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Circular and Hyperbolic Symmetry Unified in Hyper-Spacetime

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Abstract

A unified geometric framework for the full Lorentz group is introduced, based on a bounded angle parametrization of spacetime transformations within a hyper-spherical geometry. By mapping the unbounded hyperbolic angle φ to a bounded angle β using the Gudermannian function, hyperbolic, causal, and Euclidean three-spheres are brought together into a single structure: **hyper-spacetime**. This structure unifies the Euclidean \mathbb{R}^4 and Minkowski $\mathbb{R}^{1,3}$ domains, incorporates discrete symmetries in a continuous way, and removes discontinuities at the lightlike boundary. Each three-sphere carries a natural spinor set, encoding symmetry, and acting as eigen-spinors of corresponding observables. These reproduce the Dirac spectrum while confining singular behavior to a scalar factor. The bounded angle parametrization therefore provides a continuous, closed representation of the full Lorentz group and a transparent geometric basis for spacetime symmetry.

Keywords: full Lorentz group; spacetime spinors; three-sphere; hyper-spacetime; geometric algebra; Dirac equation; null spinors; geometric duality

1. Introduction

The Lorentz group $O(1, 3)$ [1–6] underlies the symmetries of spacetime and forms the mathematical backbone of both classical and quantum relativistic physics. In its standard formulation, spatial rotations are described by compact angles on the circle, while boosts are parameterized by the unbounded rapidity $\varphi \in (-\infty, +\infty)$. Although effective, this conventional parametrization exhibits several structural limitations:

- Rapidity diverges at the lightlike boundary $v \rightarrow \pm c$;
- Discrete transformations $\{P, T, PT\}$ remain algebraically disconnected from continuous Lorentz transformations;
- Euclidean and Minkowski spacetime structures are treated as fundamentally separate, linked only by analytic continuation or Wick rotation;
- Spinor solutions of the Dirac equation are usually derived analytically, rather than emerging directly from geometric structure.

These issues motivate the need for a compact, continuous, and geometrically transparent representation of Lorentz symmetry, one that unifies continuous and discrete transformations while clarifying the relationship between Euclidean and Minkowski domains.

1.1. Motivation

The divergence of rapidity at the lightlike limit is not merely a coordinate inconvenience: it obscures geometric relations between boosts and rotations, interrupts continuity at $v = \pm c$, and masks the structural similarity between hyperbolic and circular symmetries. Moreover, discrete spacetime reflections remain isolated algebraic operations in the conventional Lorentz group, preventing smooth interpolation between parity P , time reversal



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T , and the identity component I . A compact parametrization that connects these sectors continuously would offer a coherent geometric understanding of the full Lorentz group.

At the same time, many modern formalisms, from path integrals to Euclidean field theory and particle classification, rely on controlled transitions between Minkowski and Euclidean metrics [7]. Standard Wick rotation introduces imaginary time as a technical device, but its geometric meaning remains conceptually opaque [8–10]. A structural framework that embeds both domains within a single real geometry would provide a transparent foundation for analytic continuation and symmetry-based reasoning.

These motivations converge in the search for a unified geometric representation capable of handling boosts, rotations, discrete symmetries, and Euclidean/Minkowski duality in a single continuous mathematical object.

1.2. Research Gap and Objectives

Despite extensive research on Lorentz geometry [11], no previous framework provides the following:

- A bounded and continuous parametrization covering the full Lorentz group.
- A real geometric structure unifying Euclidean and Minkowski domains.
- Spinor sets that simultaneously encode symmetry, discrete operations, and Dirac eigen-spinors.
- A compact angular coordinate that regularizes the lightlike boundary.

This paper aims to fill these gaps through the following objectives:

1. Unify circular and hyperbolic symmetry via the bounded angle β .
2. Construct a triplet of three-spheres forming hyper-spacetime.
3. Develop spinor sets that derive Dirac eigen-spinors geometrically.
4. Provide a continuous representation of the full Lorentz group, including discrete reflections.
5. Bridge Euclidean and Minkowski metrics through a natural geometric duality.

1.3. Central Idea: A Bounded Angular Parametrization

The key innovation of this work is the introduction of a bounded angular variable β , obtained through the Gudermannian mapping $\varphi = \tanh^{-1}(\sin(\beta))$ [12–14] (Appendix B). This mapping transforms the unbounded rapidity parameter into a bounded angular domain, effectively compactifying the boost parameter space. In doing so, it closes the hyperbola into a hyperbolic one-sphere \mathbb{S}_H^1 and provides a continuous, finite coordinate for Lorentz boosts [15,16]. Replacing rapidity φ with the bounded angle β produces several important consequences:

- The lightlike boundary becomes a regular point of the domain.
- Divergences in boosts are isolated in a scalar density, while spinors remain finite.
- Both branches of the hyperbola are traversed continuously.
- Temporal orientation changes correspond to smooth shifts in β , turning a discrete transformation into a continuous operation.

This compactification unifies hyperbolic and circular symmetries, creating a single angular structure that links rotations, boosts and reflections across the full Lorentz group.

1.4. Hyper-Spacetime: A Unified Three-Sphere Geometry

Building on the bounded parametrization, this work introduces hyper-spacetime, a unified geometric structure constructed from a triplet of three-spheres: $\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$:

- $\mathbb{S}_H^3 \subset \mathbb{R}^{1,3}$ is a hyperbolic unit sphere representing normalized momentum.
- $\mathbb{S}_C^3 \subset \mathbb{R}^{1,3}$ is a causal sphere encoding spacetime intervals.
- $\mathbb{S}^3 \subset \mathbb{R}^4$ is a Euclidean sphere preserving the same temporal and spatial orientation.

All three share a common temporal axis and a reciprocal temporal bivector plane, establishing a correspondence between spatial vectors in the Minkowski domain and temporal bivectors in the Euclidean domain, and vice versa. This duality creates a continuous geometric bridge linking the Minkowski and Euclidean representations.

1.5. Spinors, Observables, and the Dirac Spectrum

Within this unified geometry, spinor sets $\{U_j\}$ and $\{V_j\}$ arise directly from variations in the hyper-spherical coordinates (β, θ, ϕ) . These spinor sets act simultaneously as:

1. Symmetry generators preserving the three-spheres;
2. Eigen-spinors of geometric observables $\{p, q, u_r\}$;
3. Continuous representatives of the discrete operations $\{I, P, T, PT\}$;
4. A geometric reconstruction of the four Dirac plane-wave solutions.

Crucially, the Dirac spectrum emerges without solving a differential equation: it arises geometrically from the structure of hyper-spacetime, demonstrating that relativistic spinors are intrinsic to the unified geometry.

1.6. Novelty and Contribution

The resulting framework provides the following:

- A closed, continuous, and bounded representation of the full Lorentz group $O(1,3)$;
- A unified spinor structure generating the Dirac spectrum directly;
- Continuity across discrete symmetries and the lightlike boundary;
- A new geometric interpretation of Wick rotation;
- A compactified Lorentz parameter suitable for quantization and field-theoretic formulations;
- A pedagogically transparent geometric picture of spacetime symmetry.

The remainder of this work develops this structure rigorously, derives the related spinors and observables, and discusses both physical implications and potential applications.

1.7. Layout of the Paper

The core of the paper (Sections 2–6) develops the geometric framework. Sections 2 and 3 present the bounded angular parametrization of Lorentz boosts that unifies circular and hyperbolic symmetry and applies naturally to special relativity, yielding harmonic relativistic proportionality factors [17,18]. Section 4 extends to four dimensions, a triplet of three-spheres is introduced that defines hyper-spacetime and shows how the full Lorentz group, with both continuous and discrete symmetries, emerges within its hyper-spherical coordinates. In Sections 5 and 6, spinor sets are then constructed and analyzed as eigen-spinors of geometric observables, reproducing the Dirac spectrum while confining singularities to scalar factors. The final part of the paper (Sections 7 and 8) turns to interpretation and outlook, discussing the physical implications of hyper-spacetime, and concludes with a summary of its unifying features.

1.8. Mathematical Formalism

The mathematical formalism throughout is based on the geometric algebra (GA) of spacetime (STA), as developed by Hestenes [19,20] (Appendix A). Its foundations trace back to Grassmann [21] and Clifford [22], whose work established a unified algebra of vectors, bivectors, and higher-grade multivectors. In this algebra, Lorentz transformations are expressed naturally through rotors, which generalize rotations and boosts in a coordinate-free manner [23–27]. In STA [19], a spacetime inertial frame $\{t, x, y, z\}$ is represented by four orthogonal basis vectors $\gamma_\mu = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, which satisfy the real Clifford-algebra [7,28] relation $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$, equivalent to the algebra of the Dirac gamma matrices [4].

This framework provides the natural setting for the bounded parametrization, the three-sphere geometry, and the spinor sets constructed later in the paper.

2. Hyperbolic Rotation

Lorentz boosts are conventionally expressed as hyperbolic rotations, parameterized by the unbounded angle $\varphi \in (-\infty, +\infty)$ (rapidity) (Figure 1a).

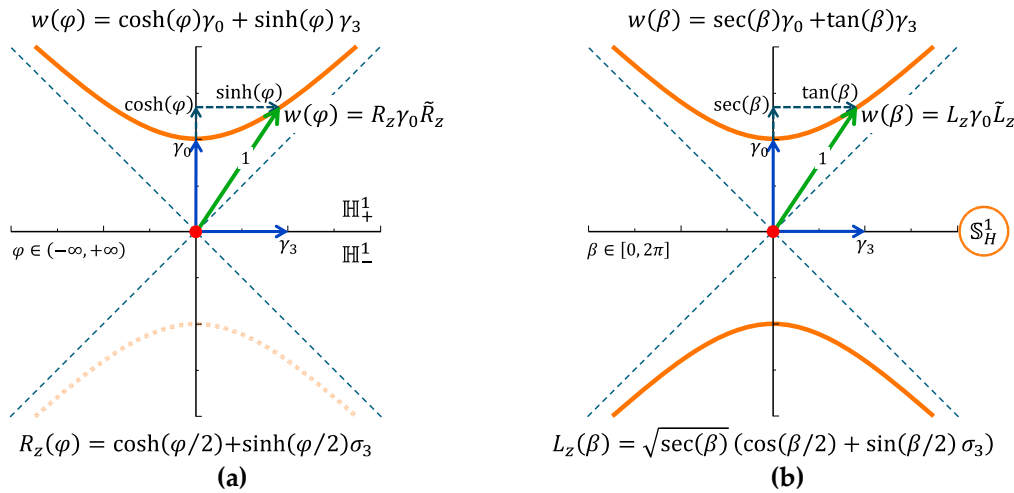


Figure 1. Comparison of hyperbolic rotation. (a) In the unbounded parametrization, the rotor $R_z(\varphi)$ covers only one branch \mathbb{H}_+^1 , with a discontinuity at lightlike boundaries. (b) Using the bounded angle β , the rotor $L_z(\beta)$ spans the full symmetry $\mathbb{S}_H^1 = \mathbb{H}_+^1 \oplus \mathbb{H}_-^1$. The Gudermannian mapping ensures smooth continuity through infinity, closing the hyperbola into a one-sphere \mathbb{S}_H^1 .

A boost along the γ_3 -axis (z -axis) is generated by the hyperbolic rotor $R_z(\varphi)$:

$$R_z(\varphi) = \exp(\varphi \sigma_3/2) = \cosh(\varphi/2) + \sinh(\varphi/2)\sigma_3, \quad \varphi \in (-\infty, +\infty), \quad R_z \tilde{R}_z = 1 \tag{1}$$

where bivector $\sigma_3 = \gamma_3\gamma_0$ represents the temporal plane of rotation. Acting on the temporal basis vector γ_0 , the rotor generates the boosted spacetime vector $w(\varphi)$:

$$w(\varphi) = R_z \gamma_0 \tilde{R}_z = \cosh(\varphi)\gamma_0 + \sinh(\varphi)\gamma_3, \quad \|w(\varphi)\|^2 = 1 \tag{2}$$

However, rotor $R_z(\varphi)$ only covers the positive branch \mathbb{H}_+^1 of the hyperbola. The negative branch \mathbb{H}_-^1 of the hyperbola is not present (Figure 1a).

To cover the full hyperbolic symmetry in a single parameter domain, the unbounded angle $\varphi \in (-\infty, +\infty)$ is related to the bounded angle $\beta \in [0, 2\pi]$ by using the Gudermannian function $\varphi = \tanh^{-1}(\sin(\beta))$ (Appendix B) [12,13]. Substituting this function into $R_z(\varphi)$ yields the rotor $L_z(\beta)$:

$$\begin{aligned} L_z(\beta) &= \exp(\tanh^{-1}(\sin(\beta))\sigma_3/2) = \sqrt{\sec(\beta)}L_{u1}(\beta), & L_z \tilde{L}_z &= 1 \\ L_{u1}(\beta) &= \cos(\beta/2) + \sin(\beta/2)\sigma_3, & L_{u1} \tilde{L}_{u1} &= \cos(\beta) \end{aligned} \tag{3}$$

The scalar factor $\sqrt{\sec(\beta)}$ ensures normalization, while $L_{u1}(\beta)$ acts as a **temporal spinor** (Appendix C). Applying rotor $L_z(\beta)$ to γ_0 generates the boosted spacetime vector $w(\beta)$:

$$w(\beta) = L_z \gamma_0 \tilde{L}_z = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3, \quad \beta \in [0, 2\pi], \quad \|w(\beta)\|^2 = 1 \tag{4}$$

where the full hyperbolic symmetry is covered continuously (Figure 1b).

In summary, the bounded angle parametrization provides several advantages. It removes singularities by isolating them in the scalar density factor $\sqrt{\sec(\beta)}$, while the associated spinor remains regular. At $\beta = \pm\pi/2$, the mapping passes smoothly through infinity, ensuring continuity across the lightlike boundary where crossing corresponds to a time reversal. The two disconnected hyperbolic branches \mathbb{H}_+^1 and \mathbb{H}_-^1 are thereby closed into a single hyperbolic one-sphere \mathbb{S}_H^1 (Figure 1b). In this way, hyperbolic symmetry is unified within a single angular cycle.

3. Harmonic Relativistic Proportionality Factors

In special relativity, the relative speed v/c is conventionally mapped to the unbounded angle $\varphi \in (-\infty, +\infty)$ (rapidity). Using instead the bounded angle $\beta \in [0, 2\pi]$ introduced in Section 2, this mapping provides a harmonic set of relativistic proportionality factors:

$$\begin{aligned} \sin(\beta) &= \pm v/c, & \cos(\beta) &= \pm \sqrt{1 - (v/c)^2} \\ \sec(\beta) &= \frac{1}{\pm \sqrt{1 - (v/c)^2}}, & \tan(\beta) &= \frac{\pm v/c}{\pm \sqrt{1 - (v/c)^2}} \end{aligned} \quad (5)$$

These factors form a bridge between circular and hyperbolic symmetry:

$$\begin{aligned} \sin^2(\beta) + \cos^2(\beta) &= 1, & \sec^2(\beta) - \tan^2(\beta) &= 1 \\ \mathbb{S}^1 \subset \mathbb{R}^2 & & \longleftrightarrow & \mathbb{S}_H^1 \subset \mathbb{R}^{1,1} \end{aligned} \quad (6)$$

Consequently, the bounded angle mapping unifies circular $\mathbb{S}^1 \subset \mathbb{R}^2$ with hyperbolic $\mathbb{S}_H^1 \subset \mathbb{R}^{1,1}$ symmetry (Figure 1b). A complete cycle on one-sphere $\mathbb{S}^1 \subset \mathbb{R}^2$ corresponds one-to-one with a complete cycle on hyperbolic one-sphere $\mathbb{S}_H^1 \subset \mathbb{R}^{1,1}$. This includes a smooth passage through infinity at the lightlike boundaries $\beta = \pm\pi/2$.

To incorporate all the symmetries of the full Lorentz group $O(1, 3)$, the framework must extend from the two-dimensional (2D) relation $\mathbb{R}^2 \longleftrightarrow \mathbb{R}^{1,1}$ to a four-dimensional (4D) relation that unifies the Euclidean domain \mathbb{R}^4 with the Minkowski domain $\mathbb{R}^{1,3}$.

4. One Unified Geometry Bridging \mathbb{R}^4 and $\mathbb{R}^{1,3}$

The generalization from two to four dimensions is realized by the introduction of a triplet of three-spheres embedded in a Euclidean $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ and a Minkowski $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ domain (Appendix A). Accordingly, three distinct but related three-spheres are defined as follows:

$$\begin{aligned} \mathbb{S}_H^3 &= \left\{ p = a_0\gamma_0 + a_k\gamma_k \in \mathbb{R}^{1,3} \mid a_\mu \in \mathbb{R}, \quad \|p\|^2 = 1 \right\}, & \text{metric} &: \{+1, -1, -1, -1\} \\ \mathbb{S}_C^3 &= \left\{ q = \gamma_0 + a_k\gamma_k \in \mathbb{R}^{1,3} \mid a_k \in \mathbb{R}, \quad \|q\|^2 = \cos^2(\beta) \right\}, & \text{metric} &: \{+1, -1, -1, -1\} \\ \mathbb{S}^3 &= \left\{ u_r = a_0\gamma_0 + a_k\sigma_k \in \mathbb{R}^4 \mid a_\mu \in \mathbb{R}, \sigma_k = \gamma_k\gamma_0, \|u_r\|^2 = 1 \right\}, & \text{metric} &: \{+1, +1, +1, +1\} \end{aligned} \quad (7)$$

Here,

- The hyperbolic three-sphere \mathbb{S}_H^3 is preserved by spacetime rotor Λ_1 (Figure 2b).
- The causal three-sphere \mathbb{S}_C^3 is preserved by spacetime spinor U_1 (Figure 3a).
- The Euclidean three-sphere \mathbb{S}^3 is preserved by Euclidean spinor V_1 (Figure 3b).

Each of these three-spheres is preserved by a specific spinor structure (as defined in Section 4.1), which ensures that the geometry remains invariant under both Lorentzian and Euclidean transformations. In this way, every three-sphere is not only defined by its metric but also carries a natural spinor representation that encodes its symmetry.

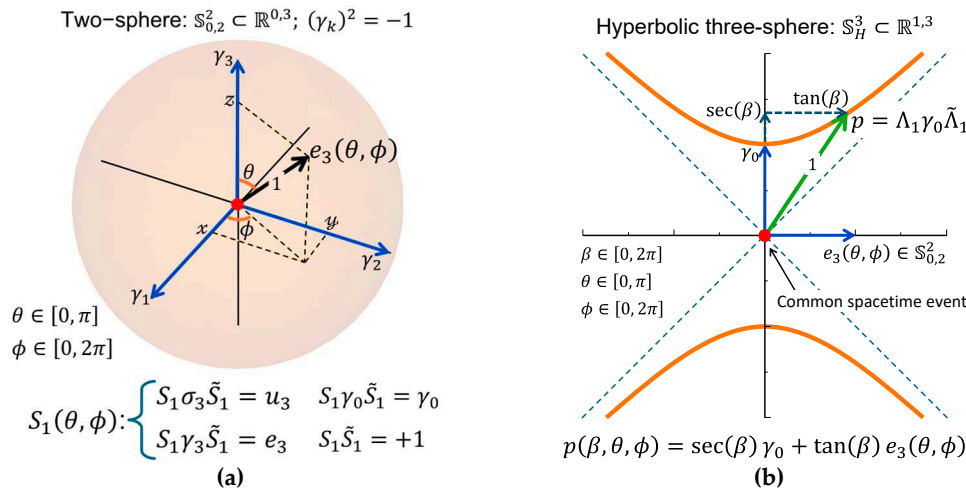


Figure 2. (a) The spatial two-sphere $\mathbb{S}_{0,2}^2 \subset \mathbb{R}^{0,3}$, spanned by vectors $e_3(\theta, \phi)$, is preserved by the spatial spinor S_1 . (b) The hyperbolic three-sphere $\mathbb{S}_H^3 \subset \mathbb{R}^{1,3}$, spanned by vectors p , is preserved by the spacetime rotor Λ_1 . The spatial vectors $e_3 \in \mathbb{S}_{0,2}^2$ are all orthogonal to the temporal axis γ_0 .

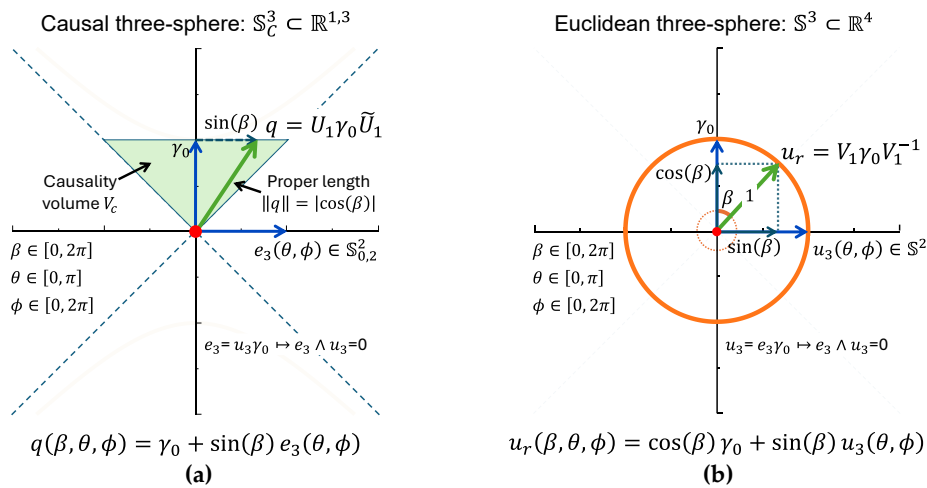


Figure 3. (a) The causal three-sphere $\mathbb{S}_C^3 \subset \mathbb{R}^{1,3}$, spanned by vectors q , is preserved by the spacetime spinor U_1 . (b) The Euclidean three-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, spanned by vectors u_r , is preserved by the Euclidean spinor V_1 . In this structure, each of the aligned vectors $e_3 \in \mathbb{S}_{0,2}^2$ and $u_3 \in \mathbb{S}^2$ is orthogonal to the temporal axis γ_0 in its respective domain, thereby fixing a reciprocal temporal bivector plane.

All three-spheres share the following: a common temporal axis γ_0 , a common origin (spacetime event), and a reciprocal temporal bivector plane. They differ in their metric and spatial subspaces $\{\gamma_k\} \subset \mathbb{R}^{1,3}$ and $\{\sigma_k\} \subset \mathbb{R}^4$ (Equation (7)), which define two complementary two-spheres:

$$\begin{aligned} \mathbb{S}_{0,2}^2 &= \left\{ e_3 = a_k \gamma_k \in \mathbb{R}^{0,3} \mid a_k \in \mathbb{R}, (e_3)^2 = -1 \right\} \\ \mathbb{S}^2 &= \left\{ u_3 = a_k \sigma_k \in \mathbb{R}^{3,0} \mid a_k \in \mathbb{R}, (u_3)^2 = +1 \right\} \end{aligned} \tag{8}$$

Both two-spheres are preserved by spatial spinor S_1 (Figure 2a):

$$S_1(\theta, \phi) = \exp(-i\sigma_3\phi/2)\exp(-i\sigma_2\theta/2), \quad S_1 \tilde{S}_1 = 1, \quad S_1 \gamma_0 \tilde{S}_1 = \gamma_0 \tag{9}$$

Applying spinor S_1 to γ_3 and σ_3 generates the spatial unit vectors e_3 and u_3 :

$$\begin{aligned} e_3(\theta, \phi) &= S_1 \gamma_3 \tilde{S}_1 = \sin(\theta) \cos(\phi) \gamma_1 + \sin(\theta) \sin(\phi) \gamma_2 + \cos(\theta) \gamma_3, & (e_3)^2 &= -1, & e_3 &= u_3 \gamma_0 \\ u_3(\theta, \phi) &= S_1 \sigma_3 \tilde{S}_1 = \sin(\theta) \cos(\phi) \sigma_1 + \sin(\theta) \sin(\phi) \sigma_2 + \cos(\theta) \sigma_3, & (u_3)^2 &= +1, & u_3 &= e_3 \gamma_0 \end{aligned} \quad (10)$$

They are orthogonal to γ_0 , aligned ($e_3 \wedge u_3 = 0$), and define spatial direction. Moreover, they span a reciprocal temporal bivector plane ($e_3 = u_3 \gamma_0 \leftrightarrow u_3 = e_3 \gamma_0$), which is a cornerstone of the unified geometry. The Minkowski vector $e_3 \in \mathbb{R}^{1,3}$ acts as a temporal bivector in the Euclidean domain, while the Euclidean vector $u_3 \in \mathbb{R}^4$ act as a temporal bivector in the Minkowski domain. This geometrical duality specifies spatial direction while simultaneously encoding for temporal orientation in the opposite domain. Their associated dual spatial bivector planes $i u_3$ belongs to the common three-dimensional even subalgebra (Appendix A), ensuring domain-independence of spatial orientation.

4.1. Definition of the Spinors $\{U_1, V_1\}$, Rotor $\Lambda_1 = \rho U_1$, and Vectors $\{p, q, u_r\}$

The two spinors $\{U_1, V_1\}$, and associated rotor $\Lambda_1 = \rho U_1$, form a spinor structure that preserves the triplet of three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ (Figures 2 and 3). Thus, every three-sphere carries a natural spinor representation that encodes its symmetry. The Euclidean spinor V_1 (preserving \mathbb{S}^3) and spacetime spinor U_1 (preserving \mathbb{S}_C^3 and \mathbb{S}_H^3) are defined as

$$\begin{aligned} V_1(\beta, \theta, \phi) &= S_1(\theta, \phi) L_{v1}(\beta), & L_{v1}(\beta) &= \cos(\beta/2) + \sin(\beta/2) \gamma_3, & V_1 V_1^{-1} &= 1 \\ U_1(\beta, \theta, \phi) &= S_1(\theta, \phi) L_{u1}(\beta), & L_{u1}(\beta) &= \cos(\beta/2) + \sin(\beta/2) \sigma_3, & U_1 \tilde{U}_1 &= \cos(\beta) \\ \Lambda_1(\beta, \theta, \phi) &= \sqrt{\sec(\beta)} U_1(\beta, \theta, \phi), & & & \Lambda_1 \tilde{\Lambda}_1 &= 1 \end{aligned} \quad (11)$$

Here S_1 acts as a spatial spinor (Equation (9)), L_{v1} and L_{u1} act as temporal spinors (Appendix C), and the scalar factor $\sqrt{\sec(\beta)}$ ensures the unitarity of rotor Λ_1 , the normalized form of U_1 . Applying the spinors U_1 and V_1 yields the vectors spanning the three-spheres:

$$\begin{aligned} p \in \mathbb{S}_H^3 : p(\beta, \theta, \phi) &= \Lambda_1 \gamma_0 \tilde{\Lambda}_1 = \sec(\beta) \gamma_0 + \tan(\beta) e_3(\theta, \phi), & \|p\|^2 &= 1 \\ q \in \mathbb{S}_C^3 : q(\beta, \theta, \phi) &= U_1 \gamma_0 \tilde{U}_1 = \gamma_0 + \sin(\beta) e_3(\theta, \phi), & \|q\|^2 &= \cos^2(\beta) \\ u_r \in \mathbb{S}^3 : u_r(\beta, \theta, \phi) &= V_1 \gamma_0 V_1^{-1} = \cos(\beta) \gamma_0 + \sin(\beta) u_3(\theta, \phi), & \|u_r\|^2 &= 1 \end{aligned} \quad (12)$$

Here, $p \in \mathbb{S}_H^3$ is a hyperbolic unit vector, $q \in \mathbb{S}_C^3$ is a causal vector with variable norm, and $u_r \in \mathbb{S}^3$ is a Euclidean unit vector. The causal three-sphere \mathbb{S}_C^3 covers the spacetime region bounded by the light cone of a past event (Figure 3a). Each causal vector $q \in \mathbb{S}_C^3$ is future pointing and its norm represents proper length, vanishing at the lightlike boundaries. In this way, \mathbb{S}_C^3 integrates a natural form of causality in the triplet of three-spheres.

4.2. Geometric Duality

A central feature of hyper-spacetime is the geometric duality between the Minkowski and Euclidean domains, expressed directly through the spacetime spinor U_1 and the Euclidean spinor V_1 (Equation (11)). This duality is governed by the temporal spinors L_{u1} and L_{v1} , which are defined by the temporal bivectors

$$\left[\gamma_3 \in \mathbb{R}^{1,3} \leftrightarrow \gamma_3 = \sigma_3 \gamma_0 \subset \mathbb{R}^4 \right] \leftrightarrow \left[\sigma_3 \in \mathbb{R}^4 \leftrightarrow \sigma_3 = \gamma_3 \gamma_0 \subset \mathbb{R}^{1,3} \right].$$

This relation shows that the temporal bivector $\sigma_3 \gamma_0 \subset \mathbb{R}^4$ acts as the spatial basis vector $\gamma_3 \in \mathbb{R}^{1,3}$, while the temporal bivector $\gamma_3 \gamma_0 \subset \mathbb{R}^{1,3}$ acts as the spatial basis vector $\sigma_3 \in \mathbb{R}^4$. With this structure in place, the only difference between U_1 and V_1 is the domain to which the temporal bivector resides. Their spatial component S_1 (Equations (9) and (11)) is identical in both cases, since the spatial bivectors $i \sigma_k$ belong to the shared three-dimensional even subalgebra common to both metrics (Appendix A).

This establishes the geometric duality transparently: a vector (spatial direction) in one domain corresponds to a bivector (orientation) in the other. The combined spinors $U_1 = S_1 L_{u_1}$ and $V_1 = S_1 L_{v_1}$, therefore, encode the same orientation data, expressed within two different metric signatures. Each spinor preserves its associated three-spheres: U_1 preserves \mathbb{S}_H^3 and \mathbb{S}_C^3 , while V_1 preserves \mathbb{S}^3 . Consequently, the same angular coordinates (β, θ, ϕ) generate parallel spinor structures in both domains. This illustrates how the unified hyper-spacetime geometry emerges from the **vector** \leftrightarrow **bivector duality** linking the Euclidean and Minkowski representations.

In summary, the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ (Equation (7)) exhibit the following features:

- A common temporal axis γ_0 with a shared origin (spacetime event).
- Unit vectors $e_3 \in \mathbb{R}^{1,3}$ and $u_3 \in \mathbb{R}^4$ that encode spatial direction and temporal orientation through the reciprocal relation $\{e_3 = u_3 \gamma_0 \leftrightarrow u_3 = e_3 \gamma_0\}$, expressing a geometric duality between domains.
- Domain-independent spatial bivector planes $i u_3 \subset \mathbb{R}^3$, common to both domains.

A complete cycle on the Euclidean three-sphere \mathbb{S}^3 corresponds one-to-one with a complete cycle on hyperbolic three-sphere \mathbb{S}_H^3 , thereby unifying \mathbb{R}^4 and $\mathbb{R}^{1,3}$. The apparent boundary at infinity ($\beta = \pm\pi/2$) is absorbed smoothly, making the entire structure closed and traversable. The unified geometry constructed from the triplet of three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ is referred to as hyper-spacetime \mathcal{B} :

$$\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\} \quad (13)$$

Its coordinates are given by the hyper-spherical set (β, θ, ϕ) , where the spatial angles (θ, ϕ) encode direction, and the temporal angle β encodes relative speed (Equation (5)). Hence, the familiar relativistic parameters of direction and speed are reformulated as angular variables in a closed domain, and continuity at the lightlike boundaries.

5. Full Lorentz Group

The full Lorentz group $O(1,3)$ [29–31] contains the symmetries of spacetime that preserve the Minkowski metric. It includes the following: (1) continuous transformations as boosts and spatial rotations, and (2) spacetime event reflections $\{I, P, T, PT\}$: identity (I), parity (P), time-reversal (T), and spacetime-reversal (PT). Thus, the full Lorentz group is defined as

$$O(1,3) = \{I, P, T, PT\} \cdot SO^+(1,3) \quad (14)$$

where $SO^+(1,3)$ is the proper orthochronous Lorentz subgroup connected to identity.

In the conventional formalism, boosts in $SO^+(1,3)$ are parameterized by the unbounded angle $\varphi \in (-\infty, +\infty)$, leading to singularities at the lightlike boundaries. Moreover, the discrete reflections $\{P, T, PT\}$ remain disconnected from identity. The standard way to extend the full Lorentz group $O(1,3)$ is through its double cover $\text{Pin}(1,3)$, as it includes both proper and improper transformations [31]. However, even within $\text{Pin}(1,3)$, the discrete transformations remain disconnected from identity, leaving a gap between reflections and continuous transformations. This gap is closed by hyper-spacetime, in which continuous and discrete symmetries arise naturally within a unified geometry.

5.1. A Unified Geometry

In the hyper-spherical coordinates (β, θ, ϕ) of hyper-spacetime (defined in Section 4), the spacetime event reflections $\{I, P, T, PT\}$ manifest naturally as smooth angular transformations:

$$\begin{aligned}
 \text{Identity} & \quad I & : & (\pm\beta, \theta, \phi) \\
 \text{Parity} & \quad P & : & (\mp\beta, \theta \pm \pi, \phi) \\
 \text{Time – reversal} & \quad T & : & (\mp\beta \pm \pi, \theta, \phi) \\
 \text{Spacetime – reversal} & \quad PT & : & (\pm\beta \pm \pi, \theta \pm \pi, \phi)
 \end{aligned} \tag{15}$$

These transformations correspond to changes in temporal orientation (\pm sign changes or $\pm\pi$ shifts in β) and spatial orientation ($\pm\pi$ shifts in θ). Crucially, all four reflections are smoothly connected to identity, eliminating discontinuities.

5.2. Spinor Representations

The spinor structure of hyper-spacetime provides explicit representations of the $\{I, P, T, PT\}$ transformations. Starting from the spacetime spinor $U_1(\beta, \theta, \phi)$, variations in the hyper-spherical coordinates (β, θ, ϕ) generate the full set of spacetime spinors $\{U_j\}$, where each of the spinors represents one of the $\{I, P, T, PT\}$ transformations:

$$\begin{aligned}
 I : \quad U_1(\beta, \theta, \phi) & = U_1(+\beta, \theta, \phi) = S_1(\theta, \phi)(+\cos(\beta/2) + \sin(\beta/2)\sigma_3) \\
 P : \quad U_2(\beta, \theta, \phi) & = U_1(-\beta, \theta + \pi, \phi) = S_2(\theta, \phi)(+\cos(\beta/2) - \sin(\beta/2)\sigma_3) \\
 T : \quad U_3(\beta, \theta, \phi) & = U_1(-\beta + \pi, \theta, \phi) = S_1(\theta, \phi)(-\sin(\beta/2) - \cos(\beta/2)\sigma_3) \\
 PT : \quad U_4(\beta, \theta, \phi) & = U_1(+\beta + \pi, \theta + \pi, \phi) = S_2(\theta, \phi)(-\sin(\beta/2) + \cos(\beta/2)\sigma_3)
 \end{aligned} \tag{16}$$

With $U_j \tilde{U}_j = \cos(\beta)\{+1, +1, -1, -1\}$. Similarly, changes in $V_1(\beta, \theta, \phi)$ generate the full set of Euclidean spinors $\{V_j\}$:

$$\begin{aligned}
 I : \quad V_1(\beta, \theta, \phi) & = V_1(+\beta, \theta, \phi) = S_1(\theta, \phi)(+\cos(\beta/2) + \sin(\beta/2)\gamma_3) \\
 P : \quad V_2(\beta, \theta, \phi) & = V_1(-\beta, \theta + \pi, \phi) = S_2(\theta, \phi)(+\cos(\beta/2) - \sin(\beta/2)\gamma_3) \\
 T : \quad V_3(\beta, \theta, \phi) & = V_1(-\beta + \pi, \theta, \phi) = S_1(\theta, \phi)(-\sin(\beta/2) - \cos(\beta/2)\gamma_3) \\
 PT : \quad V_4(\beta, \theta, \phi) & = V_1(+\beta + \pi, \theta + \pi, \phi) = S_2(\theta, \phi)(-\sin(\beta/2) + \cos(\beta/2)\gamma_3)
 \end{aligned} \tag{17}$$

With $V_j V_j^{-1} = \{+1, +1, +1, +1\}$. Note the algebraic closure: each of the spinors U_j and V_j can be a basis to generate the full set of transformations. The spatial spinors $\{S_1, S_2\}$ in U_j and V_j are generated from spatial spinor S_1 (Figure 2a) via a parity transformation:

$$P : \quad S_1(\theta + \pi, \phi) = S_2(\theta, \phi), \quad S_k \tilde{S}_i = \delta_{ki} \tag{18}$$

This orthonormal set $\{S_k\}$ corresponds to the spatial Pauli spinors [32].

Each set $\{U_j\}$ and $\{V_j\}$ forms an orthonormal basis within its respective metric, and the discrete transformations remain continuously connected to the identity. The components of both sets are identical, differing only in their temporal bivector plane: $\sigma_3 = \gamma_3\gamma_0$ in the Minkowski domain versus $\gamma_3 = \sigma_3\gamma_0$ in the Euclidean domain.

5.3. Example: Continuous Time-Reversal in Hyper-Spacetime

The importance of representing the full Lorentz group becomes evident when considering time-reversal T . In the conventional formulation, T is an isolated discrete reflection that cannot be reached continuously from the identity transformation. In hyper-spacetime, however, time-reversal corresponds to a smooth angular shift in the bounded parameter β :

$$T : (\beta, \theta, \phi) \mapsto (-\beta + \pi, \theta, \phi).$$

As β varies continuously, a boost can pass smoothly through the lightlike boundary and change temporal orientation without algebraic discontinuity. The boosted vector

$$p(\beta) = \sec(\beta)\gamma_0 + \tan(\beta)e_3,$$

thus evolves continuously into its time-reversed counterpart. This example illustrates why the full Lorentz group $O(1,3)$, —including the discrete reflections P , T , and PT — must be treated on equal geometric footing. These symmetries are essential in relativistic physics (e.g., CPT invariance [31,33,34]), and incorporating them continuously removes the traditional separation between boosts, rotations, and reflections. Hyper-spacetime provides this unified representation within a single angular domain.

In summary, hyper-spacetime provides:

- A smooth, closed, continuous angular domain for the full Lorentz group $O(1,3)$.
- Discrete reflections $\{P, T, PT\}$, continuously connected to identity.
- Spinor sets $\{U_j, V_j\}$ forming orthonormal bases in their respective metrics, unifying the Euclidean domain \mathbb{R}^4 with the Minkowski domain $\mathbb{R}^{1,3}$.
- A closed realization of the double cover $\text{Pin}(1,3)$ through the spinor sets $\{U_j, V_j\}$.
- Isolation of infinities at the lightlike boundaries $\beta = \pm\pi/2$ into the scalar density factor $\sqrt{\sec(\beta)}$ (Equation (11)), while the spinors themselves remain regular.

This prepares the ground for Section 6, where the eigen-spinor and eigenvalue structure of physical observables in hyper-spacetime is developed.

6. Eigen-Spinors and Eigenvalues in the Geometry of Hyper-Spacetime

Analogous to how the observables of $O(2)$ lie on the surface of one-sphere \mathbb{S}^1 , the observables of hyper-spacetime reside on the surfaces of the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$, each preserved by the spinor sets $\{U_j, V_j\}$. These spinor sets act not only as symmetry generators, but also as eigen-spinors of the corresponding observables $\{p, q, u_r\}$ (Equation (12)). This follows directly from the geometric algebra eigen-spinor eigenvalue relation (Appendix D):

$$\mathcal{O}\Omega_k = \lambda_k\Omega_k e_1 \mapsto \mathcal{O} = \frac{\lambda_k}{\Omega_k \tilde{\Omega}_k} \Omega_k e_1 \tilde{\Omega}_k, \quad \frac{\lambda_k}{\Omega_k \tilde{\Omega}_k} \in \mathbb{R} \quad (19)$$

Here, \mathcal{O} is an observable, e_1 is a basis vector (reference observable, e.g., γ_0, σ_3 , or γ_3), Ω_k are the eigen-spinors of \mathcal{O} , and λ_k the corresponding eigenvalues. In hyper-spacetime, however, the spinor sets $\{U_j, V_j\}$ are generated geometrically by variations in the hyper-spherical coordinates (β, θ, ϕ) (Equation (15)), rather than being obtained by solving an eigen-spinor eigenvalue equation explicitly (Appendix D).

6.1. Spatial Eigen-Spinors

The spatial spinor set $\{S_k\}$ (Equation (18)), generated from a parity transformation, act as eigen-spinors of the spatial unit vectors $\{e_3 \in \mathbb{S}_{0,2}^2\} \subset \mathbb{R}^{1,3}$ (Figure 2a) and $\{u_3 \in \mathbb{S}^2\} \subset \mathbb{R}^4$ (Equation (10)). They satisfy the GA eigen-spinor eigenvalue relations (Appendix D):

$$\begin{aligned} e_3 \in \mathbb{S}_{0,2}^2 &\mapsto e_3 S_k = \lambda_k S_k \gamma_3, & \lambda_k \in \{\pm 1\} \\ u_3 \in \mathbb{S}^2 &\mapsto u_3 S_k = \lambda_k S_k \sigma_3, & \lambda_k \in \{\pm 1\} \end{aligned} \quad (20)$$

So, e_3 and u_3 are observables with eigen-spinors S_k and eigenvalues $\lambda_k = \pm 1$. Physically, these correspond to spin-up or spin-down measurements along the spatial directions of $u_3(\theta, \phi)$ and $e_3(\theta, \phi)$.

6.2. Spacetime Eigen-Spinors

The spacetime spinor set $\{U_j\}$ (Equation (16)) and Euclidean spinor set $\{V_j\}$ (Equation (17)) act as eigen-spinors of the hyper-spacetime observables $\{p, q, u_r\}$ (Equation (12)):

$$\begin{aligned} p \in \mathbb{S}_H^3 &\mapsto pU_j = \eta_j U_j \gamma_0, & \eta_j &\in \{+1, +1, -1, -1\} \\ q \in \mathbb{S}_C^3 &\mapsto qU_j = \kappa_j U_j \gamma_0, & \kappa_j &\in \cos(\beta)\{+1, +1, -1, -1\} \\ u_r \in \mathbb{S}^3 &\mapsto u_r V_j = \eta_j V_j \gamma_0, & \eta_j &\in \{+1, +1, -1, -1\} \end{aligned} \quad (21)$$

Here, the eigenvalues η_j encodes two constant spin states combined with positive/negative temporal orientation in \mathbb{S}_H^3 and \mathbb{S}^3 , while the eigenvalues κ_j encodes two variable ($\pm\cos(\beta)$) spin states combined with positive/negative temporal orientation in \mathbb{S}_C^3 . Unlike η_j , the values of κ_j depend on relative speed via angle β (Equation (5)). Hence, a spin state in \mathbb{S}_C^3 vanishes at the lightlike boundaries $\beta = \pm\pi/2$.

6.3. Null Spinors

At the lightlike boundaries $\beta = \pm\pi/2$ ($v = \pm c$), the spacetime spinors U_j reduce to null spinors:

$$U_j(\beta, \theta, \phi) = S_k(\theta, \phi)L_{ij}(\beta) \mapsto U_j\left(\pm\frac{\pi}{2}, \theta, \phi\right) = \frac{1}{\sqrt{2}}S_k(\theta, \phi)(\pm 1 \pm \sigma_3), \quad U_j \tilde{U}_j = 0 \quad (22)$$

These spinors have zero intensity but preserve direction, lying on the light cone of \mathbb{S}_C^3 and \mathbb{S}_H^3 . For the Euclidean spinors V_j , the corresponding boundary form is

$$V_j(\beta, \theta, \phi) = S_k(\theta, \phi)L_{vj}(\beta) \mapsto V_j\left(\pm\frac{\pi}{2}, \theta, \phi\right) = \frac{1}{\sqrt{2}}S_k(\theta, \phi)(\pm 1 \pm \gamma_3), \quad V_j V_j^{-1} = 1 \quad (23)$$

Thus, the Euclidean spinors V_j keep their unit intensity, while the intensities of spacetime spinors U_j become zero. Each spinor set pass smoothly through the lightlike boundary.

6.4. Correspondence with the Dirac Equation

The eigen-spinors U_j (Equation (21)) and eigenvalues η_j of the observable $p \in \mathbb{S}_H^3$ reproduce those of the Dirac equation [23] (two spin states combined with positive/negative temporal orientation, i.e., matter/antimatter) (Appendix E). This agreement is remarkable because in hyper-spacetime these eigen-spinors are generated geometrically (Equation (16)), rather than being obtained by solving an eigen-spinor eigenvalue equation (Appendix D). This establishes a direct geometric origin for the Dirac spectrum, demonstrating that the fundamental structure of spacetime eigen-spinors emerges naturally within the hyper-spacetime framework.

In summary, the eigen-spinor structure of hyper-spacetime reproduces the Dirac spectrum, isolates singularities into a scalar density, and reveals null spinors as natural lightlike states.

7. Discussion

The hyper-spacetime framework developed in this paper provides a compact, continuous, and geometrically transparent representation of Lorentz symmetry. By replacing the unbounded rapidity φ with the bounded angular coordinate β , the structure of Lorentz boosts is reformulated in a closed domain in which both branches of the hyperbola, as well as the lightlike boundary, appear as regular geometric features. This compact parametrization enables boosts, rotations, and reflections to be expressed within a unified angular system, thereby revealing structural parallels between hyperbolic and circular symmetries that are hidden in the conventional rapidity formalism.

A central outcome is the introduction of a triplet of three-spheres (Figures 2 and 3)

$$\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_c^3, \mathbb{S}^3\},$$

which encode hyperbolic, causal, and Euclidean structure within a shared geometric framework. The three-spheres possess a common temporal axis and a reciprocal temporal bivector plane, providing a continuous geometric bridge between Minkowski and Euclidean domains (Equation (10)). This real-domain duality constitutes an alternative to analytic continuation and offers a natural geometric basis for Wick-type transformations [8,9], in which transitions between signatures correspond to smooth variations in the bounded angle β .

The unified geometry also clarifies the status of discrete symmetries. In the conventional Lorentz group, $\{P, T, PT\}$ appear as isolated, algebraically imposed reflections. Here, they arise instead as continuous angular operations within the compact domain of hyper-spherical coordinates (β, θ, ϕ) : shifts of β and θ encode changes in temporal or spatial orientation. This continuity removes the algebraic disconnection between discrete and continuous transformations and provides a closed and unified representation of the full Lorentz group $O(1, 3)$.

Spinor structure plays a central role in this unification. By treating spinors as geometric entities generated directly from variations in the hyper-spherical coordinates (β, θ, ϕ) (Equation (15)), the spinor sets $\{U_j\}$ and $\{V_j\}$ (Equations (16) and (17)) act both as symmetry generators and as eigen-spinors of geometric observables $\{p, q, u_r\}$. Remarkably, their eigenvalue structure reproduces the Dirac spectrum: two spin states combined with positive and negative temporal orientation, with null spinors emerging naturally at the lightlike boundary. In this formulation, Dirac spinors arise not from solving the Dirac differential equation but from the intrinsic geometry of hyper-spacetime.

A brief physical example illustrates this geometry in practice: the hyperbolic three-sphere \mathbb{S}_H^3 parametrizes normalized relativistic momentum in bounded form, while the causal sphere \mathbb{S}_c^3 simultaneously parametrizes normalized spacetime intervals (Appendix E). Their geometric product yields relativistic action, and the compact angular domain provides a clean representation of lightlike limits, where the action reduces to a null-spinor configuration. This highlights how the framework can be used to reformulate standard relativistic quantities in a bounded, continuous setting.

Finally, to place the work in its broader context [35,36], conventional treatments of Lorentz symmetry [37] are typically based on Lie-algebraic parametrizations [38], rapidity methods, or complex-analytic Wick rotations [8,10]. More recent studies examine compactifications or alternative parametrizations, but these typically apply only to restricted subgroups (e.g., $SO^+(1, 3)$) or require analytic continuation to imaginary time [39–41]. In contrast, hyper-spacetime combines compactification, spinor unification, and Euclidean–Minkowski duality within a single four-dimensional real geometric construction.

8. Conclusions

This paper introduces a unified geometric framework—hyper-spacetime—that provides a bounded, continuous representation of the full Lorentz group. By replacing rapidity with a bounded angular parameter β , the framework unifies circular and hyperbolic symmetry, eliminates divergences at the lightlike boundary, and reveals a continuous pathway through both branches of the Lorentz boost hyperbola. This compact domain enables discrete symmetries $\{P, T, PT\}$ to be expressed as smooth angular transformations, providing a closed geometric picture of $O(1, 3)$.

The construction of a triplet of three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ forms the core of hyper-spacetime. These spheres share a temporal axis and a reciprocal temporal bivector plane, establishing a real geometric duality between Minkowski and Euclidean representations. This duality offers a natural geometric interpretation of Wick rotation and clarifies how Euclidean and Lorentzian structures can be embedded within a single continuous geometry.

The associated spinor sets $\{U_j\}$ and $\{V_j\}$ (Equations (16) and (17)) reproduce the eigen-spinor structure of the Dirac equation, including null spinors at the lightlike boundary. In this sense, hyper-spacetime provides a geometric origin for Dirac spinors, where eigen-spinor behavior emerges directly from the geometry rather than from analytic solutions of a differential equation. Singularities are isolated in scalar density factors, allowing the spinors themselves to remain regular throughout the full angular domain.

Overall, the hyper-spacetime framework yields a closed and unified representation of Lorentz geometry, clarifies the role of discrete symmetries, and provides a continuous real-domain connection between Euclidean and Minkowski structures. Beyond its conceptual clarity, the framework suggests potential applications to geometric quantization, path-integral formulations, and the study of null-spinor behavior at the lightlike boundary. Future work may explore these directions, as well as the possibility of embedding interactions or gauge structures within the same compact geometric setting.

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Appendix A. Spacetime Algebra

Table A1. Spacetime algebra bridging \mathbb{R}^4 and $\mathbb{R}^{1,3}$.

#	Spacetime Algebra [19,23,42,43]	Description															
1	$\{t, x, y, z\} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \subset \mathbb{R}^{1,3}$ $\gamma_k = \sigma_k \gamma_0$ $\{t, x, y, z\} = \{\gamma_0, \sigma_1, \sigma_2, \sigma_3\} \subset \mathbb{R}^4$ $\sigma_k = \gamma_k \gamma_0$ Metric diagonal for $\mathbb{R}^{1,3} \rightarrow \{+1, -1, -1, -1\}$ Metric diagonal for $\mathbb{R}^4 \rightarrow \{+1, +1, +1, +1\}$ $(\gamma_0)^2 = +1$ $(\gamma_k)^2 = -1$ $(\sigma_k)^2 = +1$	The four orthogonal basis vectors $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ and $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ span the same four-dimensional (4D) spacetime axes $\{t, x, y, z\}$. The common temporal basis vector $\gamma_0 \in (\mathbb{R}^{1,3} \cap \mathbb{R}^4)$ squares to plus one (+1). The spatial basis vectors $\{\gamma_k\} \subset \mathbb{R}^{1,3}$ square to minus one (-1). The spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ square to plus one (+1).															
2	$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$ $\sigma_k \cdot \sigma_j + \sigma_j \cdot \sigma_k = 2\delta_{kj}$	The basis vectors $\{\gamma_\mu\} \subset \mathbb{R}^{1,3}$ satisfy the algebra of the Dirac gamma matrices, while the spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ satisfy the algebra of three-dimensional space.															
3	<table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Grade (#)</th> <th>Spacetime $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$</th> <th>Spacetime $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$</th> </tr> </thead> <tbody> <tr> <td>$\langle M \rangle_j$</td> <td></td> <td></td> </tr> <tr> <td>$\langle M \rangle_1 (1)$</td> <td></td> <td>1</td> </tr> <tr> <td>$\langle M \rangle_1 (4)$ $\langle M \rangle_2 (3)$</td> <td>$\{\gamma_0, \gamma_k\}$ $\sigma_k = \gamma_k \gamma_0$</td> <td>$\{\gamma_0, \sigma_k\}$ $\gamma_k = \sigma_k \gamma_0$</td> </tr> <tr> <td>$\langle M \rangle_2 (3)$ $\langle M \rangle_3 (4)$ $\langle M \rangle_4 (1)$</td> <td>$(i\sigma_k = i\gamma_k \gamma_0) \in \mathbb{R}^3$ $i\gamma_\mu = \{i\gamma_0, i\sigma_k \gamma_0\}$ $i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$</td> <td></td> </tr> </tbody> </table>	Grade (#)	Spacetime $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$	Spacetime $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$	$\langle M \rangle_j$			$\langle M \rangle_1 (1)$		1	$\langle M \rangle_1 (4)$ $\langle M \rangle_2 (3)$	$\{\gamma_0, \gamma_k\}$ $\sigma_k = \gamma_k \gamma_0$	$\{\gamma_0, \sigma_k\}$ $\gamma_k = \sigma_k \gamma_0$	$\langle M \rangle_2 (3)$ $\langle M \rangle_3 (4)$ $\langle M \rangle_4 (1)$	$(i\sigma_k = i\gamma_k \gamma_0) \in \mathbb{R}^3$ $i\gamma_\mu = \{i\gamma_0, i\sigma_k \gamma_0\}$ $i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$		The algebra of four-dimensional (4D) spacetime, unifying the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ with the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$, is a graded $\langle M \rangle_j$ linear space with dimension $2^4 = 16$. Multi-vectors M are “geometric numbers” being scalars $\langle M \rangle_0$, vectors $\langle M \rangle_1$, bivectors $\langle M \rangle_2$, trivectors $\langle M \rangle_3$, and pseudoscalars $\langle M \rangle_4$. Both domains, $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ and $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$, share the following: scalar 1, temporal basis vector γ_0 , three spatial bivectors $i\sigma_k$, four trivectors $i\gamma_\mu$, and pseudoscalar i . The spatial basis vectors $\{\gamma_k\} \subset \mathbb{R}^{1,3}$ correspond to temporal bivectors $\sigma_k \gamma_0 \subset \mathbb{R}^4$ in the Euclidean domain, while the spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ correspond to temporal bivectors $\gamma_k \gamma_0 \subset \mathbb{R}^{1,3}$ in the Minkowski domain. This geometric duality (spatial vector \leftrightarrow temporal bivector) unifies the two domains at the level of temporal bivector planes.
Grade (#)	Spacetime $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$	Spacetime $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$															
$\langle M \rangle_j$																	
$\langle M \rangle_1 (1)$		1															
$\langle M \rangle_1 (4)$ $\langle M \rangle_2 (3)$	$\{\gamma_0, \gamma_k\}$ $\sigma_k = \gamma_k \gamma_0$	$\{\gamma_0, \sigma_k\}$ $\gamma_k = \sigma_k \gamma_0$															
$\langle M \rangle_2 (3)$ $\langle M \rangle_3 (4)$ $\langle M \rangle_4 (1)$	$(i\sigma_k = i\gamma_k \gamma_0) \in \mathbb{R}^3$ $i\gamma_\mu = \{i\gamma_0, i\sigma_k \gamma_0\}$ $i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$																
4	$i = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ $i^2 = -1$ $i = \sigma_1 \sigma_2 \sigma_3 = \gamma_1 \gamma_0 \wedge \gamma_2 \gamma_0 \wedge \gamma_3 \gamma_0 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ $i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$	Pseudoscalar i in the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$. Pseudoscalar i in 3D space \mathbb{R}^3 of the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$. These pseudoscalars are equal.															

#	Spacetime Algebra [19,23,42,43]	Description								
5	$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3 + \langle M \rangle_4$ $\tilde{M} = \langle M \rangle + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3 + \langle M \rangle_4$	The form of a general spacetime multi-vector M . Revision operator—on top of a multi-vector is an invariant type of conjugation.								
6	$M_+ = \frac{1}{2}(M - iMi) = \langle M \rangle + \langle M \rangle_2 + \langle M \rangle_4$ $M_+ \tilde{M}_+ = M_+$ $M_+ \tilde{M}_+ = \langle M \rangle + \langle M \rangle_4$	The even subalgebra M_+ (see item 7 in this table). The even subalgebra M_+ is closed under a geometric product (GP). The GP of $M_+ \tilde{M}_+$ gives a scalar $\langle M \rangle$ plus a pseudoscalar $\langle M \rangle_4$.								
7	<table border="1"> <thead> <tr> <th>Grade (#)</th> <th>3D even subalgebra M_+</th> </tr> </thead> <tbody> <tr> <td>$\langle M \rangle (1)$</td> <td>1</td> </tr> <tr> <td>$\langle M \rangle_2 (6)$</td> <td> $\sigma_k = \gamma_k \gamma_0$ $i\sigma_k = \{\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2\}$ </td> </tr> <tr> <td>$\langle M \rangle_4 (1)$</td> <td>$i = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3$</td> </tr> </tbody> </table>	Grade (#)	3D even subalgebra M_+	$\langle M \rangle (1)$	1	$\langle M \rangle_2 (6)$	$\sigma_k = \gamma_k \gamma_0$ $i\sigma_k = \{\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2\}$	$\langle M \rangle_4 (1)$	$i = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3$	The three orthogonal basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ span the spatial axes $\{x, y, z\}$ of 3D space. Together with scalar 1, the spatial bivectors $i\sigma_k$, and the pseudoscalar i , they provide an eight-dimensional common 3D even subalgebra M_+ . The spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$, orthogonal to the temporal basis vector γ_0 , correspond to the temporal bivectors $\{\gamma_k\gamma_0\} \subset \mathbb{R}^{1,3}$ in the Minkowski domain. Both domains \mathbb{R}^4 and $\mathbb{R}^{1,3}$, therefore, share the three spatial bivectors $i\sigma_k \in \mathbb{R}^3$, ensuring domain-independence of spatial orientation.
Grade (#)	3D even subalgebra M_+									
$\langle M \rangle (1)$	1									
$\langle M \rangle_2 (6)$	$\sigma_k = \gamma_k \gamma_0$ $i\sigma_k = \{\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2\}$									
$\langle M \rangle_4 (1)$	$i = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3$									

Appendix B. Gudermannian Function

The Gudermannian* function [12,13] relates circular and hyperbolic functions without introducing complex arguments. In its primary form, it connects the unbounded hyperbolic angle $\varphi \in (-\infty, +\infty)$ to the bounded angle $\beta \in [-\pi/2, \pi/2]$, through the following identity:

$$\tanh(\varphi) = \sin(\beta) \leftrightarrow \varphi = \tanh^{-1}(\sin(\beta)). \tag{A1}$$

Within this interval, the associated identities

$$\operatorname{sech}(\varphi) = \cos(\beta), \quad \sinh(\varphi) = \tan(\beta), \tag{A2}$$

map the positive branch \mathbb{H}_+^1 of the Lorentz-boost hyperbola

$$w(\varphi) = \cosh(\varphi)\gamma_0 + \sinh(\varphi)\gamma_3, \quad \|w(\varphi)\|^2 = 1, \tag{A3}$$

onto the trigonometric form

$$w(\beta) = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3, \quad \beta \in [-\pi/2, \pi/2], \quad \|w(\beta)\|^2 = 1, \tag{A4}$$

which preserves the Minkowski norm $\|w(\beta)\|^2 = \sec^2(\beta) - \tan^2(\beta) = 1$.

Periodic Extension and Coverage of Both Hyperbolic Branches

The primary Gudermannian mapping $\varphi = \tanh^{-1}(\sin(\beta))$ covers only the region where $\cosh(\varphi) > 0$, corresponding to the positive branch \mathbb{H}_+^1 of the Lorentz-boost hyperbola. However, a complete representation of Lorentz boosts must include the negative branch \mathbb{H}_-^1 , where the hyperbolic functions reverse sign. This extension is achieved by exploiting the trigonometric periodicity.

$$\sin(\beta \pm \pi) = -\sin(\beta), \quad \cos(\beta \pm \pi) = -\cos(\beta), \quad \tan(\beta \pm \pi) = \tan(\beta), \tag{A5}$$

which, when inserted into the Gudermannian relations,

$$\tanh(\varphi) = \sin(\beta), \quad \operatorname{sech}(\varphi) = \cos(\beta), \quad \sinh(\varphi) = \tan(\beta), \quad \mapsto \beta \in [-\pi/2, \pi/2], \tag{A6}$$

produce precisely the negative hyperbolic functions

$$-\tanh(\varphi) = \sin(\beta \pm \pi), \quad -\operatorname{sech}(\varphi) = \cos(\beta \pm \pi), \quad -\sinh(\varphi) = \tan(\beta \pm \pi), \quad \mapsto \beta \pm \pi \in [+ \pi/2, 3\pi/2]. \tag{A7}$$

Therefore, shifting β by $\pm\pi$ maps the trigonometric functions onto the complementary branch of the hyperbola. Thus, combining the principal interval $\beta \in [-\pi/2, +\pi/2]$ with the shifted interval $\beta \pm \pi \in [+3\pi/2, 3\pi/2]$ yields a continuous coverage of both branches of the Lorentz hyperbola within the full periodic domain $\beta \in [0, 2\pi]$ (Figure A1).

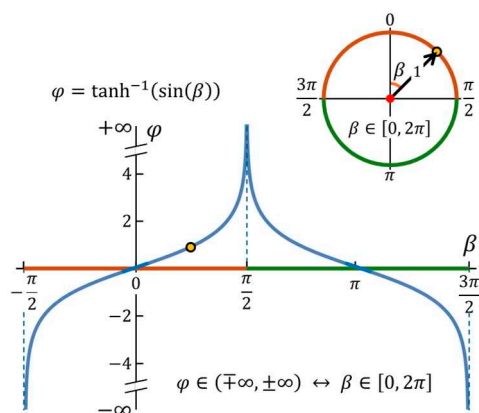


Figure A1. The Gudermannian function $\varphi = \tanh^{-1}(\sin(\beta))$ provides a smooth mapping between the unbounded angle $\varphi \in (\mp\infty, \pm\infty)$ and the bounded angle $\beta \in [0, 2\pi]$.

This periodic extension is what closes the two disconnected hyperbolic branches \mathbb{H}_+^1 and \mathbb{H}_-^1 into the hyperbolic one-sphere \mathbb{S}_H^1 , ensuring that boosts pass smoothly through the lightlike boundary ($\beta = \pm\pi/2$) and that the complete Lorentz-boost structure is represented within a single bounded angular parameter.

* Christoph Gudermann (25 March 1798–25 September 1852) introduced this function and the concept of uniform convergence. See Wikipedia.

Appendix C. Spinors and Rotors

In geometrical algebra (GA) the rotation of a multivector M by a rotor R is performed by a double-sided geometric product (GP):

$$M' = RM\tilde{R} \tag{A8}$$

This is a powerful generalization of rotation, as it applies in any dimension and to any multivector. A rotor R can be written as a scalar density ρ times a spinor S_R :

$$R = \rho S_R, \quad R\tilde{R} = 1 \mapsto \rho = \left(S_R \tilde{S}_R \right)^{-1/2} \tag{A9}$$

Here, the unitarity of the rotor determines the scalar density $\rho \in \mathbb{R}$. In general, for a spinor to define a proper rotor, its intensity must be a scalar:

$$\left(S_R \tilde{S}_R \vee S_R S_R^{-1} \right) \in \mathbb{R} \tag{A10}$$

Any nonzero additional grade contribution to the intensity would prevent normalization with a scalar density ρ (Equation (A9)).

A spinor is characterized by its bivector, i.e., the plane of rotation. In hyper-spacetime, the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ and the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ share the three spatial bivectors $i\sigma_k \in \mathbb{R}^3$ (Appendix A). Hence, a general spatial spinor is as follows:

$$S_{xyz} = \cos(\alpha/2) + i\sigma_k \sin(\alpha/2), \quad S_{xyz} \tilde{S}_{xyz} = 1 \mapsto \rho = 1, \quad (i\sigma_k)^2 = -1. \tag{A11}$$

The three temporal bivectors in the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ are $\sigma_k = \gamma_k \gamma_0$. Hence, a general Minkowski temporal spinor is as follows:

$$S_t = \cos(\alpha/2) + \sigma_k \sin(\alpha/2), \quad S_t \tilde{S}_t = \cos(\alpha) \mapsto \rho = \sqrt{\sec(\alpha)}, \quad (\sigma_k)^2 = +1. \quad (A12)$$

The three temporal bivectors in the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ are $\gamma_k = \sigma_k \gamma_0$. Hence, a general Euclidean temporal spinor is

$$S_t = \cos(\alpha/2) + \gamma_k \sin(\alpha/2), \quad S_t S_t^{-1} = 1 \mapsto \rho = 1, \quad (\gamma_k)^2 = -1. \quad (A13)$$

Both the spatial S_{xyz} and the two temporal spinors S_t are pure spinors, since their intensities reduce to scalar values.

The pure spinors $\{U_j, V_j\}$ (Equations (16) and (17)) preserving hyper-spacetime and acting as eigen-spinors, are geometrically generated from the spinors U_1 and V_1 (Equation (11)). Both are composed of a spatial spinor $S_1(\theta, \phi)$ (Equation (9)) and the temporal spinors $L_{v1}(\beta)$ and $L_{u1}(\beta)$:

$$\begin{aligned} V_1(\beta, \theta, \phi) &= S_1(\theta, \phi)L_{v1}(\beta), & L_{v1}(\beta) &= \cos(\beta/2) + \sin(\beta/2)\gamma_3, & V_1 V_1^{-1} &= 1 \\ U_1(\beta, \theta, \phi) &= S_1(\theta, \phi)L_{u1}(\beta), & L_{u1}(\beta) &= \cos(\beta/2) + \sin(\beta/2)\sigma_3, & U_1 \tilde{U}_1 &= \cos(\beta) \end{aligned} \quad (A14)$$

The associated proper rotor Λ_1 , the normalized form of U_1 , is

$$\Lambda_1(\beta, \theta, \phi) = \sqrt{\sec(\beta)}U_1(\beta, \theta, \phi), \quad \Lambda_1 \tilde{\Lambda}_1 = 1 \quad (A15)$$

The rotor associated with spinor V_1 is equal to V_1 , since its intensity satisfies $V_1 V_1^{-1} = 1$. The same holds for the spatial spinor S_1 , which also has unit intensity.

The temporal spinors L_{v1} and L_{u1} (Equation (A14)), characterized by the temporal bivectors $(\gamma_3 = \sigma_3 \gamma_0) \subset \mathbb{R}^4$ and $(\sigma_3 = \gamma_3 \gamma_0) \subset \mathbb{R}^{1,3}$, reveal the dual role of these temporal bivector planes across hyper-spacetime $\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$. In this dual relation, the temporal bivectors $(\gamma_k = \sigma_k \gamma_0)$ of the Euclidean domain appear as the spatial basis vectors of the Minkowski domain, while the temporal bivectors $(\sigma_k = \gamma_k \gamma_0)$ of the Minkowski domain play the same role in the Euclidean domain. This geometric duality unifies the two domains at the level of their temporal bivector planes. Moreover, the two domains share the three spatial bivectors $i\sigma_k \subset \mathbb{R}^3$ (Equation (A11)), which belong to a common three-dimensional even subalgebra, ensuring that the spatial spinors remain domain independent.

Appendix D. Eigen-Spinors and Eigenvalues

The hyper-spacetime spinor sets $\{U_j, V_j\}$ (Equations (16) and (17)), which preserve the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$, act also as eigen-spinors of the corresponding observables $\{p, q, u_r\}$. Their connection is described by the geometric algebra eigen-spinor eigenvalue relation:

$$\mathcal{O}\Omega_k = \lambda_k \Omega_k e_1 \mapsto \mathcal{O} = \frac{\lambda_k}{\Omega_k \tilde{\Omega}_k} \Omega_k e_1 \tilde{\Omega}_k, \quad \frac{\lambda_k}{\Omega_k \tilde{\Omega}_k} \in \mathbb{R} \quad (A16)$$

Here, \mathcal{O} is observable, e_1 is a basis vector (reference observable, e.g., γ_0, σ_3 , or γ_3), Ω_k is the eigen-spinor set of \mathcal{O} , and λ_k are the eigenvalues of \mathcal{O} . However, in hyper-spacetime, these spinor sets $\{U_j, V_j\}$ are geometrically generated (Equations (16) and (17)), rather than being obtained by solving an eigen-spinor eigenvalue equation.

To calculate eigen-spinors and eigenvalues in practice, one uses the linear algebra (LA) eigen-spinor eigenvalue equation:

$$[\mathcal{O}][\Omega_k] = \lambda_k[\Omega_k] \tag{A17}$$

where $[\mathcal{O}]$ is the matrix representation of the observable, $[\Omega_k]$ the column vector representation of the eigen-spinors, and λ_k the eigenvalues. A one-to-one conversion between GA and LA is obtained via the standard matrix representations of the basis vectors:

$$\begin{aligned} \sigma_\mu &\leftrightarrow [\sigma_\mu] = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, & i = \sqrt{-1} \\ \gamma_\mu &\leftrightarrow [\gamma_\mu] = \left\{ \begin{pmatrix} [\sigma_0] & 0 \\ 0 & -[\sigma_0] \end{pmatrix}, \begin{pmatrix} 0 & -[\sigma_k] \\ [\sigma_k] & 0 \end{pmatrix} \right\} \end{aligned} \tag{A18}$$

Using these, the observables $\{p, q, u_r\}$ become 4 by 4 matrices:

$$\begin{aligned} p \in \mathbb{S}_H^3 : p &= \sec(\beta)\gamma_0 + \tan(\beta) e_3(\theta, \phi) \leftrightarrow [p] = \begin{pmatrix} +\sec(\beta)[\sigma_0] & -\tan(\beta)[\mathcal{S}] \\ +\tan(\beta)[\mathcal{S}] & -\sec(\beta)[\sigma_0] \end{pmatrix} \\ q \in \mathbb{S}_c^3 : q &= \gamma_0 + \sin(\beta) e_3(\theta, \phi) \leftrightarrow [q] = \begin{pmatrix} +[\sigma_0] & -\sin(\beta)[\mathcal{S}] \\ +\sin(\beta)[\mathcal{S}] & -[\sigma_0] \end{pmatrix} \\ u_r \in \mathbb{S}^3 : u_r &= \cos(\beta)\gamma_0 + \sin(\beta) u_3(\theta, \phi) \leftrightarrow [u_r] = \begin{pmatrix} +\cos(\beta)[\sigma_0] & +\sin(\beta)[\mathcal{S}] \\ +\sin(\beta)[\mathcal{S}] & -\cos(\beta)[\sigma_0] \end{pmatrix} \end{aligned} \tag{A19}$$

where $[\mathcal{S}]$ is the 2 by 2 spatial spin observable matrix associated with the spatial unit vectors e_3 and u_3 (Equation (10)):

$$[\mathcal{S}] = \begin{pmatrix} +\cos(\theta) & +\sin(\theta)e^{-i\phi} \\ +\sin(\theta)e^{+i\phi} & -\cos(\theta) \end{pmatrix}, \quad i = \sqrt{-1} \tag{A20}$$

Solving the LA eigen-spinor eigenvalue Equation (A17) for $[\mathcal{S}]$ gives the spatial eigen-spinors S_k with eigenvalues $\lambda_k = \pm 1$:

$$[\mathcal{S}][S_k] = \lambda_k[S_k] \rightarrow [S_k] = \left\{ \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{+i\phi/2} \end{pmatrix}, \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ +\cos(\theta/2)e^{+i\phi/2} \end{pmatrix} \right\} \tag{A21}$$

Here, the set $[S_k]$ is the orthonormal spatial Pauli spinors [32]. Similarly, solving Equation (A17) for the observables $\{p, q, u_r\}$ yields the column vector eigen-spinor sets:

$$\begin{aligned} [p][U_j] &= \eta_j[U_j] \rightarrow [U_j] = \left\{ \begin{pmatrix} +\cos(\beta/2)[S_1] \\ +\sin(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} +\cos(\beta/2)[S_2] \\ -\sin(\beta/2)[S_2] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_1] \\ -\cos(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_2] \\ +\cos(\beta/2)[S_2] \end{pmatrix} \right\} \\ [q][U_j] &= \kappa_j[U_j] \\ [u_r][V_j] &= \eta_j[V_j] \rightarrow [V_j] = \left\{ \begin{pmatrix} +\cos(\beta/2)[S_1] \\ +\sin(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} +\cos(\beta/2)[S_2] \\ -\sin(\beta/2)[S_2] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_1] \\ -\cos(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_2] \\ +\cos(\beta/2)[S_2] \end{pmatrix} \right\} \end{aligned} \tag{A22}$$

with eigenvalues $\kappa_j \in \cos(\beta)\{+1, +1, -1, -1\}$ and $\eta_j \in \{+1, +1, -1, -1\}$.

In summary, the eigen-spinors U_j and V_j obtained by solving the LA eigen-spinor eigenvalue equation are equal to the geometrically generated (Equation (15)) spacetime spinor sets $\{U_j, V_j\}$ (Equations (16) and (17)).

Appendix E. Dirac Equation

This appendix provides the explicit correspondence between (1) the spacetime spinors U_j obtained as solutions of the Dirac equation and (2) the geometrically generated spacetime spinors U_j (Equation (16)).

The observables $\{p, q, u_r\}$, as elements of the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$, are normalized vectors and can directly be linked to relativistic dynamics. The hyperbolic three-sphere \mathbb{S}_H^3 vector p corresponds to normalized momentum:

$$p = \frac{p_m}{m_0 c} \rightarrow p_m = p_t \gamma_0 + p_x \gamma_1 + p_y \gamma_2 + p_z \gamma_3 = m_0 c (\sec(\beta) \gamma_0 + \tan(\beta) e_3(\theta, \phi)), \quad (p_m)^2 = m_0^2 c^2 \quad (\text{A23})$$

While the causal three-sphere \mathbb{S}_C^3 vector q corresponds to normalized distance:

$$q = \frac{q_t}{c \Delta t} \rightarrow q_t = t \gamma_0 + x \gamma_1 + y \gamma_2 + z \gamma_3 = c \Delta t (\gamma_0 + \sin(\beta) e_3(\theta, \phi)), \quad (q_t)^2 = c^2 \Delta t^2 \cos^2(\beta) \quad (\text{A24})$$

The geometric product of p_m and q_t yields relativistic action S [44,45]:

$$S = p_m q_t = m_0 c^2 \Delta t \cos(\beta) \mapsto S = \pm m_0 c^2 \Delta t \sqrt{1 - (v/c)^2} \quad (\text{A25})$$

Treating action as a phase S/\hbar , a free particle plane-wave state [23] is defined as

$$\psi_1(c \Delta t, \beta, \theta, \phi) = U_1(\beta, \theta, \phi) e^{-i \sigma_3 p_m q_t / \hbar} \quad (\text{A26})$$

where $p_m q_t / \hbar$ is the normalized action, and $U_1(\beta, \theta, \phi)$ is a constant spacetime spinor. This plane-wave state ψ_1 is a solution of the Dirac equation in geometric algebra [23]:

$$\hbar \nabla \psi_1(x) i \sigma_3 - m_0 c^2 \psi_1(x) \gamma_0 = 0, \quad \nabla = \gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{c \partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}, \quad \gamma^\mu \cdot \gamma_\nu = \delta_\nu^\mu \quad (\text{A27})$$

which is equivalent to the linear algebra (LA) form [4]:

$$(i \hbar [\gamma^\mu] \partial_\mu - m_0 c^2 [i]) [\psi_1(x)] = 0 \quad (\text{A28})$$

Substitution of the plane wave state ψ_1 in Equation (A27) shows that spinor U_1 satisfies the GA eigen-spinor eigenvalue relation:

$$p U_1 = U_1 \gamma_0 \quad (\text{A29})$$

Multiplying from the right with \tilde{U}_1 gives

$$p U_1 \tilde{U}_1 = U_1 \gamma_0 \tilde{U}_1 \mapsto U_1 \tilde{U}_1 = \cos(\beta) \mapsto p \cos(\beta) = q \mapsto \gamma_0 + \sin(\beta) e_3(\theta, \phi) = \gamma_0 + \sin(\beta) e_3(\theta, \phi) \quad (\text{A30})$$

Hence, a plane wave state $\psi_1(c \Delta t, \beta, \theta, \phi)$ (Equation (A26)) is a solution of the Dirac equation.

Finally, variations in the hyper-spherical coordinates (β, θ, ϕ) of ψ_1 , encoded by the spacetime event reflections $\{I, P, T, PT\}$ (Equation (15)), generate the four Dirac plane-wave solutions:

$$\begin{aligned} I: & \quad \psi_1(c \Delta t, \beta, \theta, \phi) = \psi_1(c \Delta t, +\beta, \theta, \phi) = U_1(\beta, \theta, \phi) e^{-i \sigma_3 p_m q_t / \hbar} \mapsto S = +m_0 c^2 \Delta t \cos(\beta) \\ P: & \quad \psi_2(c \Delta t, \beta, \theta, \phi) = \psi_1(c \Delta t, -\beta, \theta + \pi, \phi) = U_2(\beta, \theta, \phi) e^{-i \sigma_3 p_m q_t / \hbar} \mapsto S = +m_0 c^2 \Delta t \cos(\beta) \\ T: & \quad \psi_3(c \Delta t, \beta, \theta, \phi) = \psi_1(c \Delta t, -\beta + \pi, \theta, \phi) = U_3(\beta, \theta, \phi) e^{+i \sigma_3 p_m q_t / \hbar} \mapsto S = -m_0 c^2 \Delta t \cos(\beta) \\ PT: & \quad \psi_4(c \Delta t, \beta, \theta, \phi) = \psi_1(c \Delta t, +\beta + \pi, \theta + \pi, \phi) = U_4(\beta, \theta, \phi) e^{+i \sigma_3 p_m q_t / \hbar} \mapsto S = -m_0 c^2 \Delta t \cos(\beta) \end{aligned} \quad (\text{A31})$$

The orientation of action $S = p_m q_t$ depends on the temporal angle β , such that matter and antimatter solutions follow naturally from the spacetime event reflections $\{I, P, T, PT\}$. Each of these plane-waves ψ_j is a valid solution of the Dirac equation, demonstrating the

remarkable correspondence between the Dirac equation and the spacetime event reflections $\{I, P, T, PT\}$.

In addition to the four fermionic Dirac plane-wave solutions ψ_j , these solutions also yield a lightlike boundary solution at $\beta = \pm\pi/2$:

$$\psi_j(c\Delta t, \beta = \pm\pi/2, \theta, \phi) = \frac{1}{\sqrt{2}} S_k(\theta, \phi) (\pm 1 \pm \sigma_3) \quad (\text{A32})$$

This corresponds to a photon-like state with vanishing mass term. At the lightlike boundary, the fermionic plane-wave solutions reduce to null spinors with zero intensity while retaining directionality. Thus, the hyper-spacetime framework produces both fermionic (massive) and photonic (massless) solutions as limiting cases of a unified spinor structure.

Clifford-Algebra Structure and Lorentz-Invariant Spinors

The spacetime spinors U_j (Equation (16)) naturally belong to the even subalgebra $Cl^+(1, 3)$ of the real Clifford-algebra generated by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu},$$

which is isomorphic to $\text{Spin}(1, 3)$, the double cover of the Lorentz group $SO(1, 3)$. Since Lorentz transformations are generated by bivectors in $Cl(1, 3)$ and implemented by rotors $R = \exp(B/2)$, the spinors $U_j = S_k L_{uj}$ (Equation (16)) are automatically Lorentz-covariant: their spatial part S_k rotates within the shared spatial bivector plane $\mathbb{i}\sigma_k$ (Appendix A), while the temporal part L_{uj} rotates in the Minkowski temporal plane $\sigma_3 = \gamma_3\gamma_0$. The geometric eigenvalue relation

$$pU_j = \eta_j U_j \gamma_0,$$

is the Clifford-algebra analog of the Dirac equation's operator form

$$(\gamma^\mu p_\mu) \psi_j = \lambda_j \psi_j.$$

Thus, the four spinors generated geometrically by variations in the hyper-spherical coordinates reproduce the four standard Dirac plane-wave spinors (two spin states, each with positive or negative temporal orientation). In this sense, the Dirac spectrum is not imposed but emerges directly from the hyper-spacetime geometry.

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