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Explorations in higher-derivative gravity: holography and astrophysics

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EXPLORATIONS IN HIGHER-DERIVATIVE GRAVITY: HOLOGRAPHY AND ASTROPHYSICS

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PROGRAMA DE DOUTORAMENTO EN FÍSICA NUCLEAR E DE PARTÍCULAS

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UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

Contents

Resumo en galego

O obxectivo da investigación presentada nesta tese é estudar distintas teorías de gravidade con derivadas de orde superior, poñendo o foco nas súas aplicacións nos ámbitos da holografía e a astrofísica. Este tipo de modelos foron considerados dende fai décadas como posibles vías para unha futura unificación da gravidade coa mecánica cuántica, mais é importante antes de nada ter clara a súa orixe e relevancia.

Proposta por Einstein en 1915, a teoría da relatividade xeral describe as interaccións gravitatorias como consecuencia da xeometría do espazotempo, que á súa vez depende das distribucións de materia e enerxía no mesmo. Como sucesora da lei da gravitación universal de Newton, foi capaz de superar multitude de probas experimentais dende o seu nacemento, destacando por exemplo as predicións clásicas da precesión do perihelio de Mercurio ou a deflexión das traxectorias da luz, así como as recentes deteccións de ondas gravitacionais.

Pero esta non deixa de ser unha teoría clásica, e de algún xeito debería ser posible reconciliala co outro grande logro da física teórica durante o século pasado: a teoría cuántica de campos. Esta é capaz de describir cunha precisión extrema todas as interaccións microscópicas que observamos, e polo tanto é de agardar que a estas escalas a gravidade sexa gobernada polas mesmas normas. Porén, se un intenta cuantizala na súa forma actual atopa que non é posible, posto que a teoría non é renormalizable, indicando a necesidade de atopar unha descrición alternativa para a gravidade cuántica. Unha posibilidade extensamente considerada é a teoría de cordas, que predí a aparición a baixas enerxías dunha serie de termos con derivadas superiores na acción, pero cuxa forma exacta non é coñecida.

Esta tese trata o problema descrito cun enfoque "de abaixo cara arriba", construíndo a forma dos termos adicionais que suplementan a acción de Einstein-Hilbert, a correspondente á relatividade xeral, en base a certos requirimentos físicos. Estas contribucións serán máis relevantes a enerxías altas, ou distancias pequenas, e van conter derivadas de orde superior dos campos da teoría. Durante a maior parte deste traballo consideramos unicamente correccións construídas con derivadas da métrica, o cal se traduce en contraccións de tensores de curvatura, aínda que na Parte II incluiremos tamén contribucións de campos adicionais.

A construción destas teorías trátase en certo detalle no Capítulo 1. Como primeiro exemplo podemos mencionar as teorías de Lovelock [1, 2], que son os Lagranxianos máis xerais que posúen ecuacións do movemento de segunda orde en calquera métrica, de forma similar á relatividade xeral. Porén, en catro dimensións redúcense trivialmente á acción de Einstein-Hilbert, e polo tanto non producen modificacións na dinámica gravitacional. Debido a isto, neste traballo estamos máis interesados na familia das "Generalized Quasitopological gravities", identificada máis recentemente [3–11]. Estas manteñen certas características desexables das teorías de Lovelock, como a existencia de ecuacións do movemento de segunda orde ou a propagación só do gravitón sen masa de relatividade xeral, aínda que en principio só en espazos maximamente simétricos. En calquera caso, isto é suficiente para permitir a construción, polo xeral de forma numérica, de solucións de buraco negro con propiedades termodinámicas accesibles de forma exacta, o cal será importante para os estudos expostos nesta tese. Estes modelos poden ser construídos para calquera orde nas derivadas [10, 11], e de feito calquera teoría de orde superior pode ser escrita como un Lagranxiano desta familia [9], polo que serven como base para estudar os efectos de correccións cuánticas xerais. Aparte disto, son de particular interese debido a que producen modificacións non triviais tamén en 4 dimensións. En concreto, estaremos interesados na teoría máis simple con estas características, "Einteinian cubic gravity", formada por contraccións cúbicas dos tensores de curvatura e orixinalmente identificada en [3].

Ademais disto, o outro tema fundamental dunha porción importante deste traballo é a dualidade holográfica [12–14], que tamén introducimos no Capítulo 1. Esta é unha conxectura inicialmente derivada da teoría de cordas, pero que foi estendida ata ser considerada unha dualidade máis xeral entre certas teorías cuánticas e de gravidade, polo que habitualmente se denomina "dualidade gauge/gravidade". Na súa forma máis habitual, a correspondencia AdS/CFT, propón unha descrición dunha teoría cuántica de campos con simetría conforme (CFT) en termos dun modelo de gravidade cuántica nun espazotempo anti de-Sitter (AdS) cunha dimensión adicional, de xeito que a CFT estaría localizada na fronteira asintótica deste espazo AdS. Isto apóiase no principio holográfico, proposto a partir do resultado da entropía dun buraco negro, que en relatividade xeral é proporcional á área do horizonte, en lugar do volume encerrado como cabería agardar. Isto implicaría que de algún xeito en gravidade a información dos graos de liberdade contidos nunha rexión estaría codificada na fronteira da mesma. Ademais, a correspondencia resulta particularmente interesante no límite en que a CFT ten acoplamento forte e moitos graos de liberdade, que é imposible de tratar coas ferramentas habituais da mecánica cuántica. Na teoría de gravidade dual isto corresponde ao límite de baixas enerxías, que en principio está descrito pola relatividade xeral. Polo tanto, esta construción permite calcular magnitudes da CFT estudando campos clásicos na teoría gravitatoria, poñendo por medio un "dicionario holográfico" que relaciona certas cantidades a cada lado.

Volvendo ao tema anterior, podemos pensar en incluír correccións cuánticas na acción da gravidade clásica, na forma de termos con derivadas superiores, o cal permitiría desprazarse deste réxime altamente cuántico da CFT. De novo, non coñecemos a forma exacta das correccións axeitadas na acción da gravidade, pero podemos considerar diversas teorías como modelos de xoguete con constantes de acoplamento libres, que producen modificacións en certas cantidades da CFT en función destes parámetros. Polo tanto, estas teorías de orde superior permiten estudar CFTs con propiedades distintas ás descritas pola gravidade de Einstein, incrementando o rango de aplicacións da dualidade holográfica.

Este punto é estudado na Parte I, onde consideramos unicamente a adición de termos con contraccións dos tensores de curvatura na teoría gravitatoria, e analizamos a súa influencia en distintos aspectos da dualidade holográfica.

O Capítulo 2 trata a cuestión da renormalización e regularización da acción da gravidade en holografía. En concreto, hai dous problemas a resolver. Por unha banda, a acción é infinita preto da fronteira de AdS debido ao volume deste espazotempo. Isto corresponde a diverxencias a distancias pequenas na teoría de campos dual, e é necesario eliminalas para poder identificar os valores de distintas cantidades na CFT. En relatividade xeral, isto conséguese mediante a prescrición da renormalización holográfica [15–17], que consiste en eliminar os infinitos que aparecen sumando os contratermos necesarios. Porén, estes non se poden escribir de forma pechada, senon que hai que calculalos explicitamente identificando as diverxencias en cada dimensión. Pola outra banda, é necesario formular un problema variacional ben definido, no sentido de que a variación da acción produza as ecuacións do movemento para a métrica coa única condición de que o valor desta estea fixo na fronteira. Para relatividade xeral, o termo de Gibbons-Hawking-York logra precisamente isto. En teorías con correccións de orde superior na curvatura as diverxencias presentes serán distintas, co cal os contratermos anteriores non funcionarán. Propoñemos polo tanto empregar unha extensión da prescrición dos *Kounterterms* [18, 19], que son termos obtidos a partir dunha expresión pechada para dimensións xerais que involucra as curvaturas extrínseca e intrínseca da fronteira. Estes reproducen a forma dos contratermos habituais, incluíndo a contribución de GHY, en dimensións baixas, e polo tanto resolven de forma natural os dous problemas explicados anteriormente. Para teorías xerais de gravidade, propoñemos sumar estes *Kounterterms* multiplicados por unha constante global, que dependerá dos parámetros de acoplamento da propia teoría. Ao longo do Capítulo mostramos explicitamente que estes termos son capaces de eliminar as diverxencias e proporcionar un problema variacional ben definido en dimensión igual ou menor a 5. O razoamento é válido para teorías xerais que admitan solucións AdS, exceptuando un subconxunto para o cal a asintótica do espazotempo difire da obtida en relatividade xeral, e que tipicamente corresponden con teorías críticas de algunha forma. Ademais, para o problema variacional requirimos que a métrica da fronteira sexa conformalmente plana, o cal se satisfai automaticamente para dimensións menores a 5.

No Capítulo 3 exploramos os efectos de termos de orde superior na curvatura nun sistema de materia condensada que pode ser descrito a través da correspondencia AdS/CFT: o "supercondutor holográfico" [20–22]. Un material supercondutor pódese caracterizar pola existencia dunha transición de fase a unha temperatura crítica, baixo a cal un parámetro de orde toma un valor non nulo producindo unha ruptura espontánea de simetría. Nas teorías de materia condensada habituais este parámetro de orde corresponde coa densidade de portadores de carga supercondutores, co cal o sistema pasa a opoñer unha resistencia nula ao transporte de corrente eléctrica continua. Na descrición holográfica modelamos este sistema introducindo un campo escalar na teoría gravitatoria, cargado baixo unha simetría gauge U(1). A temperatura do sistema, pola súa parte, é proporcionada por un buraco negro no interior do espazotempo, que actúa como un baño térmico como é habitual na dualidade holográfica. Ademais, consideramos un espazotempo de catro dimensións na teoría de gravidade, co cal a teoría de campos ten tres, é dicir, dúas espaciais. Polo tanto o modelo sería axeitado para describir materiais cunha estrutura cristalina laminar, de modo que a interacción entre os graos de liberdade dentro de cada lámina se produza cun acoplamento forte. Este sistema está ben estudado na literatura, pero interésanos explorar o efecto que as correccións cuánticas na dinámica gravitatoria poden ter no mesmo. Para isto, incluímos na acción gravitatoria os termos de Einsteinian cubic gravity, pois son as correccións máis simples coñecidas que admiten ecuacións do movemento de segunda orde e non son triviais en 4 dimensións, polo que modificarán a métrica de AdS co buraco negro. De feito, esta é a primeira ocasión en que se consideran correccións de orde superior para un supercondutor holográfico con esta dimensión. Empregando técnicas numéricas estudamos tanto o réxime no que a métrica do espazotempo está fixa, como aquel no cal os campos de materia a modifican. Podemos estudar como cambian cantidades como a magnitude deste condensado ou a temperatura crítica co parámetro de acoplamento das correccións cúbicas, e tamén coa resposta desta métrica aos valores dos campos. Introducindo unha perturbación nunha compoñente espacial do potencial electromagnético calculamos tamén a condutividade, atopando igualmente un valor infinito cando a frecuencia desta perturbación é cero, correspondente coa resistividade infinita para corrente continua. O principal efecto dos termos cúbicos que podemos identificar é unha diminución da temperatura crítica, mais a interpretación disto dende o punto de vista da teoría de materia condensada non está clara. En calquera caso, observamos que a tendencia coincide coa observada en estudos anteriores, que inclúen termos correspondentes a teorías de Lovelock en dimensións superiores.

Ata este punto consideramos unicamente correccións construídas a partir dos tensores de curvatura, mentres que a dinámica dos demais campos que introducimos é a habitual, coa única diferenza de que a métrica do espazotempo no que viven vese modificada. Pero é natural pensar que estes campos tamén poden contribuír ás correccións con derivadas de orde superior. Este é o punto central da Parte II, onde engadimos certos termos deste tipo na acción gravitatoria, que en última instancia corresponderán a un potencial químico con acoplamento non minimal na teoría dual.

Comezamos construíndo estas teorías no Capítulo 4, onde estendemos a familia das denominadas "Electromagnetic Quasitopological gravities", introducidas para 4 dimensións en [23], a dimensión xeral. Estes son modelos de gravidade cunha $(d-2)$-forma diferencial, sendo d a dimensión da fronteira. Para construír as teorías requirimos que teñan solucións de buraco negro accesibles de forma analítica, considerando que o valor da forma diferencial é o correspondente a unha solución magnética. Mediante unha transformación de dualidade este modelo describe un campo electrostático na teoría de gravidade, que será equivalente a un potencial químico na fronteira. Con estes requisitos, obtemos a forma dos Lagranxianos para calquera potencia da curvatura e da forma diferencial, e en dimensión arbitraria.

A partir de aquí, no resto do Capítulo só nos centramos nas teorías de orde máis baixa, para as cales realizamos unha serie de cálculos de relevancia en AdS/CFT. Empezamos establecendo varias entradas do dicionario holográfico, primeiro atopando os valores das cargas centrais que gobernan distintas funcións de correlación na teoría dual, e que difiren das correspondentes á gravidade de Einstein. Calculamos tamén a distribución angular do fluxo de enerxía resultante de introducir unha perturbación da corrente electromagnética localizada na CFT, determinada unicamente por un parámetro que depende da teoría. Nos dous casos obtemos expresións exactas en termos das constantes de acoplamento, é dicir, non perturbativas. Ademais, como a forma destas cantidades difire das obtidas coa gravidade de Einstein, vemos que estas teorías son útiles para describir CFTs que pertencen a clases de universalidades distintas ás descritas pola relatividade xeral. Estes resultados tamén serven para acoutar os parámetros dos termos de orde superior, ao considerar requirimentos de unitariedade e positividade do fluxo de enerxía na CFT, e comprobamos que isto é equivalente a impoñer a propagación causal de perturbacións dos campos na teoría de gravidade. Continuamos cun estudo das propiedades termodinámicas da CFT dual, que son equivalentes ás dos buracos negros que coloquemos no espazo AdS. Atopamos que estas teorías posúen un espazo de fases máis rico que o correspondente á gravidade de Einstein, pero estas fases adicionais son desfavorecidas ao considerar as cotas anteriores nas constantes de acoplamento. Rematamos obtendo o cociente entre a viscosidade de cizalladura e a densidade de entropía correspondentes a un plasma na teoría dual. Este é un cálculo habitual na correspondencia AdS/CFT, que orixinalmente levou a propoñer a famosa "cota KSS" como un valor mínimo que esta cantidade podía tomar para calquera fluído. Coas teorías desta familia atopamos que non só non se cumpre esta cota, coma en moitas outras gravidades de orde superior, senón que é posible conseguir que esta cantidade sexa menor que cero respectando as restricións coñecidas nas constantes de acoplamento.

Continuamos co estudo destas teorías no Capítulo 5, neste caso calculando entropías de entretecemento e de Rényi. A entropía de entretecemento é unha medida da cantidade de entretecemento cuántico que hai entre unha rexión espacial da CFT e o seu complementario, mentres que a entropía de Rényi é unha xeneralización deste concepto. Ámbalas dúas se definen en mecánica cuántica en termos da matriz densidade, pero para as teorías que nos interesan pódense obter de forma máis sinxela empregando as ferramentas da holografía. Isto reduce o cálculo a un problema xeométrico na teoría de gravidade, de xeito similar a como a entropía dun buraco negro é proporcional á área do horizonte en relatividade xeral. Desta maneira calculamos as entropías de Rényi para as teorías introducidas no Capítulo anterior, considerando unha superficie de entretecemento esférica na CFT. Observamos que o potencial químico tende a aumentar o valor desta entropía, sempre que se cumpran os requirimentos físicos obtidos previamente para os valores dos parámetros de acoplamento. Os resultados atopados permiten confirmar que estes modelos cumpren unhas relacións universais previamente enunciadas para teorías xerais, apoiando así o seu estudo. Do mesmo xeito, calculamos a dimensión conforme e a resposta magnética dos *twist operators* asociados a estas entropías, e comprobamos que tamén verifican certas relacións con distintos correladores da CFT anteriormente coñecidas. A partir do resultado obtido, propoñemos unha expresión universal para a entropía de entretecemento dunha rexión esférica en teorías cun potencial químico pequeno, en termos de constantes que caracterizan as correspondentes funcións de correlación. Probamos que esta identidade é certa en xeral, en base ás relacións coñecidas para os *twist operators*, e apoiamos a proposta cun cálculo explícito para as Electromagnetic Quasitopological gravities de orde xeral que atopamos no Capítulo anterior.

Se ben todo o traballo previo se centrou en considerar correccións con derivadas superiores en aplicacións holográficas, é lóxico pensar en estudar tamén o efecto das mesmas noutros escenarios. Este é o tema central da Parte III, a última do traballo, onde poremos o foco no seu papel en astrofísica.

No Capítulo 6 estudamos distintos problemas de acreción de materia por un buraco negro, engadindo á acción gravitatoria o escalar de Einsteinian cubic gravity. Isto modifica a forma da solución de buraco negro en catro dimensións, cambiando tanto o radio do horizonte como a xeometría no exterior, o cal influirá no comportamento da materia nas súas inmediacións. Coma sempre, os termos cúbicos aparecen multiplicados por unha constante de acoplamento, cuxo valor non coñecemos, e aínda que existen cotas experimentais estas non son moi restritivas. En calquera caso agardariamos que o parámetro fose pequeno, pois estas teorías só están ben fundamentadas no límite perturbativo, mais para ilustrar o efecto destes termos consideramos valores relativamente grandes. As correccións de orde superior son sempre máis importantes en rexións de curvatura alta. Isto fai que sexa interesante estudar o escenario de acreción, no cal o buraco negro absorbe continuamente materia da rexión preto do horizonte. Consideramos xeneralizacións relativistas de dous modelos clásicos de acreción, analizando a influencia dos termos cúbicos nos mesmos. O primeiro deles considera o escenario en que existe un movemento relativo entre o buraco negro e a materia, habitualmente coñecido como acreción de vento, mentres que no segundo están en repouso un con respecto ao outro, o cal se denomina acreción esférica. Nos dous casos realizamos cálculos semianalíticos, empregando a solución de buraco negro obtida de forma numérica para a teoría cúbica. Esta é sempre a xeneralización da solución de Schwarzschild, por simplicidade, xa que non é posible construír solucións con rotación para Einsteinian cubic gravity fóra dalgúns límites concretos. Os resultados do primeiro modelo, que considera a materia como un conxunto de partículas masivas que non interactúan entre si, son respaldados por outros obtidos a partir de simulacións numéricas de hidrodinámica relativista. Nos dous casos atopamos que os termos cúbicos producen de forma consistente un aumento da taxa de acreción. De feito, o efecto é maior que o incremento do radio do horizonte, co cal este cambio non é suficiente para explicar o resultado, indicando que o comportamento da solución fóra do buraco negro posúe relevancia. Ademais, as diferenzas son maiores en réximes de enerxías altas, nos cales as correccións relativistas tenden a ser máis importantes. En calquera caso, o cambio non é relevante para buracos negros de tamaño astrofísico, pero si para outros de tamaño máis pequeno. Isto permite especular coa aplicación deste estudo aos hipotéticos buracos negros primordiais, propostos como unha contribución á densidade de materia escura, pois podería influír nas cotas existentes sobre a súa presenza no Universo.

Finalmente, a tese pecha cun Capítulo breve no que expoñemos unhas conclusións xerais sobre o traballo, así como posibles futuras direccións. Isto serve como complemento ás discusións incluídas ao final de cada un dos Capítulos principais.

Overview of the thesis

This thesis is a compilation of the results obtained during the last four years of work, in collaboration with the supervisor, José Edelstein, and several other authors: Ignacio Araya, Pablo Bueno, Pablo Cano, Nicolás Grandi, Ángel Murcia, Gabriel Rodríguez, Emilio Tejeda, David Vázquez, Alejandro Vilar and Xuao Zhang. This document incorporates material from the works [24–28], and an additional, related article has been authored separately [29].

The purpose of the current Chapter is to provide a concise overview of the contents within this thesis. We commence by clarifying the motivations and objectives that drive the research, which might get diluted as we dig into more intricate details in the following Chapters, as well as a general description of the methodology and tools employed to attain the desired results. Afterwards, we provide a preview of the reserach presented here, with the aim to assist the reader by offering a brief insight into the topics and problems covered in each Chapter.

Motivation and objectives

The main goal of this work is to explore the consequences that incorporating higher-derivative terms in the gravitational action can have in different physical scenarios. The appearance of these terms is a general feature of proposals for a quantum theory of gravity, and in this work they are obtained from a bottom-up approach. This considers the Einstein-Hilbert action as the lowest-energy contribution in an infinite series of terms, where higher-derivative corrections become important in regimes of large curvature. We are primarily interested in theories belonging to the Generalized Quasitopological class, which provide non-trivial corrections in 4 dimensions while retaining some of the desirable features of former models such as Lovelock gravity. Originally, these theories are built by supplementing the action of general relativity with certain combinations of higher-order contractions of the curvature tensors, and their actual form can be known to arbitrary orders.

Among their wide range of applications we must highlight their role in holography, since they serve as toy models to study dual quantum theories with features that can not be described with Einstein gravity alone. This will be the main focus during a significant portion of this work. In particular, we intend to propose an extension for general theories of the holographic renormalization procedure, which produces some counterterms for the action that are able to cancel divergences while providing a well-posed variational problem in the bulk theory. The AdS/CFT correspondence is also known to be a powerful tool to study problems in condensed matter physics, and it is interesting to investigate the role of higher-derivative terms in this setup, which are traditionally assumed to correspond to finite N or finite coupling effects in the dual theory. We will do so by studying the addition of such corrections in what is perhaps the most famous example: the holographic superconductor.

Beside the pure gravity corrections, it is reasonable to think about higher-derivative corrections built with contractions of other fields, which can provide even richer holographic toy models. In this regard, we can wonder about the effects of a non-minimally coupled chemical potential in the boundary CFT. We will model this by adding to the bulk action contractions of the curvature tensors with a differential form, that can be related to an electromagnetic field by means of a duality transformation. Therefore, this will provide a description for more general setups involving a chemical potential in the CFT, and in particular can be useful to check and propose universal features of such models.

Lastly, it is possible to depart from the holographic framework and study the implications of these corrections in other contexts. A well-established scenario where higher-curvature terms can play a significant role is that of cosmology, and in fact some of these theories are known to provide useful models for inflation. Given this, it is reasonable to examine the impact of these corrections on astrophysical systems and processes as well. A specific objective of this thesis is to delve into the interaction between matter and black holes, which leads us to the study of problems of accretion.

Methodology

Similarly to any work in theoretical high-energy physics, carrying out this thesis involved delving into an extensive body of existing literature. This is essential in order to gain knowledge about the current status of the field, enabling the identification of relevant open questions, but also serves as a valuable resource for acquiring concepts and techniques that aid in the actual computations. Additionally, the procedure of work is further supported by engaging in discussions with fellow researches, which can prove useful to find intuition about the interpretation of the results, as well as decide what avenues are worth pursuing.

With respect to tools used for the actual computations, aside from the traditional pen and paper the problems pursued often require the help of different pieces of computer software. Wolfram's *Mathematica* plays a pivotal role in this regard, as we use it extensively for both symbolic and numerical computations. Its capabilities are further enhanced by the suite of packages for tensor computer algebra *xAct* [30], which is able to manipulate complex expressions involving contractions of tensors in an efficient manner. For a specific part of the thesis we also rely on the GPL software *aztekas* [31, 32], to simulate the dynamics of relativistic fluids on a fixed background. Finally, custom implementations of numerical algorithms, for problems such as integrating differential equations or finding roots, are at times indispensable. For these cases we chose to work with the programming language C++, due to its performance.

Outline

All the work presented here revolves around incorporating higher-derivative corrections into the gravity action, and examining their implications in various scenarios. Therefore, before delving into the results, it is crucial to establish the necessary background by presenting this kind of gravitational theories, and in particular those that hold our primary interest: the family of Generalized Quasitopological gravities. This is treated in Chapter 1, where we also provide an explanation of some relevant concepts in black hole thermodynamics, as well as an introduction to the gauge/gravity duality, which is another central topic that will be explored throughout a significant portion of the thesis.

The different results of the research carried out are presented after this, grouping the main Chapters of the text in three distinct Parts according to the topics that they address. Each individual Chapter features its own introduction and conclusions, but we present a brief overview of their contents in what follows.

First, in Part I we study the consequences of adding higher-curvature corrections to the gravity action in the context of the AdS/CFT duality. In Chapter 2 we treat the problem of holographic renormalization for this kind of theories. We propose a method based on the Kounterterms introduced in [18, 19], and we show that it is enough to cancel the near-boundary divergences, while also providing a well-posed variational problem, for generic higher-curvature theories of gravity in an AdS bulk with 5 dimensions or less.

Following that, in Chapter 3 we consider one of the most celebrated applications of the AdS/CFT correspondence in the realm of condensed matter physics: the holographic superconductor. Again, we are interested in the role that higher-curvature terms in the bulk action can play in this description, so we add to the action the contribution of Einsteinian cubic gravity, which is the simplest extension to Einstein gravity in 4 dimensions that fulfills certain physical requirements. We perform the usual computations of the condensation and conductivity in the superconducting phase, which requires us to implement some complex numerical methods, and find that the net effect of the cubic terms is to decrease the critical temperature, thus making the superconducting phase harder to reach. The results are compared with similar studies performed for higher-dimensional supercondutors with Lovelock corrections.

After that we move to Part II, where we extend the previous construction of higher-derivative theories of gravity to include also certain matter fields. In particular, we introduce the family of "Electromagnetic Quasitopological gravities" for arbitrary dimension in Chapter 4. These involve contractions of the curvature tensors with a $(d-2)$-form field, which can be dualized to form a theory with a non-minimally coupled electromagnetic field, that corresponds to a chemical potential in the holographic dual theory. We provide the form of these Lagrangians for any power of the curvature and the differential form, and study extensively the lowest order representatives in the holographic setup. We carry out different calculations, such as computing correlators or the thermodynamic phase space of the boundary theory, that allow us to characterize the dual CFT as well as place bounds on the couplings of the gravity theory.

This study of holographic models with a non-minimally coupled chemical potential is continued in Chapter 5, focusing on the study of Rényi and entanglement entropies for a spherical entangling region. We prove that the theories considered respect a set of relations involving their Rényi entropies and the corresponding twist operators, which were previously known to be universal. The results obtained allow us to propose, and also prove, another relation about the dependence of this entanglement entropy on the chemical potential for general theories.

Finally, in Part III, we shift our focus towards a rather different topic: the role of higher-curvature corrections in astrophysics. This is composed only of Chapter 6, where we consider the problem of accretion of matter by a black hole. Since we want to characterize the qualitative effects that the higher-curvature terms can have in this setup, we again supplement the 4-dimensional action with the relatively simple scalar of Einsteinian cubic gravity. We study different settings in the background of the black hole, most importantly those of wind and spherical accretion, and find that all the results obtained point towards an increase in the rate of accretion due to the quantum corrections, whose effect becomes more important in the higher-energy regimes.

The main body of this thesis is closed with a short Chapter where we provide some global conclusions about this work, and mention possible future directions. Finally, we include several Appendices to complete different discussions in the main text.

Notation and conventions

Throughout this thesis we will include some very technical computations and discussions. The text intends to be self-consistent, and while most conventions used are more or less standard in the literature, it can be useful for the reader to find them collected here.

As is usual in the field of high-energy theoretical physics, we will in natural units where the speed of light is $c = 1$, and when required we also set the reduced Planck constant to be $\hbar = 1$. Newton's gravitational constant G_N is written explicitly in most of our expressions, and its value will be clear otherwise.

Dimensions, indices and curvature

We always consider the spacetime described by our gravity theories to be D-dimensional. In most cases however this is an anti-de Sitter spacetime, which has a boundary where the dual CFT resides. The dimension of this boundary, and thus of the dual theory, is $d = D - 1$. The coordinates of the total D-dimensional manifold are always denoted by Greek indices, μ, ν, ... In Chapter 2 we will split the coordinates between those normal and tangent to the boundary, labeling the latter with Latin indices i, j, ..., but this will be emphasized again when necessary.

As is customary in this area of research, we use the "mostly plus" signature, meaning for example that the Minkowski metric takes the form

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, \dots, +1)\,. \tag{1}$$

We follow also the usual prescription for the components of the Riemann tensor,

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho}\,, \tag{2}$$

and the Ricci tensor and curvature scalar are computed from these as

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}\,, \qquad R = g^{\mu\nu} R_{\mu\nu}\,. \tag{3}$$

Differential forms and the generalized Kronecker delta

In Part II (this is, Chapters 4 and 5) we will introduce a $(d-2)$-form field that generalizes the electromagnetic potential to higher dimensions. Therefore, we need to establish some notions and conventions on differential forms, for which we follow [33].

Let us start by defining the completely antisymmetric Levi-Civita symbol on a D-dimensional manifold, as

$$\tilde{\epsilon}_{\mu_1\mu_2\ldots\mu_D} := \begin{cases} +1 & \text{if } \mu_1\mu_2\ldots\mu_D \text{ is an even permutation of } 01\ldots(D-1), \\ -1 & \text{if } \mu_1\mu_2\ldots\mu_D \text{ is an odd permutation of } 01\ldots(D-1), \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The components of this symbol are defined in this way for any right-handed coordinate system, but from this we can construct the Levi-Civita tensor, which does transform as a tensor. Its covariant and contravariant components are given by

$$\epsilon_{\mu_1\mu_2\ldots\mu_D} := \sqrt{|g|}\tilde{\epsilon}_{\mu_1\mu_2\ldots\mu_D}\,, \qquad \epsilon^{\mu_1\mu_2\ldots\mu_D} := \frac{1}{\sqrt{|g|}}\tilde{\epsilon}_{\mu_1\mu_2\ldots\mu_D}\,, \tag{5}$$

where g is the determinant of the metric of the manifold. Note that the position of the indices (upstairs or downstairs) in the Levi-Civita symbol $\tilde{\epsilon}$ is irrelevant.

We can sometimes find expressions with contractions of Levi-Civita tensors, which can be simplified using the identity

$$\epsilon^{\mu_1\ldots\mu_p\rho_{p+1}\ldots\rho_D}\epsilon_{\nu_1\ldots\nu_p\rho_{p+1}\ldots\rho_D} = (-1)^s p!(D-p)!\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_p}_{\nu_p]}\,, \tag{6}$$

where $s=0$ if the manifold is Euclidean, and $s=1$ if it is Lorentzian.

Differential forms

We can define a p-form α, which is a $(0,\ p)$-tensor that is totally antisymmetric. In general, it can be expressed in terms of the basis vectors of the co-tangent space of the manifold, dx^μ, as

$$\alpha := \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,, \tag{7}$$

where $\wedge$ denotes the wedge product, an antisymmetrized product of differential forms. In particular, given a p-form α and a q-form β, the wedge product $\alpha\wedge\beta$ is a $(p+q)$-form whose components are given by

$$(\alpha\wedge\beta)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\alpha_{[\mu_1\ldots\mu_p}\beta_{\mu_{p+1}\ldots\mu_{p+q}]}\,, \tag{8}$$

where as usual indices between brackets are antisymmetrized.

The exterior derivative d is the natural extension of the derivative operator to differential forms. For a p-form α, it is defined as

$$\mathrm{d}\alpha := \frac{1}{p!}\partial_\nu\alpha_{\mu_1\ldots\mu_p}dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,, \tag{9}$$

so $\mathrm{d}\alpha$ is a $(p+1)$-form whose components are given by

$$(\mathrm{d}\alpha)_{\nu\mu_1\ldots\mu_p} = (p+1)\partial_{[\nu}\alpha_{\mu_1\ldots\mu_p]}\,. \tag{10}$$

Notice that, because of commutativity of partial derivatives, it follows that

$$\mathrm{d}(\mathrm{d}\alpha) = 0\,. \tag{11}$$

The final operator that we need to introduce is the Hodge star $\star$, which acts as a map from p-forms to $(D-p)$-forms, as

$$\star\alpha \;:= \frac{1}{p!(D-p)!}\alpha_{\mu_1\dots\mu_p}\epsilon^{\mu_1\dots\mu_p}{}_{\mu_{p+1}\dots\mu_D} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_D}\,, \tag{12}$$

or in components

$$(\star\alpha)_{\mu_1\dots\mu_{D-p}} = \frac{1}{p!}\epsilon^{\nu_1\dots\nu_p}{}_{\mu_1\dots\nu_{D-p}}\alpha_{\nu_1\dots\nu_p}\,. \tag{13}$$

One can also apply this operator twice, recovering the original form α with an additional sign, as

$$\star\star\,\alpha = (-1)^{s+p(D-p)}\alpha\,, \tag{14}$$

where again $s=0$ or $s=1$ for Euclidean or Lorentzian signature, respectively. This identity can be checked by applying the Hodge star operator twice and replacing products of the Levi-Civita symbol using Eq. (6).

Of course, differential forms play a role in many problems, and they fulfill some other relations that could be useful in different contexts. However, the ones reviewed here should be enough for our purposes, and we refer the reader to [33] for a more in-depth but also pedagogical introduction that follows our same notation.

The generalized Kronecker delta

Both in Chapter 2 and in Part II of this thesis, although in different contexts, we will come upon expressions with contractions of antisymmetrized indices. These can lead to some cumbersome computations, which are greatly simplified if one introduces the generalized Kronecker delta. Following the conventions of [34], this is defined as the antisymmetrized product of p Kronecker deltas,

$$\delta^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_p} = \det\left[\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_p}_{\nu_p}\right] = p!\,\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_p}_{\nu_p]}\,. \tag{15}$$

Some useful properties of this object in D dimensions are the following:

$$\delta^{\mu_1\dots\mu_p\mu_{p+1}\dots\mu_q}_{\nu_1\dots\nu_p\nu_{p+1}\dots\nu_q}\delta^{\nu_{p+1}\dots\nu_q}_{\mu_{p+1}\dots\mu_q} = (q-p)!\frac{(D-p)!}{(D-q)!}\delta^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_p}\,, \tag{16a}$$

$$\delta^{\mu_1\dots\mu_p\mu_{p+1}\dots\mu_q}_{\nu_1\dots\nu_p\mu_{p+1}\dots\mu_q} = \frac{(D-p)!}{(D-q)!}\delta^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_p}\,, \tag{16b}$$

$$\delta^{\mu_1\dots\mu_p}_{\rho_1\dots\rho_p}\delta^{\rho_1\dots\rho_p}_{\nu_1\dots\nu_p} = p!\,\delta^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_p}\,. \tag{16c}$$

List of abbreviations

(A)dS	(Anti-)de Sitter
AlAdS	Asymptotically locally AdS
CFT	Conformal field theory
ECG	Einsteinian cubic gravity
EE	Entanglement entropy
EFT	Effective field theory
EH	Einstein-Hilbert
EM	Einstein-Maxwell
EoS	Equation of state
EQG	Electromagnetic Quasitopological gravity
FG	Fefferman-Graham
GB	Gauss-Bonnet
GHY	Gibbons-Hawking-York
GQG	Generalized Quasitopological gravity
GR	General relativity
HDG	Higher-derivative gravity
HR	Holographic renormalization
IR	Infrared
ISCO	Innermost stable circular orbit
MSS	Maximally symmetric spacetime
PBH	Penrose-Brown-Henneaux
QFT	Quantum field theory
QTG	Quasitopological gravity
RE	Rényi entropy
SSS	Static and spherically symmetric
UV	Ultraviolet
WGC	Weak gravity conjecture

1

Introduction

Our fundamental understanding of Nature rests upon two cornerstone theories developed during the 20th century: the Standard Model of particle physics and Einstein's general relativity. The former is based on the postulates of quantum mechanics, and describes accurately the behavior of particles and their interactions on microscopic distances, providing unparalleled precision in its predictions. In contrast, general relativity is an elegant explanation of gravity in terms of curvature of spacetime, excelling on cosmic scales.

However, when attempting to rejoin these two descriptions we have to face several challenges. The Standard Model unifies in a robust manner the electromagnetic, weak and strong nuclear forces, but gravity lies completely outside its domain. In fact, trying to describe the gravitational interaction with the language of quantum mechanics one finds that it is non-renormalizable [35–40], which implies that it does not admit a quantization in its current form.

Another puzzle posed by the current understanding of the large-scale structure of the Universe is the ΛCDM model. This is built on the foundations of GR, but relies on the presence of two cryptic components dubbed "dark energy" and "dark matter" to match experimental measurements of the accelerated expansion of the Universe, the cosmic microwave background and dynamics of galaxies. The first of these is a constant vacuum energy density with negative pressure, and the second is believed to be some kind of matter that is only subject to gravitational interaction, and thus can not be observed directly. The existence of these two entities cannot be accounted for by the field content of the Standard Model, and several alternatives for their origin have been proposed throughout the years [41–45], each with its own perspective on the cosmic puzzle. However, ideally we would expect a corrected theory of gravity to provide a better description for these empirical observations, perhaps without the need to include such components.

Lastly let us turn out attention to black holes, whose existence is one of the most successful predictions of general relativity and for which there is extensive experimental evidence. Still, these objects bring some fundamental puzzles on their own, such as the curvature singularity behind the horizon and the unknown microscopic nature of their entropy. Besides, the issue of black hole evaporation deserves special mention. While black holes are predicted to behave as thermal objects and emit radiation, from a semiclassical treatment it is not clear how these outgoing particles can account for the information encoded in matter that could have been absorbed before, which would lead to a loss of information that is forbidden in a quantum theory that respects unitarity. This is known as the "information paradox" (a review can be found in [46]), and is tightly related to the problem of the nature of the black hole entropy.

The fact that all the issues mentioned arise at the intersection of gravity and quantum field theory points towards the need to join both descriptions in a theory of "quantum gravity." This has occupied many theoretical physicists during the last few decades, and among the most promising approaches we should mention loop quantum gravity [47–50] and string theory [51–54]. This thesis, however, fits into a distinct but not unrelated line of research. We will consider a bottom-up approach, which consists on adding corrections to the action of gravity that become relevant only at high enough energies, and that must fulfill certain physical requirements. These higher-order terms are agnostic about the actual form of the UV-complete theory, and thus allow us to perform computations and identify behaviors that should be expected from a quantum description of gravity.

In what follows, we offer some background knowledge that is crucial for motivating and comprehending the findings presented throughout the rest of the thesis. Each subsequent Chapter is featured with its own introduction, that delves into more specific and technical concepts. We begin by reviewing general relativity and higher-derivative extensions to it, which pose the underlying theme of this thesis, placing particular emphasis on the family of Generalized Quasitopological gravities. Then we move on to some general notions on black holes and their thermodynamic properties, and finish with a short overview of the AdS/CFT correspondence and the role of higher-derivative corrections in this duality.

1.1 General relativity and higher-derivative extensions

Since its original proposal more than a century ago, Einstein's general relativity has stood out as one of the most remarkable achievements in the history of physics. It provides a novel description for the gravitational interaction that departs from the classical notion of it as a force between masses. Instead, it emerges as a consequence of the geometry of the cosmos, with massive entities inducing curvature which shapes the trajectories of other objects that move in this landscape.

Originally,[1] the theory is formulated in terms of a symmetric tensor $g_{\mu\nu}$, which is to be interpreted as the metric of a torsion-free D-dimensional manifold $\mathcal{M}$. The dynamics of this field is determined by the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G_{\text{N}}} \int_{\mathcal{M}} d^D x \sqrt{-g} R \,, \tag{1.1}$$

where R is the Ricci scalar associated with $g_{\mu\nu}$. This is in fact the simplest non-trivial covariant action that can be constructed from the metric tensor. The Einstein-Hilbert action is usually supplemented by a cosmological constant term, Λ, which is interpreted as a vacuum energy density. Also, there can be matter fields in the system, and in this case the entire action reads

$$S = \frac{1}{16\pi G_{\text{N}}} \int_{\mathcal{M}} d^D x \sqrt{-g} \,(R - 2\Lambda) + S_{\text{matter}} \,. \tag{1.2}$$

[1] An alternative approach to general relativity is the Palatini formulation, which considers the metric and the affine connection to be independent fields, thus obtaining first order equations of motion. A review of this formalism can be found in [55].

The equations of motion are obtained by varying this total action with respect to the metric tensor, finding Einstein's field equations

$$\mathcal{E}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_{\rm N} T_{\mu\nu}\,, \tag{1.3}$$

where $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm matter}}{\delta g_{\mu\nu}}$ is the stress-energy tensor associated with the matter distribution. Besides, due to diffeomorphism invariance these equations fulfill the Bianchi identity $\nabla_\mu \mathcal{E}^{\mu\nu} = 0$, which is consistent with the conservation of the stress-energy tensor, $\nabla_\mu T^{\mu\nu} = 0$.

If the manifold $\mathcal{M}$ has a boundary, however, it is necessary to specify certain boundary conditions in the metric when performing the variation. For that, one fixes the value of the metric there with Dirichlet boundary conditions,

$$\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0\,. \tag{1.4}$$

However, taking this into account when computing the variation of Eq. (1.2) one is left with an undesirable boundary term, that should be cancelled. This is achieved by supplementing the action with the Gibbons-Hawking-York term

$$S_{\rm GHY} = \frac{\epsilon}{8\pi G_{\rm N}} \int_{\partial\mathcal{M}} d^{D-1}x\,\sqrt{-h} K\,, \tag{1.5}$$

where h_{ij} is the induced metric on the boundary $\partial\mathcal{M}$, ϵ is the norm of the unit vector normal to that boundary and K is the trace of the extrinsic curvature of h_{ij}, given by Eq. (A.9).

Einstein's theory of gravity has undergone rigorous scrutiny through a multitude of experiments, each supporting its accuracy as a description for the gravitational dynamics of the Universe. Among the classical tests, we should mention the prediction of the perihelion precession of Mercury's orbit, the deflection of light by the Sun, gravitational redshift, and the Shapiro time delay. However, one of its most outstanding predictions are black holes, whose thermodynamic properties are reviewed in Section 1.2. While at the beginning they were believed to be mere mathematical constructs, substantial evidence for their existence has been accumulated over time. This is in fact a very promising area of research, particularly with recent breakthroughs such as the direct measurement of gravitational waves produced during mergers [56–67], which are actually another striking prediction of the theory, and the images obtained by the Event Horizon Telescope collaboration [68–71].

However, all these tests prove that general relativity is accurate at cosmological scales, since the available experiments are only able to measure regimes of small curvature. In fact, trying to extrapolate the results of this theory to shorter distances one is faced with the issues mentioned before, such as the non-renormalizability and the appearance of singularities. One possible approach to this problem, and the one that we are interested in, is to consider the Einstein-Hilbert Lagrangian (1.1) as the first term in an effective field theory expansion of an (unknown) UV-complete theory of gravity. Therefore, it would be natural to include terms in the action that become relevant at higher energies, and in order to respect the symmetries of the theory they must be made of scalar contractions of the curvature tensors. Schematically, this action would read

$$S = \frac{1}{16\pi G_{\rm N}} \int_{\mathcal{M}} d^D x\,\sqrt{-g}\left(R - 2\Lambda + \frac{\alpha_2}{M_*^2}\mathcal{R}^{(2)} + \frac{\alpha_3}{M_*^4}\mathcal{R}^{(3)} + \ldots\right)\,, \tag{1.6}$$

where $\{\alpha_i\}$ are coupling constants of $\mathcal{O}(1)$, and the new scale M_* determines when these new terms become relevant. The object $\mathcal{R}^{(n)}$ represents a term with contractions of n tensors of curvature, but whose actual form is not known. The series would continue to infinity, but in practice it needs to be truncated.

The appearance of such higher-curvature corrections is motivated by string theory, as they appear in the low-energy expansion of the action of different models weighted by the string length α' [72, 73], which would be proportional to M_*^{-2} in our notation. But this is not the only reason why such terms are desirable from a theoretical point of view. Indeed, it has been known for some time that adding contributions that are quadratic in the curvature to the Einstein-Hilbert action is sufficient to obtain a renormalizable theory [74], although at the expense of producing ghost excitations in general.

Following a more phenomenological approach, higher-curvature models have been considered in different physical scenarios. One of these is cosmology, and we should mention Starobinsky's model [75], which improves the Big Bang singularity by introducing a R^2 term in the action. In the same line, it was recently shown that certain higher-curvature gravities can produce an inflationary behavior without the need to introduce the hypothetical inflaton field [76–82]. Finally, these models are relevant in holography, since they provide descriptions for dual field theories with properties that are not achievable with Einstein gravity, as we will see later.

1.1.1 General notions on higher-curvature gravities

As mentioned above, in this thesis we are mostly interested in theories of gravity whose Lagrangian contains corrections made of different contractions of the curvature tensors.[2] In the case of pure models of gravity, the action can be written in general as

$$S = \int_{\mathcal{M}} d^D x \sqrt{-g}\mathcal{L}(g_{\mu\nu},\, R^{\mu}{}_{\nu\rho\sigma})\,, \tag{1.7}$$

and they are typically dubbed "higher-curvature" theories. In Part II of this thesis we will consider also corrections made with contractions of the curvature tensors and an additional field, and we will refer to these generically as "higher-derivative" gravities. However, for now we stick to theories of the form (1.7), assuming in particular that they admit an expansion as that given in Eq. (1.6). Also, in this work we will be concerned about the first few terms of the series, since contributions of higher order would become relevant only at increasingly large energy scales.

The variation of a general action of this form was first performed in [83], and it results in

$$\delta S = \int_{\mathcal{M}} d^D x \sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \int^{\partial\mathcal{M}} d^{D-1}x \sqrt{-g}\nabla_\mu \delta v^\mu\,, \tag{1.8}$$

where the first term produces the equations of motion

$$\mathcal{E}_{\mu\nu} \equiv P_{\mu}{}^{\rho\sigma\lambda}R_{\nu\rho\sigma\lambda} - \frac{1}{2}g_{\mu\nu}\mathcal{L} - 2\nabla^\sigma\nabla^\rho P_{\mu\rho\sigma\nu} = \frac{1}{2}T_{\mu\nu}\,, \tag{1.9}$$

[2] It is possible to consider also terms with explicit covariant derivatives of the curvature tensors. In fact, we treat them in [29], where we show that they generally suffer from the appearance of ghost excitations. However, since these corrections are not relevant for the work presented in this document we do not mention them explicitly.

while the boundary term is

$$\delta v^{\mu} \equiv 2P^{\rho\sigma\mu\nu}\nabla_{\sigma}\delta g_{\nu\rho} - 2\delta g_{\nu\rho}\nabla_{\sigma}P^{\rho\mu\sigma\nu} \,. \tag{1.10}$$

These expressions are written in terms of the tensor

$$P^{\mu\nu\rho\sigma} \equiv \left.\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\right|_{g^{\alpha\beta}} , \tag{1.11}$$

which by definition inherits the symmetries of the Riemann tensor. Also, the equations of motion (1.9) also fulfill the Bianchi identity $\nabla_{\mu}\mathcal{E}^{\mu\nu} = 0$ [83]. This is not straightforward to see from the form of the equations, but it is a consequence of diffeomorphism invariance.

In principle, if the manifold has a border one should add a boundary term that cancels the one produced in the variation of the action (1.8), provided that suitable boundary conditions are imposed. For Einstein gravity this is achieved by supplementing the action with the Gibbons-Hawking-York term, written in Eq. (1.5), but this will not be valid for most higher-curvature theories. While the equivalent term is known for some particular models, for generic theories it is still an open problem, and indeed it is one of the main questions treated in Chapter 2 of this thesis.

We will deal mostly with spacetimes that have a large amount of symmetries. These are typically counted in terms of the Killing vectors ξ^{μ}, which represent directions on the manifold along which the metric remains unchanged, and fulfill the Killing equation

$$\nabla_{(\mu}\xi_{\nu)} = 0 \,. \tag{1.12}$$

Of particular interest are vacuum solutions with the maximum number of isometries or independent Killing vectors, which for a D-dimensional manifold is $D(D+1)/2$. These are known as maximally symmetric spacetimes (MSS), and their Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = \frac{\kappa}{L^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) , \tag{1.13}$$

where L and κ are related to the cosmological constant in Einstein gravity as

$$\Lambda = \kappa\frac{(D-1)(D-2)}{L^2} \,. \tag{1.14}$$

Here, L has units of length, while κ determines the global topology of the manifold. In particular, if $\kappa = 0$ the spacetime is said to be flat or Minkowski, while $\kappa = 1$ or $\kappa = -1$ correspond to de Sitter and anti-de Sitter solutions, respectively. Hence, L is usually known as the (A)dS radius. Notice that this fulfills the field equations of Einstein gravity, Eq. (1.3), with $T_{\mu\nu} = 0$. However, when higher-curvature terms are taken into account this is not exactly true, but it is enough to replace the length scale L in Eq. (1.13) with a new one, $\tilde{L}$, which will be related to that appearing in the cosmological constant with a proportionality factor that is obtained by plugging it into Eq. (1.9).

When classifying higher-curvature theories of gravity, we will be concerned primarily about static and spherically symmetric (SSS) configurations. The metric of a static spacetime does not change in time[3] and is also irrotational. In this case we want it to admit also a spherical

[3] This means that the manifold has an asymptotically timelike Killing vector, which is the defining property of a stationary spacetime. Hence, it is a special case of a stationary metric.

symmetry, and thus the metric can be written in general as

$$ds^2_{\text{SSS}} = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{(D-2)}\,, \tag{1.15}$$

where $d\Omega^2_{(D-2)}$ is the line element of a $(D-2)$-sphere of unit radius. This also describes black hole solutions, in which case there is some value of r for which $f(r) = 0$, which is known as the event horizon. It is possible to consider also spacetimes with different topologies, for which one would replace this by $d\Sigma^2_{k,(D-2)}$, where $k = -1$, 0 or +1 corresponding to a hyperbolic, flat or spherical horizon, respectively. General relativity has solutions of this form with $N(r) = 1$, or in other words $g_{tt}g_{rr} = -1$, which are usually known as single-function black holes, since the entire solution is determined by $f(r)$. Therefore, it is possible to classify higher-curvature theories of gravity depending on whether they admit such solutions or not.

A general higher-curvature theory will not admit single-function solutions of this form (this is, with constant $N(r)$). Even if it does, it might happen that the equation of motion of $f(r)$ is up to fourth order in derivatives, due to the term $\nabla^\sigma\nabla^\rho P_{\mu\rho\sigma\nu}$ in Eq. (1.9), which would mean that it is impossible to obtain the solution for $f(r)$ without specifying additional boundary conditions. Besides, that same term in the general equations of motion (1.9) points to another important problem that higher-curvature gravities typically possess: they propagate ghost excitations on MSS backgrounds. This is a consequence of Ostrogradsky's theorem [84, 85], which implies that the Hamiltonian of any system whose equations of motion are more than second order in derivatives will present instabilities, which in this case manifest as these negative-energy modes. Indeed, for any theory of the form (1.7) these can be found explicitly using the linearization procedure for MSS backgrounds introduced in [3, 86].

While these are generic features of such theories, they can be avoided for certain combinations of the higher-curvature corrections. In this thesis we will focus on those belonging to the recently proposed family known as "Generalized Quasitopological gravities" [3, 6], which we will introduce shortly. However, before that let us discuss a few other models that are relevant in this regard.

1.1.2 Early examples of higher-curvature theories

1.1.2.1 $f(R)$ gravity

The simplest possible corrections to the Einstein-Hilbert model are those given by $f(R)$ theories (see the reviews [87–89]). In this case, the gravitational part of the action takes the form

$$S_{f(R)} = \frac{1}{16\pi G_{\text{N}}}\int_{\mathcal{M}} d^D x\,\sqrt{-g}\,f(R)\,, \tag{1.16}$$

where $f(R)$ is some smooth scalar function involving only the scalar curvature R. The equations of motion of such a theory can be obtained explicitly, and read [87]

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right]f'(R) = 8\pi G_{\text{N}}T_{\mu\nu}\,, \tag{1.17}$$

where $T_{\mu\nu}$ is the stress-energy tensor associated with some matter fields that could appear in the action. Due to the derivative operators between brackets, it is clear that this equation is fourth order in derivatives of the metric, which in principle implies the appearance of the negative-energy excitations.

In this case, however, it is possible to get rid of these instabilities. For that let us introduce a new scalar field ϕ, and rewrite the action in terms of it as [89]

$$S_{f(R)} = \frac{1}{16\pi G_{\mathrm{N}}} \int_{\mathcal{M}} d^D x \sqrt{-g} \left[\phi R - V(\phi)\right] , \qquad V(\phi) = \psi(\phi)\phi - f(\psi(\phi)) , \tag{1.18}$$

where ψ is given implicitly by $\phi = f'(\psi)$. The transformation is valid as long as $f''(R) \neq 0$, and leaves us with a scalar-tensor theory of the Brans-Dicke type [90]. This means that the pure gravity action (1.16) can be translated into an equivalent model containing the usual massless graviton of GR, plus an additional scalar field ψ, thus avoiding the appearance of the ghost excitations in the gravitational sector.

Because of this, $f(R)$ models have been extensively considered in particular for studies of cosmology, for example for explaining inflation or the current accelerated expansion of the Universe (see again [87, 88] for reviews of the topic). The counterpart of this is that, since the modified behavior is "absorbed" by the scalar degree of freedom, these theories do not introduce new gravitational dynamics, and in fact, all vacuum solutions of the equations of general relativity also solve the equations of motion of $f(R)$ gravities. Therefore, while these theories are simple to study and can be useful for certain setups, they are not interesting for our purposes.

1.1.2.2 Lanczos-Lovelock theories

The other extension of Einstein gravity that we would like to talk about are the Lanczos-Lovelock, or simply Lovelock, theories [1, 2] (see also [91]). For a spacetime with dimension D, the action is given by

$$S_{\mathrm{Lovelock}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_{\mathcal{M}} d^D x \sqrt{-g} \sum_{k=0}^{\lfloor D/2 \rfloor} \alpha_n \ell^{2n-2} \chi_{2n} , \tag{1.19}$$

where $\{\alpha_n\}$ are some coupling constants, ℓ is a length scale and $\lfloor x \rfloor$ is the floor operator, which produces the largest integer that is smaller than or equal to x. The scalars χ_{2n} are the dimensionally continued Euler densities, given by

$$\chi_{2n} = \frac{1}{2^n} \delta^{\mu_1 \cdots \mu_{2n}}_{\nu_1 \cdots \nu_{2n}} R^{\nu_1 \nu_2}{}_{\mu_1 \mu_2} \cdots R^{\nu_{2n-1} \nu_{2n}}{}_{\mu_{2n-1} \mu_{2n}} . \tag{1.20}$$

Due to the antisymmetry in the generalized Kronecker delta, it is clear that χ_{2n} vanishes trivially for $D < 2n$, while at the particular dimension $D = 2n$ it is a topological quantity by virtue of the Chern-Gauss-Bonnet theorem, which does not modify the equations of motion. This is manifest in the upper limit of the sum in Eq. (1.19).

This theory is a direct generalization of general relativity, in the sense that the lowest order terms produce the Einstein-Hilbert Lagrangian. In fact, by convention we consider $\chi_0 = 1$, corresponding to the cosmological constant term in Eq. (1.2), and it is possible to compute $\chi_2 = R$. The next term in the series is quadratic in the curvature, and reads

$$\chi_4 = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} . \tag{1.21}$$

This is usually known as the Gauss-Bonnet density, and has been extensively studied in the literature. However, as mentioned above it is topological in $D = 4$, and one would have to consider spacetimes with more dimensions in order for it to become non-trivial.

The equations of motion obtained from the action (1.19) read

$$\sum_{n=0}^{\lfloor D/2\rfloor} \alpha_n \ell^{2n-2}\mathcal{E}_{\mu\nu}^{(n)} = 0\,, \qquad \mathcal{E}_{\mu\nu}^{(n)} = -\frac{1}{2^{n+1}} g_{\mu\lambda}\delta^{\lambda\mu_1\cdots\mu_{2n}}_{\nu\nu_1\cdots\nu_{2n}} R^{\nu_1\nu_2}{}_{\mu_1\mu_2}\cdots R^{\nu_{2n-1}\nu_{2n}}{}_{\mu_{2n-1}\mu_{2n}}\,, \tag{1.22}$$

where we can see clearly that the term of order $2n = D$, whose action is topological, does not contribute to the equations of motion. However, the most important aspect of these expressionw is that they are algebraic in the curvature tensors, and thus these theories produce second order equations of motion for any metric. From the discussion before, this means that these models do not suffer from the Ostrogradsky instability, and thus propagate the same degrees of freedom as GR. In fact, Lovelock's theorem states that these are the most general theories with this feature that can be constructed, implying that Einstein gravity is the only non-trivial theory of gravity in 4 dimensions with second order equations of motion for any background.

Finally, let us mention that the equivalent to the Gibbons-Hawking-York boundary term is known for these particular theories [92, 93]. Also, contrary to what we discussed for $f(R)$ models, Lovelock gravities do modify the solutions of the field equations of GR, and the form of different metrics such as black holes have been obtained (see e.g. [94–97]). Besides, certain Lovelock terms have been found explicitly in EFT expansions of string theory (as is the case for Gauss-Bonnet [98, 99]), which makes them interesting from the point of view of the AdS/CFT correspondence [100–106].

1.1.3 (Generalized) Quasitopological gravities

One consequence of Lovelock's theorem [1, 2], as already stated, is the fact that the only non-trivial theory of gravity that has second order equations of motion in $D = 4$ is Einstein's general relativity. This means that we need to drop that requirement if we want to find a higher-curvature theory that modifies the gravitational dynamics in four spacetime dimensions, but we can still think about enforcing it only in some particular setups.

This reasoning led to the discovery of Quasitopological gravity [107, 108], which is a cubic theory in $D \geq 5$ that has second order traced field equations and propagates only the massless graviton around maximally symmetric backgrounds. For general D, its Lagrangian is given by the combination

$$\begin{aligned}\mathcal{Z}_D = {}& R_{\mu\ \nu}^{\ \rho\ \sigma} R_{\rho\ \sigma}^{\ \alpha\ \beta} R_{\alpha\ \beta}^{\ \mu\ \nu} + \frac{1}{(2D-3)(D-4)}\left(\frac{3(3D-8)}{8} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R - 3(D-2)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\alpha}R^{\sigma\alpha}\right.\\ &\left. + 3DR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 6(D-2)R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} - \frac{3(3D-4)}{2}R_{\mu}{}^{\nu}R_{\nu}{}^{\mu}R + \frac{3D}{8}R^3\right).\end{aligned} \tag{1.23}$$

Besides those mentioned above, the most important feature of this theory is that it admits static black hole solutions with different horizon topologies characterized by one single function, whose equation of motion is algebraic. The same construction has been generalized to quartic [109] and quintic [110] orders in the curvature, thus it is natural to define the family of "Quasitopological gravities" (QTGs) as those satisfying the same set of properties. However, as appealing as they might be, QTGs have the same problem as Lovelock theories for our purposes: they do not modify the gravitational dynamics in $D = 4$.

More recently, a new type of theories sharing some of the same features have been identified. These are dubbed "Generalized Quasitopological gravities" (GQGs), and the main difference with QTGs is that the equation of motion for the function $f(r)$ in the black hole metric can be of second order instead of algebraic. Also, they provide non-trivial corrections in four dimensions. This is therefore a much broader family than that of Lovelock or GQG theories, which are of course contained within it.

The first Lagrangian belonging to this class was found in [3] through the analysis of metric perturbations on a maximally symmetric background. Indeed, the authors developed a procedure to compute the spectrum of propagating modes for any $\mathcal{L}(R^{\mu\nu}_{\rho\sigma})$ theory on such a background (see also [86]), and applied it to the most general theory built with cubic contractions of the Riemann tensor. Alongside the usual massless graviton, this unveiled the existence of two massive modes: a ghost graviton and a scalar, whose masses depend on the relative couplings of the cubic scalars. Therefore, it is possible to fine-tune these constants to make the masses of the undesired additional modes infinitely large, which prevents their propagation. This results in six different linearly independent combinations, which reduce to only two after imposing that their coefficients do not depend on the spacetime dimension. One of these is cubic Lovelock (which results from setting $n = 3$ in Eq. (1.20), and is trivial for $D < 7$), while the other is a novel Lagrangian, dubbed "Einsteinian cubic gravity," that reads

$$\mathcal{P} = 8R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} - 12R^{\mu\rho}R^{\nu\sigma}R_{\mu\nu\rho\sigma} + 12R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta}R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu} + R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} \,. \tag{1.24}$$

The main attractive of this theory is the fact that it produces non-trivial contributions in the equations of motion in $D = 4$, while preserving the spectrum of Einstein gravity. Shortly after the discovery of this theory it was found that it admits simple generalizations of the Schwarzschild black hole, and also static solutions with planar and hyperbolic horizons [4, 5]. These are determined by a single function $f(r)$ whose form is given by a second order equation of motion, while the thermodynamic properties of the solution are accessible by solving a system of algebraic equations.

Later, it was proved that the requirements that the theory admits single-function black hole solutions and has Einstein spectrum are equivalent [7]. This results in a more powerful and straightforward procedure to identify theories of this kind, which we outline now. For this it is necessary to introduce the reduced Lagrangian $L_{N,f}$, which is nothing but the original higher-curvature Lagrangian multiplied by $\sqrt{-g}$ and evaluated on the SSS metric (1.15),

$$L_{N,f}(r,\, f,\, N,\, f',\, N',\, ...) \equiv N(r)r^{D-2}\mathcal{L}|_{N,f} \,. \tag{1.25}$$

Also, we will denote by L_f this reduced Lagrangian after setting $N(r) = 1$. Using the chain rule, it is possible to show that the equations of motion (1.9) on the background (1.15) are equal to combinations of the Euler-Lagrange equations of $L_{N,f}$ with respect to each function, as[4]

$$\frac{1}{r^{D-2}}\frac{\delta \mathcal{L}_{N,f}}{\delta N} = \frac{2\mathcal{E}_{tt}}{N^2 f} \,, \qquad \frac{1}{r^{D-2}}\frac{\delta \mathcal{L}_{N,f}}{\delta f} = \frac{\mathcal{E}_{tt}}{N f^2} + N\mathcal{E}_{rr} \,. \tag{1.27}$$

[4] As usual, the equations of motion for a general gravitational theory are defined as

$$\mathcal{E}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\mu\nu}} \,. \tag{1.26}$$

The fact that these relations hold is a consequence of the symmetries of the spacetime, and this procedure is not valid for general metrics. In this case we only have two unknown functions, $N(r)$ and $f(r)$, so the solution must be determined by two equations of motion. In fact, using the Bianchi identity $\nabla^\mu \mathcal{E}_{\mu\nu} = 0$ it can be shown that the angular components of $\mathcal{E}_{\mu\nu}$ are satisfied whenever $\mathcal{E}_{tt} = \mathcal{E}_{rr} = 0$, while the off-diagonal components of $\mathcal{E}_{\mu\nu}$ are identically zero due to the symmetries of the metric.[5]

As shown in [7], the condition for the theory to belong to the GQG class is simply

$$\frac{\delta L_f}{\delta f} \equiv \frac{\partial L_f}{\partial f} - \frac{d}{dr}\frac{\partial L_f}{\partial f'} + \frac{d^2}{dr^2}\frac{\partial L_f}{\partial f''} - \ldots = 0\,, \quad \forall\, f(r)\,. \tag{1.28}$$

This means that the Euler-Lagrange equation for L_f vanishes identically for any form of $f(r)$, implying that L_f is a total derivative, $L_f = L_1'$. Therefore, the reduced Lagrangian can be expanded as [7]

$$L_{N,f} = N(r)L_1' + N'(r)L_2 + N''(r)L_3 + \mathcal{O}\left(N'^2/N\right)\,, \tag{1.29}$$

where L_1, L_2 and L_3 depend only on the function $f(r)$ and its derivatives. After integrating by parts, the corresponding action reads

$$S_{N,f} = \Omega_{D-2}\int dt \int dr \left[N(r)\left(L_1 - L_2 + L_3'\right)' + \mathcal{O}\left(N'^2/N\right)\right]\,, \tag{1.30}$$

in addition to some boundary terms. The equation of motion for $f(r)$ is obtained by taking the variation of this with respect to $N(r)$ and then setting $N(r) = 1$. From the current form of the action it is clear that this equation is a total derivative, which after integrating once reads

$$L_1 - L_2 + L_3' = C\,, \tag{1.31}$$

where C is a constant that turns out to be proportional to the ADM mass of the solution [111–114]. Therefore, obtaining the equation of motion for $f(r)$ in a GQG is greatly simplified by this formalism, since it is enough to evaluate the reduced Lagrangian $L_{N,f}$ and then identify L_1, L_2 and L_3 through Eq. (1.29).

As argued in [7], Eq. (1.31) will involve at most the first two derivatives of $f(r)$. This is indeed the defining property of the Generalized Quasitopological gravities,[6] and means that it admits black hole solutions that are characterized only by the ADM mass. The actual form of $f(r)$ can be obtained by employing numerical methods, and it is always a continuous deformation of the equivalent solution in GR, which means that the theory has a well-defined Einstein gravity limit (this is, the limit in which the couplings of the higher-curvature terms are set to zero). Besides, the thermodynamic properties of these metrics are accessible through a system of algebraic equations [4–8,115], which make these GQGs very appealing for holographic setups.

5 A discussion of this point for $D = 4$, but that can easily be extended to general dimensions, can be found at Appendix B of [23].

6 Quasitopological gravity theories also belong to the Generalized Quasitopological family, and they correspond to the limiting case in which Eq. (1.31) is algebraic in $f(r)$, this is, it does not involve derivatives of this function. It is conjectured [7] that for these theories the reduced Lagrangian takes the exact form

$$L_{N,f} = N(r)L_1' + N'(r)L_2 + N''(r)L_3\,, \tag{1.32}$$

following the same notation as before.

Based on this construction, theories belonging to this class can be obtained for any order and dimension [10, 11]. In fact, for $D \geq 5$ there exist $n-1$ inequivalent densities of order n in the curvature, and only one for $D = 4$. It was also shown [9, 10] that any higher-curvature gravity theory can be mapped to a GQG via field redefinitions of the metric tensor, which means that these models are able to capture the effects of general higher-curvature terms in gravitational dynamics.

Throughout much of this work we will focus on this kind of theories, or extensions thereof. In particular, we will be interested mostly in the simplest non-trivial correction of Einstein gravity in $D = 4$. This is given by Einsteinian cubic gravity, whose Lagrangian density is written in Eq. (1.24), but also the cubic scalars $\mathcal{C}$ and $\mathcal{C}'$ identified later in [6]. These also belong to the GQG family, and read

$$\mathcal{C} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\alpha}R^{\sigma\alpha} - \frac{1}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R - 2R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\rho} + \frac{1}{2}R_{\mu}{}^{\nu}R_{\nu}{}^{\mu}R\,, \tag{1.33}$$

$$\mathcal{C}' = R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} - \frac{3}{4}R_{\mu}{}^{\nu}R_{\nu}{}^{\mu}R + \frac{1}{8}R^3\,, \tag{1.34}$$

and it can be checked that they fulfill the relation

$$4\left(\mathcal{C} - 2\mathcal{C}'\right) = \epsilon_{\mu\nu\rho\sigma\alpha}\epsilon^{\beta\gamma\delta\kappa\lambda}R^{\mu}{}_{\beta}{}^{\nu}{}_{\gamma}R^{\rho}{}_{\delta}{}^{\sigma}{}_{\kappa}R^{\alpha}{}_{\lambda}\,. \tag{1.35}$$

Notice that for $D = 4$ the Levi-Civita tensor with five indices is identically zero, so $\mathcal{C} = 2\mathcal{C}'$, meaning that in this particular case it is enough to consider only one of these Lagrangian densities.

The scalars $\mathcal{C}$ and $\mathcal{C}'$ were not studied at first, since they do not modify the equations of motion of the SSS metric given in Eq. (1.15). However, they gained relevance when these theories were considered in the cosmological setup. This was treated for the first time in [76], where it was found that the particular combination $\mathcal{P} - 8\mathcal{C}$ produces second order equations of motion also when evaluated on the Friedmann-Lemaître-Robertson-Walker metric, which is time-dependent and thus not stationary. Therefore, the evolution of the scale factor of the universe can be computed without supplying any additional initial condition at the Big Bang, and indeed it was found that this model can produce an exponential expansion of the universe without the need to introduce the inflaton field. Another point in favor of considering the density $\mathcal{P} - 8\mathcal{C}$ is the fact that it can be derived from 5-dimensional Quasitopological gravity, as shown in [77]. The construction was generalized afterwards by including corrections with arbitrary powers of the curvature tensors [78, 80], and the inflaton field was reintroduced in [81, 82]. This, however, suffers from a series of problems, the most important being those arising in the study of perturbations [82, 116]. In any case, for consistency with these results, and even though this will not affect most of our results, in this work we will work with the cosmological version of ECG, $\mathcal{P} - 8\mathcal{C}$.

Due to their interesting properties, Generalized Quasitopological gravities attracted a lot of attention in the last few years, and have been considered in the study of a wide array of problems, most of them related to holography and black holes [117–140]. However, in a recent paper [141] it was argued that these theories should only be trusted in the EFT regime, which is the setup in which they were originally proposed. In any case, they provide very useful models to study the effects of higher-curvature corrections to different problems.

1.2 Black hole thermodynamics

Black holes are one of the most remarkable and intriguing predictions of general relativity. They are asymptotically flat solutions[7] that contain an "event horizon," a null hypersurface which encloses a region that is not in the backward-directed lightcone of future timelike infinity. In more layman terms, no causal signal can come out from behind the horizon and reach an observer sitting infinitely far away.

Let us introduce also the concept of the Killing horizon, which is a null hypersurface Σ_h whose normal vector ξ is also a Killing vector of the spacetime. This is interesting for us, since the event horizon of a stationary and asymptotically symmetric black hole is typically also a Killing horizon.[8] Since $\xi_\mu \xi^\mu = 0$ on the hypersurface, its gradient $\nabla_\mu(\xi \cdot \xi)$ must be normal to it, and thus proportional to ξ^μ itself. Taking into account the Killing equation (1.12), this can be stated as

$$\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu |_{\Sigma_h} , \tag{1.36}$$

where the proportionality factor κ is known as the surface gravity, and is constant at the horizon (a proof can be found in Section 12.5 of [142]). Its value is equal to the acceleration that an observer would experience while falling through the horizon of a static and asymptotically flat black hole, assuming that the Killing vector is normalized as $\xi_\mu \xi^\mu = -1$ at infinity. Also, we will see later that this quantity is proportional to the Hawking temperature of the black hole.

A black hole in general relativity is fully characterized by only three conserved quantities: mass, electric charge and angular momentum, and also magnetic charge if it were found to exist. This idea is commonly referred to as the "no-hair conjecture," and poses quite a few challenges when viewing the black hole as a quantum object, as it implies that the information of the matter that formed it was somehow lost. However, this is very advantageous when constructing the actual solutions in GR, as they are fairly simple. Particularly relevant are the static and spherically symmetric neutral solutions of the field equations, which is given for general dimensions by Eq. (1.15) with $N(r) = N_0$ ($N_0 = 1$ for asymptotically flat spacetimes) and

$$f(r) = 1 - \frac{16\pi G_{\rm N} M}{(D-2)\Omega_{(D-2)} r^{D-3}} - \frac{2\Lambda r^2}{(D-1)(D-2)} , \quad \text{where} \quad \Omega_{(D-2)} = \frac{2\pi^{(D-1)/2}}{\Gamma\left[(D-1)/2\right]} . \tag{1.37}$$

This is known as the Schwarzschild-Tangherlini solution, and is fully determined by the ADM mass M. Besides, it will have different asymptotics depending on the sign of the cosmological constant Λ, as explained below Eq. (1.14).

The form of this solution will change due to the higher-curvature corrections that we are interested in, and indeed in order to find the form of $f(r)$ for Generalized Quasitopological theories one generally needs to rely on numerical techniques. Trying to add a non-vanishing angular momentum for theories of this kind is usually a rather complex task, and such solutions could only be found in some approximate regimes, such as slow rotation and small coupling constants [125, 127, 131]. Therefore, in this thesis we will stick to static solutions, which should be enough to identify different effects of higher-curvature terms.

7 Although not trivial, most of the intuition and results presented here can be extended to (A)dS black holes, which are actually of relevance in a large part of this thesis.

8 We should stress out the fact that the Killing and event horizon are two different concepts, and they are not equal for general metrics. A discussion on this point can be found at Section 6.3 of [33].

1.2.1 The laws of black hole mechanics

Black holes are now understood as thermodynamic systems that have a macroscopic temperature and thus radiate part of their mass away. While this is by no means measured or observed, it is widely accepted due to the amount of theoretical clues and arguments in this regard. In order to understand where this association comes from, let us start by summing up the four laws of black hole mechanics [143].

- Zeroth law: The surface gravity κ is constant over the event horizon of a stationary black hole.

- First law: If a stationary black hole of mass M, charge Q and angular momentum J is perturbed and settles down in such a way that these quantities change respectively in δM, δQ and δJ, then the change in the mass must be given by

$$\delta M = \frac{\kappa}{8\pi G_{\rm N}}\delta A_H + \Phi_H \delta Q + \Omega_H \delta J \,, \tag{1.38}$$

  where A_H, Φ_H and Ω_H are respectively the area, electrostatic potential and angular velocity of the horizon.

- Second law: The area of the event horizon can not decrease in any physical process,

$$\Delta A_H \geq 0 \,. \tag{1.39}$$

- Third law: It is not possible, by any physical process, to reduce κ to zero in a finite sequence of operations.[9]

These resemble very closely the usual four laws of thermodynamics, and in particular point towards the identification of the surface gravity with a temperature, and the area of the horizon with some notion of entropy.

The actual expression of the former was derived by Hawking, who studied semiclassically the behavior of quantum fields in the vicinity of an event horizon [144]. It reads

$$T_{\rm BH} = \frac{\kappa}{2\pi} \,, \tag{1.40}$$

and corresponds to the thermal distribution of particles emitted by the black hole that an asymptotic observer would measure. This expression is valid for any metric, with the only unknown parameter being the surface gravity κ. For a SSS solution as the one given in Eq. (1.15) this takes the value $\kappa = f'(r_h)/2$, where r_h is the horizon radius, $f(r_h) = 0$. Then the temperature can be obtained simply by evaluating

$$T_{\rm BH} = \frac{f'(r_h)}{4\pi} \,, \tag{1.41}$$

which will prove useful throughout the following Chapters.

The first law of black hole mechanics, as written in Eq. (1.38), can be very easily compared to the first law of thermodynamics $dE = TdS + \ldots$. This is consistent with the identification of the

9 This law is only proposed as a conjecture in [143].

horizon area with an entropy, and the actual proportionality constant can be obtained by taking into account Eq. (1.40), thus finding the formula for the Bekenstein-Hawking entropy [145, 146],

$$S_{\rm BH} = \frac{A_H}{4G_{\rm N}}\,. \tag{1.42}$$

While it might seem surprising at first, the fact that black holes themselves carry entropy is required in order for the second law of thermodynamics to not be violated as external matter falls through the horizon. However, this poses an interesting problem, since we do not know what set of microscopic states can provide this entropy. This has been an open question for a few decades now, and one would expect a full quantum theory of gravity to answer this, which is another argument in favor of studying black holes with higher-curvature corrections.

Let us make one final comment about the Bekenstein-Hawking entropy formula. Since the entropy is a measure of the degrees of freedom in a given region, one would expect it to scale with its volume, as usually happens in quantum field theory. In this case, however, it is proportional to the area of the horizon, suggesting that the dynamics inside the horizon should be described in terms of some degrees of freedom located at the boundary, which might actually be a general feature of gravity theories [147, 148]. This is known as the "holographic principle" (see e.g. [149] for a review), and we will come back to it in Section 1.3, as it is one of the foundations on which the AdS/CFT correspondence stands.

1.2.2 Black hole thermodynamics in higher-derivative gravities

A natural question that arises when considering higher-curvature terms is whether the laws of black hole mechanics are satisfied. It turns out that the first law is not directly fulfilled as written in Eq. (1.38), since the derivation of that expression relies on the particular form of Einstein's equations [150, 151]. Therefore, in order to reformulate it we need to find more general expressions for the thermodynamics quantities.

The horizon temperature can be computed in different theory-agnostic manners, which only require the existence of the horizon, and one always arrives at Hawking's result given in Eq. (1.40). Besides, it has been shown that the surface gravity is constant on the horizon of any static or axisymmetric black hole with purely geometrical arguments [151–153], so we conclude that the zeroth law of black hole mechanics holds in general HCGs, at least for this kind of solutions.

Therefore, the only way to obtain a relation equal to the first law of thermodynamics is that the entropy is not proportional to the area of the horizon in general, but a more complicated function of the parameters of the black hole. The existence of such a first law was shown by Wald [154], and it is a consequence of the diffeomorphism invariance of the theory. It replaces Eq. (1.38) by

$$\delta M = \frac{\kappa}{2\pi}\delta S_{\rm Wald} + \Phi_H \delta Q + \Omega_H \delta J\,, \tag{1.43}$$

and $S_{\rm Wald}$ is given by [150, 154, 155]

$$S_{\rm Wald} = -2\pi \int_{\Sigma_h} d^{D-2}x\sqrt{h}\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{1.44}$$

where h is the determinant of the induced metric at the bifurcation surface of the horizon, Σ_h, and $\epsilon_{\mu\nu}$ is its binormal, which is antisymmetric and normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. The functional derivative is computed by considering the Riemann tensor as a field independent to the metric. For a general Lagrangian that can contain also explicit covariant derivatives of the curvature, it is computed as

$$\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial (\nabla_\alpha R_{\mu\nu\rho\sigma})} \right) + \dots \,, \tag{1.45}$$

but for our purposes only the first term is relevant. Of course, if we plugged in the Einstein-Hilbert Lagrangian $\mathcal{L} = R/(16\pi G_\mathrm{N})$ we would recover the Bekenstein-Hawking result given in Eq. (1.42).

Let us close this discussion with a short comment on the second law. In general relativity it states that the area of the horizon is monotonically increasing, which is a consequence of Hawking's area theorem [156]. This however does not take into account the process of evaporation, and in that case one should formulate the second law in terms of a generalized entropy $S_\mathrm{gen} = S_\mathrm{BH} + S_\mathrm{out}$, where S_out is the contribution corresponding to the quantum fields outside the horizon (see [157] for a review). For higher-curvature theories we would expect a modification of the first law, also implying the monotonicity of some entropy evaluated at the bifurcation surface of the horizon. One possibility is that proposed by [158–160], which considers the Camps-Dong functional introduced in [161, 162] to construct $S_\mathrm{gen} = S_\mathrm{Camps\text{-}Dong} + S_\mathrm{out}$. This quantity is shown to be monotonically increasing, $\Delta S_\mathrm{gen} > 0$, for small perturbations of the metric around stationary black holes. Therefore, since we are only interested on this type of solutions, we can conclude that Wald's formula (1.44) is enough for our purposes.

1.3 Holography and higher-derivative gravities

The other foundational pillar upon which a significant portion of this thesis is built is the gauge/gravity duality, which is a relation between the dynamics of a strongly-correlated quantum system and a classical theory of gravity with one dimension more [12–14]. This is perhaps the most fruitful manifestation within the vast net of dualities inherent to string theory, and has focused the efforts of many theoretical physicists since it was first proposed more than two decades ago.

Moreover, since the degrees of freedom of the quantum theory are localized at the boundary of some manifold in the gravity side, this duality aligns with the holographic principle mentioned at the end of Section 1.2.1. Because of this, it is often called the "holographic duality." In this Section we will introduce some basic concepts behind it,[10] and comment on the expected consequences of adding higher-derivative terms in the gravitational action.

1.3.1 The gauge/gravity duality

The most important embodiment of the gauge/gravity duality is known as the "AdS/CFT correspondence," which establishes connections between the dynamics of a theory of quantum gravity in a $(d+1)$-dimensional anti-de Sitter spacetime and those of a conformal field theory that resides in its d-dimensional boundary. Nonetheless, this duality is believed to be more

[10] Some detailed reviews of the holographic duality can be found at [163–169], while other references focusing on its application to condensed matter physics are [170–172].

general, linking different pairs of theories that might not necessarily come from string theory, and hence both denominations of the correspondence are often used interchangeably.

The original formulation of the duality was proposed by Maldacena in [12], and it states an equivalence between Type IIB string theory on an $\mathrm{AdS}_5 \times \mathbb{S}^5$ spacetime and 4-dimensional Super Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry and gauge group $\mathrm{SU}(N)$.[11] In the string theory side, N is the number of units of flux of a 5-form on the $\mathbb{S}^5$, we denote by L the radius of both the AdS_5 and $\mathbb{S}^5$ spacetimes, and there are two additional constants: the string length $l_s = \sqrt{\alpha'}$ and the coupling g_s. On the other hand, in the QFT side the only free parameters are the coupling g_{YM} and the degree of the gauge group N. While N appears explicitly in both theories, the different constants are related as[12]

$$g_{\mathrm{YM}}^2 = 2\pi g_s \,, \qquad 2 g_{\mathrm{YM}}^2 N = \frac{L^4}{\alpha'^2} \,. \tag{1.46}$$

The backbone of the duality is the equality between the two partition functions,

$$\mathcal{Z}_{\mathrm{CFT}} = \mathcal{Z}_{\mathrm{string}} \,, \tag{1.47}$$

which is conjectured to hold independently of the values of the parameters. However, we intend to focus on a regime that is actually tractable. For that, let us start by defining the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$. As a first step we choose $g_s \to 0$ to suppress contributions from string diagrams with loops. In view of Eq. (1.46) this implies that $g_{\mathrm{YM}} \to 0$, but we do so keeping the 't Hooft coupling λ constant, so $N \gg 1$ and the SYM theory has a large number of degrees of freedom. Then we go to the strong-coupling limit of the gauge theory, $\lambda \to \infty$, which means that the length of the string becomes negligible, as $L^4/\alpha'^2 \to \infty$. Therefore, the string theory reduces to classical gravity in an AdS_5 background, which is under control.

The partition function of the string theory is greatly simplified in this case by means of the saddle-point approximation, which takes into account the fact that in the classical regime quantum corrections are fully suppressed in the path integral. Therefore, the equality between path integrals reads

$$\mathcal{Z}_{\mathrm{CFT}}\big|_{N\to\infty,\,\lambda\to\infty} \simeq e^{-S_{\mathrm{E,\,gravity}}} \,, \tag{1.48}$$

where $S_{\mathrm{E,\,gravity}}$ is the Euclidean action of classical (super)gravity evaluated on the solution of its equations of motion.

For our purposes, the original duality presented until now serves as a motivation for a more ambitious form of the AdS/CFT correspondence. Indeed, the holographic duality between classical gravity in AdS spaces and highly-quantum field theories, in the form written in Eq. (1.48), is now regarded as a general feature of these theories. Therefore, it is possible to consider theories with different dimensions, field content, black holes in the bulk of AdS, etc. In general cases we do not know the exact form of the dual QFT, but as before we expect it to be strongly coupled and have a large number of degrees of freedom, and we can learn about it by studying the dynamics of the fields in the AdS spacetime. This is in fact the philosophy followed in this thesis and a large variety of other works, and is interesting for several reasons. For instance,

[11] In this case we are relating a 4-dimensional QFT to a gravity theory in 10 dimensions. This contrasts with the previous description of the AdS/CFT correspondence, which implies that the gravitational theory should have only one additional dimension. However, the 5 compact dimensions on the sphere are reduced via a Kaluza-Klein procedure, producing an infinite tower of massive modes, and thus the dual spacetime is effectively AdS_5.

[12] A pedagogical review of the two theories involved in this duality can be found in [167].

the gauge/gravity duality is a powerful computational tool, since it allows us to easily compute quantities in the CFT in a regime that is not accessible through the standard perturbative methods, and it can also be used to capture generic features of such conformal field theories.

In order to actually perform computations using the gauge/gravity duality, we need to establish a set of equivalences between quantities in both sides, which is commonly known as the "holographic dictionary." These are derived from the equivalence of the partition functions, Eq. (1.48), and in order to illustrate this we outline now the computation of expectation values and correlators of fields in the boundary.

We denote collectively by $\varphi(x)$ the fields in the CFT, while $\psi(x)|_{\partial\text{AdS}}$ are those living in the bulk of AdS. These are integrated subject to certain conditions at the boundary, which are identified with the values of the fields of the dual CFT. According to the saddle-point approximation, the partition function of the CFT in the large-N and strong coupling limits is given by

$$\mathcal{Z}_{\text{CFT}}[\varphi(x)]\big|_{N\to\infty,\,\lambda\to\infty} \simeq \exp\left\{-S_{\text{E}}^{\text{ren}}[\psi_0(x)|_{\partial\text{AdS}}]\right\}, \tag{1.49}$$

where $S_{\text{E}}^{\text{ren}}[\psi(x)|_{\partial\text{AdS}}]$ is the classical action of Euclidean gravity, properly renormalized in order to cancel the divergences due to the infinite volume of the bulk near the boundary, which correspond to short-distance divergences in the dual CFT (see e.g. [173] for a review on holographic renormalization). The action is evaluated on the field configuration $\psi_0(x)|_{\partial\text{AdS}}$, which is a solution of the classical equations of motion with the boundary behavior[13]

$$\lim_{z\to 0} z^{\Delta-d}\psi_0(z,\,x) = \varphi(x)\,, \tag{1.50}$$

for some constant Δ. So we see explicitly how the CFT field $\varphi(x)$ is related to the asymptotic value of the bulk fields at the conformal boundary. In the CFT action, $\varphi(x)$ would appear together with an operator $\mathcal{O}(x)$, in the schematic form $\int d^dx\varphi(x)\mathcal{O}(x)$. Hence, $\varphi(x)$ can be interpreted as the source for that operator, and Δ corresponds to its mass scaling dimension.

Let us consider now a general n-point function of the operator $\mathcal{O}(x)$. As usual, this can be computed by taking functional derivatives of its corresponding generating functional with respect to the source and then setting said source to zero. By virtue of Eq. (1.49), for a holographic theory this becomes

$$\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = -\left.\frac{\delta^n S_{\text{E, ren}}^{\text{on-shell}}[\psi_0|_{\partial\text{AdS}}]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}\right|_{\varphi=0}. \tag{1.51}$$

But $\varphi(x)$ is nothing but the (regularized) boundary value of a bulk field, as given in Eq. (1.50). Thus, we see how this CFT correlator can be computed uniquely from quantities in the bulk, and by considering different field contents in the bulk (such as scalars, vectors or even perturbations of the metric), it is possible to compute correlators of the corresponding fields in the CFT side.

We can also think about modifying the metric in the gravity theory, and particularly interesting is the case of an asymptotically AdS spacetime containing an event horizon. As we argued in Section 1.2, black holes are thermodynamic systems, in the sense that they have an associated temperature and entropy that fulfill the usual laws of thermodynamics. In the holographic setup, this spacetime is dual to a thermal state in the boundary theory, whose temperature is given by Hawking's result in Eq. (1.40), while its von Neumann entropy is equal to the Bekenstein-Hawking (or Wald, in HCGs) entropy of the horizon.

[13] Customarily, z is the coordinate corresponding to the direction normal to the asymptotic boundary of AdS, and it is defined in such a way that this boundary is located at $z = 0$.

This last identification is intimately related to the proposal by Ryu and Takayanagi to compute entanglement entropies of holographic CFTs [174, 175]. Loosely speaking, the entanglement entropy of a region A in a QFT, $S_{\text{EE}}(A)$, is a measure of the amount of entanglement between that region and its complementary $\bar{A}$. In the language of quantum mechanics this could, in principle, be computed in terms of the matrix density of the system. However, for holographic theories in the highly-quantum regimes, it can be evaluated by means of the Ryu-Takayanagi formula[14]

$$S_{\text{EE}}(A) = \frac{\text{Area}(\Gamma_A)}{4G_{\text{N}}}\,, \tag{1.52}$$

where Γ_A is a hypersurface with a minimal area that penetrates into the bulk and ends in ∂A. If we choose A to cover the entire boundary, then the homologous surface Γ_A would be nothing but the event horizon of the black hole, thus recovering the result for the Bekenstein-Hawking entropy given in Eq. (1.42).

The Ryu-Takayanagi proposal is considered to be one of the most important achievements of the gauge/gravity duality, as it directly relates concepts of quantum information to the geometry of a spacetime in a rather unexpected manner. The prescription has been extended to take into account also the entropy of quantum fields that may be present in the bulk [176, 177], and this in turn has been explored to tackle the problem of loss of information during black hole evaporation [178, 179], which has attracted a lot of attention in the recent years. Furthermore, this intuition has led to the suggestion that gravity and geometry in the bulk could appear as a consequence of entanglement in the dual theory [180, 181]. All in all, while some of these proposals might be speculative, it is clear that this all points to the existence of a deeper relation between gravity and quantum entanglement, which is one of the most groundbreaking outcomes the gauge/gravity duality.

1.3.2 Higher-derivative corrections in holography

In the holographic setup, we are typically interested in the strong-coupling and large-N regime of the boundary theory, and we know this to be equivalent to the low-energy limit of the dual string theory, which reduces to Einstein gravity. However, as mentioned in Section 1.1, this effective action can be expanded with terms of higher orders in the curvature, weighted by powers of the string length $\sqrt{\alpha'}$. Thus, it is legitimate to consider the higher-curvature corrections in the gravity action that we introduced earlier, which following this logic would correspond to finite λ and N effects in the dual CFT.

In general, we do not know the actual quantum model dual to a given higher-curvature gravity. An interesting approach therefore is to consider the corrected theories of gravity as toy models, that allow us to describe CFTs that are not dual to general relativity, and thus belong to different universality classes. In this thesis we will follow this philosophy, by considering higher-curvature corrections with desirable physical properties, such as the Generalized Quasitopological gravities reviewed in Section 1.1.3, and compute different quantities of their corresponding dual theories.

[14] This formula is valid assuming the CFT is dual to Einstein gravity, and if we were to consider higher-curvature corrections it should be replaced, in the same spirit that the Wald entropy (1.44) generalizes the Bekenstein-Hawking result. For general $\mathcal{L}(R^{\mu\nu}_{\rho\sigma})$ theories one should consider the Camps-Dong functional [161, 162], which is equal to the Wald functional evaluated on the minimal surface Γ_A plus an anomaly term, that vanishes on a Killing horizon.

The first thing that one must study when considering higher-curvature terms in holography is the existence of an AdS vacuum. This is a MSS solution of the equations of motion with a negative cosmological constant, given by Eq. (1.14) with $\kappa = -1$. In Einstein gravity, the metric of this spacetime can be written in Poincare coordinates simply as

$$ds^2_{\text{AdS}} = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) , \tag{1.53}$$

where z is the coordinate that penetrates into the bulk, in such a way that the asymptotic boundary is located at $z = 0$. For a general higher-curvature gravity this solution is not guaranteed to exist, and if it does the length scale will not necessarily be equal to the one appearing in the cosmological constant. Then, in Eq. (1.53) we would change

$$L \longrightarrow \tilde{L} = \frac{L}{\sqrt{f_\infty}} , \tag{1.54}$$

and similarly in the corresponding Riemann tensor, given by Eq. (1.13). Here, f_∞ is a constant whose value can be computed from a polynomial equation obtained from the corresponding equations of motion. In principle this will have several solutions, depending on the order of the higher-curvature theory considered, and some of them (or all) might be complex or negative. Among the positive solutions, we are always interested in the one that connects smoothly with the value obtained in Einstein gravity, $f_\infty = 1$, when the couplings of the higher-curvature terms are taken to zero.

Once the existence of a well-behaved AdS vacuum is ensured, we can move on to the characterization of the dual CFTs through the computation of different quantities, and see how these differ from those dual to Einstein gravity. As an example of this, let us comment on the coefficients a and c of the trace anomaly in 4 dimensions. Indeed, while the stress-energy tensor of a CFT should be traceless due to conformal symmetry, this might not be the case in the presence of anomalies that arise due to placing the theory in a curved background. In $d = 4$ we write in general [117]

$$\langle T_a{}^a \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2}\chi_4 , \tag{1.55}$$

where χ_4 is the Gauss-Bonnet scalar written in Eq. (1.21) and I_4 is the square of the Weyl tensor, $I_4 = W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$, both evaluated in the 4-dimensional metric of the CFT. a and c are known as the central charges, and for a theory that is dual to Einstein gravity they take the same value [15, 16, 182],

$$a = c = \frac{\pi L^3}{8 G_{\text{N}}} . \tag{1.56}$$

But it is possible to break this degeneracy in the central charges by adding higher-curvature terms on the gravity side of the duality. Indeed, supplementing the action on the 5-dimensional bulk by the Gauss-Bonnet term, as $\mathcal{L} = R - 2\Lambda + \lambda L^2 \chi_4/2$, one finds [117]

$$a = \frac{\pi \tilde{L}^3}{8G_{\text{N}}}\left(1 - 6\lambda f_\infty\right) , \qquad c = \frac{\pi \tilde{L}^3}{8G_{\text{N}}}\left(1 - 2\lambda f_\infty\right) , \tag{1.57}$$

where $\tilde{L} = L/\sqrt{f_\infty}$, with f_∞ given by

$$f_\infty = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 4\lambda}\right) . \tag{1.58}$$

Thus, we see clearly how higher-derivative theories can be used to explore holographic CFTs that would not be possible to access with Einstein gravity alone. A similar conclusion would be reached by studying other quantities of the dual CFT that can be computed holographically, such as the parameters t_2 and t_4 that characterize the angular distribution of the energy flux after a local insertion of the stress-energy tensor [183], and are also related to the 2- and 3-point functions of the stress-energy tensor. These constants are zero for theories dual to Einstein gravity, but a non-vanishing value can be achieved by introducing higher-curvature corrections [100–102, 117], thus describing theories that belong to wider universality classes.

Based on this, one can think on using higher-curvature theories to propose or check universal relations that can not be tested with Einstein gravity alone, and whose determination from first principles is sometimes not possible. The paradigmatic example of this is the computation of the ratio between the shear viscosity and the thermal entropy density of a plasma, which for a CFT dual to Einstein gravity takes the constant value [184–186]

$$\left.\frac{\eta}{s}\right|_{\rm GR} = \frac{1}{4\pi}\,, \tag{1.59}$$

independently of the dimension. Based on experimental results, this was conjectured to be the minimum value of this quantity that any fluid in nature could achieve, which is known as the Kovtun-Son-Starinets (KSS) bound [186]. However, computations of the ratio η/s in different higher-curvature extensions of Einstein gravity have proved that this value can in fact be lowered [100, 105, 117, 119, 122, 123, 187–196], so it might not be a universal bound after all. We will come back to this in Section 4.6 of this work, where we compute it explicitly for a certain family of higher-derivative theories.

There are other universal relations that have been proposed following this approach, such as the c-theorem established in [197, 198], the behavior of corner contributions to the entanglement entropy [199, 200], or the universal relation between the free energy of a CFT in a squashed sphere and the coefficients of the 3-point function $\langle TTT \rangle$ [121, 128], to name a few. Some other interesting examples of relations obtained with this procedure can be found in [201–204]. We see therefore that, even without knowing the actual form of the dual theory, the gauge/gravity duality proves to be a powerful tool to inspect whether different features of holographic CFTs dual to Einstein gravity are general or not.

Part I

Holographic applications of higher-curvature gravity

2

Renormalization in higher-curvature gravities with $D \leq 5$

The AdS/CFT correspondence provides a method for computing correlation functions in a quantum field theory, using the action of a dual gravity theory in an asymptotically AdS spacetime as a generating functional. These correlators in a QFT will typically suffer from UV divergences, due to the infinite number of degrees of freedom present. On the gravity side, due to the UV/IR relation [205], these manifest as IR divergences in the action that are produced by the infinite volume of AdS near the boundary. So in order to make sense of the holographic duality one needs to get rid of these divergences appearing on both sides in a consistent manner, and identify the remaining results. Apart from this one also requires that the variational problem is well posed, in the sense that the variation of the action produces the Einstein equations subject only to the condition that the metric (and not its derivatives) is fixed at the boundary.

In standard GR and Lovelock gravity, both requirements are achieved through the holographic renormalization prescription, first introduced in [15, 16], and further developed in [17] (see also [173, 206–209]). A similar method for Lovelock gravity theories has been proposed in [210]. The basic idea is to first supplement the action with the standard Gibbons-Hawking-York term (or the Myers term for Lovelock [211]), and then expand the on-shell action in a series of powers of a regulator near the boundary in order to identify and isolate the diverging terms. Finally, these divergences are removed by adding boundary terms that depend only on the induced metric at the boundary and its curvature, while preserving the well-posedness of the variational principle. We will review this procedure for GR in Section 2.1.

Being it the central topic of this thesis, we are interested in exploring these ideas of holographic renormalization in theories of gravity with higher-derivative corrections. For arbitrary HCGs, the boundary terms required for fixing the variational problem and renormalizing the action on AlAdS spacetimes are not known. However, it has been suggested that the action of GR with the corrections of Einsteinian cubic gravity can be rendered finite with the same combination of the GHY and the HR counterterms used in GR, up to a coupling-dependent overall factor [119]. Hence, it is worthwile to consider the possibility of formulating in a similar manner a universal prescription for renormalization, that would work for arbitrary HCGs and in any dimension, allowing to cancel the divergences of the action and pose the variational principle properly.

In this Chapter, we will propose such a renormalization scheme for generic HCGs, in spacetimes with dimension $D \leq 5$. In order to achieve this, we study the radial decomposition of the equations of motion for an arbitrary higher-order theory of gravity, expanded in the Poincare coordinate z by means of the Fefferman-Graham expansion of the metric [212], which is standard for AlAdS manifolds. From these we will find that the coefficients of order z and z^3 in this expansion are zero in general, while the third coefficient (of order z^2) has a universal form. Also, we will see that the terms which contribute to the divergences of the on-shell action at the boundary, for bulks with $D \leq 5$, depend only on the first four FG coefficients of the metric. Using these facts, we will be able to show that the extrinsic curvature counterterms introduced in [18, 19], with theory-dependent overall constants, successfully implement the renormalization and fix the variational principle[1] at the asymptotic boundary.

The research presented in this Chapter has been published previously in [24]. The information of this article can be found in page 210.

2.1 Holographic renormalization in general relativity

Before getting into the more general prescription, we review in this Chapter the standard holographic renormalization procedure valid for general relativity. Let us start by considering the Einstein-Hilbert action in D dimensions, supplemented by the standard Gibbons-Hawking-York term

$$S_{\text{EH}} + S_{\text{GHY}} = \frac{1}{16\pi G_{\text{N}}} \int_{\mathcal{M}} d^D x \sqrt{-G} \left(R + \frac{(D-1)(D-2)}{L^2} \right) + \frac{\epsilon}{8\pi G_{\text{N}}} \int_{\partial\mathcal{M}} d^d x \sqrt{-h} K \,. \tag{2.1}$$

The D-dimensional global manifold $\mathcal{M}$ has AdS asymptotics, hence the value $\Lambda = -(D-1)(D-2)/(2L^2)$ for the cosmological constant. We denote its metric by[2] $G_{\mu\nu}$ and its coordinates X^μ, while its boundary ∂M has $d = D-1$ dimensions and an induced metric h_{ij}. K is the extrinsic curvature of the boundary, which can be computed from h_{ij} using Eq. (A.9). The constant ϵ is the norm of the vector normal to the submanifold $\partial\mathcal{M}$, as defined in Eq. (A.3). In this case, since the boundary of AdS is spacelike, $\epsilon = 1$.

The GHY boundary term in (2.1) is enough to make the variational problem well posed when subject to Dirichlet boundary conditions, but we still need to identify and cancel out the divergences that appear when expanding the bulk action close to the boundary. For this, let us write the AlAdS metric in the Fefferman-Graham form, as [15, 16, 212]

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = \frac{L^2}{4\rho^2} d\rho^2 + h_{ij}(x,\, \rho) dx^i dx^j \,, \tag{2.2}$$

where ρ is the holographic radial coordinate, defined in such a way that the boundary is located at $\rho = 0$, and the coordinates x^i, $i = 0, \ldots, d$, correspond to the directions tangent to the

[1] In the particular case $D = 5$ the well-posedness of the variational principle will require the boundary to be conformally (or Weyl-) flat, as will be made clear later on. For lower dimensions this requirement is automatically fulfilled, since any manifold with $d \leq 3$ has a vanishing Weyl tensor. In general relativity the prescription is valid regardless of this.

[2] Although the notation of the metric of the total spacetime by $G_{\mu\nu}$ can conflict with the rest of the thesis, we make this choice to be consistent with the standard literature on holographic renormalization. g_{ij} is reserved for the conformally rescaled version of the induced metric at the boundary, as will be clear later.

boundary. h_{ij} is the tangent part of the metric, and it can be rescaled and expanded in a power series in ρ as

$$\rho\, h_{ij}(x, \rho) \equiv g_{ij}(x, \rho) = g^{(0)}_{ij}(x) + \rho g^{(2)}_{ij}(x) + \ldots + \rho^{d/2} g^{(d)}_{ij} + \rho^{d/2} \log \rho\, \tilde{h}^{(d)}_{ij}(x) + \ldots\,, \tag{2.3}$$

where the term proportional to $\rho^{d/2} \log \rho$ only appears if d is even, and ultimately corresponds to the metric variation of the holographic conformal anomaly [15]. The term $g^{(0)}_{ij}$ is the metric at the conformal boundary, and by expanding Einstein's equations order by order in ρ it is possible to compute the coefficients $g^{(2)}_{ij}, \ldots, g^{(d-2)}_{ij}, \tilde{h}^{(d)}_{ij}$, and the trace and covariant divergence of $g^{(d)}_{ij}$ as algebraic expressions of the first coefficient (explicit expressions for some of these can be found in Appendix A of [17]). Also, Einstein's equations forbid terms with non-integer powers of ρ from appearing in the expansion (2.3). Whether or not this is the case for more general theories of gravity will be a central point in the discussion presented in this Chapter.

The renormalization procedure requires the identification of the divergent terms of the gravitational action (2.1) close to the boundary. For this, we restrict the integration range to $\rho > \epsilon$, where $\epsilon > 0$ is a cutoff, and expand the action in powers of this. Schematically, it takes the form [15–17]

$$S_{\text{EH}} + S_{\text{GHY}} = \frac{1}{16\pi G_{\text{N}}} \int_{\rho=\epsilon} d^d x \sqrt{-g^{(0)}} \left(\epsilon^{d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-2)} + \log \epsilon\, a_{(d)} + \mathcal{O}(\epsilon) \right)\,, \tag{2.4}$$

where the coefficients $a_{(n)}$ are local covariant expressions involving the boundary metric $g^{(0)}_{ij}$ and its curvatures, whose explicit forms can be found in Appendix B of [17].

The renormalized action is obtained by subtracting these divergent terms explicitly, whcih is achieved through the addition of some counterterms. The actual form of these depend on the dimension of the spacetime, but for $d \leq 6$ it is given by [207]

$$\begin{aligned} S_{\text{ct}} = \frac{1}{16\pi G_{\text{N}}} \int_{\rho=\epsilon} d^d x \sqrt{-h} \Bigg[& \frac{d-1}{L} + \Theta(d-3) \frac{L}{d-2} \mathcal{R} \\ & + \Theta(d-5) \frac{L^3}{(d-4)(d-2)^2} \left(\mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{d}{4(d-1)} \mathcal{R}^2 \right) + \ldots \Bigg]\,, \end{aligned} \tag{2.5}$$

where $\Theta(x)$ is the Heaviside step function, which is equal to one for $x > 0$ and zero otherwise, and is introduced to make sure that the higher order terms only appear for large enough dimensions. In other words, the amount of divergent terms in (2.4) depends on the dimension, and therefore the series of counterterms in S_{ct} needs to be truncated correspondingly. This action is written in terms of the curvature of the induced metric, $\mathcal{R} = \mathcal{R}(h) \sim \rho \mathcal{R}(g^{(0)})$, and this combined with the determinant factor h produce the different divergent powers of the regulator that are found in Eq. (2.4). The renormalized action is finally computed as

$$S_{\text{ren}} = S_{\text{EH}} + S_{\text{GHY}} + S_{\text{ct}}\,. \tag{2.6}$$

The counterterm action (2.5) depends only on the boundary geometry, in the sense that all the relevant contributions of the FG expansion (2.3), and thus the curvature $\mathcal{R}_{ijkl}(h)$, can be written in terms of $g^{(0)}_{ij}$ and its derivatives with respect to the tangent coordinates. Therefore, the well-posedness of the variational principle achieved by adding the GHY term in Eq. (2.1) is preserved.

Even though the procedure for computing the counterterms (2.5) can be more or less systematized, they can not be written in a closed form for arbitrary dimensions. However, an alternative proposal for the regularization appeared in [18, 19, 213–216], and it requires the addition of the so-called *Koutnerterms*. These are topological quantities that can be written in any arbitrary dimension as expressions involving the extrinsic curvature of the boundary, and have the advantage that they cancel the divergences and provide a well-posed variational principle for the EH action at the same time. These will be the main ingredient of the renormalization procedure for general HCGs proposed in this Chapter, and we will introduce them explicitly in Section 2.3.

2.2 Projected equations of motion in HCGs

In order to be able to test the method for renormalization of the action that we will propose later for a general higher-curvature theory of gravity, we need to understand whether or not our theory admits a Fefferman-Graham expression equivalent to that given in Eqs. (2.2) and (2.3), which is the goal of the current Section.

Let us start by considering a HCG in $D \leq 5$, whose action that can contain any possible term constructed from arbitrary contractions of the Riemann tensor and the metric, with the only condition that it admits vacuum AdS solutions. We denote this as

$$S = \int_{\mathcal{M}} d^D X \sqrt{-G} \mathcal{L}\left(R_{\mu\nu}^{\rho\sigma}\right) . \tag{2.7}$$

As already written in Eq. (1.9), the equations of motion for such a theory are given in general by [83]

$$\mathcal{E}_{\mu}^{\nu} = P_{\mu\alpha}^{\beta\gamma} R_{\beta\gamma}^{\nu\alpha} - \frac{1}{2}\delta_{\mu}^{\nu}\mathcal{L} - 2\nabla^{\alpha}\nabla_{\beta}P_{\mu\alpha}^{\beta\nu} , \tag{2.8}$$

where the tensor $P_{\rho\sigma}^{\mu\nu}$ is defined as

$$P_{\rho\sigma}^{\mu\nu} = \frac{\partial L}{\partial R_{\mu\nu}^{\rho\sigma}} . \tag{2.9}$$

Since we are interested in asymptotically locally AdS (AlAdS) backgrounds, we consider that the Riemann tensor near the boundary behaves as

$$R_{\mu\nu}^{\rho\sigma} \longrightarrow -\frac{1}{L^2}\left(\delta_{\mu}^{\rho}\delta_{\nu}^{\sigma} - \delta_{\nu}^{\rho}\delta_{\mu}^{\sigma}\right) , \tag{2.10}$$

where L is the effective AdS length scale, related to the one appearing in the cosmological constant, $\Lambda_0 = -(D-1)(D-2)/(2L_0^2)$, through a relation $L = L(L_0)$ that depends on the particular theory. Notice that by construction $P_{\rho\sigma}^{\mu\nu}$ has the same symmetries as the Riemann tensor, and therefore close to the boundary it becomes

$$P_{\rho\sigma}^{\mu\nu} \longrightarrow \frac{1}{2}C(L)\left(\delta_{\mu}^{\rho}\delta_{\nu}^{\sigma} - \delta_{\nu}^{\rho}\delta_{\mu}^{\sigma}\right) . \tag{2.11}$$

The constant $C(L)$ introduced here can be obtained by replacing these expressions in the field equations, and it is given by [86]

$$C(L) = -\frac{L^2}{2(D-1)}\mathcal{L}|_{\mathrm{AdS}} = \frac{L^3}{2D(D-1)}\frac{\partial \mathcal{L}|_{\mathrm{AdS}}}{\partial L} , \tag{2.12}$$

where $\mathcal{L}|_{\text{AdS}}$ is the Lagrangian evaluated in the AdS vacuum solution. In the case of Einstein-Hilbert gravity, this constant is nothing but $1/(16\pi G_{\text{N}})$.

Our goal is to identify and cancel the divergences of the general action (2.7), similarly to the standard HR procedure described in Section 2.1. For this we need to consider the FG expansion of the bulk metric, which requires splitting the coordinates into those normal and tangent to the boundary, as $X^\mu = (x^i, z)$, where z is the holographic coordinate. The line element in the D-dimensional manifold is

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = \frac{L^2}{z^2} dz^2 + h_{ij}(x, z) dx^i dx^j \,. \tag{2.13}$$

The induced metric h_{ij} can be expanded in powers of z as

$$h_{ij}(x, z) = \frac{1}{z^2} g_{ij}(x, z) = \frac{1}{z^2} \left(g^{(0)}_{ij}(x) + z g^{(1)}_{ij}(x) + z^2 g^{(2)}_{ij}(x) + z^3 g^{(3)}_{ij}(x) + \ldots \right) \,, \tag{2.14}$$

where we are omitting the logarithmic term that would appear for even dimensions of the boundary, since it is unimportant for our purpose. The Poincare coordinate z is related to ρ introduced in Eq. (2.2) as $\rho = z^2$. While the odd terms in the expansion (2.14) vanish in GR, $g^{(1)}_{ij} = g^{(3)}_{ij} = 0$, in more general higher-curvature gravities this might not be true, so it makes sense to write our expressions in terms of z.

The coefficients $g^{(n)}_{ij}(x)$ with $n < D - 1$ are determined completely in GR as expressions of $g^{(0)}_{ij}(x)$, by means of the projected equations of motion $\mathcal{E}^z_z = 0$, $\mathcal{E}^i_j = 0$ and $\mathcal{E}^z_i = 0$. Since we are considering $D \leq 5$, we want to scrutinize the terms in the FG expansion up to order z^3, in particular to see whether they are the same as those in Einstein's gravity,

$$g^{(1)}_{ij} = 0 \,, \tag{2.15a}$$

$$g^{(2)}_{ij} = -\frac{L^2}{D-3} \left(\mathcal{R}_{ij}\left(g^{(0)}\right) - \frac{1}{2(D-2)} \mathcal{R}\left(g^{(0)}\right) g^{(0)}_{ij} \right) \,, \tag{2.15b}$$

$$g^{(3)}_{ij} = 0 \,, \tag{2.15c}$$

or not. In these expressions, $\mathcal{R}_{ij}(g^{(0)})$ and $\mathcal{R}(g^{(0)})$ are respectively the Ricci tensor and curvature scalar computed from $g^{(0)}_{ij}$, the metric at the conformal boundary, and thus do not depend on z. In the form (2.13) the metric is naturally decomposed in the parts that are normal (zz component) and tangent (ij components) to the boundary. The tangent indices are raised with the inverse metric $h^{ij}(x, z) = z^2 g^{ij}(x, z)$, which has an expansion in z such that $h_{ij} h^{jk} = \delta^j_i$. The vector normal to the boundary, n, is defined as

$$n = n^\mu \partial_\mu = n^z \partial_z = -\frac{z}{L} \partial_z \,, \qquad n^\mu n_\mu = 1 \,, \tag{2.16}$$

where the minus sign appears due to the fact that the boundary is located at the lowest limit of the range of values of the radial coordinate z, and the norm being positive means that the boundary is a spacelike hypersurface. We can also define a covariant derivative compatible with the tangent part of the metric, $\tilde{\nabla}_i$, such that

$$\tilde{\nabla}_i h_{jk} = \tilde{\nabla}_i g_{jk} = 0 \,, \tag{2.17}$$

since $h_{jk} = g_{jk}/z^2$, and $\tilde{\nabla}_i z = 0$ by definition. As h_{ij} and g_{ij} depend on z, the tangent covariant derivative will also admit an expansion in this normal coordinate. Apart from this, the usual covariant derivative is compatible with the global metric, $\nabla_\mu G_{\rho\sigma} = 0$. For completeness, we write the explicit form of the different non-vanishing Christoffel symbols obtained from the metric (2.13),

$$\Gamma^z_{zz}(\nabla) = -\frac{1}{z}\,, \qquad \Gamma^i_{zj}(\nabla) = \Gamma^i_{jz}(\nabla) = \frac{1}{2}h^{ik}\partial_z h_{jk}\,, \qquad \Gamma^z_{ij}(\nabla) = -\frac{1}{2}\frac{z^2}{L^2}\partial_z h_{ij}\,, \tag{2.18}$$

whereas $\Gamma^k_{ij}(\nabla) = \Gamma^k_{ij}(\tilde{\nabla}) + \mathcal{O}(z)$. Again, since $h_{ij} = h_{ij}(x, z)$, most of these can be written as an expansion in powers of z, in terms of the different coefficients in (2.14). We further need to know the form of the extrinsic curvature of the induced metric at the boundary. It can be computed using Eq. (A.9), and in this case it reads

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} = \frac{1}{2}n^z\partial_z h_{ij} = -\frac{1}{2}\frac{z}{L}\partial_z\left(\frac{1}{z^2}g_{ij}\right)\,. \tag{2.19}$$

We can now use the Gauss-Codazzi equations presented in Appendix A (with $\epsilon = 1$, since the normal vector is spacelike) to write the different components of the Riemann tensor of the bulk metric in terms of the induced metric,

$$R_{izjz} = \frac{L^2}{z^2}\left(-\mathcal{L}_n K_{ij} + K_{ik}K^k_j - a_i a_j + \tilde{\nabla}_{(i}a_{j)}\right)\,, \tag{2.20a}$$

$$R_{ijkz} = -n_z\left(\tilde{\nabla}_i K_{jk} - \tilde{\nabla}_j K_{ik}\right)\,, \tag{2.20b}$$

$$R_{ijkl} = \mathcal{R}_{ijkl}(h) + K_{il}K_{jk} - K_{ik}K_{jl}\,. \tag{2.20c}$$

In these expressions, $a_\mu = n^\alpha\nabla_\alpha n_\mu$, and $\mathcal{R}_{ijkl}(h)$ is the usual Riemann tensor of the tangent metric $h_{ij}(x, z)$, which can be expanded also in powers of z,

$$\mathcal{R}_{ijkl}(h) = \frac{1}{z^2}\mathcal{R}_{ijkl}(g) = \frac{1}{z^2}\left(\mathcal{R}_{ijkl}\left(g^{(0)}\right) + \mathcal{O}(z)\right)\,. \tag{2.21}$$

Notice that the indices of $\mathcal{R}_{ijkl}(h)$ are raised with the inverse tangent metric h^{ij}, while those of $\mathcal{R}_{ijkl}(g)$ must be raised with g^{ij}. This implies, for example, the relations

$$\mathcal{R}^{ij}_{kl}(h) = z^2\mathcal{R}^{ij}_{kl}\left(g^{(0)}\right) + \mathcal{O}(z)\,, \qquad \mathcal{R}^{ijkl}(h) = z^4\mathcal{R}^{ijkl}\left(g^{(0)}\right) + \mathcal{O}(z^3)\,, \tag{2.22}$$

which one needs to take into account in the computations.

With these ingredients, we are now in position to project and evaluate the equations of motion (2.8), in order to compute the form of the coefficients of the FG expansion for a general higher-curvature theory of gravity. In what follows we will do so order by order in z, up to the third power required for $D \leq 5$.

2.2.1 Vanishing of $g^{(1)}_{ij}$ in a general HCG

First, let us expand the equations of motion to next-to-leading order in the holographic coordinate, to see whether or not $g^{(1)}_{ij} = 0$. By explicitly computing the different components of the Riemann tensor, we find that they match the expression

$$R^{\mu\nu}_{\rho\sigma} = -\frac{2}{L^2}\delta^{[\mu}_\rho\delta^{\nu]}_\sigma + z\frac{2}{L^2}\delta^{[\mu}_\rho g^{(1)\nu]}{}_\sigma + \mathcal{O}(z^2)\,, \tag{2.23}$$

where by definition only the components of ${g^{(1)}}^{\nu}{}_{\sigma}$ with both indices in the tangent directions can be non-zero. Regarding $P^{\mu\nu}_{\rho\sigma}$, we can use its definition (2.9) together with the expression of the Riemann tensor (2.23) to constrain its tensorial form,

$$P^{\mu\nu}_{\rho\sigma} = C(L)\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} + z\left(A^{(1)}(L)\delta^{[\mu}_{\rho}{g^{(1)}}^{\nu]}{}_{\sigma} + B^{(1)}(L)\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma}\text{Tr}\, g^{(1)}\right) + \mathcal{O}(z^2)\,, \tag{2.24}$$

where $C(L)$ was given in Eq. (2.12), whereas $A^{(1)}(L)$ and $B^{(1)}(L)$ are scalar functions depending on the couplings of the particular theory and the effective AdS radius L. Also, $\text{Tr}\, g^{(1)} := {g^{(1)}}^{i}{}_{i}$.

The Lagrangian itself also appears in the general expression of the equations of motion (2.8), so we expand it symbolically to first order in z as

$$\mathcal{L} = \mathcal{L}^{(0)} + P^{\rho\sigma}_{\mu\nu}\delta R^{\mu\nu}_{\rho\sigma} + \ldots = \mathcal{L}^{(0)} + z\mathcal{L}^{(1)} + \mathcal{O}(z^2)\,, \tag{2.25}$$

where $\mathcal{L}^{(0)} := \mathcal{L}|_{\text{AdS}}$ is the Lagrangian evaluated in the background solution (2.10), and we denoted by $\delta R^{\mu\nu}_{\rho\sigma}$ the deviation of the Riemann tensor with respect to this background. Therefore, by taking into account Eq. (2.23) we see that $\delta R^{\mu\nu}_{\rho\sigma} \sim z$, which allows us to compute

$$\mathcal{L}^{(1)} = \frac{D-1}{L^2}C(L)\text{Tr}\, g^{(1)}\,. \tag{2.26}$$

Using these expressions, the equations of motion decomposed in their radial and tangential components and expanded in powers of z read

$$\mathcal{E}^{z}_{z} = \left(\frac{(1-D)C(L)}{L^2} - \frac{\mathcal{L}^{(1)}}{2}\right) + \frac{z}{2L^2}\left(a^{(1)}(L) + (D-1)b^{(1)}(L)\right)\text{Tr}\, g^{(1)} + \mathcal{O}(z^2)\,, \tag{2.27}$$

$$\mathcal{E}^{i}_{j} = \left(\frac{(1-D)C(L)}{L^2} - \frac{\mathcal{L}^{(1)}}{2}\right)\delta^{i}_{j} + \frac{z(D-2)}{2L^2}\left(a^{(1)}(L){g^{(1)}}^{i}{}_{j} + b^{(1)}(L)\delta^{i}_{j}\text{Tr}\, g^{(1)}\right) + \mathcal{O}(z^2)\,, \tag{2.28}$$

$$\mathcal{E}^{z}_{i} = \frac{1}{2(D-2)L^2}\left(a^{(1)}(L)\tilde{\nabla}_{j}{g^{(1)}}^{j}{}_{i} + b^{(1)}(L)\tilde{\nabla}_{i}\text{Tr}\, g^{(1)}\right) + \mathcal{O}(z^3)\,, \tag{2.29}$$

where

$$\begin{aligned} a^{(1)}(L) &:= C(L) - A^{(1)}(L)\,, \\ b^{(1)}(L) &:= -C(L) - A^{(1)}(L) - 4B^{(1)}(L)\,. \end{aligned} \tag{2.30}$$

The zeroth order gives the aforementioned result for $C(L)$, Eq. (2.12), and all the information of the equations at the next lowest order is encoded in $a^{(1)}(L)$ and $b^{(1)}(L)$. The main feature of these is that, except in some particular cases, they imply

$$g^{(1)}_{ij} = 0\,. \tag{2.31}$$

More specifically, the equations of motion fix the $\mathcal{O}(z)$ coefficient of the FG expansion to be zero, as found in GR, except in some particular cases:

- If $a^{(1)}(L) \neq 0$ and $b^{(1)}(L) = -a^{(1)}(L)/(D-1)$, the off-diagonal components of $g^{(1)}_{ij}$ are fixed to zero and the elements in the diagonal are fixed to be equal to each other, but their actual value is free.
- If $a^{(1)}(L) = 0$ and $b^{(1)}(L) \neq 0$, $\text{Tr}\, g^{(1)}$ is fixed to zero, but the rest of the components of $g^{(1)}_{ij}$ are free.

- And finally, if both $a^{(1)}(L) = 0$ and $b^{(1)}(L) = 0$, $g^{(1)}_{ij}$ is completely free and not restricted by the equations of motion.

This is interesting as it means that, for a generic HCG theory, the equations of motion require the $\mathcal{O}(z)$ coefficient in the FG expansion of the metric to vanish, just like for Einstein-AdS gravity. This is analogous to the universality of the $\mathcal{O}(z^2)$ coefficient as implied by the PBH transformation [217], which will be reviewed in the next subsection, and both are central to the applicability of the generic renormalization procedure presented in this work.

An example: quadratic curvature gravity

Let us illustrate this further by working out explicitly the case of quadratic curvature gravity. Its action can be written as

$$S = \int_{\mathcal{M}} d^D X \sqrt{-G} \left[\frac{1}{16\pi G_{\rm N}} (R - 2\Lambda_0) + \alpha_1 R_{\mu\nu} R^{\mu\nu} + \alpha_2 R^2 + \alpha_3 \chi_4 \right] , \tag{2.32}$$

where χ_4 is the Gauss-Bonnet combination written in Eq. (1.21). By explicitly computing $P^{\mu\nu}_{\rho\sigma}$ for this theory we can obtain the values of $C(L)$, $a^{(1)}(L)$ and $b^{(1)}(L)$, which read

$$C(L) = \frac{1}{16\pi G_{\rm N}} - \frac{2(D-1)}{L^2} \left[\alpha_1 + D\alpha_2 + \frac{(D-2)(D-3)}{D-1} \alpha_3 \right] , \tag{2.33}$$

$$a^{(1)}(L) = \frac{1}{16\pi G_{\rm N}} - \frac{1}{L^2} \left[(3D-4)\alpha_1 + 2D(D-1)\alpha_2 + 2(D-3)(D-4)\alpha_3 \right] , \tag{2.34}$$

$$b^{(1)}(L) = -\frac{1}{16\pi G_{\rm N}} + \frac{1}{L^2} \left[(D-4)\alpha_1 + 2(D-1)(D-4)\alpha_2 + 2(D-3)(D-4)\alpha_3 \right] . \tag{2.35}$$

The equations of motion imply $g^{(1)}_{ij} = 0$ unless one of the conditions discussed after Eq. (2.31) is met. Some examples of quadratic curvature gravity theories where $a^{(1)}(L) = b^{(1)}(L) = 0$, such that $g^{(1)}_{ij}$ is not fixed by the equations of motion, include:

- Einstein-Lanczos-Gauss-Bonnet gravity at the (dimensionally continued) Chern-Simons point. In [218], the authors find that in this theory the coefficients of the FG expansion are not fixed by the equations of motion, which vanish identically.

- Conformal gravity in 4 dimensions [219–221], where it is found again that the equations of motion at the lowest orders vanish for any form of the coefficients in the expansion.

- New massive gravity [222, 223] at the special point (in the language of [222]).

A feature all these cases share is that their AdS vacua are degenerate. Indeed, we checked that this is true for any quadratic or cubic gravity fulfilling $a^{(1)}(L) = b^{(1)}(L) = 0$, but not the other way around. However, while these particular cases allow for $g^{(1)}_{ij} \neq 0$, they do not enforce it. In particular, $g^{(1)}_{ij}$ cannot be given as an expression in terms of $g^{(0)}_{ij}$, thereby its value must be fixed as a boundary condition [221]. Therefore, in general we could pick $g^{(1)}_{ij} = 0$ for any theory in vacuum, and build the rest of our discussion on top of this assumption.

We have also repeated this analysis for general theories of gravity with cubic contractions of the curvature tensors. The results are written in Appendix B.

2.2.2 Universality of $g^{(2)}_{ij}$ from the PBH transformations

We now turn to the term of second order in the FG expansion, $g^{(2)}_{ij}$. In general relativity it takes the value given in Eq. (2.15b), which is fixed by symmetry under a certain type of conformal transformations [217]. As will be shown, and was already assumed in [217], these arguments carry on with minimal modifications to a general higher-curvature theory. While it should be possible to perform a similar analysis to what is shown in Section 2.2.1 for $g^{(1)}_{ij}$, expanding the equations of motion $\mathcal{E}^{\mu}_{\nu}$ to order z^2, the current approach is simpler and produces a more robust result.

The argument is based on the invariance of the bulk metric under PBH (Penrose-Brown-Henneaux) transformations, which are a subset of the bulk diffeomorphisms that act as Weyl transformations on the boundary, while leaving the form of the bulk metric unchanged [217]. In particular, these act on the first coefficient of the FG expansion (2.14) as

$$g^{(0)}_{ij}(x) \longrightarrow e^{2\sigma(x)/L} g^{(0)}_{ij}(x) \,. \tag{2.36}$$

Let us consider the ansatz for the transformation of the coordinates

$$z = z' e^{-\sigma(x')} \simeq z' \left(1 - \sigma(x')\right) \,, \qquad x^i = x'^i + a^i(x', z') \,, \tag{2.37}$$

where primes denote the transformed coordinates and the $a^i(x', z')$ are infinitesimal and will be restricted by requiring that the form of the metric is invariant. Working to the lowest order in the parameters of the transformation σ and a^i, the line element becomes

$$\begin{aligned} ds^2 = & \frac{L^2}{z'^2} \left(dz'^2 - 2\sigma dz'^2 - 2z' \partial_k \sigma dz' dx'^k\right)(1 + 2\sigma) \\ & + \frac{1}{z'^2}(1 + 2\sigma)\left(g'_{ij} - z'\sigma\partial_z g_{ij} + a^k \partial_k g_{ij}\right)\left(dx'^i dx'^j + 2\partial_l a^i dx'^j dx'^l + 2\partial_z a^i dx'^j dz'\right) \,. \end{aligned} \tag{2.38}$$

Comparing this expression with the original form of the metric (2.13) and combining some terms, we find the change of the tangent metric

$$\delta g_{ij} = \sigma(2 - z\partial_z) g_{ij} + \tilde{\nabla}_i a_j + \tilde{\nabla}_j a_i \,. \tag{2.39}$$

As we said, these transformations should leave the form of the bulk metric unchanged, and in particular the off-diagonal components $dx'^i dz'$ in Eq. (2.38) must vanish in the FG gauge. This relates the parameters of the diffeomorphisms as

$$\partial_z a^i = L^2 z g^{ij} \partial_j \sigma \,. \tag{2.40}$$

It is also possible to integrate the given expression, subject to the boundary condition $a^i(x, z = 0) = 0$, but the differential form is enough for our computation. $a^i(x, z)$ admits its own expansion in powers of z near the boundary,

$$a_i(x, z) = \sum_{n=1}^{\infty} z^n a^{(n)}_i(x) \,, \tag{2.41}$$

where the term at order $\mathcal{O}(z^0)$ does not appear due to the mentioned boundary condition. Replacing this and the FG expansion of the tangent metric (2.14) in Eq. (2.40), and studying separately the terms with different powers in z, we find

$$\begin{aligned} a^i_{(1)} &= 0\,, \\ a^i_{(2)} &= \frac{L^2}{2} g^{(0)ij}\partial_j\sigma\,, \\ a^i_{(3)} &= -\frac{L^3}{3} g^{(1)ij}\partial_j\sigma\,, \end{aligned} \tag{2.42}$$

and equivalent relations at higher orders, which are irrelevant for us. Applying the same procedure on the expression of δg_{ij}, Eq. (2.39), we find the variation of the different coefficients in the transverse metric due to the Weyl transformation of the boundary metric (2.36),

$$\begin{aligned} \delta g^{(0)}_{ij} &= 2\sigma g^{(0)}_{ij}\,, \\ \delta g^{(1)}_{ij} &= \sigma g^{(1)}_{ij}\,, \\ \delta g^{(2)}_{ij} &= \tilde{\nabla}^{(0)}_i a^{(2)}_j + \tilde{\nabla}^{(0)}_j a^{(2)}_i\,, \\ \delta g^{(3)}_{ij} &= -\sigma g^{(3)}_{ij} + \tilde{\nabla}^{(0)}_i a^{(3)}_j + \tilde{\nabla}^{(0)}_j a^{(3)}_i + g^{(0)}_{jk}\tilde{\nabla}^{(1)}_i a^{(2)k} + g^{(0)}_{ik}\tilde{\nabla}^{(1)}_j a^{(2)k} \\ &\quad + g^{(1)}_{jk}\tilde{\nabla}^{(0)}_i a^{(2)k} + g^{(1)}_{ik}\tilde{\nabla}^{(0)}_j a^{(2)k}\,. \end{aligned} \tag{2.43}$$

In these expressions, $\tilde{\nabla}^{(n)}_i$ refer to the term of order z^n that appear when expanding the covariant derivatives of the tangent space, $\tilde{\nabla}_j$, being $\tilde{\nabla}^{(0)}_i$ the one compatible with $g^{(1)}_{ij}$.

We want expressions for the coefficients $g^{(n)}_{ij}$ that depend covariantly only on the data at the boundary. The transformations under study do not fix the coefficient $g^{(1)}_{ij}$ in terms of $g^{(0)}_{ij}$, so in principle it could be left as free data, and all odd coefficients in the series can be constructed with contractions of $g^{(1)}_{ij}$ with tensors of the boundary metric. However, as we have seen before, most theories require $g^{(1)}_{ij} = 0$, which we will assume from now on.

However, the variation $\delta g^{(2)}_{ij}$ is independent of $g^{(1)}_{ij}$, which means that it takes the same form in any theory. In particular, it contains terms with two derivatives in the transverse coordinates x, as can be seen more clearly replacing $a^{(2)}_i$ using Eq. (2.42). Therefore, $g^{(2)}_{ij}$ must be a linear combination of terms proportional to the Riemann tensor of the boundary metric $g^{(0)}_{ij}$, the most general form being

$$g^{(2)}_{ij} = \alpha\mathcal{R}_{ij}\left(g^{(0)}\right) + \beta\mathcal{R}\left(g^{(0)}\right)g^{(0)}_{ij}\,. \tag{2.44}$$

If we compute $\delta g^{(2)}_{ij}$, taking into account the form of the Weyl transformation of the curvature tensors, and compare it with Eq. (2.43), we can read off the values of the constants α and β and thus the form of $g^{(2)}_{ij}$, which matches that found in Einstein gravity,

$$g^{(2)}_{ij} = -\frac{L^2}{D-3}\left(\mathcal{R}_{ij}\left(g^{(0)}\right) - \frac{1}{2(D-2)}\mathcal{R}\left(g^{(0)}\right)g^{(0)}_{ij}\right)\,. \tag{2.45}$$

Following the same procedure it is possible to obtain the form of the coefficients that appear at higher even orders in z, as done in [217], although for our purposes it is enough to stop at quadratic order.

We should note that, since this procedure builds upon invariance under the boundary Weyl transformations (2.36), it fails to capture further contributions made of contractions of the Weyl tensor of the boundary that might appear.[3] However, for 2- and 3-dimensional

3 Of course, $\mathcal{W}^{ij}_{ik}(g^{(0)})$ vanishes identically due to symmetry, so this term is ruled out. However, at this order there could appear contractions with two free indices, for example with the schematic form $\sqrt{\mathcal{W}^{(0)}\mathcal{W}^{(0)}}$, as seen for the case of Chern-Simons gravity in [218].

boundaries, which correspond respectively to $D = 3$ and $D = 4$ bulk dimensions, the Weyl tensor is identically zero and thus these contributions do not occur. While this is not true for $D = 5$, when treating the well-posedness of the variational problem we will need to assume Asymptotic Conformal Flatness [224], which implies that the Weyl tensor of the boundary metric must be zero. Therefore, we see that under these assumptions the coefficient $g^{(2)}_{ij}$ takes the same form in a general HCG as in Einstein gravity.

2.2.3 Vanishing of $g^{(3)}_{ij}$ in a general HCG

As a final step before getting into the problem of renormalization of the gravity action, we should discuss the term of third order in z in the FG expansion (2.14). In particular, we will see that in general it should take the same value as in Einstein gravity, which is $g^{(3)}_{ij} = 0$. This is the last relevant coefficient for the dimensions that we are interested in, since terms of higher order do not contribute to the divergent part of the action for $D \leq 5$, which will be clear later on when analyzing the cancellation of these divergences.

Following the same logic as in Section 2.2.1, we expect the coefficient $g^{(3)}_{ij}$ to be fixed, in general, by the contributions of the equations of motion multiplied by z^3. In order to obtain this we need to expand the different objects that appear in $\mathcal{E}^{\mu}_{\nu}$ as written in Eq. (2.8), and in particular we use symbolic expansions for both $\mathcal{L}$ and $P^{\mu\nu}_{\rho\sigma}$. Since we have already fixed $g^{(1)}_{ij} = 0$, and we know the form of the zeroth order coefficients in these two objects through Eqs. (2.11) and (2.12), these can be written as

$$\mathcal{L} = -\frac{2(D-1)}{L^2}C(L) + z^2\mathcal{L}^{(2)} + z^3\mathcal{L}^{(3)} + \ldots\,, \tag{2.46}$$

$$P^{\mu\nu}_{\rho\sigma} = C(L)\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} + z^2 P^{(2)\mu\nu}{}_{\rho\sigma} + z^3 P^{(3)\mu\nu}{}_{\rho\sigma} + \ldots\,. \tag{2.47}$$

We are interested only on the third order terms in the equations of motion, so we should understand the form of $P^{(3)\mu\nu}{}_{\rho\sigma}$. This will depend on the theory, of course, but we can follow the same reasoning as before and use its tensorial structure to write the components $P^{(3)ij}{}_{kl}$ and $P^{(3)iz}{}_{jz}$ as the combinations

$$\begin{aligned} P^{(3)ij}{}_{kl} &= A^{(3)}(L)\delta^{[i}_{[k}g^{(3)j]}{}_{l]} + B^{(3)}(L)\delta^{[i}_{[k}\delta^{j]}_{l]}\mathrm{Tr}\, g^{(3)}\,, \\ P^{(3)iz}{}_{jz} &= D^{(3)}(L)g^{(3)i}{}_{j} + E^{(3)}(L)\delta^{i}_{j}\mathrm{Tr}\, g^{(3)}\,, \end{aligned} \tag{2.48}$$

where the constants $A^{(3)}(L)$, $B^{(3)}(L)$, $D^{(3)}(L)$ and $E^{(3)}(L)$ depend on the effective AdS radius L and the higher-curvature couplings. Since we already assume that $g^{(1)}_{ij} = 0$, terms of the form $g^{(1)i}{}_{j}g^{(2)k}{}_{l}$ will not appear in these general expressions. Also, the components with one index in the normal direction, such as $P^{(3)ij}{}_{zk}$, could have terms proportional to $\tilde{\nabla}g^{(2)}$, but these appear in the equations of motion at higher orders in z, since they must be contracted with $n^z = z/L$.

Regarding the expansion of the Lagrangian, $\mathcal{L}^{(3)}$ can not have contributions of the form $\tilde{\nabla}_j g^{(2)j}{}_{k}$, as there are no objects with an odd number of tangent indices to contract it producing a term of order z^3. Therefore, it can only contain terms that are proportional to $\mathrm{Tr}\, g^{(3)}$, and if we expand it as in Eq. (2.25) we see that they can only be produced by $P^{(0)\mu\nu}{}_{\rho\sigma}R^{(3)\rho\sigma}{}_{\mu\nu}$. Plugging in the third order terms of the components of the Riemann tensor, which for the metric (2.13) are

$$R^{(3)ij}{}_{kl} = \frac{6}{L^2}\delta^{[i}_{[k}g^{(3)j]}{}_{l]}\,, \qquad R^{(3)zi}{}_{zj} = -\frac{3}{2L^2}g^{(3)i}{}_{j}\,, \tag{2.49}$$

we find exactly

$$\mathcal{L}^{(3)} = \frac{3(D-3)}{L^2} C(L) \mathrm{Tr}\, g^{(3)} \,. \tag{2.50}$$

With all of this, we can compute the terms of third order in the projections of the equations of motion, which read

$$\mathcal{E}^{(3)z}{}_z = \frac{1}{2L^2} \left(a^{(3)}(L) + (D-1) b^{(3)}(L) \right) \mathrm{Tr}\, g^{(3)} \,, \tag{2.51}$$

$$\mathcal{E}^{(3)i}{}_j = \frac{D-4}{2L^2} \left(a^{(3)}(L) g^{(3)i}{}_j + b^{(3)}(L) \delta^i_j \mathrm{Tr}\, g^{(3)} \right) \,, \tag{2.52}$$

where the constants $a^{(3)}(L)$ and $b^{(3)}(L)$ are related to those introduced in Eq. (2.48) as

$$\begin{aligned} a^{(3)}(L) &= 3C(L) + 4(D-6) D^{(3)}(L) - (D-3) A^{(3)}(L) \,, \\ b^{(3)}(L) &= -3C(L) - A^{(3)}(L) - 2(D-2) B^{(3)}(L) + 4(D-6) E^{(3)}(L) \,. \end{aligned} \tag{2.53}$$

For completeness, we give the values of the constants $a^{(3)}(L)$ and $b^{(3)}(L)$ for general quadratic and cubic theories of gravity in Appendix B.

From the discussion above on the different contributions to $\mathcal{L}^{(3)}$ and $P^{(3)\mu\nu}{}_{\rho\sigma}$, it is clear that the equations $\mathcal{E}^{(3)z}{}_z = 0$ fixes, for general HCGs,[4] $\mathrm{Tr}\, g^{(3)} = 0$. This in turn means, when we consider the equation $\mathcal{E}^{(3)i}{}_j = 0$, that

$$g^{(3)}_{ij} = 0 \,. \tag{2.54}$$

As we commented when fixing $g^{(1)}_{ij} = 0$, there are families of theories for which the value of $g^{(3)}_{ij}$ is not determined by the equations of motion. In particular, the same analysis discussed after Eq. (2.31) applies in this case as well, but considering the coefficients $a^{(3)}(L)$ and $b^{(3)}(L)$ instead. All the quadratic and cubic theories considered in Section 2.2.1 and Appendix B, at the particular points mentioned, allow for $g^{(3)}_{ij} \neq 0$ even when choosing $g^{(1)}_{ij} = 0$ as a boundary condition.[5] However, the conditions that leave $g^{(3)}_{ij}$ undetermined do not imply degeneracy of the different AdS vacua for the quadratic and cubic theories considered in the Appendix.

2.3 Counterterms for generic HCGs in $D \leq 5$

Now that we understand the behavior of the bulk metric near the AdS boundary, by means of the Fefferman-Graham expansion (2.13) and (2.14), we are in position to tackle the problem of regularization of the action.

It was suggested in [119], for Einsteinian cubic gravity, that one can renormalize the action using the same boundary terms that are used in the holographic renormalization of Einstein-AdS gravity, reviewed in Section 2.1, multiplied by a coupling-dependent overall coefficient. This was introduced with the objective of the cancellation of divergences of the gravity action, leaving aside the well-posedness of the variational principle, since for higher-curvature gravities

4 Notice that $\mathrm{Tr}\, g^{(3)}$ must vanish for $D = 4$, while $g^{(3)}_{ij}$ is left undetermined. This is expected, given that in three boundary dimensions $g^{(3)}_{ij}$ is dual to the stress-energy tensor and there is no conformal anomaly.

5 Except for new massive gravity, since it is a 3-dimensional theory, ant therefore the coefficient $g^{(3)}_{ij}$ is sub-normalizable and should not be considered in the expansion (2.14).

(except for Lovelock theories [211]) this is an open problem. As explained in what follows, this idea for generating counterterms based on the Einstein-AdS case can be generalized to arbitrary HCGs considering the asymptotic behavior of AlAdS spaces.

When considering pure AdS vacua, a minimal requirement for the renormalization procedure is to render the Euclidean on-shell action equal to either zero or the vacuum energy of the maximally-symmetric configuration. The vacuum energy appears if the bulk is odd-dimensional, and in the context of AdS/CFT it is related to the Casimir energy of the CFT side. One can then assume that the boundary term for HCGs is equal to the one for Einstein gravity with a coupling-dependent overall factor, that can then be fixed by requiring the cancellation of divergences in the action for the maximally-symmetric solution. Said action evaluated in the vacuum solution is proportional to the AdS volume, with an overall constant that depends on the couplings of the theory, and one can then check if the same boundary term works for other AlAdS solutions besides the pure AdS configuration. This will be performed explicitly in Section 2.4, making use of the FG expansion reviewed above.

A similar approach was pursued in [225] and [226], where the authors considered some counterterms with a multiplicative constant —that matches the prescription in [119]— in order to compute the Noether-Wald charges for quadratic curvature gravities in even-dimensional AlAdS spacetimes. In [227] the same additional terms are introduced to obtain finite entanglement entropies.

The counterterms considered in these last three references are, however, different from the usual HR proposal. The latter prescription produces a series of terms, whose complexity depends on the dimension and can not be expressed in any closed form. The alternative approach consists on adding to the action some topological quantities dubbed *Kounterterms*, because they can be naturally written in terms of the extrinsic curvature of the boundary. They were originally proposed in [18, 19, 213–216] to renormalize the Einstein-Hilbert action and obtain a well-posed variational principle, which then allows to compute finite conserved charges in AdS gravity. Moreover, this method has been considered also for the computation of renormalized entanglement entropies [224, 228, 229].

In the present work, we aim to expand this prescription to more general theories of gravity admitting AlAdS solutions in up to 5 dimensions, whose metric can be expanded in terms of the radial coordinate as in Eqs. (2.13) and (2.14), with the coefficients written in Eq. (2.15). In this Section we will simply introduce the form of the Kounterterms for even and odd dimensional bulks, as given in the literature. The only modification that we propose is to multiply these boundary terms by the constant $C(L)$ defined in Eq. (2.12), which is the only theory-dependent part of the entire expression. In Sections 2.4 and 2.5 we will see that this constant appears naturally in the terms that need to be cancelled, thus motivating our prescription.

2.3.1 Kounterterms for even bulk dimensions

The Kounterterms for $D = 2n$ dimensions are given by [18]

$$S_{\mathrm{Kt}} = c_{2n-1} \int_{\partial\mathcal{M}} d^{2n-1}x\, B_{2n-1}[h,\, K,\, \mathcal{R}]\,, \tag{2.55}$$

where B_{2n-1} is the n-th Chern form[6]

$$B_{2n-1} = -2n\sqrt{-h}\int_0^1 dt\, \delta^{i_1\dots i_{2n-1}}_{j_1\dots j_{2n-1}} K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3} - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3}\right) \times \cdots \times \left(\frac{1}{2}\mathcal{R}^{j_{2n-2} j_{2n-1}}_{i_{2n-2} i_{2n-1}} - t^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}}\right), \tag{2.56}$$

and we write the constant c_{2n-1} as

$$c_{2n-1} = -\frac{\left(-L^2\right)^{n-1}}{n(2n-2)!}C(L)\,. \tag{2.57}$$

This recovers the usual value of the constant for Einstein gravity, presented for example in [230], since in that case $C(L) = 1/(16\pi G_{\mathrm{N}})$ with our conventions. However, we claim that this boundary term is suitable for more general theories of gravity whose Lagrangian is made of arbitrary contractions of the Riemann tensor, in particular, whose bulk is 4-dimensional.

As shown in [230], for Einstein gravity the Kounterterm (2.55) is exactly equivalent to the usual HR prescription in $D = 4$ and, at least, in $D = 6$ as long as the boundary is conformally flat. We will see this explicitly in Section 2.4, when we show that it cancels the divergences of the on-shell action in 4 dimensions.

Besides, let us mention that by means of Euler's theorem this even-dimensional Kounterterm can be written also as a bulk integral. In particular,

$$\int_{\mathcal{M}_{2n}} d^{2n}x\,\mathcal{E}_{2n} = (4\pi)^n n!\,\chi\left(\mathcal{M}_{2n}\right) + \int_{\partial\mathcal{M}_{2n}} d^{2n-1}x B_{2n-1}\,, \tag{2.58}$$

where $\chi\left(\mathcal{M}_{2n}\right)$ is the Euler characteristic of the manifold $\mathcal{M}_{2n}$, and $\mathcal{E}_{2n}$ is the $2n$-dimensional Euler density

$$\mathcal{E}_{2n} = \frac{\sqrt{-G}}{2^n}\delta^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}} R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}}\,. \tag{2.59}$$

This alternative form, albeit interesting, will not be necessary for our purposes.

2.3.2 Kounterterms for odd bulk dimensions

For $D = 2n+1$ bulk dimensions, the Kounterterm reads [19]

$$S_{\mathrm{Kt}} = c_{2n}\int_{\partial\mathcal{M}} d^{2n}x\, B_{2n}[h,\, K,\, \mathcal{R}]\,, \tag{2.60}$$

where the integrand B_{2n} is given by

$$B_{2n} = -2n\sqrt{-h}\int_0^1 dt\int_0^t ds\,\delta^{i_1\dots i_{2n}}_{j_1\dots j_{2n}} K^{j_1}_{i_1}\delta^{j_2}_{i_2}\left(\frac{1}{2}\mathcal{R}^{j_3 j_4}_{i_3 i_4} - t^2 K^{j_3}_{i_3}K^{j_4}_{i_4} + \frac{s^2}{L^2}\delta^{j_3}_{i_3}\delta^{j_4}_{i_4}\right) \times\cdots\times\left(\frac{1}{2}\mathcal{R}^{j_{2n-1} j_{2n}}_{i_{2n-1} i_{2n}} - t^2 K^{j_{2n-1}}_{i_{2n-1}}K^{j_{2n}}_{i_{2n}} + \frac{s^2}{L^2}\delta^{j_{2n-1}}_{i_{2n-1}}\delta^{j_{2n}}_{i_{2n}}\right), \tag{2.61}$$

6 In these expressions, $\delta^{i_1\dots i_n}_{j_1\dots j_n}$ is the generalized Kronecker delta defined in Eq. (15).

and the overall constant c_{2n} is

$$c_{2n} = -\frac{\left(-L^2\right)^{n-1}}{2^{2n-2}n(n-1!^2)}C(L)\,. \tag{2.62}$$

As in the even-dimensional case, we recover the values of this constant presented in [230] if we set $C(L) = 1/(16\pi G_{\text{N}})$, as corresponds to Einstein gravity. Also, this is equivalent to the counterterms derived with the HR proposal, up to logarithmic divergent terms, in $D = 3$, 5 and 7, as long as the boundary is conformally flat [230].

In this case, B_{2n} can not be written as the pullback of a topological quantity in the $D = 2n+1$ manifold, contrasting with what is found for even dimensions. Also, the fact that B_{2n} depends on the AdS radius L, while B_{2n-1} does not, is an indicative of the topological origin of the latter.

2.4 Divergence cancellation in HCGs up to $D = 5$

We now address the problem of renormalizing the action of a general higher-curvature gravity when evaluated on an AlAdS background. For this matter, we will first find the form of the divergent terms at the boundary, with a general expression valid for $D \leq 5$. This restriction is motivated by the fact that we are interested on holographic applications in realistic situations, with strongly coupled gauge theories in at most four dimensions. Then, we will analyze the divergent terms explicitly for $D = 3$, 4 and 5, and show that they are indeed cancelled by the Kounterterm presented in Section 2.3.

2.4.1 Divergent terms in the on-shell action

Let us consider the action of a general higher-curvature theory of gravity in $D \leq 5$,

$$S = \int_{\mathcal{M}} d^D X \sqrt{-G}\mathcal{L}\left(R^{\rho\sigma}_{\mu\nu}\right)\,, \tag{2.63}$$

and evaluate it on an asymptotically locally AdS spacetime. We want to find the divergent terms that appear in each dimension, so we write the metric as in Eq. (2.13), with the coefficients of the FG expansion given by Eq. (2.15). Since in these coordinates the asymptotic boundary is located at $z \to 0$, we can identify the divergences in this region simply by looking at the terms with negative powers of z in the expansion of the action. Given the form of the metric (2.13), to leading order near $z = 0$ the square root of the determinant behaves as

$$\sqrt{-G} \sim \frac{1}{z^D}\,, \tag{2.64}$$

plus additional higher-order contributions that decay faster near the boundary. However, this leading behavior is enough to identify the terms in the Lagrangian that will produce divergences. As shown in Section 2.2, the odd coefficients in the FG expansion of the tangent metric (2.14) up to the order that we are interested in vanish, so the on-shell Lagrangian can only have terms of the form z^{2i} with $i \in \mathbb{Z}^+$. In the action, these result in

$$\int dz \sqrt{-G} z^{2i} \sim \int dz\, z^{2i-D} \sim z^{2i-(D-1)}\,. \tag{2.65}$$

So such a term can produce three different behaviors as $z \to 0$:

- If $i < (D-1)/2$ the term is divergent, and it needs to be subtracted.
- If $i > (D-1)/2$ the term vanishes at the boundary.
- For odd spacetime dimensions, there can be contributions with $i = (D-1)/2$. In this case the integral above is not correct, as it produces a logarithmic divergence at the boundary. This is universal and related to the conformal anomaly of the dual CFT [223, 230], and it is not cancelled by the topological Kounterterms.

Therefore, depending on the dimensionality of the spacetime the last term that produces divergences will be different. In our case, for up to 5 dimensions we will have to look at terms with $i \leq 2$. This is the reason why the higher-order contributions in the counterterm of standard HR, given in Eq. (2.5), only appear for large enough D.

In order to isolate the divergent terms, first of all we have to obtain an expansion of the Lagrangian $\mathcal{L}\left(R_{\mu\nu}^{\rho\sigma}\right)$ close to the boundary. This can be written as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(0)} + P_{\rho\sigma}^{\mu\nu}\delta R_{\mu\nu}^{\rho\sigma} + \dots \\ &= \mathcal{L}^{(0)} + P_{kl}^{ij}\delta R_{ij}^{kl} + 4P_{jk}^{zi}\delta R_{zi}^{jk} + 4P_{zi}^{zj}\delta R_{zj}^{zi} + \dots \, , \end{aligned} \tag{2.66}$$

where $\delta R_{\mu\nu}^{\rho\sigma}$ denotes the terms in the components of the Riemann tensor that are different from the background value (2.10), this is, those that depend on z. They can be computed using the Gauss-Codazzi equations (2.20), finding

$$\begin{aligned} \delta R_{kl}^{ij} &= \frac{z^2}{(D-2)(D-3)}\mathcal{R}^{(0)}\delta_{[k}^{i}\delta_{l]}^{j} - \frac{4z^2}{D-3}\mathcal{R}^{(0)[i}_{\ \ [k}\delta_{l]}^{j]} + z^2\mathcal{R}^{(0)ij}_{\ \ kl} + \mathcal{O}\left(z^4\right) \, , \\ \delta R_{zi}^{jk} &= \frac{L^2 z^3}{D-3}\left(\tilde{\nabla}^{(0)j}\mathcal{R}^{(0)k}_{\ \ i} - \tilde{\nabla}^{(0)k}\mathcal{R}^{(0)j}_{\ \ i}\right) + \frac{L^2 z^3}{2(D-3)(D-2)}\left(\delta_i^j\tilde{\nabla}^{(0)k}\mathcal{R}^{(0)} - \delta_i^k\tilde{\nabla}^{(0)j}\mathcal{R}^{(0)}\right) \\ &\quad + \mathcal{O}\left(z^5\right) \, , \\ \delta R_{zi}^{zj} &= \mathcal{O}\left(z^4\right) \, , \end{aligned} \tag{2.67}$$

where we introduced the shorthand notation $\tilde{\nabla}^{(0)} \equiv \tilde{\nabla}\left(g^{(0)}\right)$ and $\mathcal{R}^{(0)} \equiv \mathcal{R}\left(g^{(0)}\right)$. Therefore, since $P^{(0)zi}_{\ \ \ jk} = 0$, the lowest order contributions in the expansion of the Lagrangian are

$$\mathcal{L} = \mathcal{L}^{(0)} + P^{(0)ij}_{\ \ \ kl}\delta R_{ij}^{kl} + \mathcal{O}\left(z^4\right) \, , \tag{2.68}$$

where $\mathcal{O}\left(z^4\right)$ includes terms coming from $P_{jk}^{zi}\delta R_{zi}^{jk}$, and others with higher powers of z from the expansion of $P_{\rho\sigma}^{\mu\nu}$.

Also, although we are not writing them explicitly, in this expression there should be terms with higher derivatives of the Lagrangian with respect to the Riemann tensor. However, since any such derivative is, to the lowest possible order, constant in z, a term of the form $(\partial^n\mathcal{L}/(\partial R)^n)(\delta R)^n$ will be at least $\mathcal{O}\left(z^{2n}\right)$. Therefore, second or higher derivatives are unimportant when looking for the divergences in low dimensions.

If we now compute $P^{(0)ij}_{\ \ \ kl}\delta R_{ij}^{kl}$ explicitly with the expressions above for $\delta R_{\mu\nu}^{\rho\sigma}$ and $P^{(0)\mu\nu}_{\ \ \ \rho\sigma}$ given by Eq. (2.11), we find that it vanishes to the lowest order in z. So we conclude that the only divergent part of $\mathcal{L}\left(R_{\mu\nu}^{\rho\sigma}\right)$ is

$$\mathcal{L} = \mathcal{L}^{(0)} + z^4\mathcal{L}^{(4)} \, , \tag{2.69}$$

where the term $\mathcal{L}^{(4)}$ will contribute only to the logarithmic divergence in $D = 5$. This contains all the terms of order z^4 mentioned in the paragraph above, but its particular form is not relevant for our computations, since we assume that our method of renormalization will not cancel divergences of this type.

Now that we have expanded the Lagrangian in the coordinate z, we need to do the same with the determinant factor that appears in the action (2.63). Using again the FG expansion of the metric given by Eqs. (2.13) and (2.14), this is

$$\sqrt{-G} = \frac{L}{z}\sqrt{-h} = \frac{L\sqrt{-\mathrm{g}^{(0)}}}{z^D}\left(1 + \frac{z^2}{2}\mathrm{Tr}\, \mathrm{g}^{(2)} + \mathcal{O}\left(z^4\right)\right). \tag{2.70}$$

Plugging everything in, the divergent terms of the general action (2.63) for $D \leq 5$ are

$$\begin{aligned} S_{\text{diver}} &= \int_{\mathcal{M}} d^D X \sqrt{-G}\left(\mathcal{L}^{(0)} + z^4 \mathcal{L}^{(4)}\right) \\ &= L\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-\mathrm{g}^{(0)}}\int_{z=z_0} dz\left(\frac{1}{z^D}\mathcal{L}^{(0)} + \frac{1}{2z^{D-2}}\mathcal{L}^{(0)}\mathrm{Tr}\, \mathrm{g}^{(0)} + \mathcal{O}\left(z^{4-D}\right)\right), \end{aligned} \tag{2.71}$$

where we introduced the cutoff z_0 in the lower limit of integration, which must be taken to zero once the divergences have been cancelled. The actual form of these divergent contributions once integrated in z depend on the dimension of the spacetime, and in particular the second and third terms produce the aforementioned logarithmic divergences at $D = 3$ and $D = 5$, respectively. However, the terms that we want to cancel, the power-law divergences, always appear multiplied by $\mathcal{L}^{(0)}$ at these low dimensions. This is proportional to the constant $C(L)$ appearing at the lowest order in $P^{\mu\nu}_{\rho\sigma}$ through Eq. (2.12), or equivalently

$$\mathcal{L}^{(0)} = -\frac{2(D-1)}{L^2}C(L). \tag{2.72}$$

Therefore, this supports our claim that the Kounterterms which cancel these divergences are the same as those introduced for Einstein gravity, with the general prefactor $C(L)$.

2.4.2 Explicit analysis in different dimensions

We will now show how the Kounterterms introduced in Section 2.3 are able to cancel the divergences in Eq. (2.71), explicitly in 3-, 4- and 5-dimensional spacetimes. Notice that the computations carried out here were already done in Section 3.4 of [230], and the only difference in our results is the generic constant $C(L)$ that multiplies both the Kounterterms and the divergent terms in the on-shell action. In order to see that, we will have to write the objects in (2.55) and (2.60) in terms of the intrinsic curvature of the boundary metric $g^{(0)}_{ij}$.

In particular, provided the coefficients of the FG expansion are given by Eq. (2.15), the extrinsic curvature reads

$$\begin{aligned} K^i_j = h^{ik}K_{kj} &= \frac{1}{L}\delta^i_j - \frac{z^2}{L}{g^{(2)}}^i{}_j + \frac{z^4}{L}\left({g^{(2)}}^{ik}g^{(2)}_{jk} - 2{g^{(4)}}^i{}_j\right) \\ &= \frac{1}{L}\delta^i_j + z^2\frac{L}{D-3}\left(\mathcal{R}^i_j - \frac{1}{2(D-2)}\mathcal{R}^{(0)}\delta^i_j\right) + \mathcal{O}\left(z^4\right). \end{aligned} \tag{2.73}$$

The determinant of the tangent metric also needs to be expanded in powers of the radial coordinate, and by means of Eq. (2.15b) it can be written as

$$\sqrt{-h} = \frac{\sqrt{-g^{(0)}}}{z^{D-1}} \left(1 - z^2 \frac{L^2}{4(D-2)} \mathcal{R}^{(0)} + \mathcal{O}\left(z^4\right)\right) . \tag{2.74}$$

2.4.2.1 3 *bulk dimensions*

In $D = 3$, the divergent terms in (2.71) become, after integrating in z,

$$S_{\text{diver}} = -C(L) \int d^2x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^2} + L \log z_0 \mathcal{R}^{(0)}\right] , \tag{2.75}$$

where we used Eq. (2.15b) to rewrite $\operatorname{Tr} g^{(2)}$, and Eq. (2.72) for $\mathcal{L}^{(0)}$. We need to find whether the Kounterterm (2.60), particularized for $D = 2n + 1 = 3$, cancels the divergences found here. In this case, the constant c_2 and the function B_2 are equal to

$$c_2 = -C(L) , \qquad B_2 = -\sqrt{-h} K . \tag{2.76}$$

Therefore, replacing the determinant h and the extrinsic curvature K in terms of the intrinsic curvature of $g_{ij}^{(0)}$, we find that the total Kounterterm for $D = 3$ dimensions reads

$$S_{\text{Kt}} = c_2 \int d^2x\, B_2 = C(L) \int d^2x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^2} + \mathcal{O}\left(z_0^2\right)\right] , \tag{2.77}$$

which cancels the power-law divergence found in Eq. (2.75), but not the logarithmic one, as we had anticipated in Section 2.3.

2.4.2.2 4 *bulk dimensions*

The divergent terms (2.71) in *d* bulk dimensions become

$$S_{\text{diver}} = -C(L) \int d^3x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^3} - \frac{3L}{4z_0} \mathcal{R}^{(0)}\right] . \tag{2.78}$$

The coupling of the Kounterterm that should cancel this and the second Chern form read

$$c_3 = \frac{L^2}{4} C(L) , \qquad B_3 = -4\sqrt{-h}\,\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} K^{j_1}_{i_1} \left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3} - \frac{1}{3} K^{j_2}_{i_2} K^{j_3}_{i_3}\right) . \tag{2.79}$$

Writing it all together, the Kounterterm for a general theory in $D = 4$ is

$$S_{\text{Kt}} = C(L) \int d^3x \sqrt{-g^{(0)}} \left[\frac{2}{Lz_0^3} - \frac{3L}{4z_0} \mathcal{R}^{(0)} + \mathcal{O}\left(z_0\right)\right] , \tag{2.80}$$

which cancels exactly the divergent terms as written in Eq. (2.78).

2.4.2.3 5 *bulk dimensions*

In this case, the divergent part of the action will have an additional logarithmic term, which depends on $\mathrm{Tr}\, g^{(4)}$ and $\mathcal{L}^{(4)}$,

$$S_{\text{diver}} = -C(L)\int d^4x\sqrt{-g^{(0)}}\left[\frac{2}{Lz_0^4} - \frac{L}{3z_0^2}\mathcal{R}^{(0)} + \mathcal{O}(\log z_0)\right]. \tag{2.81}$$

As before, this should be regularized by the Kounterterm (2.60) particularized for $D = 2n+1 = 5$. The value of the constant c_4 and the function B_4 are

$$c_4 = \frac{L^2}{8}C(L), \qquad B_4 = -\sqrt{-h}\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3}K^{j_1}_{i_1}\left(\mathcal{R}^{j_2 j_3}_{i_2 i_3} - K^{j_2}_{i_2}K^{j_3}_{i_3} + \frac{1}{3L^2}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right). \tag{2.82}$$

Then the total Kounterterm in this dimension is

$$S_{\text{Kt}} = C(L)\int d^4x\sqrt{-g^{(0)}}\left[\frac{2}{Lz_0^4} - \frac{L}{3z_0^2}\mathcal{R}^{(0)} + \mathcal{O}(1)\right], \tag{2.83}$$

which cancels the divergences (2.81) except for the logarithmic one, as in the case $D = 3$. As mentioned before, these divergences are universal terms, proportional to the conformal anomaly of the dual field theory, and therefore they were not expected to be cancelled out by this renormalization procedure.

2.5 Variational principle in HCGs up to $D = 5$

The other problem that we need to take into account is that of the well-posedness of the variational principle, which ensures that the equations of motion for the metric are obtained with the condition that the value of the metric, and not its derivatives, is fixed at the boundary. In practice, this requires that the boundary terms that appear when varying the action depend only on the variation of the metric of the conformal boundary, $g^{(0)}_{ij}$.

We will show here that the Kounterterms presented in Section 2.3 are also able to achieve this in the dimensions that we are interested in. First we will obtain the form of the boundary terms that we need to cancel for general dimensions up to 5, and then particularize the analysis to $D = 3$, 4 and 5 as done in the previous Section to treat the divergences of the on-shell action.

2.5.1 Divergences in the boundary term of the variation of a general HCG

Let us consider again the action (2.63) for a general theory of gravity with higher-order contractions of the Riemann tensor. Its variation produces two contributions [83, 86],

$$\delta S = \int_{\mathcal{M}} d^DX\sqrt{-G}\mathcal{E}_{\mu\nu}\delta G^{\mu\nu} + \epsilon\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-h}n_\mu\delta v^\mu. \tag{2.84}$$

The first of these is proportional to the equations of motion (2.8), and thus vanishes on-shell, while the second one is a contraction of the vector normal to the boundary $\partial\mathcal{M}$ (normalized such that $n_\mu n^\mu = \epsilon = \pm 1$) and the quantity[7]

$$\delta v^\mu = -2P^{\mu\rho\sigma}{}_\nu \delta\Gamma^\nu_{\rho\sigma} - 2\nabla_\nu P^{\mu\rho\sigma\nu}\delta G_{\rho\sigma} \,. \tag{2.86}$$

We want to evaluate the boundary term in a solution of the equations of motion with AdS asymptotics, so we consider the metric to be given by the usual FG expansion given by Eqs. (2.13), (2.14) and (2.15). The vector normal to the boundary is given in Eq. (2.16), and therefore $\epsilon = 1$ from now on.

In order to evaluate the boundary term in δS we need expressions for the variation of the Christoffel symbols. In particular, the ones we need can be written in terms of variations of the extrinsic curvature (2.19) as

$$\delta\Gamma^i_{jk} = \frac{1}{2}h^{il}\left(\nabla_k\delta h_{jl} + \nabla_j\delta h_{kl} - \nabla_l\delta h_{jk}\right)\,, \qquad \delta\Gamma^z_{ij} = \frac{z}{L}\delta K_{ij}\,, \qquad \delta\Gamma^i_{zj} = -\frac{L}{z}\delta K^i_j\,. \tag{2.87}$$

Also, we expand the tensor $P^{\mu\nu}_{\rho\sigma}$ asymptotically as

$$P^{\mu\nu}_{\rho\sigma} = \frac{C(L)}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right) + \delta P^{\mu\nu}_{\rho\sigma}\,, \tag{2.88}$$

where $\delta P^{\mu\nu}_{\rho\sigma}$ is not a variation, but a symbolic way of writing all the terms in $P^{\mu\nu}_{\rho\sigma}$ that contain powers of z (see Eq. (2.90) below). As explained, when evaluated on-shell the variation of the action is equal to the boundary term proportional to (2.86), which after replacing the expressions above for $\delta\Gamma^\mu_{\rho\sigma}$ and $P^{\mu\nu}_{\rho\sigma}$ becomes

$$\begin{aligned}\delta S = -\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-h}\Big[&C(L)\left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i\right) + 2\delta P^{zi}_{zj}\left(2\delta K^j_i + \left(h^{-1}\delta h\right)^j_k K^k_i\right)\\ &- 4n_z\nabla_l\delta P^{zi}_{jk}h^{jl}\left(h^{-1}\delta h\right)^k_i - 2n^z h^{jk}\nabla_z\delta P^{zi}_{zk}\delta h_{ij}\Big]\,,\end{aligned} \tag{2.89}$$

where $\left(h^{-1}\delta h\right)^i_j \equiv h^{ik}\delta h_{kj}$, and the variation $\delta h_{ij} = \delta g_{ij}/z^2$ can be written as a variation of the metric on the conformal boundary $\delta g^{(0)}_{ij}$ on-shell, due to the relations (2.14) and (2.15). If we restricted the analysis to Einstein gravity, we would have $C(L) = 1/(16\pi G_{\rm N})$ and $\delta P^{\mu\nu}_{\alpha\beta} = 0$, independently of the background. Therefore, only the first term in these expressions would contribute, recovering the results in Appendix D of [224].

The expressions above are written in terms of variations of the metric and the extrinsic curvature. However, they are related on-shell through Eqs. (2.15) and (2.19), and therefore the variations of K_{ij} can be written as variations of the metric $g^{(0)}_{ij}$, thus leading to a well-posed Dirichlet problem once we get rid of the divergences. The next step is to expand $\delta P^{\mu\nu}_{\rho\sigma}$ in (2.89) in powers of z. We are not interested in more than $D = 5$ bulk dimensions, so it is enough to keep only the terms with powers up to z^4, as the behavior of the determinant in the integrand to leading order is $\sqrt{-h} \sim z^{-(D-1)}$. Knowing that $g^{(1)}_{ij} = 0$, we can expand $\delta P^{\mu\nu}_{\rho\sigma}$ as

$$\delta P^{\mu\nu}_{\rho\sigma} \equiv z^2 P^{(2)\mu\nu}{}_{\rho\sigma} + z^3 P^{(3)\mu\nu}{}_{\rho\sigma} + z^4 P^{(4)\mu\nu}{}_{\rho\sigma} + \ldots\,. \tag{2.90}$$

7 Notice that this is equal to the boundary term written in Eq. (1.10), which can be checked if one takes into account that the variation of the affine connection is given by

$$\delta\Gamma^\nu_{\rho\sigma} = \frac{1}{2}G^{\nu\lambda}\left(\nabla_\rho\delta G_{\lambda\sigma} + \nabla_\sigma\delta G_{\rho\lambda} - \nabla_\lambda\delta G_{\rho\sigma}\right)\,. \tag{2.85}$$

Plugging this into (2.89) and evaluating the covariant derivatives explicitly yields

$$\begin{aligned}\delta S = -\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{-h}\Big[C(L)\left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i\right)\\ + 2\frac{z_0^2}{L}\left(-(2D-7){P^{(2)}}^{zk}_{zi} + 2{P^{(2)}}^{lk}_{li}\right)\left(h^{-1}\delta h\right)^i_k\\ + 4\frac{z_0^3}{L}\left(-(D-4){P^{(3)}}^{zk}_{zi} + {P^{(3)}}^{lk}_{li}\right)\left(h^{-1}\delta h\right)^i_k + z_0^4\mathcal{O}\left(\delta g^{(0)}_{ij}\right) + \ldots\Big].\end{aligned} \tag{2.91}$$

The term of order z_0^3 inside the brackets contains the contractions ${P^{(3)}}^{zk}_{zi}$ and ${P^{(3)}}^{lk}_{li}$, which can be seen to vanish[8] provided $g^{(3)}_{ij} = 0$, following the reasoning of Section 2.2.3. So these divergences do not appear in the general theories that we are interested in.[9]

The contribution at order z_0^2 requires some attention, but we will see that the two contractions of ${P^{(2)}}^{\mu\nu}_{\rho\sigma}$ appearing here vanish when evaluated on-shell. The tensor $P^{\mu\nu}_{\rho\sigma}$ is defined as the derivative of the Lagrangian $\mathcal{L}\left(R^{\mu\nu}_{\rho\sigma}\right)$ with respect to the Riemann tensor, so its components will be given by contractions of the curvatures with four free indices that fulfill the symmetries of the Riemann tensor itself. Since we are interested in the form of the terms at order z^2 in this tensor, ${P^{(2)}}^{\mu\nu}_{\rho\sigma}$, we need to study the components of the Riemann up to this order, which are given on-shell by

$$\begin{aligned} R^{ij}_{kl} &= -\frac{2}{L^2}\delta^{[i}_k\delta^{j]}_l + z^2\left(\frac{4}{L^2}\delta^{[i}_{[k}{g^{(2)}}^{j]}_{l]} + \mathcal{R}^{ij}_{kl}\right) + \mathcal{O}\left(z^4\right),\\ R^{zi}_{zj} &= -\frac{1}{L^2}\delta^i_j + \mathcal{O}\left(z^4\right),\\ R^{zi}_{jk} &= \mathcal{O}\left(z^3\right).\end{aligned} \tag{2.92}$$

As said, also contractions of the curvature can contribute to $P^{\mu\nu}_{\rho\sigma}$ in a general theory, and in particular in this case it is enough to consider R^{ik}_{jk} and R^{zi}_{zi}. If we impose that the equations of motion are fulfilled, $g^{(2)}_{ij}$ is given by Eq. (2.15b) and thus the form of these contractions is found to be

$$R^{ik}_{jk} = -\frac{D-2}{L^2}\delta^i_j + \mathcal{O}\left(z^4\right), \qquad R^{zi}_{zi} = -\frac{D-1}{L^2} + \mathcal{O}\left(z^4\right). \tag{2.93}$$

Thereby we see that the uncontracted components R^{ij}_{kl} are the only ones that can contribute to $P^{\mu\nu}_{\rho\sigma}$ on-shell at order z^2. This means that we can write

$${P^{(2)}}^{ij}_{kl} = C^{(2)}\left(\frac{4}{L^2}\delta^{[i}_{[k}{g^{(2)}}^{j]}_{l]} + \mathcal{R}^{ij}_{kl}\right), \qquad {P^{(2)}}^{zi}_{zj} = {P^{(2)}}^{zi}_{jk} = 0, \tag{2.94}$$

where $C^{(2)}$ is a constant depending upon the parameters of the particular theory that we consider. However, these expressions are enough to see that, once $g^{(2)}_{ij}$ is replaced by its form

8 As seen before, in $D = 4$ only $\mathrm{Tr}\, g^{(3)} = 0$ is fixed, while the other components of $g^{(3)}_{ij}$ (the off-trace part) are free and correspond to the holographic stress-energy tensor of the dual CFT. In this case it can happen that ${P^{(3)}}^{lk}_{li} \neq 0$, thus inducing a constant term at the boundary, which is standard.

9 The expansion of ${P^{(3)}}^{\mu\nu}_{\rho\sigma}$ also contains terms of the form $\nabla g^{(2)}$. However, these are only present in the components ${P^{(3)}}^{zi}_{jk}$ or ${P^{(3)}}^{ij}_{zk}$, which are absent at this order, although they might appear at higher orders in the expansion of the boundary term of the variation (2.89).

given in Eq. (2.15b),

$$P^{(2)}{}^{zk}_{zi} = P^{(2)}{}^{lk}_{li} = 0\,, \tag{2.95}$$

and thus the terms at order z^2 in the boundary term of the variation of the action, written in Eq. (2.91), are zero on-shell in general.

Gathering everything up, we see that the boundary term of the variation relevant for $D \leq 5$ reads

$$\delta S = -\int_{\partial\mathcal{M}} d^{D-1}x\,\sqrt{-h}\Big[C(L)\left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i\right) + z_0^4\mathcal{O}\left(\delta g^{(0)}_{ij}\right) + \ldots\Big]\,. \tag{2.96}$$

By simple inspection, we observe that the only divergent part in up to 5 bulk dimensions is the same as that of Einstein gravity, presented for example in [224], multiplied by the constant $C(L)$ which depends on the particular details of the theory. Therefore, we will assume that the divergences in this object can be regularized by adding to the original action (2.63) the usual boundary Kounterterms that are known to work for Einstein gravity, multiplied by the constant $C(L)$ as given in Section 2.3, and which we already know that are enough to cancel the divergences of the on-shell action in these dimensions.

2.5.2 Explicit analysis in different dimensions

We will now see how the Kounterterms presented in Section 2.3 can be used to cancel the divergences in the boundary term that appears when varying the action on-shell, given by Eq. (2.96). Although the following computations are carried out more generally in Appendix D of [224], here we will show them more explicitly in up to 5 bulk dimensions.

2.5.2.1 3 *bulk dimensions*

In this case the determinant factor is $\sqrt{-h} \sim z^{-2}$, so the only non-vanishing terms of the variation (2.96) are simply

$$\delta S = -C(L)\int_{\partial\mathcal{M}} d^2x\,\sqrt{-h}\left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i\right)\,. \tag{2.97}$$

In order to regularize this, we add the Kounterterm (2.60) particularized to $D = 3$, which reads

$$S_{\text{Kt}} = C(L)\int_{\partial\mathcal{M}} d^2x\,\sqrt{-h}K\,. \tag{2.98}$$

It is straightforward to compute its variation, finding

$$\delta S_{\text{Kt}} = C(L)\int_{\partial\mathcal{M}} d^2x\,\sqrt{-h}\left(\delta K^i_i + \frac{1}{2}\left(h^{-1}\delta h\right)^i_i K\right)\,. \tag{2.99}$$

Then, adding this to δS above we get the variation of the regularized action in $D = 3$,

$$\delta S_{\text{reg}} = \delta S + \delta S_{\text{Kt}} = C(L)\int_{\partial\mathcal{M}} d^2x\,\sqrt{-h}\left(\frac{1}{2}\left(h^{-1}\delta h\right)^i_i K - \left(h^{-1}\delta h\right)^i_j K^j_i - \delta K^i_i\right)\,. \tag{2.100}$$

To the lowest orders in z, the extrinsic curvature behaves as given in Eq. (2.73), which can be written as

$$K^i_j = \frac{1}{L}\delta^i_j + z^2 L S^i_j\left(g^{(0)}\right) + \ldots\,, \tag{2.101}$$

where $\mathcal{S}^i_j\left(g^{(0)}\right)$ is the Schouten tensor of the boundary metric $g^{(0)}_{ij}$,

$$\mathcal{S}_{ij}\left(g^{(0)}\right) = \frac{1}{D-3}\left(\mathcal{R}_{ij}\left(g^{(0)}\right) - \frac{1}{2(D-2)}\mathcal{R}\left(g^{(0)}\right) g^{(0)}_{ij}\right) . \tag{2.102}$$

Using this, we see that the terms in the parenthesis start at order z^2, which is finite when multiplied by the determinant factor. Therefore the Kounterterm cancels the divergences in the variation for this dimension of the spacetime, and it also allows the variation to be written only in terms of variations with respect to $g^{(0)}_{ij}$, thus leading to a well-posed variational problem.

2.5.2.2 4 *bulk dimensions*

The divergent boundary terms of the variation of the action in $D = 4$ are

$$\delta S = -C(L)\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i\right) . \tag{2.103}$$

These should be cancelled by the Kounterterm (2.55) with $n = 2$, which is

$$S_{\text{Kt}} = -L^2C(L)\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\,\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3} - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}\right) . \tag{2.104}$$

Its variation can be evaluated explicitly term by term, and the final result reads [224]

$$\begin{aligned}\delta S_{\text{Kt}} = C(L)&\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i\right)\\ &+L^2C(L)\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Big[W^{il}_{jl}\left(\left(h^{-1}\delta h\right)^j_k K^k_i + 2\delta K^j_i\right)\\ &\qquad - \delta^{ijk}_{lmn}K^l_i\tilde{\nabla}^n\tilde{\nabla}_j\left(h^{-1}\delta h\right)^m_k\Big] .\end{aligned} \tag{2.105}$$

The first term in this expression cancels exactly the divergent terms in the variation (2.103). The second integral in δS_{Kt} is finite if we assume Asymptotic Conformal Flatness. Indeed, in this case the relevant components of the bulk Weyl tensor behave as[10] $W^{il}_{jl} \sim z^{D-1} = z^3$, and since $\delta K^i_j \sim z^2$ due to Eq. (2.101), only the first term in that parenthesis contributes. The last term of Eq. (2.105) can also be shown to vanish for this number of dimensions. Assuming that the boundary submanifold is infinite, we can integrate by parts without adding a boundary term,

$$\sqrt{-h}\,\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}\tilde{\nabla}^{j_3}\tilde{\nabla}_{i_2}\left(h^{-1}\delta h\right)^{j_2}_{i_3} \longrightarrow \sqrt{-h}\,\delta^{i_1i_2i_3}_{j_1j_2j_3}\left(h^{-1}\delta h\right)^{j_2}_{i_3}\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1} . \tag{2.106}$$

But since $\tilde{\nabla}_l K^{j_1}_{i_2} \sim z^2$ at least (to zeroth order of K^i_j is proportional to δ^i_j) and the indices of the covariant derivative are raised with the metric $h^{ij} = z^2 g^{ij}$, we have

$$\sqrt{-h}\,\delta^{i_1i_2i_3}_{j_1j_2j_3}\left(h^{-1}\delta h\right)^{j_2}_{i_3}\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1} \sim z\,, \tag{2.107}$$

[10] The Asymptotic Conformal Flatness (ACF) condition [224] implies that the leading behavior of the bulk Weyl tensor with all indices in the directions tangent to the boundary is that of a normalizable mode, $W^{ij}_{kl} \sim z^{D-1}$. As shown in Appendix C of [224], ACF is equivalent to the boundary being conformally flat. In the case $D = 4$ this is fulfilled directly, since any 3-dimensional manifold is conformally flat, but in higher dimensions it is necessary to specifically require the Weyl tensor of the boundary metric to vanish.

so this term vanishes as $z \to 0$. Therefore, the boundary term of the variation of the regularized action in $D = 4$ reads

$$\delta S_{\text{reg}} = \delta S + \delta S_{\text{Kt}} = L^2 C(L) \int_{\partial\mathcal{M}} d^3x \sqrt{-h} W^{il}_{jl} \left(h^{-1}\delta h\right)^j_k K^k_i \,, \tag{2.108}$$

which is finite and can be written as depending only on the variations $\delta g^{(0)}_{ij}$, thus leading to a well-posed variational problem with Dirichlet boundary conditions and no divergences.

2.5.2.3 5 *bulk dimensions*

In 5 bulk dimensions the form of the divergent terms is the same as before, with the difference that now also the terms of order z_0^4 contribute,

$$\delta S = -\int_{\partial\mathcal{M}} d^{D-1}x \sqrt{-h} \Big[C(L) \left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i \right) + z_0^4 \mathcal{O}\left(\delta g^{(0)}_{ij}\right) + \ldots \Big] \,. \tag{2.109}$$

In this case only the first term inside the brackets produces divergences, and the second one is constant in z_0, so it does not need to be subtracted and its actual form will not be relevant for our purposes. The boundary Kounterterm that should cancel these divergences is (2.60) with $n = 2$,

$$S_{\text{Kt}} = -\frac{L^2}{8} C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h}\, \delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} K^{j_1}_{i_1} \left(\mathcal{R}^{j_2 j_3}_{i_2 i_3} - K^{j_2}_{i_2} K^{j_3}_{i_3} + \frac{1}{3L^2} \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) . \tag{2.110}$$

Obtaining the variation of this Kounterterm entails a rather involved computation, which again can be carried out following Appendix D of [224]. The final result reads

$$\delta S_{\text{Kt}} = C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h} \left(2\delta K^i_i + \left(h^{-1}\delta h\right)^i_j K^j_i \right) + \delta S^{(W)} + \delta S^{(0)} + \delta S^{(\tilde\nabla)} \,, \tag{2.111}$$

where the first term cancels exactly the divergent part of (2.109), and we have defined

$$\delta S^{(W)} = -\frac{L^2}{8} C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h}\, \delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} W^{j_2 j_3}_{i_2 i_3} \left(2\delta K^{j_1}_{i_1} + \left(h^{-1}\delta h\right)^{j_1}_k K^k_{i_1} \right) , \tag{2.112}$$

$$\begin{aligned} \delta S^{(0)} = \frac{L^2}{16} C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h}\, \delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} & \left(\mathcal{R}^{j_3 j_4}_{i_3 i_4} - K^{j_3}_{i_3} K^{j_4}_{i_4} + \frac{1}{L^2} \delta^{j_3}_{i_3} \delta^{j_4}_{i_4} \right) \\ & \times \left(\left(h^{-1}\delta h\right)^{j_1}_k \left(K^k_{i_1} \delta^{j_2}_{i_2} - \delta^k_{i_1} K^{j_2}_{i_2} \right) + 2\delta^{j_1}_{i_1} \delta K^{j_2}_{i_2} \right) , \end{aligned} \tag{2.113}$$

$$\delta S^{(\tilde\nabla)} = \frac{L^2}{4} C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h}\, \delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} \left(h^{-1}\delta h\right)^{j_2}_{i_3} \tilde\nabla_{i_2} \tilde\nabla^{j_3} K^{j_1}_{i_1} \,. \tag{2.114}$$

With this, the variation of the total regularized action is

$$\delta S_{\text{reg}} = \delta S^{(W)} + \delta S^{(0)} + \delta S^{(\tilde\nabla)} + \delta S^{(z_0^4)} \,, \tag{2.115}$$

where $\delta S^{(z_0^4)}$ corresponds to the terms of order z_0^4 in δS that produce a constant in the integrand, and whose particular form depends on the theory. In order to show that this variation of the regularized action is finite we should count the powers of z appearing in each of the terms, which we do now.

- The variation $\delta S^{(W)}$ can be rewritten by expanding the sum in the indices of the antisymmetric delta, and using $W_{ij}^{ij} = 0$, which follows from $W_{\mu\nu}^{\mu\nu} = W_{\mu i}^{\mu i} = 0$. We find

$$\delta S^{(W)} = \frac{L^2}{2} C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h}\, W_{jl}^{il} \left(2\delta K_i^j + \left(h^{-1}\delta h\right)^j_k K_i^k \right) . \tag{2.116}$$

But now recall that Eq. (2.101) implies that $\delta K_i^j \sim z^2$, and under the assumption of conformal flatness $W_{jl}^{il} \sim z^{D-1} = z^4$. Then, since $\sqrt{-h} \sim z^{-4}$, the term with δK_i^j in the parenthesis does not contribute, and we can simply write

$$\delta S^{(W)} = \frac{L^2}{2} C(L) \int_{\partial\mathcal{M}} d^4x \sqrt{-h}\, W_{jl}^{il} \left(h^{-1}\delta h\right)^j_k K_i^k . \tag{2.117}$$

- The first parenthesis in $\delta S^{(0)}$ can be rewritten in terms of the Weyl tensor of the bulk metric, using the Gauss-Codazzi equation (A.10),

$$R_{kl}^{ij} = \mathcal{R}_{kl}^{ij} - 2K^i_{[k} K^j_{l]} , \tag{2.118}$$

and the definition of the Weyl tensor, which to the lowest order in z yields

$$W_{kl}^{ij} = R_{kl}^{ij} + \frac{2}{L^2} \delta^i_{[k} \delta^j_{l]} . \tag{2.119}$$

These two expressions can be combined in the form

$$\mathcal{R}_{kl}^{ij} = W_{kl}^{ij} + 2K^i_{[k} K^j_{l]} - \frac{2}{L^2} \delta^i_{[k} \delta^j_{l]} . \tag{2.120}$$

The first parenthesis in $\delta S^{(0)}$, taking into account the prefactor $\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4}$ can now be rewritten as

$$\mathcal{R}^{j_3 j_4}_{i_3 i_4} - K^{j_3}_{i_3} K^{j_4}_{i_4} + \frac{1}{L^2} \delta^{j_3}_{i_3} \delta^{j_4}_{i_4} = W^{j_3 j_4}_{i_3 i_4} + K^{j_3}_{i_3} K^{j_4}_{i_4} - \frac{1}{L^2} \delta^{j_3}_{i_3} \delta^{j_4}_{i_4} , \tag{2.121}$$

and since $W_{kl}^{ij} \sim z^4$ and $K^i_j = \delta^i_j / L + \mathcal{O}\left(z^2\right)$, we see that to the lowest order

$$\mathcal{R}^{j_3 j_4}_{i_3 i_4} - K^{j_3}_{i_3} K^{j_4}_{i_4} + \frac{1}{L^2} \delta^{j_3}_{i_3} \delta^{j_4}_{i_4} \sim z^2 . \tag{2.122}$$

In the second parenthesis of $\delta S^{(0)}$ in Eq. (2.113) we have

$$K^k_{i_1} \delta^{j_2}_{i_2} - \delta^k_{i_1} K^{j_2}_{i_2} \sim z^2 , \qquad \delta K^{j_2}_{i_2} \sim z^2 . \tag{2.123}$$

Therefore, the whole integrand starts at order z^4, and when integrated with $d^4x \sqrt{-h}$ it produces a term that is constant and thus non-divergent when $z \to 0$.

- Performing a naive power counting in the term $\delta S^{(\tilde{\nabla})}$, we could find that it produces a constant at the boundary $z \to 0$. Indeed, as was shown in $D = 4$, $\tilde{\nabla}_l K^{j_1}_{i_2} \sim z^2$. Therefore, one might be tempted to think that

$$\sqrt{-h}\, \delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} \tilde{\nabla}_{i_2} \tilde{\nabla}^{j_3} K^{j_1}_{i_1} \sim 1 . \tag{2.124}$$

However, this expression vanishes if we impose the boundary to be conformally flat. Indeed, since we can expand K^i_j in terms of the Schouten tensor as in Eq. (2.101), to leading order in z we can write

$$\begin{aligned}\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1} &= z^2 L\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}\mathcal{S}^{(0)}{}^{j_1}_{i_1} + \ldots = z^4 L\tilde{\nabla}^{(0)}_{i_2}\tilde{\nabla}^{(0)j_3}\mathcal{S}^{(0)}{}^{j_1}_{i_1} + \ldots \\ &= z^3\frac{L}{2}\tilde{\nabla}^{(0)}_{i_2}\left(\tilde{\nabla}^{(0)j_3}\mathcal{S}^{(0)}{}^{j_1}_{i_1} - \tilde{\nabla}^{(0)j_1}\mathcal{S}^{(0)}{}^{j_3}_{i_1}\right) + \ldots\,,\end{aligned} \tag{2.125}$$

where quantities with the superscript (0) correspond to the boundary metric $g^{(0)}_{ij}$, and their indices are raised using that instead of h^{ij}, thus the extra z^2 factor in the second step. To get to the last line we used the fact that this quantity is contracted with a generalized Kronecker delta, and hence it is antisymmetrized in the indices j_1 and j_3. We can now use the definition of the Cotton tensor,

$$C^{(0)}_{ijk} = \tilde{\nabla}^{(0)}_k\mathcal{S}^{(0)}_{ij} - \tilde{\nabla}^{(0)}_j\mathcal{S}^{(0)}_{ik}\,, \tag{2.126}$$

in order to rewrite the expression above as

$$\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1} = z^4\frac{L}{2}\tilde{\nabla}^{(0)}_{i_2}C^{(0)}{}_{i_1}{}^{j_1 j_3} + \ldots\,. \tag{2.127}$$

But the Cotton tensor of $g^{(0)}$ is related to its Weyl tensor as [230]

$$\tilde{\nabla}^{(0)}_{i_2}C^{(0)}{}_{i_1}{}^{j_1 j_3} = \frac{1}{D-4}\tilde{\nabla}^{(0)}_{i_2}\tilde{\nabla}^{(0)l}\mathcal{W}^{(0)}{}_{i_1}{}^{j_3 j_1}{}_l\,, \tag{2.128}$$

which is zero if we impose the metric to be conformally flat, $\mathcal{W}^{(0)}{}^{ij}_{kl} = 0$. Therefore, the term of order z^4 in $\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1}$ vanishes, and $\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1} \sim z^6$, which means that the total integrand in $\delta S^{(\tilde{\nabla})}$ is zero in $D = 5$, since

$$\sqrt{-h}\,\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3}\left(h^{-1}\delta h\right)^{j_2}_{i_3}\tilde{\nabla}_{i_2}\tilde{\nabla}^{j_3}K^{j_1}_{i_1} \sim z^2 \tag{2.129}$$

vanishes at the boundary $z \to 0$.

Gathering everything up, we find that the boundary term of the variation of the regularized action in 5 bulk dimensions is

$$\delta S_{\text{reg}} = \int_{\partial\mathcal{M}} d^4x\,\sqrt{-h}\left[\frac{L^2}{2}C(L)W^{il}_{jl}\left(h^{-1}\delta h\right)^j_k K^k_i + z_0^4\mathcal{O}\left(g^{(0)}_{ij}\right)\right]\,, \tag{2.130}$$

which again corresponds to a well-posed Dirichlet variational problem. The last term between brackets contains the contributions of order z_0^4 appearing in the original variation (2.109), as well as those coming from $\delta S^{(0)}$.

2.6 Discussion

In this Chapter, we have proposed a renormalization procedure that can be applied to arbitrary higher-curvature gravity theories, evaluated on asymptotically locally AdS manifolds and with up to 5 bulk dimensions.[11] This method uses the counterterms proposed in [18, 19], whose expressions involve the extrinsic curvature of the boundary, but with a theory dependent coupling constant, as given in Eqs. (2.57) and (2.62).

In order to show the universality of the method, first we decompose the equations of motion of an arbitrary higher-curvature theory into their components normal and tangential to the boundary, and expand them in powers of the holographic Poincare coordinate. By means of these, we are able to argue that in general the relevant odd coefficients of the Fefferman-Graham expansion of the bulk metric, $g_{ij}^{(1)}$ and $g_{ij}^{(3)}$, are zero. Furthermore, requiring symmetry under the PBH transformations [217] we argue that $g_{ij}^{(2)}$ is constrained to have the universal form given in Eq. (2.15b). Then, considering these general features of the FG expansion, we verify that the proposed procedure is enough to ensure the cancellation of divergences on the on-shell action (Section 2.4) and the well-posedness of the variational principle (Section 2.5).

The argument presented fails for particular theories (discussed in Section 2.2.1 and Appendix B), for which the equations of motion do not constrain the form of the coefficients $g_{ij}^{(1)}$ and/or $g_{ij}^{(3)}$ of the expansion. Even though these theories correspond to zero-measure submanifolds in the theory space spanned by the couplings of the higher-curvature terms, they are interesting on their own, since they include theories displaying degenerate AdS vacua and modified AdS asymptotics. While one usually wants to avoid such behaviors, it can be interesting to use the conditions obtained in Section 2.2.1 to look for new exotic theories of gravity.

Expanding the method to higher dimensions would be a natural continuation of this work, but this entails some additional difficulties. Besides the need to extend further the FG expansion (2.14), in $D > 5$ one would need to consider also the divergent terms of $\mathcal{O}\left(z^{4-D}\right)$ and higher in Eq. (2.71), which would become relevant. However, with our analysis we have no reason to think that these will be proportional to $\mathcal{L}^{(0)}$, as required for the current approach to work. In any case, we can not make any claim in this respect, and an explicit study would be needed for larger dimensions.

Finally, while the study presented here is very abstract, in the sense that we only propose and show the validity of the method, it can be considered for more practical purposes such as obtaining finite asymptotic charges for black hole solutions in higher-curvature theories. Also, in the context of the AdS/CFT correspondence, one would use the method for renormalizing holographic entanglement entropies, as in [224, 227–229, 231]. In fact, our prescription has been used to compute entanglement entropies and some related quantities, as anomaly coefficients and central charges, for general cubic theories in [232, 233].

[11] In the case $D = 5$, one needs to explicitly require Asymptotic Conformal Flatness, which is equivalent to the boundary being conformally flat [224]. This is needed to guarantee that $g_{ij}^{(2)}$ has the universal form given in Eq. (2.45) and for the variational principle to be well-posed, as discussed in Section 2.5.2.3.

3

Holographic superconductor in Einsteinian cubic gravity

The holographic duality relates the dynamics of a strongly-coupled QFT in its large N limit to those of a classical gravitational system [12], much easier to treat. Therefore, it can be regarded as a tool to study strongly correlated quantum systems, that are untractable by conventional methods. Although it was originally formulated in the realm of high-energy physics, the AdS/CFT correspondence has been applied also to different condensed matter setups, the resulting branch of research being known as AdS/CMT, for condensed matter theory [234].

The simplest example of one such system is the "holographic superconductor," whose existence was first established in [20, 21] (see also [22]). This is a $(2+1)$-dimensional system with a U(1) symmetry, which is spontaneously broken below a critical temperature by the condensation of a scalar operator $\mathcal{O}$, thus giving rise to a transition into the superconducting phase [235]. At the other side of the duality, this corresponds to a planar black hole in $(3+1)$-dimensional AdS spacetime, with a scalar field which develops an instability below the critical temperature. This holographic setup reproduces several known features of high-T_c superconductors [236], which are not described by the usual BPS theory, and whose physical mechanism is not known. The crystalline structure of these materials is often layered, with a strong coupling inside each of them, so it is reasonable to study them as this $(2+1)$-dimensional holographic model.

As mentioned before, the classical regime of the gravity system corresponds strictly to the infinite N limit of the QFT. However, this is not realistic enough, and one would try to improve this description by introducing $1/N$ corrections into the boundary setup. A subset of these diagrams —which also include $1/\sqrt{\lambda}$ effects, being λ the 't Hooft coupling— can be resummed producing higher-curvature terms in the bulk action [72]. So classical holography in the higher-order gravity background would take into account some such corrections to the dual field theory.

In a top-down approach it could in principle be possible to find the form of these higher-curvature terms, provided we know the D-brane configuration, field content and regularization scheme at hand. This however is not feasible in general, so we will stick to the bottom-up construction that we always follow in this thesis, choosing our gravitational Lagrangian based on generic physical requirements alone. As always, we want a theory that admits black hole solutions with a single function, does not propagate any modes other than the massless graviton and is non-trivial in 4 dimensions, in particular for an AdS background. The lowest order theory fulfilling these conditions that we know of is Einsteinian cubic gravity, introduced in Section 1.1.3. In particular, the gravitational part of the action will be

$$S_{\text{Gravity}} = S_{\text{EH}} + S^{(3)}_{\text{GQG}} = \frac{1}{16\pi G_{\text{N}}} \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{\tilde{\beta} L^4}{54} \left(\mathcal{P} - 8\mathcal{C} \right) \right] , \tag{3.1}$$

where G_{N} is Newton's constant, L the AdS length scale and $\tilde{\beta}$ the coupling parameter of the cubic terms, normalized in a convenient way for this Chapter. $\mathcal{P}$ and $\mathcal{C}$ are the cubic Lagrangian densities, whose form is given in Eqs. (1.24) and (1.33), and the combination $\mathcal{P} - 8\mathcal{C}$ allows the theory to have also a well posed cosmological scenario [76]. Although this construction could be generalized to any higher order in the curvature [78], for simplicity we will stick to the cubic case, which should be enough to identify some features of generic corrected models.

This gravitational action can be used to describe a dual CFT with a non-zero value for the stress tensor three-point function t_4 —which is a function of the coupling $\tilde{\beta}$ [119]—, but it can also produce causality issues [103, 104, 237, 238] for relatively large values of the coupling. If we further required the positivity of energy fluxes as in [183], we would need $|t_4| < 4$, which results in a bound on the coupling as restrictive as $0 \leq \tilde{\beta} \leq 0.0211$ [119]. However, this would lead to very small corrections on the system and we did not find any noticeable effect when this bound is violated. Therefore, since our aim is to explore the effect of the cubic terms we will ignore this constraint.

The Coleman-Mermin-Wagner theorem rules out the spontaneous breaking of a continuous symmetry in $(2+1)$-dimensional systems at finite temperature, due to the presence of long-wavelength fluctuations that prevent the order parameter from having a non-zero expectation value [239, 240]. In a holographic setting, these fluctuations would be suppressed in the large N limit [241], since the Goldstone fluctuations only show up at subleading order. Said regime corresponds to Einstein gravity in the holographic framework, and this is the reason why the scalar field is able to condense during a phase transition.

In order to restore the Coleman-Mermin-Wagner theorem, one could naively think that it is enough to consider corrections away from the large N limit. However, since it is a large distance effect it is not restored by simply adding these $1/N$ contributions, and it would indeed require the calculation of Witten loops in the bulk [242] (see also [236]). In spite of this, it has been observed that different higher-curvature corrections make the scalar condensation more difficult, delaying the superconducting phase transition to lower temperatures [243–246]. However, in all these cases the system is higher-dimensional, and thus the theorem cannot be directly applied. The interest of studying the superconductor in Einsteinian cubic gravity resides in the fact that it is the first theory that allows us to explore the effects of such higher-curvature corrections on a $(2+1)$ dimensional boundary theory.

The contents of the current Chapter have been publised in [25]. The details of that article can be found in page 211 of this document.

3.1 The model

Let us start the study by defining the setting of our problem. We want to describe a $(2+1)$-dimensional s-wave superconductor,[1] so we consider a holographic model in $(3+1)$ dimensions.

[1] A superconductor system in condensed matter physics is characterized by an order parameter, which is typically the wavefunction that determines the density of superconducting charge carriers. For the s-wave models that

The boundary has a global U(1) symmetry, which we extend into the bulk by adding a U(1) gauge field, and a charged scalar that will correspond to the condensate that will break the symmetry in the superconducting phase transition. The entire action of our bulk system is

$$S = S_{\text{Gravity}} + S_{\text{Maxwell}} + S_{\text{Scalar}}\,, \tag{3.2}$$

where S_{Gravity} is the gravitational action given by Eq. (3.1), and the other terms correspond to the electromagnetic and charged scalar fields,

$$S_{\text{Maxwell}} + S_{\text{Scalar}} = -\int d^4x\,\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |\partial\psi - iqA\psi|^2 + m^2|\psi|^2\right)\,. \tag{3.3}$$

Here, m and q are respectively the mass and the electric charge of the scalar.

The action has a well-defined probe limit, in which the scalar and electromagnetic fields do not curve the background geometry. This corresponds to the regime in which the fields ψ and A and their derivatives are small, since the energy momentum tensor is quadratic in those fields, while their equations of motion contain linear terms. However, in order to keep some interaction between the two fields in this limit, one should take also $q \to \infty$ while keeping the products qA and $q\psi$ finite. In this work we will consider both this probe limit and the fully backreacting regime.

We are interested in describing a spatially infinite system at equilibrium at the boundary, so we look for solutions that are static and have AdS asymptotics, and with a $(2+1)$-dimensional boundary. This is realized by the planar ansatz

$$ds^2 = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}\left(dx^2 + dy^2\right)\,, \tag{3.4}$$

$$\psi = \psi(r)\,, \qquad A = \phi(r)dt\,, \tag{3.5}$$

which requires the cosmological constant appearing in Eq. (3.1) to take the value

$$\Lambda = -\frac{3}{L^2}\,. \tag{3.6}$$

In these expressions, L is a constant with units of length, usually referred to as the AdS length scale.

The equations of motion for the coupled fields can be obtained by using the reduced action approach described in Section 1.1.3, which consists on evaluating the action with the ansatz and then computing the functional derivative with respect to each of the functions in the fields. After setting $16\pi G_{\text{N}} = 1$, which we will keep for the entire Chapter, we end up with

$$\psi'' + \left(\frac{2}{r} + \frac{N'}{N} + \frac{f'}{f}\right)\psi' + \frac{1}{f}\left(\frac{q^2\phi^2}{N^2 f} - m^2\right)\psi = 0\,, \tag{3.7a}$$

$$\phi'' + \left(\frac{2}{r} - \frac{N'}{N}\right)\phi' - \frac{2q^2\psi^2}{f}\phi = 0\,, \tag{3.7b}$$

$$fNr^3\left[2rn^2\left(6\frac{r}{L^2} - m^2r\psi^2 - 2f'\right) - 2fN^2\left(r^2\psi'^2 + 2\right) - r^2\phi'^2\right] - 2q^2r^5N\psi^2\phi^2$$

we are interested on this carries no angular momentum and is spherically symmetric, so it can be considered a scalar function.

$$
\begin{aligned}
&+ \frac{2}{27}\tilde{\beta}L^4\Big\{24f^2N^3f'(rf'-f) - 24f^4N'\big(2(r^2N'^2-N^2)+rNN'\big)\\
&+ 6rff'N'\Big[8fN^2(rf'-2f) - r^2Nf'(5fN'+2Nf') + 4rf^2N'(4N+rN')\Big]\\
&+ 6rNf^2f''\Big[4f\big(-rN^2f' + (N^2+rNN'-r^2N'^2)\big) + r^2N^2f''\Big]\\
&+ 12rf^2NN''\Big[2f^2(3rN'-N) + rf'\big(N(2f-rf') - 3rN'f\big) + r^2Nff''\Big]\\
&+ 3r^2N^2f^3f^{(3)}\big(4(rN'-N) + 2rf^2Nf'\big)\Big\} = 0\,,
\end{aligned}
\tag{3.7c}
$$

$$
\begin{aligned}
&Nr^3\Big(2f^2NN' - rf^2N^2\psi'^2 - q^2r\psi^2\phi^2\Big)\\
&+ \frac{2}{27}\tilde{\beta}L^4\Big\{3N'\Big[4N^2f^2f'(rf'-2f) + f^2N'\big(4f^2(N-2rN') + 3r^2f'(2fN'-Nf')\big)\Big]\\
&+ 3Nf^2N'f''\Big[Nr(8f-5rf') - 6r^2fN'\Big]\\
&+ 3f^2N''\Big[2N'f\big(2Nr(f-3rf') + r^2fN'\big) + N^2\big(2f(4rf'-f) - 3r^2f'^2\big)\Big]\\
&- 3r^2Nf^3N''\big(2fN''+Nf''\big) - 3rNf^3N^{(3)}\big(2f(rN'-N) + rNf'\big)\Big\} = 0\,.
\end{aligned}
\tag{3.7d}
$$

Near the AdS boundary, which corresponds to $r \to \infty$ with our choice of coordinates, the functions in the metric can be expanded as

$$
f(r) = \frac{r^2}{L^2}f_\infty + \mathcal{O}\left(r^{-1}\right)\,, \tag{3.8}
$$

$$
N(r) = N_\infty + \mathcal{O}\left(r^{-2}\right)\,, \tag{3.9}
$$

where f_∞ and N_∞ are constants. By simple inspection of Eq. (3.4), one can notice that if we rescale the bulk time coordinate t in such a way that $N_\infty^2 f_\infty = 1$, then it also measures the boundary time. On the other hand, a horizon at a finite $r = r_h$ will provide a finite temperature to both the bulk and boundary systems. The metric functions can be expanded around that point as

$$
f(r) = \frac{4\pi T}{N_h}(r-r_h) + \mathcal{O}\left((r-r_h)^2\right)\,, \tag{3.10}
$$

$$
N(r) = N_h + \mathcal{O}(r-r_h)\,, \tag{3.11}
$$

where N_h is a constant that sets the time unit at the horizon and T is the Hawking temperature of the black hole. These expressions, supplemented by suitable expansions of the fields $\psi(r)$ and $\phi(r)$, can be plugged into the equations of motion to relate the different constants and obtain the subleading terms. This computation will be performed numerically in what follows, for the different phases and limits.

3.2 The normal phase

In order to study this system we must first introduce the normal, or non-superconducting, phase. This is the vacuum configuration, corresponding to a zero value of the condensate and the electrostatic potential, $\psi = \phi = 0$. In this case the equations of motion (3.7c) and (3.7d) are simplified and can be partially integrated as for any Generalized Quasitopological theory of gravity, obtaining

$$\frac{r^3}{L^2} - rf + \frac{L^4}{27}\tilde{\beta}\left(3ff'f'' - f'^3 + \frac{6}{r^2}f^2\left(f' - rf''\right)\right) = M\,, \tag{3.12}$$

$$N(r) = N\,, \tag{3.13}$$

where we introduced the integration constants M, which is proportional to the mass of the black hole [119], and N, which sets the time units.

Plugging the asymptotic expansions (3.8) and (3.9) into these equations of motion we obtain the relation

$$\frac{4}{27}\tilde{\beta} f_\infty^3 - f_\infty + 1 = 0\,, \tag{3.14}$$

and as said before we choose $N = N_\infty = 1/\sqrt{f_\infty}$ in order for t to correspond to the time coordinate of the boundary.

Notice that Eq. (3.14) is solved by up to three different values of f_∞ for a given $\tilde{\beta}$. However, in order for the metric to have the correct signature we need $f_\infty > 0$, which is only possible for $\tilde{\beta} \leq 1$. Besides, imposing that the terms of higher order in the asymptotic expansion of $f(r)$ are well-behaved, so that black hole solutions exist, requires $\tilde{\beta} \geq 0$ [119]. Thus, we are left with

$$0 \leq \tilde{\beta} \leq 1\,. \tag{3.15}$$

In this range the equation still has two possible roots, but only one of them produces a positive effective Newton constant for the gravitational perturbations. That solution flows into Einstein AdS when $\tilde{\beta}$ goes to zero [97], namely $\lim_{\tilde{\beta}\to 0} f_\infty = 1$. This can be solved analytically, finding

$$f_\infty\left(\tilde{\beta}\right) = \frac{3}{\sqrt{\tilde{\beta}}}\sin\left[\frac{1}{3}\arcsin\left(\sqrt{\tilde{\beta}}\right)\right]\,, \tag{3.16}$$

which takes values in the range $f_\infty \in [1,\, 3/2]$, the largest value corresponding to $\tilde{\beta} = 1$.[2] In consequence, the form of the metric is completely determined by the integration constant M and the cubic coupling $\tilde{\beta}$.

We should now turn to the behavior of the metric at the horizon. Plugging the expansions (3.10) and (3.11) in the equation of motion (3.12), separating terms with different powers of $r - r_h$ and imposing them to vanish independently, we find the temperature and radius of the horizon,

$$T = \frac{3r_h}{4\pi L^2\sqrt{f_\infty}}\,, \qquad r_h = \left(\frac{ML^2}{1-\tilde{\beta}}\right)^{1/3}\,. \tag{3.18}$$

[2] The effect of the higher-curvature terms at the boundary can be interpreted as a rescaling of the AdS radius. Indeed, by looking at the asymptotic expansion (3.8) we see that it is natural to absorb f_∞ in a redefinition

$$L \longrightarrow \tilde{L} = \frac{L}{\sqrt{f_\infty}}\,, \tag{3.17}$$

and since $f_\infty \leq 1$ we conclude that the cubic terms decrease the effective AdS radius. This is completely analogous to what we find in Section 4.2, when studying the effect of the Gauss-Bonnet term on the AdS asymptotics.

By looking at these expressions, we see that the limit $\tilde{\beta} = 1$ is of particular interest. Even though the horizon radius seems to diverge, one can easily check that Eq. (3.12) with $M = 0$ is solved by the analytical function

$$f(r) = \frac{3}{2L^2}\left(r^2 - r_h^2\right) , \tag{3.19}$$

for any value of r_h. Therefore, in this limit the sense of scale of the solution is somewhat lost, since neither the horizon radius nor the temperature are determined by the equations of motion. For simplicity, in the numerical computations we will set $r_h = 1$ when using this critical solution.

We still need to obtain the form of $f(r)$ for general values of $\tilde{\beta}$ inside the allowed range (3.15). In the GR limit $\tilde{\beta} = 0$, it takes the form

$$f(r) = \frac{r^2}{L^2}\left(1 - \frac{ML^2}{r^3}\right) , \tag{3.20}$$

while in the critical one $\tilde{\beta} = 1$ it is given by Eq. (3.19). However, for any other value of the coupling we need to solve Eq. (3.12) numerically. In principle this could be achieved by implementing the numerical shooting procedure described in [5, 119]. While this method is able to produce numerically accurate solutions, the calculation becomes too complex when trying to obtain the backreacted solution. In that case $\psi(r) \neq 0$ and $\phi(r) \neq 0$, so we need to consider the entire higher-order set of equations of motion (3.7), and in general $N(r) \neq$ constant.

In this work we instead employ a numerical relaxation method [247], described in Appendix C. The main advantage of this approach is that it is able to naturally implement boundary conditions at both ends of the integration interval, which makes it very appealing for our purposes. It requires a discretization of the space of values of the independent variable, so the first step is to make the range $r \in [r_h, \infty)$ finite by introducing the inverse coordinate $z = L^2/r$. Then we need to specify a seed for the unknown function, which is iteratively modified until it converges to the solution. In practice we solve for $L^2 f(r)/r^2$, and choose the seed to be a linear function in z, equal to 0 at the horizon and to f_∞ at the boundary.

The solutions found with the relaxation method for several values of $\tilde{\beta}$ are shown in Figure 3.1. The numerical accuracy of this procedure was checked by plugging the solution back into the equation of motion, and it was found to yield significantly lower errors than the shooting procedure, at least for the first few derivatives of $f(r)$, which are enough for our purposes. The numerical solution also matches the exact forms known for the cases $\tilde{\beta} = 0$ and $\tilde{\beta} = 1$.

3.3 The superconducting phase

At low enough temperatures, the near horizon region of the 4-dimensional AdS black hole metric (3.4) is equal to an AdS_2 geometry. As in the standard holographic superconductor, a choice of the mass of the scalar field that is stable in the asymptotic AdS_4 regime can violate the two-dimensional Breitenlohner-Freedman (BF) bound in the near horizon region [22], leading to an instability that results in the development of a charged scalar hair, which represents the superconducting phase.

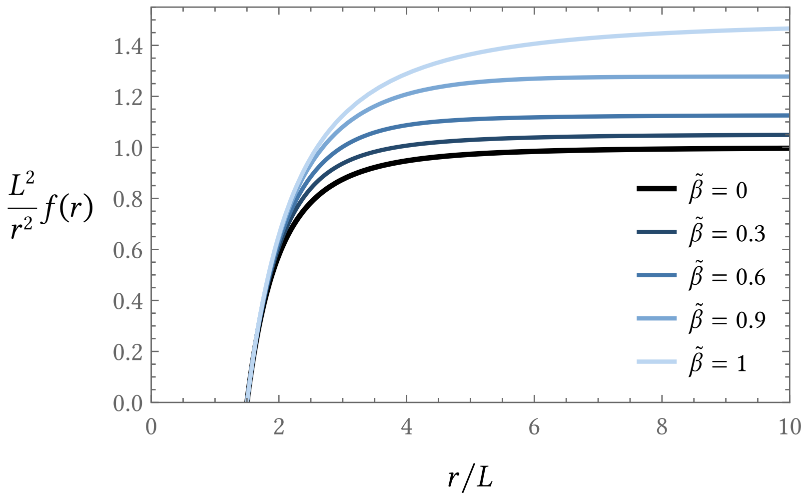


Figure 3.1: Function $f(r)$ of the black hole metric, for different values of $\tilde{\beta}$ and $r_h = 1.5L$. This was obtained by solving numerically Eq. (3.12) in terms of the coordinate $z = L^2/r$, using the relaxation method with 40 points in the range $z \in \left[0,\, L^2/r_h\right]$.

3.3.1 Instability of the scalar field at low temperature

In this Section we will review the procedure that allows the scalar field to develop a non-zero value at low enough temperatures, which is nothing but the mechanism of spontaneous symmetry breaking that produces the superconducting phase. We will also take into account the role of the cubic terms in this phenomenon.

Let us consider the scalar field ψ to have a mass m and a vanishing value initially, with an electrostatic potential $\phi(r)$ that modifies the spacetime metric, now corresponding to an AdS Reissner-Nordström black hole. Since $\psi = 0$, the equation of motion of the potential (3.7b) becomes

$$\frac{d}{dr}\left(\frac{r^2\phi'(r)}{N(r)}\right) = 0\,, \tag{3.21}$$

which is solved by

$$\phi'(r) = Q\frac{L^2}{r^2}N(r)\,, \tag{3.22}$$

where Q is a constant proportional to the electric charge of the black hole. This ansatz also admits a single-function solution in our cubic theory, so we can set $N(r) = N_\infty$, whose value will be fixed later. Then, the equation of motion for the function $f(r)$ (3.7c) reduces to

$$\frac{r^3}{L^2} - rf + \frac{4\pi G L^4 Q^2}{r} + \frac{L^4}{27}\tilde{\beta}\left(3ff'f'' - f'^3 + \frac{6}{r^2}f^2\left(f' - rf''\right)\right) = M\,. \tag{3.23}$$

The asymptotic limit of this equation also corresponds to the same expansion as before, Eq. (3.8), with f_∞ given by (3.16), so we fix $N_\infty = 1/\sqrt{f_\infty}$. Let us now expand $f(r)$ near the horizon as

$$f(r) = \frac{4\pi T}{N_\infty}(r - r_h) + \frac{1}{\ell^2}(r - r_h)^2 + \mathcal{O}\left((r - r_h)^3\right)\,, \tag{3.24}$$

where ℓ is a constant with units of length. Plugging this in Eq. (3.23) we find

$$T = \frac{3r_h^4 - 4\pi G_N L^6 Q^2}{4\pi L^2 r_h^3 \sqrt{f_\infty}}\,, \qquad M = \frac{r_h^3}{L^2} + \frac{4\pi G_N L^4 Q^2}{r_h} - \frac{\tilde{\beta}}{L^2}\left(\frac{3r_h^4 - 4\pi G_N L^6 Q^2}{3r_h^3}\right)^3 . \tag{3.25}$$

Since we are interested on the behavior of the system at very low temperatures, we will take the extremal limit $T = 0$, which corresponds to

$$Q^2 = \frac{3r_h^4}{4\pi G_N L^6}\,, \qquad r_h = \left(\frac{ML^2}{4}\right)^{1/3} . \tag{3.26}$$

In this case it is easy to obtain the value of ℓ in the expansion (3.24), by replacing it again in Eq. (3.23) and taking into account the values of Q and r_h above. One finds

$$\ell^2 = \frac{L^2}{6}\,, \tag{3.27}$$

independently of the value of $\tilde{\beta}$.

Let us now introduce the coordinates τ and ρ, through the relations

$$r - r_h = \epsilon\frac{\ell^2}{L^2}\rho\,, \qquad t = \frac{L^2}{\ell^2}\frac{\tau}{\epsilon N_\infty}\,, \tag{3.28}$$

where ϵ is a small parameter. These are chosen in such a way that the near-horizon metric becomes, to first order in ϵ,

$$ds^2 \simeq -\frac{\rho^2}{\ell^2}d\tau^2 + \frac{\ell^2}{\rho^2}d\rho^2 + r_h^2\left(dx^2 + dy^2\right)\,, \tag{3.29}$$

which corresponds to the product of AdS_2 with the real plane. The electrostatic potential ϕ is also modified, and it reads[3]

$$\phi_\tau(\rho) \simeq Q\frac{L^2}{r_h^2}\rho\,. \tag{3.30}$$

The equation of motion of a scalar field $\psi(\rho)$ of mass m in this background is

$$\psi'' + \frac{2}{\rho}\psi' + \frac{\ell^2}{\rho^2}\left(\frac{\phi_\tau^2\ell^2}{\rho^2} - m^2\right)\psi = 0\,, \tag{3.31}$$

so it has an effective mass (squared) given by the second parenthesis In the extremal limit (3.26) it becomes

$$m_{\text{eff}}^2 = m^2 - \frac{3\ell^2}{4\pi G_N L^2}\,, \tag{3.32}$$

which is independent of $\tilde{\beta}$, since ℓ^2 is given by Eq. (3.27). In order for the scalar field to become unstable and condense at these low temperatures it would need to violate the BF bound, which in the AdS_2 spacetime of the near-horizon region reads [167]

$$m_{\text{eff}}^2 \geq -\frac{1}{4\ell^2}\,. \tag{3.33}$$

3 This can be seen by writing the potential as a differential form $A = \phi dt = \phi_\tau d\tau$. The coordinates t and τ are related as in Eq. (3.28), and $\phi(r)$ can be expanded near $r = r_h$ as $\phi(r) \simeq \phi'(r_h)(r - r_h)$.

Equivalently, for the original mass m this bound implies

$$m^2 \geq \frac{3\ell^2}{4\pi G_{\rm N} L^2} - \frac{1}{4\ell^2}\,. \tag{3.34}$$

This effective BF bound is independent of the coupling of the cubic terms $\tilde{\beta}$. However, the mass should be taken with respect to some scale, and since the cubic terms modify the effective AdS scale it is natural to measure it with respect to $L/\sqrt{f_\infty}$, so we should formulate this inequality in terms of m^2L^2/f_∞. If we set $L = 16\pi G_{\rm N} = 1$, as in the upcoming numerical computations, it reads

$$\frac{m^2}{f_\infty} \geq \frac{1}{2f_\infty}\,. \tag{3.35}$$

Later we will fix the value of m^2/f_∞ in such a way that this inequality is violated, so the field becomes unstable at the horizon and the superconducting phase appears. But since f_∞ increases monotonically with the coupling of the cubic terms, for a given value of m^2/f_∞ the violation of the inequality becomes smaller as $\tilde{\beta}$ grows. Therefore, we can expect that the higher-curvature terms in the gravity action will make the condensation harder, and decrease the critical temperature at which the phase transition takes place.

3.3.2 Probe limit

Let us now turn to the numerical study of the system, first in the probe limit in which the matter fields do not backreact on the metric. This is the regime in which the scalar field ψ and the electrostatic potential ϕ, as well as their derivatives, are small enough that they can be discarded in the equations of $f(r)$ and $N(r)$, (3.7c) and (3.7d) respectively, where they appear squared. The gravitational background is thus the vacuum black hole studied in Section 3.2, on which the equations for the scalar (3.7a) and gauge potential (3.7b) need to be solved. As mentioned before, in order to keep a non-trivial coupling between the gauge and scalar fields we need to take the limit $q \to \infty$, while keeping $q\psi$, $q\phi \approx$ constant.

In order to perform the numerical computation, we must first understand the expansion of the matter fields at the limits of the integration interval. At the horizon we impose regularity conditions in such a way that $A_\mu A^\mu$ does not diverge. It can be checked with the equations of motion (3.7a) and (3.7b) that this implies

$$\phi = \mathcal{O}(r - r_h)\,, \tag{3.36}$$
$$\psi = \psi_h + \mathcal{O}(r - r_h)\,, \tag{3.37}$$

where ψ_h is a constant. On the other hand, at large r we find from the same equations the asymptotic expansions

$$\phi = \mu + \frac{\rho}{r} + \mathcal{O}\left(r^{-2}\right)\,, \tag{3.38}$$
$$\psi = \psi_+\left(1 + \mathcal{O}\left(r^{-1}\right)\right) r^{-\Delta_+} + \psi_-\left(1 + \mathcal{O}\left(r^{-1}\right)\right) r^{-\Delta_-}\,, \tag{3.39}$$

where μ, ρ and $\psi_\pm$ are constants, and the exponents $\Delta_\pm$ are given by

$$\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{L^2 m^2}{f_\infty}}\,. \tag{3.40}$$

In order for the solution to be stable near the boundary, the discriminant inside this square root needs to be positive, resulting in the well known Breitenlohner-Freedman bound on the scalar mass. Also, since f_∞ increases with the coupling $\tilde{\beta}$, the bound is lowered by the cubic curvature terms, this is, the field can have a larger tachyonic mass while still being stable.

In what follows we will fix the mass of the scalar field as $m^2 = -2f_\infty/L^2$, producing the convenient values $\Delta_+ = 2$ and $\Delta_- = 1$ in order to make contact with the standard literature. This value fulfills the Breitenlohner-Freedman bound in 4 dimensions, but violates the effective near-horizon bound given in Eq. (3.34), thus allowing the superconducting phase transition to take place. It also renders both terms in the expansion (3.39) normalizable [167], and any of its coefficients $\psi_\pm$ is proportional to the expectation value of a boundary operator $\langle \mathcal{O}_\pm \rangle$, which we define as

$$\langle \mathcal{O}_\pm \rangle = \sqrt{2}\psi_\pm \,. \tag{3.41}$$

The other coefficient can be identified with the corresponding source $\mathcal{J}_\mp$, but since we are interested in spontaneous symmetry breaking we will set such source to zero.

3.3.2.1 Numerical method and boundary conditions

For the numerical computations we employ again the relaxation method explained in Appendix C. This requires the range of the independent variable to be finite, so in practice we work with the coordinate $z = L^2/r$, which satisfies $0 < z < L^2/r_h$. Besides, we also solve for the function $P(r) \equiv r\psi(r)$, which goes to either 0 or a finite value at large r, depending on the dimension of the condensate that is turned on.

This procedure has the advantage that it lets us apply boundary conditions at both ends of the interval in a natural way, so we will take advantage of this feature. In all cases, at the horizon $z = z_h \equiv L^2/r_h$ we fix the values

$$\phi(z_h) = 0 \,, \qquad P(z_h) = P_h \,, \qquad P'(z_h) = \frac{m^2L^2 - zf'(z_h)}{z^2 f'(z_h)} \,, \tag{3.42}$$

where P_h is a constant that will ultimately determine the temperature of the system, and the last expression can be obtained by studying the equation of motion (3.7a) near the horizon, taking into account the other two conditions. Furthermore, we will impose one condition at the boundary $z = 0$, depending on the condensate that we want to study. Since $P(z) \sim \psi_- + z\psi_+$, these are

$$\begin{aligned} P'(0) = 0 \quad &\text{such that} \quad \langle \mathcal{O}_- \rangle = \sqrt{2}\psi_- \neq 0 \,, \\ P(0) = 0 \quad &\text{such that} \quad \langle \mathcal{O}_+ \rangle = \sqrt{2}\psi_+ \neq 0 \,. \end{aligned} \tag{3.43}$$

The relaxation method also requires an initial seed for the fields that we want to compute, which we choose to be a linear function interpolating from the known conditions at the horizon, and some values at the boundary that will change as the algorithm converges. For the field $P(z)$ this last number can either be zero (if we want $\langle \mathcal{O}_+ \rangle \neq 0$) or a constant (for $\langle \mathcal{O}_- \rangle \neq 0$), which we take to be equal to P_h. On the other hand, for the Maxwell potential we take $\phi(0) = -\rho_c z_h/L^2$, where ρ_c is the critical value of the charge density at which the condensation begins.

This critical charge density can be estimated by solving a Sturm-Liouville eigenvalue problem, as explained in [248]. Let us review this computation here for our case.

The equation of motion for the scalar field (3.7a) can be rewritten in terms of the coordinate z and the fields

$$\tilde{f}(z) = \frac{z^2}{L^2} f(z)\,, \qquad F(z) = \frac{\sqrt{2}}{\langle \mathcal{O}_\pm \rangle} \left(\frac{L^2}{z}\right)^{\Delta_\pm} \psi(z)\,, \tag{3.44}$$

resulting in

$$F'' + \left(\frac{\tilde{f}'}{\tilde{f}} + \frac{2(\Delta_\pm - 1)}{z}\right) F' + \left(\frac{\Delta_\pm \tilde{f}'}{z\tilde{f}} + \frac{\Delta_\pm(\Delta_\pm - 3)}{z^2} - \frac{m^2 L^2}{z^2 \tilde{f}}\right) F + \frac{f_\infty q^2 \phi^2}{\tilde{f}^2} F = 0\,. \tag{3.45}$$

Since we want to study the onset of the superconducting phase transition, we can consider $F \approx 0$, so the Maxwell potential is equal to its background value $\phi(z) \approx \rho_c(z - z_h)$. Plugging this into the equation, we see that it now takes the form of a Sturm-Liouville eigenvalue problem

$$\frac{d}{dz}\left[k(z)\frac{dy}{dz}\right] - q(z)y(z) + \lambda\rho(z)y(z) = 0\,, \qquad K(z)y(z)y'(z)\Big|_{z_0}^{z_1}\,, \tag{3.46}$$

where the eigenvalue λ and the rest of the functions are identified as

$$\lambda = \rho_c^2\,, \qquad k(z) = \tilde{f} z^{2\Delta_\pm - 2}\,, \qquad q(z) = -z^{2\Delta_\pm - 2}\left(\frac{\Delta_\pm \tilde{f}'}{z} + \frac{\Delta_\pm(\Delta_\pm - 3)\tilde{f}}{z^2} - \frac{m^2 L^2}{z^2}\right)\,,$$
$$\rho(z) = z^{2\Delta_\pm - 2} \frac{f_\infty q^2 (z - z_h)^2}{\tilde{f}}\,, \qquad y(z) = F(z)\,, \tag{3.47}$$

and the limits of the interval are $z_0 = 0$ and $z_1 = z_h$. The eigenvalue of the differential equation, which in our case will give us the value of the critical charge density, can be obtained by solving the following minimization problem [248]

$$\lambda = \left.\frac{\int_{z_0}^{z_1} dz\,\left[k(z)y'(z)^2 + q(z)y(z)^2\right]}{\int_{z_0}^{z_1} dz \rho(z) y(z)^2}\right|_{y = y_{\min}}\,. \tag{3.48}$$

In order to minimize the quotient of integrals with respect to the function $y(z) = F(z)$, we introduce the relatively simple ansatz $F(z) = 1 - az^2 + bz^3 + cz^4$. Then, the problem reduces to finding numerically the values of the constants a, b and c that minimize that quantity.

Finally, we should comment on how temperatures are treated in this problem. Notice that the Hawking temperature of the black hole is constant, given in Eq. (3.18), but the numerical results will be displayed as functions of the temperature of the system. The way to do this is by finding another energy scale to compare it with, in this case the charge density ρ that can be read from the asymptotic expansion of $\phi(z)$, given in Eq. (3.38). In the numerical procedure, we solve the system for one value of the constant $P(z_h) = P_h$ each time, as said before. This will produce a profile for both $P(z)$ and $\phi(z)$, from where we can read the expectation value of the condensate $\langle \mathcal{O}_\pm \rangle$ and the corresponding charge density ρ, which we use to measure the critical temperature as

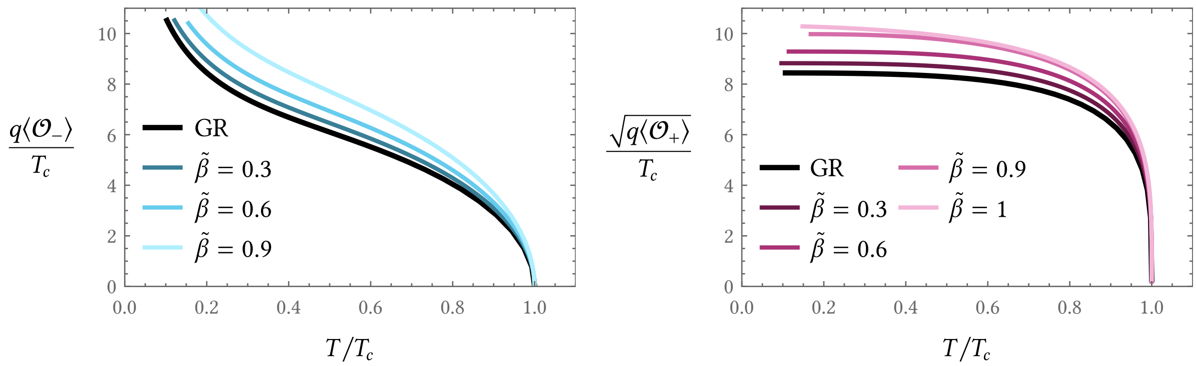
$$T_c = \frac{3 r_h}{4\pi L^2 \sqrt{f_\infty}} \sqrt{\frac{\rho}{\rho_c}}\,. \tag{3.49}$$

Here ρ_c is the critical charge density, obtained from the results of the onset of the superconductivity. Therefore, even though the temperature of the black hole does not change, since there is another energy scale involved we can obtain results corresponding to different temperatures.

3.3.2.2 *Results for the condensation*

Let us now show the numerical results for the expectation value of the two condensates $\langle \mathcal{O}_\pm \rangle$ as a function of the temperature, obtained as explained above. Let us remind the reader that in the numerical computations we always set $L = 16\pi G_{\rm N} = 1$, for simplicity.

We checked in each case that the solution found for $\psi(r)$ does not have nodes, this is, points where $\psi(r) = 0$, although it is possible to achieve profiles with these features using our numerical procedure and different boundary conditions. Such solutions correspond to excited states of the condensate [249–252], whose study can be interesting on its own, but lies outside the scope of the present work.



(a) Condensate $\langle \mathcal{O}_- \rangle$. (b) Condensate $\langle \mathcal{O}_+ \rangle$.

Figure 3.2: Condensation of the operators of dimensions 1 and 2 with respect to the temperature in the probe limit, $q \to \infty$, for different values of $\tilde{\beta}$.

In Figure 3.2 we see how the condensate value $\langle \mathcal{O}_\pm \rangle$ in each of the possible quantizations changes as a function of the temperature. In general, we observe that increasing values of the cubic coupling $\tilde{\beta}$ leads to larger condensates at low temperatures, when normalized by the critical temperature. This is analogous to what was previously reported in other higher-curvature theories in higher dimensions [243, 245]. As in the standard GR case, there is a divergence on the condensate $\langle \mathcal{O}_- \rangle$ as T goes to zero, which spoils the decoupling limit but disappears when backreaction on the metric is considered, as we will see later.

In Figure 3.3 we see the effect of the cubic-curvature terms in the critical temperature for the superconducting phase transition. It decreases monotonically as a function of the cubic coupling, until values close to its upper bound $\tilde{\beta} = 1$ are approached, where there is a qualitative difference between the two quantizations that can be understood as follows. In this limit the function $f(r)$ is given exactly by Eq. (3.19), where the radius of the horizon can take any positive value. On the other hand, the asymptotic form of $\phi(r)$ and $\psi(r)$ is still given by Eqs. (3.38) and (3.39). However, if we look at the condensate $\langle \mathcal{O}_- \rangle$, this is, we set $\psi(r) = \psi_-/r$ and $\phi(r) = \mu + \rho/r$, the equations of motion (3.7a) and (3.7b) imply that $\psi_- = \rho = 0$, which means that both fields are equal to zero in this case. Thus, it makes sense that the critical temperature diverges as seen in Figure 3.3, since the energy scale with respect to which it is measured, $\sqrt{\rho}$, vanishes. This does not happen if one studies the other condensate, $\langle \mathcal{O}_+ \rangle$, in which case ψ_+, μ and ρ are not determined analytically by the equations of motion and the entire numerical

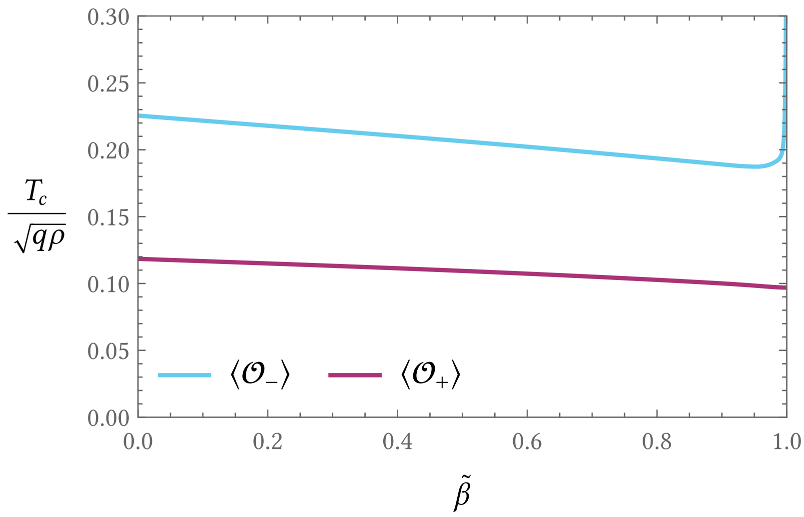


Figure 3.3: Dependence of the critical temperature of the two operators with the coupling $\tilde{\beta}$, in the probe limit. Obtained from the critical charge density ρ_c as given in Eq. (3.49). The divergence on the blue curve at $\tilde{\beta} \to 1$ is due to the vanishing of ρ in that limit.

computation needs to be carried out, finding a non-vanishing value of the condensate as shown in Figure 3.2.

3.3.2.3 *Electric conductivity*

It is possible to evaluate the electric conductivity of the dual theory using standard holographic methods. To that end, we turn on a perturbation on the spatial component of the gauge field, such that the Maxwell potential is

$$A = \phi(r)dt + e^{-i\omega t}\delta A_x(r)dx\,, \tag{3.50}$$

and the equation of motion of this perturbation reads

$$\delta A_x'' + \left(\frac{f'}{f} + \frac{N'}{N}\right)\delta A_x' + \frac{1}{f}\left(\frac{\omega^2}{N^2 f} - 2q^2\psi^2\right)\delta A_x = 0\,. \tag{3.51}$$

This would excite a perturbation in the tx component of the metric, $e^{-i\omega t}\delta g_{tx}$, but in the probe limit it is safe to turn it off. By expanding this equation for $r \to \infty$ we find that, near the boundary, δA_x behaves as

$$\delta A_x(r) = \delta A_x^{(0)} + \frac{\delta A_x^{(1)}}{r} + \mathcal{O}\left(r^{-2}\right)\,. \tag{3.52}$$

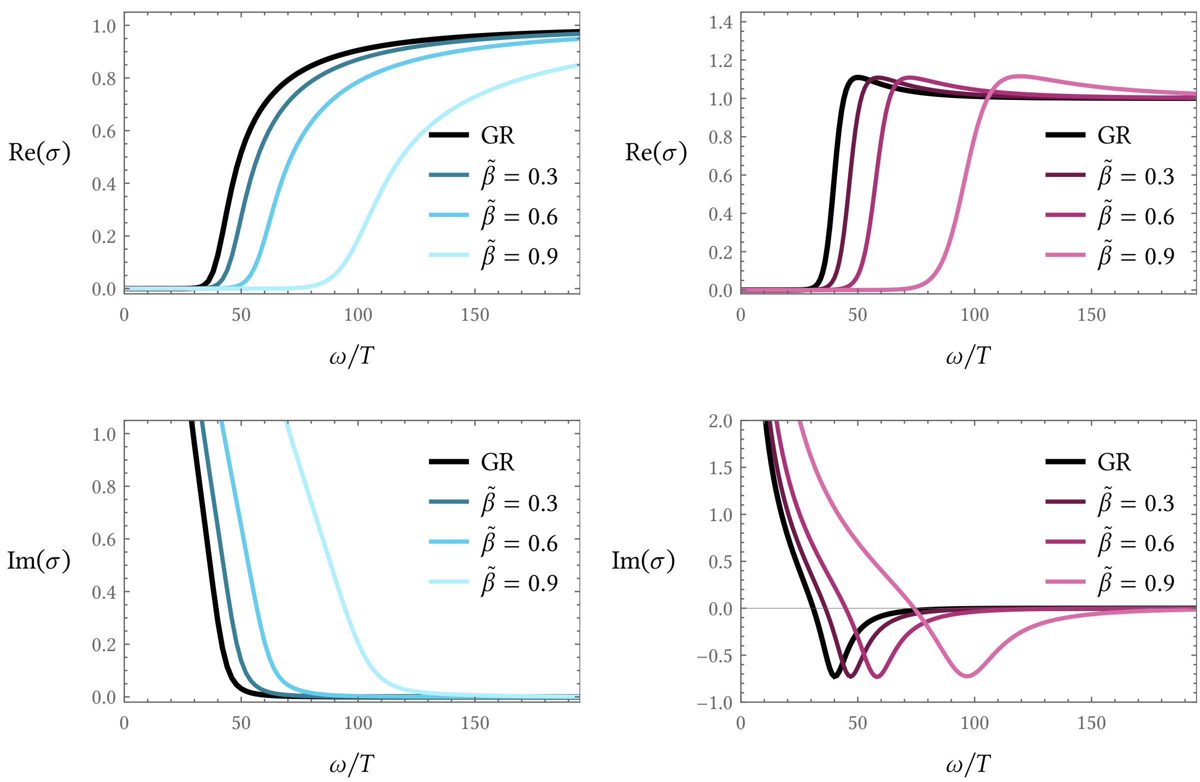
If we now look at the behavior of Eq. (3.51) near the horizon, and plug in the previous expansions for the different fields, we find

$$\delta A_x(r) \simeq \delta A_{\rm out}(r - r_h)^{i\omega/4\pi T} + \delta A_{\rm in}(r - r_h)^{-i\omega/4\pi T}\,, \tag{3.53}$$

where $\delta A_{\rm out}$ and $\delta A_{\rm in}$ are outgoing and ingoing modes at the horizon, respectively. Then, in order for the propagation of the perturbations to respect causality we need to impose $\delta A_{\rm out} = 0$, while the actual value of $\delta A_{\rm in}$ is not relevant as long as it is finite, and in practice we set it to 1.

Now that we know the form of δA_x at the horizon, it is straightforward to solve the equation of motion (3.51) for any given value of ω, plugging in also the value of the field $\psi(r)$ obtained previously which we assume to be unchanged by the perturbation. The equation can be integrated with a standard numerical procedure, starting from the horizon and towards the boundary, in order to read the values of the coefficients in the expansion (3.52). Following the usual holographic procedure we can identify the leading term with a perturbation of the electric potential, $\delta A_x^{(0)} = \delta E_x \sqrt{f_\infty}/i\omega$, and the subleading one with its linear response on the electric current, $\delta A_x^{(1)} = \delta J_x$. Then, the conductivity can be obtained according to the Kubo formula

$$\sigma = \frac{\delta J_x}{\delta E_x} = \frac{\sqrt{f_\infty}}{i\omega}\frac{\delta A_x^{(1)}}{\delta A_x^{(0)}}\,. \tag{3.54}$$



(a) Condensate $\langle \mathcal{O}_- \rangle$. (b) Condensate $\langle \mathcal{O}_+ \rangle$.

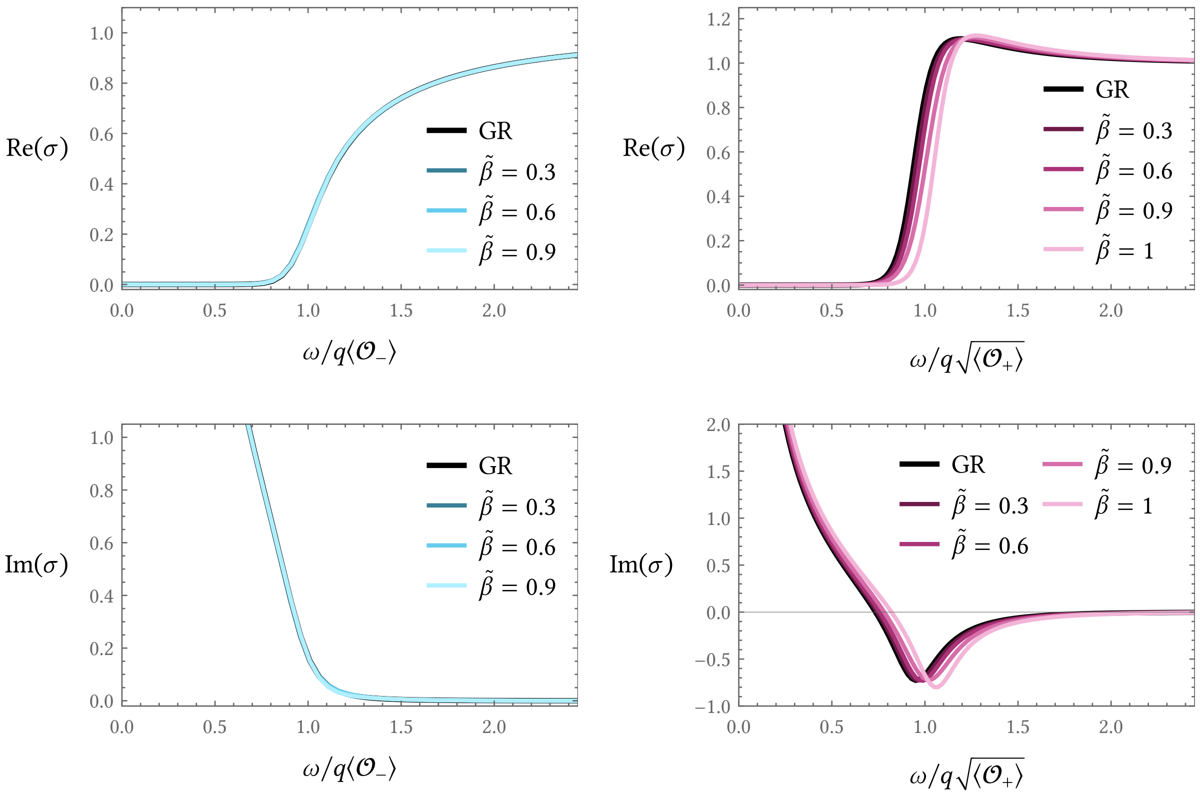
Figure 3.4: Real and imaginary parts of the conductivity as a function of the frequency, for different values of $\tilde{\beta}$ and $T/T_c = 0.2$, in the probe limit. One can observe that the gap becomes larger as $\tilde{\beta}$ increases, while the asymptotic value of σ is independent of this coupling.

Plots of the resulting conductivities as functions of the frequency in the superconducting phase are shown in Figure 3.4, where one can observe two main features that are shared with the results from Einstein gravity. First of all, there is a frequency gap above which the conductivity goes to a constant. This could be a hint of the existence of a microscopic mechanism that is responsible for the creation of superconducting charge carriers, which would be broken by external excitations with large enough energy, resulting in a finite conductivity that corresponds to a normal phase.

The other important feature is a delta in $\mathrm{Re}(\sigma)$ for $\omega = 0$. While this is not observed directly in the plots, due to the finite resolution of the numerics, it can be inferred from the divergence of $\mathrm{Im}(\sigma)$ and the Kramers-Kronig relation

$$\mathrm{Im}\left[\sigma(\omega)\right] = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left[\sigma(\omega')\right]}{\omega'-\omega}d\omega' , \tag{3.55}$$

which follows from causality [21]. In this probe limit the delta implies an infinite DC conductivity [21, 22], which is a defining feature of a superconducting system.



(a) Condensate $\langle\mathcal{O}_-\rangle$.

(b) Condensate $\langle\mathcal{O}_+\rangle$.

Figure 3.5: Plots of the conductivity with respect to a rescaled frequency for $T/T_c = 0.2$, in the probe limit. The dependence on the coupling $\tilde{\beta}$ is mostly due to the change in the value of the condensate, with a residual contribution that is larger in the dimension-2 case.

In the same Figure 3.4 we see that the main effect of the cubic terms is to make the energy gap larger. However, the dependence of this gap on $\tilde{\beta}$ mimics almost completely that of the condensate found in Section 3.3.2.2. This is confirmed by rescaling the frequency with the appropriate power of the condensate value, as can be seen in Figure 3.5. The residual dependence can be attributed to f_∞ in Eq. (3.54), and it is stronger in the $\langle\mathcal{O}_+\rangle$ case. This could have been expected on the grounds of the larger conformal dimension of the operator, as in Eq. (3.51) the rescaling of ω makes the condensate appear linearly as $\langle\mathcal{O}_+\rangle/f^2 \sim \langle\mathcal{O}_+\rangle/f_\infty^2$, as compared to the quadratic behavior $\langle\mathcal{O}_-\rangle^2/f_\infty^2$ for the other quantization. In both cases, the asymptotic value for large frequencies is independent of the cubic coupling.

3.3.3 Backreaction

As the final part of this Chapter, let us turn to the fully backreacted case. As before, we will analyze the dependence of the condensation of the two operators $\langle \mathcal{O}_\pm \rangle$ on the parameter of the cubic terms $\tilde{\beta}$, but also the electromagnetic coupling q. As explained in Section 3.1, the probe limit already studied corresponds to $q \to \infty$, so now we will consider finite values of this coupling, the smaller ones corresponding to a stronger backreaction of the fields on the metric.

However, we will not study the electric conductivity with backreaction. Introducing a perturbation δA_x in the gauge field would require turning on also a perturbation δg_{tx} in the metric, whose equation of motion should be considered. In GR this equation is simple and can be replaced in that of δA_x, so in the end one does not need to actually solve for δg_{tx}. On the contrary, in the cubic gravity case its equation of motion is modified, and therefore one would need to actually solve for this perturbation, providing an appropriate set of boundary conditions. This is an interesting problem on its own, but lies outside the scope of this work.

3.3.3.1 Numerical method and boundary conditions

The main challenge that we are faced with is the complexity of the full system of equations of motion (3.7), since now the metric is not simply the one studied in Section 3.2 but will depend on the values of the matter fields. These have a total of ten derivatives, so a naive power counting would result in ten constants being required to fully determine the solution. However, several constraints can be found in the system, thus reducing such number.

In the standard GR case the equations of motion for the metric are first order, as can be seen by setting $\tilde{\beta} = 0$ in Eqs. (3.7c) and (3.7d), so solving the entire system would require 6 constants. Two of the remaining constants are fixed by demanding $N_\infty^2 f_\infty = 1$ and $f(r_h) = 0$, and another one setting $\phi(r_h) = 0$ to avoid a singularity in the norm of the potential. One further reduction is obtained by evaluating at the horizon the equation of motion for $\psi(r)$, which fixes the value of $\psi'(r_h)$ in terms of known quantities. The remaining two constants can be identified with the charge density ρ in the expansion (3.38) and the value of the condensate $\langle \mathcal{O}_+ \rangle \propto \psi_+$ or $\langle \mathcal{O}_- \rangle \propto \psi_-$. We reduce the number of required constants to one by imposing the value of the boundary source $\mathcal{J}_- \propto \psi_-$ or $\mathcal{J}_+ \propto \psi_+$ in (3.39) to zero, in order to study the spontaneous condensation of the corresponding operator. Finally, by considering different values of this remaining constant, we obtain a functional relation between $\langle \mathcal{O}_\pm \rangle$ and ρ, or more precisely the dimensionless combination $T/\sqrt{\rho}$.

A similar counting holds in the probe limit studied before, in which the function $N(r)$ is a constant and the equation for $f(r)$ reduces to (3.12), which was solved independently of the rest of the system. In the end, the solutions for $\psi(r)$ and $\phi(r)$ where characterized by a single constant (P_h in Section 3.3.2.1), and by varying this number we obtained relations between the condensates $\langle \mathcal{O}_\pm \rangle$ and the ratio $T/\sqrt{\rho}$.

The cubic curvature case is more complex, since the equations that govern the dynamics of the spacetime, (3.7c) and (3.7d), have up to six derivatives of the functions of the metric. However, by expanding the functions in a power series close to the horizon one can find three relations between these terms, reducing to a total of seven the number of constants required by the system. There are three more conditions that we can impose in the metric functions: $N_\infty^2 f_\infty = 1$, $f(r_h) = 0$ and regularity of $f(r)$ at infinity, which further reduce the number of integration constants to four. Finally, we need to impose the same boundary and regularity conditions on $\psi(r)$ and $\phi(r)$ as in the other cases, and in the end we are led to the usual relation between $\langle\mathcal{O}_\pm\rangle$ and $\sqrt{\rho}/T$.

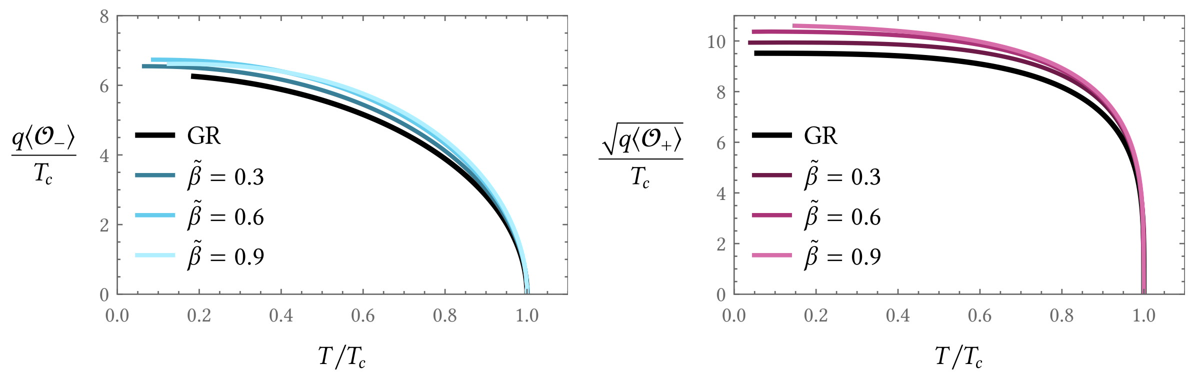
For the numerical computations, we resort again to the relaxation method explained in Appendix C. As in the non-backreacted case we work in practice with the inverse coordinate $z = L^2/r$, and we need to specify some initial seeds for the functions to compute. For the matter fields $\phi(r)$ and $\psi(r)$ we do exactly the same as before, described in Section 3.3.2.1. For the function $f(r)$ we use the solution obtained in vacuum, computed in Section 3.2, while we set initially $n(r) = 1/\sqrt{f_\infty}$ in the entire range of the radial coordinate.

Finally, as explained in Section 3.3.2.1 for the probe limit, we always plot the temperature divided by the energy scale of the system in each case, which we take to be $\sqrt{\rho}$, where ρ is the charge density obtained in the numerical procedure. Also, since now the form of the metric depends on the field content the temperature is not given by Eq. (3.18), but instead we need to compute it from the solutions of $f(r)$ and $N(r)$ as

$$T = \frac{f'(r_h)N(r_h)}{4\pi}\,. \tag{3.56}$$

3.3.3.2 *Results for the condensation*

Let us show now the numerical results for the condensation in the backreacted case, computed as explained above.

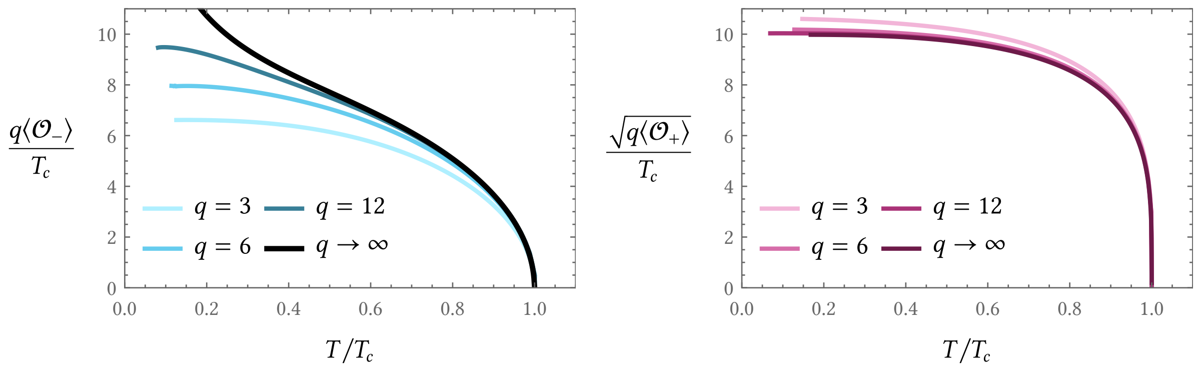


(a) Condensate $\langle\mathcal{O}_-\rangle$. (b) Condensate $\langle\mathcal{O}_+\rangle$.

Figure 3.6: Condensation of the operators of dimensions 1 and 2 with respect to the temperature in the backreacted case, with $q = 3$ and different values of $\tilde{\beta}$.

In the plots, one can see the profiles of the condensates $\langle\mathcal{O}_\pm\rangle$ as a function of the temperature for the backreacted case. First we show the results for different values of the cubic coupling in Figure 3.6, where we observe that a large cubic coupling leads again to a larger condensate, as in the probe limit. Then, in Figure 3.7 we compare the condensation curves for different values of the charge q, which controls the strength of the backreaction. Similarly to what is found in the Einstein gravity case [22], the condensate $\langle\mathcal{O}_+\rangle$ gets larger due to backreaction, while $\langle\mathcal{O}_-\rangle$ gets smaller. As advanced, the divergence of $\langle\mathcal{O}_-\rangle$ at low temperatures disappears.

Finally, in Figure 3.8 we show the dependence of the critical temperature on the two parameters. It again decreases monotonically as a function of the cubic coupling, which resembles the



(a) Condensate $\langle\mathcal{O}_-\rangle$.

(b) Condensate $\langle\mathcal{O}_+\rangle$.

Figure 3.7: Condensation of the operators with dimensions 1 and 2 with respect to the temperature, for $\tilde{\beta} = 0.9$ and different strengths of the backreaction, controlled by the parameter q.

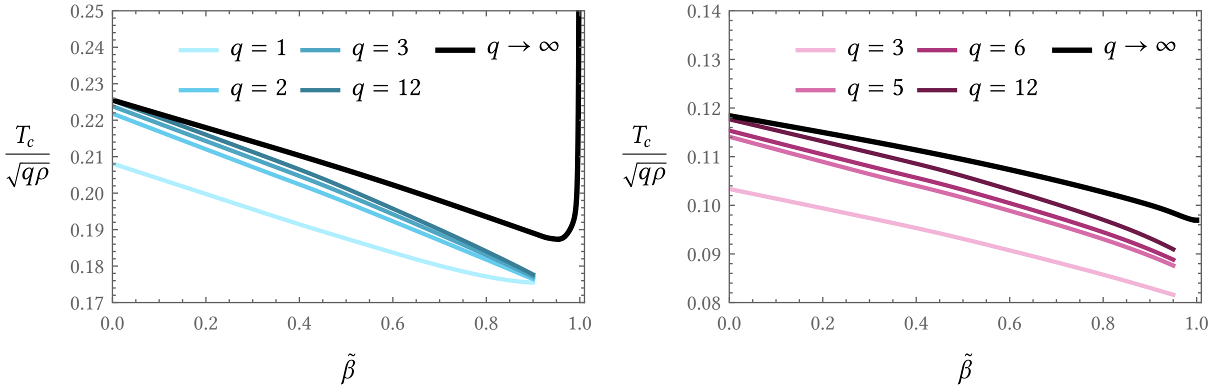
behavior of other models of higher-curvature superconductors in higher dimensions [243, 245]. Like in the standard Einstein case [22], the critical temperature is reduced as backreaction becomes more important.

3.4 Discussion

The main goal of this Chapter has been to study a condensed matter system using the tools of holography, and with higher-curvature corrections to the gravity action. In particular we investigated the already known holographic superconductor in 2 + 1 dimensions, with the gravity side of the duality being governed by the action of Einsteinian cubic gravity given in Eq. (3.1), aiming to explore the effects of finite N and 't Hooft coupling corrections in the physics of the boundary. We focused on the dependence of the critical temperature and the strength of the resulting condensate on the coupling of the cubic terms, and computed also the electric conductivity.

We found that, as previously reported in the higher-dimensional Gauss-Bonnet and Quasitopological cases [243–246], the presence of higher-curvature corrections to the bulk action makes the condensation of the scalars more difficult, lowering the critical temperature both in the non-backreacting and backreacting cases (see Figures 3.3 and 3.8). When the critical temperature is reached and the phase transition occurs, the resulting condensate grows with the cubic coupling $\tilde{\beta}$ in both cases. We also studied the response to electric perturbations in the probe limit, finding like in the Einstein gravity case that there is a gap in the conductivity as a function of the frequency, whose size is compatible with the value of the condensate. A residual dependence of the conductivity gap on $\tilde{\beta}$ can be attributed to the asymptotic value of the black hole radial function, given by the constant f_∞. Besides, a delta for zero frequency perturbations is found in the real part of the conductivity, which corresponds to the infinite DC conductivity that is characteristic of a superconducting phase.

An interesting point about the results found is the following. The decrease of the critical temperature as the higher-curvature coupling is increased could be naively attributed to a gradual restoration of the validity of the Coleman-Mermin-Wagner theorem, as the higher-curvature terms correspond to $1/N$ corrections in the boundary theory. In fact, in contrast



(a) Condensate $\langle \mathcal{O}_- \rangle$. (b) Condensate $\langle \mathcal{O}_+ \rangle$.

Figure 3.8: Critical temperature for the superconducting transition with respect to the cubic coupling $\tilde{\beta}$, for different values of the parameter q which controls the strength of the backreaction. These were chosen differently for each plot, in order to make the differences between the curves more noticeable.

with the previously known higher-curvature superconductors [243–246, 253], in our case the boundary theory is $(2+1)$-dimensional, satisfying one of the main hypotheses of the theorem. However, higher-curvature corrections to the gravitational action arise from bulk loop diagrams which are finite in the IR, while the Coleman-Mermin-Wagner result is due to an IR divergence on the Goldstone boson correlator [254], which can only be captured via the calculation of loop diagrams in the bulk as done in [242]. Thus, the decrease of the critical temperature as higher-curvature corrections are added must be attributed to different physics. However, since this seems to be a generic behavior found for different dimensions and corrections, it deserves further investigation.

The present result could be straightforwardly expanded by including terms of higher order in the bulk action, both in the curvature tensors and the Maxwell field. This would allow us to study richer dual theories in the boundary, and based on the results presented here one would expect the behavior to be qualitatively similar.

Part II

Holography with a non-minimally coupled chemical potential

4

Electromagnetic Quasitopological gravities in any dimension

As already argued in Chapters 1 and 3, the interest of considering higher-derivative gravity theories in the context of the AdS/CFT correspondence resides in the fact that they could capture finite N and 't Hooft coupling effects in the boundary CFT. In addition, they serve as toy models to describe a wider range of dual field theories, which may exhibit features that differ from those of CFTs dual to Einstein gravity. This can also be useful to explore whether known relations of holographic CFTs are universal or exclusive to those dual to general relativity.

However, these analyses are usually performed perturbatively in $1/N$ and $1/\lambda$, since the higher-order corrections in the gravity action are taken to be small. In this second Part of the thesis we will consider theories that allow us to go past this regime, and in particular we are interested in higher-derivative theories that contain not only the metric, but also a vector field, which according to the holographic dictionary couples to a current operator J^i in the boundary. As in the case of the $\mathcal{L}\left(R^{\mu\nu}_{\rho\sigma}\right)$ theories considered up until now, these higher-derivative terms will allow us to study more general classes of dual CFTs. An important quantity in this regard is the 3-point correlator $\langle TJJ \rangle$, whose form is fixed in Einstein-Maxwell theory, but can contain an additional structure for a general CFT. This is encoded in the energy-flux parameter a_2 as written in [183], which can acquire a non-vanishing value for higher-derivative theories with non-minimal couplings. It is interesting to note that for QCD $a_2 \approx -3/2$ [255–257], so if one wanted to provide a holographic approximation to this theory it would be necessary to consider higher-derivative operators with couplings of order one.

This vector field in the gravity theory will allow us to explore the effect of a chemical potential in the CFT, again by means of the holographic duality. It is then interesting to study how the predictions for certain properties of the CFT, such as the hydrodynamics of charged plasmas [258–260] or entanglement and Rényi entropies [261], change depending on the couplings of the higher-order terms. Some of these questions have already been explored in the literature, but most of the analyses so far have followed a perturbative approach [262–267], or have stuck to particular models [194, 268, 269]. The goal of the current study is to perform a non-perturbative analysis of this type of theories, taking into account a wider range of interactions between gravity and electromagnetism, in particular including higher-derivative contractions of the schematic form RF^2. We will see that these are interesting additions to the Einstein-Maxwell action, due to their effects on the dual CFT.

Besides the non-minimal couplings of the Maxwell field and the geometry, in order to build a general theory to the given order we will need to consider also contractions of curvature tensors. Since we are interested in carrying an exact exploration rather than a perturbative one, we require these pure gravity corrections to be amenable to analytic computations, which is typically not the case when higher derivatives are involved. Of course, we want the theory to have other desirable features, such as admitting single-function black hole solutions and propagating only a massless graviton on maximally symmetric backgrounds, as explained in Chapter 1. Among these, if we further require the corrections to the black hole function $f(r)$ to be analytic we are restricted to the families of Lovelock and Quasitopological gravity theories.

The mentioned theories can be minimally coupled to a Maxwell field while keeping all of their properties. However, this is not a sufficiently general theory, as it misses higher-derivative terms involving the vector field. For this reason, the family of Electromagnetic Quasitopological gravities (EQGs) was introduced for $D = 4$ in [23], as extensions of the Generalized Quasitopological gravity theories with a vector field that can be coupled to gravity in many forms. These theories were also recently studied in $D = 3$ in [139], and among their features is the fact that they are able to provide black hole solutions without singularities [23, 134, 270].

In this Chapter we generalize this construction to an arbitrary dimension $D = d + 1$, by writing them in terms of a $(d-2)$-form B and requiring the existence of black hole solutions magnetically charged under this field, whose metric depends on a single function that can be computed analytically. The differential form B can then be dualized into a vector field, in terms of which the solutions are electrically charged. Most of the time we will focus on the lowest order theory, which contains four-derivative corrections, but expressions for EQGs at any order in the field strength and the curvature will also be provided.

After that we will establish several basic entries of the holographic dictionary for the resulting theories. We start by computing different 2- and 3-point functions, obtaining in particular the central charge C_J of $\langle JJ \rangle$, as well as the parameter a_2 that controls the angular distribution of energy radiated after a local insertion of the current operator J. Then we will obtain constraints for the couplings of the theory from arguments of causality and positivity of the energy fluxes, as well as other requirements derived from the mild form of the weak gravity conjecture [271–273].

Finally, the main interest of these EQGs is the fact that they can be used to learn different aspects about holography in the presence of a chemical potential, beyond the Einstein-Maxwell regime. In this line, we will study the thermodynamic properties of the dual CFT, finding a richer phase structure as a function of the chemical potential μ that can even allow for zeroth order transitions. However, these additional phases are disfavored by the previous constraints on the couplings. Another interesting quantity to compute is the ratio of the shear viscosity to the entropy density, and we will find that its behavior with μ is different depending on the sign of the parameter a_2. Interestingly, for $a_2 > 0$ one can get $\eta/s = 0$ for a sufficiently large chemical potential, even if all the constraints are fulfilled.

The study of Rényi and entanglement entropies in these EQG theories is very interesting on its own, and for this reason it is left for the following Chapter.

The research presented in this Chapter is published in [26], and its detailed information can be found in page 212.

4.1 Construction of the theories

4.1.1 Gravity, $(d-2)$-forms and their electromagnetic dual

In this Chapter we consider $(d+1)$-dimensional theories of gravity with a $(d-2)$-form field B, whose action is given in general by[1]

$$I = \int_{\mathcal{M}} d^{d+1}x \sqrt{|g|}\mathcal{L}\left(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, H_{\mu_1\ldots\mu_{d-1}}\right) , \tag{4.1}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the metric $g_{\mu\nu}$, and the $(d-1)$-form H is the field strength of B, $H = \mathrm{d}B$. The Lagrangian must be a scalar function built with contractions of these tensors, and we implicitly assume that it can be written or expanded as a polynomial in these. In particular, we are interested in theories that reduce to the standard Einstein-$(d-2)$-form theory for small curvatures and field strengths,

$$\mathcal{L} = \frac{1}{16\pi G_{\mathrm{N}}}\left[R + \frac{d(d-1)}{L^2} - \frac{2}{(d-1)!}H_{\mu_1\ldots\mu_{d-1}}H^{\mu_1\ldots\mu_{d-1}} + \ldots\right] . \tag{4.2}$$

Such a theory is invariant under diffeomorphisms and gauge transformations $B \to B + \mathrm{d}\Lambda$, where Λ is a $(d-3)$-form. The equations of motion can be obtained by varying the action with respect to the fields $g_{\mu\nu}$ and H, and they read in general

$$\mathcal{E}_{\mu\nu} = P_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{L} + 2\nabla^{\sigma}\nabla^{\rho}P_{(\mu|\sigma|\nu)\rho} - (d-1)\mathcal{M}_{(\mu}{}^{\alpha_1\ldots\alpha_{d-2}}H_{\nu)\alpha_1\ldots\alpha_{d-2}} = 0\,, \tag{4.3}$$

$$\mathcal{E}^{\nu_1\ldots\nu_{d-2}} = \nabla_{\mu}\mathcal{M}^{\mu\nu_1\ldots\nu_{d-2}} = 0\,, \tag{4.4}$$

where

$$P^{\alpha\beta\gamma\delta} = \frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\gamma\delta}}\,, \qquad \mathcal{M}^{\alpha_1\ldots\alpha_{d-1}} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial H_{\alpha_1\ldots\alpha_{d-1}}}\,. \tag{4.5}$$

Notice that the equation of motion obtained by varying the metric, Eq. (4.3), is equal to the Padmanabhan equation presented before (see Eqs. (1.9) and (2.8)) supplemented by an additional term, which is due to the field H.

Our interest in these theories lies on the fact that they allow for black hole solutions magnetically charged under the differential form B, as explained below. Furthermore, the $(d-2)$-form can be related to a 1-form (a vector field) by means of a duality transformation. This means that it is possible to map any of these models to a higher-derivative extension of Einstein-Maxwell, which is the interpretation that we are most interested in.

Let us quickly review this process of dualization. Starting from the theory (4.1), we can dualize the $(d-2)$-form B into a 1-form by introducing the Bianchi identity $\mathrm{d}H = 0$ explicitly in the action,[2]

$$\tilde{I} = \int_{\mathcal{M}} d^{d+1}x \sqrt{|g|}\left[\mathcal{L}\left(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, H_{\mu_1\ldots\mu_{d-1}}\right) + \frac{1}{4\pi G_{\mathrm{N}}(d-1)!}A_{\alpha_1}\partial_{\alpha_2}H_{\alpha_3\ldots\alpha_{d+1}}\epsilon^{\alpha_1\ldots\alpha_{d+1}}\right] . \tag{4.6}$$

[1] A gravity action such as (4.1) needs to be supplemented with Gibbons-Hawking-York boundary terms in order to make the variational problem well posed [274, 275]. This was actually the problem treated in Chapter 2 of this thesis, where it is argued that the counterterms should be the same as those of Einstein gravity multiplied by an overall constant, as long as we restrict to asymptotically AdS spacetimes. We will addressed this again in Section 4.5.1.

[2] The factor $1/(4\pi G_{\mathrm{N}})$ is introduced taking into account that the Lagrangian density $\mathcal{L}$ will contain an overall normalization $1/(16\pi G_{\mathrm{N}})$.

Here A_μ is introduced as a Lagrange multiplier, whose variation yields the Bianchi identity of H, and is now a considered a fundamental variable instead of B. We can integrate this dual action by parts, to express it as

$$\begin{aligned}\tilde{I} = &\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\left[\mathcal{L}\left(g_{\mu\nu},\, R_{\mu\nu\rho\sigma},\, H_{\mu_1\ldots\mu_{d-1}}\right) + \frac{1}{4\pi G_{\rm N}(d-1)!}(\star F)_{\alpha_1\ldots\alpha_{d-1}}H^{\alpha_1\ldots\alpha_{d-1}}\right]\\ &+\frac{1}{4\pi G_{\rm N}}\int_{\partial\mathcal{M}} d^dx\sqrt{|h|}n^\mu A^\nu(\star H)_{\mu\nu}\,,\end{aligned} \tag{4.7}$$

where we have defined the field strength $F = \mathrm{d}A$. The variation with respect to A_μ still yields the Bianchi identity of H, but now it becomes clear that the variation with respect to H produces an algebraic relation between this field and F, namely

$$F = 4\pi G_{\rm N}(d-1)! \star \frac{\partial\mathcal{L}}{\partial H}\,. \tag{4.8}$$

This should be inverted in order to get $H(F)$, and inserting it back in the action $\tilde{I}$ we would get the dual theory for the vector A_μ. Note that this process also produces a boundary contribution, which is precisely the term needed for making the variational principle for A_μ well posed, and that corresponds to working in the canonical ensemble (with fixed electric charge) when computing the Euclidean action.

The dual Lagrangian $\tilde{\mathcal{L}}$ is the Legendre transform of $\mathcal{L}$ with respect to H. Then, by means of the properties of this transform and Eq. (14) with $s = 1$, one can write the inverse relation between H and F as

$$H = -8\pi G_{\rm N} \star \frac{\partial\tilde{\mathcal{L}}}{\partial F}\,. \tag{4.9}$$

This relation is useful since it allows us to identity the electric and magnetic charges in both descriptions. In the frame of the $(d-2)$-form we will have solutions with magnetic charge, which in the frame of the vector field correspond to electrostatic solutions. This charge can be defined in either frame as

$$q = \frac{1}{4\pi G_{\rm N}}\int_{\mathcal{S}_{d-1}} H = -2\int_{\mathcal{S}_{d-1}} \star\frac{\partial\tilde{\mathcal{L}}}{\partial F}\,, \tag{4.10}$$

where the integral is performed over any spacelike hypersurface of codimension 1 that encloses the charge source. In the case of black hole solutions, $\mathcal{S}_{d-1}$ can be any surface that encloses the black hole horizon.

It is not possible in general to invert Eq. (4.8) in order to write the dual Lagrangian. However, an important type of theories that will be considered in this work are those quadratic in H, which can be written as

$$I = \frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\left[\mathcal{L}_{\rm grav} - \frac{2}{(d-1)!}\left(H^2\right)_{\mu\nu}{}^{\rho\sigma}Q^{\mu\nu}{}_{\rho\sigma}\right]\,, \tag{4.11}$$

where $\mathcal{L}_{\rm grav} = R - 2\Lambda + \ldots$ only depends on the geometry, $Q^{\mu\nu}{}_{\rho\sigma}$ can depend on the curvature tensors and is antisymmetric in both pairs of indices, and we are introducing the notation[3]

$$\left(H^2\right)^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_n} \equiv H^{\mu_1\ldots\mu_n\mu_{n+1}\ldots\mu_{d-1}}H_{\nu_1\ldots\nu_n\mu_{n+1}\ldots\mu_{d-1}}\,. \tag{4.12}$$

3 It is possible to prove that the most general quadratic Lagrangian can be written using only the object $\left(H^2\right)_{\mu\nu}{}^{\rho\sigma}$, this is, with only four free indices, by expressing it first in terms of $\star H$.

In this case, it is possible to find the dual theory explicitly. The relation (4.8) can be written as

$$(\star F)_{\alpha_1\ldots\alpha_{d-1}} = Q^{\mu\nu}{}_{[\alpha_1\alpha_2}H_{\alpha_3\ldots\alpha_{d-1}]\mu\nu}\,. \tag{4.13}$$

In order to invert this, let us first introduce the tensor

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \frac{12}{(d-1)(d-2)}Q^{[\alpha\beta}{}_{\alpha\beta}\delta^{\mu}{}_{\rho}\delta^{\nu]}{}_{\sigma}\,, \tag{4.14}$$

and its inverse, that we denote by $\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma}$, and which is defined by means of the equation

$$\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\alpha\beta}\tilde{Q}^{\alpha\beta}{}_{\rho\sigma} = \delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,. \tag{4.15}$$

With this, one can check that Eq. (4.13) is inverted by

$$H_{\alpha_1\ldots\alpha_{d-1}} = \frac{1}{2}\epsilon_{\alpha_1\ldots\alpha_{d-2}\rho\sigma}\left(\tilde{Q}^{-1}\right)^{\rho\sigma}{}_{\alpha\beta}F^{\alpha\beta}\,, \tag{4.16}$$

and therefore the dual action takes the form

$$\begin{aligned}\tilde{I} = &\frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\left[\mathcal{L}_{\rm grav} - F_{\mu\nu}F^{\rho\sigma}\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma}\right]\\ &+\frac{1}{4\pi G_{\rm N}}\int_{\partial\mathcal{M}} d^{d}x\sqrt{|h|}n^{\mu}A^{\nu}\left(\tilde{Q}^{-1}\right)^{\alpha\beta}{}_{\mu\nu}F_{\alpha\beta}\,.\end{aligned} \tag{4.17}$$

If the Lagrangian contained terms beyond quadratic order in H, such as $\left(H^2\right)^2$, Eq. (4.8) would become a tensorial polynomial equation, whose resolution is more involved. One could nevertheless find an approximate solution by considering a series expansion in terms of F, which would be valid as long as these fields take small values.

4.1.2 General definition of EQGs

Since we are interested in constructing theories of the form (4.1) suitable for holography, we need them to admit charged static solutions with spherical, planar or hyperbolic sections. A general metric ansatz for these configurations reads

$$ds^2_{N,f} = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{k,(d-1)}\,, \tag{4.18}$$

where the metric $d\Sigma^2_{k,(d-1)}$ of the submanifold normal to the radial direction is given by[4]

$$d\Sigma^2_{k,(d-1)} = \begin{cases} d\Omega^2_{(d-1)} & \text{for } k=1 \text{ (spherical)},\\ \frac{1}{L^2}dx^2_{(d-1)} & \text{for } k=0 \text{ (flat)},\\ d\Xi^2_{(d-1)} & \text{for } k=-1 \text{ (hyperbolic)}.\end{cases} \tag{4.20}$$

[4] This can be written in a more general manner as

$$d\Sigma^2_{k,(d-1)} = \frac{d\rho^2}{1-k\rho^2/L^2} + \rho^2 d\Omega^2_{(d-2)}\,, \tag{4.19}$$

but in practice we don't need it, as the package *xAct* used for the computations is able to deal with symmetric submanifolds in a natively.

In addition, we assume the magnetic ansatz for the field H,

$$H_Q = Q\omega_{k,(d-1)}\,, \tag{4.21}$$

where Q is a constant related to the magnetic charge, and $\omega_{k,(d-1)}$ is the volume form of $d\Sigma^2_{k,(d-1)}$. This is given explicitly by

$$\omega_{k,(d-1)} = \frac{1}{(d-1)!}\sqrt{|h|}\tilde{\epsilon}_{a_1\dots a_{d-1}} dx^{a_1}\wedge\cdots\wedge dx^{a_{d-1}}\,, \tag{4.22}$$

where $\tilde{\epsilon}$ is the Levi-Civita symbol defined in Eq. (4), h is the determinant of the metric $d\Sigma^2_{k,(d-1)}$ and the indices $a_1, \dots, a_{d-1}$ label these tangent directions. The integral of this object yields the volume of the corresponding submanifold, which we denote by $V_{k,d-1} = \int \omega_{k,(d-1)}$.

Clearly, the form of H given in Eq. (4.21) satisfies the Bianchi identity $\mathrm{d}H = 0$, but one can also check that, for any theory of the form (4.1), it also solves the equations of motion (4.4) when the metric takes the form (4.18).[5] Since we do not have to worry about the "modified Maxwell equation" anymore, the problem of finding the solutions becomes simpler. Indeed, we only have to solve the equations for the metric functions $N(r)$ and $f(r)$ that, as shown in Section 1.1.3, can be obtained by means of the reduced Lagrangian

$$L_{N,f} = \sqrt{|g|}\mathcal{L}|_{ds^2_{N,f},H_Q}\,. \tag{4.24}$$

The desired equations of motion can be obtained simply by varying this Lagrangian with respect to the functions in the metric,

$$\mathcal{E}_N = \frac{\delta L_{N,f}}{\delta N}\,, \qquad \mathcal{E}_f = \frac{\delta L_{N,f}}{\delta f}\,. \tag{4.25}$$

As shown in Section 1.1.3, $\mathcal{E}_N = \mathcal{E}_f = 0$ imply that the field equations (4.3) are satisfied, taking into account that H_Q solved its own equations (4.4).

So far the analysis is completely general, but typically one would not be able to solve these equations for a generic Lagrangian. For this reason, we will restrict to a subset of theories, introduced as Electromagnetic Quasitopological gravities in [23] for $d+1=4$, that make it possible to perform analytic computations. These are characterized by the condition that

$$\left.\frac{\delta L_{N,f}}{\delta f}\right|_{N=\text{const.}} = 0 \;\; \forall\; f(r)\,, \tag{4.26}$$

which means that for these theories the reduced Lagrangian $L_{N,f}$ is a total derivative when $N(r) = $ constant. As already known, in the purely gravitational case this definition gives rise to the Generalized Quasitopological gravities [3, 6–8, 10], which include Quasitopological [107–110] and Lovelock gravities [1, 2, 91] as particular cases. The construction presented here extends the definition of those theories to include a $(d-2)$-form (or equivalently, a vector

5 This can be checked by following the reasoning of Section 3.1 of [23], with the generalized magnetic ansatz

$$H = \chi'(\theta)\omega_{k,(d-1)}\,, \tag{4.23}$$

where θ is one of the coordinates tangent to the boundary. As in the reference, the equation of motion for $\chi(\theta)$ can be obtained with the reduced action approach, and one finds $\chi'(\theta) = $ constant, which matches our form for H given in Eq. (4.21).

field upon dualization), allowing one to study charged black hole solutions with corrections in the action of the Maxwell field. Let us note that the standard two-derivative theory (4.2) satisfies the condition (4.26), and therefore belongs to the EQG class. In general, all theories in this family satisfy a number of properties, which are the same as for their four-dimensional counterparts [23] and similar to those explained in Section 1.1.3 for GQGs. Let us summarize them here:

1. They propagate the same degrees of freedom in maximally symmetric backgrounds as the two-derivative theory. This is relevant in particular in the gravity sector, since general higher-curvature gravities typically have a massive ghost-like graviton and a scalar mode in their spectrum, along with the standard massless graviton. The condition (4.26) guarantees that these modes are absent on MSS vacua.

2. The theory allows for charged solutions of the form (4.18) and (4.21), with $N(r) = N_k =$ constant, i.e, characterized by a single function $f(r)$.

3. The equation for $f(r)$, obtained from $\mathcal{E}_N|_{N=N_k} = 0$, can be integrated once, with an integration constant that is proportional to the total mass of the spacetime.

4. For some theories the integrated equation for $f(r)$ is algebraic and can be solved trivially, and in this case the theory if of the "Quasitopological" subclass. Other times the integrated equation is a second order ODE for $f(r)$, meaning that the theory is of the "Generalized Quasitopological" subclass.

5. The thermodynamic properties of charged black holes can be studied analytically.

In this work we are only interested on dealing with the Quasitopological class of Lagrangians, whose solution for $f(r)$ can be obtained analytically, thus leading to exact results.

4.1.3 Four-derivative EQGs

Now that we know the features that define the EQG family of theories, let us obtain their explicit form. The first non-trivial corrections will contain four derivatives, and at this order there are four types of terms, schematically R^2, RH^2, H^4 and $(\nabla H)^2$, although we will be interested mostly on the first two. Regarding the pure gravity Lagrangian, the most general form of the quadratic-curvature corrections is

$$\mathcal{L}^{R^2} = \lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \,, \tag{4.27}$$

although there exists only one such combination that satisfies the condition given by Eq. (4.26): the Gauss-Bonnet, or quadratic Lovelock, density,

$$\chi_4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \,. \tag{4.28}$$

It is well known that Lovelock theories possess single-function solutions of the form (4.18) [94, 96–98], so let us turn our attention to the terms coupling the $(d-2)$-field H to gravity.

There exist three independent scalars of the form RH^2, which can be written as[6]

$$\mathcal{L}_{RH^2} = \alpha_1 H^2 R + \alpha_2 \left(H^2\right)^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + \alpha_3 \left(H^2\right)^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} \,, \tag{4.29}$$

6 There exists a four contraction of the form $\left(H^2\right)^{\mu\nu}{}_{\rho\sigma} R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}$, but this is related to the term multiplied by α_3 in Eq. (4.29), as can be checked using the Bianchi identity of the Riemann tensor.

where again we are using the notation introduced in Eq. (4.12). Evaluating this Lagrangian on the ansatz given by Eqs. (4.18) and (4.21) with $N(r)$ equal to a constant value N_k, we obtain

$$r^{d-1}\mathcal{L}_{RH^2}\big|_{N_k,f} = -\frac{Q^2(d-1)!}{r^{d+1}}\Big[\big(2\alpha_3+\alpha_1(d-1)(d-2)+\alpha_2(d-2)\big)(f-k) + f'\big(2\alpha_1(d-1)r+\alpha_2 r\big)+\alpha_1 r^2 f''\Big]\,, \tag{4.30}$$

where we included the factor r^{d-1} that comes from the volume element $\sqrt{|g|}$. In order for this Lagrangian to belong to the EQG family we need to apply the condition (4.26), which implies that the quantity above should be a total derivative. Computing the functional derivative of this reduced Lagrangian with respect to f, we find that there is a single relation for the couplings that make it vanish identically, and it can be expressed as

$$\alpha_3 = -(2d-1)(d-1)\alpha_1 - (d-1)\alpha_2\,. \tag{4.31}$$

Therefore, there are two independent contractions of the form RH^2 that can be added to the two-derivative Lagrangian while maintaining single-function solutions.

Moving to the next kind of terms, in general dimensions there are two operators of the form H^4 that do not violate parity, which can be chosen as[7]

$$\mathcal{L}_{H^4} = \beta_1\left(H^2\right)^2 + \beta_2\left(H^2\right)^{\mu}{}_{\nu}\left(H^2\right)^{\nu}{}_{\mu}\,. \tag{4.32}$$

When evaluated on the ansatz (4.18) and (4.21) we see that both on-shell densities are independent of $f(r)$, so they both belong to the EQG class independently. However, since both terms contribute to the black hole solutions with any k in the same way, it will be enough for our purposes to keep only one of them, and for simplicity we choose the $\left(H^2\right)^2$ operator. Finally, we checked explicitly that there are no terms of the form $(\nabla H)^2$ that belong to the EQG class.[8]

Therefore, introducing some normalization factors for convenience, we can write the general four-derivative EQG theory that we will consider in most of this Chapter as

$$\begin{aligned} I_{\mathrm{EQG},4} = \frac{1}{16\pi G_{\mathrm{N}}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\Bigg[& R+\frac{d(d-1)}{L^2}-\frac{2}{(d-1)!}H^2+\frac{\lambda L^2}{(d-2)(d-3)}\mathcal{X}_4 \\ & +\frac{2\alpha_1 L^2}{(d-1)!}\left(H^2R-(d-1)(2d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right) \\ & +\frac{2\alpha_2 L^2}{(d-1)!}\left(R^{\mu}{}_{\nu}\left(H^2\right)^{\nu}{}_{\mu}-(d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right) \\ & +\frac{\beta L^2}{(d-1)!^2}\left(H^2\right)^2\Bigg]\,. \end{aligned} \tag{4.33}$$

Notice that we included also different powers of the length scale L, in such a way that the couplings λ, α_1, α_2 and β are dimensionless.

This theory with four independent parameters should be enough from the point of view of effective field theory. As shown in [262, 265], an EFT extension of Einstein-Maxwell theory only

7 This is easier to see if one works in terms of the 2-form $G=\star H$. Indeed, it is possible to write only two inequivalent quartic contractions of this form: $\left(G_{\mu\nu}G^{\mu\nu}\right)^2$ and $G_{\mu}{}^{\nu}G_{\nu}{}^{\alpha}G_{\alpha}{}^{\beta}G_{\beta}{}^{\mu}$.

8 This might be connected to the fact that there are no well-behaved theories with terms of the form $(\nabla\,\mathrm{Riemann})^2$, as we showed in [29]. However, a rigorous exploration should be performed before making such claim.

requires four independent parity-preserving terms, and the rest of higher-derivative operators can be removed via field redefinitions. We have checked that our Lagrangian above indeed spans the basis of four independent operators in the references, which means that we can capture any parity-preserving four-derivative correction to Einstein-Maxwell theory. This is not the case for the parity-breaking Chern-Simons term that appears in five dimensional supergravity theories [276–278], but we would not expect it to modify the results presented in this Chapter.

However, as opposed to what is common in the EFT approach, our theory will allow us to perform a fully non-perturbative analysis, since we will obtain quantities that are exact in the couplings of the action (4.33). Of course, one can always produce perturbative expressions by expanding linearly in the couplings, but the exact result is more interesting and can serve as an educated guess for the behavior of these theories and their holographic duals beyond the perturbative regime.

Let us close this Section by discussing the electromagnetic dual theory of (4.33). The fact that we have an H^4 term makes it difficult to invert Eq. (4.8) explicitly using the procedure explained in Section 4.1.1. However, it is easy to obtain the dual action by performing a expansion in the order of derivatives. In that case, we can write $H(F) = H_0(F) + L^2 H_2(F) + \mathcal{O}\left(L^4\right)$, and the inversion of Eq. (4.8) at each order in L is straightforward. We find that the dual theory to fourth order in derivatives reads

$$\begin{aligned}\tilde{I}_{\mathrm{EQG},4} = \frac{1}{16\pi G_{\mathrm{N}}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\Bigg[& R + \frac{d(d-1)}{L^2} - F^2 + \frac{\lambda L^2}{(d-2)(d-3)}\mathcal{X}_4 \\ & + \frac{L^2}{d-2}RF^2\left(3d\alpha_1 + \frac{d\alpha_2}{d-1}\right) \\ & - \frac{2L^2}{d-2}F_{\mu\alpha}F_{\nu}{}^{\alpha}R^{\mu\nu}\left(4(2d-1)\alpha_1 + \frac{(3d-2)\alpha_2}{d-1}\right) \\ & + \frac{2L^2}{d-2}F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma}\left((2d-1)\alpha_1 + \alpha_2\right) + \frac{\beta}{4}\left(F^2\right)^2 + \mathcal{O}\left(L^4\right)\Bigg]\,,\end{aligned} \tag{4.34}$$

where $\mathcal{O}(L^4)$ denotes an infinite series of terms that could in principle be computed in the same manner.

In Section 4.2 we study the black hole solutions of the theory (4.33), but let us first show how analogous versions of this theory can be written at higher orders in the derivatives.

4.1.4 EQGs at all orders

Although in this Chapter we are mostly interested in the four-derivative EQG theory given in Eq. (4.33), it is possible to use the same procedure to construct theories of this family at arbitrary order in the curvature tensor and the field strength, in any spacetime dimension $D = d + 1$. In the case of pure gravity, Quasitopological and Generalized Quasitopological theories at all orders have been obtained in [10], so we focus on the case of non-minimally coupled theories. In analogy with the four-dimensional theories identified in [23], we have been able to find an infinite family of EQGs with the Lagrangian densities

$$\begin{aligned}\mathcal{L}^{(a)}_{d,s,m} &= \left(sR\left(R^{s-1}\right)^{\mu\nu}{}_{\rho\sigma} + \kappa_{d,s,m}\left(R^{s}\right)^{\mu\nu}{}_{\rho\sigma} + 2s(s-1)R^{\mu}{}_{\gamma}R^{\beta}{}_{\rho}\left(R^{s-2}\right)^{\gamma\nu}{}_{\beta\sigma}\right)\left(H^2\right)_{\mu\nu}{}^{\rho\sigma}\left(H^2\right)^{m-1},\\ \mathcal{L}^{(b)}_{d,s,m} &= \frac{1}{2}\left(2sR^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} + g_{d,s,m}R^{\alpha\beta}{}_{\mu\nu}\right)\left(R^{s-1}\right)^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\alpha\beta}\left(H^2\right)^{m-1},\end{aligned} \tag{4.35}$$

where

$$g_{d,s,m} = -d(s-1) - 2(d-1)m\,, \qquad \kappa_{d,s,m} = \frac{1}{2}g_{d,s,m}\left(1 - g_{d,s,m}\right)\,. \tag{4.36}$$

Here, $\left(H^2\right)^{\mu\nu}{}_{\rho\sigma}$ is the contraction given in Eq. (4.12), and we have introduced the notation

$$(R^n)^{\mu\nu}{}_{\rho\sigma} \equiv R^{\mu\nu}{}_{\mu_1\nu_1}R^{\mu_1\nu_1}{}_{\mu_2\nu_2}\cdots R^{\mu_{n-1}\nu_{n-1}}{}_{\rho\sigma}\,. \tag{4.37}$$

Evaluating the previous Lagrangian densities on the ansatz given by Eqs. (4.18) and (4.21), we find

$$\begin{aligned}\mathcal{L}^{(a)}_{d,s,m} &= \frac{2^{s-2}Q^{2m}\left((d-1)!\right)^m}{r^{2(d-1)m}}\psi_k^{s-2}\left[4\kappa_{d,s,m}\psi_k^2 - 2s\left((d-1)\mathcal{H}_{k,d} + \mathcal{G}_d\right)\psi + 2s(s-1)\mathcal{H}^2_{s,d}\right],\\ \mathcal{L}^{(b)}_{d,s,m} &= \frac{2^{s-1}Q^{2m}\left((d-1)!\right)^m}{r^{2(d-1)m}}\psi_k^{s-1}\left[-s\mathcal{H}_{k,d} + g_{d,s,m}\psi_k\right],\end{aligned} \tag{4.38}$$

where we defined the scalar functions

$$\psi_k = \frac{k-f}{r^2}\,, \tag{4.39}$$

$$\mathcal{H}_{k,d} = \frac{(d-2)f - (d-2)k + rf'}{r^2} + \frac{fN'}{rN}\,, \tag{4.40}$$

$$\mathcal{G}_d = \frac{2(d-1)fN' + 4rfN'' + 6rf'N' + N\left(2(d-1)f' + 2rf''\right)}{2Nr}\,. \tag{4.41}$$

Since these theories belong to the EQG class, the reduced Lagrangians (this is, taking into account the volume element) become a total derivative when evaluated on $N(r) =$ constant. Indeed, by computing these explicitly we find

$$\begin{aligned}r^{d-1}\mathcal{L}^{(a)}_{d,s,m}\big|_{ds^2_{1,f},H_Q} &= \frac{d}{dr}\mathcal{I}^{(a)}_{d,s,m}\,,\\ r^{d-1}\mathcal{L}^{(b)}_{d,s,m}\big|_{ds^2_{1,f},H_Q} &= \frac{d}{dr}\mathcal{I}^{(b)}_{d,s,m}\,,\end{aligned} \tag{4.42}$$

where

$$\begin{aligned}\mathcal{I}^{(a)}_{d,s,m} &= 2^{s-1}Q^{2m}\left((d-1)!\right)^m r^{d-2m(d-1)}\psi_k^{s-1}\left[\left(1-2m+d(2m+2n-1)\right)\psi_k + sr\psi_k'\right],\\ \mathcal{I}^{(b)}_{d,s,m} &= 2^{s-1}Q^{2m}\left((d-1)!\right)^m r^{d-2m(d-1)}\psi_k^{s}\,.\end{aligned} \tag{4.43}$$

Therefore, it is possible to construct infinite examples of EQGs at any order in the curvature and the field strength by considering linear combinations of $\mathcal{L}^{(a)}_{d,s,m}$ and $\mathcal{L}^{(b)}_{d,s,m}$. In general, these can be written as

$$I_{\text{EQG, gen}} = \frac{1}{16\pi G_{\text{N}}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\left[R + \frac{d(d-1)}{L^2} - \frac{2}{(d-1)!}H^2 + \mathcal{L}^{\text{EQG}}\right], \tag{4.44}$$

where

$$\mathcal{L}^{\rm EQG}=\frac{2}{(d-1)!}\sum_{s=0}^{\infty}\sum_{m=1}^{\infty}L^{2(s+m-1)}\left(\alpha_{1,s,m}\mathcal{L}^{(a)}_{d,s,m}+\alpha_{2,s,m}\mathcal{L}^{(b)}_{d,s,m}\right). \tag{4.45}$$

Of course, one could add to this action that of the pure (Generalized) Quasitopological gravities found in [10]. As an example, the four-derivative theory given in Eq. (4.33) corresponds to

$$\begin{aligned}I_{\rm EQG,4}=\frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}}d^{d+1}x\sqrt{|g|}\Bigg[R+\frac{d(d-1)}{L^2}-\frac{2}{(d-1)!}H^2+\frac{\lambda L^2}{(d-2)(d-3)}\mathcal{X}_4\\+\frac{2L^2\alpha_1}{(d-1)!}\mathcal{L}^{(a)}_{d,1,1}+\frac{2L^2\alpha_2}{(d-1)!}\mathcal{L}^{(b)}_{d,1,1}-\frac{2L^2\beta}{(3d-4)(d-1)!}\mathcal{L}^{(b)}_{d,0,2}\Bigg].\end{aligned} \tag{4.46}$$

Since by construction the theory (4.44) is an EQG, it has solutions of the form (4.18) and (4.21) with $N(r)=N_0=$ constant. The dynamics in this case is only determined by the equation of motion for $f(r)$, which after integrating in the radial variable reads

$$\begin{aligned}k-f-\frac{m}{(d-1)r^{d-2}}+\frac{2Q^2}{(d-1)(d-2)r^{2(d-2)}}+\frac{r^2}{L^2}\\+\sum_{s,p=1}^{\infty}\frac{2^sL^{2(s+p-1)}Q^{2p}\left((d-1)!\right)^{p-1}\psi_k^{s-1}}{(d-1)r^{2(d-1)p}}\left(\beta_{s,p}k+\gamma_{s,p}\right)=0,\end{aligned} \tag{4.47}$$

where m is an integration constant proportional to the mass of the black hole that we have already defined, and

$$\begin{aligned}\beta_{s,p}&=(d-1)(2s+2p-1)\alpha_{1,s,p}+\alpha_{2,s,p},\\ \gamma_{s,p}&=\left(d-1-(4d-2)s+2ds^2+2p(d-1)(2s-1)\right)\alpha_{1,s,p}+(s-1)\alpha_{2,s,p}.\end{aligned} \tag{4.48}$$

Notice that the equation of motion (4.47) is algebraic in $f(r)$, and therefore these theories do in fact belong to the Quasitopological subclass. However, theories of the Generalized Quasitopological type are expected to exist as well, although their study lies outside the scope of the present work.

During the rest of this Chapter we will focus on the four-derivative model given by Eq. (4.33). The more general theories considered in this Section, however, will be revisited in Chapter 5, where we will show that the entanglement entropy of their dual CFTs fulfill a universal relation that will be proposed.

4.2 AdS vacua and black hole solutions

Let us now move to discussing the solutions for the four-derivative theory (4.33), and we start by determining its anti-de Sitter vacua. As usual, the higher-derivative terms modify the AdS length scale, which no longer coincides with the scale L appearing in the cosmological constant. This is often written as

$$\tilde{L}^2=\frac{L^2}{f_\infty}, \tag{4.49}$$

for a dimensionless constant f_∞, so that the Riemann tensor of pure AdS takes the form

$$R^{\mu\nu}{}_{\rho\sigma}=-\frac{2f_\infty}{L^2}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}. \tag{4.50}$$

Plugging this into the equations of motion for the metric (4.3), one finds that f_∞ must satisfy

$$1 - f_\infty + \lambda f_\infty^2 = 0\,, \tag{4.51}$$

where we also set $H = 0$, since we are computing the vacuum solution. The polynomial equation (4.51) is due entirely to the Gauss-Bonnet contribution and well known in the literature [117]. It has two real roots if $\lambda \leq 1/4$, but only one of them is continuously connected to the Einstein gravity vacuum when $\lambda \to 0$, and it reads

$$f_\infty = \frac{1}{2\lambda}\left(1 - \sqrt{1-4\lambda}\right)\,. \tag{4.52}$$

There are no solutions with $\lambda > 1/4$, so this is the maximum value that this coupling can take. As corresponding to Lovelock gravity, and also to the family of Generalized Quasitopological gravities, the linearized gravitational equations around this vacuum are identical to the linearized Einstein equations with an effective Newton's constant that determines the coupling to matter [86]. For GB gravity, this effective Newton's constant is

$$G_{\text{eff}} = \frac{G_{\text{N}}}{1 - 2\lambda f_\infty}\,. \tag{4.53}$$

The denominator in this expression is the slope of the AdS vacuum equation (4.51), as happens for all theories with an Einstein-like spectrum [121, 279] . Also, G_{eff} is divergent in the limit $\lambda \to 1/4$, which is known as the critical theory [280, 281] and is one of the singular theories mentioned in Section 2.2.

Let us now obtain the symmetric solutions of the theory (4.33) with different topologies. This theory belongs to the EQG class by construction, so it allows for solutions of the form (4.18) and (4.21) with $N(r) = N_k =$ constant. In fact, the equation $\delta L_{N,f}/\delta f = 0$ implies that $N'(r) = 0$, so these are the only possible solutions, and we only need to solve for $f(r)$ by considering the equation $\delta L_{N,f}/\delta N|_{N=N_k} = 0$. This takes the form of a total derivative, as expected based on [7], and it reads explicitly

$$\begin{aligned}\frac{\delta L_{N,f}}{\delta N} = \frac{d}{dr}\Bigg[&(d-1)\frac{r^d}{L^2}\left(1 - \frac{L^2}{r^2}\big(f(r)-k\big) + \lambda\frac{L^4}{r^4}\big(f(r)-k\big)^2\right)\\ &+ \frac{2Q^2}{d-2}\frac{1}{r^d}\Big(r^2 + (d-1)(d-2)L^2\alpha_1 f(r) + k(d-2)L^2\big(3(d-1)\alpha_1 + \alpha_2\big)\Big)\Bigg] = 0\,.\end{aligned} \tag{4.54}$$

As anticipated, this integrated equation is algebraic in $f(r)$ instead of differential, which characterizes this theory as belonging to the Quasitopological subclass. Let us remark that in $d = 3$ we should take $\lambda = 0$ in this equation, as in that case the GB invariant does not contribute to the equations of motion. Equating the argument of the derivative to a constant m, which will be proportional to the physical mass of the black hole, and introducing

$$X \equiv \frac{L^2}{r^2}\big(f(r) - k\big)\,, \tag{4.55}$$

we can rewrite the equation as

$$\lambda X^2 - \Gamma(r)X + 1 + \Upsilon(r) = 0\,, \tag{4.56}$$

where we defined

$$\Gamma(r) = 1 - \frac{2\alpha_1 L^2 Q^2}{r^{2(d-1)}}\,, \tag{4.57}$$

$$\begin{aligned} \Upsilon(r) = & -\frac{mL^2}{(d-1)r^d} + \frac{2L^2Q^2}{(d-1)(d-2)r^{2(d-1)}}\left(1 + k(d-2)\frac{L^2}{r^2}\left(4(d-1)\alpha_1 + \alpha_2\right)\right) \\ & - \frac{\beta L^4 Q^4}{(3d-4)(d-1)r^{4(d-1)}}\,. \end{aligned} \tag{4.58}$$

Written in this form, Eq. (4.56) is simply a quadratic polynomial in X, which can be straightforwardly solved obtaining

$$f(r) = k + \frac{r^2}{2\lambda L^2}\left[\Gamma(r) \pm \sqrt{\Gamma^2(r) - 4\lambda\,(1 + \Upsilon(r))}\right]. \tag{4.59}$$

The two roots found correspond to two solutions connected to different AdS vacua at $r \to \infty$, but if we want the one that reduces to the Einstein gravity result in the limit $\lambda \to 0$ we should choose the "$-$" sign. When $\lambda = 0$, which is always the case for $d \leq 3$, this solution simply becomes

$$f(r) = k + \frac{r^2(1 + \Upsilon(r))}{L^2\Gamma(r)}\,. \tag{4.60}$$

Let us now identify the physical features of this solution. For $r \to \infty$, $f(r)$ can be expanded as

$$f(r) = f_\infty \frac{r^2}{L^2} + k - \frac{m}{(d-1)(1 - 2\lambda f_\infty)r^{d-2}} + \mathcal{O}\left(\frac{1}{r^{2(d-2)}}\right) + \ldots\,, \tag{4.61}$$

where f_∞ is given by Eq. (4.51). Therefore, it approaches asypmtotically the AdS vacuum that we have determined above, with the Riemann tensor given by Eq. (4.50). On the other hand, the mass M can be identified by looking at the next term in the expansion of $f(r)$ [114, 282–285],

$$-\frac{16\pi G_{\rm eff} M}{(d-1)N_k V_{k,\,d-1}}\frac{1}{r^{d-2}} \in f(r)\,, \tag{4.62}$$

where $G_{\rm eff}$ is the effective Newton's constant and N_k takes into account the normalization of the time coordinate at infinity, which sets the units of the problem. Note also that, in the cases in which the volume of the transverse section $V_{k,\,d-1}$ is infinite, one would instead define a mass density $\rho \equiv M/V_{k,\,d-1}$.

Taking into account the value of $G_{\rm eff}$ given in Eq. (4.53), we get that the physical mass of the black hole in this solution is

$$M = \frac{N_k V_{k,\,d-1}}{16\pi G_{\rm N}} m\,, \tag{4.63}$$

proportional to the integration constant m, as mentioned before. On the other hand, as given in Eq. (4.10), the magnetic charge of the $(d-2)$-form B is defined as

$$q = \frac{1}{4\pi G_{\rm N}}\int_{\mathcal{S}_{d-1}} H\,, \tag{4.64}$$

where the integral is performed over any spacelike codimension-two hypersurface $\mathcal{S}_{d-1}$ that encloses the origin $r = 0$. As we discussed around Eq. (4.10), this quantity is also the electric charge of the dual theory. With the form of H given in Eq. (4.21), it is straightforward to compute

$$q = \frac{V_{k,d-1}}{4\pi G_{\rm N}} Q\,, \tag{4.65}$$

and again in the cases $k = 0$ and -1 one should define instead a charge density $q/V_{k,d-1}$.

Later on, it will be important to know the electrostatic potential of the dual theory. The field strength of the dual vector A_μ is obtained according to Eq. (4.8), and evaluating that expression on the forms of the metric and H given by Eqs. (4.18) and (4.21) we find that it corresponds to a pure electric field,

$$F = dt \wedge dr N_k Q\Bigg[-\frac{1}{r^{d-1}} - \frac{L^2\alpha_1}{r^{d+1}}\big(3d(d-1)k - 3d(d-1)f(r) + 2(d-1)rf'(r) + r^2 f''(r)\big) - \frac{L^2\alpha_2}{r^{d+1}}\big(dk - df(r) + rf'(r)\big) + \frac{L^2Q^2\beta}{r^{3(d-1)}}\Bigg]. \tag{4.66}$$

This can be written explicitly as a total derivative, $F_{tr} = -\Phi'(r)$, from where we can identify the electrostatic potential of the dual theory,

$$\Phi(r) = -N_k Q\Bigg[\frac{1}{(d-2)r^{d-2}} + \frac{L^2\alpha_1}{r^d}\big(3(d-1)k - 3(d-1)f(r) - rf'(r)\big) + \frac{L^2\alpha_2}{r^d}\big(k - f(r)\big) - \frac{L^2Q^2\beta}{(3d-4)r^{3d-4}}\Bigg] + \Phi_\infty\,. \tag{4.67}$$

In this expression we added an integration constant Φ_∞, which represents the value of this potential at infinity.

The solution given by Eq. (4.59) represents a black hole if the function $f(r)$ has a zero, $f(r_+) = 0$, which is smoothly connected to infinity (this is, there are no singularities between $r = r_+$ and $r \to \infty$). The point $r = r_+$ corresponds to the event horizon, and it is easier to look for its position by inspecting Eq. (4.56). In fact, at the horizon we have $X(r_+) = -kL^2/r_+^2$, and hence we get the equation

$$\lambda\frac{k^2L^4}{r_+^4} + \Gamma(r_+)\frac{kL^2}{r_+^2} + 1 + \Upsilon(r_+) = 0\,. \tag{4.68}$$

We can not obtain the value of r_+ explicitly from this, but it will prove useful to express instead different quantities associated to the solution as functions of r_+ and Q. In particular, the mass of the black hole is given by

$$M = \frac{N_k V_{k,d-1}}{16\pi G_{\rm N}}\Bigg[(d-1)\left(kr_+^{d-2} + \frac{r_+^d}{L^2} + \lambda k^2L^2r_+^{d-4}\right) + \frac{2Q^2}{(d-2)r_+^{d-2}}\left(1 + k(d-2)\frac{L^2}{r_+^2}\big(3(d-1)\alpha_1 + \alpha_2\big)\right) - \frac{\beta L^2Q^4}{(3d-4)r_+^{3d-4}}\Bigg]. \tag{4.69}$$

The Hawking temperature of the black hole can be computed as $T = N_k f'(r_+)/4\pi$. This can be easily evaluated by differentiating Eq. (4.56) with respect to r and evaluating at r_+, which yields

$$T = \frac{N_k}{4\pi r_+\left(1 - 2L^2Q^2\alpha_1 r_+^{-2(d-1)} + 2kL^2\lambda r_+^{-2}\right)}\left[\left((d-2)k + d\frac{r_+^2}{L^2} + (d-4)k^2\lambda\frac{L^2}{r_+^2}\right) - \frac{2Q^2}{(d-1)r_+^{2(d-1)}}\left[r_+^2 + dkL^2\left(3(d-1)\alpha_1 + \alpha_2\right)\right] + \frac{\beta L^2Q^4}{(d-1)r_+^{2(2d-3)}}\right]. \tag{4.70}$$

On the other hand, the electrostatic potential given by Eq. (4.67) must vanish at the horizon.[9] This is achieved by fixing the integration constant Φ_∞, which is the asymptotic value of the potential, to the value

$$\Phi_\infty = N_kQ\left[\frac{1}{(d-2)r_+^{d-2}} + \frac{L^2\alpha_1}{r_+^d}\left(3(d-1)k - r_+\frac{4\pi T}{N_k}\right) + \frac{L^2\alpha_2 k}{r_+^d} - \frac{L^2Q^2\beta}{(3d-4)r_+^{3d-4}}\right]. \tag{4.71}$$

Finally, we will need the value of the entropy of the black hole. For a general theory of gravity, this is given by the Iyer-Wald formula [150, 154] that we introduced in Section 1.2.2,

$$S = -2\pi\int_{\Sigma_+} d^{d-1}x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{4.72}$$

where h is the determinant of the induced metric at the horizon Σ_+, and $\epsilon_{\mu\nu}$ is the binormal, normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. Evaluating this expression, one finds the value of the entropy

$$S = \frac{r_+^{d-1}V_{k,d-1}}{4G_\mathrm{N}}\left(1 + \frac{2L^2Q^2\alpha_1}{r_+^{2d-2}} + \frac{2kL^2(d-1)\lambda}{(d-3)r_+^2}\right). \tag{4.73}$$

The thermodynamic properties of these black holes will be further discussed in Section 4.5.

4.3 Holographic dictionary

The family of Electromagnetic Quasitopological gravity theories introduced in Section 4.1.2 is most naturally written in terms of a $(d-2)$-form field. However, as explained in Section 4.1.1, the differential form can be dualized into a vector field making these theories equivalent to higher-derivative extensions of Einstein-Maxwell. Therefore, while most computations presented here are performed in the frame of the $(d-2)$-form, their holographic aspects are better understood in the "Maxwell frame."

A vector field in the bulk of AdS couples to a current in the boundary theory. In our case we are working with a dimensionless[10] gauge field A_μ, but the holographic dictionary requires that the vector has dimensions of energy. Thus, the field that couples to the dual current J^a is not A_μ, but rather

$$\tilde{A}_\mu = \frac{1}{\ell_*}A_\mu\,, \tag{4.74}$$

where ℓ_* is a length scale that should be fixed in each particular case. This implies that, for instance, the chemical potential in the dual CFT is identified as

$$\mu = \lim_{r\to\infty}\tilde{A}_t = \lim_{r\to\infty}\frac{1}{\ell_*}A_t\,. \tag{4.75}$$

[9] This is a regularity condition, which avoids the divergence of the norm $\sqrt{A_\mu A^\mu}$ at the horizon.

[10] The fact that A_μ is dimensionless can be seen, for example, from a simple dimensional analysis of the dual action (4.34).

The goal of this Section is to compute other entries of the holographic dictionary of the theory (4.33). In particular, we will compute the 2-point function $\langle JJ\rangle$ and the energy flux after an insertion of J^a, which is equivalent to the 3-point function $\langle TJJ\rangle$. We will also review the form of the correlators $\langle TT\rangle$ and $\langle TTT\rangle$, which depend entirely on the purely gravitational part of the action.

The term H^4 will not play any role in these computations, since in order to compute $\langle JJ\rangle$ and $\langle TJJ\rangle$ we only need the quadratic contributions. Thus, we can ignore this term for now, which is equivalent to setting $\beta = 0$. In addition, in this Section we do not really need to stick to the EQG family, so for generality we can consider the action

$$I = \frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\left[R + \frac{d(d-1)}{L^2} + \frac{\lambda L^2}{(d-2)(d-3)}\chi_4 - \frac{2}{(d-1)!}\left(H^2\right)_{\mu\nu}{}^{\rho\sigma} Q^{\mu\nu}{}_{\rho\sigma}\right], \quad (4.76)$$

where $Q^{\mu\nu}{}_{\rho\sigma}$ contains the three possible couplings at linear order in the curvature,

$$Q^{\mu\nu}{}_{\rho\sigma} = \delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\left(1 - \alpha_1 L^2 R\right) - \alpha_2 L^2 R^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} - \alpha_3 L^2 R^{\mu\nu}{}_{\rho\sigma}\,. \quad (4.77)$$

The tensor $\tilde{Q}$ defined in Eq. (4.14) can be easily computed from the form of $Q^{\mu\nu}{}_{\rho\sigma}$, finding

$$\begin{aligned}\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = &\left[1 - L^2 R\left(\alpha_1 + \frac{\alpha_2}{d-1} + \frac{2\alpha_3}{(d-1)(d-2)}\right)\right]\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} \\ &+ 2\left(\frac{\alpha_2}{d-1} + \frac{4\alpha_3}{(d-1)(d-2)}\right)R^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} - \frac{2\alpha_3}{(d-1)(d-2)}L^2 R^{\mu\nu}{}_{\rho\sigma}\,,\end{aligned} \quad (4.78)$$

and we can write the dual theory using the inverse of this tensor as

$$\tilde{I} = \frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\left[R + \frac{d(d-1)}{L^2} + \frac{\lambda L^2}{(d-2)(d-3)}\chi_4 - \left(\tilde{Q}^{-1}\right)^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]. \quad (4.79)$$

The quantities corresponding to the EQG action given in Eq. (4.33) are then obtained by setting

$$\alpha_3 = -(2d-1)(d-1)\alpha_1 - (d-1)\alpha_2\,. \quad (4.80)$$

4.3.1 Stress tensor 2- and 3-point functions

It is well known that higher-curvature corrections in the gravity action modify the structure of the correlators of the dual stress-energy tensor. For our action (4.79) only the Gauss-Bonnet scalar χ_4 will contribute to this quantity, and its effect has been extensively studied in the literature [100–102].

The form of the 2-point function of the stress-energy tensor for any CFT is fixed by the constraints that arise from conformal symmetry and energy conservation [101, 286, 287]. It reads

$$\langle T_{ab}(x)T_{cd}(0)\rangle = \frac{C_T}{|x|^{2d}}\mathcal{I}_{ab,cd}(x)\,, \quad (4.81)$$

where we introduced the tensorial structures

$$\mathcal{I}_{ab,cd} = \frac{1}{2}\left(I_{ac}(x)I_{bd}(x) + I_{ad}(x)I_{bc}(x)\right) - \frac{1}{d}g_{ab}g_{cd}\,, \quad (4.82)$$

and

$$I_{ab}(x) = g_{ab} - 2\frac{x_a x_b}{x^2}\,, \tag{4.83}$$

with the Latin indices corresponding to the boundary submanifold, and g_{ab} being the metric there, which is typically flat. The constant C_T in Eq. (4.81) is known as the central charge, and it is the only part that depends on the theory at hand. Holographically, this is determined by studying linearized gravitational fluctuations around the AdS vacuum and evaluating the action on this solution. But since the linearized equations of GB gravity are identical to those of Einstein gravity with a renormalization of Newton's constant, the value of C_T can be obtained from the one in GR replacing $G_{\rm N}$ by $G_{\rm eff}$ given in Eq. (4.53). Therefore, it reads

$$C_T = \frac{(1-2\lambda f_\infty)\Gamma(d+2)}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}}\frac{\tilde{L}^{d-1}}{G_{\rm N}}\,, \tag{4.84}$$

where we also included the effective AdS radius $\tilde{L} = L/\sqrt{f_\infty}$, with f_∞ given by Eq. (4.52).

The next quantity that we are interested in is the 3-point function $\langle TTT\rangle$ which is characterized by only three constants in theories that preserve parity, usually denoted $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ [286]. The Ward identity of the stress-energy tensor provides a relation between one of these and the central charge C_T, so only two additional parameters are necessary, and these can be chosen to be the coefficients t_2 and t_4 that measure the energy fluxes at infinity after an insertion of the stress tensor [183]. The explicit relation between those sets of parameters has been found in [101], but it is not important for our computations.

In Einstein gravity one finds $t_2 = t_4 = 0$, while higher-order gravities allow us to explore more general universality classes of dual CFTs. In particular, in Gauss-Bonnet gravity the coefficient t_2 is non-vanishing for $d > 3$, and it reads [101]

$$t_2 = \frac{4\lambda f_\infty}{1-2\lambda f_\infty}\frac{d(d-1)}{(d-2)(d-3)}\,, \tag{4.85}$$

while $t_4 = 0$. A non-vanishing t_4 can be achieved by introducing other higher-derivative terms such as Quasitopological [117] and Generalized Quasitopological gravity [119, 128], or more general theories with and Einstein-like spectrum [288]. However, since the focus of this Chapter is the presence of non-minimal coupled gauge fields, it will be enough to stick to the case of the Gauss-Bonnet correction.

4.3.2 Current 2-point function

Let us now turn to correlators involving the current operator J^a, which will be modified by the non-minimal couplings of our theory (4.79). In a CFT, the 2-point function of any pair of operators is constrained by conformal symmetry up to a proportionality constant. In the case of a vector current, we have

$$\langle J_a(x) J_b(y)\rangle = \frac{C_J}{|x-y|^{2(d-1)}} I_{ab}(x-y)\,, \tag{4.86}$$

where the quantity $I_{ab}(x)$ has already been introduced in Eq. (4.83), and the constant C_J is the central charge of the current J.

As a first example, let us compute this quantity for a CFT dual to the theory

$$I_{\text{example}} = \frac{1}{16\pi G_{\text{N}}} \int_{\mathcal{M}} d^{d+1}x \sqrt{|g|} \left[R + \frac{d(d-1)}{L^2} - F^2 + \epsilon_1 L^2 R F^2 + \epsilon_2 L^2 R_{\mu\nu} F^{\mu\alpha} F^{\nu}{}_{\alpha} \right. \\ \left. + \epsilon_3 L^2 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right] . \tag{4.87}$$

In terms of $\tilde{A}_\mu = A_\mu/\ell_*$, the Maxwell term in the action can be written as $-\tilde{F}^2/(4g^2)$, from where we identify the gauge coupling constant g,

$$g^{-2} = \frac{\ell_*^2}{4\pi G_{\text{N}}} . \tag{4.88}$$

In order to compute C_J we have to consider a small perturbation of A_μ around pure AdS space, and evaluate the action in the corresponding solution with appropriate boundary conditions. Since in this example we do not consider a GB term in the action (if we did, we should simply change $L \to \tilde{L}$), the AdS curvature is

$$R^{\mu\nu}{}_{\rho\sigma} = -\frac{2}{L^2} \delta^{[\mu}{}_{[\rho} \delta^{\nu]}{}_{\sigma]} , \tag{4.89}$$

and the quantity appearing in the action is

$$\left[F^2 - \epsilon_1 L^2 R F^2 - \epsilon_2 L^2 R_{\mu\nu} F^{\mu\alpha} F^{\nu}{}_{\alpha} - \epsilon_3 L^2 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right]_{\text{AdS}} = \left(1 + d(d+1)\epsilon_1 + d\epsilon_2 + 2\epsilon_3 \right) F^2 . \tag{4.90}$$

Then, around pure AdS spacetime the only effect of the non-minimal couplings is to rescale the gauge coupling constant to an effective one, which in this case reads

$$g_{\text{eff}}^{-2} = g^{-2} \left(1 + d(d+1)\epsilon_1 + d\epsilon_2 + 2\epsilon_3 \right) . \tag{4.91}$$

Therefore, the central charge C_J in the example theory (4.87) is the same one as in Einstein-Maxwell theory, replacing g by g_{eff}. This yields

$$C_J^{\text{example}} = \left(1 + d(d+1)\epsilon_1 + d\epsilon_2 + 2\epsilon_3 \right) C_J^{\text{EM}} , \tag{4.92}$$

where the Einstein-Maxwell central charge C_J^{EM} reads[11]

$$C_J^{\text{EM}} = \frac{\Gamma(d)}{\Gamma(d/2-1)} \frac{\ell_*^2 \tilde{L}^{d-3}}{4\pi^{d/2+1} G_{\text{N}}} , \tag{4.93}$$

and in this case $\tilde{L} = L$. Note also that unitarity requires $C_J > 0$, which sets a bound in the couplings ϵ_i. This constraint will be exploited later for our theory.

Let us now turn to the case that we are interested in, the theory for the $(d-2)$-form written in Eq. (4.76), which is expressed in the Maxwell frame in Eq. (4.79). The most difficult part of the treatment is that it involves computing the inverse of the tensor $\tilde{Q}^{\mu\nu}_{\rho\sigma}$, but this becomes trivial when working on a vacuum AdS background. Due to the GB term, the AdS radius in

[11] This charge is that of [261] multiplied by 4, to account for the different normalization of the vector field in the action.

this case is $\tilde{L} = L/\sqrt{f_\infty}$, and when evaluated on the curvature given in Eq. (4.50) the tensor (4.78) takes the form

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \alpha_{\text{eff}} \delta^{[\mu}{}_{[\rho} \delta^{\nu]}{}_{\sigma]} \,, \tag{4.94}$$

where

$$\alpha_{\text{eff}} = 1 + d(d+1) f_\infty \alpha_1 + d f_\infty \alpha_2 + 2 f_\infty \alpha_3 \,. \tag{4.95}$$

Thus, the inverse of $\tilde{Q}$ is simply

$$\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma} = \frac{1}{\alpha_{\text{eff}}} \delta^{[\mu}{}_{[\rho} \delta^{\nu]}{}_{\sigma]} \,, \tag{4.96}$$

and around the AdS vacuum the quadratic term of the field $\tilde{A}_\mu = A_\mu/\ell_*$ in the action (4.79) is given by

$$\mathcal{L}_{\tilde{F}^2} = -\frac{1}{4 g_{\text{eff}}^2} \tilde{F}^2 \,, \qquad g_{\text{eff}}^2 = \frac{4\pi G_{\text{N}}}{\ell_*^2} \alpha_{\text{eff}} \,. \tag{4.97}$$

Following the same logic as in the previous example, we conclude that the central charge C_J is the same as for Einstein-Maxwell theory, but rescaled by the constant α_{eff},

$$C_J = \frac{C_J^{\text{EM}}}{\alpha_{\text{eff}}} \,, \tag{4.98}$$

with C_J^{EM} given in Eq. (4.93). Notice that, since the duality transformation has the effect of inverting the effective gauge coupling, the combination α_{eff} appears in the denominator instead of the numerator of C_J. This means that the 2-point function $\langle JJ \rangle$ can diverge for finite values of the couplings α_i, while it vanishes if we take any of these to infinity. In any case, due to unitarity we have to impose the constraint

$$\alpha_{\text{eff}} > 0 \,, \tag{4.99}$$

which will later be used to set bounds on the parameters α_i.

The Electromagnetic Quasitopological gravity (4.33) is a particular case of the theory considered here, with the couplings related according to Eq. (4.31). This means that the effective coupling takes the value

$$\alpha_{\text{eff}}^{\text{EQG}} = 1 - (3d^2 - 7d + 2) f_\infty \alpha_1 - (d-2) f_\infty \alpha_2 \,, \tag{4.100}$$

and the central charge is given similarly in terms of this through Eq. (4.98).

4.3.3 Energy fluxes

We now want to perform the conformal collider thought experiment introduced for the first time in [183], and eventually compute the value of the parameter a_2 that characterizes the energy flux after the insertion of a current operator for our theory.

Let us consider a d-dimensional CFT in flat space $ds^2 = -(dx^0)^2 + \delta_{ij} dx^i dx^j$ and in its vacuum state, that we denote by $|0\rangle$. The bulk geometry dual to this CFT is pure AdS in the Poincare patch, expressed as

$$ds^2 = \frac{\tilde{L}^2}{z^2} \left[-(dx^0)^2 + \delta_{ij} dx^i dx^j + dz^2 \right] \,, \tag{4.101}$$

with $x^0 = t$. We want to perform an insertion of a current operator of the form $\epsilon_i J^i$, where ϵ_i is a constant polarization tensor, and obtain the energy flux measured at infinity. More precisely, we consider an operator of the form

$$\mathcal{O}_E = \int d^d x \epsilon_i J^i e^{-iEx^0} \psi(x/\sigma)\,, \tag{4.102}$$

where $\psi(x/\sigma)$ is a distribution function that localizes the insertion at $x^a = 0$ for $\sigma \to 0$, and E is the energy. In terms of the cartesian coordinates x^a, the operator for the energy flux in the direction $\vec{n}$ is given by

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^{d-2} \int_{-\infty}^{\infty} dx^0 T^0{}_i \left(x^0, r\vec{n}\right) n^i\,, \tag{4.103}$$

where $r^2 \equiv \delta_{ij} x^i x^j$. We are interested in the expectation value for the energy flux after the insertion of the operator $\mathcal{O}_E$,

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{\langle 0|\mathcal{O}_E^\dagger \mathcal{E}(\vec{n}) \mathcal{O}_E|0\rangle}{\langle 0|\mathcal{O}_E^\dagger \mathcal{O}_E|0\rangle}\,. \tag{4.104}$$

Taking advantage of the O$(d-1)$ symmetry of the problem, one can see that the expectation value of this energy flux takes the form[12] [183]

$$\langle \mathcal{E}(\vec{n}) \rangle_J = \frac{E}{\Omega_{(d-2)}} \left[1 + a_2 \left(\frac{|\epsilon \cdot n|^2}{|\epsilon|^2} - \frac{1}{d-1}\right)\right]\,, \tag{4.106}$$

where $\Omega_{(d-2)}$ is the volume of the $(d-2)$-sphere of unit radius and a_2 is a constant depending on the theory.

By construction, it is clear that $\langle \mathcal{E}(\vec{n}) \rangle$ involves an integrated $\langle TJJ \rangle$ correlator over an integrated 2-point function $\langle JJ \rangle$. The 3-point function $\langle TJJ \rangle$ is constrained by conformal symmetry up to two constants, so the parameter a_2 must be a function of these. The Ward symmetry of the stress-energy tensor provides an additional relation between these constants and C_J, so it is clear that the correlator $\langle TJJ \rangle$ is fully determined by the central charge C_J and the parameter a_2. This relation will be shown explicitly in the next Section.

Holographically, the energy fluxes can be obtained by evaluating the gravitational action on the background of a shock wave, given by the metric

$$ds^2 = \frac{\tilde{L}^2}{u^2} \left[\delta(y^+) \mathcal{W}\left(y^i, u\right) (dy^+)^2 - dy^+ dy^- + \sum_{i=1}^{d-2} (dy^i)^2 + du^2\right]\,. \tag{4.107}$$

The coordinates (y^a, u) introduced here are not the same as the original cartesian coordinates (x^a, z) of Eq. (4.101), but they are related as

$$y^+ = -\frac{1}{x^+}\,, \qquad y^- = x^- - \frac{\sum_{i=1}^{d-2}(x^i)^2}{x^+} - \frac{z^2}{x^+}\,, \qquad y^i = \frac{x^i}{x^+}\,, \qquad u = \frac{z}{x^+}\,, \tag{4.108}$$

[12] Eq. (4.106) is a generalization for arbitrary dimensions of the expression given for $d = 4$ in [183]. The factor multiplying a_2 ensures that the energy flux integrated over the entire solid angle is equal to E. Indeed, it can be checked that for a vector perturbation,

$$\int \frac{|\epsilon \cdot n|^2}{|\epsilon|^2} d\Omega_{(d-2)} = \frac{\Omega_{(d-2)}}{d-1}\,. \tag{4.105}$$

for $i = 1, 2, \ldots, d-2$, and where $x^{\pm} = x^0 \pm x^{d-1}$. Additional details on this construction can be found in [117, 183]. This metric is a solution of the gravitational field equations if $\mathcal{W}$ satisfies the equation

$$\partial_u^2 \mathcal{W} - \frac{d-1}{u}\partial_u \mathcal{W} + \sum_{i=1}^{d-2} \partial_i^2 \mathcal{W} = 0\,, \tag{4.109}$$

which holds for Einstein gravity and general higher-derivative extensions of it [237, 289]. The solution of the previous equation that we are interested in reads

$$\mathcal{W}(y^i, u) = \frac{\mathcal{W}_0 u^d}{\left(u^2 + \sum_{i=1}^{d-2}(y^i - y_0^i)^2\right)^{d-1}}\,, \tag{4.110}$$

where $\mathcal{W}_0$ is a normalization constant and $y_0^i = n^i/(1+n^{d-1})$, being n^i the components of the vector $\vec{n}$ in the system of coordinates x^i. This solution is localized at $u = 0$ and $y^i = y_0^i$, and also at $y^+ = 0$ due to the $\delta(y^+)$ in the metric.

Since we want to measure energy fluxes of an excited state, we must consider a perturbation of the vector field A_μ on top of this background. In particular, an insertion of the operator (4.102) at the boundary is dual to a non-normalizable perturbation of the vector field, and if we choose for instance a constant polarization in the direction x^1 this means that we must consider a vector determined by the boundary condition $A_{x^1} \propto z^0 e^{-iEx^0}$ when $z \to 0$. When extended into the bulk and expressed in the (y^i, u) coordinate system, this kind of perturbation behaves near $y^+ = 0$ as [183]

$$A_{y^1}(y^+ \approx 0, y^-, y^i, u) \sim e^{iEy^-/2}\delta(y^1)\cdots\delta(y^{d-2})\delta(u-1)\,. \tag{4.111}$$

This is important, since as mentioned the shockwave is localized at $y^+ = 0$, and we will eventually have to evaluate A_μ at that point.

Working directly in the coordinates (y^a, u), we may simply consider a perturbation of the form

$$A = A_{y^1} dy^1 + A_{y^+} dy^+\,, \tag{4.112}$$

where we included a component A_{y^+}, which will prove to be necessary shortly. The non-vanishing components of its field strength tensor are

$$F_{\mu\nu} = 2\partial_{[\mu|}A_{y^1}\delta^1{}_{|\nu]} + 2\partial_{[\mu|}A_{y^+}\delta^+{}_{|\nu]}\,. \tag{4.113}$$

The dynamics of the field A is determined in principle by the action with higher-order corrections, in the background (4.107). However, if we ignore contact terms (this is, terms of the form $A\mathcal{W}$) in its equations of motion, they reduce simply to Maxwell's equations

$$\nabla_\mu F^{\mu\nu} = 0\,. \tag{4.114}$$

This is equivalent to the dual Lagrangian on vacuum AdS being equal to the Maxwell Lagrangian with a modified coupling constant, as we saw in Section 4.3.1. We can now impose the transverse gauge condition, $\nabla_\mu A^\mu = 0$, which in our system of coordinates implies that

$$\partial_- A_{y^+} = \frac{1}{2}\partial_{y^1} A_{y^1}\,. \tag{4.115}$$

Then, the Maxwell equations are reduced to

$$-4\partial_+\partial_- A_{y^1} + \partial_u^2 A_{y^1} - \frac{d-3}{u}\partial_u A_{y^1} + \sum_{i=1}^{d-2}\partial_i^2 A_{y^1} = 0\,. \tag{4.116}$$

The component A_{y^+} included in Eq. (4.112) was necessary to arrive to this equation, as it allows us to enforce the transverse gauge condition. The solution to this equation with the boundary conditions discussed above develops the behavior given in Eq. (4.111).[13]

In order to compute the energy flux, we have to evaluate the on-shell action and extract the part proportional to $\mathcal{W}A^2$, since this is the piece that couples to TJJ. For our theory (4.79), this requires that we evaluate first the tensor $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$ and then compute the components of its inverse $\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma}$ using the relation (4.15). The former is given by Eq. (4.78), and taking into account the fact that the shockwave background (4.107) is an Einstein space that satisfies

$$R_{\mu\nu} = -\frac{d f_\infty}{L^2} g_{\mu\nu}\,, \tag{4.117}$$

we find that

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \alpha_{\text{eff}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} - \frac{2\alpha_3}{(d-1)(d-2)}L^2 W^{\mu\nu}{}_{\rho\sigma}\,. \tag{4.118}$$

Here, the constant α_{eff} is given by Eq. (4.100), and $W^{\mu\nu}{}_{\rho\sigma}$ is the Weyl tensor, whose non-vanishing components in this metric read

$$\begin{aligned} W^{-i}{}_{+j} &= \delta(y^+)\frac{f_\infty u}{L^2}\left[u\partial_i\partial_j\mathcal{W} - \delta^i{}_j\partial_u\mathcal{W}\right]\,, \\ W^{-i}{}_{u+} &= W^{u-}{}_{+i} = -\delta(y^+)\frac{f_\infty}{L^2}u^2\partial_i\partial_u\mathcal{W}\,, \\ W^{u-}{}_{u+} &= -\delta(y^+)\frac{f_\infty u}{L^2}\left[u\partial_i\partial_i\mathcal{W} - (d-2)\partial_u\mathcal{W}\right]\,, \end{aligned} \tag{4.119}$$

plus those obtained interchanging indices. These expressions have been simplified using the equation of motion (4.109), since they will be employed to evaluate the on-shell action. Also, we note that this Weyl tensor satisfies

$$W^{\mu\nu}{}_{\rho\sigma}W^{\rho\sigma}{}_{\alpha\beta} = 0\,, \tag{4.120}$$

and therefore the inverse of $\tilde{Q}$ simply reads

$$\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma} = \frac{1}{\alpha_{\text{eff}}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} + \frac{2\alpha_3}{(d-1)(d-2)\alpha_{\text{eff}}^2}L^2 W^{\mu\nu}{}_{\rho\sigma}\,. \tag{4.121}$$

The next step is to evaluate the dual action (4.79) on-shell. The only relevant terms are those of the form $\mathcal{W}A^2$, which come from

$$\begin{aligned} \tilde{I}_{\mathcal{W}A^2} &= -\frac{1}{16\pi G_{\text{N}}}\int d^{d+1}x\sqrt{|g|}\left(\tilde{Q}^{-1}\right)^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &= -\frac{1}{16\pi G_{\text{N}}}\int d^{d+1}x\sqrt{|g|}\left[\frac{1}{\alpha_{\text{eff}}}F^2 + \frac{2\alpha_3 L^2}{(d-1)(d-2)\alpha_{\text{eff}}^2}W^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]\,. \end{aligned} \tag{4.122}$$

[13] The reasoning behind this is very similar to the one leading to Eq. (4.26) in [117]. The main difference is that we consider a perturbation of the Maxwell field, which behaves as $A_{x^1} \sim z^0$ near the boundary, as fixed by Eq. (4.116), while the authors of the reference have a perturbation of the metric with asymptotic behavior $h_{x^1x^2} \sim z^{-2}$.

Since the only component of the inverse metric that depends on $\mathcal{W}$ is g^{--}, we have

$$F^2 = 2(F_{-1})^2 g^{--} g^{11} + \ldots = -\frac{8 f_\infty^2 u^4 \delta(y_+)\mathcal{W}}{L^4}(\partial_- A_{y^1})^2 + \ldots \,, \tag{4.123}$$

where the ellipsis denotes contributions that do not depend on $\mathcal{W}$, and therefore are irrelevant for our computation. On the other hand, for the second term we have

$$W^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 4W^{-1-1}(F_{-1})^2 = -\delta(y^+)\frac{8 f_\infty^3 u^6}{L^6}\left[\partial_1^2 \mathcal{W} - \frac{1}{u}\partial_u \mathcal{W}\right](\partial_- A_{y_1})^2 \,, \tag{4.124}$$

plus again other contributions that are unimportant. Then, putting everything together, replacing also the determinant $\sqrt{|g|} = \tilde{L}^{d-1}/(2u^{d+1})$ and integrating by parts, we find

$$\tilde{I}_{\mathcal{W}A^2} = -\frac{1}{4\pi G_{\rm N} \alpha_{\rm eff}} \int du\, d^d y \frac{\tilde{L}^{d-3}}{u^{d-3}} \delta(y^+) \mathcal{W} A_{y^1} \partial_-^2 A_{y_1} \left[1 + \frac{2 f_\infty \alpha_3}{(d-1)(d-2)\alpha_{\rm eff}} T_2\right] \,, \tag{4.125}$$

where we defined

$$T_2 = \frac{u\,(u\partial_1\partial_1 \mathcal{W} - \partial_u \mathcal{W})}{\mathcal{W}} \,. \tag{4.126}$$

The shockwave localizes the integral to $y^+ = 0$, where A_{y^1} behaves as (4.111), so we have to evaluate the integrand at $u = 1$ and $y^i = 0$. This can be done in a straightforward manner by plugging in the solution for $\mathcal{W}$ given in Eq. (4.110), and taking into account that the perturbation in Eq. (4.112) has a polarization $\epsilon = (\epsilon_1, 0, \ldots, 0)$ we find

$$T_2\big|_{u=1,\, y^i=0} = d(d-1)\left(n_1^2 - \frac{1}{d-1}\right) = d(d-1)\left(\frac{|\epsilon\cdot n|^2}{|\epsilon|^2} - \frac{1}{d-1}\right) \,. \tag{4.127}$$

Therefore, comparing the expressions of the energy flux (4.106) and the on-shell action (4.125), with the value of T_2 written above, we can read off the coefficient a_2,

$$a_2 = \frac{2d f_\infty \alpha_3}{(d-2)\alpha_{\rm eff}} = \frac{2d f_\infty \alpha_3}{(d-2)\big(1 + d(d+1) f_\infty \alpha_1 + d f_\infty \alpha_2 + 2 f_\infty \alpha_3\big)} \,, \tag{4.128}$$

where we replaced $\alpha_{\rm eff}$ using Eq. (4.95). If the theory belongs to the EQG class, in which case the couplings are related by Eq. (4.31), this result reduces to

$$a_2^{\rm EQG} = -\frac{2d(d-1) f_\infty \big((2d-1)\alpha_1 + \alpha_2\big)}{(d-2)\big(1 - (3d^2 - 7d + 2) f_\infty \alpha_1 - (d-2) f_\infty \alpha_2\big)} \,. \tag{4.129}$$

4.3.4 3-point functions $\langle TJJ\rangle$

Let us finish this Section by discussing the 3-point correlator $\langle TJJ\rangle$. In a CFT, its form in position space is constrained to be [286, 287]

$$\langle T_{ab}(x_1) J_c(x_2) J_d(x_3)\rangle = \frac{t_{abef}(X_{23}) g^{eg} g^{fh} I_{cg}(x_{21}) I_{dh}(x_{31})}{|x_{12}|^d |x_{13}|^d |x_{23}|^{d-2}} \,, \tag{4.130}$$

where $I_{ab}(x)$ is the structure introduced in Eq. (4.83), and

$$\begin{aligned} t_{abcd}(X^a) &= \hat{a}h^{(1)}_{ab}(\hat{X}^a)g_{cd} + \hat{b}h^{(1)}_{ab}(\hat{X}^a)h^{(1)}_{cd}(\hat{X}^a) + \hat{c}h^{(2)}_{abcd}(\hat{X}^a) + \hat{e}h^{(3)}_{abcd}\,, \\ h^{(1)}_{ab}(\hat{X}^a) &= \hat{X}_a\hat{X}_b - \frac{1}{d}g_{ab}\,, \\ h^{(2)}_{abcd}(\hat{X}^a) &= 4\hat{X}_{(a}g_{b)(d}\hat{X}_{c)} - \frac{4}{d}\hat{X}_a\hat{X}_b g_{cd} - \frac{4}{d}\hat{X}_c\hat{X}_d g_{ab} + \frac{4}{d^2}g_{ab}g_{cd}\,, \\ h^{(3)}_{abcd} &= g_{ac}g_{bd} + g_{ad}g_{bc} - \frac{2}{d}g_{ab}g_{cd}\,. \end{aligned} \tag{4.131}$$

Here we also introducd

$$x^a_{12} = x^a_1 - x^a_2\,, \qquad X^a_{12} = \frac{x^a_{13}}{|x_{13}|^2} - \frac{x^a_{23}}{|x_{23}|^2}\,, \qquad \hat{X}^a_{12} = \frac{X^a_{12}}{|X_{12}|}\,, \tag{4.132}$$

and so on with their corresponding permutations.

The expression for $\langle TJJ\rangle$ is thus determined by four theory-dependent constants, $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{e}$. However, requiring current conservation imposes the constraints

$$d\hat{a} - 2\hat{b} + 2(d-2)\hat{c} = 0\,, \qquad \hat{b} - d(d-2)\hat{e} = 0\,, \tag{4.133}$$

so only two of them are free parameters. Following [261], we will work in terms of $\hat{c}$ and $\hat{e}$. In addition, there is a Ward identity that relates the central charge C_J to these coefficients as

$$C_J = \frac{2\pi^{d/2}}{\Gamma(d/2+1)}(\hat{c}+\hat{e})\,. \tag{4.134}$$

This reduces the number of independent parameters to just one, which can be related to the coefficient a_2 entering into the expectation value of the energy flux, Eq. (4.106). As is clear from Eq. (4.103), this flux involves an integrated $\langle TJJ\rangle$ correlator, and therefore it is possible to obtain the desired relationship by means of a somewhat straightforward field theory computation. This was performed for general dimensions in [290], finding

$$a_2 = \frac{(d-1)\big(d(d-2)\hat{e} - \hat{c}\big)}{(d-2)(\hat{c}+\hat{e})}\,. \tag{4.135}$$

With these two expressions, we can fully determine the 3-point function $\langle TJJ\rangle$ in terms of C_J and a_2. Inverting the two equations, it is possible to write

$$\hat{c} = \frac{C_J(d-2)\Gamma(d/2+1)}{2\pi^{d/2}(d-1)^3}\big(d(d-1) - a_2\big)\,, \tag{4.136}$$

$$\hat{e} = \frac{C_J\Gamma(d/2+1)}{2\pi^{d/2}(d-1)^3}\big(d-1+(d-2)a_2\big)\,. \tag{4.137}$$

Finally, replacing the values of a_2 and C_J found for our four-derivative EQG theory, these coefficients read

$$\hat{c}^{\rm EQG} = \frac{d(d-2)d!L^{d-3}\ell_*^2\big[d-2-(d-1)(3d^2-10d+2)f_\infty\alpha_1 - (d^2-4d+2)f_\infty\alpha_2\big]}{32(d-1)^2\pi^{d+1}f_\infty^{(d-3)/2}G_{\rm N}\big[1-(d-2)f_\infty\big((3d-1)\alpha_1+\alpha_2\big)\big]}\,, \tag{4.138}$$

$$\hat{e}^{\rm EQG} = \frac{d(d-2)(d-2)!L^{d-3}\ell_*^2\big[1-(d-1)(7d-2)f_\infty\alpha_1 - (3d-2)f_\infty\alpha_2\big]}{32(d+1)\pi^{d+1}f_\infty^{(d-3)/2}G_{\rm N}\big[1-(d-2)f_\infty\big((3d-1)\alpha_1+\alpha_2\big)\big]}\,. \tag{4.139}$$

This result will be important for us in the next Chapter, particularly in Section 5.1.2.

4.4 Causality, unitarity and weak-gravity-conjecture constraints

The four-derivative theory (4.33), which is the focus of the holographic explorations performed in this Chapter, depends on four coupling constants. These are free parameters which, as seen in the previous Section, modify several entries of the holographic dictionary allowing us to study CFTs belonging to more general universality classes than those dual to Einstein-Maxwell theory. However, if we want to obtain sensible answers from holography we must demand that the dual theory satisfies reasonable physical properties, such as unitarity and causality, which in the end will constrain the values of these couplings.

Before considering these, there is one more fundamental condition that our theory must meet: the existence of an AdS vacuum. The effective AdS length scale is $\tilde{L} = L/\sqrt{f_\infty}$, with f_∞ given by Eq. (4.52). Requiring that $\tilde{L}$ is real thus provides a bound in the Gauss-Bonnet coupling λ, namely

$$\lambda \leq \frac{1}{4}\,, \quad \text{for} \quad d > 3\,, \tag{4.140}$$

which we take into account from now on.

4.4.1 Unitarity and positivity of energy flux in the boundary

Several constraints can be found by demanding that the different correlators and energy fluxes of the boundary theory, defined in the previous Section, respect unitarity. We will explore these in what follows.

4.4.1.1 Constraints from $\langle TT\rangle$ and $\langle TTT\rangle$

We start by considering the correlators $\langle TT\rangle$ and $\langle TTT\rangle$, which are determined exclusively by the purely gravitational terms of the action. Hence, by studying those we expect to find bounds only on the GB coupling λ. The constraints explained here are not new, and have been known for some time [100–102].

The first condition can be obtained from demanding that the central charge of the stress-energy tensor 2-point function is positive, $C_T > 0$, as this sets the norm of the states in the CFT created by $T_{\mu\nu}$. This can also be interpreted as a unitarity condition in the bulk, since it is equivalent to imposing $G_{\text{eff}} > 0$, which prevents the graviton from having a negative energy. This central charge is modified by the Gauss-Bonnet term, and is given by Eq. (4.84). Imposing it to be positive produces the condition

$$1 - 2\lambda f_\infty > 0\,, \tag{4.141}$$

and this also ensures that $G_{\text{eff}} > 0$, as given by Eq. (4.53). By considering the value of f_∞ written in Eq. (4.52), it is easy to see that the inequality above is always fulfilled for the allowed values of λ, and thus this condition does not provide any additional constraint.

A stronger bound can be achieved by demanding positivity of the energy 1-point function. Analogously to what we explained in Section 4.3.3, the expectation value of the energy flux produced after an insertion of the stress-energy tensor of the form $\epsilon_{ij}T^{ij}$ in general reads [183]

$$\langle \mathcal{E}(\vec{n})\rangle_T = \frac{E}{\Omega_{(d-2)}}\left[1+t_2\left(\frac{\epsilon^*_{ij}\epsilon_{il}n^j n^l}{\epsilon^*_{ij}\epsilon_{ij}}-\frac{1}{d-1}\right)+t_4\left(\frac{|\epsilon_{ij}n^i n^j|}{\epsilon^*_{ij}\epsilon_{ij}}-\frac{2}{d^2-1}\right)\right]. \tag{4.142}$$

As explained in Section 4.3.1, for holographic CFTs dual to our theory (4.33) we have $t_4 = 0$, while t_2 is given by Eq. (4.85). This energy flux must be positive in any direction $\vec{n}$ and for any choice of the polarization tensor ϵ_{ij}. The resulting conditions are clearly analyzed in general dimensions first in [102] and then in Section 3.3 of [101], finding that the value of λ must be in the interval

$$-\frac{(3d+2)(d-2)}{4(d+2)^2}\leq \lambda \leq \frac{(d-2)(d-3)(d^2-d+6)}{4(d^2-3d+6)^2}. \tag{4.143}$$

Note that $\lambda = 1/4$ is not allowed by the upper bound in any dimension (that quantity indeed tends to $1/4$ for $d\to\infty$), while the lower bound prevents λ from becoming too negative.

4.4.1.2 Constraints from $\langle JJ\rangle$ and $\langle TJJ\rangle$

Let us turn now to the correlators involving the current operator J, which should provide constraints on the parameters of the non-minimally coupled terms in the action, α_1 and α_2. The arguments are very similar to the ones of the gravitational case, and are based on the unitarity of $\langle JJ\rangle$ and the positivity of the energy 1-point function $\langle \mathcal{E}(\vec{n})\rangle_J$.

The central charge of the current 2-point function, C_J, is given by Eq. (4.98), and its positivity implies that

$$\alpha_{\rm eff}^{\rm EQG} = 1-(3d^2-7d+2)f_\infty\alpha_1-(d-2)f_\infty\alpha_2 > 0. \tag{4.144}$$

This quantity is, up to a constant, the coupling of the Maxwell field in vacuum. Therefore, its positivity is equivalent to demanding that photons in the bulk carry positive energy.

More interesting bounds can be obtained from the energy flux created after an insertion of the current operator, given by Eq. (4.106). By demanding that the energy flux is positive in any direction, we find that the parameter a_2 must satisfy[14]

$$-\frac{d-1}{d-2}\leq a_2\leq d-1. \tag{4.145}$$

Plugging in the value of a_2 for the four-derivative Electromagnetic Quasitopological theory, given by Eq. (4.129), this translates into

$$-1\leq -\frac{2df_\infty\big((2d-1)\alpha_1+\alpha_2\big)}{1-(3d^2-7d+2)f_\infty\alpha_1-(d-2)f_\infty\alpha_2}\leq d-2. \tag{4.146}$$

But notice that the denominator of this expression is precisely $\alpha_{\rm eff}^{\rm EQG}$, which is assumed to be positive. Then, if we multiply the whole inequality by $\alpha_{\rm eff}^{\rm EQG}$, the two constraints can be written as

$$1-(7d^2-9d+2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2\geq 0, \tag{4.147}$$

[14] In order to find the bounds in a_2, we need to consider the different types of perturbations possible for the vector field J. For this, the simplest approach is to choose a system of coordinates in which only $\epsilon_1\neq 0$ and $\vec{n}=(n_1, 0, \ldots, 0, n_{d-1})$, normalized as $\vec{n}\cdot\vec{n}=1$. Then, the lower bound in Eq. (4.145) is obtained from scalar perturbations ($n_1=1, n_{d-1}=0$), while the upper bound comes from vector perturbations ($n_1=0, n_{d-1}=1$).

$$1-\frac{(d-1)(3d^2-14d+4)}{d-2}f_\infty\alpha_1-\frac{d^2-6d+4}{d-2}f_\infty\alpha_2\geq 0\,, \tag{4.148}$$

Again, these constraints are only true once (4.144) is imposed, so it is important to consider also that one.

The second inequality, Eq. (4.148), has a different character depending on the dimension: the coefficient multiplying α_1 is positive for $d=3$ and 4, and negative for $d\geq 5$, while that of α_2 is positive for $d=3,\ 4$ and 5, and changes sign for $d\geq 6$. For instance, if $\alpha_2=0$ we find that α_1 must be in the interval

$$\frac{d-2}{(d-1)(3d^2-14d+4)}\leq f_\infty\alpha_1\leq\frac{1}{7d^2-9d+2}\,,\quad \text{for}\quad d=3,\,4,\quad \alpha_2=0\,, \tag{4.149}$$

but the lower bound disappears for $d\geq 5$.

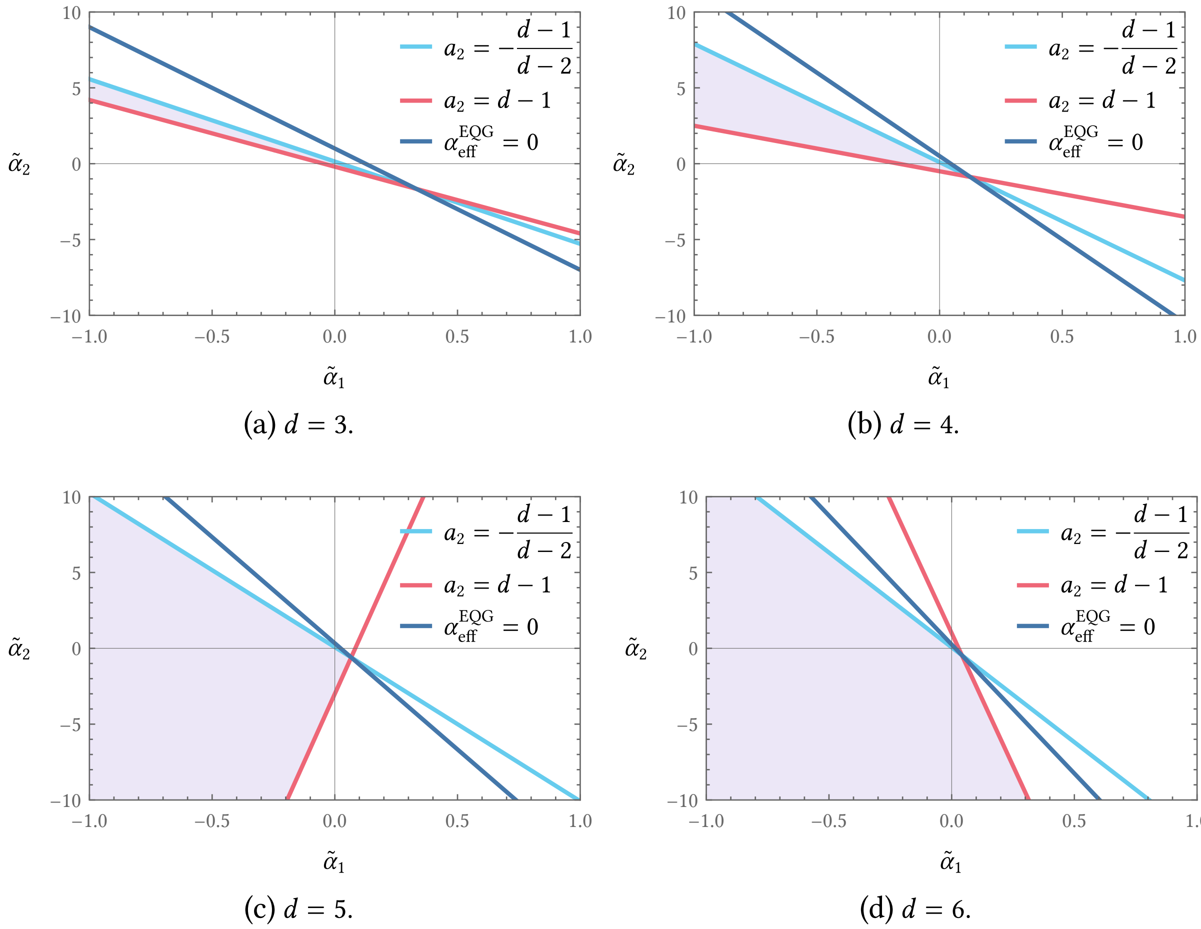


Figure 4.1: Bounds in the couplings $\tilde\alpha_1=f_\infty\alpha_1$ and $\tilde\alpha_2=f_\infty\alpha_2$ obtained from imposing unitarity and positivity of energy fluxes, as given in Eqs. (4.144), (4.147) and (4.148). The allowed region is shaded in each case, and it is infinite. For $d>6$ this region looks qualitatively very similar to the one shown for $d=6$.

Note that the bounds are imposed directly on the "renormalized" couplings $f_\infty\alpha_i$ rather than on the original ones. However, the value of f_∞ is always close to one for the allowed values of

λ in Eq. (4.143), and $f_\infty = 1$ always in $d = 3$. The different constraints are shown in Figure 4.1, where we can see that the allowed region in the plane $(f_\infty \alpha_1,\ f_\infty \alpha_2)$ becomes bigger for larger dimensions. Also, for $d = 3$, 4 and 5 there is an absolute upper bound for α_1, regardless of α_2. This is found at the intersection of the three constraints, and it reads

$$f_\infty \alpha_1 \leq \frac{1}{d(d-2)}\,, \quad \text{for} \quad d = 3,\ 4,\ 5. \tag{4.150}$$

Similarly, there is an absolute lower bound for α_2 in $d = 3$ and 4,

$$f_\infty \alpha_2 \geq -\frac{2d-1}{d(d-2)}\,, \quad \text{for} \quad d = 3,\ 4. \tag{4.151}$$

For higher dimensions the two parameters can take values in the entire real line, but they cannot both be too positive at the same time. In fact, only small values are allowed in that case, as can be seen in Figure 4.1d.

4.4.2 Causality in the bulk

It is reasonable to expect physically consistent bulk theories to give rise to consistent dual CFTs, and vice versa. Therefore, the unitarity constraints discussed above should have a counterpart in the bulk. For the constraints coming from the 2-point functions $\langle TT \rangle$ and $\langle JJ \rangle$ the interpretation is direct, as the positivity of the central charges is related to that of the energy of gravitational and electromagnetic waves in the bulk. However, the bulk interpretation of the constraints arising from the positivity of the energy 1-point function is more subtle. At least for Lovelock gravity, it is known that demanding $\langle \mathcal{E}(\vec{n}) \rangle_T \geq 0$ is equivalent to imposing the bulk theory to respect causality [102, 104, 106, 291], by avoiding superluminal propagation of gravitational waves [191, 193, 268, 291].[15] Here we investigate the analogous connection between causality of electromagnetic waves and the positivity of the energy flux $\langle \mathcal{E}(\vec{n}) \rangle_J$, given by Eq. (4.106).

We start with the neutral planar black hole (or black brane) solution of the theory (4.33). The form of the metric is

$$ds^2 = -\frac{f(r)}{f_\infty} dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2} dx^2_{(d-1)}\,, \tag{4.152}$$

and the function $f(r)$ is given by

$$f(r) = \frac{r^2}{2\lambda L^2} \left(1 - \sqrt{1 - 4\lambda + \frac{4\lambda L^2 m}{(d-1) r^d}} \right). \tag{4.153}$$

This is nothing but the metric (4.18) with $k = 0$, and we have set $N_0^2 = 1/f_\infty$ so that the speed of light at the boundary is one. Also, the solution for $f(r)$ is the one studied in Section 4.2 with $Q = 0$, since we are interested in the propagation of perturbations over the neutral background.

In order to study the speed of electromagnetic waves in our EQG theory (4.33), we can either use its formulation in terms of the $(d-2)$-form B or in terms of the dual vector field A. The results obtained in both frames must hold simultaneously.

[15] This connection is not that well understood in other theories not belonging to the Lovelock family [117, 237].

Let us consider first a perturbation of the $(d-2)$-form in this black hole background. At linear order, the equation for B can be written as

$$\nabla_{\alpha_1}\left(\tilde{Q}^{[\mu\nu}{}_{\mu\nu}H^{\alpha_1\ldots\alpha_{d-1}]}\right)=0\,, \tag{4.154}$$

where $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$ takes the value (4.78). Particularizing it to the EQG case, which requires replacing α_3 by its value given in Eq. (4.31), this tensor becomes

$$\begin{aligned}\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = &\left[1+\left(\frac{3d\alpha_1}{d-2}+\frac{d\alpha_2}{(d-1)(d-2)}\right)L^2R\right]\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\\ &-2\left(\frac{4(2d-1)\alpha_1}{d-2}+\frac{(3d-2)\alpha_2}{(d-1)(d-2)}\right)R^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}+\frac{2}{d-2}\big((2d-1)\alpha_1+\alpha_2\big)L^2R^{\mu\nu}{}_{\rho\sigma}\,.\end{aligned} \tag{4.155}$$

When evaluated on the metric (4.152), it takes the form

$$\tilde{Q}_{\mu\nu}{}^{\rho\sigma}=\gamma_1\rho_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]}+2\gamma_2\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}+\gamma_3\sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,, \tag{4.156}$$

where

$$\rho^{\alpha}{}_{\beta}\equiv\delta^{\alpha}{}_{t}\delta^{t}_{\beta}+\delta^{\alpha}{}_{r}\delta^{r}_{\beta}\,,\qquad \sigma^{\alpha}{}_{\beta}\equiv\sum_{i=1}^{d-1}\delta^{\alpha}{}_{i}\delta^{i}_{\beta} \tag{4.157}$$

are the projectors in the (t, r) and transverse spatial directions, respectively, and γ_i are the functions

$$\begin{aligned}\gamma_1 &= 1-\frac{\alpha_1L^2}{r^2}\left[3d(d-1)f-2(d-1)rf'-r^2f''\right]-\frac{\alpha_2L^2}{r^2}\left[df-rf'\right]\,,\\ \gamma_2 &= 1-\frac{\alpha_1L^2}{r^2}\left[(3d^2-11d+4)f+2drf'-r^2f''\right]\\ &\quad-\frac{\alpha_2L^2}{2(d-1)r^2}\left[2(d^2-4d+2)f+(d+1)rf'-r^2f''\right]\,,\\ \gamma_3 &= 1-\frac{\alpha_1L^2}{(d-2)r^2}\left[(d-5)(3d^2-10d+4)f+2(3d^2-11d+4)rf'+3dr^2f''\right]\\ &\quad-\frac{\alpha_2L^2}{(d-1)(d-2)r^2}\left[(d-3)(d^2-6d+4)f+2(d^2-4d+2)rf'+dr^2f''\right]\,.\end{aligned} \tag{4.158}$$

Let us now consider a fluctuation of the field B with a polarization orthogonal to the direction x^1,

$$B=\psi(r)e^{-i\omega t+ikx^1}dx^2\wedge\cdots\wedge dx^{d-1}\,. \tag{4.159}$$

Its field strength $H=\mathrm{d}B$ is

$$\begin{aligned}H=e^{-i\omega t+ikx^1}\Big(&\psi'(r)dr\wedge dx^2\wedge\cdots\wedge dx^{d-1}-i\omega\psi(r)dt\wedge dx^2\wedge\cdots\wedge dx^{d-1}\\ &+ik\psi(r)dx^1\wedge dx^2\wedge\cdots\wedge dx^{d-1}\Big)\,,\end{aligned} \tag{4.160}$$

and with this ansatz the equations of motion (4.154) are reduced to a single component, corresponding to the indices $\alpha_2\ldots\alpha_{d-1}=x^2\ldots x^{d-1}$, so it is not necessary to activate other components of B. We want to study the small wavelength limit $\omega, k\to\infty$, so we only need to keep the derivatives with respect to t and x^1. With this approximation, we get

$$\nabla_\alpha\left(\tilde{Q}^{[\mu\nu}{}_{\mu\nu}H^{\alpha x^2\ldots x^{d-1}]}\right)\propto\frac{L^{2(d-2)}}{r^{2(d-2)}}\left(-\frac{f_\infty}{f(r)}(i\omega)^2\gamma_2+\frac{L^2}{r^2}(ik)^2\gamma_1\right)B^{x^2\ldots x^{d-1}}\,, \tag{4.161}$$

and equating this to zero we obtain the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{L^2 f(r)\gamma_1}{r^2 f_\infty \gamma_2}\,. \tag{4.162}$$

By plugging in the value of $f(r)$ in Eq. (4.153) and expanding near the boundary, it becomes

$$\frac{\omega^2}{k^2} = 1 - \frac{L^2 m\big(1-(7d^2-9d+2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2\big)}{(d-1)(2-f_\infty)\alpha_{\rm eff}^{\rm EQG} r^d} + \mathcal{O}\left(\frac{1}{r^{2d}}\right)\,. \tag{4.163}$$

This quantity is the square of the phase velocity of the wave front, and consistency with causality requires it to be smaller than the speed of light, $\omega/k \leq 1$. Since $f_\infty < 2$ and we take $\alpha_{\rm eff}^{\rm EQG} > 0$, this condition implies

$$1-(7d^2-9d+2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2 \geq 0\,, \tag{4.164}$$

which matches exactly the constraint (4.147) computed from the lower bound in the allowed range of values of a_2, Eq. (4.145).

Obtaining the exact value of the dispersion relation (4.163) would require to consider all terms of higher order in $1/r$, which become important when going further into the bulk. However, we have tried different values of the couplings satisfying this constraint, and found that in all cases $\omega/k \leq 1$ everywhere inside the bulk. So the bound in Eq. (4.164) appears to be enough to ensure causality in the propagation of these perturbations everywhere, but a more in-depth analysis would be required in order to make such a claim.

A different constraint in the couplings can be obtained by choosing inequivalent polarizations of the B field, and in particular this means that we need to study a perturbation polarized along the r direction. However, instead of this it is easier to consider a perturbation of the dual vector A_μ of the form

$$A = \phi(r) e^{-i\omega t + ikx^2} dx^1\,. \tag{4.165}$$

Indeed, it can be seen that the H field obtained from the dualization of this vector is not of the form (4.160), but instead has a term $\sim k dt \wedge dr \wedge dx^3 \wedge \ldots \wedge dx^{d-1}$, corresponding to a polarization of B along the r direction. The linearized equations of motion for this field read

$$\nabla_\mu\left(\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma} F^{\rho\sigma}\right) = 0\,, \tag{4.166}$$

where $\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma}$ is the inverse of the tensor given in Eq. (4.156). With that form of $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$, its inverse is simply

$$\left(\tilde{Q}^{-1}\right)_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\gamma_1}\rho_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]} + \frac{1}{\gamma_2}\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + \frac{1}{\gamma_3}\sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,. \tag{4.167}$$

The linearized Maxwell equations for the perturbation written in Eq. (4.165) are reduced to a single component $\nu = x^1$, which reads

$$\left(\frac{\omega^2 L^2 f_\infty}{r^2 f \gamma_2} - \frac{k^2 L^4}{r^4 \gamma_3}\right) + \frac{(d-1) f L^2}{r^3 \gamma_2}\phi' + \frac{d}{dr}\left(\frac{f L^2 \phi'}{r^2 \gamma_2}\right) = 0\,. \tag{4.168}$$

In the short-wavelength limit ω, $k \to \infty$ this becomes

$$\frac{\omega^2}{k^2} = \frac{L^2 f(r)\gamma_2}{r^2 f_\infty \gamma_3}\,, \tag{4.169}$$

and expanding near infinity we get

$$\frac{\omega^2}{k^2} = 1 - \frac{L^2 m\big(d-2-(d-1)(3d^2-14d+4)f_\infty\alpha_1-(d^2-6d+4)f_\infty\alpha_2\big)}{(d-1)(d-2)(2-f_\infty)\alpha_{\rm eff}^{\rm EQG} r^d} + \mathcal{O}\left(\frac{1}{r^{2d}}\right) , \quad (4.170)$$

where we plugged in the values of γ_2 and γ_3 given in Eq. (4.158), and the function $f(r)$ in Eq. (4.153). In order for this perturbation to respect causality, it is necessary that $\omega^2/k^2 \leq 1$ as we move away from the boundary, which results in the constraint

$$d-2-(d-1)(3d^2-14d+4)f_\infty\alpha_1-(d^2-6d+4)f_\infty\alpha_2 \geq 0 . \quad (4.171)$$

This is precisely the condition (4.148), obtained from the upper bound in the value of a_2 given in Eq. (4.145). Again, we checked that $\omega/k \leq 1$ in the entire bulk for different sets of values of the couplings that respect this constraint.

While the analysis performed is relatively simple, we can argue that it is the most general. Indeed, one can be convinced that there are no other inequivalent polarizations by counting the number of them captured by the perturbations (4.159) and (4.165). For a fixed direction of propagation, (4.159) is the only possible form of the field B that is orthogonal to the propagation and with no t or r components, while there are $d-2$ polarizations of the type (4.165) for A obtained by exchanging dx^1 with dx^i, $i \neq 2$. So in total we can describe $d-1 = D-2$ different polarizations, which is the number of degrees of freedom of a massless vector field (and of a $(D-3)$-form) in D dimensions. Therefore, we conclude that there are no additional constraints that can be found from the study of causality in the background of a neutral black brane.

Another interesting problem would be the study of perturbations around charged black branes, which would be relevant if one wishes to perform holography in such backgrounds. In that case gravitational and electromagnetic perturbations would be linearly coupled, making the analysis of the speed of propagation more complex. However, this could perhaps lead to stronger constraints than the ones found here.

To conclude this discussion, let us mention that there are other types of causality violations, like the ones involving the graviton 3-point vertex found in [238]. Indeed, one of the implications of that study in the holographic context is that the Gauss-Bonnet coupling (in units of the AdS scale) must be very small, $|\lambda| \ll 1$. These bounds would be applicable, in principle, to any theory of gravity that modifies the 3-point function structure of Einstein gravity. There exist, however, non-trivial higher-curvature terms that do not modify this 3-point function, and one can not apply these results to them. In any case, we do not know of similar constraints for the terms RH^2 and H^4 of our theory (4.33). In fact, there are theories that have a large value of a_2, such as QCD, and in order to capture these holographically one needs bulk models with non-minimal higher-derivative terms with $\mathcal{O}(1)$ couplings [183].

4.4.3 WGC and positivity of entropy corrections

So far we have obtained constraints for three of the four coupling constants of the EQG theory (4.33) by imposing unitarity of the boundary theory, which we found to be equivalent to causality in the bulk theory. However, the parameter β multiplying the term H^4 is still unconstrained, as it does not affect any 2- or 3-point function. Also, while the existing constraints prevent the couplings from being too large, they do not say anything about the sign of these parameters.

In this Section we will find additional bounds by applying the mild form of the weak gravity conjecture (WGC) [271, 272], which has recently received some attention in the context of higher-derivative theories of gravity [292–302]. For an AdS spacetime, the implications of the WGC were studied in [273] (see also [303–305]). One of the heuristic ideas behind it is that extremal black holes should be able to decay, which can happen if there exists a particle whose charge-to-mass ratio is larger than the one of an extremal black hole. This is the standard form of the WGC [306, 307]. The mild form, however, involves only black holes and claims that the decay of an extremal black hole into a set of smaller black holes should be possible, at least from the point of view of energy and charge conservation. Since an extremal black hole has a fixed mass for a given value of the charge, $M_{\rm ext}(Q)$, such decay process is only possible if

$$M_{\rm ext}(Q_1+Q_2)\geq M_{\rm ext}(Q_1)+M_{\rm ext}(Q_2)\,. \tag{4.172}$$

For asymptotically flat black holes in Einstein-Maxwell theory we have $M_{\rm ext}(Q)\propto|Q|$, so the inequality is saturated. However, higher-derivative corrections modify the charge-to-mass relation, and by demanding that the deviations respect the property (4.172) we should obtain a constraint on the coefficients of the higher-derivative terms. In all cases, it is clear that in order to preserve (4.172) the corrections to the extremal mass must be negative, $\delta M_{\rm ext}<0$ [308].

The reasoning above does not carry on directly to an asymptotically anti-de Sitter spacetime. As noted in [273], the bound (4.172) is no longer saturated for extremal black holes, since the relation $M_{\rm ext}(Q)$ is not linear, and hence perturbative (arbitrary small) higher-derivative corrections cannot violate it.[16] Instead, that reference makes use of the proposal of [271], which claims that the corrections to the entropy of black holes of arbitrary charge and mass should be positive as long as these are thermodynamically stable. It is known [309, 310] that, when applied to near-extremal black holes, the positivity of corrections to the entropy is connected to the negativity of the corrections to the extremal mass. Therefore, one can still apply the condition $\delta M_{\rm ext}<0$ to bound the higher-order couplings, just like in the asymptotically flat case. However, the conditions studied in [273] are more ambitious, as they demand $\delta S>0$ for arbitrary charge and mass not only for near-extremal black holes, as long as the specific heats are positive. Let us apply these requirements to our theory (4.33).

The Wald entropy of static black holes is given in Eq. (4.73), and we repeat its value here for convenience,

$$S=\frac{r_+^{d-1}V_{k,d-1}}{4G_{\rm N}}\left(1+\frac{2L^2Q^2\alpha_1}{r_+^{2d-2}}+\frac{2kL^2(d-1)\lambda}{(d-3)r_+^2}\right)\,. \tag{4.173}$$

This expression, together with the relation $M(r_+,\,Q)$ given in Eq. (4.69), can be used to obtain the exact value of the entropy $S(M,\,Q)$. Here however we only need the perturbative correction to the entropy at fixed charge and mass. In order to simplify the analysis let us introduce the variable

$$x=\frac{r_+^{(0)}}{L}\,, \tag{4.174}$$

where $r_+^{(0)}$ is the zeroth-order value of the horizon radius, which can be obtained from Eq. (4.69) by setting the higher-order couplings to zero. Also, the extremal value of the charge in the two derivative theory reads

$$Q_{\rm ext}^{(0)}=(Lx)^{d-2}\sqrt{\frac{d-1}{2}}\sqrt{k(d-2)+dx^2}\,, \tag{4.175}$$

[16] In the small size limit AdS black holes behave as asymptotically flat ones, and in this case the inequality (4.172) could still be applied to constrain the higher-derivative terms.

so it is also convenient to define the variable

$$\xi = \frac{Q}{Q_{\text{ext}}^{(0)}}\,, \tag{4.176}$$

which takes values in the range from 0 to 1. Since we are working at fixed M and Q, we can use Eq. (4.69) to obtain the correction to the horizon radius as a series

$$r_+ = r_+^{(0)} + r_+^{(1)} + \ldots\,, \tag{4.177}$$

with the first term being

$$r_+^{(1)} = \frac{k^2\lambda L}{(\xi^2-1)x\big(k(d-2)+dx^2\big)} + \frac{3\alpha_1 kL(d-1)\xi^2}{(\xi^2-1)x} + \frac{\alpha_2 kL\xi^2}{(\xi^2-1)x} - \frac{\beta(d-1)L\xi^4\big(k(d-2)+dx^2\big)}{4(3d-4)(\xi^2-1)x}\,. \tag{4.178}$$

Inserting this into the expression for the entropy (4.173), we get the correction at linear order

$$\begin{aligned}\delta S(M,Q) = \frac{(d-1)L^{d-1}x^{d-3}V_{k,d-1}}{4G_{\text{N}}}\Bigg[&k\lambda\left(\frac{k}{(\xi^2-1)\big(k(d-2)+dx^2\big)} + \frac{2}{d-3}\right)\\ &+\alpha_1\xi^2\left(\frac{k\big((d-2)\xi^2+2d-1\big)}{\xi^2-1} + dx^2\right)\\ &+\frac{\alpha_2 k\xi^2}{\xi^2-1} - \frac{\beta(d-1)\xi^4\big(k(d-2)+dx^2\big)}{4(3d-4)(\xi^2-1)}\Bigg]\,.\end{aligned} \tag{4.179}$$

Following [273], we should demand this to be positive for any black hole that is thermodynamically stable at zeroth order. We will focus on spherically symmetric black holes, with $k = 1$. The case $k = 0$ is obtained as the limit of large size of these spherical black holes, while the case $k = -1$ is somewhat different and we will comment on it at the end of the Section.

Let us consider first neutral black holes, $\xi = 0$, for which only the Gauss-Bonnet term is relevant. The correction to the entropy is thus

$$\delta S(M,Q)\big|_{\xi=0} = \frac{(d-1)L^{d-1}x^{d-3}V_{1,d-1}}{4G_{\text{N}}}\lambda\left(-\frac{1}{d-2+dx^2} + \frac{2}{d-3}\right)\,. \tag{4.180}$$

The variable x defined in Eq. (4.174) can range from 0 to infinity, and for any of these values the quantity in the parenthesis is positive if $d \geq 3$.[17] Neutral large black holes are known to be stable in AdS, and therefore the WGC would imply that the GB coupling must be non-negative,

$$\lambda \geq 0\,. \tag{4.181}$$

This makes sense, as the Gauss-Bonnet coupling arises explicitly from string theory effective actions and in many instances[18] it indeed has a positive coupling [311–314] (see also [315] and Appendix B of [189]).

[17] For $d = 3$ one should define $\hat{\lambda} = \lambda/(d-3)$ and take the limit $d \to 3$ with fixed $\hat{\lambda}$. The correction to the entropy found is topological, and identical for any spherical black hole.

[18] It was shown in [277] that a negative λ can also be achieved, indicating that this coupling can actually have different signs depending on the setup.

Next we can look at the case of near-extremal black holes, which are also stable in the two-derivative theory. This corresponds to the limit $\xi \to 1$, and hence the dominant contribution in Eq. (4.179) is

$$\delta S(M,Q)\big|_{\xi\to 1} = \frac{(d-1)L^{d-1}x^{d-3}V_{1,d-1}}{4G_{\rm N}(1-\xi^2)}\left[-\frac{\lambda}{d-2+dx^2}-3(d-1)\alpha_1-\alpha_2 + \frac{\beta(d-1)(d-2+dx^2)}{4(3d-4)}\right]. \tag{4.182}$$

This has a non-trivial dependence on the radius of the black hole through the variable x, and therefore we can find several constraints on the couplings by imposing it to be positive. For large black holes $x \gg 1$ the correction proportional to β dominates, and $\delta S \geq 0$ implies

$$\beta \geq 0\,. \tag{4.183}$$

On the other hand, in the limit of small black holes $x \to 0$ we obtain

$$-\frac{\lambda}{d-2}-3(d-1)\alpha_1-\alpha_2+\frac{\beta(d-1)(d-2)}{4(3d-4)} \geq 0\,. \tag{4.184}$$

This may be the most reliable constraint that we can produce from the WGC arguments of [273], since small black holes behave as asymptotically flat ones, for which the inequality (4.172) must hold. The condition above implies that the shift in the extremal mass is negative, hence ensuring that Eq. (4.172) is satisfied for black holes much smaller than the AdS scale.

Finally, another condition comes from studying large black holes (or equivalently black branes, with $k = 0$) of arbitrary charge. This corresponds to $x \to \infty$, and therefore the shift in the entropy is

$$\delta S(M,Q)\big|_{x\to\infty} = \frac{d(d-1)L^{d-1}x^{d-1}V_{1,d-1}}{4G_{\rm N}}\left(\alpha_1\xi^2+\frac{\beta(d-1)\xi^4}{4(3d-4)(1-\xi^2)}\right)\,. \tag{4.185}$$

In order for this quantity to remain positive for any value $\xi \in [0,\,1)$ we must impose not only $\beta \geq 0$ as found in Eq. (4.183), but also

$$\alpha_1 \geq 0\,. \tag{4.186}$$

This is a very strong constraint since, when combined with the bounds from unitarity shown in Figure 4.1, it implies that α_1 and α_2 can only lie in a small compact set of the plane for $d = 3$, 4 and 5. The Gauss-Bonnet coupling λ must also lie in a small interval given by Eqs. (4.143) and (4.181), so only β can take arbitrarily high values with the constraints that we have found. It would be interesting to explore whether different constraints could impose an upper bound on β, and indeed the results of the next Section suggest that this coupling should not be too large.

To close this Section, let us discuss what happens if one attempts to apply the WGC bounds to hyperbolic black holes, for which $k = -1$. We will consider them neutral for simplicity, $\xi = 0$. One can check that all these solutions are thermodynamically stable in the two-derivative theory, and therefore we should impose $\delta S \geq 0$. From Eq. (4.179) we obtain in this case

$$\delta S(M,Q)\big|_{k=-1,\,\xi=0} = \frac{(d-1)L^{d-1}x^{d-3}V_{-1,d-1}}{4G_{\rm N}}(-\lambda)\left(\frac{1}{dx^2-(d-2)}+\frac{2}{d-3}\right)\,, \tag{4.187}$$

and since for hyperbolic black holes $dx^2 - (d-2) \geq 0$,[19] the positivity of δS in this case implies that $\lambda \leq 0$, which is the opposite to what we found for spherical and planar black holes in Eq. (4.181). If these two bounds were to hold at the same time for any choice of the boundary geometry, we would be led to the conclusion that $\lambda = 0$, which seems an unreasonably strong claim. Likewise, if we combine the results for the cases $k = 1$ and $k = -1$ it is possible to find similarly stringent constraints for the other constants. Although it is not clear how to solve this issue, we are more inclined to trust the constraints for spherical black holes and ignore those for $k = -1$ for a couple of reasons. On the one hand, spherical black holes make direct connection with the original motivation of the WGC regarding black hole evaporation, while the evaporation of a hyperbolic black hole is not a well-defined problem, as they are always stable [316]. On the other, as mentioned above, a positive Gauss-Bonnet coupling is actually realized in many explicit string models. This suggests that the positivity-of-entropy bounds might not be applicable directly to hyperbolic black holes, but it would be interesting to understand why. Based on these arguments, for the rest of this work we will only make use of the constraints found for $k = 1$.

4.5 Thermodynamic phase space

In Section 4.2 we studied charged black hole solutions of the four-derivative theory (4.33). These can be used to describe, in the context of the AdS/CFT correspondence, CFT plasmas at finite temperature and chemical potential, and have different properties and interpretations depending on the geometry of the horizon. The form of the metric and the $(d-1)$-form field H in these solutions is

$$\begin{aligned} ds^2 &= -N_k^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{k,(d-1)}\,, \\ H &= Q\omega_{k,(d-1)}\,, \end{aligned} \tag{4.188}$$

where N_k is a constant, $f(r)$ is the function whose solution is written in Eq. (4.59) and $\omega_{k,(d-1)}$ is the volume form of the metric tangent to the boundary $d\Sigma^2_{k,(d-1)}$, given for the different topologies in Eq. (4.20).

On the other hand, in the electromagnetic dual frame we have a Maxwell field strength that can be computed through Eq. (4.8), which leads to the vector potential

$$A = \Phi(r) dt\,, \tag{4.189}$$

where the electrostatic potential $\Phi(r)$ is given by Eq. (4.67).

We already computed some thermodynamic quantities associated to these solutions in Section 4.2, such as the temperature and the Wald entropy, but in order to make contact with the dual CFT we must first obtain the free energy from the on-shell Euclidean action, which is done in what follows.

[19] This is required for the temperature of the neutral black hole in the two-derivative theory, given by Eq. (4.70) with all higher-order couplings set to zero and $Q = 0$, to be non-negative.

4.5.1 Euclidean action and free energy

Let us work in the frame of the $(d-2)$-form B. We first perform a Wick rotation of the black hole solutions by writing $t = i\tau$, and the Euclidean time τ has a periodicity $\tau \sim \beta_T + \tau$, where $\beta_T = 1/T$, the inverse of the temperature given by Eq. (4.70). In order to compute the free energy we must first evaluate the Euclidean action, whose bulk part reads

$$\begin{aligned} I_{\rm E}^{\rm bulk} = -\frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\Bigg[&R+\frac{d(d-1)}{L^2}-\frac{2}{(d-1)!}H^2+\frac{\lambda L^2}{(d-2)(d-3)}\chi_4 \\ &+\frac{2\alpha_1 L^2}{(d-1)!}\left(H^2 R-(d-1)(2d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right) \\ &+\frac{2\alpha_2 L^2}{(d-1)!}\left(R^{\mu}{}_{\nu}\left(H^2\right)^{\nu}{}_{\mu}-(d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right) \\ &+\frac{\beta L^2}{(d-1)!^2}\left(H^2\right)^2\Bigg]\,. \end{aligned} \tag{4.190}$$

On top of this, we need to add the generalized Gibbons-Hawking-York boundary term to make the variational problem in the gravity sector well posed [274, 275], as well as counterterms to make the action finite [207]. The generalized GHY term for the Gauss-Bonnet density is known [92, 93], as well as the appropriate counterterms [210]. However, for simplicity we will consider the effective boundary terms proposed in [119], which match the renormalization prescription described in Chapter 2,

$$I_{\rm E}^{\rm bdry} = -2C\int_{\partial\mathcal{M}} d^d x\sqrt{h}\left(K-\frac{d-1}{\tilde{L}}-\frac{\tilde{L}\Theta(d-3)}{2(d-2)}\mathcal{R}+\ldots\right)\,. \tag{4.191}$$

Here, K is the trace of the extrinsic curvature and $\mathcal{R}$ the Ricci scalar of the boundary metric, while $\Theta(d-3) = 1$ for $d \geq 3$ and 0 otherwise. Additional $\mathcal{O}(\mathcal{R}^n)$ terms appear for $d \geq 5$. These terms are simply the same as those necessary for Einstein gravity, but with a proportionality constant given by (see Eq. (2.12))

$$C = -\frac{\tilde{L}^2}{2d}\mathcal{L}|_{\rm AdS}\,, \tag{4.192}$$

where $\mathcal{L}|_{\rm AdS}$ is the Lagrangian evaluated on the AdS vacuum to which the solution asymptotes. For our Lagrangian, this takes the value

$$C = \frac{1}{16\pi G_{\rm N}}\left(1-\frac{2(d-1)}{d-3}f_\infty\lambda\right)\,. \tag{4.193}$$

On the other hand, the variation of the terms RH^2 with respect to the metric decays very fast at infinity, so one does not need to include additional boundary terms to ensure the well-posedness of the variational problem. Also, they behave near the boundary as the H^2 term, so no counterterms are needed either.

In order to compute the Euclidean action, we note that the Lagrangian becomes an explicit total derivative when evaluated on the solution (4.192), which is indeed the defining property of the family of Electromagnetic Quasitopological theories. We find explicitly

$$16\pi G_{\rm N}\mathcal{L}|_{\text{on-shell}} = \frac{1}{r^{d-1}}\frac{d\mathcal{I}(r)}{dr}\,, \tag{4.194}$$

where

$$
\begin{aligned}
\mathcal{I}(r) = & -r^{d-1}f'(r) - (d-1)r^{d-2}(f(r)-k) + \frac{(d-1)r^d}{L^2} + \frac{2Q^2r^{2-d}}{d-2} \\
& - 2\alpha_1 L^2 Q^2 r^{-d}\big(3(d-1)(f(r)-k) + rf'(r)\big) - 2\alpha_2 L^2 Q^2 r^{-d}(f(r)-k) \\
& + \frac{d-1}{d-3}\lambda L^2 r^{d-4}(f(r)-k)\big((d-3)(f(r)-k) + 2rf'(r)\big) + \frac{\beta L^2 Q^4 r^{4-3d}}{4-3d}\,.
\end{aligned}
\tag{4.195}
$$

Therefore, the bulk part of the Euclidean action is computed by integrating this, and reads

$$
I_{\mathrm{E}}^{\mathrm{bulk}} = -\beta_T N_k V_{k,d-1}\int_{r_+}^{\infty} dr\, r^{d-1}\mathcal{L} = \frac{\beta_T N_k V_{k,d-1}}{16\pi G_{\mathrm{N}}}\left[\mathcal{I}(r_+) - \mathcal{I}(r\to\infty)\right], \tag{4.196}
$$

where the inverse of the temperature β_T appears due to the integration over the Euclidean time. The evaluation $\mathcal{I}(r\to\infty)$ is infinite, but one can check that all these divergences are exactly cancelled by the boundary contribution (4.191), which do not introduce any meaningful finite terms to the on-shell action.[20] Hence, we get for the total Euclidean action

$$
I_{\mathrm{E}} = I_{\mathrm{E}}^{\mathrm{bulk}} + I_{\mathrm{E}}^{\mathrm{bdry}} = \frac{\beta_T N_k V_{k,d-1}}{16\pi G_{\mathrm{N}}}\mathcal{I}(r_+)\,. \tag{4.197}
$$

The fact that we are computing this in the frame of the field B has a non-trivial effect. Indeed, when varying the action we must fix the value of this field at the boundary, which for its holographic dual corresponds to working at fixed charge. This implies that the CFT is described by the canonical ensemble, and the gravitational Euclidean action we computed is equal to the Helmholtz free energy of the system, as $F = TI_{\mathrm{E}}$, which is a function of the temperature and the charge. From the result above, we have

$$
\begin{aligned}
F = \frac{N_k V_{k,d-1}}{16\pi G_{\mathrm{N}}}\Bigg[& \frac{(d-1)r_+^d}{L^2} - r_+^{d-1}\frac{4\pi T}{N_k} + k(d-1)r_+^{d-2} + \frac{2Q^2 r_+^{2-d}}{d-2} \\
& + 2\alpha_1 L^2 Q^2 r_+^{-d}\left(3k(d-1) - \frac{4\pi T r_+}{N_k}\right) + 2k\alpha_2 L^2 Q^2 r_+^{-d} \\
& + (d-1)k\lambda L^2 r_+^{d-4}\left(k - \frac{8\pi T r_+}{(d-3)N_k}\right) + \frac{\beta L^2 Q^4 r_+^{4-3d}}{4-3d}\Bigg],
\end{aligned}
\tag{4.198}
$$

where the temperature is given in terms of Q and r_+ by Eq. (4.70). It can be checked that this quantity fulfills the definition of the Helmholtz free energy,

$$
F = M - TS\,, \tag{4.199}
$$

where M, T and S are given respectively by Eqs. (4.69), (4.70) and (4.73). We also introduce the chemical potential $\mu = \lim_{r\to\infty} A_t/\ell_*$, which from Eq. (4.71) reads

$$
\mu = \frac{N_k Q}{\ell_*}\left[\frac{1}{(d-2)r_+^{d-2}} + \frac{\alpha_1 L^2}{r_+^d}\left(3(d-1)k - \frac{4\pi T r_+}{N_k}\right) + \frac{\alpha_2 L^2 k}{r_+^d} - \frac{\beta L^2 Q^2}{(3d-4)r_+^{3d-4}}\right]. \tag{4.200}
$$

[20] In odd d some counterterms can introduce contributions of the form $I_{\mathrm{E}} \to I_{\mathrm{E}} + c\beta_T$, for some constant c. But this simply represents a global shift in the free energy, and we will assume that these finite counterterms have been chosen in such a way that pure AdS has zero free energy.

The free energy (4.198) satisfies the usual first law of thermodynamics,

$$dF = -SdT + \mu d\mathcal{N} \,, \tag{4.201}$$

where $\mathcal{N} = q\ell_*$ represents the number of charged particles under the current J in the boundary theory, and it reads (replacing the physical charge q by its value in Eq. (4.65))

$$\mathcal{N} = \frac{V_{k,d-1}\ell_* Q}{4\pi G_{\rm N}} \,. \tag{4.202}$$

We wish to work in the grand canonical ensemble, which corresponds to fixed chemical potential, so instead of F we are interested in the grand potential (or grand free energy). This is defined as

$$\Omega = F - \mu\mathcal{N} \,, \tag{4.203}$$

and it can also be obtained directly from the Euclidean action by adding or removing the appropriate boundary terms, depending on whether we are in the Maxwell or B-field frames. By construction, this quantity satisfies

$$d\Omega = -SdT - \mathcal{N} d\mu \,, \tag{4.204}$$

so Ω is to be understood as a function of T and μ. Its explicit form is

$$\begin{aligned}\Omega = \frac{N_k V_{k,d-1}}{16\pi G_{\rm N}} \Bigg[& \frac{(d-1)r_+^d}{L^2} - \frac{2Q^2 r_+^{2-d}}{d-2} + r_+^{d-2}\left((d-1)k - \frac{4\pi T r_+}{N_k}\right) \\ & - 2\alpha_1 L^2 Q^2 r_+^{-d}\left(3(d-1)k - \frac{4\pi T}{N_k}\right) - 2k\alpha_2 L^2 Q^2 r_+^{-d} \\ & + \frac{3\beta L^2 Q^4 r_+^{4-3d}}{3d-4} + (d-1)k\lambda L^2 r_+^{d-4}\left(k - \frac{8\pi T r_+}{N_k(d-3)}\right)\Bigg] \,.\end{aligned} \tag{4.205}$$

In the next Section we study the properties of these thermal states in the case of a flat boundary. The hyperbolic case will be considered in Section 5.1, in the context of the computation of Rényi entropies.

4.5.2 Flat boundary: black branes

We will perform now a thorough study of the thermodynamic phase structure of planar black holes, $k = 0$. The geometry of the boundary is Minkowski spacetime, and therefore these solutions are useful to probe the properties of thermal CFTs in flat space. More precisely, the boundary metric is conformal to

$$ds^2_{\rm bdry} = -N_0^2 f_\infty dt^2 + dx^2_{(d-1)} \,, \tag{4.206}$$

so it is natural to set

$$N_0 = \frac{1}{\sqrt{f_\infty}} \,, \tag{4.207}$$

which is equivalent to working in units such that the speed of light at the boundary is one. Note also that, taking into account the definition of the transverse space in Eq. (4.20), the volume is

$$V_{0,d-1} = \frac{V_{\mathbb{R}^{d-1}}}{L^{d-1}} \,, \tag{4.208}$$

where now $V_{\mathbb{R}^{d-1}}$ is the volume of the constant-t spatial slices of the boundary metric (4.206), which can be infinite. Therefore, it makes sense to work in terms of the grand potential, entropy, mass and number densities, defined as

$$\omega = \frac{\Omega}{V_{\mathbb{R}^{d-1}}}\,,\qquad s = \frac{S}{V_{\mathbb{R}^{d-1}}}\,,\qquad \rho = \frac{M}{V_{\mathbb{R}^{d-1}}}\,,\qquad N = \frac{\mathcal{N}}{V_{\mathbb{R}^{d-1}}}\,. \tag{4.209}$$

It will also be useful to replace the charge parameter Q by a new dimensionless variable p, as

$$Q = p\frac{r_+^{d-1}}{L}\,. \tag{4.210}$$

With this, the expressions for the grand potential density, entropy density, temperature and chemical potential read

$$\omega = -\frac{r_+^d}{16\pi G_{\rm N} L^{d+1}\sqrt{f_\infty}(d-1)}\left(\frac{d^2-3d+2+2p^2}{d-2}-\frac{\beta p^4}{3d-4}\right)\,, \tag{4.211}$$

$$s = \frac{r_+^{d-1}}{4L^{d-1}G_{\rm N}}\left(1+2\alpha_1 p^2\right)\,, \tag{4.212}$$

$$T = \frac{r_+}{4\pi\sqrt{f_\infty}L^2}\frac{d(d-1)-2p^2+\beta p^4}{(d-1)(1-2\alpha_1 p^2)}\,, \tag{4.213}$$

$$\mu = \frac{r_+}{\sqrt{f_\infty}\ell_* L}\frac{p}{(d-1)(1-2\alpha_1 p^2)}\left[(d-1)\left(\frac{1}{d-2}-d\alpha_1\right)-\left(\frac{2\alpha_1}{d-2}+\frac{(d-1)\beta}{3d-4}\right)p^2 - \frac{(d-2)\alpha_1\beta p^4}{3d-4}\right]\,, \tag{4.214}$$

and they are all independent of λ and α_2 in this planar case.[21] Each of these quantities scale in a definite way with r_+, and in particular $T \propto r_+$. In the neutral limit $p = 0$, this gives rise to the well-known relation

$$s|_{\mu=0} = C_S T^{d-1}\,, \tag{4.215}$$

where C_S is the thermal entropy charge, which in our case reads

$$C_S = \frac{\left(4\pi L\sqrt{f_\infty}\right)^{d-1}}{4G_{\rm N} d^{d-1}}\,. \tag{4.216}$$

It is known that for holographic Gauss-Bonnet gravity, and Lovelock gravity in general, this charge is not modified with respect to its value in Einstein gravity, besides the appearance of the factor f_∞. But other theories, such as Einsteinian cubic gravity [119] and Generalized Quasitopological gravities [7, 122, 123], do introduce non-trivial corrections to C_S.

Using this constant, and replacing r_+ in terms of T and p, we can write

$$\omega = -\frac{C_S}{d}T^d\left(\frac{1-2\alpha_1 p^2}{1-\frac{2p^2-\beta p^4}{d(d-1)}}\right)^d\left(1+\frac{2p^2}{(d-1)(d-2)}-\frac{\beta p^4}{(d-1)(3d-4)}\right)\,, \tag{4.217}$$

[21] There is an implicit dependence of λ through f_∞, but it only produces a trivial rescaling of some thermodynamic potentials.

$$s = C_S T^{d-1}\left(\frac{1-2\alpha_1 p^2}{1-\frac{2p^2-\beta p^4}{d(d-1)}}\right)^{d-1}\left(1+2\alpha_1 p^2\right)\,. \tag{4.218}$$

The variable p can be considered to be a function of the dimensionless ratio between the chemical potential and the temperature, given implicitly by the relation

$$\begin{aligned}\hat{\mu} \equiv \frac{\ell_* \mu}{4\pi L T} = \frac{p}{d(d-1)-2p^2+\beta p^4}\Bigg[&(d-1)\left(\frac{1}{d-2}-d\alpha_1\right)-\left(\frac{2\alpha_1}{d-2}+\frac{(d-1)\beta}{3d-4}\right)p^2\\ &-\frac{(d-2)\alpha_1\beta p^4}{3d-4}\Bigg]\,.\end{aligned} \tag{4.219}$$

Note the appearance in this expression of the ratio ℓ_*/L, whose value is an input of the particular holographic duality. As mentioned, the expression above defines p implicitly as a function of $\hat{\mu}$, which means that the quantity

$$\hat{\omega} \equiv \frac{\omega}{T^d} \tag{4.220}$$

is only a function of this variable. Therefore, studying the dependence of $\hat{\omega}$ on $\hat{\mu}$ is the same as analyzing how the grand free energy varies as we change the chemical potential and keep the temperature fixed, or equivalently, studying the free energy as a function of the temperature at fixed chemical potential. In fact, due to the scaling properties of the problem (μ, $T \sim r_+$), only the ratio $\hat{\mu}$ is relevant, and we can explore in particular the monotonicity of the grand canonical potential or the existence of phase transitions in terms of it.

Let us remark a few points before performing a more in-depth analysis. First, in most cases we will not be able to invert Eq. (4.219) explicitly, so it is useful to use p in order to obtain the curves ($\hat{\mu}$, $\hat{\omega}$) parametrically. Second, note that the temperature (4.213) must be non-negative in order to have black hole solutions. The extremal limit is thus reached for

$$d(d-1)-2p^2+\beta p^4 = 0\,, \tag{4.221}$$

which is a second order equation for p^2, and has real solutions only for $\beta \leq 1/(d(d-1))$. We can argue also that black hole solutions only exist if $1-2\alpha_1 p^2 \geq 0$. Even though we could have $T > 0$ with $1-2\alpha_1 p^2 < 0$, it turns out that this implies the existence of a naked singularity for some $r > r_+$, as can be seen by studying explicitly the solution for $f(r)$ given in Eq. (4.59), so this case must be ruled out.

For Einstein-Maxwell theory, it is possible to invert Eq. (4.219) explicitly to obtain[22]

$$p = \frac{1}{4\hat{\mu}}\left[-\frac{d-1}{d-2}+\sqrt{\left(\frac{d-1}{d-2}\right)^2+8d(d-1)\hat{\mu}^2}\,\right]\,. \tag{4.222}$$

This is a one-to-one relation $p(\hat{\mu})$, with $p(\mu \to \pm\infty) = \pm\sqrt{d(d-1)/2}$, and it is possible to check that the relation $\hat{\omega}(\hat{\mu})$ is also monotonic. Therefore, in holographic Einstein-Maxwell theory there are no phase transitions for charged plasmas.

For the full four-derivative EQG considered here, the thermodynamic phase space will depend on the values of the parameters β and α_1 in a non-trivial manner. Therefore, in order to simplify the exploration, in the following Sections we consider separately the cases $\beta = 0$ and $\alpha_1 = 0$, and after that we will study a particular case in which both couplings take non-vanishing values.

[22] There is one additional solution with a negative square root, but this would result in p and $\hat{\mu}$ having opposite signs.

4.5.2.1 *Phase space with $\alpha_1 \neq 0$ and $\beta = 0$*

First we want to isolate the contribution of α_1 to the phase space of the system, so we set $\beta = 0$. A negative α_1 is ruled out by the weak gravity conjecture as discussed in Section 4.4.3, so we consider only $\alpha_1 \geq 0$. As argued above, in order to obtain meaningful solutions we must have $T > 0$ and $1 - 2\alpha_1 p^2 \geq 0$, so the range of allowed values of p is

$$|p| \leq \min\left\{p_{\text{ext}}, \frac{1}{\sqrt{2\alpha_1}}\right\}, \quad \text{where} \quad p_{\text{ext}} = \sqrt{\frac{d(d-1)}{2}}. \tag{4.223}$$

p_{ext} is the value of the parameter p that makes the black hole extremal, $T = 0$, and the transition between both bounds happens for $\alpha_1 = 1/(d(d-1))$. We have to distinguish four different cases.

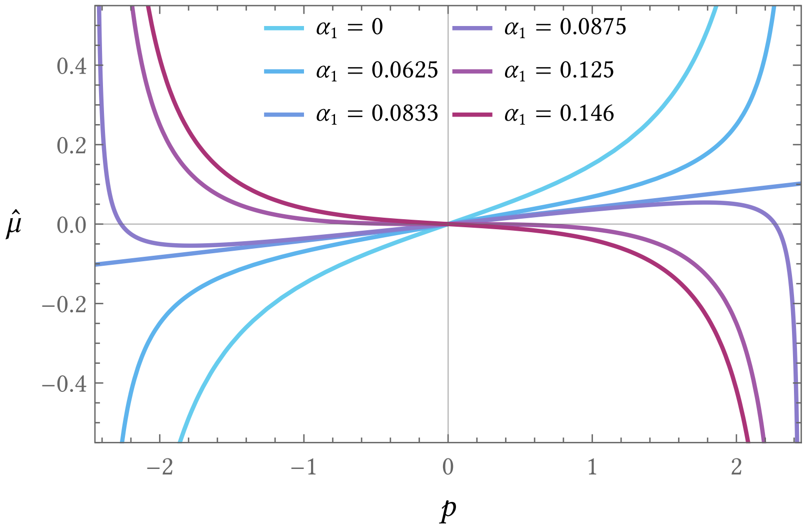


Figure 4.2: Dimensionless chemical potential $\hat{\mu}$ as a function of p for different values of the coupling α_1, with $\beta = 0$ and $d = 4$. In this case, the range where $\hat{\mu}(p)$ becomes non-invertible is $\alpha_1 \in (1/12, 1/8) \approx (0.0833, 0.125)$, as given in Eq. (4.226).

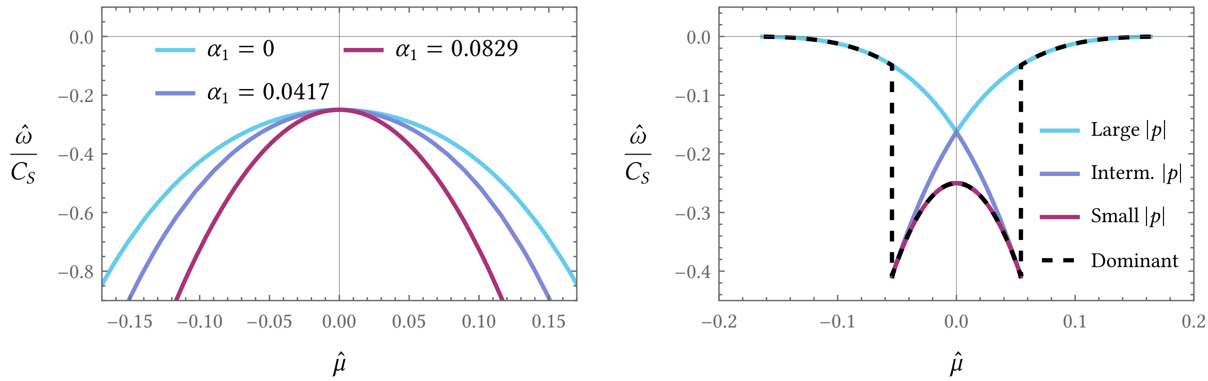
1. For $\alpha_1 < 1/(d(d-1))$, the relation $\hat{\mu}(p)$ is one-to-one and $\hat{\mu}$ takes values in the entire real line, as shown in Figure 4.2. It diverges to $\pm\infty$ for $p = \pm p_{\text{ext}}$, which corresponds to the extremal limit. The curves for $\hat{\omega}(\hat{\mu})$ are shown in Figure 4.3a, and they behave qualitatively as those in Einstein-Maxwell theory.

2. The particular case $\alpha_1 = 1/(d(d-1))$ is special, in the sense that here the relation $\hat{\mu}(p)$ becomes linear, and the temperature is independent of p,

$$\hat{\mu} = \frac{p}{d(d-1)(d-2)}, \qquad T = \frac{r_+ d}{4\pi\sqrt{f_\infty}L^2}. \tag{4.224}$$

Thus, there exist solutions for arbitrary values of $\hat{\mu}$, but no extremal limit. The expression for the grand potential density also simplifies in this case, and it reads

$$\omega = -\frac{C_S}{d}T^d\left(1 + 2d^2(d-1)(d-2)\left(\frac{\ell_*\mu}{4\pi TL}\right)^2\right). \tag{4.225}$$



(a) α_1 outside the range (4.226).

(b) α_1 inside the range (4.226).

Figure 4.3: Grand canonical potential $\hat{\omega}$ as a function of $\hat{\mu}$, for $\beta = 0$ and $d = 4$. In the left plot we take $\alpha_1 < 1/12$, so there is only one phase for each value of $\hat{\mu}$. In the right plot α_1 is inside the interval (4.226), so there are three phases with different values of p, the dominant one being denoted with a dashed black line. We can observe the zeroth-order phase transition mentioned in the main text, and there are no solutions for $|\hat{\mu}| > \mu_{\text{max}} = \sqrt{\alpha_1/2}$.

3. A new and interesting behavior appears for $\alpha_1 > 1/(d(d-1))$. Since the extremal value p_{ext} cannot be reached in this case, we see from Eq. (4.219) that only a finite range of values of $\hat{\mu}$ is allowed, and in particular it can be seen that $|\hat{\mu}| < \sqrt{\alpha_1/2}$, which is reached for $p = \pm 1/\sqrt{2\alpha_1}$.

   Besides, we find that, if α_1 is within the interval

$$\alpha_1 \in \left(\frac{1}{d(d-1)}, \frac{1}{d(d-2)} \right) , \tag{4.226}$$

   then the relation $\hat{\mu}(p)$ becomes non-invertible. In this case the function $\hat{\mu}(p)$ develops a local maximum and minimum (see Figure 4.2), and some values of $\hat{\mu}$ correspond to three distinct values of p, meaning that there are three different phases with either small, intermediate or large values of p.

   To explore the existence of phase transitions, we must look at the diagram of $\hat{\omega}$ vs $\hat{\mu}$ for all the branches of solutions. This is shown in Figure 4.3b in the case of $d = 4$ for a representative value of α_1 inside the interval (4.226). For small $\hat{\mu}$ there are three phases, and the one with the smallest p has the lowest free energy and hence dominates, while also presenting the usual quadratic behavior $\partial^2\omega/\partial\mu^2 < 0$, as can be seen in the plot. However, at a certain value $\hat{\mu}_{\text{crit}}$ this phase and the one with intermediate values of p merge, and cease to exist for larger $\hat{\mu}$. At that point, a zeroth-order phase transition takes place towards the solution with large p, as illustrated in Figure 4.3b. The new phase is somewhat exotic, since it has $N = -\partial\omega/\partial\mu < 0$ for $\hat{\mu} > 0$, so a positive chemical potential generates a negative number density, and vice versa, although it still satisfies $\partial N/\partial\mu = -\partial^2\omega/\partial\mu^2 > 0$. This phase disappears at $|\hat{\mu}| = \hat{\mu}_{\text{max}} = \sqrt{\alpha_1/2}$, and no solutions exist beyond that point.

4. If we set $\alpha_1 = 1/(d(d-2))$, the upper bound in the range (4.226), the maximum and the minimum of $\hat{\mu}(p)$ collapse at $p = 0$. Therefore, for $\alpha_1 \geq 1/(d(d-2))$ there exists a single phase, and it always has $N < 0$ for $\hat{\mu} > 0$, but $\partial N/\partial\mu > 0$. Also, this only exists for a limited range of values of $\hat{\mu}$, due to the bounds for the value of p in Eq. (4.223).

While this theory seems to allow for a more complex phase space than Einstein-Maxwell, we need to ask ourselves whether these new phases, and the special features that happen for $\alpha_1 > 1/(d(d-1))$, are allowed by the physical constraints derived in Section 4.4. If only α_1 is active, i.e., $\alpha_2 = \lambda = 0$, then according to Eq. (4.149) we have

$$\alpha_1 \leq \frac{1}{7d^2 - 9d + 2}\,, \tag{4.227}$$

which is smaller than $1/(d(d-1))$ for $d \geq 3$, thus disallowing these phase transitions.

However, it is reasonable to wonder what happens if we consider non-vanishing α_2 and λ, which do not affect directly the thermodynamic quantities for the planar black hole, but change the allowed range of values for α_1. In $d = 3$, 4 and 5, the unitarity constraints impose an upper bound in α_1 given by Eq. (4.150), which still allows for the phase transition to take place. A stronger bound can be found by taking into account the constraint (4.184) obtained from the WGC. In $d = 3$, in which case $\lambda = 0$, combining Eq. (4.184) with Eqs. (4.147) and (4.148) one can see that α_1 is bound from above by

$$\alpha_1\big|_{d=3,\,\beta=0} \leq \frac{1}{8}\,, \tag{4.228}$$

while phase transitions can take place above $\alpha_1 = 1/(d(d-1)) = 1/6$, so these are forbidden.

The $d = 4$ case is more involved, as now the Gauss-Bonnet term must be taken into account. The upper bound in α_1 can be found from the intersection of (4.148) and (4.184) at saturation, and it yields

$$\alpha_1\big|_{d=4,\,\beta=0} \leq \frac{1 - f_\infty\lambda}{12 f_\infty}\,. \tag{4.229}$$

But since $\lambda \geq 0$ due to the WGC, this value is always smaller or equal to $1/12$, which is precisely the threshold value to produce phase transitions given by Eq. (4.226). Thus, the different physical constraints seem to conspire to forbid the existence of phase transitions, at least in the most relevant cases $d = 3$ and 4. This also avoids the non-physical situation of absence of solutions for large $\hat{\mu}$.

In $d = 5$, the same constraints lead to an absolute maximum $\alpha_1|_{d=5,\,\beta=0} \leq 1/16$, which is not enough to rule out the phase transition, whose threshold value is $\alpha_1 = 1/20$. For $d \geq 6$ the value of α_1 can be arbitrarily large, as long as α_2 takes a large negative value, so these exotic phase transitions can not be avoided for large dimensions given our current constraints.

4.5.2.2 *Phase space with $\alpha_1 = 0$ and $\beta \neq 0$*

Let us now study the effects of the term H^4 in the structure of the phase space, which we isolate by setting $\alpha_1 = 0$. We also consider $\beta \geq 0$, as implied by the WGC constraints obtained in Section 4.4.3. We distinguish two main scenarios depending on the roots of the polynomial

$$d(d-1) - 2p^2 + \beta p^4 = 0\,, \tag{4.230}$$

which determine extremality, as can be seen in Eq. (4.213). There are four different solutions for this equation, $p = \pm p^{\pm}_{\text{ext}}$ (the two signs are independent), where

$$p^{\pm}_{\text{ext}} = \sqrt{\frac{1}{\beta}\left(1 \pm \sqrt{1 - d(d-1)\beta}\right)}\,. \tag{4.231}$$

There are different possible scenarios similarly to the previous case, now depending on the value of β, which we treat separately.

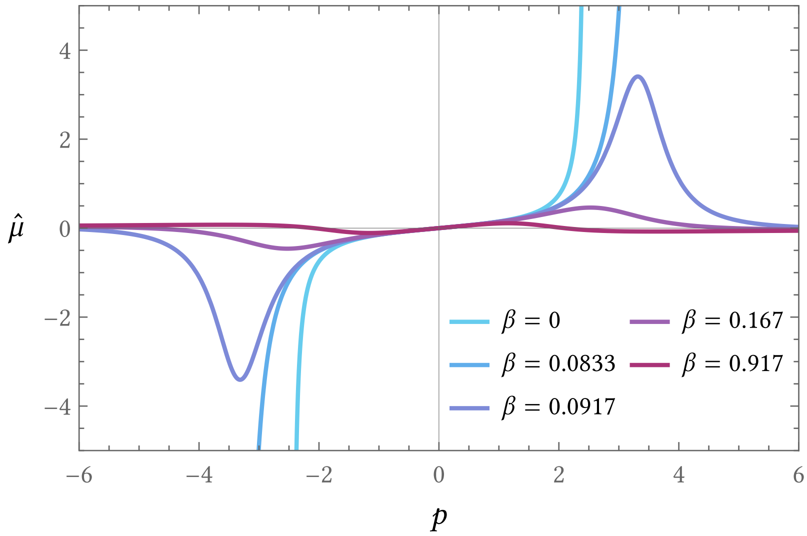


Figure 4.4: Dimensionless chemical potential $\hat{\mu}$ with respect to p for different values of the coupling β, with $\alpha_1 = 0$ and $d = 4$. In this case, $\hat{\mu}(p)$ is non-invertible for $\beta > 1/(d(d-1)) \approx 0.0833$.

1. If $\beta \leq 1/(d(d-1))$ the two roots in Eq. (4.231) are real, and there are three families of solutions. The first of them has $|p| \leq p^{-}_{\text{ext}}$ and is connected to the solutions of Einstein-Maxwell theory. It exists for arbitrary values of $\hat{\mu}$ and reaches the extremal limit for $p = p^{-}_{\text{ext}}$. The temperature is negative for $p^{-}_{\text{ext}} < |p| < p^{+}_{\text{ext}}$, and a second branch with $T \geq 0$ happens for

$$p^{+}_{\text{ext}} \leq |p| \leq \sqrt{\frac{3d-4}{(d-2)\beta}}\,, \tag{4.232}$$

the upper limit corresponding to $\hat{\mu} = 0$ while keeping $T \neq 0$. This one also exists for arbitrary $\hat{\mu}$ and has an extremal limit for $p = p^{+}_{\text{ext}}$. However, it can be seen that it is not possible to find a value of p such that $\hat{\mu}'(p) = 0$ in this range, and hence there is only one thermodynamic phase. Finally, there is the third type of solutions with $|p| > \sqrt{(3d-4)/((d-2)\beta)}$, which only exist for very small values of $\hat{\mu}$. However, these do not produce phase transitions either, as the solution in the Einstein-Maxwell branch always has the smallest grand-canonical free energy.

2. For $\beta > 1/(d(d-1))$ the roots $p^{\pm}_{\text{ext}}$ become complex, implying that the extremal limit does not exist. In this case, as shown in Figure 4.4, the relation $\hat{\mu}(p)$ has a maximum and a minimum for $p > 0$ and vice versa for $p < 0$, instead of the divergence found

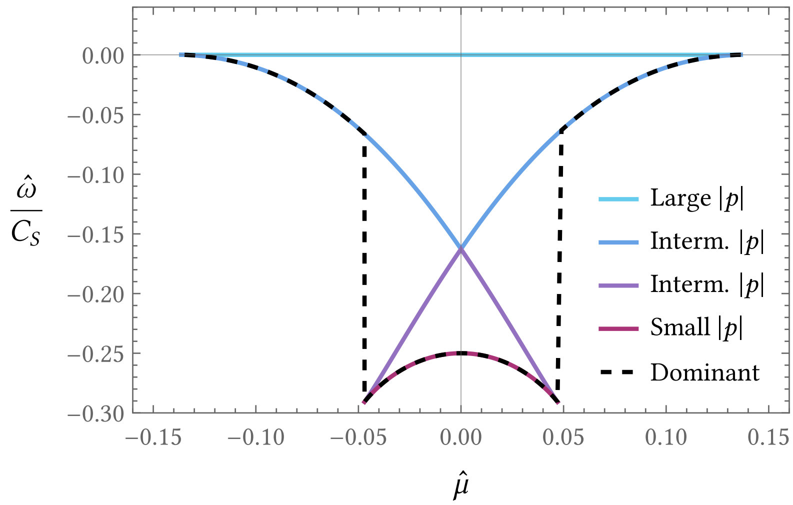


Figure 4.5: Grand canonical potential $\hat{\omega}$ with respect to $\hat{\mu}$ for $\beta = 0.433 > \beta_{\text{thr}}$, with $\alpha_1 = 0$ and $d = 4$. One can see that there is a region where four phases coexist with different values of p, and the dominant one is outlined with a black dashed line. There exists a zeroth-order phase transition, and a maximum value of $|\hat{\mu}|$ beyond which no solutions exist.

for smaller β. This means that there are up to four different phases for small $\hat{\mu}$, and it is also clear that there exists a maximum value of $\hat{\mu}$ for which any phase exists. If we keep increasing β we find that there is a point where the maximum value of $\hat{\mu}$ for $p > 0$ becomes smaller than the maximum for $p < 0$ (and respectively with the minima). This happens when β takes the value

$$\beta_{\text{thr}} \equiv \frac{(3d-4)^2}{d(d-1)(d-2)^2}\,. \tag{4.233}$$

If $1/(d(d-1)) < \beta < \beta_{\text{thr}}$ there are no phase transitions, and the dominant phase exists up to a certain $\hat{\mu}_{\text{max}}$, beyond which simply there are no solutions. But when $\beta > \beta_{\text{thr}}$, on the other hand, there is another solution that extends beyond the dominant phase, and hence when the latter finds its endpoint there is a zeroth-order phase transition. This is illustrated in Figure 4.5 for $d = 4$, where one can see also that the phase with large $|p|$ is always subdominant.

Although it is the most interesting one, the scenario with $\beta > 1/(d(d-1))$ looks rather unphysical, as we would expect the theory to allow for solutions of arbitrary $\hat{\mu}$. However, unlike in the case of α_1, we have no additional constraints on β that could rule out this case from basic principles. In fact, the coupling β is the least constrained one in our theory, since it does not affect the linearized equations on neutral backgrounds or the correlators $\langle JJ\rangle$ and $\langle TJJ\rangle$. Its first appearance will take place at $\langle JJJJ\rangle$, and it should provide non-trivial corrections when studying the propagation of electromagnetic waves on charged backgrounds. It would be interesting to study those cases and investigate possible unitarity or causality constraints on β, to see whether these could forbid the unwanted values $\beta > 1/(d(d-1))$.

4.5.2.3 *Phase space with $\alpha_1 \neq 0$ and $\beta \neq 0$*

As we have seen in the previous two Sections, the couplings α_1 and β separately have similar effects: they produce simultaneous phases that can lead to zeroth-order transitions for certain values of the couplings. However, the most undesirable feature is the fact that, whenever this happens, there exist no solutions beyond a maximum value of $|\hat{\mu}|$.

When $\beta = 0$ this situation is forbidden in $d = 3$ and 4 by the physical constraints found in Section 4.4, and if one further assumes $\alpha_2 = 0$ then this behavior is ruled out in every dimension. However, when $\beta \neq 0$ we can not avoid it, as this coupling can in principle be arbitrarily large, although we have argued that it is likely that there are other constraints that will bound the possible values of β to a finite interval.

If both β and α_1 are allowed to be non-zero we find again these exotic situations, and even more involved ones. However, in view of our previous discussion, we are interested in the cases in which $\hat{\mu}$ can take arbitrarily large values. Let us first determine the values of α_1 and β that satisfy this property. For that, note that p has a maximum value whenever $\alpha_1 > 0$, since we must fulfill $1 - 2\alpha_1 p^2 \leq 0$, as we argued below Eq. (4.221). Then, from Eqs. (4.213) and (4.219) it follows that, in order for $\hat{\mu}$ to take arbitrarily high values, the extremal limit must exist. This means that the roots (4.231) must be real, which requires

$$\beta \leq \frac{1}{d(d-1)} \,. \tag{4.234}$$

In addition, in order for $1 - 2\alpha_1 p^2 \geq 0$ to hold we must demand

$$\alpha_1 \leq \frac{1}{2(p^-_{\text{ext}})^2} \,. \tag{4.235}$$

Whenever these two conditions are satisfied, it is guaranteed that there exist solutions for any value of $\hat{\mu}$. So the question we want to answer now is whether there are any special features in the phase space within this reasonable set of couplings.

First, note that the solutions with $|p| \leq p^-_{\text{ext}}$ always exist, and that for these $\hat{\mu}$ takes values in the entire real line. In order to see whether there can be multiple phases, we need to find out if there exist points for which $\hat{\mu}'(p) = 0$ within this interval. However, it is possible to check that this never happens as long as β and α_1 are constrained by Eqs. (4.234) and (4.235), respectively. Therefore, $|p| \leq p^-_{\text{ext}}$ generates a unique phase for which $\hat{\mu}$ ranges from $-\infty$ to $+\infty$. Furthermore, this phase has the same qualitative behavior as the one found for Einstein-Maxwell theory, shown in Figure 4.3a.

However, there can be other phases if α_1 is small enough. This happens for $\alpha_1 \leq 1/(2(p^+_{\text{ext}})^2)$, in which case there exist additional solutions for

$$p^+_{\text{ext}} \leq |p| \leq \frac{1}{2\sqrt{\alpha_1}} \,. \tag{4.236}$$

In general, these have a quite similar profile to those in the interval $|p| \leq p^-_{\text{ext}}$, but they never dominate. The conclusion seems to be that, whenever we have the reasonable situation that $\hat{\mu}$ is unbounded, then no phase transition can take place and everything is qualitatively similar to what is found for Einstein-Maxwell theory in the interval $|p| \leq p^-_{\text{ext}}$, but they never dominate. The conclusion seems to be that, whenever we have the reasonable situation that $\hat{\mu}$ is unbounded, then no phase transition can take place and everything is similar to what is found for Einstein-Maxwell theory.

4.6 Holographic hydrodynamics: shear viscosity

In Section 4.5.2 above we have explored the thermodynamic properties of charged flat black holes (or black branes), which describe holographic CFTs at finite temperature and chemical potential. In this state the theory behaves as a plasma, and in the hydrodynamic regime, which corresponds to low momentum, it can be studied as a fluid [184, 317–319]. Then, we can ask ourselves about the propagation of sound waves or different transport coefficients, which determine the response of the plasma under perturbations.

In this respect, one of the quantities that attracted more attention from the early times of holographic hydrodynamics was the shear viscosity η. The ratio between this quantity and the entropy density takes a constant value in Einstein gravity [184–186],

$$\left.\frac{\eta}{s}\right|_{\rm GR} = \frac{1}{4\pi}\,, \tag{4.237}$$

which holds also in the presence of a chemical potential [258–260,320]. This led to the conjecture that it is an universal result in holographic conformal field theories, and to conjecturing the Kovtun-Son-Starinets (KSS) bound [186], which claims that $\eta/s \leq 1/(4\pi)$ for any fluid in nature.

But this holographic prediction is modified if one considers higher-curvature corrections in the bulk theory, showing that it might not be a truly universal result [187–192]. The ratio η/s has even been computed in a non-perturbative manner in theories such as Gauss-Bonnet [100, 191, 193], Lovelock gravity [105, 194, 195], cubic and quartic Quasitopological gravity [117, 196] and Generalized Quasitopological gravities [119, 122, 123], among others. Those examples showed that, even when the theory is constrained by physical requirements, the KSS bound can be lowered, although without reaching zero.[23] Nevertheless, the status of the question of how much this bound can be consistently lowered is not clear, since there are arguments, such as those in [238], that constrain the higher-order couplings to be perturbatively small. However, the arguments of this reference can not be directly applied to all types of higher-derivative interactions, and the common belief is that there exists a bound for η/s lower than the KSS one [321].

The effect of a chemical potential on the shear viscosity to entropy density ratio has been less explored. As we mentioned, η/s is independent of μ in holographic Einstein-Maxwell theory, but this is no longer true if one introduces higher-derivative corrections. The effect of perturbative corrections on the viscosity was computed in [264,265] for $N = 2$ supergravity, and in a general $d = 4$ EFT (see also [267]). Regarding the non-perturbative calculations, the case of Lovelock gravity minimally coupled to a vector field has been considered in [194, 268, 269].

However, a more general analysis has not yet been performed, and in particular the effect of non-minimally coupled operators has not been studied at the non-perturbative level. Here we perform the first exact and analytic computation of η/s with a chemical potential in a higher-derivative theory, with non-minimal interactions. The goal is to get a broader perspective on the effect of the higher-derivative terms on this ratio, and investigate how much the KSS bound can be lowered while satisfying the physical constraints found in Section 4.4.

Let us start by writing the charged planar black hole solutions, which are given by

$$ds^2 = -\frac{f(r)}{f_\infty}dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}dx^2_{(d-1)}\,, \tag{4.238}$$

[23] In Lovelock gravity it can be made arbitrarily small by taking $d \to \infty$ [105].

$$H = \frac{Q}{L^{d-1}} dx^1 \wedge \cdots \wedge dx^{d-1} \,, \tag{4.239}$$

where we fixed the value of the constant $N_0 = 1/\sqrt{f_\infty}$ so that the speed of light at the conformal boundary is one, and where $f(r)$ is given by Eq. (4.59) with $k = 0$. Let us introduce the new radial coordinate

$$z = 1 - \frac{r_+^2}{r^2} \,, \tag{4.240}$$

defined in such a way that the horizon is located at $z = 0$ and the boundary at $z = 1$. Also, we rescale the radial function to $\tilde{f}(r)$, defined by

$$f(r) = \frac{r^2}{L^2} \tilde{f}(r) \,, \qquad \lim_{r\to\infty} \tilde{f}(r) = f_\infty \,. \tag{4.241}$$

Using these, the metric (4.238) can be expressed as

$$ds^2 = \frac{r_+^2}{L^2(1-z)} \left(-\frac{\tilde{f}(z)}{f_\infty} dt^2 + dx^2_{(d-1)} \right) + \frac{L^2}{4\tilde{f}(z)} \frac{dz^2}{(1-z)^2} \,. \tag{4.242}$$

Since the horizon is located at $z = 0$, the function $\tilde{f}(z)$ can be expanded around that point as

$$\tilde{f}(z) = \tilde{f}'_+ z + \frac{1}{2} \tilde{f}''_+ z^2 + \frac{1}{6} \tilde{f}'''_+ z^3 + \ldots \,, \tag{4.243}$$

which will prove useful shortly.

The shear viscosity can be computed from the Kubo formula

$$\eta = -\lim_{\omega\to 0} \frac{1}{\omega} G^{\mathrm{R}}_{12,12}(\omega, \vec{k} = 0) \,, \tag{4.244}$$

where $G^{\mathrm{R}}_{12,12}(\omega, \vec{k} = 0)$ is a component of the retarded Green function of the stress-energy tensor, given at zero momentum by

$$G^{\mathrm{R}}_{ab,cd}(\omega, \vec{k} = 0) = -i \int dt\, d^{d-1}x\, e^{i\omega t} \Theta(t) \langle [T_{ab}(\vec{x}), T_{cd}(0)] \rangle \,. \tag{4.245}$$

This correlator can be computed holographically by considering a perturbation of the metric $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, with $h_{12} \neq 0$ [317, 322]. A rigorous computation of this quantity for higher-derivative theories at perturbative order can be found, for instance, in [265]. However, the method can be summarized in the much simpler procedure explained in [323], which has been applied to several cases [117, 119, 122, 123, 196]. Nonetheless, we will compare our results in the perturbative limit with those in [265].

According to [323], it is enough to consider a metric perturbation obtained by performing

$$dx^2 \;\to\; dx^2 + \varepsilon e^{-i\omega t} dx^1 \,, \tag{4.246}$$

where ε is a small parameter, on the background (4.242). By evaluating the Lagrangian $\sqrt{|g|}\mathcal{L}$ with this perturbed metric and expanding to second order in ε, one finds that a pole appears at the horizon $z = 0$. Then, the shear viscosity can be read off from the residue of this pole,

$$\eta = -8\pi T \lim_{\omega,\,\varepsilon\to 0} \frac{\mathrm{Res}\left[\sqrt{|g|}\mathcal{L},\; z = 0\right]}{\omega^2 \varepsilon^2} \,. \tag{4.247}$$

The computation for our theory (4.33) is more or less straightforward, and one obtains

$$\eta = \frac{1}{16\pi G_{\rm N}(d-2)L^{d-1}r_+^{d+1}}\left[r_+^{2d}\left(d-2-4\tilde{f}_+'\lambda\right)+2L^2Q^2r_+^2\left((5d-4)\alpha_1+2\alpha_2\right)\right]\,. \tag{4.248}$$

Here we find explicitly the coefficient $\tilde{f}_+'$ of the expansion (4.243). This is related to the temperature in Eq. (4.70), and it reads

$$\tilde{f}_+' = \frac{2\pi\sqrt{f_\infty}L^2}{r_+}T = \frac{d(d-1)r_+^{4d}-2L^2Q^2r_+^{2(d+1)}+L^4Q^4r_+^4\beta}{2(d-1)r_+^{2d}\left(r_+^{2d}-2L^2Q^2r_+^2\alpha_1\right)}\,. \tag{4.249}$$

On the other hand, the entropy density in the boundary theory has already been computed, and is given in Eq. (4.212). Therefore, gathering everything together and simplifying, the ratio of the shear viscosity to the entropy density can be written as

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1+\frac{4p^2\left((2d-1)\alpha_1+\alpha_2\right)}{(d-1)(1+2\alpha_1p^2)}-\lambda\frac{2\left(d(d-1)-2p^2+\beta p^4\right)}{(d-1)(d-2)(1-4\alpha_1^2p^4)}\right]\,, \tag{4.250}$$

where we replaced Q by the parameter p introduced in Eq. (4.210). This is a function of the radio between the chemical potential and the temperature, $\hat{\mu} = \mu\ell_*/(4\pi LT)$, which is related to p through Eq. (4.219).

Let us first compare our result with that of [265]. Working at linear order in the couplings, it is enough to invert Eq. (4.219) at zeroth order, and with this we get, in $d=4$,

$$\left.\frac{\eta}{s}\right|_{d=4} = \frac{1}{4\pi}\left[1-4\lambda+\frac{8(8\hat{\mu})^2}{\left(1+\sqrt{1+2(8\hat{\mu})^2/2}\right)^2}\left(7\alpha_1+\alpha_2+\frac{\lambda}{3}\right)+\ldots\right]\,. \tag{4.251}$$

Now we have to take into account that our $\hat{\mu}$ is related to $\bar{\mu}$ in the reference by $\bar{\mu} = 8\hat{\mu}$, one factor of 4 coming from our definition as $\hat{\mu} = \mu\ell_*/(4\pi LT)$ and the remaining factor of 2 being due to the different normalization of the vector field in the action. In addition, λ is related to c_1 in [265] as $\lambda = 2c_1$, since these are the coefficients of the Riemann square term. We also see, by inspecting the action in the Maxwell frame given by Eq. (4.34), that the coefficient of the term $R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, called c_2 in [265], is precisely $4c_2 = 7\alpha_1+\alpha_2$, where again the factor of 4 accounts for the different normalization of the vector field. Therefore, we reproduce Eq. (3.25) of [265], which serves as a consistency check of the pole method that we have employed. Now that this is settled, we can move on to discussing the properties of the ratio η/s in a fully non-perturbative level.

From Eq. (4.250), it is clear that there are two contributions to η/s: one proportional to λ and another one proportional to the combination $(2d-1)\alpha_1+\alpha_2$, which is also the quantity that appears in the numerator of a_2 in Eq. (4.129). The Gauss-Bonnet coupling also determines the energy flux parameter t_2 as given in Eq. (4.85), and therefore for these theories it follows that

$$a_2 = t_2 = 0 \implies \frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{4.252}$$

One could speculate about a possible relation between having "trivial" 3-point functions $\langle TTT\rangle$ and $\langle TJJ\rangle$, and the absence of corrections to η/s for more general CFTs. However, this could be just an accident due to the fact that our theory has only a few parameters. In fact, note that the combination $(2d-1)\alpha_1+\alpha_2$ is the coupling of the term $\left(H^2\right)^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}$ in the action (4.33), so it makes sense that it controls both a_2 and the corrections to η/s. Then, one would expect additional higher-order terms to spoil this connection, but it is quite remarkable nevertheless that Eq. (4.252) holds non-perturbatively for our theories. Note also that in the extremal limit, $d(d-1)-2p^2+\beta p^4=0$, the GB contribution to η/s vanishes and only the contribution coming from the non-minimal couplings remains, as was already noticed in [265, 269].

Let us study now the dependence on the chemical potential of the shear viscosity to entropy density ratio. We focus on the values of the parameters α_1 and β that give rise to a single phase with an extremal limit, as studied in Section 4.5.2.3, while taking into account also the constraints in the parameters found in Section 4.4. For convenience, we rewrite Eq. (4.250) as

$$\frac{\eta}{s}=\frac{1}{4\pi}\left[1+\left((2d-1)\alpha_1+\alpha_2\right)f_1(p)-\lambda f_2(p)\right], \tag{4.253}$$

where the dependence on p is encoded in the functions

$$f_1(p)=\frac{4p^2}{(d-1)(1+2\alpha_1p^2)}, \qquad f_2(p)=\frac{2\left(d(d-1)-2p^2+\beta p^4\right)}{(d-1)(d-2)(1-4\alpha_1^2p^4)}. \tag{4.254}$$

Since $\alpha_1\geq 0$, it is straightforward to see that $f_1(p)$ grows monotonically with $|p|$, ranging from $f_1(0)=0$ to its maximum value at extremality $p=\pm p_{\text{ext}}$. On the other hand, $f_2(p)$ has the opposite behavior, since it takes its maximum value at $p=0$ and then it decreases to zero at extremality.[24]

As a consequence of the WGC we have $\lambda\geq 0$, meaning that the GB contribution to η/s is always negative but monotonically increasing with the chemical potential. Then, the global behavior of the ratio η/s will depend strongly on the sign of a_2, and we have to distinguish the two possible cases.

4.6.0.1 Case 1: $a_2\leq 0$

Whenever $a_2\leq 0$, so that $(2d-1)\alpha_1+\alpha_2\geq 0$, the shear viscosity to entropy ratio is a growing function of $\hat{\mu}$. It therefore reaches its minimum value for $\hat{\mu}=0$,

$$\left.\frac{\eta}{s}\right|_{\hat{\mu}=0}=\frac{1}{4\pi}\left[1-\lambda\frac{2d(d-1)}{(d-1)(d-2)}\right]. \tag{4.255}$$

The largest value of λ is given by the upper bound in Eq. (4.143), and therefore we find that the absolute minimum value of η/s that can be reached with $a_2\leq 0$ is

$$\min\left[\frac{\eta}{s}\right]=\frac{1}{4\pi}\left[1-\frac{d(d-3)(d^2-d+6)}{2(d^2-3d+6)^2}\right]. \tag{4.256}$$

This is the lower bound for GB gravity [100, 101, 191, 193], and one can see that η/s never reaches zero, but its minimum value takes place in $d=8$, giving $4\pi\eta/s\geq\frac{219}{529}\approx 0.41399$.

The maximum value of η/s as a function of the chemical potential for $a_2\leq 0$ is reached at extremality, which corresponds to

$$p_{\text{ext}}=\sqrt{\frac{1}{\beta}\left(1-\sqrt{1-d(d-1)\beta}\right)}. \tag{4.257}$$

[24] We recall that we always take α_1 to satisfy $1-2\alpha_2p_{\text{ext}}^2>0$.

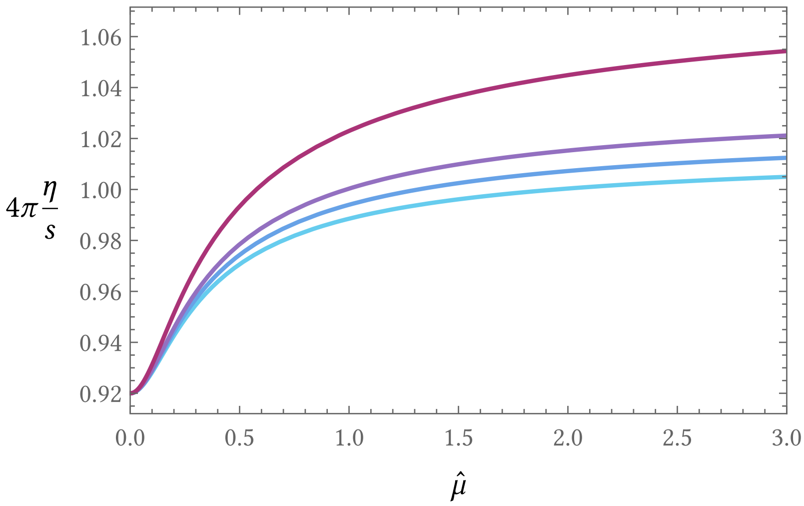


Figure 4.6: Ratio of the shear viscosity over the entropy density as a function of the chemical potential, with $\alpha_2 < 0$. We set $d = 4$ and $\lambda = 0.02$ in all cases, and from blue to red the different curves correspond to $\{\alpha_1, \alpha_2, \beta\} = \{0.0024, -0.016, 0.082\}, \{0.0019, -0.012, 0.08\}, \{0.0006, -0.0023, 0.077\}$ and $\{0, 0.0038, 0.083\}$, which satisfy all the physical constraints studied in Section 4.4. In general, η/s is grows with $\hat{\mu}$, and its value does not depart much from $1/(4\pi)$.

This gives

$$\left.\frac{\eta}{s}\right|_{\text{ext}} = \frac{1}{4\pi}\left[1 + \big((2d-1)\alpha_1 + \alpha_2\big) f_1(p_{\text{ext}})\right], \tag{4.258}$$

and since $f_1(p)$ is a growing function, this value will be larger if p is larger. Now, p_{ext} always grows with β, and its maximum value is reached for $\beta = 1/(d(d-1))$, after which the extremal limit can not exist. In this case we obtain $p_{\text{ext}}^2 = d(d-1)$, and plugging it in the expression for η/s above we get an upper bound for that ratio in our theories,

$$\frac{\eta}{s} \leq \frac{1}{4\pi}\left[1 + \frac{4d(d-1)\big((2d-1)\alpha_1 + \alpha_2\big)}{(d-2)\big(1 + 2d(d-1)\alpha_1\big)}\right]. \tag{4.259}$$

We can now explore the values of α_1 and α_2 that maximize this quantity while being compatible with the constraints found in Section 4.4. We have first the unitarity constraints (4.147) and (4.148), that involve the rescaled couplings $f_\infty \alpha_{1,2}$. Since Eq. (4.259) has no local maximum inside the region delimited by these inequalities, the maximum must occur at the boundary, this is, where the constraints are saturated. By studying what happens for a few values of d, we are drawn to the conclusion that the maximum is reached at the intersection of $\alpha_1 = 0$ and the boundary given by Eq. (4.147). We therefore obtain

$$\left.\frac{(2d-1)\alpha_1 + \alpha_2}{1 + 2d(d-1)\alpha_1}\right|_{(4.147)} \leq \frac{1}{(3d-2)f_\infty}, \tag{4.260}$$

and the maximum happens for $\alpha_2 = \frac{1}{(3d-2)f_\infty}$. Since $f_\infty \geq 1$, this value is larger when $\lambda = 0$, in which case $f_\infty = 1$. On the other hand, we can consider the WGC constraint given by Eq. (4.184).

The looser bound on α_1 and α_2 is reached precisely for $\lambda = 0$ and the maximum allowable value of β, $\beta = 1/(d(d-1))$. This yields

$$3(d-1)\alpha_1 + \alpha_2 \leq \frac{d-2}{4d(3d-4)}\,, \tag{4.261}$$

and when we combine this together with $\alpha_1 \geq 0$ we find

$$\left.\frac{(2d-1)\alpha_1 + \alpha_2}{1 + 2d(d-1)\alpha_1}\right|_{(4.262)} \leq \frac{d-2}{3d(3d-4)}\,, \tag{4.262}$$

the maximum being reached for $\alpha_1 = 0$ and $\alpha_2 = \frac{d-2}{4d(3d-4)}$. We can see that, in every dimension $d \leq 3$, the bound (4.262) is stronger than (4.260), so the former is the relevant one. Therefore, we conclude that the maximum possible value of η/s is

$$\max\left[\frac{\eta}{s}\right] = \frac{1}{4\pi}\left[1 + \frac{d-1}{3d-4}\right]\,, \tag{4.263}$$

and among these the largest value happens for $d = 3$, which gives $4\pi\eta/s \leq 7/5$.

In conclusion, in the case $a_2 \leq 0$ the shear viscosity to entropy density ratio is a growing function of the dimensionless chemical potential, and it has absolute lower and upper bounds given by

$$\frac{219}{529} \leq 4\pi\frac{\eta}{s} \leq 75\,. \tag{4.264}$$

As shown, these constraints hold in any dimension, for arbitrary chemical potential and for any value of the higher-derivative couplings that satisfy the physical conditions discussed in Section 4.4. Thus, whenever $a_2 \leq 0$, the ratio η/s cannot depart a lot from the Einstein-Maxwell prediction $1/(4\pi)$. This is illustrated in Figure 4.6, where we show the profile of η/s as a function of $\hat{\mu}$ for several values of the parameters compatible with all the physical constraints.

4.6.0.2 Case 2: $a_2 > 0$

The situation becomes very different when one considers $a_2 > 0$, this is, $(2d-1)\alpha_1 + \alpha_2 < 0$. This means that both corrections to η/s are negative, so this quantity is always smaller than $1/(4\pi)$. On the other hand, the a_2 contribution to η/s is now a decreasing function of $\hat{\mu}$, but the GB contribution is still growing, so the overall character of η/s will depend on the particular case. By expanding Eq. (4.253) near $p = 0$, one can see that this becomes a decreasing function of the chemical potential, at least for small $\hat{\mu}$, if

$$(2d-1)\alpha_1 + \alpha_2 + \frac{\lambda}{d-1} < 0\,. \tag{4.265}$$

Since the GB correction is already able to lower the KSS bound, whenever the inequality above is satisfied we may obtain an even lower bound by turning on the chemical potential. In order to answer how much the value of η/s can be diminished we need to take into account all physical constraints on the higher-derivative couplings. However, a general analysis would be more complicated than before, since we do not know anymore for which value of $\hat{\mu}$ the ratio η/s reaches its minimum.

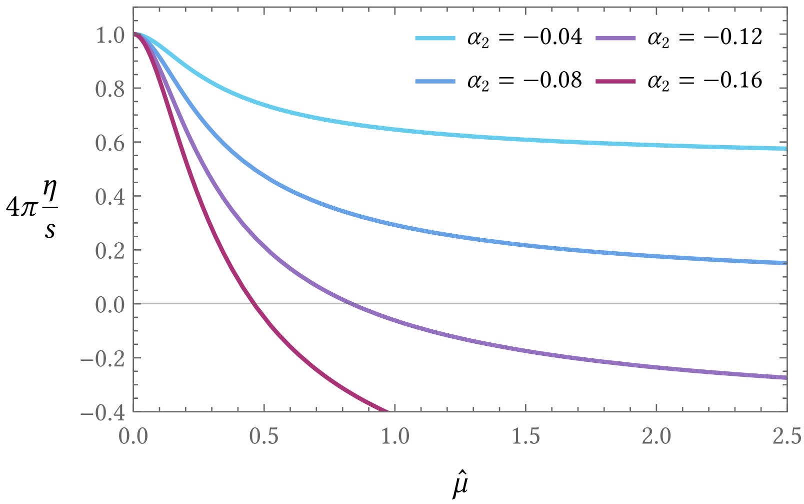


FIGURE 4.7: Ratio of the shear viscosity over the entropy density as a function of the chemical potential, with $a_2 > 0$. For this plot we chose $d = 4$ dimensions and the values of the couplings $\lambda = \alpha_1 = \beta = 0$. Even though these respect all the physical constraints that we found, it is possible to obtain $\eta/s = 0$ for a sufficiently large chemical potential.

In spite of this, it is enough to consider a simple example to show that we can lower the value of η/s, and therefore also that of η, all the way down to zero and below. Let us take $\alpha_1 = \lambda = \beta = 0$, so the only active coupling is α_2. Due to the WGC bound (4.184), this parameter has to be negative, $\alpha_2 \leq 0$. In that case the shear viscosity to entropy ratio reads simply

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 + \frac{4p^2}{d-2}\alpha_2\right], \tag{4.266}$$

and we show it as a function of $\hat{\mu}$ in Figure 4.7. The parameter $|p|$ ranges from 0 to $p_{\text{ext}} = \sqrt{d(d-1)/2}$ when $\hat{\mu}$ goes from 0 to infinity, and therefore the minimum value of η/s is reached at extremality,

$$\left.\frac{\eta}{s}\right|_{\text{ext}} = \frac{1}{4\pi}\left[1 + \frac{2d(d-1)}{d-2}\alpha_2\right]. \tag{4.267}$$

So now we have to look at the bounds on α_2, given by Eqs. (4.147) and (4.148), besides Eq. (4.184) which we are already taking into account. Eq. (4.147) does not impose any additional constraint, since it is always satisfied for $\alpha_2 \leq 0$, while Eq. (4.148) sets a lower bound only for $d = 3,\ 4$ and 5, reading

$$\alpha_2 \geq -\frac{d-2}{6d-d^2-4}, \quad \text{for} \quad d = 3,\ 4,\ 5. \tag{4.268}$$

In higher dimensions there is no limit on how negative α_2 can be, so η/s can certainly be taken to zero. Focusing on these lower dimensions, which are usually the most relevant, we see that the minimum value of η/s in this example is

$$\min\left[\frac{\eta}{s}\right] = \frac{1}{4\pi}\left[1 - \frac{2d(d-1)}{6d-d^2-4}\right], \tag{4.269}$$

which is in fact negative for $d = 3$, 4 and 5. A negative η is meaningless, and it indicates that the plasma is unstable, so we can define a critical value of $\hat{\mu}$ at which the viscosity vanishes and above which it makes no sense to talk about hydrodynamics. Taking into account Eq. (4.219), we see that this is

$$\hat{\mu}_{\text{crit}} = -\frac{(d-1)\sqrt{-(d-2)\alpha_2}}{(d-2)\big(d-2+2d(d-1)\alpha_2\big)}\,, \tag{4.270}$$

and its minimum value is precisely reached for the smallest α_2 in Eq. (4.268),

$$\hat{\mu}_{\text{crit}}^{\text{min}} = \frac{(d-1)\sqrt{6d-d^2-4}}{(d-2)^2(3d-2)}\,, \quad \text{for} \quad d = 3,\ 4,\ 5. \tag{4.271}$$

This result can probably be lowered by considering other couplings besides α_2, the only non-zero one in our simple example. In any case, this result means that the viscosity cannot vanish as long as $\hat{\mu} < \hat{\mu}_{\text{crit}}^{\text{min}}$ for any choice of parameters satisfying the physical constraints. For $d \geq 6$ we have $\hat{\mu}_{\text{crit}}^{\text{min}} = 0$ as α_2 is in principle allowed to be arbitrarily negative, so it is possible to achieve a vanishing η with an arbitrarily small chemical potential.

These results found for $a_2 > 0$ are certainly surprinsing, and possibly pathological. Therefore, it would be interesting to investigate whether other types of constraints, such as those coming from plasma instabilities [1, 268, 324] or causality deep into the bulk [105], could be used to rule out values of η/s very close to zero.

4.7 Discussion

In this Chapter of the thesis we have extended the construction of Electromagnetic Quasitopological gravity theories, originally proposed in $D = 4$ by [23], to general dimensions. These are models containing a $(d-2)$-form field H non-minimally coupled to gravity, which can be dualized into theories with vector fields, thus providing an appealing setup for studies of holography with a chemical potential. The interaction between this generalized Maxwell field and gravity is encoded in higher-derivative corrections to the gravity action, made of contractions of H and the curvature tensors, which are chosen in such a way that single function black hole solutions of the form (4.18), with H given by Eq. (4.21) and $N(r) =$ constant, are allowed.

The non-minimal couplings in these theories affect the central charge of the 2-point function $\langle JJ \rangle$, and give rise to a non-vanishing parameter a_2 (see Eq. (4.129)) that controls the angular distribution of the energy 1-point function according to Eq. (4.106). This in turn means that the boundary theory has a more general $\langle TJJ \rangle$ correlator, and in the end implies that the EQG models allow us to probe holographic CFTs beyond the universality class dual to Einstein-Maxwell theory. In addition, the special properties of these gravity theories allow us to carry out a fully analytic and exact study of many of their holographic aspects, so we do not need to restrict to the perturbative regime.

Most of the analysis presented here is focused on the four-derivative theory whose action is given in Eq. (4.33), and which contains couplings of the form RH^2 and H^4. However, in Section 4.1.4 we also provide expressions for EQG theories at any higher order in the curvature and the $(d-2)$-field H, which will be considered in the following Chapter.

One of the main questions we tried to answer is that of how the physics of the dual CFT can change while satisfying physically reasonable conditions. Thus, in Section 4.4 we have constrained the couplings of our bulk theory by demanding that the boundary CFT respect unitarity, which in turn means that the central charges C_T and C_J, as well as the energy fluxes $\langle\mathcal{E}(\vec{n})\rangle_J$ and $\langle\mathcal{E}(\vec{n})\rangle_T$ (see respectively Eqs. (4.106) and (4.142)), have to remain positive. We also studied the constraints coming from demanding causality in the bulk on the background of a planar neutral black hole. In the case of gravitational fluctuations, it is known that these causality constraints imply the positivity of the energy flux $\langle\mathcal{E}(\vec{n})\rangle_T$ [102, 104, 106, 291], and here we have shown that demanding that electromagnetic waves do not propagate faster than light is equivalent to constraining $\langle\mathcal{E}(\vec{n})\rangle_J$ to be positive. These causality bounds follow from looking at the phase velocity of waves close to the AdS boundary, and while we have not observed additional constraints when extending these conditions away from the boundary, a more thorough search of causality bounds inside the bulk as in [105] would be needed.

Besides these, in our analysis we include constraints derived from the weak gravity conjecture. As proposed by [271] and recently explored by [273] in the case of AdS, the so-called mild form of the WGC demands that the corrections to the entropy of thermally stable black holes be positive in the microcanonical ensemble. In particular, this implies that the charge-to-mass ratio of extremal black holes should be corrected positively [309], which is the most familiar form of the WGC for asymptotically flat black holes [272, 308]. However, demanding the entropy corrections to be positive is a more general condition than that, and is amenable to the AdS case. When applied to spherical and planar black holes we obtain constraints on the couplings, which become powerful when combined with unitarity/causality bounds. In fact, taking all these into account we find that the couplings α_1, α_2 and λ of the theory (4.33) can only lie in a very small compact set of $\mathbb{R}^3$ in $d = 3$, 4 and 5. The only parameter that can take arbitrarily large values is β, which is simply required to be positive. However, we suspect that additional causality or unitarity conditions, involving higher order correlators, should provide an upper limit for this coupling.

When the positivity-of-entropy bounds are considered instead for hyperbolic black holes we find something quite remarkable: some bounds become incompatible with those coming from spherical black holes. For example, this would require that the Gauss-Bonnet coupling is vanishing, $\lambda = 0$, which seems an unreasonably strong constraint, since a positive GB coupling (compatible with the WGC bounds imposed by spherical black holes) is explicitly realized in string theory effective actions [311, 312]. This calls into question the validity of the WGC bounds for hyperbolic black holes, so we decided to trust only those imposed by the spherical case. As we will see in the next Chapter, this leads also to reasonable physics even with hyperbolic black holes involved, in particular when computing Rényi entropies. In any case, this discussion would require further investigations.

Next, in Section 4.5 we studied the thermodynamics of charged plasmas for these holographic models. Knowing that a single phase exists for any value of the temperature and chemical potential for Einstein-Maxwell, we focused on the question of whether new phases could appear in our higher-derivative theory (4.33). We have seen that several branches of solutions could appear, and for large enough couplings we even find zeroth-order phase transitions from the usual Einstein-Maxwell-like branch to a new exotic phase.

However, this behavior always comes accompanied by the quite unphysical phenomenon of not having a large μ regime, this is, no black hole solutions exist if μ is too large. It is thus natural to ask ourselves whether this scenario can be ruled out by the physical constraints for the couplings obtained before. Since the coupling β is poorly constrained by our analysis, we focused first on the case $\beta = 0$. In this situation, we were able to show that the exotic absence-of-solutions behavior (as well as the existence of phase transitions) is ruled out in $d = 3$ and 4 dimensions, due in particular to the WGC bounds, but not in $d \geq 5$. When $\beta \neq 0$ this phase transition is not ruled out by our constraints, but as we argued this coupling is not properly constrained. In general, we tend to think that these exotic phase transitions are unphysical, and it would be interesting to investigate whether additional consistency requirements could rule all of them out.

Finally, in Section 4.6 we studied the shear viscosity of these plasmas, focusing on the case where a phase exists for arbitrary values of the chemical potential. Our result for the shear viscosity to entropy density ratio, η/s, provides a non-perturbative and d-dimensional generalization of the computation of [265]. This ratio is in general a function of the chemical potential, as given by Eq. (4.253), and the departure from the value $1/(4\pi)$ is controlled by two terms that are proportional, respectively, to the GB coupling λ and to the parameter a_2.

Taking into account the unitarity and WGC bounds, we find that the behavior is quite different depending on the sign of a_2. A negative value $a_2 \leq 0$ leads to very reasonable results: the shear viscosity to entropy ratio is always a growing function of μ, and it has absolute minimum and maximum values given by Eq. (4.264). It is worth mentioning that QCD belongs to this class, $a_2^{\rm QCD} < 0$, so it would be interesting to study if the quark-gluon plasma in QCD shares any of the qualitative properties observed in our holographic models. On the other hand, we have seen that $a_2 > 0$ can lead to $\eta = 0$ for large enough values of the chemical potential, without violating any of the available physical conditions, and in fact we were able to achieve this in a simple model with only $\alpha_2 \neq 0$. However, it would be interesting to understand if other mechanisms could prevent the vanishing of the shear viscosity. A typical argument to rule out large corrections to η/s in holographic higher-order gravities is that of [238]. However, this is based on corrections to the graviton 3-point function, and it is clear that the terms with α_2 in the action (4.33) cannot modify this quantity, so we do not have a concrete argument by which this coupling should be perturbatively small. Still, it would be convenient to investigate other constraints that might avoid reaching too small values of η, such as causality on charged backgrounds in the bulk interior, or the existence of plasma instabilities [194, 268, 324]. Otherwise, the example that we found could indicate that in certain CFTs it would be possible to reach an arbitrarily low viscosity by turning on a chemical potential.

Besides those points already mentioned above, there are some holographic aspects and applications of these theories that we did not address here and could be worth pursuing. In particular, this includes studying the thermodynamic phase space of CFTs in a sphere or carrying out a more general hydrodynamic analysis, including for instance the study of conductivities. Of course, it would be natural to consider these EQG corrections in the holographic superconductor of Chapter 3. Indeed, the basic ingredients of that model are gravity, a Maxwell field and a charged scalar field, as given by the action (3.2), so one would expect the dynamics of the system to change if one included non-minimal couplings as proposed in this Chapter.

Let us close this discussion by commenting on some new results that appeared based on the theories proposed here and in [23]. In Ref. [325], the construction presented here is generalized by allowing the function $f(r)$ that characterizes single-function black hole solutions to be determined by a second order ODE, rather than a strictly algebraic equation. This leads to finding theories that belong to the family of "Electromagnetic Generalized Quasitopological gravities," particularly in 3 bulk dimensions and at any order in the curvature. Also, Ref. [326] proves some universal properties of charge transport for generic CFTs described by 4-dimensional theories of the type treated here, but that satisfy the property of self-duality.

In the upcoming Chapter, we extend the holographic study of EQGs performed here by computing Rényi and entanglement entropies. This will serve to characterize further these theories, and we will be able to derive one universal relation for the entanglement entropy in the presence of a chemical potential.

5

Charged Rényi and entanglement entropies

Entanglement entropy [327] and Rényi entropies [328, 329], as well as their holographic counterparts [174, 175, 330], constitute a very useful way to probe the amount of entanglement between a region and its complementary in a quantum field theory [331, 332]. These quantities are able to capture interesting universal information from the vacuum state of a CFT, such as the Virasoro central charge c for two-dimensional theories [327, 333], the Euclidean partition function on the sphere in odd dimensions [334, 335], the coefficients of the trace anomaly in even dimensions [336–339], the stress-energy tensor 2-point function charge C_T [199, 201, 340, 341], and the thermal entropy coefficient C_S [119, 342, 343], among others [344–346]. It has also been suggested that the full CFT data might be accessible from a long-distance expansion of the mutual N-partite information [347–353].

For a bipartition of the Hilbert space of a quantum system in two subspaces A and B, the Rényi entropies are defined as

$$S_n(A) = \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n \,, \tag{5.1}$$

where $\rho_A \equiv \operatorname{Tr}_B \rho$ is the reduced density matrix of the subsystem A, obtained by taking the partial trace over the complementary B of the total density matrix ρ. For a QFT, we are interested in the case in which the subsystems A and B correspond to two spatial regions, at a fixed time and separated by an entangling surface Σ.

The Rényi index n is usually considered an integer, which allows one to compute these entropies by using the replica trick [327]. However, if one is able to continue n to an arbitrary real number, then it is possible to recover the entanglement entropy as the limit $n \to 1$,

$$S_{\text{EE}}(A) = -\operatorname{Tr}\left[\rho_A \log \rho_A\right] = \lim_{n\to 1} S_n(A) \,. \tag{5.2}$$

In this Chapter we will study these quantities for quantum field theories charged under a global symmetry, which as argued can be described using the Electromagnetic Quasitopological gravities studied in Chapter 4, by means of the holographic duality. However, the usual definition for the Rényi entropy given in Eq. (5.1) is not enough when a chemical potential is involved, so it needs to be extended. The appropriate generalization is proposed in [261], and it reads

$$S_n(\mu) = \frac{1}{1-n} \log \mathrm{Tr}\left[\bar{\rho}_A(\mu)\right]^n \,, \qquad \bar{\rho}_A(\mu) = \frac{\rho_A e^{\mu Q_A}}{\mathrm{Tr}\left[\rho_A e^{\mu Q_A}\right]} \,, \tag{5.3}$$

where Q_A is the charge conjugate to μ that is enclosed in the region A. It is clear from this definition that $S_n(\mu)$ reduces to S_n given by Eq. (5.1) in the limit $\mu \to 0$.

A very important feature of $S_n(\mu)$ is that, for spherical entangling surfaces Σ, it admits a generalization of the conformal map of [330, 334], which provides a way to evaluate this quantity from the thermal entropy in the hyperbolic cylinder [261]. This allows one to perform explicit holographic calculations, as will be explained below, and it is a fundamental point in our analysis. Some additional studies of charged Rényi entropies and closely related notions can be found in [354–362].

Regarding the work presented here, in Section 5.1 we continue the study of the four-derivative EQG given by Eq. (4.33), by computing the charged Rényi entropies $S_n(\mu)$ and the generalized twist operators. We will see that, as long as the physical constraints considered before are met, a small chemical potential always increases the amount of entanglement. Also, the Rényi entropies satisfy a set of standard inequalities as a function of the index n, as long as the WGC bounds are also respected. Finally, we will compute the scaling dimension of the generalized twist operators defined in [261], and check that several relations to 2- and 3-point functions shown in that reference also hold for our theory.

Afterwards, in Section 5.2, building upon the results of the four-derivative EQG studied before, we identify a relation between the expansion of the entanglement entropy for small chemical potential and the coefficients C_J and a_2 of the theory. We conjecture that this might be a universal identity for CFTs with a spherical entangling region, and show that it holds true given the known relations for the generalized twist operators. This claim will be supported by an explicit computation of these quantities for the arbitrary-order EQGs introduced in Section 4.1.4.

In this Chapter we include contents from two publications, [26] and [27], whose detailed information can be found in pages 212 and 213 of this thesis, respectively.

5.1 Rényi entropies and twist operators in the four-derivative EQG

The goal of this Section is to compute the charged Rényi entropies (5.3) for the four-derivative Electromagnetic Quasitopological gravity theories studied in the previous Chapter, whose action is given by Eq. (4.33). After that we will also analyze a couple of related quantities: the scaling dimension and the magnetic response of generalized twist operators [261], which will be introduced later on.

In the case of interest the quantum field theory is defined in flat space, and the entangling surface Σ is a sphere of radius R, namely $\Sigma = \mathbb{S}^{d-2}(R)$. For a CFT one can prove, by using the Casini-Huerta-Myers map [334], that these Rényi entropies are related to the thermal entropy of the same theory placed on a hyperbolic cylinder, $\mathbb{S}^1 \times \mathbb{H}^{d-1}(R)$. The precise relation reads [261, 330]

$$S_n(\mu) = \frac{n}{n-1} \frac{1}{T_0} \int_{T_0/n}^{T_0} S_{\mathrm{thermal}}(T,\, \mu)\, dT \,, \tag{5.4}$$

where

$$T_0 = \frac{1}{2\pi R}\,. \tag{5.5}$$

While this is formulated in the frame of the CFT, it is also clear how to compute the Rényi entropy holographically from it. Indeed, the thermal entropy of a holographic CFT on $\mathbb{S}^1 \times \mathbb{H}^{d-1}(R)$ is nothing but the Wald's entropy of a black hole with a hyperbolic horizon in the dual gravity theory.

5.1.1 Rényi entropies

As argued above, in order to compute the charged Rényi entropies for the holographic CFTs dual to (4.33), we have to consider charged black hole solutions with hyperbolic horizon topology. For our theory, these take the form

$$ds^2 = -N_{-1}^2 f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Xi^2_{(d-1)}\,, \tag{5.6}$$

where N_{-1} is a constant, $d\Xi^2_{(d-1)}$ is the line element of the hyperbolic space of unit radius and $f(r)$ is given by Eq. (4.59) with $k = -1$. This function behaves asymptotically as $f(r) \sim r^2 f_\infty/L^2$, so we set the constant N_{-1} to

$$N_{-1} = \frac{L}{\sqrt{f_\infty}R}\,. \tag{5.7}$$

This is a rescaling of the time coordinate, which makes the boundary metric conformal to

$$ds^2_{\text{bdry}} = -dt^2 + R^2 d\Xi^2_{(d-1)}\,, \tag{5.8}$$

whose spatial slices are hyperbolic spaces of radius R, as intended. The Rényi entropies across a spherical entangling region are then computed through the integral (5.4), where S_{thermal} is the Wald entropy of the black hole, given by Eq. (4.73). Notice that it is important that S_{thermal} is considered as a function of T and μ, so that the integration is carried out at constant μ. Although this integration seems tricky at first sight, it can be greatly simplified by taking into account the first law of thermodynamics (4.204), which implies

$$S = -\frac{\partial\Omega(T,\,\mu)}{\partial T}\,. \tag{5.9}$$

Therefore, plugging this in Eq. (5.4) we obtain

$$S_n(\mu) = \frac{n}{n-1}\frac{1}{T_0}\left[\Omega(T_0/n,\,\mu) - \Omega(T_0,\,\mu)\right]. \tag{5.10}$$

The form the grand canonical potential Ω in terms of the horizon radius and the charge was given before in Eq. (4.205). Setting $k = -1$ in that expression, introducing the parameters x and p defined by

$$x = \frac{r_+}{L}\,, \qquad Q = p x^{d-1} L^{d-2}\,, \tag{5.11}$$

and replacing $T = T_0/n = 1/(2\pi R n)$, the expression for $\Omega(T_0/n,\ \mu)$ becomes

$$\Omega = \frac{L^{d-1}V_{-1,d-1}}{16\pi G_{\rm N}\sqrt{f_\infty}R}\left[(d-1)x^d - \frac{2p^2x^d}{d-2} - x^{d-2}\left(d-1+\frac{2x\sqrt{f_\infty}}{n}\right) + \frac{3\beta p^4 x^d}{3d-4}\right.$$
$$\left. + 2\alpha_1 p^2 x^{d-2}\left(3(d-1)+\frac{2x\sqrt{f_\infty}}{n}\right) + 2\alpha_2 p^2 x^{d-2} + (d-1)\lambda x^{d-4}\left(1+\frac{4x\sqrt{f_\infty}}{n(d-3)}\right)\right]. \tag{5.12}$$

However, in order to replace this in Eq. (5.10) it needs to be written in terms of n and μ, so we have to find the relations $x = x(n,\ \mu)$ and $p = p(n,\ \mu)$. For that, it is convenient to present the expressions of n and μ in terms of x and p, which follow from Eqs. (4.70) and (4.200) after setting $k = -1$,

$$\frac{1}{n} = \frac{1}{2x\sqrt{f_\infty}(1-2\alpha_1 p^2 - 2\lambda x^{-2})}\left[-(d-2) + dx^2 + (d-4)\lambda x^{-2}\right. \tag{5.13}$$
$$\left. - \frac{2p^2}{d-1}\left(x^2 - 3d(d-1)\alpha_1 - d\alpha_2\right) + \frac{\beta x^2 p^4}{d-1}\right],$$

$$\mu = \frac{Lp}{\ell_*\sqrt{f_\infty}R}\left[\frac{x}{d-2} - \frac{\alpha_1}{x}\left(3(d-1)+\frac{2x\sqrt{f_\infty}}{n}\right) - \frac{\alpha_2}{x} - \frac{xp^2\beta}{3d-4}\right]. \tag{5.14}$$

In practice, it does not seem possible to invert these equations analytically to obtain explicit expressions for $x(n,\ \mu)$ and $p(n,\ \mu)$. Therefore, to circumvent this in the next Sections we focus on two limiting regimes, namely, small μ and $\mu \to \infty$.

Before getting on to the actual computations, let us introduce the notation

$$\bar{\mu} = \frac{\ell_* R\sqrt{f_\infty}}{L}\mu\,, \tag{5.15}$$

as this combination will appear repeatedly in our expressions. Notice also that this is a dimensionless quantity.

5.1.1.1 *Small μ*

Here we consider the case in which $\bar{\mu} \ll 1$, so that it is enough to carry out the inversion procedure of Eqs. (5.13) and (5.14) perturbatively in $\bar{\mu}$. Furthermore, as an attempt to make explicit computations and capture the effects produced by the non-minimal couplings we are going to set $\lambda = 0$, or equivalently $f_\infty = 1$. This makes sense, since the effect of the GB coupling on (uncharged) Rényi entropies is already known [330] — see also [119, 363–365] for other studies of holographic RE in higher-order gravities.

We therefore expand $x(n,\ \mu)$ and $p(n,\ \mu)$ in this regime as

$$x(n,\ \mu) = \hat{x}_n + \delta\hat{x}_n\bar{\mu}^2 + \mathcal{O}\left(\bar{\mu}^4\right)\,, \qquad p(n,\ \mu) = \delta p_n\bar{\mu} + \mathcal{O}\left(\bar{\mu}^3\right)\,. \tag{5.16}$$

By plugging these into Eqs. (5.13) and (5.14) we find the values of the coefficients, which are

$$\hat{x}_n = \frac{n^{-1} + \sqrt{n^{-2} + d(d-2)}}{d}\,, \tag{5.17}$$

$$\delta\hat{x}_n = -\frac{2(d-2)^2\hat{x}_n^3\left[2(d^2-1)\alpha_1+\hat{x}_n^2\big(d(d-1)\alpha_1-1\big)+d\alpha_2\right]}{(d-1)\big(d(\hat{x}_n^2+1)-2\big)\left[\hat{x}_n^2\big(d(d-2)\alpha_1-1\big)+(d-2)\big((2d-1)\alpha_1+\alpha_2\big)\right]^2}\,, \tag{5.18}$$

$$\delta p_n = \frac{(d-2)\hat{x}_n}{\alpha_{\text{eff}}^{\text{EQG}}-(\hat{x}_n^2-1)\big(d(d-2)\alpha_1-1\big)}\,. \tag{5.19}$$

We recall that $\alpha_{\text{eff}}^{\text{EQG}}$, given in Eq. (4.100), is the combination that appears in the denominator of the central charge C_J in Eq. (4.98). Taking this perturbative expansion into account, the grand canonical potential for given values of n and μ takes the form

$$\Omega_n(\mu) = -\frac{L^{d-1}V_{-1,d-1}}{16\pi G_{\text{N}}R}\left[\hat{x}_n^{d-2}(\hat{x}_n^2+1)+\frac{2(d-2)\hat{x}_n^d}{\alpha_{\text{eff}}^{\text{EQG}}-(\hat{x}_n^2-1)\big(d(d-2)\alpha_1-1\big)}\bar{\mu}^2\right]+\mathcal{O}\left(\bar{\mu}^4\right)\,. \tag{5.20}$$

Now, noting that $\hat{x}_1=1$, we can take the limit $n\to 1$ of this quantity to obtain

$$\Omega_n(\mu) = -\frac{L^{d-1}V_{-1,d-1}}{8\pi G_{\text{N}}R}\left[1+\frac{d-2}{\alpha_{\text{eff}}^{\text{EQG}}}\bar{\mu}^2\right]+\mathcal{O}\left(\bar{\mu}^4\right)\,, \tag{5.21}$$

and replacing this in Eq. (5.10) we find the form of the n-th Rényi entropy for small μ,

$$S_n = \frac{nL^{d-1}V_{-1,d-1}}{4(n-1)G_{\text{N}}}\left[\frac{2-\hat{x}_n^{d-2}(\hat{x}_n^2+1)}{2}+\frac{d-2}{\alpha_{\text{eff}}^{\text{EQG}}}\left(1-\frac{\hat{x}_n^d}{1-(\hat{x}_n^2-1)\big(d(d-2)\alpha_1-1\big)/\alpha_{\text{eff}}^{\text{EQG}}}\right)\bar{\mu}^2\right] + \mathcal{O}\left(\bar{\mu}^4\right)\,. \tag{5.22}$$

Let us remark at this point that the volume $V_{-1,d-1}$ is a divergent function of the ratio between the radius of the entangling surface R and a cutoff δ. In fact, the leading term gives an area law,

$$V_{-1,d-1} = \frac{V_{\mathbb{S}^{d-2}}}{d-2}\frac{R^{d-2}}{\delta^{d-2}}+\ldots\,,\quad \text{where}\quad V_{\mathbb{S}^{d-2}} = \frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]}\,, \tag{5.23}$$

since the quantity $V_{\mathbb{S}^{d-2}}R^{d-2}$ is indeed the area of the spherical entangling surface.

It is interesting to keep only the universal part in this expansion, which can be extracted from the subleading terms and will provide us with the regularized RE [334]. The form of this universal part depends on the dimension: for even d it is logarithmic in the cutoff, while for odd d it is simply a constant. In any case, it reads [334]

$$V_{-1,d-1}^{\text{universal}} = \frac{v_{d-1}}{4\pi}V_{\mathbb{S}^{d-1}}\,,\quad \text{where}\quad v_{d-1} = \begin{cases}(-1)^{(d-2)/2}4\log(R/\delta) & \text{for } d \text{ even,}\\ (-1)^{(d-1)/2}2\pi & \text{for } d \text{ odd,}\end{cases} \tag{5.24}$$

and we will use this regularized volume from now on. It is also useful to introduce the quantity

$$a^* = \frac{L^{d-1}}{8G_{\text{N}}}\frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\,, \tag{5.25}$$

which represents the universal contribution to the regularized EE in theories dual to Einstein gravity. This parameter can also be easily computed for higher-curvature gravities [197, 198], and in general it coincides with the a-type trace-anomaly charge for even d, while in odd dimensions it is proportional to the free energy of the corresponding theory evaluated on $\mathbb{S}^d$ [334]. In terms of this parameter, we can write the holographic Rényi entropies as

$$S_n = \frac{n a^* v_{d-1}}{n-1}\left[\frac{2-\hat{x}_n^{d-2}(\hat{x}_n^2+1)}{2}+\frac{d-2}{\alpha_{\rm eff}^{\rm EQG}}\left(1-\frac{\hat{x}_n^d}{1-(\hat{x}_n^2-1)\big(d(d-2)\alpha_1-1\big)/\alpha_{\rm eff}^{\rm EQG}}\right)\bar{\mu}^2\right] \quad (5.26)$$
$$+\mathcal{O}\left(\bar{\mu}^4\right) .$$

Let us now explore the properties of these entropies, starting with the most relevant one: the entanglement entropy, corresponding to $n \to 1$. This limit yields

$$S_{\rm EE} = \lim_{n\to 1} S_n = a^* v_{d-1}\left[1+\frac{(d-2)^2\big(1-3d(d-1)\alpha_1-d\alpha_2\big)}{(d-1)\left(\alpha_{\rm eff}^{\rm EQG}\right)^2}\bar{\mu}^2\right]+\mathcal{O}\left(\bar{\mu}^4\right) . \quad (5.27)$$

It is interesting to think about the sign of the coefficient multiplying $\bar{\mu}^2$ in this expression, or more precisely the quantity

$$\left.\frac{\partial_{\bar{\mu}}^2 S_{\rm EE}}{S_{\rm EE}}\right|_{\mu=0} = \frac{2(d-2)^2\big(1-3d(d-1)\alpha_1-d\alpha_2\big)}{(d-1)\left(\alpha_{\rm eff}^{\rm EQG}\right)^2} . \quad (5.28)$$

In Einstein-Maxwell theory this is clearly positive, so that the holographic entanglement grows when we turn on a chemical potential, but we should check if this is also true in our family of theories. For doing so we need to take into account the physical constraints on the parameters derived in Section 4.4, and in fact it is enough to consider only the unitarity constraints. Let us first note that the bound (4.147) can be expressed as

$$-\frac{2}{d-2}+2d\alpha_1+\frac{3d-2}{d(d-2)}\alpha_{\rm eff}^{\rm EQG} \geq 0 . \quad (5.29)$$

Then, we have

$$1-3d(d-1)\alpha_1-d\alpha_2 = -\frac{2}{d-2}+2d\alpha_1+\frac{d}{d-2}\alpha_{\rm eff}^{\rm EQG} > -\frac{2}{d-2}+2d\alpha_1+\frac{3d-2}{d(d-2)}\alpha_{\rm eff}^{\rm EQG} \geq 0 , \quad (5.30)$$

where we took into account that $\alpha_{\rm eff}^{\rm EQG} > 0$ and $\frac{3d-2}{d(d-2)} < \frac{d}{d-2}$ for $d \geq 3$. Note that the result we obtain is strictly an inequality, since $\alpha_{\rm eff}^{\rm EQG} = 0$ is not allowed, and thus it implies that, for all unitary CFTs dual to our theories,

$$\left.\frac{\partial_{\bar{\mu}}^2 S_{\rm EE}}{S_{\rm EE}}\right|_{\mu=0} > 0 . \quad (5.31)$$

Given the robustness of this result, it is tempting to conjecture that the entanglement entropy should always grow with the chemical potential for any unitary CFT at zero temperature.[1]

It is possible to extend this result to prove that the coefficient of μ^2 for all Rényi entropies (5.26) with $n \geq 1$ is strictly positive. For that, let us note that for $n > 1$ we have $\sqrt{(d-2)/d} < x_n < 1$, and also the inequality

$$1-\frac{3d-2}{2d}\left(1-\hat{x}_n^2\right) > \hat{x}_n^d , \quad \text{for} \quad d \geq 3,\, n > 1, \quad (5.32)$$

[1] This is in line with the results of [366] for the holographic EE of an infinite rectangular strip in the case $\mu \neq 0$ and $T = 0$.

which can be checked by plugging in the value of $\hat{x}_n$ given in Eq. (5.17). From these we observe that, defining $\xi = d(d-2)\alpha_1 - 1$, for $n > 1$ we have

$$1 - \frac{\hat{x}_n^d}{1-(\hat{x}_n^2-1)\xi/\alpha_{\rm eff}^{\rm EQG}} > 1 - \frac{1-\frac{3d-2}{2d}(1-\hat{x}_n^2)}{1-(\hat{x}_n^2-1)\xi/\alpha_{\rm eff}^{\rm EQG}} = \frac{\frac{3d-2}{2d}(1-\hat{x}_n^2)\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi}{\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi} \geq 0\,. \quad (5.33)$$

The last inequality here follows from the fact that both the numerator and the denominator are positive, as can be checked explicitly,

$$\frac{3d-2}{2d}(1-\hat{x}_n^2)\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi = \frac{(1-\hat{x}_n^2)(d-2)}{2}\left(\frac{3d-2}{d(d-2)}\alpha_{\rm eff}^{\rm EQG}+2d\alpha_1-\frac{2}{d-2}\right)\geq 0\,,$$
$$\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi > \frac{3d-2}{2d}(1-\hat{x}_n^2)\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi \geq 0\,, \quad (5.34)$$

where we have used the inequality (5.29) and taken into account that $1 > 1-\hat{x}_n^2 \geq 0$ and $\frac{3d-2}{2d}(1-\hat{x}_n^2) < 1$ for every $d \geq 3$. Therefore, by applying (5.33) in Eq. (5.26) while also taking into account (5.31), it follows that

$$\left.\frac{\partial_{\bar{\mu}}^2 S_n}{S_n}\right|_{\mu=0} > 0\,, \quad \text{for} \quad n \geq 1\,. \quad (5.35)$$

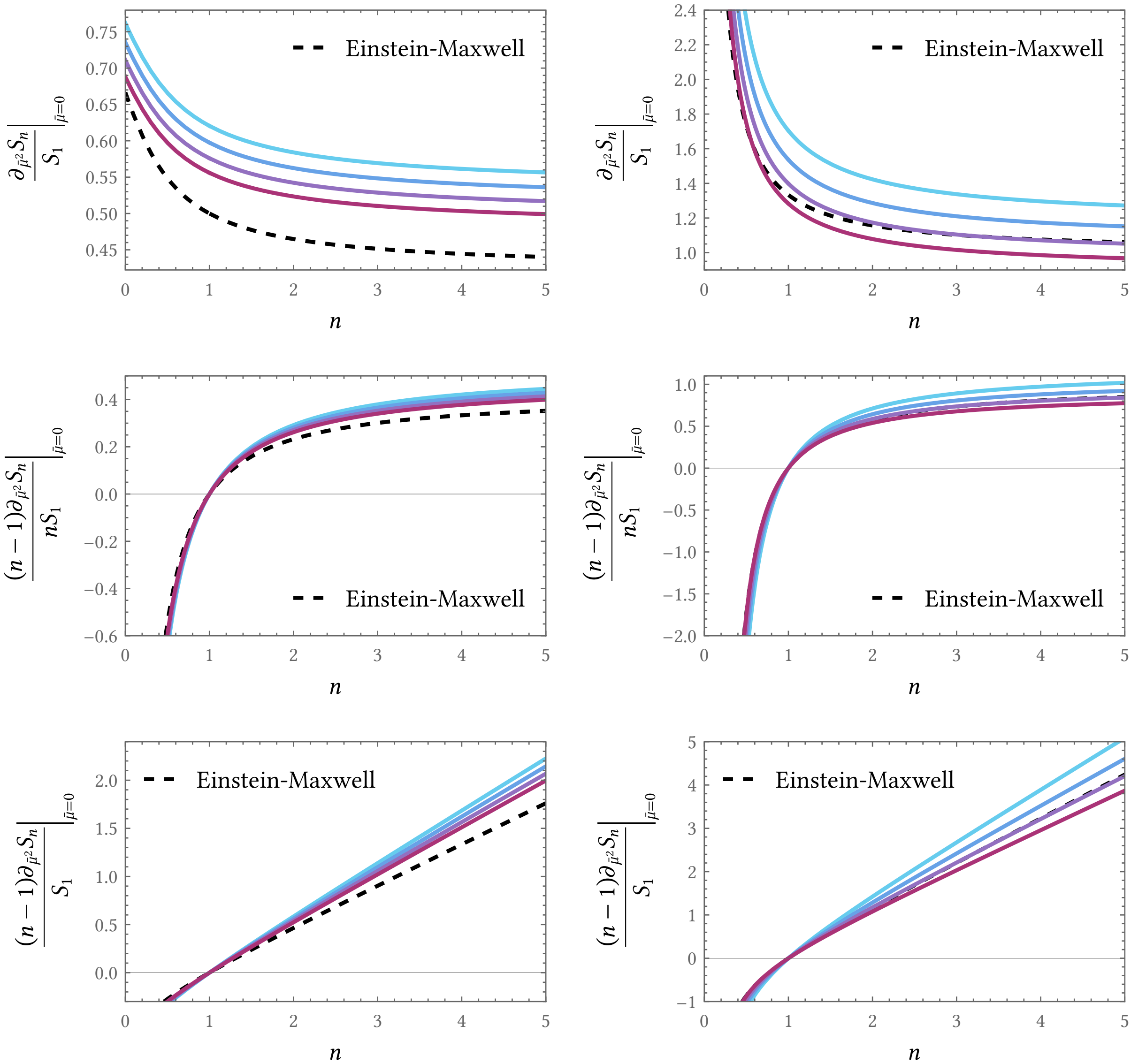
Therefore we have proven that, as long as unitarity is respected, the Rényi entropies with $n \geq 1$ always grow when a chemical potential is turned on.

We can now study the dependence of the Rényi entropies on the index n. It is known that standard REs, this is, at zero chemical potential, must satisfy the inequalities [330]

$$\frac{\partial}{\partial n}S_n \leq 0\,, \qquad \frac{\partial}{\partial n}\left(\frac{n-1}{n}S_n\right) \geq 0\,,$$
$$\frac{\partial}{\partial n}\big((n-1)S_n\big) \geq 0\,, \qquad \frac{\partial^2}{\partial n^2}\big((n-1)S_n\big) \leq 0\,. \quad (5.36)$$

It was shown in [261] that these are also fulfilled by the holographic charged Rényi entropies in Einstein-Maxwell theory. Therefore, it is interesting to check whether they still hold for our holographic higher-derivative theories, assuming that the values of the couplings satisfy the constraints found in Section 4.4. Since the uncharged Rényi entropies for Holographic Einstein gravity, obtained by setting $\bar{\mu} = 0$ in Eq. (5.26), already satisfy such inequalities [330], it is enough to check that the coefficient multiplying $\bar{\mu}$ in Eq. (5.26) fulfills them. This will guarantee that the charged REs also satisfy them, at least in the regime where the $\mathcal{O}(\mu^4)$ terms are subleading. To this aim, we show in Figure 5.1 the profile of $\partial_{\bar{\mu}}^2 S_n/S_1|_{\mu=0}$ for a few values of α_1 and α_2 allowed by the physical constraints, in $d=3$ and 4. We check that all the previous inequalities seem to hold for our EQG theories, at least in this small μ regime.

This result is indeed quite impressive, as all the properties one expects to find for Rényi entropies are satisfied whenever the parameters of the bulk theory are taken to satisfy a minimal set of physical requirements. In fact, we have been able to observe that choosing values of the couplings that do not verify all the constraints obtained might lead to different behaviors, and even divergences, in the REs. Instead, for the physically sensible values of these parameters the chemical potential always increases the amount of entanglement, and the REs have the same qualitative features found for Einstein-Maxwell theory.



(a) $d = 3$. (b) $d = 4$.

Figure 5.1: Coefficient of $\hat{\mu}^2$ in the Rényi entropies as a function of n, represented in such a way that allows to check that the inequalities (5.36) are fulfilled. The curves for $d = 3$ correspond, from blue to red, to $\{\alpha_1, \alpha_2\} = \left\{\frac{1}{24}, -\frac{19}{60}\right\}, \left\{\frac{1}{32}, -\frac{21}{80}\right\}, \left\{\frac{1}{48}, -\frac{5}{24}\right\}, \left\{\frac{1}{96}, -\frac{37}{240}\right\}$. In the same way, the curves for $d = 4$ correspond to $\{\alpha_1, \alpha_2\} = \left\{\frac{1}{24}, -\frac{1}{2}\right\}, \left\{\frac{1}{32}, -\frac{7}{16}\right\}, \left\{\frac{1}{48}, -\frac{3}{8}\right\}, \left\{\frac{1}{96}, -\frac{5}{16}\right\}$. In all cases, $\lambda = \beta = 0$, and the different constraints on the couplings imposed by unitarity and the WGC are fulfilled.

5.1.1.2 Large μ

Let us now study the opposite limit, $\mu \to \infty$. For this it is convenient to revise this limit first in the case of Einstein-Maxwell theory [261], and later generalize the study for our four-derivative theory (4.33).

In this particular case, Eqs. (5.13) and (5.14) reduce to

$$\left.\frac{1}{n}\right|_{\rm EM} = \frac{1}{2x}\left[dx^2-(d-2)-\frac{2p^2x^2}{d-1}\right], \qquad \bar{\mu}|_{\rm EM} = \frac{px}{d-2}. \tag{5.37}$$

These equations can be solved for x, finding

$$x|_{\rm EM} = \frac{1}{nd}\left[1+\sqrt{1+d(d-2)n^2+\frac{2d(d-2)^2n^2}{d-1}\bar{\mu}^2}\right]. \tag{5.38}$$

In the limit that we are interested in now, $\bar{\mu}\to\infty$, we get

$$\begin{aligned} x|_{\rm EM} &= (d-2)\sqrt{\frac{2}{d(d-1)}}\bar{\mu}+\frac{1}{nd}+\mathcal{O}\left(\frac{1}{\bar{\mu}}\right),\\ p|_{\rm EM} &= \sqrt{\frac{d(d-1)}{2}}-\frac{d-1}{2n(d-2)\bar{\mu}}+\mathcal{O}\left(\frac{1}{\bar{\mu}^2}\right). \end{aligned} \tag{5.39}$$

Given this structure for the perturbative expansions of x and p as $\bar{\mu}\to\infty$ in Einstein-Maxwell theory, it is reasonable to expect them to look similar for our four-derivative EQG. Therefore, we try the ansatzes,

$$x = x_1\bar{\mu}+x_0+\mathcal{O}\left(\frac{1}{\bar{\mu}}\right), \qquad p = p_0+\frac{p_{-1}}{\bar{\mu}}+\mathcal{O}\left(\frac{1}{\bar{\mu}^2}\right), \tag{5.40}$$

and after plugging these into Eqs. (5.13) and (5.14) we find

$$\begin{aligned} x_0 &= \frac{\left((3d-8)p_0^2-3d(d-1)(d-2)\right)p_{-1}x_1+2(3d-4)(d-2)\sqrt{f_\infty}p_0^3\alpha_1/n}{dp_0\left(p_0^2+d(d-3)+2\right)},\\ x_1 &= \frac{(d-2)(3d-4)p_0}{d\left(p_0^2+(d-1)(d-2)\right)},\\ p_0 &= \pm\sqrt{\frac{1-\sqrt{1-d(d-1)\beta}}{\beta}}, \qquad p_{-1} = -\frac{(d-1)\sqrt{f_\infty}(1-2\alpha_1p_0^2)p_0}{2nx_1\left(d(d-1)-p_0^2\right)}, \end{aligned} \tag{5.41}$$

where the sign of p_0 matches that of $\bar{\mu}$. Plugging these expansions in Eq. (5.12) and replacing that in Eq. (5.10), the Rényi entropy for large μ turns out to be

$$\lim_{\mu\to\infty} S_n = v_{d-1}\frac{(\ell_* R\mu)^{d-1}\pi^{(d-2)/2}}{8G_{\rm N}\Gamma(d/2)}(1+2\alpha_1p_0^2)\left(\frac{(d-2)(3d-4)p_0\sqrt{f_\infty}}{d\left(p_0^2+(d-1)(d-2)\right)}\right)^{d-1}. \tag{5.42}$$

Similarly to what is found in the Einstein-Maxwell case [261], we observe that Rényi entropies are independent of n as $\mu\to\infty$, and they scale with μ^{d-1}. Also, since the dependence on n becomes trivial for large μ, it is likely that the standard inequalities (5.36), that we showed to hold for small μ, are actually satisfied for every μ.

Regarding the sign of the corrections, we note that it is not definite. Since $\alpha_1>0$ due to the WGC, this coupling always has the effect of increasing the value of the Rényi entropy. But on the other hand, we see that x_1 is a decreasing function of β (which must also be non-negative) for $d = 3$ and 4, and non-monotonic for $d \geq 5$. Hence, the higher-derivative corrections can either decrease or increase the value of the RE, depending on the dimension and the values of these couplings. In spite of this, we notice that this quantity is always positive provided that the WGC is respected. Otherwise, if one were not to impose the WGC bounds, then α_1 could get arbitrarily negative, which would allow the RE to become negative for large enough values of the chemical potential.

5.1.1.3 *An exact example in connection to the WGC*

To finish off the study of the Rényi entropies, let us look at an illustrative example for some particular values of the couplings. We start by setting $d = 4$, since this is the most interesting case. Then, for a given choice of the couplings $\{\lambda, \alpha_1, \alpha_2, \beta\}$ we need to solve Eqs. (5.13) and (5.14) in order to obtain $S_n(\mu)$ according to Eq. (5.10). However, it can happen that these equations have several admissible solutions for the same n and μ. If this is the case, it denotes the existence of multiple phases and we should choose the one with the smallest value of Ω, as that is the dominant one. Therefore, we wish to study the profile of $S_n(\mu)$ when we take into account the physical constraints found in Section 4.4, as well as those in Section 4.5.2.3 that ensure the existence of a large μ limit. For this, we choose two sets of random values of the couplings: one that does satisfy both the WGC and unitarity constraints, and a second one that only satisfies unitarity. Then we study the properties of the Rényi entropy for each set.

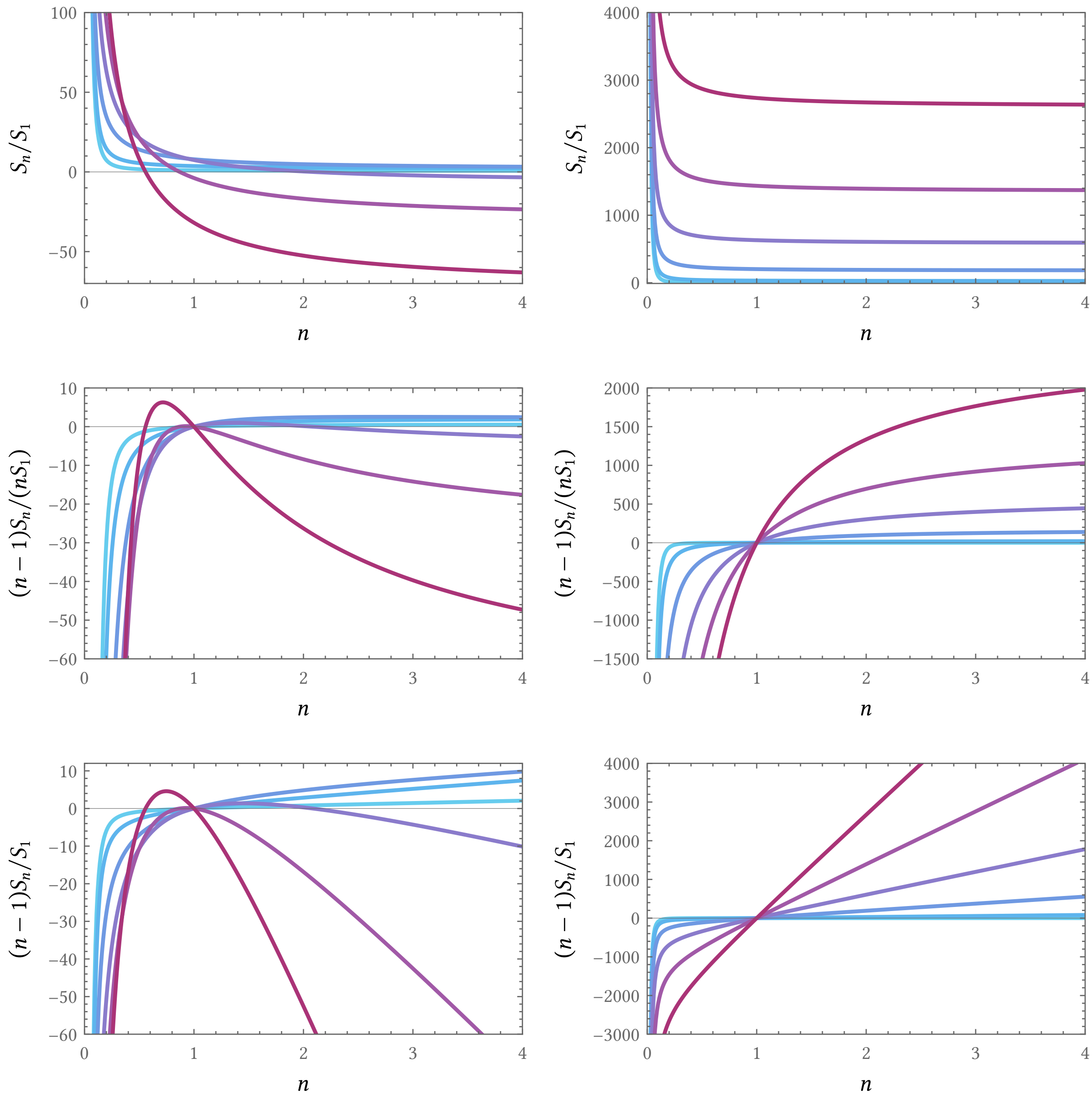
The actual values of the Rényi entropies depend on the particular choice of the couplings, but an illustrative example is shown in Figure 5.2. In the left column we represent S_n/S_1 and related quantities as a function of n, for several values of $\bar{\mu}$ and for a set of couplings that do not satisfy the WGC but do respect unitarity. In the right column we show the same quantities for a different choice of couplings that satisfy all the constraints considered. We find that, while in the latter case S_n is always positive and respects the inequalities (5.36), the RE for the theory that breaks the WGC violates the second and third of them when $\bar{\mu}$ becomes large enough, and can even become negative.

Of course, this is only an example, but looking at randomly generated couplings we have not found any instance of a theory that satisfies the WGC and unitarity and behaves as that in the left column of Figure 5.2. In fact, in all those cases we obtain plots similar to those in the right column of the Figure. Thus, these results are one additional argument pointing towards the fact that the WGC bounds might be key to produce a sensible CFT.

5.1.2 Generalized twist operators

A very interesting notion in the context of the Rényi entropies is that of twist operators, which possess a great deal of information about the CFT.

The Rényi entropy for some spatial region A can be computed using the replica trick [327], which requires evaluating the partition function Z_n of an n-fold cover of Euclidean flat space, with cuts introduced on the region A. The k-th geometry is glued to the $(k+1)$-th copy along these cuts, with the entangling surface Σ being the branch point. From this construction, the



(a) WGC constraints violated. (b) WGC constraints respected.

FIGURE 5.2: Rényi entropies as a function of n for two choices of the couplings that satisfy or not the WGC constraints. The different curves correspond, from blue to red, to $\bar{\mu}/\sqrt{f_\infty} = 0,\ 2,\ 4,\ 6,\ 8$ and 10. In both cases we work in $d = 4$ dimensions, and the values of the couplings for the plot on the left are $\{\alpha_1, \alpha_2, \beta, \lambda\} = \{0.005,\ -0.1,\ 0.8,\ 0.005\}$, while for those on the right $\{\alpha_1, \alpha_2, \beta, \lambda\} = \{0.08,\ 0.05,\ -0.6,\ 0.02\}$. It is clear that the general relations (5.36) are only fulfilled in the case where the couplings fulfill the constraints arising from the WGC.

quantity $\mathrm{Tr}\,\rho_A^n$ needed for the Rényi entropy (5.1) is then computed, for integer values of n, as

$$\mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z_1}\,. \tag{5.43}$$

In order to glue together two consecutive copies of the geometry at the cuts one needs to implement some appropriate boundary conditions, and an alternative way to achieve this is by introducing some $(d-2)$-dimensional operators σ_n that extend over the replicated geometry, known as the “twist operators” [327, 330, 340, 367]. In the presence of a chemical potential, however, one should consider instead the generalized twist operators $\sigma_n(\mu)$ defined in [261], and which are constructed by attaching a Dirac sheet carrying a “magnetic flux” $-in\mu$ to the original operators σ_n. With these, the path integral over the replicated geometry can be replaced by a path integral for the symmetric product of n copies of the theory, with the σ_n inserted, on a single copy of the geometry. Then the trace of ρ_A^n can be obtained as the expectation value of these twist operators, $\mathrm{Tr}\,\rho_A^n = \langle \sigma_n \rangle$, computed in the n-fold symmetric product CFT.

It is possible to define a notion of conformal dimension for these twist operators by performing an insertion of the stress-energy tensor T_{ab} at a small distance y from Σ. In particular, the leading singularity of the correlator $\langle T_{ab}\sigma_n(\mu)\rangle$ takes the form [330, 340]

$$\langle T_{ab}\sigma_n(\mu)\rangle = -\frac{h_n(\mu)}{2\pi}\frac{b_{ab}}{y^d}\,, \tag{5.44}$$

where $h_n(\mu)$ is the conformal dimension of σ_n and b_{ab} is a fixed tensorial structure given, e.g., in [261, 340]. In the case of a spherical entangling surface and with a finite chemical potential, the conformal mapping from flat space to the hyperbolic cylinder allows one to show that [261, 330]

$$h_n(\mu) = \frac{2\pi n}{d-1}R^d\big(\mathcal{E}(T_0,\,\mu=0) - \mathcal{E}(T_0/n,\,\mu)\big)\,, \tag{5.45}$$

where $\mathcal{E}(T,\,\mu)$ is the thermal energy density of the theory placed on $\mathcal{S}^1\times\mathcal{H}^{d-1}(R)$.

Similarly, when a chemical potential is present we also have at hand its associated current J^a, and one can therefore study the correlator $\langle J_a\sigma_n(\mu)\rangle$. In this case, the leading singularity takes the form [261]

$$\langle J_a\sigma_n(\mu)\rangle = \frac{ik_n(\mu)}{2\pi}\frac{\epsilon_{ab}n^b}{y^{d-1}}\,, \tag{5.46}$$

where n^b is a unit vector normal to J_a, and ϵ_{ab} is the volume form of the two-dimensional space orthogonal to the entangling surface. The coefficient $k_n(\mu)$ is the magnetic response of the generalized twist operators, and for a spherical entangling surface it can be computed as

$$k_n(\mu) = 2\pi n R^{d-1}\rho(n,\,\mu)\,, \tag{5.47}$$

where $\rho(n,\,\mu)$ is the charge density of the theory on $\mathbb{S}^1\times\mathbb{H}^{d-1}(R)$, at temperature $T=T_0/n$ and with chemical potential μ.

We will be more interested in the expansions of $h_n(\mu)$ and $k_n(\mu)$ around $n=1$ and $\mu=0$, which can be written as

$$h_n(\mu) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{l!m!}h_{lm}(n-1)^l\mu^m\,, \tag{5.48}$$

$$k_n(\mu) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{l!m!}k_{lm}(n-1)^l\mu^m\,, \tag{5.49}$$

where the coefficients of the expansion are obtained as

$$h_{lm} = (\partial_n)^l(\partial_\mu)^m h_n(\mu)|_{n=1,\,\mu=0}\,,\qquad k_{lm} = (\partial_n)^l(\partial_\mu)^m k_n(\mu)|_{n=1,\,\mu=0}\,. \tag{5.50}$$

As shown in [261] (and in [203, 330, 340] in the case of h_n for $\mu = 0$), these coefficients involve integrated correlators of the form $\langle T \cdots T J \cdots J\rangle$. In particular, the first few coefficients are related to 2- or 3-point functions of T and J, and therefore have a universal form for any CFT. These relations were derived in [261, 330, 340] from first principles, but here we will see that they can also be obtained by using holography with higher-derivative terms.

5.1.2.1 *Conformal dimension of generalized twist operators*

We start by studying the conformal dimension of the generalized twist operators, given by Eq. (5.45). The energy density $\mathcal{E}$ can be computed holographically as the mass of a hyperbolic black hole over the volume of the boundary,

$$\mathcal{E}(T,\,\mu) = \frac{M(T,\,\mu)}{V_{-1,\,d-1}R^{d-1}}\,. \tag{5.51}$$

This can be obtained from Eq. (4.69) by setting $k = -1$, and replacing this together with $M(T_0,\,\mu = 0) = 0$ in Eq. (5.45) we have

$$\begin{aligned} h_n(\mu) = -\frac{nL^{d-1}}{8(d-1)G_{\rm N}\sqrt{f_\infty}}\Bigg[&(d-1)\left(x^d - x^{d-2} + \lambda x^{d-4}\right) \\ &+\frac{2p^2x^d}{d-2}\left(1-\frac{d-2}{x^2}\big(3(d-1)\alpha_1+\alpha_2\big)\right) - \frac{\beta p^4 x^d}{3d-4}\Bigg]\,, \end{aligned} \tag{5.52}$$

where as usual we introduced the variables $x = r_+/L$ and $p = QL/r_+^{d-1}$, which depend on n and μ through Eqs. (5.13) and (5.14). These are solved for $n = 1$ and $\mu = 0$ by $x = 1/\sqrt{f_\infty}$ and $p = 0$, so we perform an expansion around these values to find

$$\begin{aligned} x &= \frac{1}{\sqrt{f_\infty}} - \frac{n-1}{(d-1)\sqrt{f_\infty}} + \left(\frac{\mu \ell_* R}{L}\right)^2 \frac{(d-2)^2 f_\infty^{3/2}\big(1-(3d+2)(d-1)f_\infty\alpha_1 - d f_\infty \alpha_2\big)}{(d-1)(2-f_\infty)\left(\alpha_{\rm eff}^{\rm EQG}\right)^2} + \ldots\,, \\ p &= \left(\frac{\mu\ell_* R}{L}\right)\left[\frac{(d-2)f_\infty}{\alpha_{\rm eff}^{\rm EQG}} + (n-1)\frac{(d-2)f_\infty\big(1+(d-1)(d-2)f_\infty\alpha_1 + (d-2)f_\infty\alpha_2\big)}{(d-1)\left(\alpha_{\rm eff}^{\rm EQG}\right)^2} + \ldots\right] \\ &\quad + \ldots\,, \end{aligned} \tag{5.53}$$

where we only show the terms that will be relevant for our computations. From these expressions it is straightforward to obtain the expansion of h_n in Eq. (5.48) and read off the values of the derivatives. In the first place, we find

$$h_{10} = \frac{1-2\lambda f_\infty}{4(d-1)G_{\rm N}}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-1}\,. \tag{5.54}$$

Comparing this with the value of the central charge C_T for our theory written in Eq. (4.84), it can be written as

$$h_{10} = 2\pi^{d/2+1}\frac{\Gamma(d/2)}{\Gamma(d+2)}C_T\,, \tag{5.55}$$

which is precisely the relation found in [330]. In a similar way, the second derivative of h_n at vanishing μ, that is, h_{20}, is completely determined in terms of C_T and the 3-point function coefficients t_2 and t_4 [203, 340], and those relations have been shown to be identically satisfied for holographic higher-curvature gravities [119, 128, 203].

Let us turn our attention now to the derivatives of h_n with respect to μ. From Eqs. (5.52) and (5.53) we find

$$h_{02} = -\frac{(d-2)\ell_*^2 R^2}{2(d-1)^2 G_{\rm N}} \left(\frac{L}{\sqrt{f_\infty}}\right)^{d-3} \frac{2d-3-(d-2)f_\infty\big((6d-1)(d-1)\alpha_1+(2d-1)\alpha_2\big)}{\left(\alpha_{\rm eff}^{\rm EQG}\right)^2} \,. \tag{5.56}$$

Now, looking at Eqs. (4.98), (4.100) and (4.129), we see that this expression can be written in terms of the central charge C_J and the flux parameter a_2 as

$$h_{02} = -(2\pi R)^2 \frac{C_J \pi^{d/2-1}\Gamma(d/2)}{(d-1)^3\Gamma(d+1)}\big(d(d-1)(2d-3)+a_2(d-2)^2\big)\,. \tag{5.57}$$

Finally, we can write this in terms of the coefficient of $\langle TJJ\rangle$, $\hat{c}$ and $\hat{e}$, using the relations (4.134) and (4.135),

$$h_{02} = -(2\pi R)^2 \frac{4\pi^{d-1}}{\Gamma(d+1)}\left(\frac{2}{d}\hat{c}+\hat{e}\right)\,, \tag{5.58}$$

which is precisely the result in Eq. (2.45) of [261], and which applies to any CFT.[2]

5.1.2.2 *Magnetic response of generalized twist operators*

Let us now take a look at the magnetic response of the same twist operators, $k_n(\mu)$, which can be computed using Eq. (5.47). For this we need the charge density of the boundary theory, which is simply

$$\rho(n,\,\mu) = \frac{\ell_* q}{R^{d-1}V_{-1,\,d-1}}\,, \tag{5.59}$$

with q given by Eq. (4.65). Using this, we get for the magnetic response

$$k_n(\mu) = \frac{n\ell_* Q}{2G_{\rm N}} = \frac{\ell_* L^{d-2}}{2G_{\rm N}} n p x^{d-1}\,. \tag{5.60}$$

Again, we should replace both x and p in terms of n and μ, which can be done in an approximate manner around $n=1$ and $\mu=0$ using the expansions (5.53). This is enough to read off the first derivatives close to this point, and in particular the first derivative with respect to μ ends up being

$$k_{01} = \frac{(d-2)\ell_*^2 R}{2G_{\rm N}\alpha_{\rm eff}^{\rm EQG}}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-3} = 8\pi^{d/2+1}R\frac{\Gamma(d/2+1)}{\Gamma(d+1)}C_J\,, \tag{5.61}$$

where in the second equality we used Eq. (4.98). This can also be written in terms of the coefficients of the $\langle TJJ\rangle$ correlator, $\hat{c}$ and $\hat{e}$, related to C_J through Eq. (4.134), as

$$k_{01} = \frac{16\pi^{d+1}R}{\Gamma(d+1)}\left(\hat{c}+\hat{e}\right)\,. \tag{5.62}$$

[2] Note that we have an additional factor of $(2\pi R)^2$ with respect to the expression in [261], which comes from the fact that they normalize the chemical potential with a factor $1/(2\pi R)$ with respect to our convention.

Again, up to a factor of $2\pi R$ arising from the different normalization conventions, this matches Eq. (2.57) of [261].

In the same manner, we can compute the mixed derivative k_{11}, which reads

$$k_{11} = \frac{(d-2)\ell_*^2 R\left[1+(d-2)\big((d-1)\alpha_1+\alpha_2\big)f_\infty\right]}{4(d-1)G_{\rm N}\left(\alpha_{\rm eff}^{\rm EQG}\right)^2}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-3}. \tag{5.63}$$

This can be written in terms of C_J and a_2, given respectively by Eqs. (4.98) and (4.129), in the form

$$k_{11} = \frac{4R\pi^{d/2+1}\Gamma(d/2)}{(d-1)^2\Gamma(d+1)}C_J\left(d(d-1)-a_2(d-2)^2\right), \tag{5.64}$$

or in terms of the coefficients of $\langle TJJ\rangle$, given for our theory in Eqs. (4.138) and (4.139), as

$$k_{11} = \frac{16\pi^{d+1}R}{d\Gamma(d+1)}\left(2\hat{c}-d(d-3)\hat{e}\right). \tag{5.65}$$

So in this case we also reproduce the result of [261], in particular their Eq. (2.56). Let us remark that the authors of the reference check that these relations hold for holographic Einstein-Maxwell theory, which is a more restricted case, as the dual theory has $a_2 = 0$. In our computation, on the other hand, we were able to show these universal relationships for a theory with a general 3-point function $\langle TJJ\rangle$.

5.2 A universal feature of charged entanglement entropy

In this second part of the Chapter we will conjecture and prove a universal relation for the charged entanglement entropy with a spherical entangling region. More explicitly, we will see that for a general d-dimensional CFT, this quantity is, to the lowest order in μ,

$$\frac{S_{\rm EE}(\mu)}{v_{d-1}} = a^* + \frac{\pi^d C_J}{(d-1)^2\Gamma(d-2)}\left[1+\frac{(d-2)a_2}{d(d-1)}\right](\mu R)^2, \tag{5.66}$$

where v_{d-1} is proportional to the non-divergent part of the volume, as given in Eq. (5.24), and a^* is the value of this entanglement entropy in the uncharged case. Alternatively, this relation can be formulated as a statement involving the first two derivatives of the Rényi entropy $S_n(\mu)$ with respect to μ evaluated at $n=1$ and $\mu=0$.

Notice that all the information regarding the boundary theory in the first correction with respect to the uncharged entanglement entropy is encoded in the constants C_J and a_2. We recall that C_J appears in the correlator of the current associated to the global symmetry, given by Eq. (4.86), and it is the only part of that object that is theory-dependent and not fixed by conformal symmetry. As for a_2, it is the parameter controlling the effect of the non-minimal couplings on the energy flux measured at infinity after the insertion of a current operator $\epsilon_i J^i$, given in general by Eq. (4.106). Furthermore, as discussed in Section 4.3.4, the 3-point function $\langle TJJ\rangle$ for a CFT has a fixed form up to two theory-dependent coefficients, which can indeed be taken to be C_J and a_2. Therefore, these two parameters carry a great deal of information about the particular CFT, and it makes sense that they play a role in certain universal identities.

The relation (5.66) was found in the first place from the results shown in Section 5.1.1.1 for the four-derivative theory (4.33), in particular by looking at Eq. (5.27). While that corresponds to $\lambda = 0$, it is straightforward to repeat the computation allowing for a non-vanishing Gauss-Bonnet coupling. In that case, the entanglement entropy across a spherical entangling region reads

$$\frac{S_{\rm EE}^{\rm EQG}}{v_{d-1}} = a_{\rm GB}^* + \frac{(d-2)^2\pi^{(d-2)/2}\left(1-3d(d-1)f_\infty\alpha_1 - d f_\infty\alpha_2\right)}{8(d-1)\Gamma(d/2)\left(\alpha_{\rm eff}^{\rm EQG}\right)^2}\frac{\tilde{L}^{d-3}\ell_*^2}{G_{\rm N}}(\mu R)^2 + \mathcal{O}\left(\mu^4\right) , \tag{5.67}$$

where $\alpha_{\rm eff}^{\rm EQG}$ is given in terms of α_1 and α_2 by Eq. (4.100). The constant term $a_{\rm GB}^*$ also corresponds to the same entanglement entropy without a chemical potential, but it is now modified by the Gauss-Bonnet terms and it reads [198]

$$a_{\rm GB}^* = \frac{\tilde{L}^{d-1}}{8G_{\rm N}}\frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\left[1-\frac{2(d-1)}{d-3}f_\infty\lambda\right] . \tag{5.68}$$

The leading correction to Eq. (5.67) has a complicated non-polynomial dependence on the gravitational couplings α_1 and α_2. However, this conspires to produce a linear combination of the charges $C_J^{\rm EQG}$ and $C_J^{\rm EQG}\cdot a_2^{\rm EQG}$, as can be checked using Eqs. (4.98) and (4.129), thus obtaining the relation proposed in Eq. (5.66).

5.2.1 Charged EE in EQGs of any order

We will now extend the previous result and show that the conjectured universal relation (5.66) indeed holds for the arbitrary-order family of Electromagnetic Quasitopological theories proposed in Section 4.1.4. For this, we consider the general action

$$\begin{aligned}I_{\rm EQG,\,gen} = \frac{1}{16\pi G_{\rm N}}\int_{\mathcal{M}} d^{d+1}x\sqrt{|g|}\Bigg[R + \frac{d(d-1)}{L^2} - \frac{2}{(d-1)!}H^2 + \frac{\lambda L^2}{(d-2)(d-3)}\chi_4 \\ + \frac{2}{(d-1)!}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty} L^{2(s+m-1)}\left(\alpha_{1,s,m}\mathcal{L}_{d,s,m}^{(a)} + \alpha_{2,s,m}\mathcal{L}_{d,s,m}^{(b)}\right)\Bigg] ,\end{aligned} \tag{5.69}$$

where χ_4 is the Gauss-Bonnet scalar in Eq. (4.28), and $\mathcal{L}_{d,s,m}^{(a)}$ and $\mathcal{L}_{d,s,m}^{(b)}$ are the densities written in Eq. (4.35).

In order to check the relation (5.66) we need to compute separately the parameters a_2 and C_J of the theory, as well as the entanglement entropy across an spherical region. But first we need to discuss the hyperbolic black hole solutions of this general theory.

5.2.1.1 *Hyperbolic black hole solutions*

By construction, the theories (5.69) admit charged black hole solutions with spherical, planar or hyperbolic horizons, described by a single function $f(r)$ which fulfills the equation of motion (4.47). Here we are interested only in solutions with hyperbolic sections, so we restrict ourselves to that case. The line element of such a spacetime can be written as

$$ds^2 = -\frac{L^2}{f_\infty R^2}\left(\frac{r^2}{L^2}h(r)-1\right)dt^2 + \frac{L^2dr^2}{r^2h(r)-L^2} + r^2d\Xi^2_{(d-1)} , \qquad H = Q\omega_{-1,(d-1)} , \tag{5.70}$$

where $d\Xi^2_{(d-1)}$ corresponds to the unit hyperbolic space $\mathbb{H}^{d-1}$, $\omega_{-1,(d-1)}$ is the associated volume form and f_∞ is given in terms of the Gauss-Bonnet coupling by Eq. (4.52). The constant multiplying the term dt^2 has been chosen in such a way that the metric at the boundary $r \to \infty$ becomes conformally equivalent to the hyperbolic cylinder with radius R.

The equation of motion for the function $h(r)$ in the metric can be obtained with the usual reduced Lagrangian approach, or simply performing a change of variables in Eq. (4.47), yielding the algebraic equation

$$\begin{aligned} 0 =& \frac{r^2}{L^2}(1-h) - \frac{16\pi R\sqrt{f_\infty}G_{\rm N}M}{(d-1)LV_{-1,d-1}r^{d-2}} + \frac{2Q^2}{(d-1)(d-2)r^{2(d-2)}} + \frac{\lambda r^2}{L^2}h^2 \\ &+ \sum_{s,m}\frac{(-2)^s Q^{2m}L^{2m}\Gamma(d)^{m-1}}{r^{2m(d-1)}}h^{s-1}\Bigg[-\frac{2s}{d-1}\Big((1-2m)(d-1)+1-ds\Big)\alpha_{1,s,m} + \frac{s\alpha_{2,s,m}}{d-1} \\ &-\left(\left(1-2m-4s+4ms+\frac{2s(ds-1)}{d-1}\right)\alpha_{1,s,m} + \frac{s-1}{d-1}\alpha_{2,s,m}\right)\frac{r^2}{L^2}h\Bigg], \end{aligned} \tag{5.71}$$

where M is an integration constant that should be identified with the mass of the solution, as explained in Section 4.2, and $\sum_{s,m} \equiv \sum_{s=0}^{\infty}\sum_{m=1}^{\infty}$.

Even though this equation is algebraic, it is not possible to find the exact form of $h(r)$ in general. We can, however, study the thermodynamic properties of the spacetime. For that, let us assume that g_{tt} has some zero along the positive real axis, and let us denote the one corresponding to the horizon by $r_+ = \max\{r \in \mathcal{R}^+ \,|\, h(r) = L^2/r^2\}$. Defining $x \equiv r_+/L$ and $p \equiv QL^{2-d}x^{1-d}$ as several times before, and evaluating Eq. (5.71) at $r = r_+$, the mass of the black hole solution can be seen to be

$$\begin{aligned} \frac{16\pi R\sqrt{f_\infty}G_{\rm N}}{L^{d-1}V_{-1,d-1}}M =& (d-1)x^{d-2}(x^2-1) + \frac{2p^2x^d}{d-2} + (d-1)\lambda x^{d-4} \\ &+ \sum_{s,m}\frac{(-2)^s\Gamma(d)^{m-1}p^{2m}}{x^{2s-d}}\Big(-(d-1)(1-2m-2s)\alpha_{1,s,m} + \alpha_{2,s,m}\Big). \end{aligned} \tag{5.72}$$

Similarly, taking into account that the Hawking temperature is given by

$$4\pi R\sqrt{f_\infty}T = \frac{r_+^2}{L}h'(r_+) + \frac{2L}{r_+}, \tag{5.73}$$

we find

$$\begin{aligned} 4\pi R\sqrt{f_\infty}T =& \frac{(d-1)\big(2+d(x^2-1)+(d-4)\lambda x^{-2}\big) - 2p^2x^2}{(d-1)(x-2\lambda x^{-1}) + \sum_{s,m}(-2)^s\Gamma(d)^{m-1}p^{2m}\big(\gamma_{s,m}-(s-1)\beta_{s,m}\big)x^{3-2s}} \\ &- \frac{\sum_{s,m}(-2)^s\Gamma(d)^{m-1}p^{2m}x^{-2(s-1)}\big(2s-2m+d(2m-1)\big)\beta_{s,m}}{(d-1)(x-2\lambda x^{-1}) + \sum_{s,m}(-2)^s\Gamma(d)^{m-1}p^{2m}\big(\gamma_{s,m}-(s-1)\beta_{s,m}\big)x^{3-2s}}, \end{aligned} \tag{5.74}$$

where the constants $\beta_{s,m}$ and $\gamma_{s,m}$ are given by

$$\begin{aligned} \beta_{s,m} &= (d-1)(2s+2m-1)\alpha_{1,s,m} + \alpha_{2,s,m}, \\ \gamma_{s,m} &= \big(d-1-(4d-2)s+2ds^2+2m(d-1)(2s-1)\big)\alpha_{1,s,m} + (s-1)\alpha_{2,s,m}. \end{aligned} \tag{5.75}$$

The computation of the black hole entropy S is carried out as before using the Iyer-Wald formula written in Eq. (4.72). It can be applied to the general theory (5.69) on the black hole metric (5.70) in a somewhat straightforward manner, finding

$$S = \frac{V_{-1,\,d-1} L^{d-1} x^{d-1}}{4G_{\rm N}} \left[1 - \frac{2(d-1)}{(d-3)x^2}\lambda - \sum_{s,m} \frac{(-2)^s \Gamma(d)^{m-1} s p^{2m}}{x^{2(s-1)}} \alpha_{1,\,s,\,m} \right] . \tag{5.76}$$

Interestingly, the entropy does not receive any corrections from the densities $\mathcal{L}^{(b)}_{d,\,s,\,m}$.

As explained in Section 4.3, the chemical potential μ is defined as the asymptotic value of the electrostatic potential A_t, $\mu = \lim_{r\to\infty} A_t/\ell_*$, after demanding that $A_t|_{r=r_+} = 0$. This is the only active component of the dual vector field A for the magnetic configuration (5.70), whose field strength tensor $F = \mathrm{d}A$ is given by

$$F = 4\pi G_{\rm N}(d-1)! \star \frac{\partial \mathcal{L}}{\partial H} . \tag{5.77}$$

However, an easier way to compute this chemical potential is by means of the first law of black hole thermodynamics,

$$dM = TdS + \mu d\mathcal{N} , \quad \text{where} \quad \mathcal{N} = \frac{V_{-1,\,d-1}\ell_* Q}{4\pi G_{\rm N}} . \tag{5.78}$$

Indeed, this equation is equivalent to

$$\frac{V_{-1,\,d-1}\ell_* L^{d-2} x^{d-1}}{4\pi G_{\rm N}} \mu = \frac{\partial M}{\partial p} - T\frac{\partial S}{\partial p} , \tag{5.79}$$

and by plugging in the values of M and T given respectively by Eqs. (5.72) and (5.74) one can obtain the form of the chemical potential for the theories (5.69). It can be checked that this value and the one obtained by means of Eq. (5.77) are actually the same, thus confirming the validity of the first law for our black hole solutions.

5.2.1.2 Parameters C_J and a_2 of the dual CFT

Our ultimate goal is to express the entanglement entropy corresponding to an spherical region in terms of the charges C_J and a_2 of the CFT dual to the theories (5.69), as written in Eq. (5.66). Therefore, we need to compute both the entropy and these parameters in an independent manner. Let us begin with the latter.

If we denote by F the dual field strength of H, then C_J is obtained by working out the effective gauge coupling of F^2 when evaluating the action on a pure AdS background [261], as done in Section 4.3.2 for the four-derivative theory. For this computation (and also for a_2) it is enough to restrict ourselves to terms quadratic in H in the action (5.69), this is, those with $m = 1$. Following the procedure described in Section 4.1.1 we find the relevant part of the dual action to be

$$\tilde{I}_{F^2} = \frac{1}{16\pi G_{\rm N}} \int_{\mathcal{M}} d^{d+1}x \sqrt{|g|} \left[R + \frac{d(d-1)}{L^2} + \frac{\lambda L^2}{(d-2)(d-3)} \mathcal{X}_4 - \left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma} F_{\mu\nu} F^{\rho\sigma} \right] . \tag{5.80}$$

As before, $\tilde{Q}^{-1}$ is defined as the inverse of $\tilde{Q}$,

$$\left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\alpha\beta} \tilde{Q}^{\alpha\beta}{}_{\rho\sigma} = \delta^{[\mu}{}_{[\rho} \delta^{\nu]}{}_{\sigma]} , \tag{5.81}$$

which is given in this case by

$$
\begin{aligned}
\tilde{Q}^{\mu\nu}{}_{\rho\sigma} &\equiv \frac{12}{(d-1)(d-2)} Q^{[\alpha\beta}{}_{\alpha\beta}\delta^{\mu}{}_{\rho}\delta^{\nu]}{}_{\sigma}\,,\\
Q^{\alpha\beta}{}_{\rho\sigma} &= \delta^{\alpha\beta}{}_{\rho\sigma} - \sum_{s=0}^{\infty}\Bigg[\left(sR\left(R^{s-1}\right)^{\alpha\beta}{}_{\rho\sigma} + \kappa_{d,s,1}\left(R^{s}\right)^{\alpha\beta}{}_{\rho\sigma} + 2s(s-1)\left(R^{s-2}\right)^{\mu[\alpha}{}_{\nu[\rho|}R^{\beta]}{}_{\mu}R^{\nu}{}_{|\sigma]}\right)\alpha_{1,s}\\
&\qquad + \frac{1}{2}\left(R^{s-1}\right)^{\mu\nu}{}_{\rho\sigma}\left(2sR^{[\alpha}{}_{\mu}\delta^{\beta]}{}_{\nu} + g_{d,s,1}R^{\alpha\beta}{}_{\mu\nu}\right)\alpha_{2,s}\Bigg]\,.
\end{aligned}
\tag{5.82}
$$

Here we are introducing the shorthand notation for the couplings

$$
\alpha_{1,s} \equiv \alpha_{1,s,1}\,, \qquad \alpha_{2,s} \equiv \alpha_{2,s,1}\,. \tag{5.83}
$$

Finding the inverse $\tilde{Q}^{-1}$ is in general a rather challenging task, but it becomes manageable when we restrict ourselves to backgrounds with enough symmetry. In the case at hand, since the spacetime is pure AdS space with

$$
R^{\mu\nu}{}_{\rho\sigma} = -\frac{2}{\tilde{L}^{2}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,, \qquad \tilde{L} = \frac{L}{\sqrt{f_{\infty}}}\,, \tag{5.84}
$$

we have simply

$$
Q^{\mu\nu}{}_{\rho\sigma} = \tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \alpha_{\text{eff}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,, \qquad \left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma} = \frac{1}{\alpha_{\text{eff}}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,, \tag{5.85}
$$

where the effective coupling constant takes the value

$$
\alpha_{\text{eff}} = 1 + \sum_{s=0}^{\infty}(-2)^{s-1}f_{\infty}^{s}(2-d)\big((d-1+2ds)\alpha_{1,s} + \alpha_{2,s}\big)\,. \tag{5.86}
$$

Consequently, the coefficient of F^2 in Eq. (5.80) turns out to be $1/\alpha_{\text{eff}}$. This implies that the net effect of the higher-derivative terms is the renormalization of the gauge coupling constant, producing in turn the central charge

$$
C_J^{\text{EQG}} = \frac{C_J^{\text{EQG}}}{\alpha_{\text{eff}}}\,, \qquad C_J^{\text{EM}} = \frac{\Gamma(d)}{\Gamma(d/2-1)}\frac{\ell_*^{2}\tilde{L}^{d-3}}{4\pi^{d/2+1}G_{\text{N}}}\,, \tag{5.87}
$$

being C_J^{EM} the Einstein-Maxwell central charge, although with the effective AdS length scale $\tilde{L}$ due to the Gauss-Bonnet coupling.

The computation of a_2, on the other hand, requires the knowledge of the inverse tensor $\tilde{Q}^{-1}$ on a shock-wave background given by the metric (see Section 4.3.3 for a detailed computation)

$$
\begin{aligned}
ds^2 &= \frac{\tilde{L}^2}{u^2}\left[\delta(y^+)\mathcal{W}(y^i,u)(dy^+)^2 - dy^+dy^- + \sum_{j=1}^{d-2}(dy^i)^2 + du^2\right]\,,\\
\mathcal{W}(y^i,u) &= \frac{\mathcal{W}_0 u^d}{\left(u^2 + \sum_{j=1}^{d-2}(y^j - y_0^j)^2\right)^{d-1}}\,, \quad \text{where} \quad y_0^j \in \mathcal{R}\,.
\end{aligned}
\tag{5.88}
$$

This satisfies $R_{\mu\nu} = -d/\tilde{L}^2 g_{\mu\nu}$ and $W_{\mu\nu\rho\sigma}W^{\rho\sigma\alpha\beta} = 0$, being $W_{\mu\nu\rho\sigma}$ the corresponding Weyl tensor. Taking these properties into account one can compute

$$\begin{aligned} \tilde{Q}^{\mu\nu}{}_{\rho\sigma} &= \alpha_{\text{eff}}\delta^{\mu\nu}{}_{\rho\sigma} - \frac{\beta_{\text{eff}}L^2}{(d-1)(d-2)f_\infty}W^{\mu\nu}{}_{\rho\sigma}\,, \\ \left(\tilde{Q}^{-1}\right)^{\mu\nu}{}_{\rho\sigma} &= \frac{1}{\alpha_{\text{eff}}}\delta^{\mu\nu}{}_{\rho\sigma} + \frac{\beta_{\text{eff}}L^2}{(d-1)(d-2)f_\infty\alpha_{\text{eff}}^2}W^{\mu\nu}{}_{\rho\sigma}\,, \end{aligned} \tag{5.89}$$

where β_{eff} is the combination of the couplings given by

$$\beta_{\text{eff}} = \sum_{s=0}^{\infty}(-2f_\infty)^2(d-1)s\big((2ds-1)\alpha_{1,s} + \alpha_{2,s}\big)\,. \tag{5.90}$$

This is formally equivalent to the same tensor evaluated for the four-derivative EQG, given in Eq. (4.121), upon exchange of $2\alpha_3 = -2[2(2d-1)(d-1)\alpha_1 + 2(d-1)\alpha_2] \to \beta_{\text{eff}}/f_\infty$. Therefore, the coefficient a_2 associated to the general theory (5.69) will be equal to that given in Eq. (4.129) after making the same substitution, namely

$$a_2^{\text{EQG}} = \frac{d\beta_{\text{eff}}}{(d-2)\alpha_{\text{eff}}}\,. \tag{5.91}$$

5.2.1.3 *Entanglement entropy with a spherical entangling surface*

Let us finally get to the ultimate goal of this Section: computing the entanglement entropy for the boundary theory dual to (5.69) across a spherical entangling surface of radius R, and checking that it fulfills the conjectured relation given in Eq. (5.66). This quantity can be obtained as the $n \to 1$ limit of the Rényi entropy, which results in

$$S_{\text{EE}}(\mu) = S(T_0,\,\mu)\,,\quad \text{where}\quad T_0 = \frac{1}{2\pi R}\,, \tag{5.92}$$

as can be seen by computing explicitly the limit of Eq. (5.10), and taking into account the first law of thermodynamics in the form of Eq. (4.204). In this equation, $S(T_0,\,\mu)$ is the thermal entropy of the same theory placed on the hyperbolic cylinder $\mathbb{S}^1 \times \mathbb{H}^{d-1}(R)$, which for a holographic theory is given by the Wald entropy of the dual black hole. In this case that is given in Eq. (5.76) as $S = S(x,\,p)$, so we need to find the inverse functions $x = x(T_0,\,\mu)$ and $p = p(T_0,\,\mu)$. Since we are interested only in the correction to leading order in μ, we can carry out this procedure perturbatively from the temperature and chemical potential given by Eqs. (5.74) and (5.79), respectively. We find an expansion of the form

$$x = \hat{x} + \delta x_2(\ell_*\mu)^2 + \mathcal{O}\left(\mu^4\right)\,,\qquad p = \delta p_1(\ell_*\mu) + \mathcal{O}\left(\mu^3\right)\,, \tag{5.93}$$

with the different coefficients given by

$$\begin{aligned}
\hat{x} &= \frac{1}{\sqrt{f_\infty}}\,,\\
\delta x_2 &= -\frac{(\delta p_1)^2\left[2+\sum_{s=0}^{\infty}(-2)^s f_\infty^s\left(2-4s+d(d-3+2ds-2s+4s^2)\alpha_{1,s}+(d+2s-2)\alpha_{2,s}\right)\right]}{2(d-1)^2(f_\infty-2)\sqrt{f_\infty}}\,,\\
\delta p_1 &= \frac{2(d-2)f_\infty R}{L\left[2+(d-2)\sum_{s=0}^{\infty}(-2f_\infty)^2\left((d+2ds-1)\alpha_{1,s}+\alpha_{2,s}\right)\right]}\,.
\end{aligned} \tag{5.94}$$

Plugging these perturbative expressions into Eq. (5.76), we find that the entanglement entropy to quadratic order in μ reads

$$S_{\rm EE}(\mu)=\frac{\tilde{L}^{d-1}V_{-1,d-1}}{4G_{\rm N}}\left[1-\frac{2(d-1)}{d-3}\lambda f_\infty+\left(\frac{\sqrt{f_\infty}R}{L}\right)^2\frac{(\ell_*\mu)^2}{\alpha_{\rm eff}}\left(\frac{(d-2)^2}{d-1}+\frac{(d-2)^2\beta_{\rm eff}}{(d-1)^2\alpha_{\rm eff}}\right)\right]+\mathcal{O}\left(\mu^4\right)\,, \tag{5.95}$$

with the parameters $\alpha_{\rm eff}$ and $\beta_{\rm eff}$ given respectively by Eqs. (5.86) and (5.90). If we now take into account the values of C_J and a_2 that we computed for these theories, given in Eqs. (5.87) and (5.91), we see that this can be rewritten as

$$\begin{aligned}
S_{\rm EE}(\mu) &= \frac{\tilde{L}^{d-1}V_{-1,d-1}}{4G_{\rm N}}\left[1-\frac{2(d-1)}{d-3}\lambda f_\infty\right]\\
&+\frac{\Gamma(d/2-1)\pi^{d/2+1}V_{-1,d-1}}{\Gamma(d)}C_J^{\rm EQG}\left[\frac{(d-2)^2}{d-1}+\frac{(d-2)^3a_2^{\rm EQG}}{d(d-1)^2}\right](\mu R)^2+\mathcal{O}\left(\mu^4\right)\,.
\end{aligned} \tag{5.96}$$

Since we are interested in the universal part of this entropy, we should consider only the regularized volume of the unit hyperbolic space, which is given by Eq. (5.24). Thus, we arrive at the final result

$$\frac{S_{\rm EE}(\mu)}{v_{d-1}}=a^*_{\rm GB}+\frac{\pi^d}{(d-1)^2\Gamma(d-2)}C_J^{\rm EQG}\left[1+\frac{(d-2)a_2^{\rm EQG}}{d(d-1)}\right](\mu R)^2+\mathcal{O}\left(\mu^4\right)\,, \tag{5.97}$$

where the charge a^* of the Gauss-Bonnet theory is given in Eq. (5.68). We see therefore that this is equal to Eq. (5.66), so the conjectured universal relation is fulfilled for our infinite family of Electromagnetic Quasitopological theories of gravity.

As a final comment, one may wonder about the effect of adding arbitrary pure-gravity higher-order terms belonging to the Quasitopological class [8, 10, 107–110] to the action (5.69). Given the structure and the derivation of Eq. (5.97), we expect such terms to produce a renormalization of the constant f_∞, while leaving the result (5.97) invariant.

5.2.2 Proof for general CFTs

All the results obtained until now, first for the four-derivative theory (4.33) and then for the generalizations of these proposed in Section 4.1.4, point towards the validity of the relation (5.66) for general CFTs. However, this is by no means a proof of that conjecture, and a more generic analysis involving known universal relations fulfilled by any theory should be performed. This is actually the goal of the current Section, and for that we will use a combination of the results presented in [261] with some thermodynamic identities.

Let us consider once again the generalized twist operators that we studied in Section 5.1.2. The leading divergence of the correlator $\langle J_a \sigma_n(\mu)\rangle$ is determined by the magnetic response $k_n(\mu)$, as given in Eq. (5.46). This quantity can be computed using Eq. (5.47), which we repeat here for convenience,

$$k_n(\mu) = 2\pi n R^{d-1} \rho(n,\ \mu)\,. \tag{5.98}$$

We recall that $\rho(n,\ \mu)$ is the charge density of the CFT on the hyperbolic cylinder, at temperature $T = T_0/n$, with T_0 given by Eq. (5.5). The magnetic response also has a universal expansion around $n = 1$ and $\mu = 0$, whose leading coefficients can be expressed in terms of those characterizing the $\langle TJJ\rangle$ correlator. Namely, we have [261]

$$\begin{aligned} k_n\big|_{n=1,\,\mu=0} &= \partial_n k_n\big|_{n=1,\,\mu=0} = 0\,,\\ \partial_\mu k_n\big|_{n=1,\,\mu=0} &= \frac{16\pi^{d+1}R}{\Gamma(d+1)}\left(\hat{c}+\hat{e}\right)\,,\\ \partial_n\partial_\mu k_n\big|_{n=1,\,\mu=0} &= \frac{16\pi^{d+1}R}{d\Gamma(d+1)}\left(2\hat{c}-d(d-3)\hat{e}\right)\,, \end{aligned} \tag{5.99}$$

where the charges $\hat{c}$ and $\hat{e}$ are related to C_J and a_2 by Eqs. (4.136) and (4.137). In fact, we verified that the four-derivative EQG satisfies these relations (5.99) in Section 5.1.2.2.

Let us now consider the vacuum thermal entropy of the CFT on the hyperbolic cylinder at temperature $T = T_0/n$, which is equal to the entanglement entropy with a spherical entangling surface, see Eq. (5.92). In the grand canonical ensemble, the first law of thermodynamics reads

$$d\Omega = -SdT - \mathcal{N}d\mu\,, \tag{5.100}$$

where S is that thermal entropy, Ω is the grand canonical potential and $\mathcal{N} = V_{-1,\,d-1}R^{d-1}\rho$ is the total charge. From this form of the first law we can obtain the thermodynamic relation

$$\partial_\mu S = -\partial_\mu\partial_T\Omega = -\partial_T\partial_\mu\Omega = \partial_T\mathcal{N}\,. \tag{5.101}$$

Writing now $\mathcal{N}$ in terms of the magnetic response $k_n(\mu)$, and using that $\partial_T = -\frac{T_0}{T^2}\partial_n$, we have

$$\partial_\mu S = -\frac{T_0 V_{-1,\,d-1}}{2\pi T^2}\partial_n\left(\frac{k_n(\mu)}{n}\right)\,. \tag{5.102}$$

We can now expand the derivatives and evaluate for $n = 1$ (which corresponds to $T = T_0$) and $\mu = 0$ using Eq. (5.99). In particular, taking into account Eq. (5.92) it follows that the first derivative of the entanglement entropy with respect to μ vanishes,

$$\partial_\mu S_{\text{EE}}\big|_{\mu=0} = 0\,. \tag{5.103}$$

Taking a second derivative with respect to μ in Eq. (5.102), we find

$$\partial_\mu^2 S = -\frac{T_0 V_{-1,\,d-1}}{2\pi T^2}\partial_\mu\partial_n\left(\frac{k_n(\mu)}{n}\right)\,. \tag{5.104}$$

Evaluating again for $n = 1$ and $\mu = 0$, we have

$$\partial_\mu^2 S_{\rm EE}\big|_{\mu=0} = V_{-1,\,d-1} R\big(\partial_\mu k_n - \partial_\mu \partial_n k_n\big)\big|_{n=1,\,\mu=0}\,, \tag{5.105}$$

and using Eq. (5.99) this can be written as

$$\partial_\mu^2 S_{\rm EE}\big|_{\mu=0} = V_{-1,\,d-1} \frac{16(d-2)\pi^{d+1}R^2}{d\Gamma(d+1)}\left(\hat{c} + d\hat{e}\right)\,, \tag{5.106}$$

which, using Eqs. (4.136) and (4.137) to replace the values of $\hat{c}$ and $\hat{e}$ and Eq. (5.24) for the volume $V_{-1,/,d-1}$, reduces to Eq. (5.66). This therefore completes the proof that such relation is universally valid for a CFT in the presence of a chemical potential.

5.3 Discussion

In this Chapter we continued the study of the Electromagnetic Quasitopological theories of gravity initiated in Chapter 4. We focused in particular on the holographic Rényi and entanglement entropies for spherical entangling regions, and their associated twist operators.

First we computed the Rényi entropy in different limits and observed that, provided that the dual CFT respects unitarity, a non-zero chemical potential always increases the Rényi entropies with $n \geq 1$, and also the entanglement entropy as a particular case. Furthermore, standard Rényi entropies are known to satisfy some inequalities when considered as a function of the index n, which are written in Eq. (5.36), so we wondered whether these also held in our higher-derivative theories. As it turns out, they seem to be satisfied if one assumes all the physical constraints found in Section 4.4, while if one gives up the WGC bounds some of them can be violated, and the RE can even become negative (see Figure 5.2). This is another observation pointing towards the importance of the constraints derived from the WGC in obtaining a sensible boundary theory.

Afterwards, we computed the scaling dimension $h_n(\mu)$ and the magnetic response $k_n(\mu)$ of the generalized twist operators, as introduced in [261]. By using the entries for the holographic dictionary of the four-derivative theory (4.44), we have obtained a series of relationships between the derivatives of $h_n(\mu)$ and $k_n(\mu)$ at $n = 1$ and $\mu = 0$, and C_T, C_J and the coefficients of $\langle TJJ\rangle$ (see Eqs. (5.55), (5.58), (5.62) and (5.65)). These are actually universal relations that hold for any CFT, and they were derived from first principles in [261, 330]. Therefore, the fact that one can independently obtain them by using holographic higher-derivative theories is a proof of the power of this approach to learn about universality in CFTs.

The previous results served as a motivation to conjecture, and then prove, a universal relation involving the entanglement entropy for a spherical entangling region. This is written in Eq. (5.66), and implies that the leading correction to that entanglement entropy for a non-zero chemical potential depends only on the charges C_J and a_2 of the field theory. We have checked that this formula holds for CFTs dual to the general EQGs proposed in Section 4.1.4, and then provided a general proof based on the aforementioned universal relations involving the magnetic response $k_n(\mu)$ of the twist operators, derived in [261].

An additional check of our conjecture involving free fields can be found in [27]. There, the entanglement entropy across a spherical entangling surface is computed for a theory of free scalars or fermions in $d = 4$ by means of heat-kernel techniques, as done previously in [261]. It is found that Eq. (5.66) is also verified in this case, thus confirming the validity of that universal relation through an independent non-holographic computation.

Let us remark that we showed that universal relation (5.66) holds for $d \geq 3$, but in $d = 2$ there are various reasons to expect a different situation. First, notice that both the coefficients C_J and a_2 are pathological in that limit (see Eqs. (5.87) and (5.91)). The free field results reported in [261] also suggest a different structure in that case, including possible linear terms in μ or jumps in $S_n(\mu)$ as n and μ vary. It would be interesting to investigate these features further, and the three-dimensional holographic EQGs proposed in [139] would be natural candidates for this.

Finally, we would like to emphasize the importance of the result presented here, as it allows us to obtain predictions regarding the entanglement entropy of a CFT knowing only its parameters C_J and a_2. Understanding the extent and applications of this, however, would require further investigations.

Part III

Higher-curvature gravity and astrophysics

6

Black hole accretion in Einsteinian cubic gravity

Higher-derivative corrections are expected to appear in the low-energy dynamics of a hypothetical UV completion of general relativity, and if we assume that there exists such a theory that describes our universe, then these corrections should produce some sort of potentially observable behavior. With this motivation in mind, in this last Part of the thesis we continue the study of higher-derivative theories of gravity, focusing now on astrophysical scenarios.

General relativity has stood as a remarkably successful theory since its proposal, offering robust predictions and passing all precision tests within the weak field regime commonly explored in astrophysics. Nevertheless, it still faces several challenges, such as the need to introduce additional entities like dark matter and a non-zero cosmological constant, in order to accommodate observations that can not be accounted for by known matter and energy distributions. In fact, an alternative solution to these problems could possibly be achieved by considering certain corrected theories of gravity (see [368] for a review of these developments).

The expected higher-derivative terms in the action would be suppressed by powers of a (presumably small) length scale, and therefore should have an appreciable effect only in regions or regimes with very large curvatures, such as the first instants of our universe or the vicinity of very massive objects like black holes and neutron stars. Therefore, it also makes sense to consider HDGs for studies of cosmology, and indeed they could be able to produce an inflationary era. A review of these topics can be found in [369].

Provided diffeomorphism and Lorentz invariance hold, one would expect that the aforementioned higher-derivative corrections come in the form of contractions of the curvature tensors in the action. Among these higher-curvature gravities, the simplest known non-trivial theory in 4 dimensions is Einsteinian cubic gravity, which we introduced back in Section 1.1.3, and its cosmological version $\mathcal{P} - 8\mathcal{C}$ is of particular interest. This combination, besides sharing the desirable features that define the QTG family, also produces second order equations of motion when evaluated in an FLRW background, which when solved lead to an inflationary behavior without needing an additional scalar field, the hypothetical inflation.

As mentioned, these corrections are expected to have some importance close to a compact massive object, such as a black hole, and in order to detect any deviations from general relativity one would need to investigate the effects of gravity on the surrounding matter, including particles, gas, plasma and their interactions with radiation [368]. This interplay is nowhere else more dramatic than in an accretion scenario, where external matter continuously falls towards a gravitational potential well. The development of accretion theory over the last few decades has contributed to our understanding of astrophysical black holes and their role as central engines behind high-energy astronomical sources, such as active galactic nuclei, gamma ray bursts, tidal disruption events and some types of X-ray binaries [370].

The basic principles of non-relativistic accretion theory were established in the seminal works by Hoyle & Lyttleton [371] and Bondi & Hoyle [372]. They studied the problem now known as wind accretion, in which a massive gravitational object accretes matter as it moves through a cloud of gas at supersonic speeds. The other regime of interest is that where the relative motion between the cloud of external matter and the accretor can be neglected, known as spherical accretion and first studied by Bondi in [373]. Accretion theory has been further developed by incorporating the physics of rotating systems [374, 375], the role of turbulent motions and magnetic fields [376, 377], and the complex interplay between gas particles and radiation [378]. Additionally, infalling matter with non-zero angular momentum into an accretor may lead to the formation of an accretion disk [379, 380], a phenomenon that is expected to take place e.g. in gravitational binary systems in which matter ejected from one of its components gradually flows into the other.

The goal of this last Chapter of the thesis is to study the effect of higher-curvature corrections in the action on the problem of accretion. We will start by constructing the simplest possible spherically symmetric solution of Einsteinian cubic gravity: the Schwarzschild black hole. Similarly to the asymptotically AdS black hole studied in Chapter 3, this needs to be obtained numerically, which determines how the rest of the computations need to be performed. Then, as a warm-up, in Section 6.2 we will compute the radius of the flyby and the innermost stable circular orbit (these will be defined properly later), which will give us a hint of what can be expected from the study of accretion.

Afterwards we will consider the scenario of wind accretion, in which a cloud of dust moves towards the black hole (or alternatively the black hole moves inside the cloud) in Section 6.3. For this, we consider the relativistic model introduced in [381] and perform those computations with our solution for $f(r)$. The results for the accretion rate will then be compared with those obtained from simulations of relativistic hydrodynamics performed using the open source software *aztekas* [31, 32].

Finally, in Section 6.4 we delve deeper into the study of accretion by considering the scenario in which the black hole and the cloud of gas are at rest with respect to each other. As mentioned before, this is known as spherical accretion, and following [382] we will compute the accretion rate assuming first that the fluid satisfies a polytropic equation of state and then a fully relativistic one.

In all cases we find that the higher-derivative terms tend to increase the rate of accretion. While it was already known that they also increase the horizon radius for a given mass, this effect turns out to not be enough to account for the change in the accretion rate, and the effects of the higher-curvature coupling outside the black hole are also relevant.

6.1 Schwarzschild black hole in Einsteinian cubic gravity

Let us first introduce the black hole solution that we will consider throughout this Chapter. We choose the simplest higher-curvature theory that is non-trivial in 4 dimensions, Einsteinian cubic gravity, as a representative model to study the role of this kind of corrections, and in order for the theory to have a well-behaved cosmological scenario we work with the version introduced in [76].[1] The gravitational action is therefore given by

$$S = \frac{1}{16\pi G_{\rm N}} \int d^4x \sqrt{-g} \left[R - 2\Lambda_0 - 2\beta L_*^4 \left(\mathcal{P} - 8\mathcal{C} \right) \right] , \tag{6.1}$$

where Λ_0 is a cosmological constant, $\mathcal{P}$ and $\mathcal{C}$ are the cubic Lagrangian densities introduced in Eqs. (1.24) and (1.33), β is the corresponding (dimensionless) coupling constant and L_* is a length scale. We will assume that $\beta \geq 0$,[2] and since we are interested in asymptotically flat spacetimes we fix $\Lambda_0 = 0$. Also, for simplicity we set $G_{\rm N} = 1$.

For more detailed studies of accretion one would ideally consider rotating black holes, as are most of them in the universe. However, these metrics are typically very difficult to find for higher-curvature gravities, and in fact no analytic solutions with arbitrary rotation have been found even for Lovelock theories. The same thing happens with Einsteinian cubic gravity, for which rotating black holes could only be obtained in certain regimes such as slow rotation, near extremality or a perturbative coupling constant [125, 127, 131]. Because of this we settle with the simplest black hole solution that we are able to compute, the Schwarzschild black hole, which will be enough for our purposes.

6.1.1 Black hole ansatz and equations of motion

Let us consider the ansatz for the metric in spherical coordinates

$$ds^2 = -N^2(r) f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) , \tag{6.2}$$

where $N(r)$ and $f(r)$ are for now unknown functions. In order to find the equations of motion, we follow the reduced action approach introduced in Section 1.1.3. For this, we plug the metric ansatz (6.2) into the action (6.1), and vary the evaluated action with respect to $f(r)$ and $N(r)$. The first of these variations implies that $N(r) = 1$, as corresponds to any theory of the GQG class, and after setting this we find from the second one the equation of motion for $f(r)$,

$$-\frac{1}{3}\Lambda_0 r^3 - (f-1)r - 4\beta L_*^4 \left[f'^3 + 3\frac{f'^2}{r} - 6f(f-1)\frac{f'}{r^2} - 3ff'' \left(f' - \frac{2(f-1)}{r} \right) \right] = 2M \, . \tag{6.3}$$

Here, M is an integration constant with dimensions of mass, and it is equal to the ADM mass of the black hole in the flat case $\Lambda_0 = 0$,[3] which we will consider from now on.

This 4-dimensional black hole solution was already studied in [5], where its explicit thermodynamic properties were also found. In particular, the radius of the horizon r_h is computed

[1] Strictly speaking, the Lagrangian density $\mathcal{C}$ does not modify the equations of motion for a MSS such as the Schwarzschild black hole, but we include it for consistency with the results from cosmology.

[2] If $\beta < 0$ the black hole solution would not be asymptotically flat, as it would present non-decaying oscillations at infinity [115].

[3] The ADM mass for a static and spherically symmetric spacetime can be computed using the same formula as in GR [111–114, 279], which for a metric of the form (6.2) reads

$$M_{\rm ADM} = \frac{(D-2)\Omega_{(D-2)}}{16\pi G_{\rm N}} \lim_{r\to\infty} \left(\frac{1}{f(r)} - 1 \right) . \tag{6.4}$$

by solving the implicit equation

$$\frac{2M}{r_h} = 1 - \frac{16\beta L_*^4}{r_h^4} \frac{5 + 3\sqrt{1 + 48\beta L_*^4/r_h^4}}{\left(1 + \sqrt{1 + 48\beta L_*^4/r_h^4}\right)^3}\,, \tag{6.5}$$

which can be obtained by expanding the equation of motion (6.3) near the horizon. The actual form for $f(r)$ needs to be found numerically, and while this was done also in [5], here we will obtain it by applying again the relaxation method explained in Appendix C. This was already employed in Chapter 3 to obtain the asymptotically AdS solution, so the implementation in this flat case is straightforward.

As before, this method requires the integration range to be finite, and since we are interested in the exterior of the black hole, instead of the radial coordinate $r \in [r_h, +\infty)$ it is natural to introduce $Z = 2M/r$, such that $Z \in [0, 2M/r_h]$. We also need to specify an initial seed for the function $f(Z)$, which we take to be a linear interpolation from $f(Z = 0) = 1$ to $f(Z = 2M/r_h) = 0$. With this setup, the numerical relaxation algorithm is able to produce the solution for the function in the metric for any value of β.

6.1.2 Dimensionality and observational constraints

For the remaining of this Chapter, we will replace the coupling of the cubic terms in Eq. (6.1), β, by the combination

$$\epsilon = \beta \frac{L_*^4}{M^4}\,. \tag{6.6}$$

While both ϵ and β are dimensionless, this change and the fact that we work in terms of the ratio r/M make all our computations independent of the black hole mass M.

In principle we don't know what values these constants could have, as they are not fixed when constructing the higher-curvature corrections. However, if we expect ECG to make predictions about our universe we should find bounds on the values of the coupling so that the theory is compatible with currently available observations. To the best of our knowledge, the best classical constraint found at the moment is that proposed in [118], which relies on the experimental bound for the Shapiro time delay reviewed in [383]. In our conventions, this amounts to

$$\beta L_*^4 = \epsilon M^4 < 2.212 \times 10^{35}\,\text{m}^4\,, \tag{6.7}$$

where we assumed that $\beta \sim 1$. The constraint imposed on ϵM^4 is rather loose, since the higher-curvature terms acquire relevance in regions where the spacetime curvature becomes drastic, e.g. in the neighborhood of a black hole, so the classical tests of GR are not that useful for constraining ECG. That being said, we should always keep in mind that these theories are formulated in the context of EFT, in which case the dimensionless couplings should be of $\mathcal{O}(1)$, and we can not say anything about their validity beyond that [141]. Therefore, the extrapolation that we perform here applying these theories to the study of our universe with a relatively large coupling constant should be regarded more as an academic, and perhaps speculative, exploration.

Now, given that the best constraint for the coupling that we know of is that written in Eq. (6.7), we can allow for a large ϵ by considering black holes of small mass. This makes the model more appealing for the study of smaller objects, such as the hypothetical primordial black holes, which are theorized to have formed due to density fluctuations in the early stages of the universe and would have masses smaller than that of the Sun [43].

Of course, it is also possible that more restrictive bounds could be obtained from experimental setups currently out of reach, so this discussion should be taken with a grain of salt. However, for the remaining of this Chapter we will allow ϵ to be modestly large, and the results obtained should be interpreted as an illustration of the effects that higher-curvature terms can produce in astrophysical setups.

Before closing this discussion, let us mention that one might think of constraining the higher-curvature terms using measurements of the gravitational interaction between two test masses, as those performed in [384–388]. The authors of these references consider modifications of the Newtonian potential, obtained from the weak-field expansion of the Schwarzschild metric in GR, due to the exchange of additional hypothetical modes. For the higher-curvature theory studied here the first terms in the expansion of the metric components are the same as those in GR, so one must include higher powers of r/M to find some differences. However, this makes it impossible to find the gravitational potential in the usual way, from the tt component of the metric, and therefore the results of those references are not directly applicable to our case. In fact, the modifications expected on the Newtonian potential would be due to the appearance of additional propagating modes in the metric, which do not exist in Einsteinian cubic gravity or any other GQG (this is indeed one of the defining features of these theories), so this is not the correct interpretation for the modifications of the metric that we find. A discussion on this point can be found in Section 2.5.1 of [279].

6.1.3 Generalized Kerr-Schild coordinates

The procedure explained above allows us to obtain the form of $f(r)$ outside the horizon for any value of the cubic coupling. While this is enough for most of our computations, in order to perform the hydrodynamic simulations with the software *aztekas* presented in Section 6.3.3.1 we will need to extend the solution to some point behind the horizon, so we must find a system of coordinates that is regular at $r = r_h$. The natural candidate for a Schwarzschild black hole in GR are the Kerr-Schild coordinates (see e.g. Section 1.15 of [389]), which we generalize now for our $f(r)$.

Let us start by writing the metric in Eddington-Finkelstein coordinates. For this, we must find the radial null geodesics by imposing $d\theta = d\phi = 0$ and $ds^2 = 0$ in the metric (6.2). These curves are therefore given by

$$t = \pm r^* + \text{const}, \quad \text{where} \quad dr^* = \frac{dr}{f(r)}\,. \tag{6.8}$$

Here r^* is the tortoise coordinate, which by definition is singular at the horizon. If we define the new coordinate $\tilde{V} = t + r^*$, the metric becomes

$$ds^2 = -f(r)d\tilde{V}^2 + 2d\tilde{V}dr + r^2 d\Omega_{(2)}^2\,, \tag{6.9}$$

and we see clearly that the ingoing radial null geodesics are given by $\tilde{V} =$ constant. A further transformation of coordinates $\tilde{t} = \tilde{V} - r$ brings the metric to the Kerr-Schild form

$$ds^2 = -f(r)d\tilde{t}^2 - 2\big(f(r) - 1\big)d\tilde{t}dr + \big(2 - f(r)\big)dr^2 + r^2 d\Omega_{(2)}^2\,, \tag{6.10}$$

which is regular at the horizon. This entire change of coordinates amounts to a single transformation of the time coordinate, as

$$dt \longrightarrow d\tilde{t} = dt - \left(1 - \frac{1}{f(r)}\right) dr\,, \tag{6.11}$$

which is equal to the original one asymptotically, since in any case $\lim_{r\to\infty} f(r) = 1$. Note also that the spatial coordinates have not been changed at any point.

For the actual numerical computations we will need the 3 + 1 decomposition of the metric (6.10). This is given in general by

$$ds^2 = -\alpha^2 d\tilde{t}^2 + \gamma_{ij}\left(dx^i + \beta^i d\tilde{t}\right)\left(dx^j + \beta^j d\tilde{t}\right)\,, \tag{6.12}$$

and in this case the lapse function α and the components of the shift vector β^i and the 3-metric γ_{ij} take the values

$$\begin{aligned} \alpha &= \sqrt{\frac{1}{2 - f(r)}}\,, \\ \beta^r &= \frac{f(r) - 1}{f(r) - 2}\,, \qquad \beta^\theta = \beta^\phi = 0\,, \\ \gamma_{rr} &= 2 - f(r)\,, \qquad \gamma_{\theta\theta} = r^2\,, \qquad \gamma_{\phi\phi} = r^2 \sin^2\theta\,, \end{aligned} \tag{6.13}$$

while the off-diagonal components of γ_{ij} vanish. These functions are all regular and non-zero at the horizon, and thus allow us to continue the computation behind it. The numerical solution for $f(r)$ obtained using the relaxation method is only valid in the range $r \in [r_h, \infty)$, but it is easy to integrate Eq. (6.3) inside the horizon taking the exterior solution for $f(r)$ as initial conditions. In practice, since the equation of motion (6.3) is stiff at the horizon, we use an extrapolation of the exterior solution obtained with the relaxation method to compute the values of $f(r)$ and $f'(r)$ a short distance inside the horizon. From these, we continue the solution for $r < r_h$, employing standard numerical methods for stiff equations.

The numerical solution continued inside the horizon for different values of the cubic coupling is shown in Figure 6.1. Notice that the singularity of $f(r)$ at $r = 0$ disappears when the cubic terms are turned on. However, as explained in [5], there is still a curvature singularity at the origin. This can be seen by explicitly evaluating the Kretschmann scalar, which behaves as $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim 1/r^4$, so it diverges at the origin, although the strength of this singularity decreases with respect to the result in Einstein gravity, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim 1/r^6$. This can be taken as a hint to the fact that these higher-curvature terms might improve the behavior of these spacetime singularities, as would be expected from a UV complete quantum theory of gravity.

6.2 Preliminary studies: black hole flyby and ISCO

Before getting into the study of accretion by a black hole in ECG, let us consider the simpler problem of a massive particle moving in the background of such an object. We will contemplate two different scenarios, that we describe now.

In the first one, the probe particle is non-relativistic for distant observers, that is, it has zero kinetic energy at $r \to \infty$. We are interested in computing the minimal distance from the

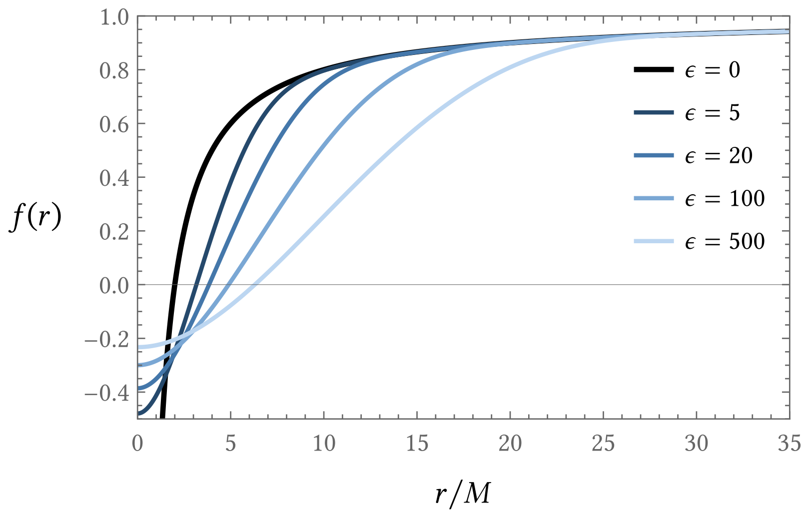


Figure 6.1: Numerical solution for the function $f(r)$ in the Schwarzschild metric (6.2), for different values of the cubic coupling ϵ introduced in Eq. (6.6). Notice that the singularity at $r = 0$ disappears for $\epsilon \neq 0$.

black hole that it can reach before flying away to infinity without being powered by external sources of energy, which is known as the flyby radius. The corresponding trajectory is also named "marginally bound orbit". This problem was treated in general relativity in [390], and here we will extend their results to our theory with cubic corrections.

The second case that we will consider has a setup somewhat similar to the previous one, but now the particle is allowed to have a non-zero kinetic energy at the asymptotic region. We are interested in computing the radius of the innermost stable circular orbit (ISCO), this is, the smallest circular orbit that the particle can follow without falling into the black hole. The radius of the ISCO has been computed before in ECG in [118] for perturbatively small values of the cubic coupling ϵ, but our solution described in Section 6.1 allows us to go beyond that regime.

In general relativity the function $f(r)$ has a simple form, so both problems can be solved in an analytic manner. In the cubic case, however, we will need to resort to some numerical root-finding methods to obtain the solutions. We are interested on the dependence of r_{flyby} and r_{ISCO} on the higher-curvature coupling, in order to gain some intuition about what we might find when studying the accretion scenario.

The initial setting in both cases is the same: we want to study the trajectory of a massive particle in the Schwarzschild metric (6.2), with $N(r) = 1$ and a generic form of the function $f(r)$. Without loss of generality, since the entire movement takes place in a plane we fix for simplicity $\theta = \pi/2$. Also, the components of the metric do not depend on t or ϕ, so we can construct the Killing vectors $\mathcal{T} = \partial_t$ and $\mathcal{R} = \partial_\phi$. Denoting by u the 4-velocity of the particle, we find the associated conserved quantities

$$u \cdot \mathcal{T} = u_t \equiv -E\,, \qquad u \cdot \mathcal{R} = u_\phi \equiv L\,, \tag{6.14}$$

which are related to the energy and angular momentum of the particle, respectively. Then, we can write the components of the 4-velocity as $u_\mu = (-E,\, g_{rr}\dot{r},\, 0,\, L)$. Since the particle is massive, it must fulfill $u \cdot u = -1$ (assuming for simplicity that it has unit mass), which leads to

$$\dot{r}^2 = -f(r) + E^2 - \frac{f(r)}{r^2} L^2 \,. \tag{6.15}$$

This can be recast as a 1-dimensional problem in classical orbital mechanics, of the form

$$\tilde{E} = \frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) \,, \tag{6.16}$$

where we defined

$$V_{\text{eff}}(r) \equiv -\frac{1}{2} + \frac{f(r)}{2} + \frac{f(r)}{2r^2} L^2 \,, \qquad \tilde{E} \equiv \frac{E^2 - 1}{2} \,. \tag{6.17}$$

As mentioned before, the constant E introduced in Eq. (6.14) is related to the energy of the particle at infinity. In order to find out how we expand Eq. (6.15) for $r \to \infty$, which results in

$$\dot{r}^2\Big|_{r\to\infty} = -1 + E^2 \,, \tag{6.18}$$

where we took into account that $\lim_{r\to\infty} f(r) = 1$. If the particle has a radial velocity at $r \to \infty$, then in this region

$$\dot{r}\Big|_{r\to\infty} = \frac{\partial r}{\partial t}\frac{\partial t}{\partial \tau}\Big|_{r\to\infty} = v_\infty E \,, \tag{6.19}$$

and plugging this above we can read off the value of the constant E,

$$E = \frac{1}{\sqrt{1 - v_\infty^2}} = \gamma_\infty \,. \tag{6.20}$$

So we see that, since the mass of the particle is 1, the conserved quantity E is equal to the total energy of the particle at infinity.

6.2.1 Black hole flyby

As explained above, the flyby radius is defined for a particle that is at rest at infinity, this is, $v_\infty = 0$. This implies, taking into account Eqs. (6.17) and (6.20), that $\tilde{E} = 0$ in the differential equation (6.16). The flyby radius is then defined as the minimum radius that this particle can reach in the orbit before escaping back to infinity. Therefore, the effective potential introduced in Eq. (6.17) must have an extremum at that point,

$$V'_{\text{eff}}(r_{\text{flyby}}) = 0 \,. \tag{6.21}$$

Combining this with Eq. (6.16) we arrive at

$$f(r_{\text{fb}}) - 1 - \frac{f(r_{\text{fb}})f'(r_{\text{fb}})}{f'(r_{\text{fb}}) - 2f(r_{\text{fb}})/r_{\text{fb}}} = 0 \,, \tag{6.22}$$

where we replaced $r_{\text{fb}} \equiv r_{\text{flyby}}$ for ease of notation. This equation can be solved analytically in general relativity [390], where $f(r) = 1 - 2M/r$, finding $r_{\text{flyby}} = 2r_h = 4M$. For the cubic theory, however, the solution has to be found numerically for a given form of $f(r)$.

The results for r_{flyby} compared with the horizon radius are shown in Figure 6.2. We see that both quantities increase with the cubic coupling ϵ, and indeed by dividing r_{flyby} by r_h we

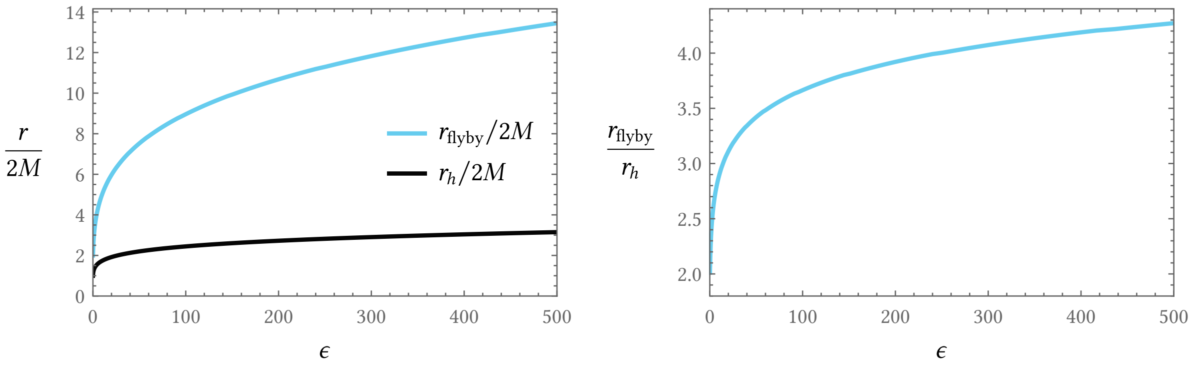


Figure 6.2: Flyby radius and horizon radius with respect to the coupling of the cubic terms ϵ, in the range $\epsilon \in [0,\ 500]$.

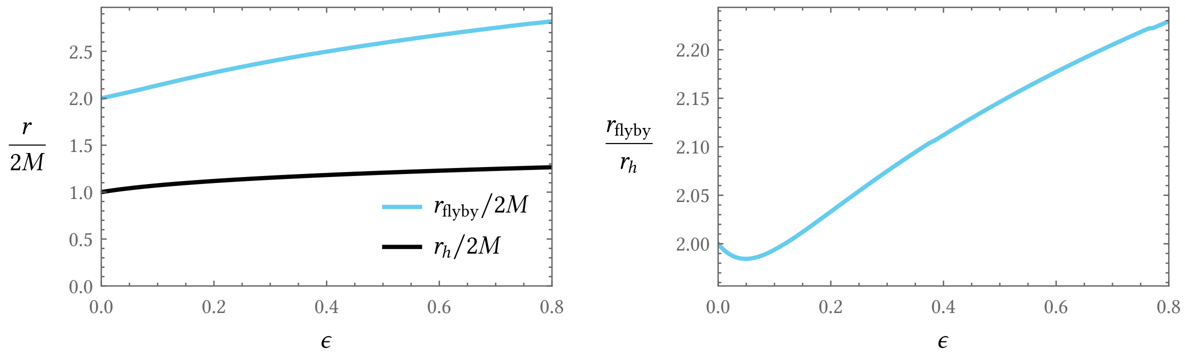


Figure 6.3: Flyby radius and horizon radius with respect to the coupling of the cubic terms ϵ, in the range $\epsilon \in [0,\ 0.8]$.

observe that the former increases faster, meaning that the change of the flyby radius is not due entirely to the larger radius of the horizon, but also to the effects of the higher-curvature terms outside the black hole. We were able to confirm this tendency by considering much larger values of the coupling, going up to $\epsilon \sim 4000$, above which our procedure presents numerical errors and should not be trusted.

For very small values of the cubic coupling, however, the effect is the opposite: the ratio r_{flyby}/r_h decreases with respect to its value of 2 found in GR. This is illustrated in Figure 6.3, where we can observe a dip in that ratio below $\epsilon \approx 0.1$. At the moment we have no satisfactory explanation for this effect, but it seems to imply that in this regime the effect of the cubic terms is greater at the horizon than in the region outside, in the sense that they enlarge the black hole horizon more than the radius of these geodesics.

6.2.2 Innermost stable circular orbit

Let us now turn to the second problem: finding the radius of the smallest circular and stable orbit that a massive particle can describe around our black hole. The setting of the problem is the same as before, but now we relax the condition that the probe particle is at rest at infinity. For a problem written in the form (6.16), finding r_{ISCO} and the corresponding angular momentum L_{ISCO} reduces to obtaining the solution of the system of equations

$$V'_{\mathrm{eff}}(r_{\mathrm{ISCO}}) = 0\,, \qquad V''_{\mathrm{eff}}(r_{\mathrm{ISCO}}) = 0\,. \tag{6.23}$$

The first of these ensures that the trajectory is circular, since the radius must be constant $\dot{r} = 0$, and this condition has to be kept during the evolution of the system. The second equation means that r_{ISCO} is an inflection point of the effective potential, which corresponds to the point at which the inequality $V''_{\mathrm{eff}}(r) \leq 0$, needed for the orbit to be stable, is saturated.

Therefore, by plugging the form of the effective potential (6.17) into Eq. (6.23) we get a system of equations involving the function $f(r)$, whose solution yields the values of both the r_{ISCO} and L_{ISCO}. Again, for GR the solution for these can be found easily, and they read $r_{\mathrm{ISCO}} = 3r_h = 6M$ and $L_{\mathrm{ISCO}} = \sqrt{12}M \approx 3.46M$. For Einsteinian cubic gravity, on the other hand, these equations need to be solved numerically resorting to some root-finding algorithm.

The quantities r_{ISCO} and L_{ISCO} for a black hole in ECG have already been computed in [118]. However, the authors consider an analytic approximation for the form of $f(r)$, which is valid only for small values of the coupling constant, in particular $\epsilon \leq 1/12 \approx 0.083$ in our conventions. The solution obtained here allows us to go beyond that perturbative regime, and we checked that our results are consistent with those in the reference.

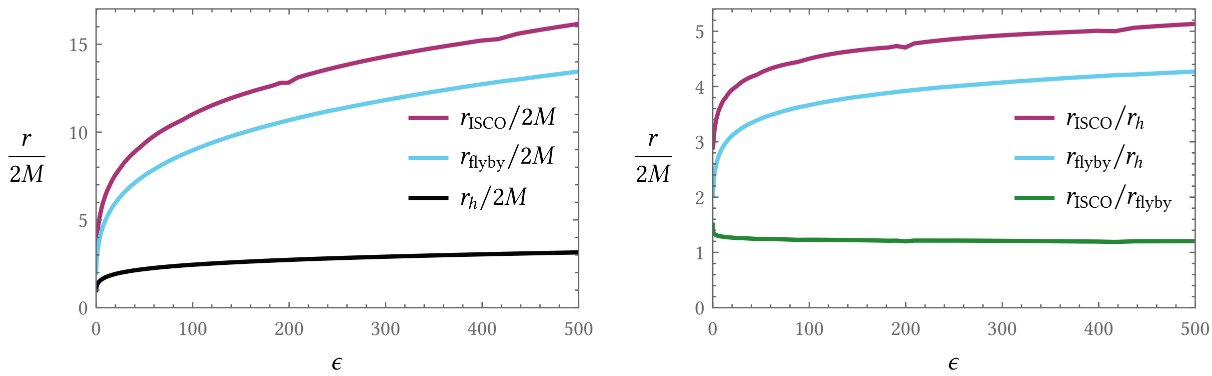


Figure 6.4: Radius of the ISCO with respect to the cubic coupling ϵ in the range $\epsilon \in [0,\ 500]$, compared with the flyby and horizon radii. The dips that appear for certian values of ϵ seem to be due to numerical inaccuracies, and should not be taken into consideration.

The numerical results for the radio of the ISCO for a wide range of values of the cubic coupling are shown in Figure 6.4, where we compare them with the flyby radius computed in Section 6.2.1. We see that r_{ISCO} also increases with ϵ faster than the horizon radius, but slower than r_{flyby}. Also, for the sake of completeness we show the dependence of the angular momentum L_{ISCO} with ϵ in Figure 6.5, which although it is not as relevant for our work, it can be if one wants to study accretion disks produced by such black holes [379, 380].

Finally, in Figure 6.6 we plot the behavior of r_{ISCO} for small values of the cubic coupling. We observe the same initial dip in the ratio r_{ISCO}/r_h that was found in Figure 6.3 for r_{flyby}/r_h, but again we could not come up with a convincing interpretation. In any case, these findings imply that the geodesics are distorted in a non-trivial manner by the cubic terms.

The results for both the radius of the ISCO and the flyby radius imply that the corrected black hole will absorb matter from a larger region than its equivalent in GR, at least in the non-perturbative regime of ϵ that we are interested in. This strongly suggests that the rate

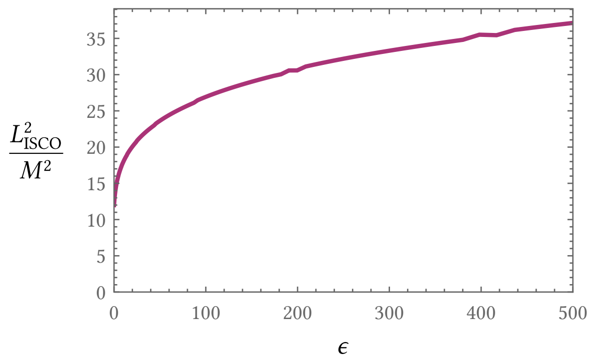


Figure 6.5: Angular momentum of the ISCO with respect to the cubic coupling ϵ in the range $\epsilon \in [0,\ 500]$.

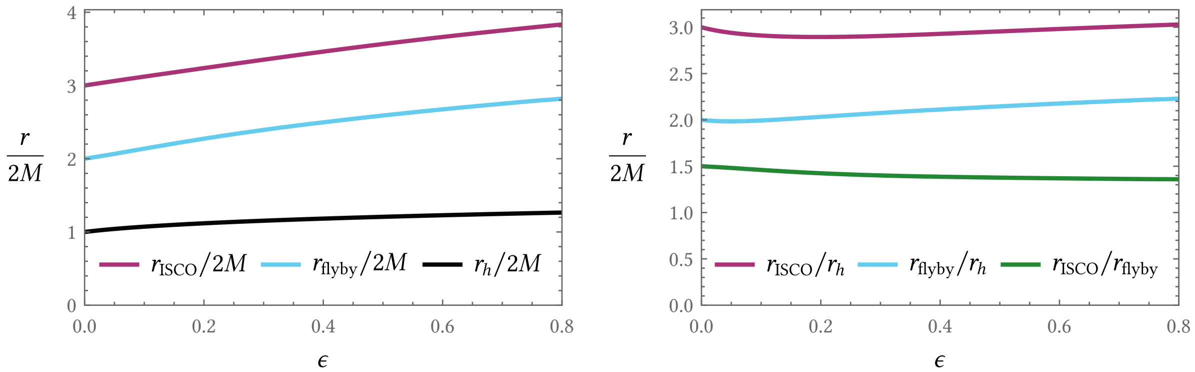


Figure 6.6: Radius of the ISCO with respect to the cubic coupling ϵ in the range $\epsilon \in [0,\ 0.8]$, compared with the flyby and horizon radii.

at which it accretes matter should increase with ϵ, and the goal of the next two Sections is to check this in different scenarios.

6.3 Accretion of wind

Let us turn now to the more complex problem of the accretion of wind by a static black hole. This has first been studied using classical gravitational physics by Hoyle and Lyttleton [391], and later refined by Bondi and Hoyle [392] (see [393] for a pedagogical introduction to these models). In this work we are interested in the relativistic extension of the setup, which was first proposed in [381], and the implications that the higher-curvature terms in the gravity action can have in the process. However, in order to get a better understanding of the scenario let us first review the classical approach.

6.3.1 Classical Hoyle-Lyttleton model

In its original form, the Hoyle-Lyttleton model [391] attempts to describe the absorption of matter by a massive object (originally a star) moving at a constant non-relativistic velocity through an infinite cloud of gas or dust. The external matter is initially at rest, and made of massive particles that are influenced only by the gravitational potential of the central attractor.

This is known as the ballistic approximation, and implies that the interaction among the particles of the gas is negligible.

Equivalently, we will consider the central object to be at rest and the external matter to be a non-relativistic "wind," characterized at infinity by a constant velocity v_∞ and density ρ_∞. Due to the symmetry of the problem, it is easier to work in spherical coordinates, and for simplicity we set the polar angle $\theta = \pi/2$. The system of equations obtained from studying the movement of a test particle in the background of the massive object can be solved analytically, finding in particular the trajectory and velocity of the particle [394]

$$\begin{aligned} r(\phi) &= \frac{b^2 v_\infty^2}{G_N M\,(1+\cos\phi) + b v_\infty^2 \sin\phi}\,, \\ \dot{r}(\phi) &= -\sqrt{v_\infty^2 + \frac{2G_N M}{r} - \frac{b^2 v_\infty^2}{r^2}}\,, \\ \dot{\phi}(\phi) &= \frac{b v_\infty}{r^2}\,, \end{aligned} \tag{6.24}$$

where M is the mass of the central accretor and we recovered Newton's constant. Notice that the solution is characterized only by b, which is the impact parameter of the particle. This setup is represented in Figure 6.7.

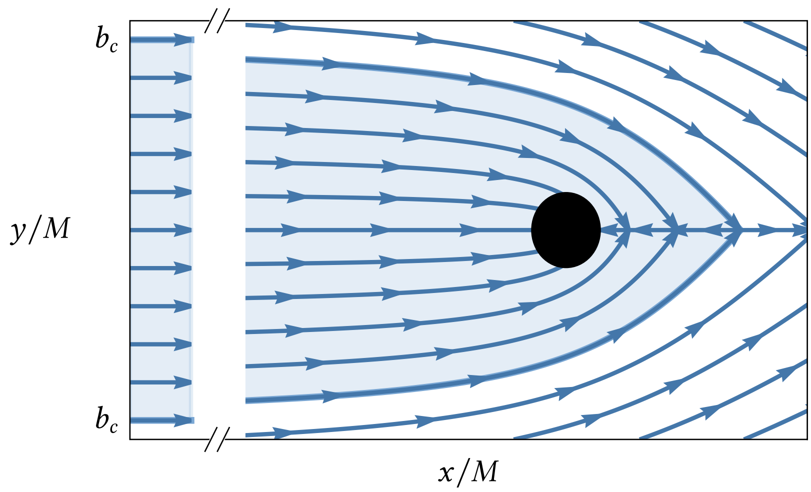


Figure 6.7: Two-dimensional representation of the accretion of wind by a massive object. The particles of matter come from the left of the diagram ($\phi = \pi$) with an asymptotic velocity v_∞, and their trajectory is characterized only by the impact parameter b and given by Eq. (6.24). The absorbed particles are those with $b < b_c$, which is represented by the shaded region in the plot.

The condition for accretion can be obtained in a simple manner by working in terms of energies. At infinity, the particle has only a kinetic energy $E_\infty = m v_\infty^2/2$, and from the analytic solution (6.24) we see that it reaches the downstream axis $\phi = 0$ at a distance $r(\phi = 0) = b^2 v_\infty^2/(2G_N M)$, and with a radial velocity $\dot{r}(\phi = 0) = v_\infty$. Here, streams of particles coming from above will collide with those coming from below, thus losing the angular component of their velocity, which is assumed to be radiated away. Then, after the collision each particle is left with a total energy

$$E' = \frac{1}{2}m\dot{r}(0) - \frac{G_N M m}{r(0)} = \frac{1}{2}mv_\infty^2 - \frac{2m(G_N M)^2}{b^2 v_\infty^2} \,. \tag{6.25}$$

The particles with negative E' become bound to the central object, and thus eventually accreted. Therefore, we can find the critical impact parameter b_c by equating this to zero, finding

$$b_c = \frac{2G_N M}{v_\infty^2} \,. \tag{6.26}$$

As depicted in Figure 6.7, the material inside the asymptotic cylinder $b < b_c$ is absorbed. Then, we can compute the flow of accreted matter, defined as the incoming volume per unit of time in this region, and multiply it by the density ρ_∞ to obtain the mass accretion rate,

$$\dot{M}_{\rm HL} = \pi b_c^2 \rho_\infty v_\infty = 4\pi\rho_\infty \frac{(G_N M)^2}{v_\infty^3} \,. \tag{6.27}$$

We will compare our results for general relativity and Einsteinian cubic gravity against this quantity later on.

The Bondi-Hoyle model [392] is a refinement of the one presented here, in which the accreted matter spreads out in a column or cone around the offstream axis $\phi = 0$. While this is a more realistic setup, the actual change of the accretion rate is of $\mathcal{O}(1)$, and therefore not very relevant for our purposes. Therefore, for ease of computations in the following we consider a relativistic extension of the simpler Hoyle-Lyttleton reviewed here.

6.3.2 Relativistic extension of the model

Let us now consider a relativistic extension of the Hoyle-Lyttleton model for accretion of wind, proposed in [381]. As before, we consider a “wind” made of massive particles that only interact with a central attractor, which now will be a Schwarzschild black hole. Its metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{6.28}$$

where the exact form of the function $f(r)$ depends on the theory at hand. Also, taking advantage of the spherical symmetry of the problem we will work in the plane $\theta = \pi/2$.

The movement of a massive particle in this background is described by the geodesic equations obtained in Section 6.2, which we rewrite here in a more convenient manner,

$$\begin{gathered} \frac{dt}{d\tau} = \frac{E}{f(r)} \,, \qquad \frac{d\phi}{d\tau} = \frac{L}{r^2} \,, \\ \frac{dr}{d\tau} = \pm\left[E^2 - f(r)\left(1 + \frac{L^2}{r^2}\right)\right]^{1/2} \,. \end{gathered} \tag{6.29}$$

Again, E and L are conserved quantities proportional respectively to the energy and angular momentum of the particle at infinity. In fact, the constant E is related to the asymptotic velocity v_∞ as given in Eq. (6.20). The last of these equations follows from $u \cdot u = -1$, and since the particles studied follow infalling trajectories we will choose the minus sign.

We can define the impact parameter b in an intuitive manner as the distance of closest approach of the particle to the origin if the spacetime were flat. In this case we would have $f(r) = 1$, and at this point Eq. (6.29) implies

$$\left.\frac{dr}{d\tau}\right|_{f(r)=1,\, r=b} = 0 = -\left[E^2 - 1 - \frac{L^2}{b^2}\right] \quad \Longrightarrow \quad L = bV_\infty\,, \tag{6.30}$$

where we replaced the value of E using Eq. (6.20), and we defined

$$V_\infty \equiv \frac{v_\infty}{\sqrt{1 - v_\infty^2}} = \gamma_\infty v_\infty\,. \tag{6.31}$$

So in what follows we write everything in terms of the parameters b and v_∞ or V_∞.

Since we want to study the trajectory of the probe particles in the plane $(r,\,\phi)$ it is natural to compute the function $r(\phi)$, this is, taking the angular coordinate to be the independent variable. Using Eqs. (6.20), (6.29), (6.30) and (6.31), we can obtain the differential equation for this function,

$$\frac{dr}{d\phi} = \pm\frac{\sqrt{\mathcal{R}(r)}}{bV_\infty}\,, \quad \text{where} \quad \mathcal{R}(r) \equiv r\left[r^2\left(1 + V_\infty^2\right) - r^3 f(r) - r f(r) b^2 V_\infty^2\right]\,. \tag{6.32}$$

Of course, this equation is generic for any theory of gravity that admits a solution of the form (6.28). The authors of [381] present exact analytical solutions for the trajectories $r(\phi)$ in GR, first derived in [395]. However, since we want to plug in the numerical solutions for $f(r)$ obtained in Section 6.1.1, these geodesic need to be computed also by means of numerical methods.[4]

There is one subtlety that needs to be taken into account when integrating Eq. (6.32): the sign of the square root must be chosen in a consistent manner. Clearly, the minus sign corresponds to infalling trajectories, while the positive sign means that the particle moves away from the central object. While all trajectories are infalling at the beginning of evolution (this is, for $r \to \infty$), implying that the integration starts with the negative sign, depending on the value of b they can reach a minimum and then go away. This minimum happens where $dr/d\phi = 0$, or equivalently $\mathcal{R}(r) = 0$, so if a point is reached where $\mathcal{R}(r) \leq 0$, the integration algorithm must go back to the value of r in the previous step, and choose the positive branch of the square root to continue the integration.

Once we know how to compute the streamlines of the wind particles for given values of v_∞ and b, we need to find which of those will be absorbed by the black hole in order to compute the accretion rate. The procedure considered is equal to what was explained for the Hoyle-Lyttleton model in Section 6.3.1: streams of particles that are symmetric to each other along the axis $\phi = 0$ collide, losing energy in such a way that some become energetically bound to the black hole. We are interested in computing the critical impact parameter, b_c, below which all the particles are eventually absorbed by the black hole.

Let us consider a unit-mass particle with constant energy E and angular momentum L before the collision, given in terms of v_∞ by Eqs. (6.20) and (6.30) respectively. When both streams meet at $\phi = 0$, the component u^ϕ of their velocities becomes zero, while $u^r = dr/d\tau$ is

4 In practice we work with the inverse coordinate $z = 1/r$, so that the range $r_h < r < \infty$ becomes finite, making the numerical treatment more amenable. The transformation of all the equations is rather straightforward, so we discuss the entire procedure in terms of the original coordinate r.

conserved. Thus, if we denote with primes the constants after the collision we have $L' = 0$, and using Eq. (6.29) in both cases,

$$\left(\frac{dr}{d\tau}\right)^2_{\phi=0} = E^2 - f\big(r(0)\big)\left(1 + \frac{L^2}{r(0)^2}\right) = E'^2 - f\big(r(0)\big) \quad \Longrightarrow \quad E'^2 = E^2 - f\big(r(0)\big)\frac{L^2}{r(0)^2}\,. \tag{6.33}$$

As in the classical case, the value of this relativistic energy determines whether the particle will escape to infinity after the collision ($E' > 1$), or if instead it will be bound to the black hole and accreted by it ($E' < 1$). The critical parameter b_c determines the limiting case $E' = 1$, which yields the simple equation

$$r(0)^2 - f\big(r(0)\big)b_c^2 = 0\,, \tag{6.34}$$

where we replaced $E^2 = 1 + L^2/b^2$, as can easily be checked using Eqs. (6.20) and (6.30).

In order to compute b_c numerically from Eq. (6.34) we had to implement a root-finding algorithm that we outline now. For some given v_∞, the program picks a value of b, solves Eq. (6.32) for the trajectory $r(\phi)$ and checks whether Eq. (6.34) is verified or not to some desired accuracy. If it is not fulfilled, it moves to a different value of b as given by a bisection algorithm, and repeats the process until the solution is found.

Once the critical impact parameter is known, it is straightforward to compute the accretion rate $\dot{M} = dM/d\tau$ as the mass per unit of proper time that enters the cylinder of radius b_c at $r \to \infty$,

$$\dot{M} = \pi b_c^2 \rho_\infty u^r\big|_{r\to\infty} = \pi b_c^2 \rho_\infty v_\infty \gamma_\infty\,, \tag{6.35}$$

where we wrote the radial velocity at infinity as

$$u^r = \frac{dr}{d\tau} = \frac{\gamma_\infty}{f(r)}\frac{dr}{dt} \quad \Longrightarrow \quad u^r\big|_{r\to\infty} = \gamma_\infty v_\infty\,. \tag{6.36}$$

Notice that the only difference between the relativistic formula for $\dot{M}$ and the classical one given in Eq. (6.27), besides the value of b_c, is the multiplicative factor γ_∞. This is due to the Lorentz contraction, which compresses the volume elements of the fluid along the direction of the wind, and will change the behavior drastically for large v_∞.

6.3.3 Numerical results with cubic corrections

In what follows we the numerical results for the model of accretion of wind explained in the previous Section, considering the function $f(r)$ that corresponds to the Schwarzschild black hole with cubic corrections obtained in Section 6.1.1. Also, one should recall that in Section 6.2 we found that the flyby radius increases with the coupling of the higher-curvature terms, and we argued that the same would be expected to happen for the accretion rate.

The results obtained for the critical impact parameter b_c are shown in Figure 6.8, where it is possible to see the behavior of this quantity with respect to the asymptotic velocity of the wind v_∞. We show curves for different values of ϵ, all of them resembling the one that corresponds to Einstein gravity, $\epsilon = 0$, which we have checked that matches the results in [381]. Also, b_c increases with the coupling of the cubic terms, and this effect is accentuated for large values of v_∞. This is shown in Figure 6.9, where we plot similar data now with ϵ in the horizontal axis. This tendency resembles the increase of r_{flyby} and r_{ISCO} that we had seen in Section 6.2,

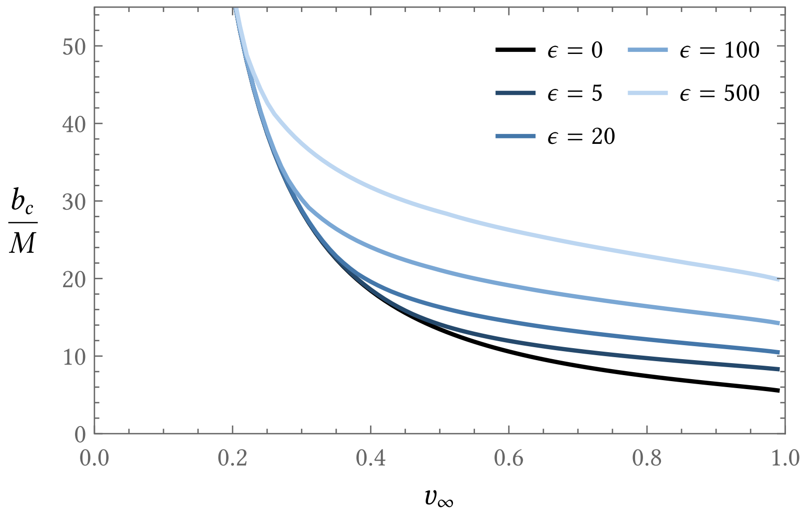


Figure 6.8: Critical impact parameter for accretion of wind with respect to v_∞ for different values of the coupling of the cubic terms ϵ.

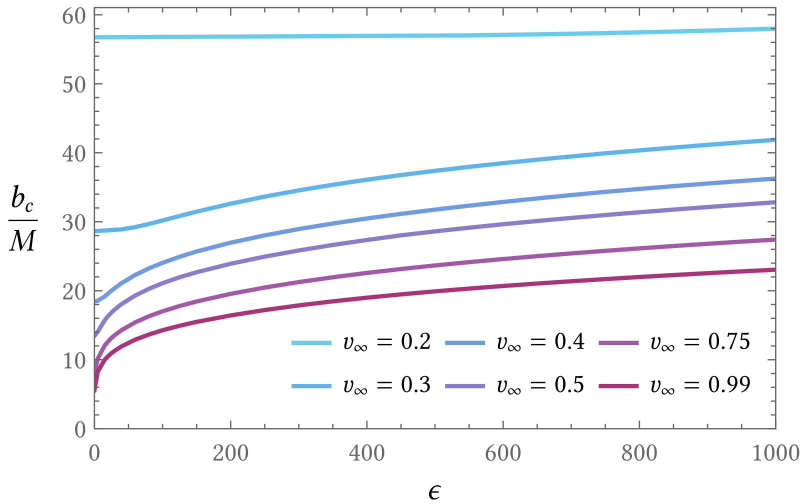


Figure 6.9: Critical impact parameter for accretion of wind with respect to ϵ, for different values of the asymptotic velocity v_∞.

although we were not able to observe a decrease in b_c/r_h for small ϵ values as we did for the other two quantities.

In Figure 6.10 we plot the critical impact parameter b_c divided, respectively, by the horizon radius and the flyby radius. By simple inspection, it is possible to conclude that the increase in these two quantities alone is not enough to explain the growth of b_c, and the effects of the cubic terms in the spacetime outside the horizon are important.

From the results for the critical impact parameter we can compute the mass accretion rate employing Eq. (6.35). We plot these results in Figure 6.11, also comparing them with the Hoyle-Lyttleton value given in Eq. (6.27). As expected, the increase of b_c is translated into an increase of the accretion rate, which becomes much more important for large values of the velocity v_∞. Note however that the scale of the vertical axis in the plot is logarithmic, which suppresses the small deviations with respect to the GR value for low v_∞.

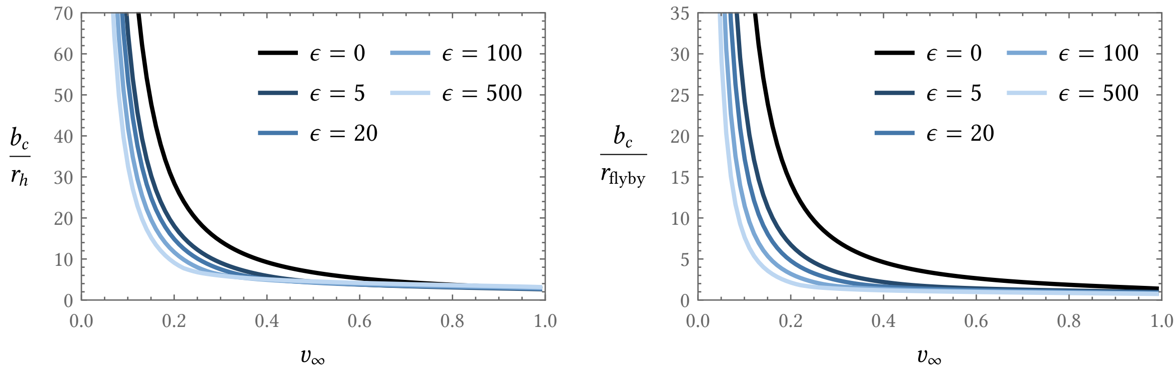


FIGURE 6.10: Critical impact parameter divided by the horizon radius or the flyby radius with respect to v_∞, for different values of the higher-curvature coupling ϵ.

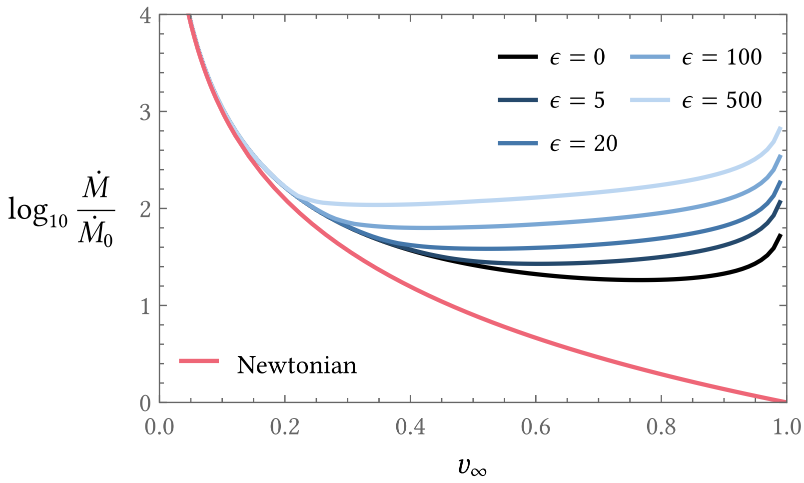


FIGURE 6.11: Mass accretion rate, normalized by $\dot{M}_0 = 4\pi M^2 \rho_\infty$, with respect to v_∞ for different values of the cubic coupling ϵ. We also show the result computed with the Hoyle-Lyttleton formula (6.27), obtained using Newtonian gravity.

As mentioned before, we can also observe that the accretion rate always blows up for $v_\infty \to 1$. This is due to the Lorentz factor γ_∞ in Eq. (6.35), which takes into account the relativistic contraction of the spacetime along the direction of movement of the fluid, so it is obviously not present in the non-relativistic Hoyle-Lyttleton approximation that we also plot in Figure 6.11.

Finally, in Figure 6.12 we show results for the same mass accretion rate, now with respect to the cubic coupling ϵ and fixing the asymptotic velocity v_∞. In this way, it becomes clearer that the effect of the cubic corrections becomes more important as the velocity v_∞ increases, which makes sense, as one would expect the higher-curvature terms to become more relevant as one approaches the ultra-relativistic regime. Also, notice that the effects are in general much more dramatic for small values of ϵ.

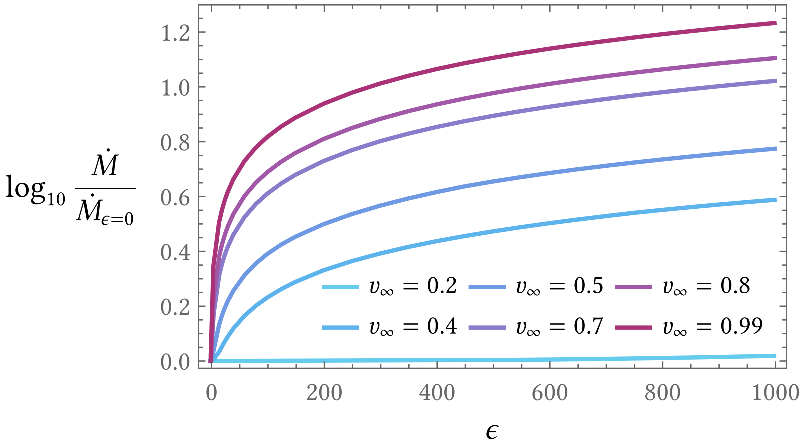


Figure 6.12: Mass accretion rate, normalized by its value in GR, with respect to ϵ. Each curve corresponds to a different value of the asymptotic velocity v_∞.

6.3.3.1 *Comparisons with simulations using* aztekas

To finish our explorations of accretion of wind in Einsteinian cubic gravity, let us compare the results of the previous model with those obtained from a full numerical simulation of the problem in the framework of relativistic hydrodynamics, which are performed using the software *aztekas* [31,32]. This is written in C under the GPL license,[5] and uses a high resolution shock-capturing scheme to solve hyperbolic partial equations in conservative form. It is able to solve the hydrodynamic equations for both non-relativistic and relativistic perfect fluids on a fixed background, and has been used extensively in the literature for different astrophysical problems, see e.g. [381, 382, 396, 397].

In order to accommodate the numerical solution for $f(r)$ computed in Section 6.1, certain aspects of the code had to be modified. In particular, we implemented a new solution for the metric which admits a general form for $f(r)$ constructed as a polynomial by parts, that interpolates between the discrete points in the solution found using the relaxation method and then continued to the interior, as previously described.

We ran simulations for $\epsilon = 0$, 5, 20, and 500, and values of v_∞ ranging from 0.2 to 0.8 in units of the speed of light. In each case, we consider a grid of 200×200 points[6] in the (r, ϕ) plane, where as always we took advantage of the spherical symmetry and fixed $\theta = \pi/2$. These two coordinates take values in the ranges

$$r \in [0.5 r_{\text{acc}},\ 10 r_{\text{acc}}] , \qquad \theta \in [0,\ \pi] , \tag{6.37}$$

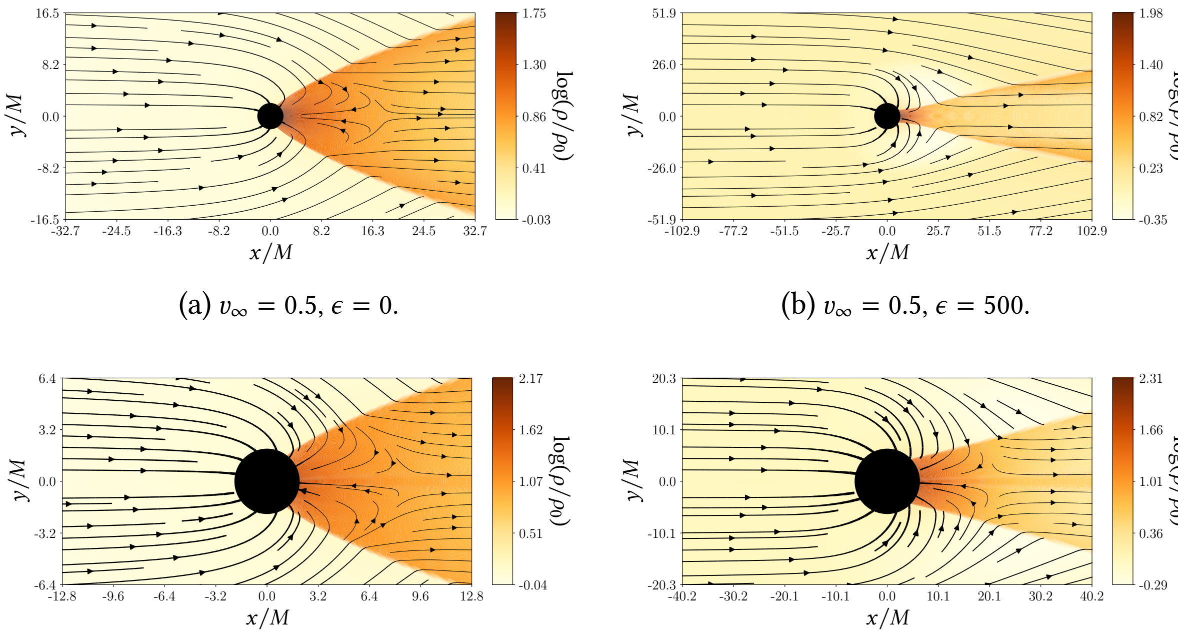
where we defined, following [381],

$$r_{\text{acc}} = \frac{r_h}{2 v_\infty^2 (1 + 1/\mathcal{M}^2)} . \tag{6.38}$$

5 The original source code can be downloaded from the repository https://github.com/aztekas-code/aztekas-main, while the forked version used in this project is located at https://github.com/AlbertoRivadulla/aztekas-main.

6 For $v_\infty = 0.8$ the resolution of the grid had to be increased in order for the simulation to converge. However, as shown in Appendix D, the results do not change in a significant manner with this.

In this expression $\mathcal{M}$ is the Mach number, related to the speed of sound at infinity, and we always take $\mathcal{M} = 5$. In Appendix D we perform some numerical tests in which we increase the size of the range (6.37), and we argue that this does not result in a relevant improvement in the simulations. Notice that for large enough values of v_∞, the minimum value of r in the range, $0.5r_{\text{acc}}$, lies inside the horizon. This means that we need to work with coordinates that are regular at $r = r_h$, so we employ the generalized Kerr-Schild system introduced in Section 6.1.3. Besides, we choose the fluid to follow a polytropic equation of state, and in all the simulations we fix the adiabatic index $\gamma = 5/3$. Finally, each integration takes place in constant time steps of size $\Delta t = C \min(\Delta r,\, r\Delta\phi)$, where C is known as the Courant factor, and its value ranges from $C = 0.4$ for $v_\infty = 0.2$ to $C = 0.1$ for $v_\infty = 0.8$, although in some cases it needed to be reduced even more to achieve convergence.



(a) $v_\infty = 0.5$, $\epsilon = 0$.

(b) $v_\infty = 0.5$, $\epsilon = 500$.

(c) $v_\infty = 0.8$, $\epsilon = 0$.

(d) $v_\infty = 0.8$, $\epsilon = 500$.

Figure 6.13: Streamlines and density on the spatial domain of the simulation, computed with *aztekas*, for representative values of v_∞ and ϵ. We see that the cubic terms tend to increase the contrast in density, while also decreasing the aperture angle of the shock cone. The width of the flow lines is proportional to the norm of the spatial velocity, $|\vec{v}| = \sqrt{v_i v^i}$.

We show the state of the fluid after some time of evolution and for different values of the parameters in Figure 6.13.[7] The incoming matter arrives from the left with an asymptotic

7 As already mentioned, the simulations are performed in the generalized Kerr-Schild coordinate system introduced in Section 6.1.3. This means that we must convert the quantities to the usual spherical coordinates for the representations, and this is relevant in particular for the radial component of the velocity, which is given by

$$v^r = \gamma^{rr}\tilde{v}_r - \frac{\beta^r}{\alpha}\,, \tag{6.39}$$

where $\tilde{v}_r$ is the velocity in Kerr-Schild coordinates computed in *aztekas*.

velocity v_∞, similarly to the models presented in Sections 6.3.1 and 6.3.2, and with an initial density ρ_0 whose value is not relevant for our computations. The mass accretion rate can be computed from these results at each step of the simulation and for any value of r, evaluating the integral [381, 398]

$$\dot{M} = 2\pi \int_0^\pi D \left(v^r - \frac{\beta^r}{\alpha} \right) r^2 \sin\phi \, d\phi \,, \tag{6.40}$$

where α and β^μ are respectively the lapse function and the shift vector in our system of coordinates given in Eq. (6.13), v^μ is the velocity of the fluid, and we introduced the combination

$$D = \rho \left(1 - \gamma_{ij} v^i v^j \right) \,. \tag{6.41}$$

Due to the conservation of the stress-energy tensor, the mass accretion rate should be the same when evaluated at any value of r outside the horizon. In the numerical computations this is not exactly true, due to the finiteness of the integration domain (see Appendix D), and in practice we compute $\dot{M}$ at each value of r and then take their average as our result.

We run the simulation for each pair of values of ϵ and v_∞ until it reaches an stationary state, for which we monitor the mass accretion rate in each step. This stabilization takes place at around $t \approx r_{\rm acc}/v_\infty$, so simulations with larger values of the asymptotic velocity of the fluid converge in less time, although the size of the time steps needs to be reduced in order to produce sensible results. The diagrams of the flow shown in Figure 6.13 correspond to this stationary state, and this is the regime that the simplified model studied earlier in this Section should be compared with.

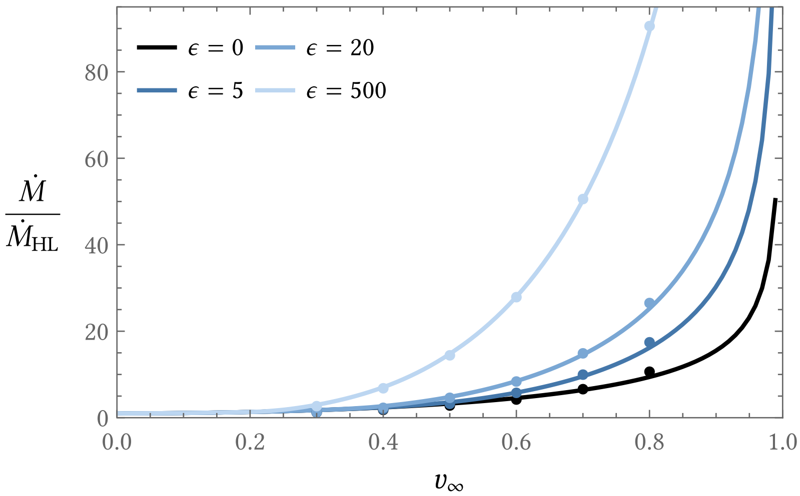


Figure 6.14: Mass accretion rate $\dot{M}$, normalized by $\dot{M}_{\rm HL}$ given in Eq. (6.27) (with $G_{\rm N} = 1$), with respect to v_∞ and for different values of ϵ. The lines correspond to the results obtained with the ballistic model presented above, while the dots are computed from the stationary flow state of each simulation performed with *aztekas*.

The results for the mass accretion rate obtained from the two models are compared in Figures 6.14 and 6.15, which show the same data in linear and logarithmic scale. Recall that for the simulations with *aztekas* we chose arbitrarily the values of the Mach number $\mathcal{M}$ and the adiabatic index γ, which do not appear in the simplified ballistic model introduced before. This

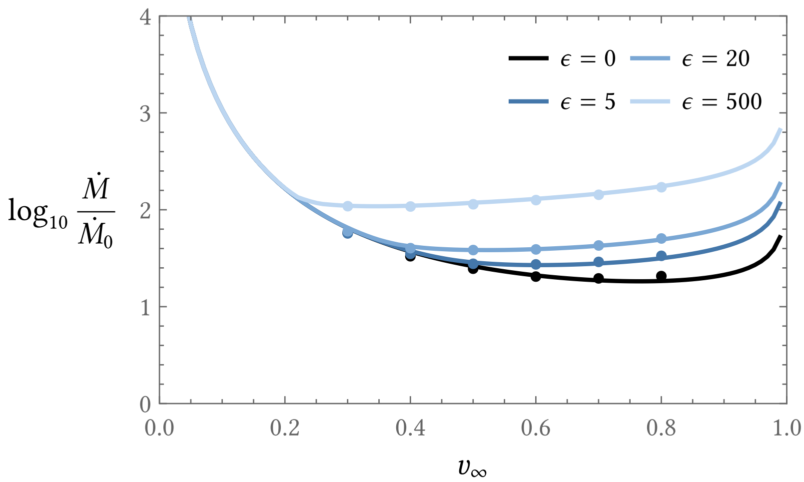


Figure 6.15: Mass accretion rate $\dot{M}$, normalized by $\dot{M}_0 = 4\pi M^2 \rho_\infty$., with respect to v_∞ and for different values of ϵ. The lines are the results obtained with the ballistic model presented above, already shown in Figure 6.11, while the dots are computed from the stationary flow state of each simulation performed with *aztekas*.

might be a source of disagreement between the two sets of results, as one would expect $\dot{M}$ to depend on those, so taking this into account we can conclude that the ballistic approximation is good enough for our purposes. Therefore, these hydrodynamic simulations provide a more robust evidence to the fact that the higher-curvature corrections, parameterized by the coupling ϵ, tend to increase the accretion rate of a black hole.

Looking at Figure 6.13, we can also notice that the cubic terms decrease the aperture angle of the shock cone, while also increasing the change in density of the accreting material. This could play a role, for example, in the mechanism of supernovae triggering due to collisions of primordial black holes with white dwarves, proposed in [399], which can give a bound on the presence of primordial black holes in the current universe.

6.4 Spherical accretion

Let us expand the study of accretion by a black hole with cubic corrections, considering now the scenario in which the black hole is immersed in an ideal gas cloud that is at rest at infinity. This is commonly referred to as "spherical accretion," and can be thought of as the $v_\infty \to 0$ limit of the previous setup, which means that we need to move away from the ballistic approximation and study explicitly the behavior of an ideal gas on the black hole background. The gas will be characterized by its asymptotic density ρ_∞ and a dimensionless temperature Θ_∞, defined as

$$\Theta = \frac{k_B T}{\bar{m}} = \frac{P}{\rho}, \tag{6.42}$$

where $\bar{m}$ is the average mass of particles in the gas, and T, P and ρ are its temperature, pressure and density, respectively. The second equality follows trivially from the classical ideal gas law.

In this Section we follow closely the treatment of [382], which considered a general relativistic extension of the Bondi model for spherical accretion, known as the Michel model [400].

First, we will consider our ideal fluid to follow a polytropic relation, which is only correct in the non-relativistic and ultra-relativistic regimes assuming we chose the correct value of the polytropic parameter. Therefore, later we will generalize the study by employing a fully relativistic equation of state, which is valid in the entire range of temperatures.

However, prior to that we review Bondi's classical model for spherical accretion of a gas to get an idea of what we should expect.

6.4.1 Bondi (classical) accretion of a polytropic fluid

The non-relativistic treatment for accretion of an ideal gas by a massive object at rest was first proposed by Bondi in [373]. The fluid is considered to follow a polytropic relation,

$$P = K\rho^{\gamma} \,, \tag{6.43}$$

where K is a constant and γ is known as the adiabatic index, which is lies in the range $1 \leq \gamma \leq 2$. Also, the gas is assumed to be in a spherically symmetric and steady-state flow, which means that its properties, such as temperature and density, do not change with time. This cloud of matter is gradually absorbed by a central attractor of mass M, due to the gravitational interaction described entirely by Newtonian gravity.

The detailed computation of the mass accretion rate in this model can be found in [382], and involves solving simultaneously the mass and momentum conservation equations of the fluid. The final result is

$$\dot{M}_{\mathrm{B}} = 4\pi\lambda_{\mathrm{B}}(GM)^2\frac{\rho_\infty}{C_\infty^3} \,, \quad \text{where} \quad \lambda_{\mathrm{B}} = \frac{1}{4}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} \,, \tag{6.44}$$

so the dependence on the adiabatic index γ is encoded entirely in the multiplicative factor λ_{B}. Notice that this expression is only valid for $\gamma \leq 5/3$. Above this λ_{B} acquires a non-zero imaginary part, implying that the Bondi treatment is not valid in that regime. The quantity C_∞ is the adiabatic speed of sound at infinity, and is defined in the classical case as

$$C \equiv \sqrt{\frac{\partial P}{\partial \rho}} = \gamma\frac{P}{\rho} = \gamma\Theta \,, \tag{6.45}$$

where we employed the polytropic relation (6.43). Therefore, it is clear that the only free parameter in the accretion rate $\dot{M}_{\mathrm{B}}$ is the asymptotic temperature of the gas, Θ_∞, once the value of the adiabatic index γ is fixed and the asymptotic density ρ_∞ is factored out.

Of course, this result is an approximation obtained with a great simplification of the gas dynamics, and should only be valid in the non-relativistic regime $\Theta_\infty \ll 1$. This will be checked explicitly later, by comparing it with the mass accretion rate obtained with the relativistic model described in the next Section.

6.4.2 Michel accretion of a polytropic fluid

We want to consider the previous problem of spherical accretion on the background a black hole with the geometry (6.28), where in principle $f(r)$ can be an unknown function. For this, we consider the Michel model of accretion, first proposed in [400], and studied in detail in [382]. A discussion of this model can also be found in Appendix G of [401].

Let us consider again an ideal gas that obeys the polytropic relation given in Eq. (6.43). The dynamics of the fluid is governed by the relativistic conservation equations

$$\begin{aligned} \nabla_\mu (\rho\, u^\mu) &= 0\,, \\ \nabla_\mu T^{\mu\nu} &= 0\,. \end{aligned} \tag{6.46}$$

Here, u^μ and $T^{\mu\nu}$ are respectively the 4-velocity and the stress-energy tensor of the fluid. Since it is ideal, the latter takes the usual form

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}\,, \tag{6.47}$$

where

$$h = 1 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = 1 + \frac{\gamma}{\gamma-1}\Theta \tag{6.48}$$

is the specific enthalpy of the fluid.[8] In this relativistic setting, the speed of sound can be defined as

$$C^2 = \frac{\rho}{h}\frac{\partial h}{\partial \rho} = \frac{\gamma}{h}\frac{P}{\rho} = \frac{\gamma}{h}\Theta\,, \tag{6.51}$$

where we used Eqs. (6.43) and (6.48). Notice that combining Eqs. (6.48) and (6.51) we can get a relation between the speed of sound and the specific enthalpy,

$$h = \frac{1}{1 - C^2/(\gamma-1)}\,, \tag{6.52}$$

which will be useful later on. With these definitions and the polytropic relation (6.43) we can write the mass density of the gas, ρ, as a function of C and h. Indeed, plugging Eq. (6.43) into Eq. (6.51) and solving for ρ we find

$$\rho(C,\, h) = \rho_\infty \left(\frac{C^2 h}{C_\infty^2 h_\infty}\right)^{1/(\gamma-1)}\,, \tag{6.53}$$

where we also imposed $\rho(C_\infty\,, h) = \rho_\infty$.

At this point, all ingredients necessary to tackle the problem are available. As said before, we consider the spacetime metric to be given by Eq. (6.2) and the flow to be spherically symmetric and steady. Then, the 4-velocity only has the components

$$u = u^t \partial_t + u^r \partial_r\,, \tag{6.54}$$

[8] The equation of conservation of the stress-energy tensor written in Eq. (6.46), with $T^{\mu\nu}$ given by Eq. (6.47), is equivalent to the relativistic version of the Euler equation. This can be expressed as [401, 402]

$$\left(\rho + \mathcal{E}' + p\right) u^\mu \nabla_\mu u^\nu = -g^{\mu\nu}\partial_\mu P - u^\mu u^\nu \partial_\mu P\,, \tag{6.49}$$

where $\mathcal{E}'$ is the internal energy density of the gas. Since it obeys the polytropic relation (6.43), this is given by [401]

$$\mathcal{E}' = \frac{P}{\gamma-1}\,, \tag{6.50}$$

and the enthalpy density is $\rho h = \rho + P + \mathcal{E}'$.

and depends only on the coordinate r. Evaluating explicitly the components of the conservation equations (6.46) we find

$$\frac{d}{dr}\left(r^2\rho\, u^r\right) = 0\,, \tag{6.55}$$

$$\frac{d}{dr}\left(r^2\rho\, h\, u_t\, u^r\right) = 0\,. \tag{6.56}$$

The second of these can be expanded in two terms using the Leibniz rule,

$$\frac{d}{dr}\left(r^2\rho\, h\, u_t\, u^r\right) = \frac{d}{dr}\left(r^2\rho\, u^r\right) h\, u_t + r^2\rho\, u^r \frac{d}{dr}\left(h\, u_t\right) = 0\,, \tag{6.57}$$

the first one being zero due to Eq. (6.55). Therefore, this implies

$$\frac{d}{dr}\left(h\, u_t\right) = 0\,. \tag{6.58}$$

Since the particles that compose the gas are massive, the two components of the velocity are related as

$$u_\mu u^\mu = -1 = -\frac{1}{f(r)}\left((u_t)^2 - (u^r)^2\right)\,, \tag{6.59}$$

where we plugged in the form of the metric (6.2). This allows us to write

$$u_t = \sqrt{u^2 + f(r)}\,, \quad \text{where} \quad u \equiv |u^r|\,. \tag{6.60}$$

Using this and integrating once, the conservation equations (6.55) and (6.58) become

$$4\pi r^2 \rho u = \dot{M} = \text{const.}, \tag{6.61}$$

$$h\sqrt{u^2 + f(r)} = h_\infty = \text{const.} \tag{6.62}$$

We are interested in a flow that is at rest asymptotically far away and regular at the black hole horizon, which implies that at some point the radial velocity of the fluid must become larger than the speed of sound. In general relativity, it is known [403, 404] that there exists a unique solution with these features, and this was shown to be true also for a broader class of metrics of the form (6.2) in [403]. However, the black hole metric that we will consider in this work, obtained by solving the equations of motion of Einsteinian cubic gravity, does not fulfill the conditions required for the proof in that article to work, in particular the condition (M4) proposed by the authors. But this does not mean that such flow does not exist in our case, and indeed we have been able to solve the equations numerically, as we will show later.

The point where the velocity of the fluid is equal to the speed of sound is known as the sonic point, and is relatively easy to identify. The mass accretion rate must be the same for any value of r, as given in Eq. (6.61), and therefore it makes sense to evaluate it at this point. Let us start by combining Eqs. (6.61) and (6.62), resulting in the expression

$$\left[1 - \frac{C^2}{u^2}\left(f + u^2\right)\right] uu' = -\frac{f'}{2} + 2\frac{C^2}{r}\left(f + u^2\right)\,. \tag{6.63}$$

At the sonic point $r = r_s$ both sides of this equation must vanish simultaneously.[9] Thus, equating them to zero and combining the resulting equations we find

$$\frac{C_s^2}{1-C_s^2} f(r_s) - r_s \frac{f'(r_s)}{4} = 0\,, \tag{6.65}$$

$$u_s^2 - \frac{C_s^2}{1-C_s^2} f(r_s) = 0\,. \tag{6.66}$$

These can be solved analytically in general relativity (see Section 2.2 of [382]), but for more general theories they need to be treated numerically, given a form of $f(r)$. In any case, with these we are able to find r_s and u_s as functions of C_s, which can then be plugged into Eq. (6.62) evaluated at the sonic point to find a relation $C_s^2 = C_s^2(h_s,\, h_\infty)$. Since C_s^2 and h_s are also related through Eq. (6.52), in the end we can find a relation $h_s(h_\infty)$, where h_∞ is given in terms of the asymptotic temperature Θ_∞ as

$$h_\infty = 1 + \frac{\gamma}{\gamma-1}\Theta_\infty\,. \tag{6.67}$$

Therefore, for given values of Θ_∞ and γ, and a solution for the function $f(r)$ in the metric, we can obtain numerically the values of h_s, C_s, u_s and r_s.

Once we know these, it is trivial to evaluate the mass accretion rate (6.61), as

$$\dot{M}_{\rm M} = 4\pi r_s^2 \rho_s u_s\,. \tag{6.68}$$

The mass density ρ_s can be written in terms of C and h at the sonic point using Eq. (6.53), so replacing this we find

$$\dot{M}_{\rm M} = 4\pi M^2 \lambda_{\rm M} \frac{\rho_\infty}{C_\infty^3}\,, \tag{6.69}$$

where we introduced the dimensionless combination[10]

$$\lambda_{\rm M} = C_\infty^3 u_s \frac{r_s^2}{M^2} \left(\frac{C_s^2 h_s}{C_\infty^2 h_\infty}\right)^{1/(\gamma-1)}\,. \tag{6.70}$$

Therefore, this mass accretion rate $\dot{M}_{\rm M}$ is determined entirely by the value of Θ_∞ (for a given adiabatic index γ), with the rest of the quantities appearing in the expressions computed numerically as explained above.

9 In order to see why this must happen, let us consider the factor multiplying u' in the LHS of Eq. (6.63) multiplied by u,

$$u^2 - C^2(f+u^2)\,. \tag{6.64}$$

At $r \to \infty$ this is equal to $-C_\infty^2$, since $u \to 0$ and $f \to 1$, whereas at $r \to r_h$ this goes to $u_h^2(1-C_h^2)$, which is positive provided $C_h^2 < 1$ as required by causality. Then, there must be a critical radius outside the horizon where this quantity is zero: the sonic point r_s. However, in order for u and u' to be regular there, we must impose the RHS of Eq. (6.63) to vanish too.

10 Notice that the constant $\lambda_{\rm M}$ does not depend on the black hole mass M, since the sonic radius r_s will be proportional to this. In practice, the numerical computations are always performed in terms of the coordinate r/M, or rather its inverse, so this scale is naturally taken out.

6.4.2.1 Numerical results with cubic corrections

In Figure 6.16 we plot the mass accretion rate for different values of the adiabatic index γ, comparing the result predicted by the classical Bondi model, with those obtained numerically from the relativistic computation presented above, varying also the cubic coupling ϵ. As mentioned below Eq. (6.44), the Bondi computation is not valid for $\gamma > 5/3$, so the corresponding value is not shown in the plot with $\gamma = 2$.

Notice that for both $\dot{M}_{\rm B}$ and $\dot{M}_{\rm M}$, given respectively by Eqs. (6.44) and (6.69), we are able to factor out the scales M and ρ_∞ dividing the accretion rate by $M^2\rho_\infty$. However, while the classical result $\dot{M}_{\rm B}$ depends trivially on the speed of sound at infinity $\mathcal{C}_\infty$, the relativistic result does not. Indeed, the quantities appearing in $\lambda_{\rm M}$ in Eq. (6.70) depend on this, which is ultimately due to the fact that in the relativistic case there exists one additional scale: the speed of light. Therefore, the mass accretion rate is plotted in Figure 6.16 with respect to Θ_∞, directly related to $\mathcal{C}_\infty$ through Eqs. (6.48) and (6.51).

In all cases it is observed that the relativistic treatment results in a larger value for the accretion rate, which increases even further when the higher-curvature terms are turned on. As expected, both models produce essentially the same result in the non-relativistic regime $\Theta_\infty \ll 1$. However, as the temperature increases the relativistic (and higher-curvature) effects become more relevant, and can indeed change the value of the mass accretion rate by more than one order of magnitude.

We can also comment on the small bump that appears at $\Theta_\infty \sim 0.1$ when the cubic coupling increases, which becomes more noticeable for larger values of γ. It seems to be a numerical artifact carrying no physical meaning, and indeed one should note that the polytropic relation (6.43) is only valid on a certain range of temperatures, depending on the value of γ. This feature is not present when considering the fully-relativistic equation of state, which is valid for any Θ_∞, as we will see in the following Section.

6.4.3 Michel accretion with a relativistic EoS

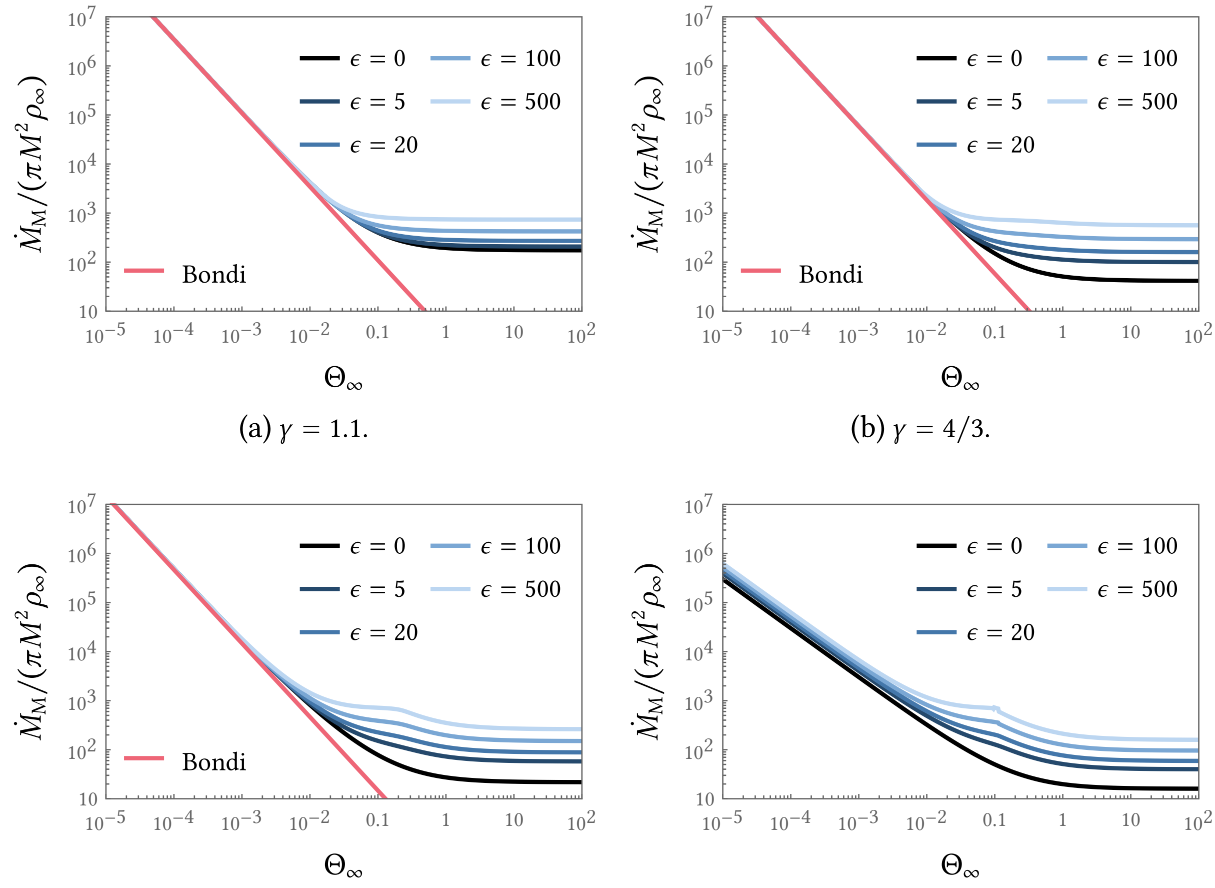
Until now, in this Section we studied the problem of a black hole accreting some ideal fluid that fulfills the polytropic relation given in Eq. (6.43). This is a simple equation of state characterized by the adiabatic index γ, whose value modifies the dynamics of the system. If the gas is monoatomic, that relation is only accurate in two relevant limits,

- non-relativistic limit $\Theta_\infty \ll 1$, described by $\gamma = 5/3$, and
- ultra-relativistic limit $\Theta_\infty \gg 1$, described by $\gamma = 4/3$,

while for intermediate values of Θ_∞ the polytropic EoS is not valid [405]. The goal of this Section is to extend the previous study of accretion to find results that are sensible in the entire range of temperatures, by replacing the polytropic relation (6.43).

The relativistic equation of state for an ideal monoatomic gas has been derived from relativistic kinetic theory, and it reads [406–408]

$$h = \frac{K_3(1/\Theta)}{K_2(1/\Theta)}, \tag{6.71}$$



(a) $\gamma = 1.1$.

(b) $\gamma = 4/3$.

(c) $\gamma = 5/3$.

(d) $\gamma = 2$.

Figure 6.16: Mass accretion rate with respect to the asymptotic temperature Θ_∞, in the setup of spherical accretion of a fluid that follows the polytropic relation (6.43), for selected values of the adiabatic index γ. In each plot, we compare the result of the non-relativistic (Bondi) model with those obtained from a relativistic treatment, with each curve corresponding to a different value of the cubic coupling ϵ.

where $K_n(x)$ is the n-th order modified Bessel function of the second kind. Imposing the adiabatic condition to the flow, it is possible to obtain the expression for the density at a given temperature,[11]

$$\frac{\rho}{\rho_\infty} = \frac{f(\Theta)}{f(\Theta_\infty)}, \qquad f(\Theta) = \Theta K_2(1/\Theta) \exp\left\{\frac{1}{\Theta}\frac{K_1(1/\Theta)}{K_2(1/\Theta)}\right\}. \tag{6.72}$$

The speed of sound is defined as in the previous case, and it reads

$$C^2 = \frac{\rho}{h}\frac{\partial h}{\partial \rho} = \frac{\bar{\gamma}}{h}\Theta, \tag{6.73}$$

[11] A derivation of Eq. (6.72) can be found at Appendix B of [409].

which can be checked using Eqs. (6.71) and (6.72), as well as some properties of the Bessel functions. By analogy to Eq. (6.51) we defined the effective adiabatic index $\bar{\gamma}$, given by

$$\bar{\gamma} = \frac{h'}{h' + \Theta^2}\,, \qquad h' \equiv \frac{d}{dx}\left[\frac{K_3(x)}{K_2(x)}\right]\,. \tag{6.74}$$

From these definitions one can find that

$$\lim_{\Theta\to 0} \bar{\gamma} = \frac{5}{3}\,, \qquad \lim_{\Theta\to\infty} \bar{\gamma} = \frac{4}{3}\,, \tag{6.75}$$

thus recovering the conditions for the validity of the polytropic equation of state, mentioned above.

The procedure to compute the accretion rate is the same as that described in Section 6.4.2: we need to identify the sonic point, where the velocity of the fluid and the local speed of sound match, and evaluate $\dot{M}$ there. This is still found by solving Eqs. (6.65) and (6.66), which were obtained from relativistic fluid dynamics alone. The EoS (6.71) will only modify the relations between the thermodynamic quantities, namely $h = h(\Theta)$ and $C = C(\Theta)$, but the computation carries on similarly. In fact, the problem becomes simpler as the adiabatic index is no longer a degree of freedom.

In practice, for given values of Θ_∞ and the cubic coupling ϵ we use Eqs. (6.65) and (6.66), together with the conservation equation (6.62), to find numerically the values of the different variables at the sonic point: Θ_s, r_s and u_s. Finally, we compute the mass accretion rate as before, evaluating Eq. (6.61) at that radius,

$$\dot{M}_M = 4\pi r_s^2 \rho_s u_s\,, \tag{6.76}$$

where ρ_s is obtained by setting $\Theta = \Theta_s$ in Eq. (6.72).

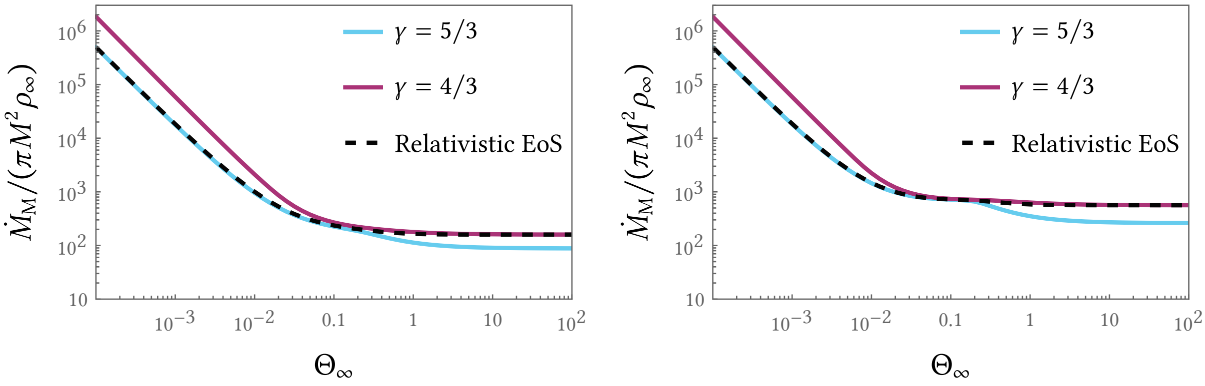
6.4.3.1 Numerical results with cubic corrections

Let us finally show and discuss the numerical results for spherical accretion of a gas with a relativistic EoS in our cubic theory. As explained in Section 6.4.2.1, the natural way to plot the results is with respect to the temperature at infinity Θ_∞, while the black hole mass M and asymptotic density ρ_∞ can be easily factored out from $\dot{M}$.

First, we compare the results for $\dot{M}_\mathrm{M}$ with a fluid that obeys either the polytropic relation (6.43) or the relativistic equation of state (6.71) in Figure 6.17. This confirms the fact that an adiabatic index $\gamma = 5/3$ is accurate for describing the fluid for low temperatures, while $\gamma = 4/3$ corresponds to higher temperatures. The relativistic EoS given in Eq. (6.48) is valid in the entire range of temperatures, and we see that its results connect smoothly those obtained in the two limiting regimes.

Next we focus on the computations corresponding to a gas whose dynamics is governed by the relativistic equation of state. First, in Figure 6.18 we plot $\dot{M}_\mathrm{M}$ with respect to the temperature Θ_∞ for different values of ϵ. We observe a similar behavior to that found in the different setups studied throughout this Chapter: the higher-curvature corrections in the action tend to increase the rate of accretion for any given temperature.

This dependence can be observed perhaps more clearly in Figure 6.19, where we plot the same data for $\dot{M}_\mathrm{M}$ now with the cubic coupling in the horizontal axis, for different values of



(a) $\epsilon = 20$.   (b) $\epsilon = 500$.

Figure 6.17: Comparison between the spherical mass accretion rate of an ideal gas described by either a polytropic or a relativistic equation of state, for two values of the cubic coupling ϵ. We can see clearly that the polytropic relation with $\gamma = 5/3$ is valid for low temperatures, while that EoS with $\gamma = 4/3$ is accurate in the ultra-relativistic regime.

Θ_∞. Notice that these curves are very similar to those shown in Figure 6.12 for accretion of wind, each corresponding to one value of the asymptotic velocity v_∞. In both cases the effect of the cubic terms becomes larger in the more relativistic regimes, corresponding respectively to higher values of Θ_∞ or v_∞.

6.5 Discussion

This last Part of the thesis is a continuation of our studies of higher-curvatures theories of gravity, but now we treated a radically different problem: the accretion of matter by an astrophysical black hole. Since we only intended to explore the implications that these higher-order corrections can have in this scenario, we chose to consider the action of cosmological Einsteinian cubic gravity, given in Eq. (6.1), as this is the simplest non-trivial modification of Einstein gravity in 4 dimensions. Also, in order to accurately describe a realistic black hole, one should allow it to have a non-zero angular momentum. However, such a solution does not exist in general for our theory, so we settled for a Schwarzschild black hole, whose construction is discussed extensively in Section 6.1.

Before delving into the problem of accretion, in Section 6.2 we took a detour to treat a couple of problems of orbital mechanics in the background of our corrected black hole. First we computed the flyby radius which, as described in [390] in the context of GR, corresponds to the minimum distance from the black hole that a probe mass initially at rest at $r \to \infty$ can reach before escaping again all the way to infinity. Then we studied the radius of the ISCO, which is the smallest stable circular orbit that a particle can describe around the black hole. This second problem had already been studied in [118] for Einsteinian cubic gravity, but here we extended their treatment to larger values of the higher-curvature coupling. The results are similar in the two cases, as both quantities tend to increase when the cubic terms are turned on. In fact, they grow with ϵ faster than the horizon radius, pointing to the relevance of higher-curvature effects outside the black hole.

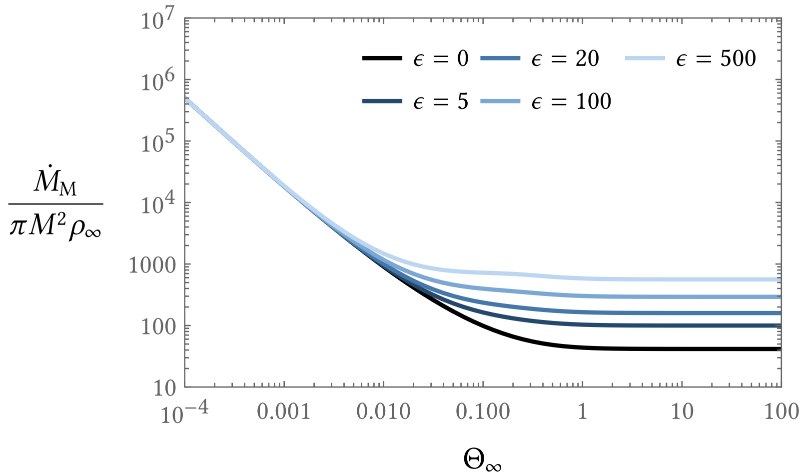


Figure 6.18: Mass accretion rate with respect to the temperature, computed with the Michel model for spherical accretion of an ideal gas that fulfills a relativistic EoS, for different values of the coupling of the cubic terms. As opposed to the results shown in Figure 6.16, obtained by considering a gas that follows a polytropic relation, the ones plotted here are valid in the entire range of temperatures.

After this discussion, we focused properly on the problem of accretion of matter. First, in Section 6.3 we considered the setup in which a black hole moves at a constant speed inside an infinite cloud of gas or dust (or equivalently, the cloud of matter moves with respect to the central object). This is usually known as accretion of wind, and we treated it by considering the simplified model of ballistic accretion, which assumes the massive particles in the gas to be subject only to the gravitational interaction, and not collide with each other. This lead us to a numerical procedure to compute the rate of accretion, which depends on the relative velocity v_∞, and the validity of the results obtained was contrasted with simulations of relativistic hydrodynamics using the software *aztekas*.

Finally, in Section 6.4 we continued the study of accretion, but now set the black hole and cloud of matter at rest with respect to each other, thus posing the scenario known as spherical accretion. We considered the exterior matter to be an ideal gas that follows a certain equation of state, and whose dynamics is determined by the asymptotic (dimensionless) temperature Θ_∞. This EoS was first taken to be a polytropic relation, which is valid only in certain regimes depending on the adiabatic index γ, and then a fully-relativistic equation of state that is correct in any case. The computation requires that there is a transonic flow solution, in which the velocity of the infalling gas reaches and surpasses the local sound speed at some point. While the existence and uniqueness of such a solution is guaranteed in GR, for a more general theory it is not, and a proof along the lines of that presented in [403] would be desired. Nonetheless, we were able to find solutions for these equations numerically in our case (although we should emphasize that the solution is not guaranteed to be unique), and from them we computed the mass accretion rate.

The results found in these last two cases are consistent with each other: the mass accretion rate increases with the magnitude of the higher-curvature corrections, controlled by the coupling constant ϵ. They also match what can be naively expected from the study of the radii of the flyby and ISCO, since one can think that increasing these would result in more matter

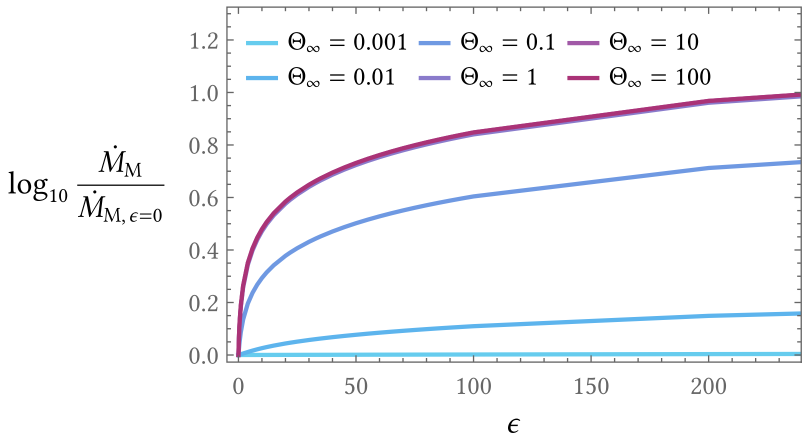


Figure 6.19: Mass accretion rate with respect to the cubic coupling, for different values of the asymptotic temperature, computed with the Michel model for an ideal gas obeying a relativistic equation of state. Notice that dependence on ϵ becomes more important as the temperature increases, and the curves for large temperatures ($\Theta_\infty = 1$, 10 and 100) are almost identical.

being absorbed by the black hole. Besides, the effect is more relevant in the ultra-relativistic limit, corresponding to having either a large v_∞ for the wind or large Θ_∞ for the spherical setup.

In order to have a noticeable deviation from the results in Einstein gravity, we need to have a large higher-curvature coupling. Although the known bounds, reviewed in Section 6.1.2, are somewhat permissive, we expect more restrictive constraints to be obtained from future experimental results. In any case, this bound is expressed in terms of the dimensionful combination ϵM^4, and therefore one could in principle increase the allowed value of the coupling ϵ by considering a smaller mass. Therefore, the study presented here is not expected to be relevant for supermassive black holes, such as those located at the centers of galaxies, but rather for smaller ones. In particular, it could be interesting to apply these results to the study of prevalence of primordial black holes, which have been theorized to contribute to the dark matter content in our universe.

Finally, a natural expansion of this study would involve adding corrections of higher orders in derivatives to the gravity action. Indeed, it has been shown in [115] that the Einstein action supplemented by an infinite series of ghost-free higher-curvature terms admits 4-dimensional black hole solutions, which actually become stable below a certain mass, as opposed to what happens in pure GR. However, we should emphasize the fact that these theories are well formulated in the EFT regime [141], and all results beyond that must be taken with caution.

Summary and conclusions

This work is mainly an exploration of the effects that certain corrections added to the the gravity action can have when considered in various physical scenarios. Throughout the Chapters that compose the main body of the thesis we have presented the different computations that have been conducted, and thoroughly commented on the results obtained. The current Chapter intends to serve as a point of convergence, summarizing those findings and offering a broader view of the presented research. This serves not only to highlight the value that this work might have, but also acknowledge its limitations, as there are some question that remain open and could be worth exploring.

It is clear that the central topic of study are higher-derivative theories of gravity. These can be constructed in different ways, but here we focused primarily on those obtained from a bottom up approach, which consists on adding general corrections to the Lagrangian and later restricting them through different physical requirements. In Chapter 1 we provided a review of theories containing contractions of the Riemann tensor that fulfill such constraints, with special emphasis on the family of Generalized Quasitopological gravities.

After that introduction, Part I focuses on applications in holography of theories with the schematic form $\mathcal{L}$(Riemann). First, in Chapter 2 we proposed a renormalization procedure that is able to regularize and provide a well-posed variational problem for any theory of this kind in an asymptotically AdS background. This is interesting, as the cancellation of divergences is necessary in order to compute quantities such as finite asymptotic charges of black holes or holographic entanglement entropies. However, we have only confirmed our prescription to be valid in up to 5 bulk dimensions since, although the construction of the counterterms is very systematic, the computations become technically complex beyond that and we have no guarantee that the undesired divergences will be cancelled out. In any case, this is something that should be considered in order to extend the range of applicability of our proposal.

Continuing the exploration of higher-curvature gravities in the context of the AdS/CFT correspondence, in Chapter 3 we studied the holographic superconductor as an illustrative example of a problem of condensed matter physics that can be treated by means of these tools. We chose the gravity action to be that of Einsteinian cubic gravity, which is the lowest-order GQG that is non-trivial in 4 dimensions. The main finding that we would like to highlight is the fact that the behavior of the system is similar to that obtained with Einstein gravity alone, which can not be taken for granted in general given the order of the equations of motion. By inspecting the results of the computations we have identified some effects of the higher-curvature terms in the system, such as the decrease in the critical temperature and the growth in the magnitude of the condensate after the phase transition. However, it is not clear what can be the actual meaning of these corrections from the point of view of the dual field theory. One could naively attribute it to a gradual restauration of the Coleman-Mermin-Wagner theorem, but this is in fact an IR effect, which would require the calculation of loops in the bulk [236, 242], and thus is not captured by the $1/N$ contributions dual to the higher-curvature terms. In any case, the study performed here should be taken as a first step towards a better understanding of such corrections in holographic condensed matter systems.

In Part II we also treat the topic of holography with higher-derivative corrections, but besides the contractions of curvature tensors considered before we now introduce a non-minimally coupled chemical potential in the dual theory. In the gravity action, this is accounted for by including contractions of a $(d-2)$-form with the curvature, as done in Chapter 4. The actual form and combination of these terms has to be restricted in order for the gravity theory to have sensible physics, and this lead us to finding the so-called Electromagnetic Quasitopological gravities. Comparing these models with the usual Einstein-Maxwell theory, we saw that they are able to describe a wider range of dual QFTs which can have modified correlators and a richer phase space, and even violate the KSS bound for the ratio of shear viscosity to density entropy. Therefore, these theories provide an interesting playground to explore different setups in holographic condensed matter physics.

The study of these theories was extended in Chapter 5, where we focused on certain holographic Rényi and entanglement entropies, and their corresponding twist operators. As a sanity check, we were able to confirm explicitly that our proposed theories fulfill certain known universal relations involving the mentioned quantities. Based on those results, we also conjectured a universal relation for the holographic entanglement entropy across a spherical entangling region in the presence of a chemical potential. We were able to test this on the general-order EQGs found in the previous Chapter, and then proved it to be true for any charged CFT. Of course, it would be interesting to compute the EE across different entangling surfaces, such as cylinders, squashed shapes or regions with corners, with these higher-order EQGs, hoping that this might help us understand better the corresponding dual CFTs.

Finally, in Part III of this thesis we departed from the domain of holography and explored a rather different scenario in the realm of astrphysics. In particular, we studied the accretion of matter by a black hole described by a gravitational action with higher-curvature terms. These hypothetical corrections would modify the dynamics of gravity most importantly in regions of very high curvature, such as near the horizon of a black hole, so it makes sense to analyze the behavior of matter in the vicinity of such an object. By proposing and treating numerically different scenarios in Chapter 6, we confirmed that higher-curvature corrections have a non-negligible impact in the process of accretion. In particular they always tend to increase the mass accretion rate, and this is more noticeable in the relativistic regimes. Even though this should affect any astrophysical massive object, due to the constraints on the higher-derivative couplings the effect is only expected to be relevant for smaller black holes, and particularly interesting for primordial ones. These are hypothetical objects that would have been formed due to density fluctuations in the early stages of the universe, and have been considered as dark matter candidates. Therefore, a corrected rate of accretion would affect the evolution of such black holes in time, thus modifying the prevalence that they might have in the current density of dark matter. In fact, one could think about combining existing bounds on their formation, which arise from the fact that they have not been observed, to provide some further constraints on the couplings of the higher-curvature terms.

To sum up, this thesis demonstrates the viability of certain higher-derivative theories in studying different physical scenarios of interest, ranging from holography to astrophysics.

While many unknowns still remain, we expect that this work can contribute to a deeper understanding of these models, with the ultimate goal to learn about an UV complete theory of gravity.

Supplementary material

A

Extrinsic geometry and the Gauss-Codazzi equations

In Chapter 2 we need to split our coordinate system into those normal and tangent to a given hypersurface (in this case the AdS boundary). This requires that we perform a consistent decomposition of the metric, and we must be able to write curvatures of the global manifold in terms of quantities defined on the lower-dimensional submanifold.

The formalism presented in this Appendix is very well explained in the textbooks [410] and [411]. The first of these introduces it in the context of the $3 + 1$ decomposition, which is suitable for numerics as it splits the metric in its temporal and spatial parts, this is, the hypersurface is spacelike and has codimension 1. Another classical reference is [412], which can serve as a guide to extend this to hypersurfaces of larger codimension, although this will not be necessary for the purposes of this thesis. Our conventions differ from those in the references, but they are consistently taken into account in the computations shown in the main text.

Let us consider a hypersurface Γ embedded in a D-dimensional Lorentzian manifold $\mathcal{M}$, with coordinates x^μ and metric $g_{\mu\nu}$. The worldvolume of Γ is parameterized by some coordinates y^i, with $i = 1, 2, \ldots, d$, being $d < D$ is the dimension of the submanifold. We will consider $d = D - 1$, this is, Γ has codimension 1 with respect to $\mathcal{M}$. The embedding can be described by a set of relations of the form $x^\mu = x^\mu(y^i)$, and the vectors that form a basis of the space tangent to Γ are given as

$$e^\mu_i = \frac{\partial x^\mu}{\partial y^i}\,, \tag{A.1}$$

which has an index in the target space $\mathcal{M}$ and another one in the submanifold Γ. With these we can compute the induced metric as the projection of $g_{\mu\nu}$ on the hypersurface,

$$h_{ij} = e^\mu_i e^\mu_j g_{\mu\nu}\,, \tag{A.2}$$

and the inverse induced metric is defined such that $h_{ik}h^{kj} = \delta^j_i$. This can also be used to compute the inverse of the basis vectors as $e^i_\mu = h^{ij} g_{\mu\nu} e^\nu_j$. In order to span the full target manifold $\mathcal{M}$ we need to find also a normal vector n^μ,[1] which fulfills

$$e^i_\mu n^\mu = 0\,, \qquad n_\mu n^\mu = \epsilon = \pm 1\,. \tag{A.3}$$

[1] If the codimension of the submanifold Γ were larger than 1 there would be more than one normal direction, and thus we would have to introduce indices for these, labeling the normal vectors as for example n^μ_A.

Once we know the form of all these vectors, the metric of the global manifold can be separated in the normal and tangent parts as

$$g_{\mu\nu} = n_\mu n_\nu + e^i_\mu e^j_\nu h_{ij}\,, \tag{A.4}$$

and similarly for the inverse metric $g^{\mu\nu}$.

Let us consider a vector field A^μ defined on the global manifold, but that only has components in the directions parallel to Γ, this is, it can be decomposed as $A^\mu = e^\mu_i A^i$ and also $A^\mu n_\mu = 0$. We can define the intrinsic covariant derivative as the projection of the covariant derivative $\nabla_\mu A_\nu$ on Γ [411],

$$\tilde{\nabla}_i A_j = e^\mu_i e^\nu_j \nabla_\mu A_\nu\,, \tag{A.5}$$

and with the ingredients that we have it can be shown that $\tilde{\nabla}_i$ is the covariant derivative compatible with the induced metric, $\tilde{\nabla}_i h_{jk} = 0$. If we instead look at the normal components of the vector $e^\nu_i \nabla_\nu A^\mu$ we find [411]

$$e^\nu_i \nabla_\nu A^\mu = e^\mu_j \tilde{\nabla}_i A^j + A^j K_{ij} n^\mu\,, \tag{A.6}$$

where we defined the extrinsic curvature, or second fundamental form of Γ,[2]

$$K_{ij} = -e^\mu_i e^\nu_j \nabla_\nu n_\mu = n_\mu e^\nu_i \nabla_\nu e^\mu_j\,, \tag{A.7}$$

where the second expression comes from the orthogonality condition $n_\mu e^\mu_j = 0$. It can be proved that this tensor is symmetric, $K_{ij} = K_{ji}$, and in the coordinates of the global manifold $\mathcal{M}$ it is given by

$$K_{\mu\nu} = -\nabla_\mu n_\nu + n_\mu a_\nu\,, \tag{A.8}$$

where $a_\nu = n^\alpha \nabla_\alpha n_\nu$ is the "acceleration" tensor. From this last expression one can also show that

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n h_{ij}\,, \tag{A.9}$$

where $\mathcal{L}_n$ is the Lie derivative with respect to the normal vector field n^μ. For simplicity, in Chapter 2 we will compute the extrinsic curvature as given in Eq. (A.9). For completion, since K_{ij} is a tensor on the hypersurface Γ, its indices have to be raised with the induced metric, and in particular its trace is given by $K = h^{ij}K_{ij}$.

Using the definitions introduced until now, we can obtain the Gauss-Codazzi equations. These relate the different components of the Riemann tensor of the global metric, $R_{\mu\nu\rho\sigma}(g_{\mu\nu})$, to those of the curvature of the induced metric, $\mathcal{R}_{ijkl}(h_{ij})$, and they read

$$e^\mu_i e^\nu_j e^\rho_k e^\sigma_l R_{\mu\nu\rho\sigma} = \mathcal{R}_{ijkl} + \epsilon\left(K_{il}K_{jk} - K_{ik}K_{jl}\right)\,, \tag{A.10}$$

$$n^\mu e^\nu_i e^\rho_j e^\sigma_k R_{\mu\nu\rho\sigma} = \tilde{\nabla}_j K_{ki} - \tilde{\nabla}_k K_{ji}\,. \tag{A.11}$$

Apart from these, by considering a projection with two normal and two tangent vectors one obtains the so-called Ricci equation,

$$n^\mu e^\nu_i n^\rho e^\sigma_j R_{\mu\nu\rho\sigma} = \mathcal{L}_n K_{ij} + K_{ik}K^k{}_j - a_i a_j + \tilde{\nabla}_j a_i\,. \tag{A.12}$$

[2] Notice that the sign of K_{ij} differs from that in [411], but it is the same as in [410]. As already emphasized, the conventions used in Chapter 2 match those presented here.

A detailed derivation for each of these expressions can be found in [410].

Let us finish by writing the contracted form of the Gauss-Codazzi equations, which are straightforward to derive from Eqs. (A.10) an (A.11) using the machinery presented here,

$$R_{ij} = \mathcal{R}_{ij} + n^\mu n^\nu R_{i\mu j\nu} + \epsilon \left(K_{ik}K^k{}_j - KK_{ij} \right) , \tag{A.13}$$

$$R = \mathcal{R} + 2n^\mu n^\nu R_{\mu\nu} + \epsilon \left(K^{ij}K_{ij} - K^2 \right) , \tag{A.14}$$

$$n^\mu R_{\mu i} = \tilde{\nabla}_i K - \tilde{\nabla}_j K^j{}_i \, . \tag{A.15}$$

While this review is not exhaustive by any means, the relations presented here should be enough to understand the computations in Chapter 2 of this thesis.

B

Conditions to leave $g_{ij}^{(1)}$ and $g_{ij}^{(3)}$ undetermined

The universal renormalization procedure proposed in Chapter 2 requires certain assumptions about the AdS asymptotics of the bulk. In particular, it can only work provided the odd terms in the Fefferman-Graham expansion of the induced metric (2.14) vanish, this is,

$$g_{ij}^{(1)}(x) = 0\,, \qquad g_{ij}^{(3)}(x) = 0\,. \tag{B.1}$$

In Section 2.2 it is shown that this is indeed the case in general, except for a measure-zero subset of theories that can be of interest in some particular scenarios. Some quadratic-curvature theories that admit $g_{ij}^{(1)} \neq 0$ are mentioned at the end of Section 2.2.1, but in this Appendix we extend that discussion to cubic theories. Afterwards we do a similar analysis for finding Lagrangians that admit $g_{ij}^{(3)} \neq 0$.

B.1 Conditions to have $g_{ij}^{(1)}$ undetermined in cubic theories

Let us consider the most general cubic theory constructed from the Einstein-Hilbert action supplemented by all independent terms that are cubic in the curvature, each of them multiplied by its own (for the moment arbitrary) coupling constant λ_i,

$$\begin{aligned} S = \int_M d^d X \sqrt{-G}\Bigg(&\frac{1}{16\pi G_{\rm N}}\,(R - 2\Lambda_0) + \lambda_1 R_{\gamma\ \delta}^{\ \mu\ \nu} R_{\mu\ \nu}^{\ \rho\ \sigma} R_{\rho\ \sigma}^{\ \gamma\ \delta} + \lambda_2 R_{\mu\nu}^{\ \ \ \gamma\delta} R^{\mu\nu\rho\sigma} R_{\rho\sigma\gamma\delta} \\ &+ \lambda_3 R^{\mu\nu} R_{\mu}^{\ \rho\sigma\gamma} R_{\nu\rho\sigma\gamma} + \lambda_4 R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \lambda_5 R^{\mu\rho} R^{\nu\sigma} R_{\mu\nu\rho\sigma} \\ &+ \lambda_6 R^{\mu\nu} R_{\nu\rho} R_{\mu}^{\ \rho} + \lambda_7 R_{\mu\nu} R^{\mu\nu} R + \lambda_8 R^3 \Bigg)\,. \end{aligned} \tag{B.2}$$

The value of the constant $C(L)$ that appears in the asymptotic behavior of $P^{\mu\nu}_{\rho\sigma}$ can be computed using Eq. (2.12),

$$\begin{aligned} C(L) = \frac{1}{16\pi G_{\rm N}} + \frac{3(D-1)}{L^4}\Bigg(&\frac{D-2}{D-1}\lambda_1 + \frac{4}{D-1}\lambda_2 + 2\lambda_3 + 2D\lambda_4 + (D-1)\lambda_5 \\ &+ (D-1)\lambda_6 + D(D-1)\lambda_7 + D^2(D-1)\lambda_8 \Bigg)\,, \end{aligned} \tag{B.3}$$

where L is the effective AdS radius, defined through the near-boundary behavior of the Riemann tensor (2.10), and depends on the couplings λ_i and the cosmological constant Λ_0. The equations of motion at order z (2.27), (2.28) and (2.29) are determined by the constants $a^{(1)}(L)$ and $b^{(1)}(L)$, which in this case are given by

$$\begin{aligned} a^{(1)}(L) = \frac{1}{16\pi G_{\mathrm{N}}} + \frac{1}{L^4}\big(6(D-3)\lambda_1 + 36\lambda_2 + 2(7D-9)\lambda_3 + 10D(D-1)\lambda_4 + (5D^2-13D+9)\lambda_5 \\ + (6D^2-15D+9)\lambda_6 + D(4D^2-9D+5)\lambda_7 + 3D^2(D-1)^2\lambda_8\big)\,, \\ b^{(1)}(L) = -\frac{1}{16\pi G_{\mathrm{N}}} + \frac{1}{L^4}\big(6\lambda_1 + 12\lambda_2 + 2(D+5)\lambda_3 - 2(D^2-17D+16)\lambda_4 - (D^2-15D+17)\lambda_5 \\ + 9(D-1)\lambda_6 - (2D^3-23D^2+37D-16)\lambda_7 - 3D(D-1)^2(D-8)\lambda_8\big)\,. \end{aligned}$$

As explained in Section 2.2.1, the equations of motion imply $g_{ij}^{(1)} = 0$ unless one of the conditions discussed below Eq. (2.31) is met. Thus, only for both $a^{(1)}(L) = b^{(1)}(L) = 0$, $g_{ij}^{(1)}$ is fully unconstrained by the equations of motion, which happens only in a zero measure region of the space of parameters λ_i. Besides, as what was found for quadratic theories of gravity, the conditions $a^{(1)}(L) = b^{(1)}(L) = 0$ end up implying that the corresponding cubic theory has degenerate AdS vacua.

In the case of cubic curvature gravity, let us quote two examples of usually well-behaved theories which have the above properties:

- Cubic Lovelock theory in general dimensions [2], which is non-trivial only for $D \geq 7$. This corresponds to setting in the action (B.2) the values of the couplings to

$$\begin{gathered} \lambda_1 = -8\mu\,, \quad \lambda_2 = 4\mu\,, \quad \lambda_3 = -24\mu\,, \quad \lambda_4 = 3\mu\,, \quad \lambda_5 = 24\mu\,, \\ \lambda_6 = 16\mu\,, \quad \lambda_7 = -12\mu\,, \quad \lambda_8 = \mu\,. \end{gathered} \tag{B.4}$$

  The particular value of μ that results in $a^{(1)}(L) = b^{(1)}(L) = 0$ is

$$\mu = -\frac{1}{16\pi G_{\mathrm{N}}}\frac{L^4}{3(D-3)(D-4)(D-5)(D-6)}\,, \tag{B.5}$$

  which corresponds to the critical value [104].

- Einsteinian cubic gravity, first proposed in [3]. We could consider the Einstein-Hilbert Lagrangian supplemented by the term $\mu_{\mathcal{P}}\mathcal{P}$, where $\mathcal{P}$ is given by Eq. (1.24). This amounts to setting

$$\lambda_1 = 12\mu_{\mathcal{P}}\,, \qquad \lambda_2 = \mu_{\mathcal{P}}\,, \qquad \lambda_5 = -12\mu_{\mathcal{P}}\,, \qquad \lambda_6 = 8\mu_{\mathcal{P}}\,, \tag{B.6}$$

  while the remaining couplings are equal to zero. The coefficient $g_{ij}^{(1)}$ becomes undetermined at the critical value of the coupling

$$\mu_{\mathcal{P}} = \frac{1}{16\pi G_{\mathrm{N}}}\frac{L^4}{12(D-3)(D-6)}\,, \tag{B.7}$$

  which corresponds to the critical value found in [119] when studying the AdS vacua of the theory in 4 dimensions. This regime is explored in Section 3.2 of this work, in the context of the holographic superconductor with a higher-curvature gravity action.

One could also consider the Lagrangian density $\mathcal{C}$, defined in [6]. In particular, the combination $\mathcal{P} - 8\mathcal{C}$ in $D = 4$, introduced in [76, 78] due to its interesting cosmological properties, also leaves the coefficient $g_{ij}^{(1)}$ undetermined for the value of the coupling (B.7). This is totally expected, since it is known that $\mathcal{C}$ does not modify the AdS vacuum in four dimensions.

Apart from these particular cases that we were able to find, it would be interesting to explore whether the condition $a^{(1)}(L) = b^{(1)}(L) = 0$ can be used to look for new theories in higher dimensions and of higher orders in the curvature, with analogous behavior to the examples discussed here.

B.2 Conditions to have $g_{ij}^{(3)}$ undetermined in quadratic and cubic theories

In what follows we simply give the values of the constants $a^{(3)}(L)$ and $b^{(3)}(L)$ introduced in Section 2.2.3 for different families of theories. These appear in the projected equations of motion at third order in z, Eqs. (2.51) and (2.52).

First let us consider the general quadratic gravity action as introduced in Eq. (2.32). These constants are expressions of the AdS radius L and the couplings in the Lagrangian, and read

$$\begin{aligned} a^{(3)}(L) &= \frac{3}{16\pi G_{\rm N}} + \frac{1}{L^2}\Big(-3(5D-14)\alpha_1 - 6D(D-1)\alpha_2 - 6(D-3)(D-4)\alpha_3 \Big)\,, \\ b^{(3)}(L) &= \frac{3}{16\pi G_{\rm N}} + \frac{1}{L^2}\Big(-3(D-6)\alpha_1 + 6(D^2-9D+24)\alpha_2 + 6(D-3)(D-4)\alpha_3 \Big)\,. \end{aligned} \tag{B.8}$$

For the general theory that contains cubic contractions of the curvature tensors, whose action is written in Eq. (B.2), these constants take the values

$$\begin{aligned} a^{(3)}(L) = \frac{3}{16\pi G_{\rm N}} + \frac{1}{L^4}\Big(& 36\lambda_1 + 36(4D-17)\lambda_2 + 6(4D^2-13D-9)\lambda_3 \\ & + 6D(D-1)(4D-15)\lambda_4 + 9(3D^2-13D+13)\lambda_5 \\ & + 9(D-1)(4D-13)\lambda_6 + 9D(D-1)(2D-5)\lambda_7 + 9D^2(D-1)^2\lambda_8 \Big)\,, \\ b^{(3)}(L) = -\frac{3}{16\pi G_{\rm N}} + \frac{1}{L^4}\Big(& 18(D-4)\lambda_1 + 36\lambda_2 + 30(D-3)\lambda_3 - 6(D^2-33D+96)\lambda_4 \\ & + 9(D^2-D-9)\lambda_5 + 9(D-1)(2D-7)\lambda_6 \\ & + 9(D-1)(9D-32)\lambda_7 - 9D(D-1)(D^2-17D+48)\lambda_8 \Big)\,. \end{aligned} \tag{B.9}$$

Studying the particular points where these constants vanish, implying that $g_{ij}^{(3)}$ could be non-zero, might lead us to theories whose dynamics differs from that of Einstein gravity. However, such thorough study is out of the scope of the current work.

C

Numerical relaxation method for differential equations

Let us explain the basics of the numerical relaxation method [247], used for computing the different fields in the holographic superconductor in Chapter 3 and to obtain the function $f(r)$ in the metric for Chapter 6. The basic goal of this procedure is to convert an arbitrarily complicated system of differential equations into a single matrix equation, which can be solved more or less easily using methods of linear algebra.

Suppose a linear differential equation of the form

$$L\left[y(x)\right] = J(x)\,, \quad \text{with} \quad x \in [a,\, b]\,. \tag{C.1}$$

The interval of values of the independent variable, $[a, b]$, must be finite, and therefore can be discretized in N points x_i, with $i = 1, \ldots, N+1$. Then we can construct a vector $\vec{y}$ such that $y_i \equiv y(x_i)$, and similarly for the source terms $J(x)$. In this discrete form, derivative operators of any order n, ∂_x^n, are expressed as $(N+1)\times(N+1)$ matrices[1] D_n, but it is also possible to replace a derivative operator of order n by the product of n first-derivative matrices,

$$\partial_x^n \;\longrightarrow\; D_n = (D_1)^n\,. \tag{C.2}$$

If the operator $L\left[y(x)\right]$ contains a term of the form $f(x)y(x)$, this $f(x)$ should be replaced also by a discrete version F, given by the matrix

$$F = \mathrm{diag}\left(f(x_1),\, \ldots,\, f(x_{N+1})\right)\,. \tag{C.3}$$

If this appears as $f(x)\partial_x^n y(x)$, it should be replaced by the matrix product FD_n, and so on. After performing all the substitutions, the total operator L is transformed into a matrix M by adding together each of the individual terms of the differential equation. Then, the final system to solve takes the form

$$M \cdot \vec{y} = \vec{J}\,. \tag{C.4}$$

We need to apply also the boundary conditions of the system. As mentioned in the main text, one advantage of this method is that it is very natural to impose conditions at any point of the interval, since one only needs to modify some components of M and $\vec{J}$. Let us see how to enforce them explicitly:

[1] These operators are constructed automatically by a function in the software *Mathematica*, which yields a matrix that computes the order n derivative at each point using an arbitrary number of nearby points. Their form can also be computed algorithmically from the coefficients of the Taylor expansion.

1. If there is a condition of the form $y(x_j) = A$, set

$$M_{ij} = 0 \quad \text{if} \quad i \neq j\,, \qquad M_{jj} = 1\,, \qquad J_j = A\,. \tag{C.5}$$

2. If the condition is of the form $\partial_x^n y(x_j) = B$, set

$$M_j = (D_n)_j\,, \qquad J_j = B\,. \tag{C.6}$$

Once the system is formulated in the form (C.4), it is straightforward to solve it for $\vec{y}$ using any standard method for linear equations.

The described implementation works for linear differential equations, as it relies on the use of linear operators. The generalization to non-linear ODEs is slightly more involved, but also possible. In order to show how, let us start by considering a non-linear differential equation

$$E\,[y(x)] = J(x)\,. \tag{C.7}$$

The solution can be found iteratively starting from an initial seed y_0, by expanding the solution as $y(x) = y_0(x) + \delta y(x)$, where $\delta y(x)$ is the change on the solution after one iteration. After replacing this in Eq. (C.7) and expanding to first order in $\delta y(x)$, it becomes

$$E\,[y_0(x)] + \delta E\,[\delta y(x)] = J(x)\,, \tag{C.8}$$

where $\delta E\,[\delta y(x)]$ is a linear differential operator acting on $\delta y(x)$, and $E\,[y_0(x)]$ can be considered part of the inhomogeneous term. Then, $\delta y(x)$ can be computed using the method described above for linear equations, and at the end one updates the total solution $y(x)$, which becomes the seed for the next iteration. This process is repeated until the solution converges, this is, $\delta y(x)$ becomes sufficiently small at every point.

The method can be straightforwardly extended for coupled differential equations. For this, it is enough to concatenate the vectors formed by discretizing every function $y^{(a)}(x)$ that we need to solve in one single vector,

$$\vec{y} = \left(y^{(1)}(x),\, y^{(2)}(x),\, \dots,\, y^{(n)}(x)\right)\,. \tag{C.9}$$

The matrix M also becomes more complex, since now each differential equation can depend on all the functions involved. It can be constructed by blocks, with each corresponding to one of the equations of the system, following the same procedure as before, and once this is done the rest of the computation carries on as usual.

Finally, one can also generalize the method for solving partial differential equations. However, since this is not necessary for the present work we do not explain it here, and we refer the reader to [247].

D

Tests of the numerical simulations with *AZTEKAS*

In this Appendix we explore the consequences of changing different parameters of the numerical simulations performed in Section 6.3.3.1. This is intended to support the previous results, since as we will show extending the integration domain or changing its resolution after some point does not produce any effect of relevant magnitude.

The data shown in the following plots were obtained in simulations with $\epsilon = 20$ and $v_\infty = 0.5$, and an integration region of 200×200 points, except those parameters whose value is explicitly stated. Also, as in the main text we set the Mach number to $\mathcal{M} = 5$ and the polytropic index $\gamma = 5/3$. In some Figures we plot the mass accretion rate with respect to the time t, which is the variable that keeps track of the progress of the simulation. This allows us to see explicitly how $\dot{M}$ stabilizes and fluctuates around a constant value for large t. In order to obtain the results in Section 6.3.3.1 we allowed it to evolve for a while after it had stabilized, and then computed an average over a small period of time.

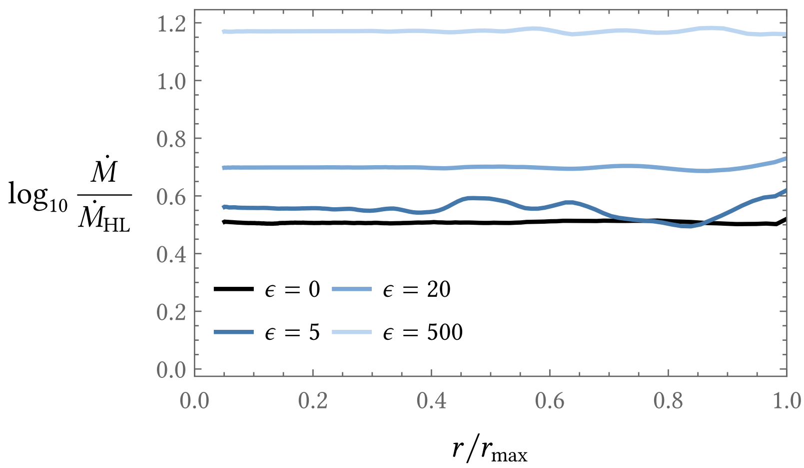


Figure D.1: Accretion rate measured over the entire domain of the radial coordinate, for $v_\infty = 0.5$ and in the stationary regime of the simulation.

First, in Figure D.1 we show that the accretion rate computed with Eq. (6.40) is not entirely constant for all values of r. As mentioned in the main text, this can be understood as a consequence of the finiteness of the integration domain, which means that there can be some matter that goes outside this, which the simulation cannot account for. Therefore, for the actual results presented in the main text we opted for computing the average of the value of $\dot{M}$ in the entire domain.

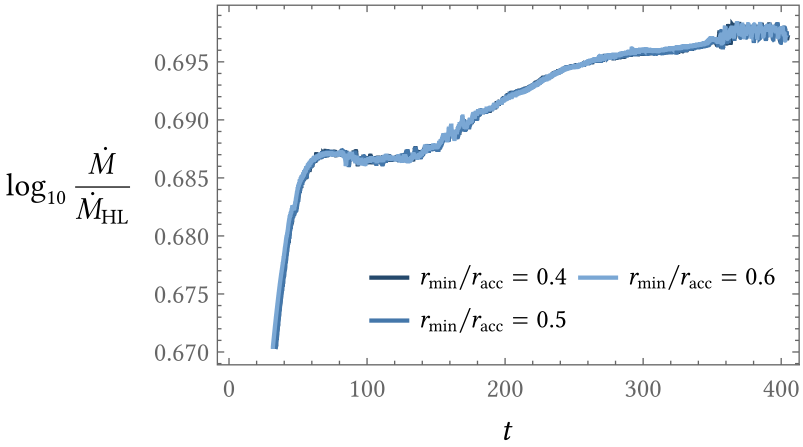


Figure D.2: Change of the accretion rate with the value of r_{min}, for $v_\infty = 0.5$ and $\epsilon = 20$. The values plotted are always measured at $r = r_h$.

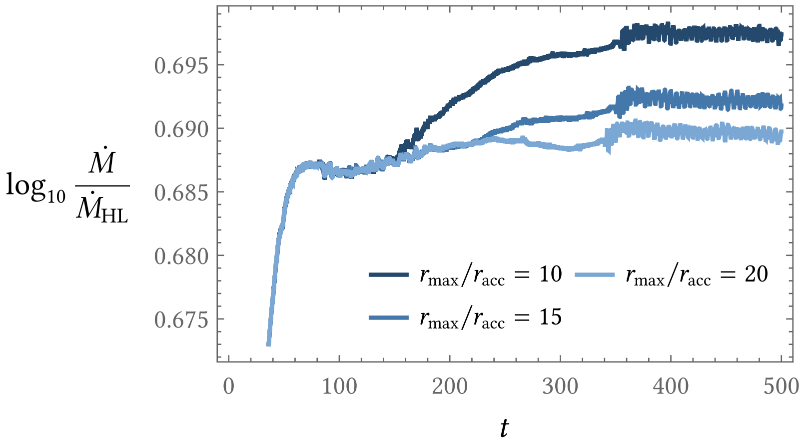


Figure D.3: Change of the accretion rate with the value of r_{max}, for $v_\infty = 0.5$ and $\epsilon = 20$. The values plotted are always measured at $r = r_h$. Although the curves separate as the simulation converges, the actual change in $\dot{M}$ is very small, so in general this effect is irrelevant.

In Figures D.2 and D.3 we show the variation of $\dot{M}$ when we change the size of the domain of integration, by moving the lower and upper limits of the range (6.37). Although the accretion rate changes more when increasing r_{max}, in practice both effects are almost negligible for our purposes.

Finally, in Figure D.4 we study the change of the accretion rate with the amount of points in the domain of integration. Of course, we expect to obtain better results with a higher resolution of this spatial grid, but again by inspecting the plot we see that these are very irrelevant overall.

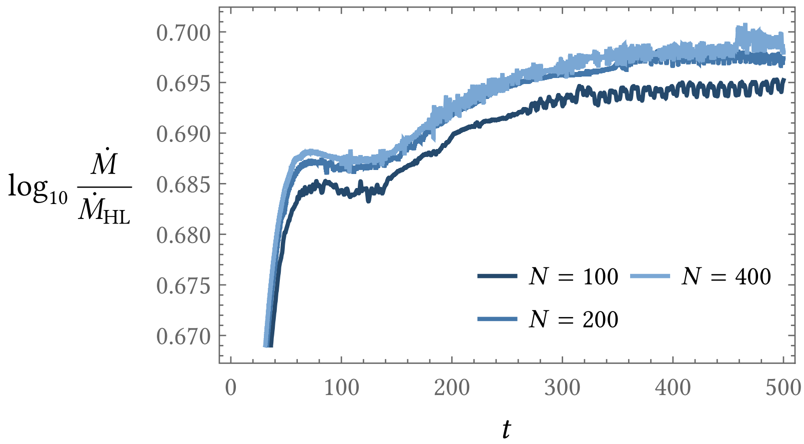


Figure D.4: Change of the accretion rate with the resolution of the numerical domain, for $v_\infty = 0.5$ and $\epsilon = 20$. In each case, the grid has $N \times N$ points in the coordinate system (r, θ).

List of reproduced publications

The results presented in this document are the outcome of four years of work, during which several scientific articles have been published. In this Chapter we collect, in chronological order, those on which this thesis is based. The content of each of them is partially reproduced in the main Chapters as mentioned.

Universal renomalization procedure for higher curvature gravities in $D \leq 5$, *JHEP* **09** (2021) 142 [24]

Authors

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PhD student contribution

Active participation in all the calculations included in the publication, discussions and meetings.

Included at

Chapter 2. Renormalization in higher-curvature gravities with $D \leq 5$.
Appendix B. Conditions to leave $g_{ij}^{(1)}$ and $g_{ij}^{(3)}$ undetermined.

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Active participation in all the calculations included in the publication, discussions and meetings.

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Appendix C. Numerical relaxation method for differential equations.

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Higher-derivative holography with a chemical potential, *JHEP* **07** (2022) 010 [26]

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PhD student contribution

Active participation in all the calculations included in the publication, discussions and meetings.

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Chapter 5. Charged Rényi and entanglement entropies.

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Universal Feature of Charged Entanglement Entropy, Phys. Rev. Lett **129** (2022) 021601 [27]

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PhD student contribution

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This thesis investigates the role of higher-derivative corrections in the gravity action across various scenarios. The first part focuses on the domain of holography, and in this context we propose a method for holographic renormalization valid for general theories in up to 5 dimensions. Subsequently, we employ the framework of AdS/CFT to examine the effect of cubic-curvature terms in a system known as the holographic superconductor. Besides the curvature tensors, it is possible to construct corrections with contractions of other fields. In particular, we consider theories that are dual to a CFT with a non-minimally coupled chemical potential. Finally, the thesis transitions into the field of astrophysics, and we study the impact of higher-curvature terms in processes of accretion of matter by black holes.