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New Approach to Glauber Theory Description of Nucleus-Nucleus Scattering

Yu. M. Shabelski[a] **and A. G. Shuvaev**[a, *]

[a] *Petersburg Nuclear Physics Institute, National Research Center "Kurchatov Institute," Gatchina, Leningrad oblast, 188300 Russia*

**e-mail: shuvaev@thd.pnpi.spb.ru*

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Abstract—A new approach to describe the nuclear scattering amplitudes in Glauber theory is proposed. The method is based on the expression for the generating function of the Glauber amplitudes. The generating function has been explicitly found in all orders of the Glauber theory.

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1. The theory of nucleus-nucleus scattering has recently acquired a modern impetus from the large number of the currently available experimental data (see e.g. [1–4]). The theoretical predictions for the scattering at the comparatively high energies more than several hundreds MeV per nucleon are standardly provided by the Glauber theory [5, 6]. It proves to be highly efficient for the hadron-nucleus collision, supplying rather simple analytical expressions for the scattering amplitudes. The case of the nucleus-nucleus scattering is much more involved. Additional simplifying approximations are commonly used to obtain an analytical expression such as the optical model or the rigid target model (see e.g. [8–10]). Apart from these models there are only numerical calculations based on Monte-Carlo method or on its modifications [11, 12].

In the present paper we propose a novel approach based on the analytically derived generating function the Glauber amplitudes for nucleus-nucleus scattering. It allows to reach the same accuracy as the numerical Monte-Carlo calculations but in much more simple way.

2. The amplitude of the elastic scattering of the incident nucleus A on the fixed target nucleus B in the Glauber theory reads [14, 15]

$$F_{AB}^{el}(q) = \frac{ik}{2\pi}\int d^2b e^{iqb}[1 - S_{AB}(b)], \qquad (1)$$

where q is the transferred momentum and k is the mean momentum carrying by a nucleon in nucleus A. The two-dimensional impact momentum b lies in the transverse plain to the vector k. The main assumption underlying the Glauber theory is that the radius of nucleon-nucleon interaction is much smaller than the typical nucleus size. Then assuming the phase shifts of the nuclear scattering to be the sum of those for each nucleon-nucleon scattering, $\chi_{NN}(b)$, the function $S_{AB}(b)$ takes the form

$$S_{AB}(b) = \int\prod_{i=1}^{A} d^2x_i \int\prod_{j=1}^{B} d^2y_j \rho_A^{\perp}(x_1 - b,\ldots,x_A - b) \times \rho_B^{\perp}(y_1,\ldots,y_B)\left\{\prod_{ij}[1 - \Gamma_{NN}(x_i - y_j)]\right\}, \qquad (2)$$

with

$$\Gamma_{NN}(b) = 1 - e^{i\chi_{NN}(b)} = \frac{1}{2\pi ik}\int d^2q e^{-iqb} f_{NN}^{\rm el}(q), \qquad (3)$$

where $f_{NN}^{\rm el}(q)$ is the nucleon-nucleon scattering amplitude. The product in (2) comprises all pairwise interactions between the nucleons from the projectile and target nuclei A and B, x_i and y_j being the nucleons' positions in the transverse plain. The nucleon densities in the transverse plain, $\rho_{A,B}^{\perp}$, are determined through three-dimensional ones integrated over longitudinal coordinates,

$$\rho_N^{\perp}(x_1,\ldots,x_N) = \int\prod_{i=1}^{N} dz_i \rho_N(z_1,x_1,\ldots,z_N,x_N),$$

$$\int\prod_{i=1}^{N} d^3r_i \rho_N(r_1,\ldots,r_N) = 1.$$

In what follows the three-dimensional nuclear densities are assumed to be the product of one-nucleon densities,

$$\rho_N(r_1,\ldots,r_N) = \prod_{i=1}^{N}\rho_N(r_i), \quad \int d^3r\rho_N(r) = 1,$$

and consequently

$$\rho_N^{\perp}(x_1,\ldots,x_N) = \prod_{i=1}^{N}\rho_N^{\perp}(x_i), \quad \int d^2x\rho_N^{\perp}(x) = 1. \tag{4}$$

Each pair $\{i, j\}$ enters the product (2) only once, meaning that each nucleon from the projectile nucleus can scatter on each nucleon on the target one but no more than once.

The total interaction cross section is

$$\sigma_{AB}^{\text{tot}} = \frac{4\pi}{k}\text{Im}\, F_{AB}^{\text{el}}(q=0) = \int d^2b[1 - S_{AB}(b)], \tag{5}$$

while the integrated elastic cross section evaluates to

$$\sigma_{AB}^{\text{el}} = \int d^2b[1 - S_{AB}(b)]^2. \tag{6}$$

The difference of these two values determines the total inelastic, or reaction, cross section,

$$\sigma_{AB}^{r} = \sigma_{AB}^{\text{tot}} - \sigma_{AB}^{\text{el}} = \int d^2b[1 - |S_{AB}(b)|^2]. \tag{7}$$

3. A main obstacle to deal with the Glauber amplitude (2) is its complicated combinatorial structure. To treat it analytically we present (2) as a functional integral. Let us consider the identity

$$\begin{aligned} C_0\int D\Phi D\Phi^* \exp\{-\int d^2xd^2y\Phi(x)\Delta^{-1}(x-y)\Phi^*(y) \\ + \sum_i \Phi(x_i) + \sum_j \Phi^*(y_j)\} \\ = \exp\Big\{\sum_{i,j}\Delta(x_i - y_j)\Big\} = \prod_{i,j} e^{\Delta(x_i - y_j)}, \end{aligned} \tag{8}$$

where C_0 is the normalization constant and the functional integral can be thought of as an infinite product of two dimensional integrals over the auxiliary variables $\Phi(x)$ at each space point x, the inverse of the propagator, $\Delta^{-1}(x-y)$, being understood as $\int d^2z\Delta^{-1}(x-z)\Delta(z-y) = \delta^{(2)}(x-y)$. If this function is chosen to obey the equation

$$e^{\Delta(x-y)}1 = -\Gamma_{NN}(x-y), \tag{9}$$

the right hand side of (8) recovers the product in (2). The function $\Delta(x-y)$ plays a role similar to that of Mayer propagator (function) in statistical mechanics, the analogy between Glauber theory and statistical mechanics has been remarked earlier (see, e.g. [16]). Then we get

$$\begin{aligned} S_{AB}(b) = C_0\int D\Phi D\Phi^* \\ \times \exp\Big\{-\int d^2xd^2y\Phi(x)\Delta^{-1}(x-y)\Phi^*(y)\Big\} \\ \times\Big[\int d^2x\rho_A^{\perp}(x-b)e^{\Phi(x)}\Big]^A\Big[\int d^2y\rho_B^{\perp}(y)e^{\Phi^*(y)}\Big]^B. \end{aligned} \tag{10}$$

This form suggests that it is natural to introduce the generating function,

$$\begin{aligned} Z(u,v) = \int D\Phi D\Phi \\ \times {}^*\exp\Big\{-\int d^2xd^2y\Phi(x)\Delta^{-1}(x-y)\Phi^*(y) \\ + u\int d^2x\rho_{\perp}(x-b)e^{\Phi(x)} + v\int d^2x\rho_{\perp}(x)e^{\Phi^*(x)}\Big\}, \end{aligned} \tag{11}$$

so that

$$S_{AB}(b) = \frac{1}{Z(0,0)}\frac{\partial^A}{\partial u^A}\frac{\partial^B}{\partial v^B}Z(u,v)\Big|_{u=v=0}. \tag{12}$$

The small interaction range, the property Glauber theory is based on, makes the complex functional integral (11) feasible. The standard parametrization of the elastic nucleon-nucleon scattering amplitude,

$$f_{NN}^{\text{el}}(q) = k\frac{\sigma_{NN}^{\text{tot}}}{4\pi}e^{-\frac{1}{2}\beta q^2}, \tag{13}$$

where σ_{NN}^{tot} is the total nucleon-nucleon cross section. It gives according to (3)

$$\Gamma_{NN}(x) = \frac{\sigma_{NN}^{\text{tot}}}{4\pi\beta}e^{-\frac{x^2}{2\beta}}, \tag{14}$$

the value $a = \sqrt{2\pi\beta}$ being of the order of the interaction radius. Assuming a to be small at the nuclear scale the nucleon-nucleon amplitude can be treated as a point-like function,

$$\Gamma_{NN}(x) \simeq \frac{1}{2}\sigma_{NN}^{\text{tot}}\delta^{(2)}(x). \tag{15}$$

If $\Delta(x-y)$ is point-like the integrals over $\Phi(x)$ fields in (11) are independent for different coordinate values. It turns the functional integral into the infinite product of finite dimension integrals, that can be separately evaluated for each x. It results into the generating function (see [7] for details)

$$\begin{aligned} Z(u,v) = Ce^{W_y(u,v)}, \\ W_y(u,v) = \frac{1}{a^2}\int d^2x \\ \times \ln\left(\sum_{M\le A, N\le B}\frac{z_y^{MN}}{M!N!}[a^2u\rho_A(x-b)]^M[a^2v\rho_B(x)]^N\right), \end{aligned} \tag{16}$$

with u- and v-independent constant C irrelevant in (12) and

$$z_y = 1 - \frac{1}{2}\frac{\sigma_{NN}^{\text{tot}}}{a^2}.$$

The sums over M and N can always be truncated up to A and B because the higher terms obviously do not contribute to the derivatives in (12). Put differently, the number of contributions to the generating function does not exceed the number of various brackets in the initial product (2).

Table 1. The reaction and the total cross sections of $^{12}C-^{12}C$ collision at the energy 950 MeV per nucleon and $R_{\rm rms} = 2.49$ fm. The first two columns present the results of the optical and rigid target approximations, the second two columns are for the results obtained with the full generating function, assuming $A \gg 1$ (third column) and exactly differentiating it (fourth column)

	Optical approximation	Rigid target approximation	Assuming $A \gg 1$	Exact differentiation
σ^r, mb	952	911	857	867
$\sigma^{\rm tot}$, mb	1572	1470	1371	1363

5. To elaborate Eq. (16), further we expand $W_y(u,v)$ into the series built of the densities overlaps,

$$t_{m,n}(b) = \frac{1}{a^2}\int d^2x[a^2\rho_A^\perp(x-b)]^m[a^2\rho_B^\perp(x)]^n. \quad (17)$$

Since $t_{0,1}(b) = t_{1,0}(b) = 1$ we have $W_y(u,v) = u + v + F(u,v)$ and the amplitude reads

$$S_{AB}(b) = \sum_{k,j\le A,B}\frac{A!B!}{(A-k)!(B-j)!} \times \frac{1}{k!}\frac{\partial^k}{\partial u^k}\frac{1}{j!}\frac{\partial^j}{\partial v^j}e^{F(u,v)}\Big|_{u=v=0}. \quad (18)$$

For $A, B \gg 1$ one may assume that $k, j \ll A, B$ and $A!/(A-k)!\,B!/(B-j)! \approx A^k B^j$, that gives

$$S_{AB}(b) \approx e^{F(A,B)}. \quad (19)$$

Really the functions $t_{m,n}(b)$ decrease as the indices m, n grow. Keeping only the lowest, $m = n = 1$, we arrive at the well known optical approximation [8]

$$F(A,B) = -\frac{1}{2}\sigma_{NN}^{\rm tot}/a^2T_{AB}(b), \quad T_{AB}(b) = ABt_{1,1}(b). \quad (20)$$

The optical approximation is equivalent to the requirement that each nucleon from one nucleus interacts with another nucleus no more than once.

Another known approximation easily reproduced here is the rigid target (or projectile) approximation [9, 10]. It allows any nucleon from the projectile to interact with several nucleons from the target whereas any target nucleon can interact no more than once. Though it seems to be rather natural when the atomic weight of the projectile is significantly smaller than that of the target nucleus, this approximation fairly good works even for the equal atomic weights [15]. It requires one density, say, $\rho_A^\perp(x)$, to be kept in Eq. (16) only in the linear order, permitting at the same time any powers of $\rho_B^\perp(x)$. It gives

$$W_y(u,v) = \frac{1}{a^2}\int d^2x\ln\left(\sum_N\frac{1}{N!}[a^2v\rho_B^\perp(x)]^N + [a^2\rho_A^\perp(x-b)]\sum_N\frac{z_y^N}{N!}[a^2v\rho_B^\perp(x)]^N\right) = v + u\int d^2x\rho_A^\perp(x-b)e^{a^2v(z_y-1)\rho_B^\perp(x)},$$

yielding the generating function

$$Z(u,v) = e^{v+uT_{rg}(v,b)}, \quad T_{rg}(v,b) = \sum_{n=0}^\infty\frac{1}{n!}t_{1,n}(b)v^n,$$

that produce for $B \gg 1$

$$S_{AB}(b) = [T_{rg}(b)]^A, \quad T_{rg}(b) = \int d^2x\rho_A^\perp(x-b)e^{-\frac{1}{2}\sigma_{NN}^{\rm tot}\rho_B^\perp(x)}. \quad (21)$$

6. Below we present the results obtained with the full generating function (16) for the $^{12}C-^{12}C$ scattering in the energy interval 800–1000 MeV per projectile nucleon, where the experimental data exist [17]. The total cross section $\sigma_{NN}^{\rm tot}C$ mb has been taken from averaging over pp and pn values, the slope value has been chosen to be $\beta = 0.2$ fm^2. The nucleon density is parameterized by harmonic oscillator distribution well suited for light nuclei with the atomic weight $A \le 20$,

$$\rho_A(r) = \rho_0\left[1 + \frac{1}{6}(A-4)\frac{r^2}{\lambda^2}\right]e^{-\frac{r^2}{\lambda^2}}, \quad (22)$$

ρ_0 being the normalization, and the factor λ adjusted to match the nuclear mean square radius, $R_{\rm rms} = \sqrt{r_A^2}$, $r_A^2 = \int d^3r r^2\rho_A(r)$.

Upon evaluating $W_y(u,v)$ through all $t_{mn}(b)$ functions (17) for $m,n \le A = 12$ in the parametrization (22) with $R_{\rm rms} = 2.49$ fm fitted for this parametrization in [12] from Monte-Carlo simulation of $^{12}C-^{12}C$ collision, we have calculated the reaction cross section (7) and the total cross section (5). The table compares their values obtained in the optical approximation (20), in the rigid target approximation (21) and with the full generating function for two cases, first assuming $A \gg 1$ and using approximate formula (19) and second by exact differentiating the generating function.

The last two numbers in the upper row of the table are in reasonable agreement with the experimental value 853 ± 6 mb [17]. One should bear in mind that the experimentally measured value actually refers to the so-called interaction cross section rather than to the reaction one. The difference between them can be at the several percents level [18]. At the same time the obtained values are close to those of the Monte-Carlo calculations with the same parameters and the density parametrization [12].

7. Concluding, the proposed generating function (16) is appropriate for any pairs of colliding nucleus regardless their atomic weight. Apart from the above considered integrated cross sections it can provide a consistent evaluation of the differential elastic cross section (1) as well. In this case, however, one has to account for the Coulomb corrections at small scattering angles.

Taking the nucleon density as the product of single particle ones (4) we thereby neglect the nucleon-nucleon correlations. The particular correlations can be in principle accounted for in our approach, provided an appropriate wavefunction is known.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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