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Quintessence Behavior of an Anisotropic Bulk Viscous Cosmological Model in Modified $f(Q)$-Gravity

Anirudh Pradhan [1,*], Archana Dixit [2] and Dinesh Chandra Maurya [3]

1 Centre for Cosmology, Astrophysics and Space Science, GLA University, Mathura 281 406, India
2 Department of Mathematics, Institute of Applied Sciences and Humanities, GLA University, Mathura 281 406, India
3 Center for Theoretical Physics and Mathematics, IASE (Deemed to be University), Sardarshahar 331 403, India
* Correspondence: pradhan.anirudh@gmail.com

Abstract: In this article, we consider an anisotropic viscous cosmological model having LRS Bianchi type I spacetime with $f(Q)$ gravity. We investigate the modified $f(Q)$ gravity with form $f(Q) = \alpha Q^2 + \beta$, where Q is the non-metricity scalar and α, β are the positive constants. From the modified Einstein's field equation having the viscosity coefficient $\xi(t) = \xi_0 H$, the scale factor is derived as $a(t) = 2\,sinh\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right)$. We apply the observational constraints on the apparent magnitude $m(z)$ using the χ^2 test formula with the observational data set such as JLA, Union 2.1 compilation and obtained the best approximate values of the model parameters m, α, H_0, ξ_0. We find a transit universe which is accelerating at late times. We also examined the bulk viscosity equation of state (EoS) parameter ω_v and derived its current value satisfying $\omega_v < -1/3$, which shows the dark energy dominating universe evolution having a cosmological constant, phantom, and super-phantom evolution stages. It tends to the Λ cold dark matter (ΛCDM) value ($\omega_v = -1$) at late times. We also estimate the current age of the universe as $t_0 \approx 13.6$ Gyrs and analyze the statefinder parameters with $(s, r) \to (0, 1)$ as $t \to \infty$.

Keywords: bulk viscosity; $f(Q)$ gravity; accelerating universe; LRS Bianchi type-I universe; observational constraints

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1. Introduction

It is well known from the recent astrophysical observations that the present-day universe is undergoing an accelerated expansion phase [1]. There has been a lot of recent research on its potential mechanisms, including extended gravity [2], the decaying cold dark matter (CDM) model [3], and the modification of EoS may account for the observed cosmic accelerated expansion. In other words, the accelerating behavior is due to an unknown mystical component having a large negative pressure termed as the dark energy (DE) [4,5]. Furthermore, the observations of the WMAP satellite [6] and large-scale structure [7] measured the cosmic microwave background (CMB) anisotropies and suggested that approximately 70% of the universe be made up of dark energy (DE), and the rest is made up of relativistic dark matter (DM) and ordinary matter [8]. The nature of DE has previously been classified using the equation of state parameter ω, which is defined as the ratio of pressure (p) to energy density (ρ), that is, $\omega = \frac{p}{\rho}$. The value of EoS parameter $\omega < -1/3$ is needed for the acceleration of cosmic expansion. The essential possibilities in this categorization are scalar field models with an EoS value $-1 < \omega < -1/3$, which articulates as the quintessence field dark energy [9,10], and $\omega < -1$ as the phantom field dark energy [11]. The phantom field has received much interest because of its interesting qualities. In the phantom model, the dark energy grows in an exciting spread leading to a future singularity in the finite time. It also violates the four energy constraints that serve to keep wormholes [12]. In addition, the combined findings of the H_0 measurements, WMAP9 [13], SNe-Ia, cosmic microwave background, and BAO demonstrate that $\omega = -1.084 \pm 0.063$ for the dark energy. This is despite the fact that the Planck team demonstrated in 2015 that $\omega = -1.006 \pm 0.045$ [14]. Cao et al. [15] demonstrated analytically that the integrability of the dynamics of the Kerr–Newman black hole is surrounded by an electromagnetic field, quintessence, and charged particles moving around it for a closed FRW cosmology model with a conformally coupled scalar field. Wu [16] studied a new interpretation of zero Lyapunov exponents in Belinskii–Khalatnikov–Lifshitz time for Mixmaster cosmology. Ma et al. [17] computationally investigated the impact of changing the cosmological constant Λ and self-interacting coefficient λ on the transition to chaos in complexified space.

Inspired by the studies on black hole thermodynamics [18], Hooft presented the holographic principle (HP) for the first time [19]. Later, the HP was applied to the DE problem, resulting in a new model of DE known as the Holographic Dark Energy (HDE), in which the energy density is based on physical quantities on the universe's boundary, such as the "reduced Planck mass and the cosmological length scale", which is chosen as the universe's future event horizon [20]. Many authors have studied the development of different cosmological models that includes viscosity. The viscosity theory of relativistic fluids was first proposed by Eckart [21] and Landau and Lifshitz [22], who considered only the first order of change from equilibrium. Israel [23] came up with the idea for the relativistic second-order theory [24]. Murphy [25] found that a zero-curvature cosmological model is perfectly solvable even in the presence of bulk viscosity. Huang [26] worked on the Bianchi type I cosmological model with bulk viscosity. Many significant problems of the standard Big-Bang cosmology may be resolved by resorting to inflationary cosmology, which may also be related with the bulk viscosity.

Several sources of inspiration for hypotheses go beyond the conventional explanation of gravity. Jimenez et al. [27] have established a new gravity theory known as the symmetric teleparallel gravity or the $f(Q)$ gravity. The non-metricity term, Q, is what causes the gravitational interaction. The theory has been investigated by Lazkoz et al. [28], along with the observational constraints to test the $f(Q)$ Lagrangian as a polynomial function of the redshift, z. Recently, Mandal et al. [29] examined the energy condition requirements that allowed for the fixation of free parameters, and thus limiting the families of $f(Q)$ models that are consistent with the accelerated expanding behavior of the universe. The $f(Q)$ gravity is the simplest modification of STEGR. The energy condition constraints and cosmography in the $f(Q)$ theory are examined [30,31], using the assumption of a power-law function. Harko et al. [32] examined the coupling matter in modified $f(Q)$ gravity. For a $f(Q)$ polynomial model, Dimakis et al. [33] addressed different issues in the quantum cosmology [34]. The above issues have been discussed in the framework of $f(Q)$ gravity enough to motivate us to work under this new framework.

The first cosmological solutions in $f(Q)$ gravity may be found in references [35,36]. Several studies have shown the profligacy of viscous fluids which includes the shear and bulk viscosity [32,36–39]. In references [40–50], the impact of bulk viscous fluids in the late-time accelerating cosmos have been examined. However, in an expanding universe, a feasible process for viscous fluid **formation** is more difficult to find. In one approach, the bulk viscosity emerges due to the interruption of the local thermodynamic equilibrium [51].

Recent studies have focused on the transit phase universe with $f(Q,T)$ gravity [52,53] and $f(Q)$ gravity [54]. Capozziello and Agostino [55] examined $f(Q)$-gravity using a model-independent approach and addressed the variety of cosmological characteristics. Anisotropic $f(Q)$ gravity with bulk viscosity has been examined by Koussour et al. [56], Maurya [57], and Dixit et al. [58]. In the present analysis, we study different mechanisms for the DE in the $f(Q)$ gravity model. In Weyl's geometry, the covariant derivative of the metric tensor is non-null. This characteristic can be displayed mathematically in terms of a novel geometric quantity named non-metricity [59]. Geometrically, the non-metricity may be described as the variation of the length of a vector during parallel transport. To better understand our universe, we are obligated to replace the curvature concept with a more general geometrical concept. We consider $f(Q)$ gravity in an anisotropic background and solve the field equations for the average scale factor $a(t)$ and the metric coefficients, which is different from what other people have accomplished in the past because most people have just assumed it. Using this scale factor, we have discussed the deceleration parameter and the EoS parameter along with other physical quantities.

The paper is organized as follows. An introduction and an overview of the literature have been given in Section 1. We present the $f(Q)$ gravity formalism in Section 2 along with the corresponding field equation for the LRS Bianchi type-I space-time. In Section 3, we used the bulk-viscosity coefficient $\xi = \xi_0 H$ to solve Einstein's modified field equations. Section 4 has been devoted to the observational constraining of the model. In Section 5, we discussed the results such as how old the universe is now and how the statefinder works. In Section 6, we list our concluding remarks.

2. Field Equations in $f(Q)$-Gravity

To find the modified Einstein's field equations in $f(Q)$-gravity, we adopted the action principle [35,36] as

$$S = \int \left[\frac{1}{2\kappa} f(Q) + L_m \right] dx^4 \sqrt{-g} \tag{1}$$

where $\kappa = 8\pi G$, $f(Q)$ is an arbitrary function of Q known as non-metricity scalar and other notations are in their usual meaning such as matter Lagrangian density L_m and metric tensor g_{ij} with determinant g. The non-metricity scalar Q is expressed as below

$$Q \equiv -(L^b_{\ell i} L^\ell_{jb} - L^b_{\ell b} L^\ell_{ij}) g^{ij} \text{ and } Q_b = Q_{b\ i}^{\ i} \tag{2}$$

where Q_b is known as the non-metricity tensor and $L^b_{\ell\gamma}$ is the deformation tensor, and these may be expressed as

$$L^b_{\ell\gamma} = -\frac{1}{2} g^{b\lambda} (\nabla_\gamma g_{\ell\lambda} + \nabla_\beta g_{\lambda\gamma} - \nabla_\lambda g_{\ell\gamma}) \tag{3}$$

The modified field equations after the variation of action (Equation (1)) with respect to the metric tensor g_{ij} is given by

$$\frac{2}{\sqrt{-g}} \nabla_\lambda (\sqrt{-g} f_Q P^\lambda_{\mu\nu}) - \frac{1}{2} f g_{\mu\nu} + f_Q (P_{\nu\rho\sigma} Q_\mu^{\rho\sigma} - 2 P_{\rho\sigma\mu} Q^{\rho\sigma}_\nu) = \kappa T_{\mu\nu} \tag{4}$$

The non-metricity tensor is defined as

$$Q_{b\mu\nu} = \nabla_b g_{\mu\nu}.$$

and the trace of the non-metricity tensor is derived as follows [35]:

$$Q_b = g^{\mu\nu} Q_{b\mu\nu} \ \ and \ \ \mathbb{Q}_b = g^{\mu\nu} Q_{\mu b\nu}.$$

In addition, we define the non-metricity conjugate tensor

$$P^b_{\mu\nu} = -\frac{1}{2} L^b_{\mu\nu} + \frac{1}{4} \left(Q^b - \mathbb{Q}^b \right) g_{\mu\nu} - \frac{1}{4} \delta^b_{(\mu Q_\nu)},$$

and using this definition, the non-metricity scalar is given as

$$Q = -Q_{b\mu\nu} P^{b\mu\nu}.$$

The stress-energy-momentum tensor $T_{\mu\nu}$ is expressed as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \tag{5}$$

The recently constructed dark energy models with bulk viscosity in $f(Q)$ gravity have a linear form $f(Q) = -\alpha Q - \beta$ in a flat FLRW universe [54]. In this paper, we derive a viscous dark energy model with a quadratic function $f(Q) = \alpha Q^2 + \beta$; α and β are arbitrary positive constants, in the LRS Bianchi Type-I space-time:

$$ds^2 = A(t)^2 dx^2 + B(t)^2(dy^2 + dz^2) - dt^2 \tag{6}$$

Here, $A(t)$ and $B(t)$ are known as the metric coefficients. For the line element (6), the non-metricity scalar Q is calculated as

$$Q = -4\frac{\dot{A}}{A}\frac{\dot{B}}{B} - 2\left(\frac{\dot{B}}{B}\right)^2 \tag{7}$$

The stress-energy-momentum tensor for the viscous fluid is considered as

$$T^{\mu}_{\nu} = diag[-\rho, \tilde{p_x}, \tilde{p_y}, \tilde{p_z}] \tag{8}$$

with the energy density ρ, anisotropic pressure for bulk viscous fluid $\tilde{p_x}$, $\tilde{p_y}$, and $\tilde{p_z}$ along the x, y and z axes, respectively. The EoS parameter for the bulk viscous fluid has been assumed as $\tilde{p_i} = \rho\tilde{\omega_i}$ for $i = x, y, z$. Thus, the diagonal terms of the stress-energy-momentum tensor becomes

$$T^{\mu}_{\nu} = diag[-1, \tilde{\omega_x}, \tilde{\omega_y}, \tilde{\omega_z}]\rho = [-1, \omega_v, \omega_v + \delta_v, \omega_v + \delta_v]\rho \tag{9}$$

where we consider the EoS parameter along the x-axis as $\tilde{\omega_x} = \omega_v$ and take a deviation δ_v in the EoS parameter along the y and z axes from the EoS parameter of the x-axis. δ_v is also known as the skewness parameter. The suffix $(_v)$ corresponds to the bulk viscous fluid. The EoS parameters ω_v and δ_v may be constant or functions of cosmic time t.

In the co-moving coordinate system, the field equations from Equations (4), (6) and (8) may be written as

$$\frac{f}{2} + f_Q\left[4\frac{\dot{A}}{A}.\frac{\dot{B}}{B} + 2\left(\frac{\dot{B}}{B}\right)^2\right] = \rho \tag{10}$$

$$\frac{f}{2} - f_Q\left[-2\frac{\dot{A}}{A}.\frac{\dot{B}}{B} - 2\frac{\ddot{B}}{B} - 2\left(\frac{\dot{B}}{B}\right)^2\right] + 2\frac{\dot{B}}{B}\dot{Q}f_{QQ} = -\tilde{p_x} \tag{11}$$

$$\frac{f}{2} - f_Q\left[-3\frac{\dot{A}}{A}.\frac{\dot{B}}{B} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2\right] + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{Q}f_{QQ} = -\tilde{p_y} = -\tilde{p_z} \tag{12}$$

Here, the over dot $(\cdot)$ denotes the derivative with respect to time t. The volume scale-factor of the considered spacetime is defined as

$$V = a(t)^3 = AB^2 \tag{13}$$

where, $a(t)$ denotes the average scale factor and A, B are the metric coefficients. The deceleration parameter (q) in terms of the average scale-factor is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{14}$$

The deceleration parameter (DP) reveals the various phases of the expanding evolution of the universe. A positive value of q depicts the decelerating phase and the negative values of q depicts the accelerating phase of the universe.

The mean Hubble parameter H is defined as

$$H = \frac{1}{3}(H_x + H_y + H_z) \tag{15}$$

where H_x, H_y and H_z are the directional Hubble parameters along the x, y and z axes, respectively. For the line element (6), the directional Hubble parameters are taken as $H_x = \frac{\dot{A}}{A}$ and $H_y = H_z = \frac{\dot{B}}{B}$.

A connection among the above three parameters H, V and a may be obtained as

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left[\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right] = \frac{\dot{a}}{a} \tag{16}$$

We define the geometrical parameters θ, σ^2, Δ called as scalar expansion, shear scalar and the mean anisotropy parameter as

$$\theta(t) = \frac{\dot{A}}{A} + 2.\frac{\dot{B}}{B} \tag{17}$$

$$\sigma^2(t) = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2 \tag{18}$$

$$\Delta = \frac{1}{3}\sum_{i=x}^{z}\left(\frac{H_i - H}{H}\right)^2 \tag{19}$$

with the directional Hubble parameters $H_i, i = x, y, z$.

3. Cosmological Solutions

We have three independent field Equations (10)–(12) in five unknowns $A, B, \rho, \omega_v, \delta_v$. Therefore, to obtain an exact solution, we may require two more constraints on these parameters, and hence, we assume ($\sigma^2 \propto \theta^2$) the shear is proportional to the expansion scalar, and this relation leads to [21]

$$A = B^m \tag{20}$$

with $m \neq 1$ as an arbitrary constant. With $m = 1$, the model represents an isotropic universe and $m \neq 1$ will give an anisotropic universe. Here is a full explanation of the primary assumptions that caused this circumstance [21]. When $\frac{\sigma^2}{\theta^2}$ is constant, it is possible to achieve isotropy in the Hubble expansion of the universe according to the observations of the velocity redshift relation for extragalactic sources. With reference to Thorne [60], it is further indicated that according to studies of the velocity-redshift relation for extragalactic sources, the Hubble expansion of the universe is currently isotropic to a degree of ≈30% [61,62]. To put it more specifically, red-shift studies set the limit for the shear-to-Hubble constant ratio in the present-day region of our Galaxy at $\frac{\sigma}{H} \leq 0.3$. Collins et al. [63] have noted that the normal congruence to the homogeneous expansion satisfies the requirement that $\frac{\sigma}{\theta}$ is constant for spatially homogeneous metrics. This circumstance has served as the beginning point for numerous studies.

Now, in the present investigation, we have considered a quadratic function of $f(Q)$ as

$$f(Q) = \alpha Q^2 + \beta, \quad \alpha > 0, \quad \beta > 0 \tag{21}$$

where α and β are arbitrary constants.

Now, using Equation (20) in (13), the metric coefficients are obtained in terms of the scale factor $a(t)$ as:

$$A = a(t)^{\frac{3m}{m+2}}, \qquad B = a(t)^{\frac{3}{m+2}} \tag{22}$$

and the directional Hubble parameters are given by

$$H_x = \frac{\dot{A}}{A}, \quad H_y = H_z = \frac{\dot{B}}{B} = \frac{1}{m}\frac{\dot{A}}{A} = \frac{1}{m}H_x \tag{23}$$

Viscous fluid pressures are defined in the x, y, and z directions as [22,56]

$$\tilde{p_x} = p - 3H_x\xi(t) \qquad \tilde{p_y} = p - 3H_y\xi(t) \qquad \tilde{p_z} = p - 3H_z\xi(t) \tag{24}$$

where p is the normal pressure and ξ is formed when the viscous fluid deviates from local thermal equilibrium. ξ may be a function of the Hubble parameter and its derivative [22,64].

Subtracting (12) from (11) and using the expression (24) gives

$$f_Q\left[\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2\right] + \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\dot{Q}f_{QQ} + 3\xi(t)(H_x - H_y) = 0 \tag{25}$$

From the expression (21),

$$f_Q = 2\alpha Q, \qquad f_{QQ} = 2\alpha \tag{26}$$

Using Equation (23) with (7) gives the non-metricity scalar as

$$Q = -\frac{2(2m+1)}{m^2}H_x^2 \tag{27}$$

Using Equation (24) for the viscous universe in Equations (10)–(12), we deduce that the parameter "bulk viscosity coefficient ξ" is connected to the matter, the Hubble parameter and its derivative. So, we assume $\xi = \xi(H)$ and consider the specific form of ξ as [44,65–72].

$$\xi(t) = \xi_0 H \tag{28}$$

where ξ_0 is an arbitrary constant.

Now, using Equations (26) to (28) in (25), we obtain

$$\dot{H_x} + \frac{(m+2)}{3m}H_x^2 - \frac{\xi_0 m(m+2)}{12\alpha(2m+1)} = 0 \tag{29}$$

Solving Equation (29), we obtain the component H_x of the Hubble parameter H as

$$H_x = \frac{m}{2}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}coth\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{30}$$

and hence, H_y is given by

$$H_y = \frac{1}{2}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}coth\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{31}$$

Integrating Equations (30) and (31), we obtain the metric coefficients as

$$A(t) = \left[2sinh\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right)\right]^{\frac{3m}{m+2}} \tag{32}$$

$$B(t) = \left[2sinh\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right)\right]^{\frac{3}{m+2}} \tag{33}$$

Hence, the average scale factor $a(t)$ and the average Hubble parameter $H(t)$ are given by

$$a(t) = 2sinh\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{34}$$

and

$$H(t) = \frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}coth\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{35}$$

The q_x and q_y components of the deceleration parameter $q(t)$ are computed, respectively, as

$$q_x = -1 + \frac{1}{3}\frac{m+2}{m}sech^2\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{36}$$

$$q_y = -1 + \frac{1}{3}(m+2)sech^2\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{37}$$

The average deceleration parameter $q(t)$ is calculated as

$$q = -1 + sech^2\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{38}$$

From the field Equations (10) to (12), the energy density ρ, the EoS parameter ω_v and the skewness parameter δ_v for bulk viscous fluid are obtained as

$$\rho(t) = \frac{\beta}{2} - \frac{6\alpha(2m+1)^2}{m^4}H_x^4 \tag{39}$$

and

$$\omega_{v_x} = -1 + \frac{\left[\frac{\xi_0^2(m+2)}{6\alpha m(2m+1)}coth^2\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) - \frac{\xi_0^2(4m-1)}{2\alpha(2m+1)}coth^4\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right)\right]}{\frac{\beta}{2} - \frac{6\alpha(2m+1)^2}{m^4}H_x^4} \tag{40}$$

and

$$\delta_{v_x} = -\frac{\frac{4\alpha(2m+1)(m-1)}{m^4}[(m+2)H_x^2 + 3m\dot{H}_x]H_x^2}{\frac{\beta}{2} - \frac{6\alpha(2m+1)^2}{m^4}H_x^4} \tag{41}$$

4. Observational Constraints

To validate the theoretical model, it is necessary to compare the derived results of the model with the observational universe. The technique of the curve-fitting of some cosmological parameters with various observational data sets is the best way to compare and validate the model. Hence, we will find the best fit curve of the Hubble parameter and apparent magnitude. For performing these curve fittings, we will require a relation between cosmic age and redshift which is taken as the relation between the redshift and average scale-factor as $\frac{a_0}{a} = 1 + z$, with $a_0 = 1$. Now, using this relationship, the cosmic age in terms of redshift z may be given as

$$t(z) = \frac{6}{m+2}\sqrt{\frac{\alpha(2m+1)}{\xi_0}}sinh^{-1}\left(\frac{1}{2(1+z)}\right) \tag{42}$$

Hence, the Hubble parameter $H(z)$ in terms of redshift z may be derived as

$$H(z) = \frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}\sqrt{1+4(1+z)^2} \tag{43}$$

Several observational datasets are presently available in the literature. Here, we use two datasets of the apparent magnitude $m(z)$ using redshift z in the derived model. The first is the 51 dataset of apparent magnitude $m(z)$ from the Joint Light Curve Analysis (JLA) [73] and the 518 data set of magnitude $m(z)$ from the Union 2.1 Compilation of SNe Ia datasets [74].

Apparent Magnitude

Luminous distance is measured by the total luminous flux of a light source and defined as $D_L = (z+1)c\int_0^z \frac{dz}{H(z)}$. We take the apparent magnitude with respect to D_L to obtain the best values of the model parameters $\{\alpha, m, \xi_0, H_0\}$ for the best-fit curve of $m(z) = 16.08 + 5log_{10}(\frac{H_0 D_L}{0.026\text{cMpc}})$. Using the 51 observed apparent magnitude dataset $m(z)$ from JLA [73] and the 518 observed apparent magnitude dataset Union 2.1 [74] to determine the best-fit curve for the apparent magnitude $m(z)$ using the SNe Ia dataset. We use the χ^2 test formula to achieve the best-fit curve for the theoretical and empirical results.
The Luminosity Distance D_L may be calculated by the formula

$$D_L = c(1+z)\int_0^z \frac{dz}{H(z)} \tag{44}$$

where c is the velocity of light and $H(z)$ is the Hubble parameter given in Equation (43).

Now, using Equation (43) in (44), the luminosity distance D_L for the above derived model may be given as

$$D_L = \frac{3c(1+z)}{m+2}\sqrt{\frac{\alpha(2m+1)}{\xi_0}}\left[sinh^{-1}(2z+2) - sinh^{-1}(2)\right] \tag{45}$$

Consequently, the apparent magnitude $m(z)$ is calculated as

$$m = 16.08 + 5\log_{10}\frac{H_0 D_L}{0.026cMpc} \tag{46}$$

We employ the χ^2 test to identify the $m(z)$ curve as

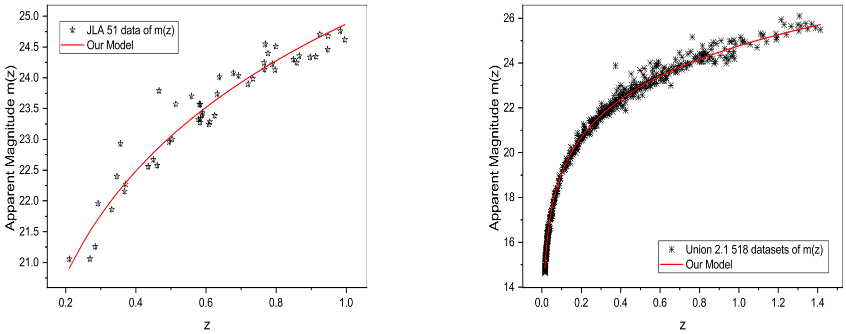
$$\chi^2_{SN} = \sum_{i=1}^{N}\left(\frac{(m_i)_{ob} - (m_i)_{th}}{(m_i)_{th}}\right)^2$$

where summation is taken from 1 to 51 data points for the JLA dataset and it taken from 1 to 518 data points for union 2.1 compilation dataset.

With this understanding of χ^2 values, we used the χ^2-test formula to obtain the best-fit curve of the apparent magnitude $m(z)$ with the observational datasets JLA and union 2.1 as well as the best fit values of the model parameters $\{\alpha, m, \xi_0, H_0\}$, which are listed in Table 1. The $\chi^2 = 0$ means that observational and theoretical values are exactly the same. Figure 1a,b show the best-fit curves of the apparent magnitude $m(z)$ with the observational values of $m(z)$ obtained from the JLA and Union 2.1 datasets which are listed in Table 1. We also observe that our theoretical model is in good agreement with these two datasets. Figures 2 and 3 demonstrate, using two different sets of data, the accuracy of our fitted model.

Table 1. The model parameters that best fit the JLA and Union 2.1 datasets.

Data	m	α	ξ_0	H_0	χ^2	R^2
JLA	2	$9.52698 \times 10^{-5} \pm 3.34447 \times 10^{-6}$	0.99998	70.00056	0.07228	0.92346
Union 2.1	2	$8.42571 \times 10^{-5} \pm 8.3134 \times 10^{-7}$	0.97455	70.00000	0.05915	0.99404



(**a**) (**b**)

Figure 1. The Joint Light–Curve Analysis and Union 2.1 datasets, respectively, were used to fit the m (z) curve.

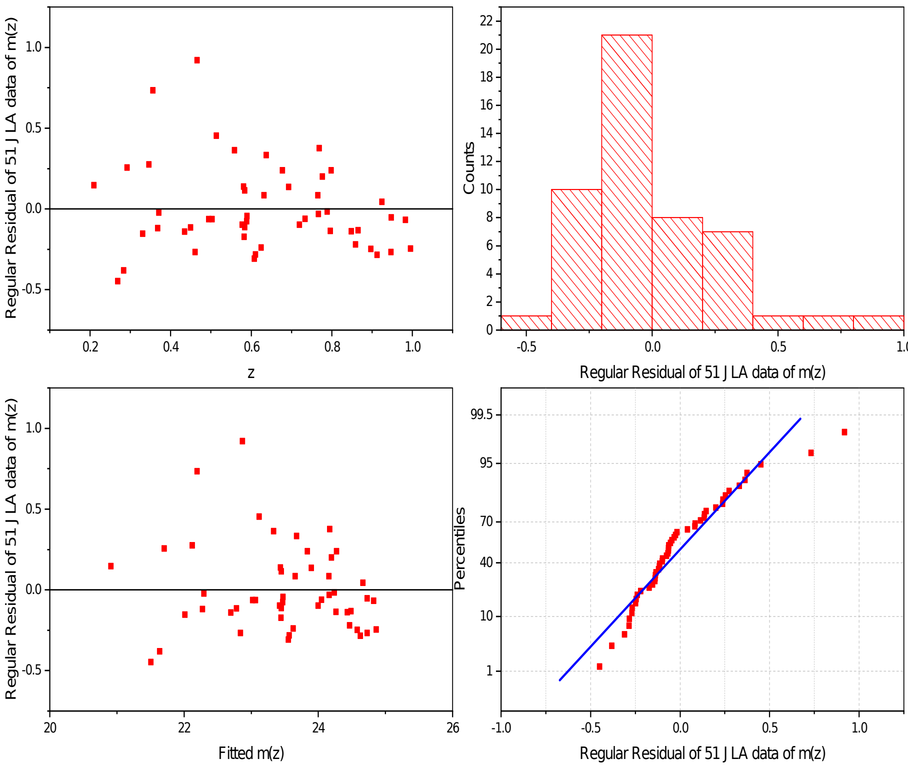


Figure 2. The residual plot of the best fit model and JLA datasets.

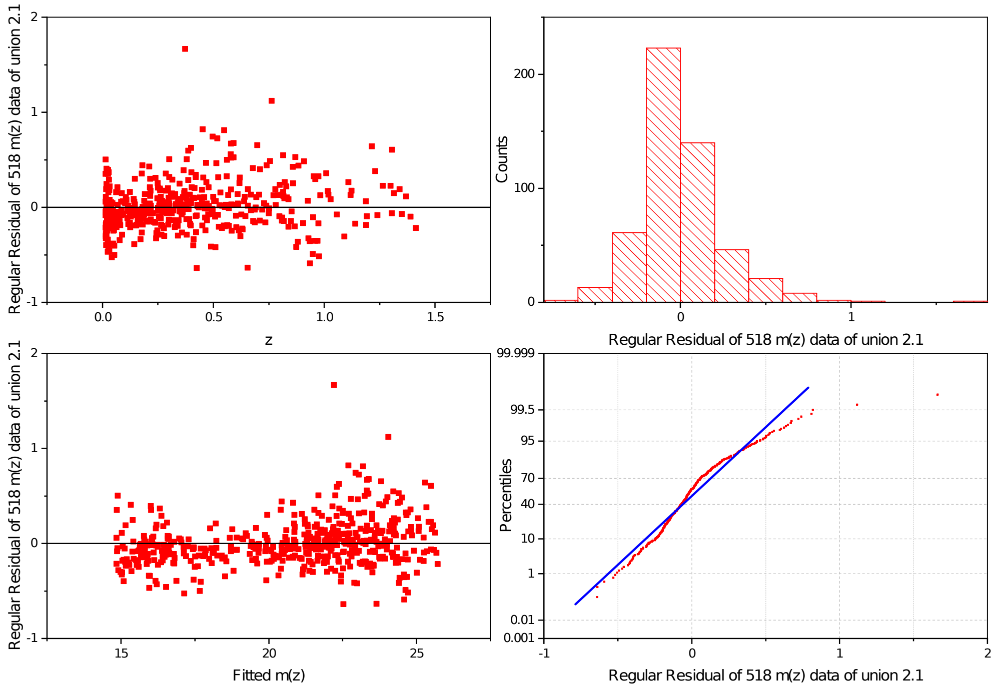


Figure 3. The residual plot of the best fit model and Union 2.1 datasets.

5. Discussion of the Results

Equation (42) represents a relationship between the cosmic time t and redshift z, and its geometrical interpretation is given by Figure 4a, from which we can say that as $z \to \infty$, the cosmic time $t \to 0$. At $z = 0$, we find the present value of $t(0) = \frac{6}{m+2}\sqrt{\frac{\alpha(2m+1)}{\xi_0}}sinh^{-1}\left(\frac{1}{2}\right)$. From Figure 4a, we can see that the cosmic time varies as $0 \leq t \leq 0.014$ over the redshift $0 \leq z \leq \infty$. The expression for the Hubble parameter $H(z)$ is **given** by Equation (42), and Figure 4b shows its geometrical behavior. From Figure 4b, we can see that H is an increasing function of redshift z, and at $z = 0$, the value of H denotes the present value of Hubble parameter H_0. Hence, at $z = 0$, $H_0 = \frac{m+2}{6}\sqrt{\frac{5\xi_0}{\alpha(2m+1)}}$. In addition, as $z \to \infty$, $H \to \infty$. From the best-fitting curve, the value of H_0 is obtained as $\approx$70, as mentioned in Table 1. From the Hubble law $v \propto d$, i.e., $\frac{v}{d}$ =constant (H_0), where v is the velocity of the moving object and d is the distance of moving object from the observer, increasing the value of H_0 with redshift z shows the acceleration in expansion.

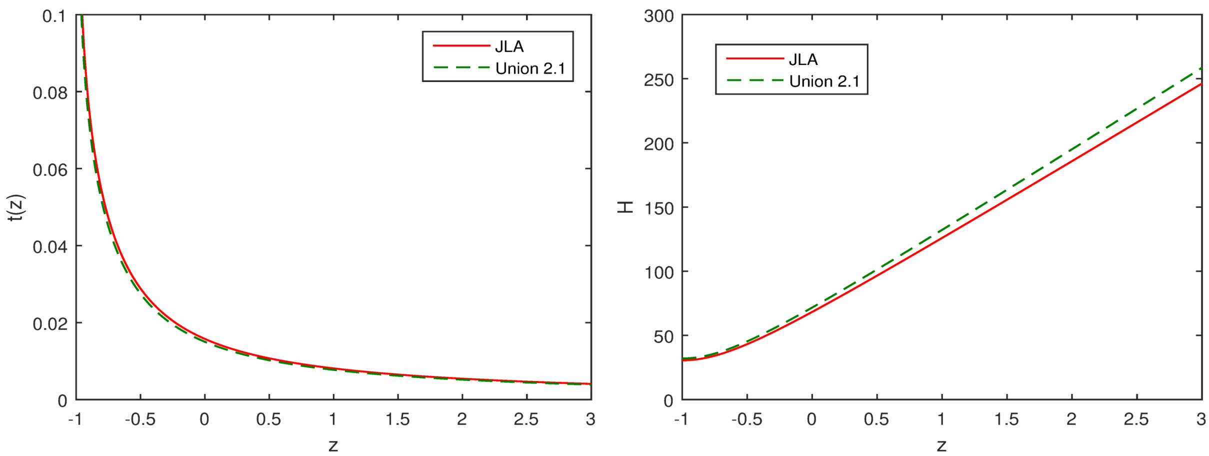


Figure 4. Cosmic time t and Hubble parameter H with z in the derived model.

Equations (36) and (37) represent the expressions for the deceleration parameter q_x and q_y along x and y axes, respectively. Equation (38) represents the expression for the average deceleration parameter $q(t)$. The geometrical behavior of these deceleration parameters is shown in Figure 5a,b for JLA and Union 2.1 compilation of SNe Ia observations. Figure 5a,b show that the values of the deceleration parameter decreases with cosmic time and as $t \to \infty$, $q \to -1$. It depicts that the universe's expansion is accelerating. We observe that the average deceleration parameter q represents a universe from static to an accelerating expanding universe, but the component of q along the x-axis q_x varies over $(-1, -0.2)$ which reveals an ever-accelerating universe and the component q_y depicts a transit phase accelerating universe. The present value of q_x is determined as $\{-0.4418, -0.4515\}$, $q_y = \{0.1164, 0.09702\}$ and $q = \{-0.1627, -0.1772\}$ along two datasets, which reveal that the present universe is accelerating. Equation (39) represents the expression for the energy density ρ, and we can observe that as $t \to \infty$, $\rho \to 0$, i.e., at the time of origin of the universe, the energy density of the universe is very high, and after the Big Bang, it has been diluted with time. For $\rho \geq 0$, we may find a constraint on β as $\beta \geq \frac{12\alpha(2m+1)^2}{m^4} H_x^4$.

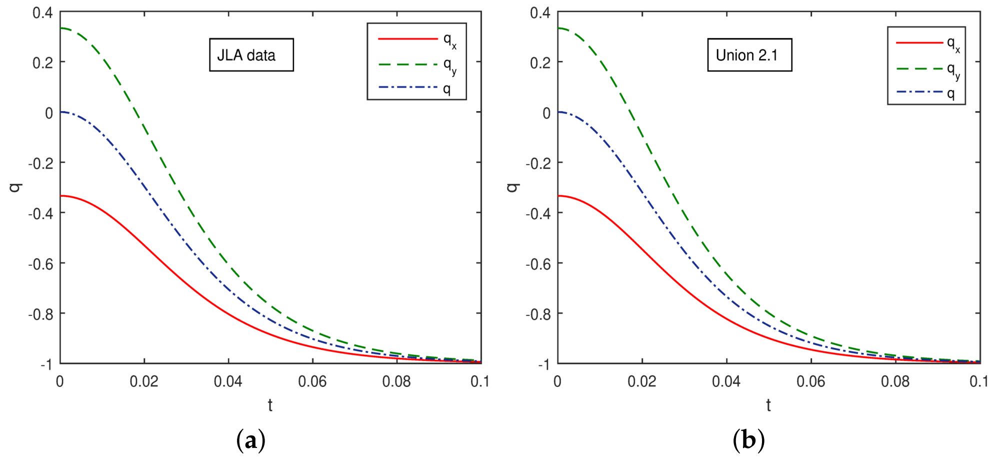
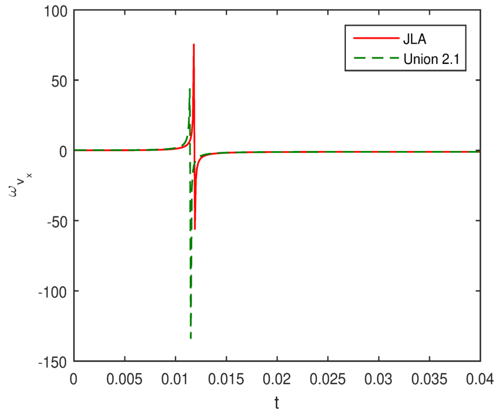
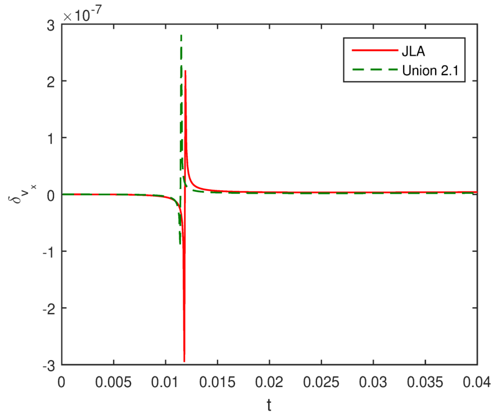


Figure 5. Behavior of deceleration parameter for two datasets, JLA and Union 2.1 Compilation of SNe Ia observations.

Equations (40) and (41) represent the expressions for the EoS parameter ω_v and skewness parameter δ_v for the bulk viscous fluid in the anisotropic universe, respectively. Figure 6a,b show their variations over cosmic time t, respectively. Figure 6a shows that the EoS parameter ω_v is an increasing function of cosmic time t with $\omega_v \geq 0$. Then, it decreases with time and tends to -1 (ΛCDM model) in the late-time universe, which is an interesting feature of the model. It depicts that the early universe was matter-dominated. As time passes, the dark energy dominates. Due to this, its volume increases, and the universe expands with acceleration. Figure 6b shows that the skewness parameter δ_v decreases with cosmic time t, and the present values of δ_v tend to zero, which is consistent with the skewness property.

Equation (28) with (35) represents the expression for the bulk-viscosity function. We have observed that the Hubble parameter H is a decreasing function of cosmic time t and estimated values of $\xi_0 = \{0.99998, 0.97455\}$ along two datasets. Hence, the bulk-viscosity function is a decreasing function of time t, which may cause the expanding and accelerating expanding universe.





(**a**) (**b**)

Figure 6. The EoS ω_v and skewness δ_v with t for JLA and Union 2.1 Compilation of SNe Ia observations.

5.1. Age of the Present Universe

The present age of the universe is estimated by

$$t_0 - t = -\int_{t_0}^{t} dt = \int_0^z \frac{dz}{(1+z)H(z)} \tag{47}$$

Using (43) in (47) and integrating, we obtain

$$t_0 - t = \frac{6}{m+2}\sqrt{\frac{\alpha(2m+1)}{\xi_0}}[tanh^{-1}(\sqrt{5}) - tanh^{-1}(\sqrt{1+4(1+z)^2})] \tag{48}$$

The geometrical behavior of $H_0(t_0 - t)$ versus redshift is represented in Figure 7. One can see that as $z \to \infty$, then $H_0(t_0 - t) \to H_0 t_0$ (i.e., $t \to 0$); here, H_0 is the present value of H and t_0 shows the age of the present universe. In our derived model, the age of the present universe is calculated as $\{13.9532, 13.2910\}$ GYrs for two datasets, respectively, the JLA and Union 2.1 compilation of supernovae Ia. These findings are supported by the recent observations [14,73,75].

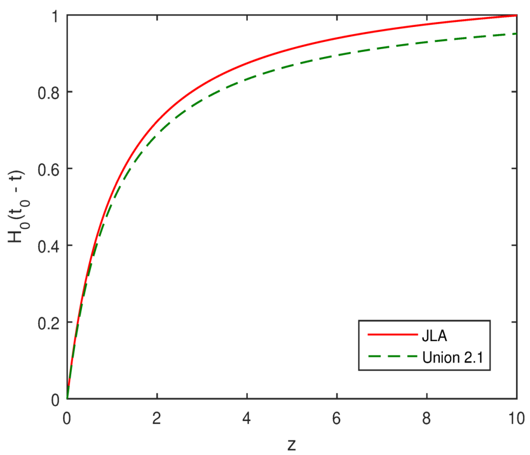


Figure 7. The cosmic age of the universe in the derived model.

5.2. Statefinder Analysis

The Hubble parameter $H = \frac{\dot{a}}{a}$ and the deceleration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2}$ are two cosmological parameters that characterize the geometrical evolution of the cosmos. These are also known as geometrical parameters. In addition to these quantities, the statefinder parameters, proposed in reference [14], depict the various phases of the universe evolution in the dark energy models [76–78]. The parameters r and s may be expressed in terms of the average scale-factor $a(t)$ and its derivative as [14]

$$r = \frac{\dddot{a}}{aH^3}, \qquad s = \frac{r-1}{3(q-\frac{1}{2})} \tag{49}$$

The statefinder r may be determined as

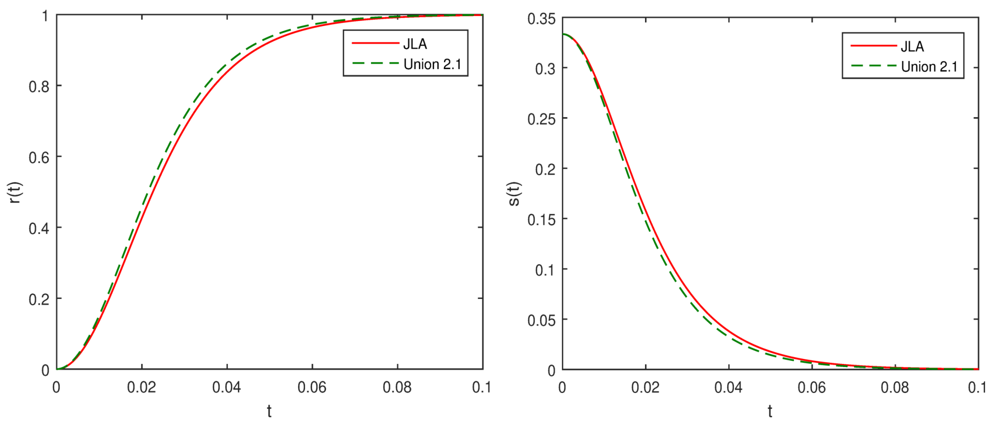
$$r = tanh^2\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right) \tag{50}$$

Using Equations (34), (42) and (43) in Equations (49) and (50), the expression for the statefinder s may be given as

$$s = \frac{2}{3}\frac{1}{3cosh^2\left(\frac{m+2}{6}\sqrt{\frac{\xi_0}{\alpha(2m+1)}}t\right)-1} \tag{51}$$

From Equations (50) and (51), we can find that as $t \to \infty$, $r \to 1$ and $s \to 0$, this implies that our derived model tends to an isotropic ΛCDM model as $t \to \infty$.

The geometrical interpretation of the statefinder parameters r, s with the variation of time t are shown in Figure 8a,b, respectively, and we have measured the present values of $(r-s)$ as $r_0 = \{0.2178, 0.1999\}$, $s_0 = \{0.2428, 0.2424\}$, respectively, for the two datasets. The evolution of (s, r) reveals different stages of the dark energy models [14,76–78]; for instance, the value $(s, r) = (0, 1)$ represents a flat FLRW ΛCDM model. It can be found that at present, $(r_0, q_0) = (0.2178, -0.2189)$, and this indicates that the present universe is a dark energy-dominated universe.



(**a**) (**b**)

Figure 8. Behavior of statefinder parameters for two datasets, JLA and Union 2.1 Compilation of SNe Ia observations.

5.3. Om Diagnostics

The *Om* diagnostic is a useful tool for categorizing different cosmic dark energy models [79]. Given that it only employs the first derivative of the cosmic scale factor, this diagnosis is precise to identify the quintessence and phantom evolution of the universe. For a spatially flat universe, $Om(z)$ is given by

$$O_m(z) = \frac{-1+\left(\frac{H(z)}{H_0}\right)^2}{-1+(1+z)^3} \tag{52}$$

where the Hubble parameter's current value is H_0. A positive slope of $Om(z)$ corresponds to the phantom evolution of dark energy, while a negative slope denotes the quintessence kind of evolution for the dark energy. The ΛCDM model is represented by the constant $Om(z)$.

In Figure 9, we found that the Om diagnostic parameter has a negative slope across a restricted set of values for the model parameter. Therefore, from the Om diagnostic test, we can conclude that the $f(Q)$ cosmological model represents quintessential behavior.

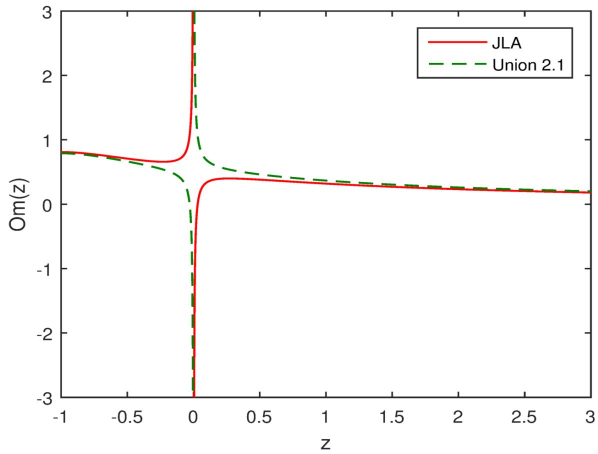


Figure 9. Behavior of Om diagnostics in our derived model.

6. Concluding Remarks

In this study, we have found a transit phase (decelerating to accelerating) cosmological model in the modified $f(Q)$-gravity theory. We have investigated anisotropic scenarios of the expanding universe with viscous fluid. We solved field equations and found an exact solution for the scale factor. The main features of our model, which we observed, are as follows:

- In the model, the average scale factor $a(t)$ is obtained by solving the field equations, and it depends on the values of model parameters $\{\alpha, m, \xi_0\}$.
- We estimated the present values of Hubble parameter H_0 as $\approx$70 Km s^{-1}/Mpc and deceleration parameter q_0 as -0.1772, which is supported by the recent observations [14,76,77].
- The derived model represents an accelerating universe evolution which has been transited from the decelerating phase.
- We have found the present value of EoS parameter $\omega_v < -1/3$ for bulk-viscous fluid and it varies over positive to negative, which indicates the existence of dark energy in the late-time universe of the model.
- The skewness parameter δ_v for bulk-viscous fluid tends to zero, which is consistent with skewness property.
- According to our calculations, the universe is currently $\approx$13.6 Gyrs old, which is supported by recent observations [14,76,77].
- We have found the bulk viscous coefficient as a decreasing function of cosmic time t, which may be a dark energy candidate.
- The model has a singularity at $t = 0$, since it is found that $a(0) = 0$.
- The value of energy density $\rho \to \infty$ as $t \to 0$ reveals the Big-Bang singularity in the model.
- The analysis of the statefinder parameters indicate that the derived model tends to the ΛCDM model as $t \to \infty$.
- In the Om diagnostic analysis of our model, we have found that the universe in the derived model will have the quintessence-like dark energy at the late times.

The present derived universe model evolves with a natural scale factor and reveals different physical and geometrical quantities compatible with observations. These results may be interesting for the researchers working in this field.

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