Comparison of Two Competing Theories of 3-Flavor Neutrino Oscillations

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Abstract. Neutrino oscillation observations are used to compare two competing theories of 3-flavor neutrino oscillations. The two theories considered here are the standard model of neutrino oscillations, and parametrized Relativistic Quantum Theory (pRQT). pRQT is a manifestly covariant quantum theory with invariant evolution parameter. Recent data and a neutrino mass model from each theory are used to calculate neutrino masses. The models yield significantly different predictions of neutrino masses.

1. Introduction

Experiments with solar neutrinos, atmospheric neutrinos, reactor neutrinos, and accelerator neutrinos have demonstrated that flavor mixing can occur between neutrino flavors composed of neutrino mass states. Observations of neutrino oscillations can be used to assess the validity of two theories of transitions between three neutrino flavor states $\{|\nu_\alpha\rangle; \alpha = e, \mu, \tau\}$ given the assumption that neutrinos are composed of up to three mass states $\{|\nu_j\rangle; j = 1, 2, 3\}$. Models of neutrino flavor oscillations based on the standard theory [1] and the parametrized Relativistic Quantum Theory (pRQT) show that significant differences exist between model results.

Parametrized Relativistic Quantum Theory (pRQT) is a manifestly covariant quantum theory with invariant evolution parameter. Introductions to pRQT are presented by Fanchi [2,3], Pavšič [4,5], and Horwitz [6]. A review of relativistic classical mechanics and electrodynamics in the parametrized framework is given by Land and Horwitz [7].

A model of neutrino oscillations by mass state mixing was developed within the context of pRQT by Fanchi [8-10]. The difference between the standard model of vacuum flavor mixing and the pRQT model of vacuum flavor mixing was studied by Rusov and Vlasenko [11]. Results of their model are analyzed here and updated using 2020 data [1].

Mass-state transitions in pRQT provide a mechanism for modeling neutrino oscillations. The single-body and N-body formulations of pRQT are outlined in Sections 2 and 3, respectively. An s-clock for quantifying the invariant evolution parameter s is constructed in Section 3 for use in the pRQT model of neutrino oscillations. The mass basis and flavor basis for three flavors are introduced in Section 4 and the form of the neutrino mass matrix used in the neutrino oscillation models is presented in Section 5. An algorithm for calculating neutrino masses is presented in Section 6. Results of the standard and pRQT models are updated and compared using 2020 data from the Particle Data Group [Zyla, et al., 2020] in Section 7. Conclusions are presented in Section 8.

2. Probabilistic Formulation of Parametrized Relativistic Quantum Theory

The probabilistic formulation of pRQT begins with the assumption that a physical system can be represented by a conditional probability density $\rho(x|s)$. The position four-vector x has components

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$\{x^\mu: \mu = 0,1,2,3\}$ where index 0 signifies the time component and by indices 1, 2, 3 signify three space components. The probability density $\rho(x|s)$ is conditioned by invariant evolution parameter s.

According to probability theory, the conditional probability density $\rho(x|s)$ must be positive definite and normalizable. The Born representation of the positive definite requirement is used to express the probability amplitude Ψ as

$$\rho(x|s) = \Psi^*(x,s)\Psi(x,s) \geq 0 \tag{1}$$

with normalization condition

$$\int_D \rho(x|s) d^4x = 1 \tag{2}$$

and D denotes spacetime volume. The probability amplitude Ψ is also referred to as the field or wave function of the system. It is specified to within the gauge transformation

$$\Psi(x,s) = \sqrt{\rho(x|s)}\, e^{i\xi(x,s)} \tag{3}$$

where ξ is a scalar function.

Equation (2) implies that a single particle is observable somewhere in space at some point in time. On the other hand, Equation (2) does not imply that the particle exists at all times. The conditional probability density $\rho(x|s)$ for a single particle can be expressed as the product of two conditional probabilities: $\rho(x|s) = \rho(x^1, x^2, x^3|x^0, s)\rho(x^0|s)$. The distribution $\rho(x^0|s)$ is the marginal probability density in time and is conditioned by the evolution parameter s. The particle cannot be detected anywhere in space when $\rho(x^0|s)$ is zero because the probability $\rho(x^0|s)$ of observing a particle at time x^0 given parameter s is zero. By contrast, when $\rho(x^0|s)$ is nonzero, there is a nonzero probability of observing a particle at time x^0 given parameter s [2,3,12].

The continuity equation for conservation of probability is

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x_\mu}(\rho V^\mu) = 0 \tag{4}$$

Equation (4) can be combined with Equation (1) to give the probability flux

$$\rho V^\mu = -\frac{i\hbar}{2m}\left[\Psi^* \frac{\partial \Psi}{\partial x_\mu} - \Psi \frac{\partial \Psi^*}{\partial x_\mu}\right] - \frac{eA^\mu}{mc}\Psi^*\Psi \tag{5}$$

for a particle with mass m, charge e, four-velocity

$$V^\mu(x,s) = \frac{\hbar}{m}\frac{\partial \xi(x,s)}{\partial x_\mu} - \frac{e}{mc}A^\mu(x,s) \tag{6}$$

and electromagnetic four-vector potential A^μ.

The parametrized field equation

$$i\hbar \frac{\partial \Psi}{\partial s} = K\Psi \tag{7}$$

is constructed from Equations (1) to (6). The mass operator K has the form

$$K = \frac{\pi^{\mu}\pi_{\mu}}{2m} + V \tag{8}$$

with potential energy V and π^{μ} is the four-momentum operator with minimal coupling

$$\pi^{\mu} = \frac{\hbar}{i}\frac{\partial}{\partial x_{\mu}} - \frac{e}{c}A^{\mu} \tag{9}$$

The term A^{μ} is the four-vector potential. Equation (7) is called the Stueckelberg equation for a single particle. An alternative construction of an s-dependent field equation in terms of a set of fundamental postulates is known as the Stueckelberg-Horwitz-Piron (SHP) theory. The SHP theory is described by Horwitz [6].

In the pRQT formulation, the definition of the expectation value of an observable Ω is

$$\langle \Omega \rangle = \int \Psi^{*}\Omega\Psi dx \tag{10}$$

with the uncertainty principle over spacetime

$$\left|\Delta x_{\mu}\right| \left|\Delta p_{\mu}\right| \geq \frac{\hbar}{2} \tag{11}$$

Summation over repeated indices is not implied in Equation (11). The uncertainty principle for both energy and three-momentum is a consequence of the manifestly covariant probabilistic formulation.

3. Formulation of pRQT for an N-Body System

Extension of the probabilistic formulation for the single-particle system outlined in Section 2 to an N-body system makes it possible to combine an experimental system with a system that can function as an s-clock, that is, a system for monitoring the invariant parameter s. Field equations for the N-body system can be written as

$$i\hbar\frac{\partial \underline{\Psi}}{\partial s} = \underline{\underline{K_N}}\,\underline{\Psi} = \left\{\sum_{a=1}^{N}\frac{\pi_a^{\mu}\pi_{a\mu}}{2m_a}\underline{\underline{I}} + \underline{\underline{V}}\right\}\underline{\Psi} \tag{12}$$

where $\underline{\Psi}$ is a column vector with elements $\underline{\Psi}^{T} = [\psi_1, \psi_2, \ldots, \psi_{\Lambda}]$ and Λ labels associated internal variables. The Hermitian operator $\underline{\underline{K_N}}$ is interpreted as an N-body mass operator with a "minimal coupling" electromagnetic interaction, $\underline{\underline{I}}$ is the identity matrix, and $\underline{\underline{V}}$ represents non-electromagnetic interactions. The four-momentum operator in the bracketed term is

$$\pi_a^{\mu} = p_a^{\mu} - \frac{e}{c}A_a^{\mu} \tag{13}$$

with

$$p_a^{\mu} = \frac{\hbar}{i}\frac{\partial}{\partial x_{a\mu}} \tag{14}$$

and subscript a labels particle a for $1 \leq a \leq N$. Particle a exists at some time and some place within a $4N$ hypervolume D^N. The expectation value of an observable Ω in the N-body formalism is

$$\langle \Omega \rangle = \int_{D^N} \underline{\Psi}^+ \Omega \underline{\Psi} dx \tag{15}$$

where $\underline{\Psi}^+$ is the conjugate transpose of $\underline{\Psi}$ and the integral is over D^N.

The N-body formulation can be used to quantify parameter s by constructing an invariant evolution parameter s-clock. The system in Figure 1 consists of an experimental system with N_e bodies and an s-clock with N_c bodies. The experimental system should not significantly interact with the s-clock. Spacetime $\{x, t\}$ coordinates of experimental system 1 and the s-clock system 2 are used to determine parameters s_1, s_2 for system 1 and system 2, respectively. The procedure. is summarized below and discussed in more detail in the literature [3,13].

An N-body system consisting of an s-clock and a model of neutrino oscillations is constructed by considering a system with two particles. The experimental system consists of Particle 1 interacting with potential V_I. Particle 2 is a free, scalar particle that serves as a simple s-clock. Particles 1 and 2 do not interact with each other. The two-particle system is represented by the field equation

$$\left\{ i\hbar \frac{\partial}{\partial s} - \left[\frac{p_1^\mu p_{1\mu}}{2m_1} + \frac{p_2^\mu p_{2\mu}}{2m_2} + V_I \right] \right\} \psi(1,2,s) = 0 \tag{16}$$

where the s-dependent eigenfunction $\psi(1,2,s)$ of the two-particle system applies to both particles 1 and 2. The eigenfunction $\psi(1,2,s)$ has a set of 4-space components y_1^μ, y_2^μ for each particle.

A solution to Equation (16) is obtained by writing $\psi(1,2,s)$ as the product of single particle eigenfunctions

$$\psi(1,2,s) = \psi(1,s)\psi(2,s) \tag{17}$$

Substituting Equation (17) into Equation (16) lets us separate Equation (16) into the two equations

$$\left\{ i\hbar \frac{\partial}{\partial s} - \left[\frac{p_1^\mu p_{1\mu}}{2m_1} + V_I + \alpha_R \right] \right\} \psi(1,s) = 0 \tag{18}$$

and

$$\left\{ i\hbar \frac{\partial}{\partial s} - \left[\frac{p_2^\mu p_{2\mu}}{2m_2} - \alpha_R \right] \right\} \psi(2,s) = 0 \tag{19}$$

with separation constant α_R. Particles 1 and 2 are physically independent so $\alpha_R = 0$.

Equation (19) with $\alpha_R = 0$ is the free particle equation

$$i\hbar \frac{\partial \psi_f(2,s)}{\partial s} = \left[-\frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^\mu \partial x_{2\mu}} \right] \psi_f(2,s) \tag{20}$$

with the particular solution

$$\psi_f(2,s) = \eta_f^{1/2} exp\left[-\frac{i\hbar}{2m_2} \left(k_2^\mu k_{2\mu} \right) s + i k_{2\mu} x_2^\mu \right] \tag{21}$$

and normalization constant η_f.

The most probable trajectory of particle 2 in the classical limit of negligible dispersion is

$$(\delta s)^2 = (s - s_0)^2 = \frac{1}{c^2}\delta\langle x_2^\mu\rangle\delta\langle x_{2\mu}\rangle \tag{22}$$

where s_0 is the invariant evolution parameter when the rest frame clocks of particles 1 and 2 are calibrated. If we synchronize the clocks at $s_0 = 0$, Equation (22) becomes

$$s^2 = \frac{1}{c^2}\delta\langle x_2^\mu\rangle\delta\langle x_{2\mu}\rangle \tag{23}$$

Now assume the motion of the free particle is linear and neglect statistical variations to obtain

$$s^2 = \delta t^2 - \frac{\delta x^2}{c^2} = \delta t^2[1 - \beta^2] \tag{24}$$

where $\{x_2^0, x_2^1\} = \{t, x\}$, and

$$\beta = \frac{v}{c} \text{ with } v \equiv \frac{\delta x}{\delta t} \tag{25}$$

The distance δx traveled by particle 2 in the interval δt can be written as L so that Equation (24) becomes

$$s = \frac{L}{v}\,[1 - \beta^2]^{1/2} = \frac{L}{c}\frac{[1 - \beta^2]^{1/2}}{\beta} \tag{26}$$

The invariant evolution parameter s is quantified by measuring the spacetime trajectory of particle 2 and using the resulting value of s in Equation (18) associated with particle 1. This s-clock has been used to model neutrino oscillations [8,11].

4. Mass Basis and Flavor Basis

Two theories of transitions between three neutrino flavor states $\{|\nu_\alpha\rangle; \alpha = e,\ \mu,\ \tau\}$ composed of up to three mass states $\{|\nu_j\rangle; j = 1,\ 2,\ 3\}$ are compared here. The basis of mass states $\{|\nu_j\rangle; j = 1,\ 2,\ 3\}$ is related to the basis of flavor states $\{|\nu_\alpha\rangle; \alpha = e,\ \mu,\ \tau\}$ by a unitary transformation:

$$\begin{bmatrix}|\nu_e\rangle\\ |\nu_\mu\rangle\\ |\nu_\tau\rangle\end{bmatrix} = U \begin{bmatrix}|\nu_1\rangle\\ |\nu_2\rangle\\ |\nu_3\rangle\end{bmatrix} \tag{27}$$

where the mass and flavor states are written as 3-component column vectors. The elements of the unitary matrix are written as

$$U = \begin{bmatrix}u_{e1} & u_{e2} & u_{e3}\\ u_{\mu1} & u_{\mu2} & u_{\mu3}\\ u_{\tau1} & u_{\tau2} & u_{\tau3}\end{bmatrix} \tag{28}$$

The inverse of the unitary matrix U is the conjugate transpose of U, thus

$$U^{-1} = (U^*)^T \tag{29}$$

with elements

$$u_{\alpha j}^{-1} = u_{j\alpha}^{*} \; ; \; j = 1,2,3 \text{ and } \alpha = e, \mu, \tau \tag{30}$$

The expanded form of the unitary transformation is

$$\begin{aligned} |\nu_e\rangle &= u_{e1}|\nu_1\rangle + u_{e2}|\nu_2\rangle + u_{e3}|\nu_3\rangle \\ |\nu_\mu\rangle &= u_{\mu 1}|\nu_1\rangle + u_{\mu 2}|\nu_2\rangle + u_{\mu 3}|\nu_3\rangle \\ |\nu_\tau\rangle &= u_{\tau 1}|\nu_1\rangle + u_{\tau 2}|\nu_2\rangle + u_{\tau 3}|\nu_3\rangle \end{aligned} \tag{31}$$

A mass basis state satisfies the temporal evolution equation

$$T|\nu_j\rangle = i\hbar \frac{\partial}{\partial \tau}|\nu_j\rangle = T_j|\nu_j\rangle \tag{32}$$

where T_j is the eigenvalue of the temporal operator $T = i\hbar \frac{\partial}{\partial \tau}$, and τ is the temporal evolution parameter. In the standard model, $(T_j)_{Std} = E_j$ where E_j is the energy of state j, and $(\tau)_{Std} = t$ with coordinate time t. In the pRQT model, $(T_j)_{pRQT} = K_j$ where K_j is the eigenvalue of state j for the mass operator K, and $(\tau)_{pRQT} = s$ with invariant evolution parameter s. Equation (32) has the formal solution

$$\begin{bmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{bmatrix} = \begin{bmatrix} e^{-i\frac{T_1\tau}{\hbar}} & 0 & 0 \\ 0 & e^{-i\frac{T_2\tau}{\hbar}} & 0 \\ 0 & 0 & e^{-i\frac{T_3\tau}{\hbar}} \end{bmatrix} \begin{bmatrix} |\nu_1(0)\rangle \\ |\nu_2(0)\rangle \\ |\nu_3(0)\rangle \end{bmatrix} \tag{33}$$

where $\nu_j(0)$ is mass state j at $\tau = 0$.

5. Neutrino Mass Matrix Model

An analysis of the competing formalisms for vacuum-favor mixing of neutrinos within the context of the standard and pRQT models requires a basis for comparison. In this case, we seek estimates of neutrino masses based on current experimental data. To achieve this goal, we present a procedure for estimating neutrino masses from a 3×3 neutrino mass matrix M_ν using a procedure introduced by Damanik [14]. Damanik's procedure is a method for obtaining phenomenological estimates of neutrino masses from experimental estimates of mass-squared differences and mixing parameters.

Damanik [14] constructed an unperturbed neutrino mass matrix $M_{\nu 0}$

$$M_{\nu 0} = \begin{bmatrix} P & Q & Q \\ Q & P & Q \\ Q & Q & P \end{bmatrix} \tag{34}$$

that is non-singular and invariant with respect to a cyclic permutation of neutrino states, that is, $\nu_1 \rightarrow \nu_2 \rightarrow \nu_3 \rightarrow \nu_1$. Damanik used the seesaw mechanism [Gell-Mann, et al., 1979; Yanagida, et al., 1979] to help motivate the form of $M_{\nu 0}$. The presentation here focuses on mathematical arguments rather than theoretical motivations to minimize the dependence of the form of the neutrino mass matrix on a specific paradigm.

Damanik [14] determined the matrix elements P, Q by finding the eigenvalues of $M_{\nu 0}$, relating the eigenvalues to neutrino masses, and then showing that the resulting masses did not correctly predict observed mass squared differences Δm_{ij}^2 where $\Delta m_{ij}^2 = m_i^2 - m_j^2, (i, j = 1,2,3)$. To resolve these

problems, Damanik introduced a parameter δ as a perturbation of diagonal elements of the neutrino mass matrix M_ν:

$$M_\nu = \begin{bmatrix} P+2\delta & Q & Q \\ Q & P-\delta & Q \\ Q & Q & P-\delta \end{bmatrix} \tag{35}$$

The form of the perturbed neutrino mass matrix M_ν includes the requirement that M_ν has the same trace as the unperturbed neutrino mass matrix $M_{\nu 0}$, thus $Tr(M_{\nu 0}) = Tr(M_\nu) = 3P$. The eigenvalues $\{\beta_1, \beta_2, \beta_3\}$ of the perturbed neutrino mass matrix M_ν are

$$\begin{gathered} \beta_1 = P + \frac{Q}{2} + \frac{\delta}{2} - \frac{\sqrt{9\delta^2 - 6Q\delta + 9Q^2}}{2} \\ \beta_2 = P + \frac{Q}{2} + \frac{\delta}{2} + \frac{\sqrt{9\delta^2 - 6Q\delta + 9Q^2}}{2} \\ \beta_3 = P - Q - \delta \end{gathered} \tag{36}$$

The value of δ is obtained by finding the angle θ that diagonalizes M_ν.

The angle θ is the angle that relates the mass basis $\{|v_j\rangle; j = 1, 2, 3\}$ to the flavor basis $\{|v_\alpha\rangle; \alpha = e, \mu, \tau\}$ by the unitary transformation in Equation (27). The unitary matrix used by Damanik [14] to relate the mass basis to the flavor basis is

$$U = \begin{bmatrix} cos\,\theta & -sin\theta & 0 \\ \frac{sin\theta}{\sqrt{2}} & \frac{cos\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{sin\theta}{\sqrt{2}} & \frac{cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{37}$$

with the inverse

$$U^{-1} = \begin{bmatrix} cos\,\theta & \frac{sin\theta}{\sqrt{2}} & \frac{sin\theta}{\sqrt{2}} \\ -sin\theta & \frac{cos\theta}{\sqrt{2}} & \frac{cos\theta}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{38}$$

The perturbation parameter δ is obtained from the relation

$$tan^2(2\theta) = \frac{8Q^2}{(Q-3\delta)^2} \tag{39}$$

where the angle θ and the associated unitary matrix U diagonalize M_ν. The solution of Equation (39) for δ gives

$$\delta = \varepsilon Q, \varepsilon = \frac{[tan(2\theta) - \sqrt{8}]}{3tan(2\theta)} \tag{40}$$

The term ε is calculated given a value of θ. In this case, the perturbation parameter δ is proportional to θ.

Neutrino masses are given by the equations

$$\begin{aligned} m_1 &= P + \frac{Q}{2} + \frac{\delta}{2} - \frac{\sqrt{9\delta^2 - 6Q\delta + 9Q^2}}{2} \\ m_2 &= P + \frac{Q}{2} + \frac{\delta}{2} + \frac{\sqrt{9\delta^2 - 6Q\delta + 9Q^2}}{2} \\ m_3 &= P - Q - \delta \end{aligned} \tag{41}$$

Damanik's procedure yields a set of neutrino masses that is an inverted hierarchy, thus

$$|m_3| < |m_1| < |m_2| \tag{42}$$

6. Algorithm for Calculating Neutrino Masses

The equations in Damanik's [14] procedure presented above can be combined with experimental measurements of mass squared differences and mixing angles to provide a phenomenological estimate of the variables $\{P, Q, \delta\}$ and corresponding neutrino masses $\{m_1, m_2, m_3\}$. We begin by writing a simplified form of the neutrino mass equations. If we use the proportionality relationship between δ and θ in Equation (40), neutrino masses may be written in the form

$$\begin{aligned} m_1 &= P + a_1 Q \\ m_2 &= P + a_2 Q \\ m_3 &= P + a_3 Q \end{aligned} \tag{43}$$

where

$$\begin{aligned} a_1 Q &= \frac{Q}{2} + \frac{\delta}{2} - \frac{\sqrt{9\delta^2 - 6Q\delta + 9Q^2}}{2} \\ a_2 Q &= \frac{Q}{2} + \frac{\delta}{2} + \frac{\sqrt{9\delta^2 - 6Q\delta + 9Q^2}}{2} \\ a_3 Q &= -Q - \delta \end{aligned} \tag{44}$$

Substituting $\delta = \varepsilon Q$ into Equation (44) gives $\{a_1, a_2, a_3\}$ as functions of ε:

$$\begin{aligned} a_1 &= \frac{1}{2}\left[1 + \epsilon - \sqrt{9\varepsilon^2 - 6\varepsilon + 9}\right] \\ a_2 &= \frac{1}{2}\left[1 + \epsilon + \sqrt{9\varepsilon^2 - 6\varepsilon + 9}\right] \\ a_3 &= -(1 + \varepsilon) \end{aligned} \tag{45}$$

We use the mass squared differences Δm_{ij}^2 and the relations in Equation (43) to solve for P, Q.

The mass squared differences are

$$\begin{aligned} \Delta m_{21}^2 &= m_2^2 - m_1^2 = (P + a_2 Q)^2 - (P + a_1 Q)^2 \\ \Delta m_{32}^2 &= m_3^2 - m_2^2 = (P + a_3 Q)^2 - (P + a_2 Q)^2 \end{aligned} \tag{46}$$

with the simplified forms

$$\Delta m_{21}^2 = 2QP(a_2 - a_1) + Q^2(a_2^2 - a_1^2) \tag{47}$$

and

$$\Delta m_{32}^2 = 2QP(a_3 - a_2) + Q^2(a_3^2 - a_2^2) \tag{48}$$

The variable Q is obtained by rearranging Equations (47) and (48) so that the QP term is on the right-hand side:

$$\Delta m_{21}^2 - Q^2(a_2^2 - a_1^2) = 2QP(a_2 - a_1) \tag{49}$$

and

$$\Delta m_{32}^2 - Q^2(a_3^2 - a_2^2) = 2QP(a_3 - a_2) \tag{50}$$

Dividing Equation (49) by (50), rearranging and solving for Q^2 gives

$$Q^2 = \frac{\Delta m_{21}^2 - \frac{(a_2 - a_1)}{(a_3 - a_2)}\Delta m_{32}^2}{(a_2^2 - a_1^2) - \frac{(a_2 - a_1)}{(a_3 - a_2)}(a_3^2 - a_2^2)} \tag{51}$$

The variable P is obtained by rearranging Equations (47) and (48) so that the Q^2 term is on the right-hand side:

$$\Delta m_{21}^2 - 2QP(a_2 - a_1) = Q^2(a_2^2 - a_1^2) \tag{52}$$

and

$$\Delta m_{32}^2 - 2QP(a_3 - a_2) = Q^2(a_3^2 - a_2^2) \tag{53}$$

Dividing Equation (52) by (53), rearranging and solving for P gives

$$P = \frac{\Delta m_{21}^2 - \frac{(a_2^2 - a_1^2)}{(a_3^2 - a_2^2)}\Delta m_{32}^2}{2Q\left[(a_2 - a_1) - \frac{(a_2^2 - a_1^2)}{(a_3^2 - a_2^2)}(a_3 - a_2)\right]} \tag{54}$$

An algorithm for calculating neutrino masses using the above relationships is summarized in Table 1. It assumes that values of $\Delta m_{21}^2, \Delta m_{32}^2, \theta$ are available.

Table 1 Algorithm for Calculating Neutrino Masses

Calculate	Using	Equation
ε	θ	40
$\{a_1, a_2, a_3\}$	ε	45
Q^2	$\{a_1, a_2, a_3\}, \Delta m_{21}^2, \Delta m_{32}^2$	51
P	$Q, \{a_1, a_2, a_3\}, \Delta m_{21}^2, \Delta m_{32}^2$	54
$\{m_1, m_2, m_3\}$	$P, Q, \{a_1, a_2, a_3\}$	43

7. Comparison of Neutrino Mass Models Using 2020 Data

Table 2 validates the algorithm using the data in the Rusov-Vlasenko [11] neutrino mass calculation. Rusov and Vlasenko [11] compared the standard and pRQT models by writing

$$\alpha_{pRQT} \approx 2\alpha_{Std} \tag{55}$$

The angle θ is given by $\theta = \theta_{12}$. The masses are inverted, that is, $|m_3| < |m_1| < |m_2|$. The sum of the absolute values of the neutrino masses is less than 0.6 eV.

Table 2 Verify Algorithm with Rusov-Vlasenko [11] Results		
	Algorithm Results	
Rusov-Vlasenko [11] Data	Standard Model	pRQT Model
$\Delta m_{21}^2 \times 10^5, eV^2$	7.50	15.00
$\Delta m_{32}^2 \times 10^5, eV^2$	-2.32	-4.64
$tan^2(\theta)$	0.452	0.452
Neutrino Masses	Standard Model	pRQT Model
$\|m_1\|, eV$	0.130855	0.185056
$\|m_2\|, eV$	0.131141	0.185461
$\|m_3\|, eV$	0.121975	0.172499
$\|m_1\| + \|m_2\| + \|m_3\|, eV$	0.38397	0.54302

Table 3 repeats the Rusov-Vlasenko [11] calculations of neutrino mass using updated data from Zyla, et al. [1]. The angle θ is given by $\theta = \theta_{12}$ and the masses are inverted. The sum of the absolute values of the neutrino masses is less than 0.6 eV.

Table 3 Rusov-Vlasenko [1] Calculations Updated with Zyla, et al. [1] Data		
	Algorithm Results	
Zyla, et al. [1] Data	Standard Model	pRQT Model
$\Delta m_{21}^2 \times 10^5, eV^2$	7.53	15.06
$\Delta m_{32}^2 \times 10^5, eV^2$	-2.546	-5.092
$tan^2(\theta)$	0.443	0.443
Neutrino Masses	Standard Model	pRQT Model
$\|m_1\|, eV$	0.125714	0.177787
$\|m_2\|, eV$	0.126013	0.178210
$\|m_3\|, eV$	0.115470	0.163300
$\|m_1\| + \|m_2\| + \|m_3\|, eV$	0.36720	0.51930

The relationship in Equation (55) used by Rusov-Vlasenko [11] is not the same as the pRQT result summarized in Appendix A. The ratio

$$\frac{\alpha_{Std}}{\alpha_{pRQT}} \approx \frac{m_2^2 - m_1^2}{m_\nu(m_2 - m_1)} = \frac{m_1 + m_2}{m_\nu} \approx 2 \tag{56}$$

is simplified by assuming that $m_\nu \approx m_1 \approx m_2$ so that

$$m_\nu \approx \frac{m_1 + m_2}{2} \tag{57}$$

Equation (57) becomes

$$\begin{aligned}\alpha_{pRQT} &= \frac{m_\nu(m_2-m_1)c^4}{4\hbar E_\nu}\frac{L}{c}\frac{1}{\beta} \\ &\approx \frac{\left(\frac{m_1+m_2}{2}\right)(m_2-m_1)c^4}{4\hbar E_\nu}\frac{L}{c}\frac{1}{\beta} \\ &= \frac{1}{2}\frac{(m_2^2-m_1^2)c^4}{4\hbar E_\nu}\frac{L}{c}\frac{1}{\beta}\end{aligned} \tag{58}$$

Recognizing that $\beta \approx 1$ for an ultrarelativistic neutrino and $\alpha_{Std} = \frac{(m_2^2-m_1^2)c^4}{4\hbar E_\nu}\frac{L}{c}$ from Equation (A.8) gives

$$\alpha_{pRQT} = \frac{1}{2}\alpha_{Std} \tag{59}$$

This can be related to observations by halving the mass squared difference $\Delta m_{ji}^2 = m_j^2 - m_i^2$, thus

$$\left(\Delta m_{ji}^2\right)_{pRQT} = \frac{1}{2}\left(\Delta m_{ji}^2\right)_{Std} \tag{60}$$

By contrast, Rusov-Vlasenko [11] doubled the mass-squared difference.

Table 4 presents neutrino mass results obtained using Equation (60) and updated data from Zyla, et al. [1]. The angle θ is given by $\theta = \theta_{12}$ and the masses are inverted. The sum of the absolute values of the neutrino masses is less than 0.6 eV.

Table 4 Neutrino Masses Updated with Equation (60) and Zyla, et al. [1] Data		
	Algorithm Results	
Zyla, et al. [1] Data	Standard Model	pRQT Model
$\Delta m_{21}^2 \times 10^5, eV^2$	7.53	3.77
$\Delta m_{32}^2 \times 10^5, eV^2$	-2.546	-1.27
$tan^2(\theta)$	0.443	0.443
Neutrino Masses	Standard Model	pRQT Model
$\lvert m_1\rvert, eV$	0.125714	0.088893
$\lvert m_2\rvert, eV$	0.126013	0.089105
$\lvert m_3\rvert, eV$	0.115470	0.081650
$\lvert m_1\rvert + \lvert m_2\rvert + \lvert m_3\rvert, eV$	0.36720	0.25965

8. Discussion

The inclusion of two temporal variables in parametrized Relativistic Quantum Theory (pRQT) yields a theory that is significantly different from theories that rely on a single temporal variable, namely the time coordinate of spacetime. The pRQT temporal variables are the time coordinate of spacetime, and an invariant evolution parameter. The difference between the temporal dependence of the standard model and the pRQT model can be directly observed in models of neutrino oscillations. This paper used 2020 data to compare neutrino mass results from the standard model of three-flavor neutrino oscillations and the pRQT model of three-flavor neutrino oscillations. Standard model neutrino masses shown in

Table 4 are in the range 0.11 eV to 0.13 eV, while pRQT neutrino masses are in the range 0.08 eV to 0.09 eV.

Neutrino mass model differences are significant for neutrino oscillation measurements. For example, the Karlsruhe Tritium Neutrino (KATRIN) Collaboration has a direct method of measuring neutrino mass based on beta decay of tritium into helium-3, an electron, and an electron antineutrino [17,18]. The KATRIN direct method has the advantage of being model independent, which makes it suitable for comparing neutrino mass predictions based on the standard model and pRQT model of three-flavor neutrino oscillations. One possible problem is that KATRIN is not currently sensitive to neutrino masses less than 0.2 eV [19,20]. The 0.2 eV limit is greater than the neutrino masses estimated here.

The neutrino mass differences shown here are dependent on the specified neutrino mass matrix and choice of variables used to conduct calculations. A more definitive comparison of competing models will depend on future studies of alternative mass matrices and associated algorithms to further incorporate all observations from all relevant experiments.

Appendix A. Outline of the Two-Flavor pRQT Model of Neutrino Oscillations

The two-flavor pRQT model of neutrino oscillations [8] is outlined here. The evolution equation in pRQT for a state may be written in terms of the evolution parameter s as

$$i\hbar\frac{\partial}{\partial s}|v_j\rangle = K_j|v_j\rangle \tag{A.1}$$

where K_j is the eigenvalue of the mass operator for mass state j. The evolution parameter dependent solution of Equation (A.1) in the mass basis for two mass states is

$$\begin{bmatrix}|v_1(s)\rangle\\|v_2(s)\rangle\end{bmatrix} = \begin{bmatrix}e^{-iK_1 s/\hbar} & 0\\ 0 & e^{-iK_2 s/\hbar}\end{bmatrix}\begin{bmatrix}|v_1(0)\rangle\\|v_2(0)\rangle\end{bmatrix} \tag{A.2}$$

where

$$K_j = \hbar^2 k_j^\mu k_{j\mu}/2m_j = \hbar^2\left[\left(\omega_j/c\right)^2 - k_j\cdot k_j\right]/2m_j \tag{A.3}$$

In pRQT, the components of the energy-momentum four-vector k_j^μ are observables and the mass m_j is a function of statistical values of k_j^μ.

In the flavor oscillation process $\nu_e \to \nu_\mu$, we begin with a pure beam of electron neutrino ν_e particles and calculate the probability for forming muon neutrino ν_μ particles. The pRQT result for the probability of forming the final state ν_μ from the initial state ν_e is

$$\begin{aligned}&P_{pRQT}(\nu_e \to \nu_\mu) = sin^2\,2\,\theta\,sin^2\left\{\frac{(m_2 - m_1)c^2}{4\hbar}s\right\}\\ &\equiv sin^2\,2\,\theta\,sin^2\,\alpha_{pRQT}\end{aligned} \tag{A.4}$$

where s is temporal duration measured by an s-clock.

Flavor oscillations may be described by quantifying the behavior of two particles as outlined in Section 3. One particle propagates without interaction or oscillation from the source to the detector and serves as a "clock" for the scalar evolution parameter s. The other particle is the oscillating particle. In this application, the source and detector are separated by a distance L. The most probable trajectory of the non-interacting s-clock particle is

$$s^2 = (\delta t)^2 - \frac{(\delta x)^2}{c^2} = (\delta t)^2[1-\beta^2], \beta = \frac{v}{c}, v = \frac{\delta x}{\delta t} \tag{A.5}$$

The distance δx traveled by the s-clock particle in the interval δt is L, so we obtain

$$s = \frac{L}{c}\frac{[1-\beta^2]^{1/2}}{\beta}, \delta x = L \tag{A.6}$$

Substituting Equation (A.6) into Equation (A.4) gives

$$\begin{aligned} &P_{pRQT}(\nu_e \to \nu_\mu) = sin^2\, 2\,\theta\, sin^2\, \alpha_{pRQT}\,, \\ &\alpha_{pRQT} = \frac{(m_2 - m_1)c^2}{4\hbar}\frac{L}{c}\frac{[1-\beta^2]^{1/2}}{\beta} \end{aligned} \tag{A.7}$$

The result for the standard theory is

$$\begin{aligned} &P_{Std}(\nu_e \to \nu_\mu) = sin^2\, 2\,\theta\, sin^2\, \alpha_{Std} \\ &\alpha_{Std} = \frac{(m_2^2 - m_1^2)c^4}{4\hbar E_\nu}\frac{L}{c} \end{aligned} \tag{A.8}$$

where E_ν is the energy of the ultrarelativistic incident neutrino

$$E_\nu = \frac{m_\nu c^2}{[1-\beta^2]^{1/2}} \tag{A.9}$$

We combine Equations (A.7) and (A.9) and rearrange to simplify comparison with Equation (A.8):

$$\begin{aligned} &P_{pRQT}(\nu_e \to \nu_\mu) = sin^2\, 2\,\theta\, sin^2\, \alpha_{pRQT}\,, \\ &\alpha_{pRQT} = \frac{(m_2 - m_1)c^2}{4\hbar}\frac{L}{c}\frac{m_\nu c^2}{E_\nu \beta} = \frac{m_\nu (m_2 - m_1)c^4}{4\hbar E_\nu}\frac{L}{c}\frac{1}{\beta} \end{aligned} \tag{A.10}$$

The ratio of the dynamical factors $\alpha_{pRQT}, \alpha_{Std}$ is

$$\frac{\alpha_{Std}}{\alpha_{pRQT}} = \frac{m_2^2 - m_1^2}{m_\nu (m_2 - m_1)}\beta = \frac{m_1 + m_2}{m_\nu}\beta \tag{A.11}$$

The ratio of probabilities in Equations (A.8) and (A.10) is

$$\frac{P_{Std}}{P_{pRQT}} = \frac{sin^2\, \alpha_{Std}}{sin^2\, \alpha_{pRQT}} \tag{A.12}$$

Comparing P_{pRQT}, P_{Std} and the dynamical factors $\alpha_{pRQT}, \alpha_{Std}$ shows that the pRQT model and the conventional theory have the same dependence on the flavor mixing angle θ, but their dependence on dynamical factors differs significantly. If the mass difference between neutrino mass and flavor states is very small and the neutrinos are ultrarelativistic, then $(m_1 + m_2)/m_\nu \approx 2$ and $\beta \approx 1$. The ratio of dynamical factors $\alpha_{Std}/\alpha_{pRQT} \approx 2$ in this case.

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