Nonmetricity plane waves in post Riemannian spacetime

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Abstract. Using the analogy with the properties of plane electromagnetic waves in Minkowski space, a definition of an affine-metric space of the plane wave type is given, which is characterized by the null action of the Lie derivative on the 40 components of the nonmetricity 1-form in the 4-dimensional affine-metric space. This leads to the conclusion that the nonmetricity of a plane wave type is determined by five arbitrary functions of delayed time. A theorem on the structure of the nonmetricity of the plane wave type is proved, which states that parts of the nonmetricity 1-form irreducible with respect to the Lorentz transformations of the tangent space, such as the Weyl 1-form, the trace 1-form, and the symmetric 1-form, are defined by one arbitrary function each, and the antisymmetric 1-form is defined by two arbitrary functions. Presence of arbitrary functions in the description of nonmetricity plane waves allows transmitting information with the help of nonmetricity waves.

1. Introduction

It is known that in the general theory of relativity, the existence of gravitational waves of the metric and curvature of space-time is predicted. It is generally accepted that this particular type of waves was recently discovered in LIGO and VIRGO experiments. However, in various generalizations of the general theory of relativity, in addition to metrics and curvature, other geometric quantities arise that determine the geometric structure of these modified gravity theories. For example, the Riemann-Cartan space, which is used in particular in the Einstein-Cartan gravitation theory, is additionally characterized by torsion, the Cartan-Weyl space is additionally characterized by torsion and Weyl type nonmetricity, and the general affine-metric space in addition to metric, curvature and torsion is characterized by nonmetricity of the general type [1,2].

Therefore, it is of considerable interest to study the properties of waves in Minkowski space and in geometrically generalized post-Riemannian spaces [3–20].

In order to investigate plane waves in spacetime, the curvilinear coordinates v, u, x, y and a basis $\vec{e}_\alpha$ ($\alpha = \mathrm{v}, u, x, y$) formed by the vectors, $\vec{e}_{\mathrm{v}} = \partial_{\mathrm{v}}$, $\vec{e}_u = \partial_u$, $\vec{e}_x = \partial_x$, $\vec{e}_y = \partial_y$, are introduced in space-time, where the first two vectors are isotropic, and the remaining two ones are space-like. The coordinate u has the meaning of delayed time and is interpreted as the phase of the wave. The coordinates x and y parametrize the wave surface $(\mathrm{v}, u) = const$. The vector $\vec{e}_{\mathrm{v}}$ is covariant constant and is tangent to the rays. The vectors $\vec{e}_x$ and $\vec{e}_y$ are orthogonal to the ray vector $\vec{e}_{\mathrm{v}}$ and commute.

The main idea in studying the structure of plane waves in the modified theories of gravitation consists in carrying out analogy between the properties of plane waves of metric, torsion, and nonmetricity, on the one hand, and of plane electromagnetic waves, on the other hand.

It is known [3] that the plane electromagnetic waves have symmetry of invariance with respect to the five-parameter group G_5 of space-time motions. The group G_5 leaves invariable the isotropic hypersurface, which describes the front of plane wave with constant amplitude and is generated by a vector field $\vec{X}$ with a structure [6],

$$\vec{X} = (a + b'x + c'y)\partial_{\mathrm{v}} + b\partial_x + c\partial_y, \tag{1}$$

where $a = const$, and $b(u)$, $c(u)$ are arbitrary functions, and $b'(u)$, $c'(u)$ are their derivatives.

The metric of plane gravitational wave in Riemann spacetime satisfies the condition $L_{\vec{X}} g_{\alpha\beta} = 0$, where $L_{\vec{X}}$ is the Lie derivative in the direction of vector $\vec{X}$. This fact defines the metric in the form

$$\breve{g} = 2H(u, x, y)du^2 + 2dud\mathrm{v} - dx^2 - dy^2, \tag{2}$$

$$H(u, x, y) = \frac{1}{2}A(u)x^2 + B(u)xy + \frac{1}{2}C(u)y^2, \tag{3}$$

with the conditions,

$$b'' + bA(u) + cB(u) = 0, \quad c'' + bB(u) + cC(u) = 0. \tag{4}$$

2. The structure of nonmetricity in the affine-metric theory of gravitation

In the general case the nonmetricity 1-form allows decomposition into a sum of four irreducible parts, which are invariant concerning the Lorentz transformations. This representation was carried out in the review [1]. Now we use the equivalent representation [21], according to which a splitting of nonmetricity 1-form into irreducible parts is realized by the following 1-forms: the Weyl 1-form $\overset{(W)}{\mathcal{Q}}_{ab}$, the trace 1-form $\overset{(t)}{\mathcal{Q}}_{ab}$, the antisymmetric 1-form $\overset{(a)}{\mathcal{Q}}_{ab}$, and the symmetric 1-form $\overset{(s)}{\mathcal{Q}}_{ab}$. In the four-dimensional Minkowski space ($n = 4$), the Weyl and trace irreducible parts have 4 components, and the antisymmetric and symmetric parts have 16 components.

The Weyl nonmetricity 1-form is defined as follows:

$$\overset{(W)}{\mathcal{Q}}_{ab} = \frac{1}{n} g_{ab} \mathcal{Q}, \quad \mathcal{Q} = g^{ab}\mathcal{Q}_{ab}, \tag{5}$$

where n is the dimension of the space, and $\mathcal{Q}$ is the Weyl trace 1-form, the component representation of which is

$$\mathcal{Q} = Q_c\theta^c = g^{ab}Q_{abc}\theta^c = Q^b_{bc}\theta^c. \tag{6}$$

The second trace nonmetricity 1-form is defined by

$$\overset{(t)}{\mathcal{Q}} = g^{ab}(\vec{e}_a \rfloor \mathcal{Q}_{bc})\theta^c - \frac{1}{n}\mathcal{Q} \tag{7}$$

(a symbol $\rfloor$ denotes a contraction) with the component representation

$$\overset{(t)}{\mathcal{Q}} = g^{ab}\left(Q_{bcd}\theta^d(\vec{e}_a) - \frac{1}{n}Q_{abc}\right)\theta^c = \left(Q^b_{cb} - \frac{1}{n}Q^b_{bc}\right)\theta^c. \tag{8}$$

Then the trace nonmetricity 1-form is equal to

$$\overset{(t)}{\mathcal{Q}}_{ab}= \frac{2n}{(n-1)(n+2)}\left(\theta_{(a}\vec{e}_{b)}\rfloor \overset{(t)}{\mathcal{Q}} - \frac{1}{n} g_{ab}\overset{(t)}{\mathcal{Q}}\right) \tag{9}$$

with the component representation (for $n = 4$)

$$\overset{(t)}{\mathcal{Q}}_{ab}= \frac{2}{9}\left(\overset{(t)}{Q}_a g_{bc}+ \overset{(t)}{Q}_b g_{ac} - \frac{1}{2}\overset{(t)}{Q}_c g_{ab}\right)\theta^c. \tag{10}$$

Further an auxiliary 1-form is introduced,

$$\overset{(0)}{\mathcal{Q}}_{ab}= \mathcal{Q}_{ab}- \overset{(t)}{\mathcal{Q}}_{ab} - \overset{(W)}{\mathcal{Q}}_{ab} . \tag{11}$$

by means of which the antisymmetric nonmetricity 1-form is defined as

$$\overset{(a)}{\mathcal{Q}}_{ab}= \frac{1}{3}\left[\vec{e}_a\rfloor\left(\theta^c\wedge \overset{(0)}{\mathcal{Q}}_{bc}\right) + \vec{e}_b\rfloor\left(\theta^c\wedge \overset{(0)}{\mathcal{Q}}_{ac}\right)\right], \tag{12}$$

where the symbol $\wedge$ denotes an external multiplication.

Finally, the symmetric nonmetricity 1-form is defined as follows:

$$\overset{(s)}{\mathcal{Q}}_{ab}= \mathcal{Q}_{ab}- \overset{(a)}{\mathcal{Q}}_{ab} - \overset{(t)}{\mathcal{Q}}_{ab} - \overset{(W)}{\mathcal{Q}}_{ab}=\overset{(0)}{\mathcal{Q}}_{ab} - \overset{(a)}{\mathcal{Q}}_{ab} . \tag{13}$$

3. Plane waves of nonmetricity

Let us introduce a non-holonomic basis of 1-forms as follows,

$$\theta^0 = H(u,x,y)du + d\mathrm{v}, \quad \theta^1 = du, \quad \theta^2 = dx, \quad \theta^3 = dy. \tag{14}$$

For this end, we choose basis vectors $\vec{e}_a (a = 0, 1, 2, 3)$ in the form

$$\vec{e}_0 = \vec{e}_\mathrm{v}, \quad \vec{e}_1 = -H(u,x,y)\vec{e}_\mathrm{v} + \vec{e}_u, \quad \vec{e}_2 = \vec{e}_x, \quad \vec{e}_3 = \vec{e}_y. \tag{15}$$

As a result the metric (2) will have the following non-zero non-holonomic components, where h^α_a is the matrix of inverse tetrad coefficients,

$$g_{ab} = \check{g}(\vec{e}_a, \vec{e}_b) = h^\alpha_a h^\beta_b g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{ab}, \tag{16}$$

The matrix of tetrad coefficients $h^a_\alpha = \theta^a(\vec{e}_\alpha)$ is defined as follows,

$$h^a_\alpha = \theta^a(\vec{e}_\alpha) = \begin{pmatrix} 1 & H & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h^\alpha_a = \begin{pmatrix} 1 & -H & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

The matrix h^α_a is defined as a matrix being inverse to the matrix h^a_α.

The non-holonomic basis vectors (15), $\vec{e}_a = \vec{e}_\alpha h^\alpha_a$ $(a = 0,1,2,3; \alpha = \mathrm{v}, u, x, y)$, satisfy the following commutative relations,

$$[\vec{e}_1, \vec{e}_2] = (\partial_x H)\vec{e}_0, \qquad [\vec{e}_1, \vec{e}_3] = (\partial_y H)\vec{e}_0. \tag{18}$$

and the remained commutative relations are equal to zero.

Definition [19, 20]. A general affine-metric space (L_4, g) is called a space of the plane-wave type, and its metric, torsion, and nonmetricity are called plane waves of metric, torsion, and nonmetricity, respectively, if the Lie derivative $L_{\vec{X}}$ acting on the metric g_{ab}, the torsion 2-form T^a, and nonmetricity 1-form $\mathcal{Q}_{ab}$ of this space satisfies the conditions, $L_{\vec{X}} g_{ab} = 0$, $L_{\vec{X}} T^a = 0$, $L_{\vec{X}} \mathcal{Q}_{ab} = 0$, where the vector field $\vec{X}$ generates the five-parameter group G_5, which is the invariance symmetry group of plane electromagnetic waves in Minkowski space.

Let us present 40 components $\mathcal{Q}_{ab}$ of nonmetricity in the space (L_4, g) at $n = 4$ as a 1-form with arbitrary coefficients depending on the delayed time u:

$$\begin{aligned}
\mathcal{Q}_{00} &= i_0\theta^0 + j_0\theta^1 + k_0\theta^2 + l_0\theta^3,\\
\mathcal{Q}_{01} &= i_1\theta^0 + j_1\theta^1 + k_1\theta^2 + l_1\theta^3,\\
\mathcal{Q}_{02} &= i_2\theta^0 + j_2\theta^1 + k_2\theta^2 + l_2\theta^3,\\
\mathcal{Q}_{03} &= i_3\theta^0 + j_3\theta^1 + k_3\theta^2 + l_3\theta^3,\\
\mathcal{Q}_{11} &= i_4\theta^0 + j_4\theta^1 + k_4\theta^2 + l_4\theta^3,\\
\mathcal{Q}_{12} &= i_5\theta^0 + j_5\theta^1 + k_5\theta^2 + l_5\theta^3,\\
\mathcal{Q}_{13} &= i_6\theta^0 + j_6\theta^1 + k_6\theta^2 + l_6\theta^3,\\
\mathcal{Q}_{22} &= i_7\theta^0 + j_7\theta^1 + k_7\theta^2 + l_7\theta^3,\\
\mathcal{Q}_{23} &= i_8\theta^0 + j_8\theta^1 + k_8\theta^2 + l_8\theta^3,\\
\mathcal{Q}_{33} &= i_9\theta^0 + j_9\theta^1 + k_9\theta^2 + l_9\theta^3.
\end{aligned} \tag{19}$$

Let us solve the equation, $L_{\vec{X}} \mathcal{Q}_{ab} = 0$. It is known that in case of nonholonomic basis the components of the Lie derivative acting on the 1-form $\sigma = \sigma_a\theta^a$ have the form [22],

$$(L_{\vec{X}}\sigma)_a = X^b(\vec{e}_b\sigma_a) + (\vec{e}_a X^b)\sigma_b + X^b(L_{\vec{e}_a}\vec{e}_b)^c\sigma_c. \tag{20}$$

Also in according to the properties of the Lie derivative, we have $L_{\vec{e}_a}\vec{e}_b = [\vec{e}_a, \vec{e}_b]$. As a consequence, components of the Lie derivative acting on the 1-form of nonmetricity $\mathcal{Q}_{ab} = Q_{abc}\theta^c$ according to equation (20) will have the following form:

$$\begin{aligned}
(L_{\vec{X}}\mathcal{Q}_{ab})_c =& X^d(\vec{e}_d Q_{abc}) + (\vec{e}_a X^d + [\vec{e}_a, \vec{e}_e]^d X^e)Q_{dbc} + \\
& (\vec{e}_b X^d + [\vec{e}_b, \vec{e}_e]^d X^e)Q_{adc} + (\vec{e}_c X^d + [\vec{e}_c, \vec{e}_e]^d X^e)Q_{abd} = 0.
\end{aligned} \tag{21}$$

After the calculations and equating the coefficients of equation (21) to zero, for arbitrary functions $b(u)$, $c(u)$ and their first derivatives, and also taking into account the conditions (4), we find the nonzero coefficients in equalities (19) and the relations between them.

For example, the result of action of the Lie derivative on a component of nonmetricity $\mathcal{Q}_{01}$ (here the prime denotes the derivative with respect to the coordinate u) has the form,

$$\begin{aligned}
L_{\vec{X}}\mathcal{Q}_{ab} =& (b'i_2 + c'i_3)d\mathrm{v} + (b'Hi_2 + b'j_2 + b'k_1 + c'Hi_3 + c'j_3 + c'l_1)du + \\
& (b'l_1 + b'k_2 + c'k_3)dx + (b'l_2 + c'_2 i_1 + c'l_3)dy = 0.
\end{aligned} \tag{22}$$

We find zero coefficients and relations between nonzero coefficients,

$$i_2 = i_3 = k_3 = l_2 = 0, \quad j_2 + k_1 = 0, \quad j_3 + l_1 = 0, \quad l_1 + k_2 = 0, \quad i_1 + l_3 = 0. \tag{23}$$

After the similar calculations of the action of the Lie derivative on all components of nonmetricity (19) and resolving the relations between the arbitrary nonzero coefficients, we obtain the following result,

$$\begin{aligned}
&\mathcal{Q}_{00} = \mathcal{Q}_{02} = \mathcal{Q}_{03} = \mathcal{Q}_{23} = 0,\\
&\mathcal{Q}_{01} = j_1\theta^1, \quad \mathcal{Q}_{11} = i_4\theta^0 + j_4\theta^1 + k_4\theta^2 + l_4\theta^3, \quad \mathcal{Q}_{22} = j_7\theta^1,\\
&\mathcal{Q}_{12} = j_5\theta^1 + k_5\theta^2, \quad \mathcal{Q}_{13} = j_6\theta^1 + l_6\theta^3, \quad \mathcal{Q}_{33} = j_9\theta^1,\\
&i_4{+}2k_5 = 0,\ k_5 = l_6,\ 2j_5 + k_4 = 0,\ 2j_6 + l_4 = 0,\ j_1 + j_7 + k_5 = 0,\ j_7 = j_9.
\end{aligned} \tag{24}$$

We receive eleven nonzero arbitrary functions,

$$j_1(u),\ i_4(u),\ j_4(u),\ k_4(u),\ l_4(u),\ j_5(u),\ k_5(u),\ j_6(u),\ l_6(u),\ j_7(u),\ j_9(u).$$

and six relations between them. Thus, we obtained that the plane waves of nonmetricity are determined by five arbitrary functions.

4. The structure of plane waves of nonmetricity

Let us prove the following theorem [19, 20].

Theorem. The nonmetricity 1-form of the affine-metric space (L_4, g) of the plane wave type has the following structure: its three irreducible parts invariant under the Lorentz transformations, namely, the Weyl 1-form, the trace 1-form, and the symmetric 1-form are defined by one arbitrary function each, and the antisymmetric nonmetricity 1-form is defined by two arbitrary functions.

Let us substitute (under $n = 4$) the solutions (24) of equation $L_{\vec{X}}\mathcal{Q}_{ab} = 0$ into (6), (8), (10) with taking into account the values of the metric tensor (16). As a result, we obtain for the Weyl trace,

$$\mathcal{Q} = g^{ab}Q_{abc}\theta^c = 2(2j_1 + k_5)\theta^1. \tag{25}$$

Therefore, due to (5) and (25), the nonzero components of the Weyl nonmetricity 1-form are equal to

$$\overset{(W)}{\mathcal{Q}}_{01} = \frac{1}{2}(2j_1 + k_5)\theta^1 = -\overset{(W)}{\mathcal{Q}}_{22} = -\overset{(W)}{\mathcal{Q}}_{33}. \tag{26}$$

They are determined only by an expression $(2j_1{+}k_5)$, which can be regarded as a single arbitrary function of u.

For the second trace nonmetricity 1-form (7) we find:

$$\overset{(t)}{\mathcal{Q}} = g^{ab}Q_{acb}\theta^c - \frac{1}{4}\mathcal{Q} = -\frac{9}{2}k_5\theta^1. \tag{27}$$

Therefore, owing to (10), the nonzero values of the trace nonmetricity 1-form will be equal to

$$\overset{(t)}{\mathcal{Q}}_{01} = -\frac{1}{2}k_5\theta^1, \quad \overset{(t)}{\mathcal{Q}}_{11} = -2k_5\theta^0, \quad \overset{(t)}{\mathcal{Q}}_{12} = k_5\theta^2, \quad \overset{(t)}{\mathcal{Q}}_{13} = k_5\theta^3, \quad \overset{(t)}{\mathcal{Q}}_{22} = \overset{(t)}{\mathcal{Q}}_{33} = -\frac{1}{2}k_5\theta^1. \tag{28}$$

Thus, the trace nonmetricity 1-form is defined by only one arbitrary function $k_5(u)$.

After calculating the nonzero values of the auxiliary 1-form (11) on the basis of formulas (24), (26) and (28), we find

$$\overset{(0)}{\mathcal{Q}}_{11} = j_4\theta^1 + k_4\theta^2 + l_4\theta^3, \quad \overset{(0)}{\mathcal{Q}}_{12} = j_5\theta^1, \quad \overset{(0)}{\mathcal{Q}}_{13} = j_6\theta^1. \tag{29}$$

Let us calculate from (12) the nonzero values of the anti-symmetric nonmetricity1-form,

$$\overset{(a)}{\mathcal{Q}}_{11} = -2(j_5\theta^2 + j_6\theta^3), \quad \overset{(a)}{\mathcal{Q}}_{12} = j_5\theta^1, \quad \overset{(a)}{\mathcal{Q}}_{13} = j_6\theta^1, \tag{30}$$

which are defined by two arbitrary functions $j_5(u)$ and $j_6(u)$.

Then using the formulas (13), (29) and (30), we find the only nonzero component of the symmetric 1-form of nonmetricity,

$$\overset{(s)}{\mathcal{Q}}_{11} = j_4\theta^1, \tag{31}$$

which is defined by only one arbitrary function $j_4(u)$.

5. Conclusion

A definition of an affine-metric space of the plane wave type is formulated as a space in which the metric, torsion and nonmetricity satisfy the conditions, $L_{\vec{X}} g_{ab} = 0$, $L_{\vec{X}} T^a = 0$, $L_{\vec{X}} \mathcal{Q}_{ab} = 0$, where the Lie derivative $L_{\vec{X}}$ acts along the vector $\vec{X}$ generating the five-parameter group of invariance of plane electromagnetic waves. Here an analogy with the properties of plane electromagnetic waves in Minkowski space has been used. Calculating the action of the Lie derivative on the nonmetricity 1-form in a 4-dimensional affine-metric space, we have obtained that nonmetricity of the plane wave type is determined only by five arbitrary functions of delayed time.

The following theorem has proved that defines the structure of plane nonmetricity waves: parts of the nonmetricity 1-form irreducible with respect to the Lorentz transformations of the tangent space, such as the Weyl 1-form, the trace 1-form, and the symmetric 1-form, are defined by one arbitrary function each, and antisymmetric nonmetricity 1-form is defined by two arbitrary functions.

This proves *the possibility of transmitting information with the help of nonmetricity waves*, since the presence of arbitrary functions that determine the propagation of nonmetricity plane waves allows them to be arbitrarily encoded in the source of these waves.

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