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Coherent states for SUSY partner Hamiltonians

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Abstract. Coherent states are constructed for systems generated by supersymmetry from an initial Hamiltonian with a purely discrete spectrum such that the levels depend analytically on their subindex. The technique is illustrated by means of the trigonometric Pöschl-Teller potentials.

1. Introduction

The beautiful attributes of the standard coherent states (CS) stems from the Heisenberg-Weyl algebra, characteristic of the harmonic oscillator. It is well known that some of these properties play the role of definitions, which can be used to built up CS for systems different from the oscillator [1–8]. In particular, if the CS are to be constructed through algebraic techniques, e.g. as eigenstates of the annihilation operator, it is fundamental to identify the algebra characterizing the system under study. Surprisingly, for Hamiltonians whose spectra consist of an infinite set of discrete energy levels $E_n = E(n)$, where $E(n)$ is a function specifying the analyzed system, the appropriate algebraic structure was just recently identified, and this allowed at the same time to perform the CS construction [9].

On the other hand, supersymmetric quantum mechanics (SUSY QM) has proved successful to generate exactly solvable potentials from an initial solvable one [10–22]. If the Hamiltonian of departure is ruled by a given algebra, it is natural to ask how much of this algebraic structure will be inherited by its SUSY partner potentials and what will be the similarities and/or differences of the corresponding CS.

Some partial answers to these question were found for the first time in 1994 [23] (see however the seminal work of Mielnik [24]). There, third order differential annihilation and creation operators were identified as the generators of the *natural algebra* (which turned out to be of polynomial type) for the simplest family of first-order SUSY partners of the oscillator (the so called Abraham-Moses-Mielnik potentials [24, 25]). The CS were also built as eigenstates of the *natural annihilation operator*, and its decomposition in terms of energy eigenstates turned out to be more involved than the standard one. In further works it was shown that simpler CS exist for the same family of potentials (working with deformations of the natural annihilation and creation operators), and it was shown that their expansion in terms of eigenstates of the Hamiltonian was essentially the same as the standard one [26, 27]. In 1999 it was identified the *natural algebra* characterizing a quite general family of SUSY partner potentials of the oscillator. The order $k \in \mathbb{N}$ of the intertwining operator was finite, and the positions of the new levels created by the transformation were arbitrary, although all of them below the ground state energy of the oscillator [28]. Once again, it was found that the *natural CS* were more involved than the standard ones. In the same paper it was shown that the standard expression for the CS can be as well recovered working with a *deformation* of the natural algebra, which mimics the Heisenberg-Weyl algebra in the subspace associated to the isospectral part of the spectrum.

In this paper we are going to perform a similar study for the SUSY partners of potentials different from the oscillator, which are not longer described by the Heisenberg-Weyl algebra. The main results to be presented here involve an initial Hamiltonians whose spectrum is composed of an infinite set of energy levels such that $E_n = E(n),\ n = 0, 1, \ldots$ For a detailed discussion of these and some other results we recommend [29].

The organization of the paper is as follows. In the next section the algebraic structures of the initial Hamiltonian H_0 will be explored, while in section 3 we are going to derive the corresponding coherent states. The SUSY partner Hamiltonians of H_0 will be discussed in section 4, while in section 5 the corresponding algebraic structures will be analyzed. In section 6 the coherent states of H_k shall be constructed. The complete procedure is going to be applied to the trigonometric Pöschl-Teller potential in section 7 while in section 8 we will finish the paper with our conclusions.

2. Algebraic structures of H_0

Let us start with an initial Schrödinger Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x), \tag{1}$$

whose eigenfunctions and eigenvalues satisfy

$$H_0|\psi_n\rangle = E_n|\psi_n\rangle, \qquad E_0 < E_1 < E_2 < \ldots \tag{2}$$

where

$$E_n \equiv E(n), \tag{3}$$

i.e., we are assuming that there is an analytical dependence, characterized by the function $E(x)$, of the energy eigenvalues with the subindex labeling them. Let us introduce now the number operator N_0 in the way:

$$N_0|\psi_n\rangle = n|\psi_n\rangle. \tag{4}$$

The *intrinsic nonlinear algebra* of H_0 is defined through the following expressions:

$$a_0^-|\psi_n\rangle = r_{\mathcal{I}}(n)|\psi_{n-1}\rangle, \quad a_0^+|\psi_n\rangle = \bar{r}_{\mathcal{I}}(n+1)|\psi_{n+1}\rangle, \tag{5}$$

$$r_{\mathcal{I}}(n) = e^{i\alpha(E_n - E_{n-1})}\,\sqrt{E_n - E_0}, \quad \alpha \in \mathbb{R}. \tag{6}$$

At the operator level the previous relationships lead to:

$$a_0^+a_0^- = E(N_0) - E_0, \quad a_0^-a_0^+ = E(N_0 + 1) - E_0. \tag{7}$$

Thus, we arrive at the following set of non-null commutation relationships characteristic of the *intrinsic algebra* of H_0:

$$[N_0, a_0^\pm] = \pm a_0^\pm, \qquad [a_0^-, a_0^+] = E(N_0+1) - E(N_0) \equiv f(N_0), \tag{8}$$

$$[H_0, a_0^\pm] = \pm a_0^\pm f\left(N_0 - \frac{1}{2} \pm \frac{1}{2}\right) = \pm f\left(N_0 - \frac{1}{2} \mp \frac{1}{2}\right) a_0^\pm. \tag{9}$$

The Hubbard representation of $a_0^\pm$ is given by:

$$a_0^- = r_{\mathcal{I}}(N_0+1) \sum_{m=0}^{\infty} |\psi_m\rangle\langle\psi_{m+1}|, \qquad a_0^+ = \bar{r}_{\mathcal{I}}(N_0) \sum_{m=0}^{\infty} |\psi_{m+1}\rangle\langle\psi_m|. \tag{10}$$

On the other hand, the *linear algebra* of H_0 is obtained from the following *deformation* of the intrinsic annihilation and creation operators $a_0^\pm$:

$$a_{0_\mathcal{L}}^- = b(N_0)\, a_0^-, \quad a_{0_\mathcal{L}}^+ = a_0^+\, b(N_0), \tag{11}$$

with

$$b(n) = \frac{r_\mathcal{L}(n+1)}{r_\mathcal{I}(n+1)} = \sqrt{\frac{n+1}{E(n+1)-E_0}}, \qquad r_\mathcal{L}(n) = e^{i\alpha f(n-1)}\sqrt{n}. \tag{12}$$

It turns out that:

$$a_{0_\mathcal{L}}^-|\psi_n\rangle = r_\mathcal{L}(n)|\psi_{n-1}\rangle, \qquad a_{0_\mathcal{L}}^+|\psi_n\rangle = \bar{r}_\mathcal{L}(n+1)|\psi_{n+1}\rangle, \tag{13}$$

which is equivalent to the following commutation relationships:

$$[N_0, a_{0_\mathcal{L}}^\pm] = \pm a_{0_\mathcal{L}}^\pm, \qquad [a_{0_\mathcal{L}}^-, a_{0_\mathcal{L}}^+] = 1, \tag{14}$$

i.e., the *linear annihilation-creation operators* $a_{0_\mathcal{L}}^\pm$ (together with N_0) satisfy the standard Heisenberg-Weyl algebra.

3. Coherent states of H_0

Along this paper we will derive the coherent states as eigenstates of an involved annihilation operator. As regards to the *intrinsic* CS of H_0, they obey:

$$a_0^-|z,\alpha\rangle_0 = z|z,\alpha\rangle_0, \qquad z \in \mathbb{C}. \tag{15}$$

An standard procedure leads to the following expression:

$$|z,\alpha\rangle_0 = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m-E_0)\frac{z^m}{\sqrt{\rho_m}}|\psi_m\rangle}, \tag{16}$$

$$\rho_m = \begin{cases} 1 & \text{if } m = 0, \\ (E_m - E_0)\ldots(E_1 - E_0) & \text{if } m > 0. \end{cases} \tag{17}$$

The completeness relationship of the *intrinsic* CS

$$\int |z,\alpha\rangle_0\, {}_0\langle z,\alpha\,| d\mu(z) = 1, \tag{18}$$

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with a measure given by

$$d\mu(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)\rho(|z|^2)\,d^2z, \tag{19}$$

is valid provided that ρ_m and $\rho(x)$ satisfy the following moment problem:

$$\int_0^{\infty} y^m \rho(y)\,dy = \rho_m, \quad m = 0, 1, \ldots \tag{20}$$

The reproducing kernel is given by

$$_0\langle z, \alpha|z', \alpha\rangle_0 = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\left(\sum_{m=0}^{\infty}\frac{|z'|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\left(\sum_{m=0}^{\infty}\frac{(\bar{z}z')^m}{\rho_m}\right), \tag{21}$$

which in particular means that two *intrinsic* CS associated to different complex eigenvalues $z \neq z'$ are non-orthogonal. It is worth to notice that a CS evolves always as a CS since

$$U_0(t)|z, \alpha\rangle_0 = \exp(-itH_0)|z, \alpha\rangle_0 = e^{-itE_0}|z, \alpha + t\rangle_0. \tag{22}$$

This property is a consequence of the phase choice of (5-6). That is why these CS sometimes are called *adapted*. Note that the eigenvalue $z = 0$ is non-degenerated:

$$|z = 0, \alpha\rangle_0 = |\psi_0\rangle. \tag{23}$$

Concerning the *linear* CS of H_0, which are eigenstates of $a^-_{0_\mathcal{L}}$, they turn out to be:

$$|z, \alpha\rangle_{0_\mathcal{L}} = e^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)}\frac{z^m}{\sqrt{m!}}\,|\psi_m\rangle, \tag{24}$$

which, up to the phase factors, have the form of the standard CS. This implies that the completeness relationship is automatically satisfied:

$$\frac{1}{\pi}\int |z, \alpha\rangle_{0_\mathcal{L}}\ {}_{0_\mathcal{L}}\langle z, \alpha|\,d^2z = 1. \tag{25}$$

The reproducing kernel is the standard one:

$$_{0_\mathcal{L}}\langle z, \alpha|z', \alpha\rangle_{0_\mathcal{L}} = \exp\left(-\frac{|z|^2}{2} + \bar{z}z' - \frac{|z'|^2}{2}\right), \tag{26}$$

and, since the phase choice for the linear is the same as for the intrinsic annihilation and creation operators, it turns out that once again a CS evolves in time into a CS.

It is worth to notice that for the linear CS it is valid

$$|z, \alpha\rangle_{0_\mathcal{L}} = D_\mathcal{L}(z)|\psi_0\rangle = \exp(za^+_{0_\mathcal{L}} - \bar{z}a^-_{0_\mathcal{L}})|\psi_0\rangle, \tag{27}$$

meaning that the *linear* CS can be obtained through the action of the displacement operator $D_\mathcal{L}(z)$ onto the ground state of the system, as for the harmonic oscillator.

4. SUSY partner Hamiltonians H_k

The SUSY partner Hamiltonians of H_0, denoted by H_k, are linked to each other through the following intertwining relationships involving two k-th order differential operators B, B^+:

$$H_k B_k^+ = B_k^+ H_0, \qquad H_0 B_k = B_k H_k, \tag{28}$$

where

$$H_k = -\frac{1}{2}\frac{d^2}{dx^2} + V_k(x). \tag{29}$$

The potential $V_k(x)$, in terms of the initial one and the solutions $\alpha_i(x, \epsilon_i)$ of a sequence of Riccati equations, is given by:

$$V_k(x) = V_0(x) - \sum_{i=1}^{k} \alpha_i'(x, \epsilon_i). \tag{30}$$

In case that all the $\epsilon_j, j = 1, \dots, k$ are different, it turns out that the i-th Riccati solution at ϵ_i can be expressed algebraically in terms of two previous ones at $\epsilon_i,\ \epsilon_{i-1}$ through the following finite-difference (Bäcklund) formula:

$$\alpha_i(x, \epsilon_i) = -\alpha_{i-1}(x, \epsilon_{i-1}) - \frac{2(\epsilon_i - \epsilon_{i-1})}{\alpha_{i-1}(x, \epsilon_i) - \alpha_{i-1}(x, \epsilon_{i-1})}, \quad i = 2, \dots k. \tag{31}$$

By iterating down this formula, it is straightforward to realize that all the α_i can be expressed in terms of the k solutions $\alpha_1(x, \epsilon_i)$ of the first Riccati equation:

$$\alpha_1'(x, \epsilon_i) + \alpha_1^2(x, \epsilon_i) = 2[V_0(x) - \epsilon_i], \quad i = 1, \dots, k. \tag{32}$$

The above formulae can be expressed in terms of k solutions $u_i(x) \propto \exp[\int_0^x \alpha_1(y, \epsilon_i) dy]$ of the associated Schrödinger equation, in particular the potential $V_k(x)$:

$$V_k(x) = V_0(x) - \{\ln[W(u_1, \dots, u_k)]\}''. \tag{33}$$

Two relevant equations resulting from this procedure are:

$$B_k^+ B_k = \prod_{i=1}^{k} (H_k - \epsilon_i), \qquad B_k B_k^+ = \prod_{i=1}^{k} (H_0 - \epsilon_i). \tag{34}$$

They show the connection between the k-th order SUSY QM and the *factorization method* [18].

Let us denote by $|\theta_{\epsilon_i}\rangle,\ i = 1, \dots, s$, the s physical states belonging to the kernel of B_k which are at the same time eigenstates of H_k, namely, $B_k|\theta_{\epsilon_i}\rangle = 0,\ H_k|\theta_{\epsilon_i}\rangle = \epsilon_i|\theta_{\epsilon_i}\rangle,\ s \leq k$. Suppose that $\text{Sp}(H_k) = \{\epsilon_1, \dots, \epsilon_q, E_0, E_1, \dots\},\ q \leq s$, i.e., $\epsilon_{q+j} = E_{m_j},\ j = 1, \dots, p = s - q,\ m_j < m_{j+1}$. We thus have $B_k^+|\psi_{m_j}\rangle = 0$.

5. Algebraic structures of H_k

Let us define in the first place the number operator N_k by its action onto the eigenstates of H_k:

$$N_k|\theta_n\rangle = n|\theta_n\rangle, \quad N_k|\theta_{\epsilon_i}\rangle = 0, \quad n = 0, 1, \dots, \quad i = 1, \dots, q. \tag{35}$$

Following [23, 24, 26–28], let us introduce now a *natural algebra* for the SUSY partner Hamiltonians H_k, whose generators are constructed from the intertwining and intrinsic annihilation and creation operators of H_0:

$$a^{\pm}_{k_{\mathcal{N}}} = B^+_k a^{\pm}_0 B_k. \tag{36}$$

It turns out that the action of the *natural annihilation and creation operators* $a^{\pm}_{k_{\mathcal{N}}}$ onto the eigenstates of H_k is given by:

$$a^{\pm}_{k_{\mathcal{N}}}|\theta_{\epsilon_i}\rangle = 0, \qquad i = 1,\dots,q, \tag{37}$$

$$a^{-}_{k_{\mathcal{N}}}|\theta_n\rangle = r_{\mathcal{N}}(n)\ |\theta_{n-1}\rangle, \qquad a^{+}_{k_{\mathcal{N}}}|\theta_n\rangle = \bar{r}_{\mathcal{N}}(n+1)\ |\theta_{n+1}\rangle, \tag{38}$$

where

$$r_{\mathcal{N}}(n) = \left\{\prod_{i=1}^{k}[E(n)-\epsilon_i][E(n-1)-\epsilon_i]\right\}^{\frac{1}{2}} r_{\mathcal{I}}(n). \tag{39}$$

An alternative expression for $a^{\pm}_{k_{\mathcal{N}}}$ is provided by the corresponding Hubbard representation:

$$a^{-}_{k_{\mathcal{N}}} \ = \ r_{\mathcal{N}}(N_k+1)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_{m+1}|, \qquad a^{+}_{k_{\mathcal{N}}} = \bar{r}_{\mathcal{N}}(N_k)\sum_{m=0}^{\infty}|\theta_{m+1}\rangle\langle\theta_m|. \tag{40}$$

On the subspace associated to the levels $\{E_n, n = 0, 1, \dots\}$ it is valid:

$$[a^{-}_{k_{\mathcal{N}}}, a^{+}_{k_{\mathcal{N}}}] = [\bar{r}_{\mathcal{N}}(N_k+1)r_{\mathcal{N}}(N_k+1) - \bar{r}_{\mathcal{N}}(N_k)r_{\mathcal{N}}(N_k)]\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{41}$$

Let us analyze now the *intrinsic algebra* of H_k, which is generated by the following annihilation and creation operators:

$$a^{-}_{k} = r_{\mathcal{I}}(N_k+1)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_{m+1}|, \qquad a^{+}_{k} = \bar{r}_{\mathcal{I}}(N_k)\sum_{m=0}^{\infty}|\theta_{m+1}\rangle\langle\theta_m|. \tag{42}$$

Their action onto the energy eigenstates of H_k becomes:

$$a^{\pm}_{k}|\theta_{\epsilon_i}\rangle = 0, \qquad i = 1,\dots,q, \tag{43}$$

$$a^{-}_{k}|\theta_n\rangle = r_{\mathcal{I}}(n)|\theta_{n-1}\rangle, \qquad a^{+}_{k}|\theta_n\rangle = \bar{r}_{\mathcal{I}}(n+1)|\theta_{n+1}\rangle. \tag{44}$$

On the subspace associated to the levels $\{E_n,\ n = 0, 1, \dots\}$, this algebra is equal to the corresponding one for H_0:

$$[a^{-}_{k}, a^{+}_{k}] = f(N_k)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{45}$$

Note that the natural and intrinsic algebras of H_k can be seen as deformations from each other, since:

$$a^{-}_{k_{\mathcal{N}}} = \frac{r_{\mathcal{N}}(N_k+1)}{r_{\mathcal{I}}(N_k+1)}\,a^{-}_{k}, \quad a^{+}_{k_{\mathcal{N}}} = \frac{r_{\mathcal{N}}(N_k)}{r_{\mathcal{I}}(N_k)}\,a^{+}_{k}, \quad a^{+}_{k_{\mathcal{N}}}a^{-}_{k_{\mathcal{N}}} = [E(N_k)-E_0]\left[\frac{r_{\mathcal{N}}(N_k)}{r_{\mathcal{I}}(N_k)}\right]^2. \tag{46}$$

Finally, let us analyze the *linear algebra* of H_k, which is generated by:

$$a_{k_{\mathcal{L}}}^{-} = r_{\mathcal{L}}(N_k+1)\sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_{m+1}|, \qquad a_{k_{\mathcal{L}}}^{+} = \bar{r}_{\mathcal{L}}(N_k)\sum_{m=0}^{\infty}|\theta_{m+1}\rangle\langle\theta_m|. \tag{47}$$

The corresponding action on the eigenstates of H_k reads:

$$a_{k_{\mathcal{L}}}^{\pm}|\theta_{\epsilon_i}\rangle = 0, \qquad i = 1,\dots,q, \tag{48}$$

$$a_{k_{\mathcal{L}}}^{-}|\theta_n\rangle = r_{\mathcal{L}}(n)|\theta_{n-1}\rangle, \qquad a_{k_{\mathcal{L}}}^{+}|\theta_n\rangle = \bar{r}_{\mathcal{L}}(n+1)|\theta_{n+1}\rangle. \tag{49}$$

At the operator level, this algebra is described by the following commutation relationships:

$$[N_k, a_{k_{\mathcal{L}}}^{\pm}] = \pm a_{k_{\mathcal{L}}}^{\pm}, \qquad [a_{k_{\mathcal{L}}}^{-}, a_{k_{\mathcal{L}}}^{+}] = \sum_{m=0}^{\infty}|\theta_m\rangle\langle\theta_m|. \tag{50}$$

Once again, the intrinsic and linear algebra are deformations from each other due to:

$$a_{k_{\mathcal{L}}}^{-} = \frac{r_{\mathcal{L}}(N_k+1)}{r_{\mathcal{I}}(N_k+1)}\,a_k^{-}, \qquad a_{k_{\mathcal{L}}}^{+} = \frac{r_{\mathcal{L}}(N_k)}{r_{\mathcal{I}}(N_k)}\,a_k^{+}, \qquad a_{k_{\mathcal{L}}}^{+}a_{k_{\mathcal{L}}}^{-} = N_k. \tag{51}$$

The previous results lead us to conclude that the algebraic structures associated to H_0, intrinsic and linear, characterize as well its SUSY partner Hamiltonians H_k on the subspace associated to the initial levels $\{E_n,\ n = 0,1,\dots\}$. It will be interesting to see what happens with the corresponding coherent states.

6. Coherent states of H_k

The *natural nonlinear coherent states* of H_k, which are eigenstates of $a_{k_{\mathcal{N}}}^{-}$, are given by:

$$|z,\alpha\rangle_{k_{\mathcal{N}}} = \left[\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\widetilde{\rho}_m}\right]^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\alpha(E_{m+m_p+1}-E_{m_p+1})}\frac{z^m}{\sqrt{\widetilde{\rho}_m}}|\theta_{m+m_p+1}\rangle, \tag{52}$$

where $\widetilde{\rho}_0 = 1$ and, for $m > 0$,

$$\widetilde{\rho}_m = \frac{\rho_{m+m_p+1}}{\rho_{m_p+1}}\prod_{i=1}^{k}(E_{m+m_p+1}-\epsilon_i)(E_{m+m_p}-\epsilon_i)^2\dots(E_{m_p+2}-\epsilon_i)^2(E_{m_p+1}-\epsilon_i). \tag{53}$$

Since the CS decomposition (52) does not involve the eigenstates $\{|\theta_{\epsilon_i}\rangle,\ |\theta_n\rangle,\ i = 1,\dots,q,\ n = 0,\dots,m_p\}$ of H_k, the completeness relationship has to be modified, namely,

$$\sum_{i=1}^{q}|\theta_{\epsilon_i}\rangle\langle\theta_{\epsilon_i}| + \sum_{m=0}^{m_p}|\theta_m\rangle\langle\theta_m| + \int |z,\alpha\rangle_{k_{\mathcal{N}}}\,{}_{k_{\mathcal{N}}}\langle z,\alpha|\,d\widetilde{\mu}(z) = 1, \tag{54}$$

where the measure reads:

$$d\widetilde{\mu}(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\widetilde{\rho}_m}\right)\widetilde{\rho}(|z|^2)\,d^2z. \tag{55}$$

The associated moment problem becomes now:

$$\int_0^{\infty} y^m\widetilde{\rho}(y)\,dy = \widetilde{\rho}_m, \quad m \geq 0. \tag{56}$$

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Note that here the degeneracy of the eigenvalue $z = 0$ can be any of the integers $\{s+1, \ldots, s+p+1\}$. The time evolution of the natural CS become:

$$U_k(t)|z,\alpha\rangle_{k_\mathcal{N}} = \exp(-itH_k)|z,\alpha\rangle_{k_\mathcal{N}} = e^{-itE_{m_p+1}}|z,\alpha+t\rangle_{k_\mathcal{N}}. \tag{57}$$

As regards the *intrinsic nonlinear coherent states* of H_k, it turns out that:

$$|z,\alpha\rangle_k = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\alpha(E_m-E_0)}\frac{z^m}{\sqrt{\rho_m}}|\theta_m\rangle. \tag{58}$$

Note that, if equation (18) is satisfied, then the completeness relationship,

$$\sum_{i=1}^{q}|\theta_{\epsilon_i}\rangle\langle\theta_{\epsilon_i}| + \int |z,\alpha\rangle_{k\,k}\langle z,\alpha|\,d\mu(z) = 1, \tag{59}$$

is also valid. Let us remark that the eigenvalue $z = 0$ is now (q+1)th degenerated.

For the *linear coherent states* of H_k we get:

$$|z,\alpha\rangle_{k_\mathcal{L}} = e^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty}e^{-i\alpha(E_m-E_0)}\frac{z^m}{\sqrt{m!}}|\theta_m\rangle. \tag{60}$$

Now the completeness relationship is simply:

$$\sum_{i=1}^{q}|\theta_{\epsilon_i}\rangle\langle\theta_{\epsilon_i}| + \frac{1}{\pi}\int |z,\alpha\rangle_{k_\mathcal{L}\,k_\mathcal{L}}\langle z,\alpha|\,d^2z = 1. \tag{61}$$

Once again, the eigenvalue $z = 0$ is $(q+1)$-th degenerated. Notice finally that:

$$|z,\alpha\rangle_{k_\mathcal{L}} = D_{k_\mathcal{L}}|\theta_0\rangle = \exp(za^+_{k_\mathcal{L}} - \bar{z}a^-_{k_\mathcal{L}})|\theta_0\rangle. \tag{62}$$

It is worth to point out that the decompositions for the intrinsic and linear CS of H_k, which involve just energy eigenstates belonging to the subspace associated to the initial levels $\{E_n,\ n = 0,1,\ldots\}$, are the same as the corresponding ones for H_0. This is due to the fact that the linear and intrinsic algebras of H_0 characterize as well the H_k on the subspace associated to the initial levels. Let us illustrate next our treatment by one simple non-trivial example.

7. Example

Let us apply the previous techniques to the trigonometric Pöschl-Teller potentials:

$$V_0(x) = \frac{(\nu-1)\nu}{2\cos^2(x)}, \qquad \nu > 1, \tag{63}$$

which are illustrated in gray in Figure 1. The eigenfunctions of the corresponding Hamiltonian, satisfying the boundary conditions $\psi_n(-\pi/2) = \psi_n(\pi/2) = 0$, as well as the eigenvalues are given by:

$$\psi_n(x) = \left[\frac{n!(n+\nu)\Gamma(\nu)\Gamma(2\nu)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})\Gamma(n+2\nu)}\right]^{1/2}\cos^\nu(x)\ C_n^\nu(\sin(x)), \tag{64}$$

$$E_n = E(n) = \frac{(n+\nu)^2}{2}, \qquad n = 0,1,2,\ldots \tag{65}$$

where $C_n^\nu(y)$ are the Gegenbauer polynomials.

The *intrinsic algebra* of H_0 is characterized by:

$$E(N_0) = \frac{(N_0+\nu)^2}{2}, \tag{66}$$

which leads to:

$$f(N_0) = E(N_0+1) - E(N_0) = N_0 + \nu + \tfrac{1}{2}. \tag{67}$$

Thus, the commutation relationships for the intrinsic algebra of H_0 become:

$$[N_0, a_0^\pm] = \pm a_0^\pm, \qquad [a_0^-, a_0^+] = N_0 + \nu + \tfrac{1}{2}, \qquad [H_0, a_0^\pm] = \pm(N_0 + \nu \mp \tfrac{1}{2})a_0^\pm. \tag{68}$$

Notice that the first two relations essentially correspond to the su(1, 1) algebra.

On the other hand, the *linear annihilation and creation operators* of H_0, which generate the Heisenberg-Weyl algebra, are given by:

$$a_{0_\mathcal{L}}^- = \sqrt{\frac{2}{N_0+2\nu+1}}\, a_0^-, \qquad a_{0_\mathcal{L}}^+ = a_0^+\sqrt{\frac{2}{N_0+2\nu+1}}, \tag{69}$$

such that

$$a_{0_\mathcal{L}}^+ a_{0_\mathcal{L}}^- = N_0, \qquad [a_{0_\mathcal{L}}^-, a_{0_\mathcal{L}}^+] = 1. \tag{70}$$

Concerning the coherent states of H_0, it turns out that the intrinsic and linear ones become respectively:

$$|z,\alpha\rangle_0 = \left[{}_0F_1(2\nu+1; 2|z|^2)\right]^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\frac{\alpha}{2}m(m+2\nu)} \sqrt{\tfrac{2^m}{m!(2\nu+1)_m}}\; z^m|\psi_m\rangle, \tag{71}$$

$$|z,\alpha\rangle_{0_\mathcal{L}} = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} e^{-i\frac{\alpha}{2}m(m+2\nu)} \tfrac{z^m}{\sqrt{m!}}\, |\psi_m\rangle. \tag{72}$$

The moment problem for the intrinsic CS involves:

$$\rho_m = \frac{m!\,(2\nu+1)_m}{2^m}, \tag{73}$$

leading to

$$\rho(y) = \frac{2^{\nu+2}y^\nu}{\Gamma(2\nu+1)} K_{2\nu}(2\sqrt{2y}). \tag{74}$$

The invariant measure is thus:

$$d\mu(z) = \frac{2^{\nu+2}|z|^{2\nu}}{\pi\Gamma(2\nu+1)}\, {}_0F_1(2\nu+1; 2|z|^2) K_{2\nu}(2\sqrt{2}\,|z|)\, d^2z, \tag{75}$$

while the reproducing kernel is:

$${}_0\langle z,\alpha|z',\alpha\rangle_0 = \left[{}_0F_1(2\nu+1; 2|z|^2)\; {}_0F_1(2\nu+1; 2|z'|^2)\right]^{-\frac{1}{2}} {}_0F_1(2\nu+1; 2\bar{z}z'). \tag{76}$$

For analyzing the SUSY partner Hamiltonians H_k, let us suppose that we use just seed solutions associated to non-physical eigenvalues of H_0, q of them becoming at the end physical

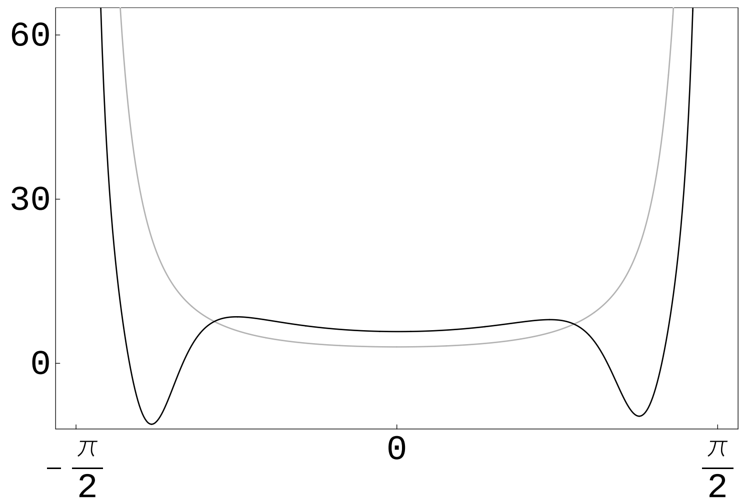

Figure 1. Illustration of the first-order SUSY partner potential $V_1(x)$ (black curve) of the Pöschl-Teller potential with $\nu = 3$ (gray curve). The potential $V_1(x)$ has an additional level, compared with $V_0(x)$, at $\epsilon = 4.4 < E_0 = 9/2$.

levels of H_k. An illustration of the corresponding potentials is plotted in black in Figure 1. The several algebraic structures of H_k are characterized by the following functions:

$$r_{\mathcal{I}}(n) = e^{i\alpha\left(n+\nu-\frac{1}{2}\right)}\,\sqrt{\frac{n(n+2\nu)}{2}}, \tag{77}$$

$$\frac{r_{\mathcal{N}}(n)}{r_{\mathcal{I}}(n)} = \frac{1}{2^k}\prod_{i=1}^{k}\sqrt{[(n+\nu-1)^2-2\epsilon_i][(n+\nu)^2-2\epsilon_i]}, \tag{78}$$

$$\frac{r_{\mathcal{L}}(n)}{r_{\mathcal{I}}(n)} = \sqrt{\frac{2}{n+2\nu}}. \tag{79}$$

The *natural CS* of H_k can now be straightforwardly evaluated:

$$|z,\alpha\rangle_{k\mathcal{N}} = \frac{1}{\sqrt{{}_0F_{4k+1}(2\nu+1,\ldots,\nu-\sqrt{2\epsilon_i},\nu-\sqrt{2\epsilon_i}+1,\nu+\sqrt{2\epsilon_i},\nu+\sqrt{2\epsilon_i}+1,\ldots;2^{2k+1}|z|^2)}} \times \sum_{m=0}^{\infty}\frac{e^{-\frac{i}{2}\alpha m(m+2\nu)}\sqrt{2^{m(2k+1)}}\,z^m|\theta_m\rangle}{\sqrt{m!(2\nu+1)_m}\prod_{i=1}^{k}\sqrt{\left(\nu-\sqrt{2\epsilon_i}\right)_m\left(\nu-\sqrt{2\epsilon_i}+1\right)_m\left(\nu+\sqrt{2\epsilon_i}\right)_m\left(\nu+\sqrt{2\epsilon_i}+1\right)_m}}. \tag{80}$$

The corresponding moment problem involves:

$$\widetilde{\rho}_m = \frac{m!(2\nu+1)_m}{2^{m(2k+1)}}\prod_{i=1}^{k}(\nu-\sqrt{2\epsilon_i})_m(\nu-\sqrt{2\epsilon_i}+1)_m(\nu+\sqrt{2\epsilon_i})_m(\nu+\sqrt{2\epsilon_i}+1)_m. \tag{81}$$

The *intrinsic and linear CS* of H_k can be obtained from the expressions (71-72) by the substitution $|\psi_m\rangle \to |\theta_m\rangle$.

8. Conclusions

In this paper we have shown that the intrinsic and linear algebras of H_0 are inherited by H_k on the subspace associated to the initial levels $\{E_n,\ n=0,1,\ldots\}$. It turns out that the linear

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algebra for both SUSY partner Hamiltonians H_0 and H_k simplifies at maximum the intrinsic algebraic structure of our systems. We have also generalized successfully the natural algebra for H_k, introduced previously for the SUSY partners of the harmonic oscillator. Once determined the algebra characterizing our systems, it is straightforward to perform the CS analysis.

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