also introduced a novel ‘comoving’ Schroedinger-like equation i.e. slowly varying relative to an actual solution ϕ_m of the Klein–Gordon equation. We have not developed this as a formal theory but this could certainly be looked at further as a complement to more established methods of quantum field theory on curved spaces [14, 42, 43]. In particular, the solutions ψ_I appear in practice to dissipate over time even for a regular black hole background. This could potentially relate to ideas for gravitational measurement, but note that this would be a very different phenomenon from gravitational decoherence [8], which applies to mixed states not pure states.

Of course, our analysis is only as good as the assumed formalism, and here we assumed the constructive approach to quantum Riemannian geometry as in [12]. As in the concluding remarks in [3], it would be fair to say that the Einstein tensor in the general set up here is not known and the proposal for Ricci is merely by analogy (a trace of Riemann) rather than springing from a more conceptual understanding. In general, in order to take a trace, the formulation of Ricci depends on a lifting map $i : \Omega^2 \rightarrow \Omega^1 \otimes_A \Omega^1$ which classically would express a two-form as an antisymmetric tensor but which in general depends on the structure of Ω^2 . Fortunately, for the models in the present paper, as in [3], there are natural basic one-forms with respect to which Ω^2 is given by skew-symmetrising, so we can take i in the standard form as classically. We also found for the FLRW model (3.15) and for the spatial geometry of the fuzzy black hole model (proposition 4.2), that the quantum Einstein tensor defined by $\text{Eins} = \text{Ricci} - \frac{S}{2}g$, where S is the Ricci scalar, led as expected to $\nabla \cdot \text{Eins} = 0$. Such vanishing of the divergence provides evidence that this ‘naive’ definition of the Einstein tensor can be useful for some classes of models. It has also been used in some other approaches, such as [4]. An actual derivation and deeper understanding from a noncommutative variational principle, remains, however, an important direction for further work. This issue is discussed further in the concluding remarks of [3].

The physics of such quantum Ricci and Einstein tensors also remains to be understood much better. For example, we took the view for the fuzzy black hole that an observer sees the event horizon at $r = r_H$, which is the physical parameter, but equated it to $2GM$ for an effective ‘Schwarzschild mass’ for the purposes of comparison. To do better, one should have a non-commutative version of ADM theory, but we saw that naively adopting its classical physical formulation in terms of the spatial Einstein tensor [5, 17, 40] while using the spatial quantum Einstein tensor gave an infinite ADM mass as a consequence of the dimension jump. We likewise found that naively adopting the classical formulation of the Komar mass as a surface

integral gave an unreasonable answer. Here again, one should have a better noncommutative version of the theory relating a ‘surface’ integral of $*d\xi$ (where ξ is the timelike Killing one-form and $*$ is the Hodge star operator) to a volume integral of the evaluated Ricci curvature (so as to be topological for a vacuum solution). This again requires a deeper understanding of the Ricci tensor in the noncommutative case. In our case, we would also have to deal with $*d\xi$ being a three-form due to the extra dimension.

It is also the case that the models in the present paper do not concern quantum gravity itself, but rather noncommutative classical gravity proposed to better model quantum gravity effects. It remains to understand mechanisms for how our class of models might indeed emerge from an underlying theory. Thus, [24, 28] gave some reasons for why the fuzzy sphere could emerge from $2 + 1$ quantum gravity, but it is unclear how such arguments might extend to the higher dimensional models proposed here. By contrast, [1] studies effects on the interior of a black hole from loop quantum gravity, albeit the considerations there are quite different. See also [6] as an example of a loop quantum gravity model of a cosmological nature. There are also many other approaches to quantum gravity, some of them implicitly or explicitly related to some kind of noncommutative geometry. Aside from Connes spectral triples, which can be applied to models of quantum gravity, e.g. [27], we also mention lattice quantum gravity [2] and causal set models [21] as approaches where spacetime is modified away from the continuum and which it would be interesting to connect better to noncommutative geometry. There are also emerging variants of the formalism we used which could be useful to consider, for example [15] studies \mathbb{Z}_n with a larger class of not necessarily edge-symmetric metrics compared to [3].

Nevertheless, the class of models studied in this paper were particularly nice as far as the quantum geometry itself is concerned and more tractable than fully noncommutative models where r, t need not be classical as they were for us. We refer to the concluding remarks of [3] for a wider discussion. Also remaining to study, even for our simple class of models, are quantum geodesics using the Schroedinger-like formalism of [10, 13]. This requires further machinery, notably the construction of a certain A – B -bimodule connection (where B is the classical geodesic-time algebra), and will be considered elsewhere. These are some directions for further work.

Data availability statement

No new data were created or analysed in this study.

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