Equivariant Cohomology and Cohomological Field Theories

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[W88] Witten 1988: YM_4^{Top}

$$N=2 \ \text{Twisted Supersymmetry} \ \rightarrow S(a,\psi,\phi;\ldots)$$

invariant under

$$\begin{array}{rcl} Q\,a &=& \psi \\ Q\,\psi &=& D_a\phi \\ Q\,\phi &=& 0 \end{array} \quad ; \quad \begin{array}{lccc} & a & \psi & \phi \\ form\ d^o & 1 & 1 & 0 \\ ghost\ \# & 0 & 1 & 2 \end{array}$$

$Q^2=0$ on gauge invariant combinations of a,ϕ,ψ in fact: $Q^2=$ gauge transformation of parameter ϕ

$$Q\ tr\,(F(a)+\psi+\phi)^2 = -d\ tr\,(F(a)+\psi+\phi)^2$$

where: $F(a)=da+\frac{1}{2}[a,a]$. Remark:

$$\int_\Sigma tr\ F\wedge F = Q\Lambda(\text{gauge inv.}(a,\psi,\phi;\ldots)+S(a,\psi,\phi;\ldots)$$

[BS88] Baulieu, Singer 1988

$$\begin{aligned}
Q &\to s \quad s^2 \equiv 0 \\
sa &= \psi - D_a\,\omega \\
s\psi &= D_a\phi + [\psi,\omega] \qquad \omega : \text{gauge Faddeev Popov ghost} \\
s\phi &= [\phi,\omega] \\
s\omega &= -\frac{1}{2}[\omega,w] + \phi \\
S(a,\psi,\phi) &= \int_\Sigma trF\wedge F + s\Lambda(a,\psi,\phi;\ldots)
\end{aligned}$$

defect: s has no cohomology ! (Observables ???)

[H89] J. Horne 1989

$S(a,\psi,\phi;\ldots)$ is not only gauge invariant but also does not depend on ω

→ Supersymmetric gauge invariance

[OSB89] S. Ouvry, R. Stora, P. van Baal (1989)

On fields a,ψ,ϕ, the action of infinitesimal gauge transformations is given by:

$$\delta_\lambda = [s, \mathcal{J}(\lambda)]_+$$

$$\mathcal{J}(\lambda)\omega = \lambda \qquad \mathcal{J}(\lambda) \text{ other } = 0$$

The cohomology of s restricted to δ_λ and $\mathcal{J}_\lambda$ invariant objects is non empty and contains Witten's example.

This set up has been known since the 50's under the name of "basic cohomology" or "equivariant cohomology".

[K93] J. Kalkman (1993)

has analysed OSB in abstract terms and made a very nice algebraic discovery possibly burried in the mathematical literature but certainly not widely known.

There, s is interpreted as follows:

$a \in \mathcal{A}$	:	space of connections on some $\mathcal{G}$-bundle $P(\Sigma, G)$(with structure group G)
δ	=	differential on $\mathcal{A}$
ω,ϕ	:	generators of a Weil algebra for $\mathcal{G}$, the gauge group:

$$\delta_W \omega = -\frac{1}{2}[\omega, \omega] + \phi$$
$$\delta_W \phi = [\phi, \omega]$$

$i(\lambda), l(\lambda)$, action of $\mathcal{G}$ on $\mathcal{A}$ inherited from the action of $\mathcal{G}$ on P one may interpret

$$s = \delta + \delta_W + l(\omega) - i(\phi)$$
$$\psi = \delta a$$

If $\mathcal{J}(\lambda), \mathcal{L}(\lambda)$ is the action on the Weil algebra:

$$\mathcal{J}(\underline{\lambda})\omega = \lambda \quad \mathcal{J}(\lambda)\phi = 0$$
$$\mathcal{L}(\underline{\lambda})\omega = [\lambda, w] \quad \mathcal{L}(\lambda)\phi = [\lambda, \phi]$$
$$l(\lambda) + \mathcal{L}(\underline{\lambda}) = [s, \mathcal{J}(\lambda)]$$

There is another interpretation, the so-called Weil interpretation [K93]

$$s = \delta + \delta_W \qquad \psi = \delta_a - D_a\omega$$

$$I(\lambda) = i(\lambda) + \mathcal{J}(\lambda) \qquad L(\lambda) = l(\lambda) + \mathcal{L}(\lambda)$$
$$L(\lambda) = [s, I(\lambda)]$$

(compare BS 88-91).

This set up generalizes to any situation where $\mathcal{A}$ is replaced by a space of fields defined on Σ and $\mathcal{G}$ by some group acting on $\mathcal{A}$. Other example: 2d gravity

$$\begin{array}{lll} & \Sigma, & \\ \mathcal{A} & \to \mathcal{M}(\Sigma) & \text{metrics on } \Sigma \\ \mathcal{G} & \to \mathrm{Diff}\Sigma & \end{array}$$

One question is: how to find equivariant cohomology classes.

One general method which involves equivariant cohomology again goes as follows: find a family of H-bundles over Σ on which the action of $\mathcal{G}$ lifts and find a $\mathcal{G}$ invariant connection Γ. H-characteristic classes of the equivariant curvature of Γ define equivariant cohomology classes of the basis B of the H-bundle family (e.g. $\Sigma \times \mathcal{A}$ in the Yang-Mills case, $\Sigma \times \mathcal{M}(\Sigma)$ or $\Sigma \times \mathcal{CM}(\Sigma)$ in the 2d gravity case). ($\mathcal{CM}(\Sigma)$ = conformal classes of metrics on Σ.)

Next, "theorem 3 of Cartan" allows to replace ω, the generator of the Weil algebra by a connection $\tilde{\omega}$ for the action of $\mathcal{G}$ on the above manifold B, thus yielding basic cohomology classes of B for the action of $\mathcal{G}$.

It follows from general properties of characteristic classes that these cohomology classes do not depend on the various connections used to define them (eg $\Gamma, \bar{\omega}$), hence can be obtained by integrating or averaging out over families of those (e.g., in the gauge case $\Gamma = a$ in the Gr_2^{top} case $\Gamma =$ an extension to the linear frame bundle of the Levi-Civita connection). This, we believe, is the reason why such cohomology classes can be expressed as "functional integrals" (to be properly defined).

This interpretation allows compact representations of those cohomology classes both in the YM_4^{top} case where it explains the role of $F + \psi + \phi$ (cf. Baulieu, Singer 88-91) and in the Gr_2^{top} case (cf. Becchi, Collina, Imbimbo 94) which remained rather mysterious for some time.

It also sheds some light on the $N = 2$ twisted supersymmetry approach. Introducing the Faddeev Popov charge $Q^{\phi\pi}$ (not to be confused with the Q in the beginning of the lecture), (and, the corresponding currents, if one wants), one has the following graded commutation relations:

$$\begin{aligned}
[s, \mathcal{J}(\lambda)]_+ &= \mathcal{L}(\lambda) \\
[\mathcal{L}(\lambda), \mathcal{L}(\lambda')]_- &= \mathcal{L}([\lambda, \lambda']) \\
[\mathcal{L}(\lambda), \mathcal{J}(\lambda')]_- &= \mathcal{J}([\lambda, \lambda']) \\
[\mathcal{J}(\lambda), \mathcal{J}(\lambda')]_+ &= 0 \\
[Q, s]_- &= s \\
[Q, \mathcal{J}(\lambda)]_- &= \mathcal{J}(\lambda) \\
[Q, \mathcal{L}(\lambda)]_- &= 0
\end{aligned}$$

Ex.: $N = 2$ superconformal set up: $\lambda \in$ Vir

$$Q \sim \int J, \qquad \mathcal{J}(\cdot) = G^- \qquad s \sim \int G^+ \qquad \mathcal{L}(\cdot) = T$$

This return to the origin requires the introduction of the operation $\mathcal{J}(\cdot)$ associated to the introduction of the Faddeev Popov ghost ω.

Conclusion

If you are an addict of BRST games, find the correct cohomology first !