An Invitation to Celestial Holography

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Abstract

The *celestial holography* research programme concerns itself with the conjecture asserting the existence of a *two–dimensional celestial conformal field theory* (CCFT), supposed to be defined on the celestial sphere $\mathbb{CS}^2$ at null infinity $\mathscr{I}$, for which the set of correlation functions contains a subset the members of which are holographically dual to the scattering amplitudes of a four–dimensional quantum field theory on Minkowski spacetime $\mathbb{R}^{(1,3)}$, referred to as the *bulk*. Thus, the problem of classifying the properties of the mapping assigning to each bulk field ϕ_Δ with conformal dimension Δ on $\mathbb{R}^{(1,3)}$ a local boundary operator $\hat{\mathcal{O}}_\Delta(w, \bar{w})$ on the putative CCFT on $\mathbb{CS}^2$ is *the* fundamental question in celestial holography. In this talk, with the aim of explaining the basic ideas of this research program, we begin by recalling some elementary concepts regarding the notions of asymptotic flatness and the BMS group, and then we present a pedagogic exposition of a construction of the on–shell massive scalar conformal primary wavefunctions $\phi_{\Delta,m}$ on the four–dimensional Minkowski spacetime $\mathbb{R}^{(1,3)}$ that transforms covariantly under the $SL(2;\mathbb{C})$ realisation of the Lorentz group on the celestial sphere $\mathbb{CS}^2$. Finally, we demonstrate that (for certain mass values of the bulk field) the 3–point amplitudes reduces to the holomorphic form which would be expected for any primary 3–point correlation function of a two–dimensional conformal field theory, hence illustrating one of the fundamental tools of the celestial holography program.

1 Asymptotic Flatness and the Bondi–Metzner–Sachs Group[1]

Let us open our discussion by recalling some well–known definitions regarding asymptotically flat spacetimes and their associated symmetry groups. In the Einstein's theory of General Relativity (GR), the Galileo's principle of equivalence between gravitational and inertial masses, on the one hand, and the Mach's principle concerning the intimate relation between gravity and inertia, on

[1]For further information concerning the material presented in this section, we recommend Refs. ([1, 4]).

the other hand, are elegantly unified with the pseudo–Riemannian geometry of spacetime. It is due to the intrinsically geometric nature of gravitation that black holes and gravitational waves exists and the Universe expands. Nonetheless, the interplay of geometry and physics makes it difficult to develop the mathematical tools that are necessary to fully comprehend the general relativistic effects.

For instance, in the study of non–gravitational (e.g., electromagnetic) interactions, we have at our disposal a non–dynamical Minkowski background, which allow us to introduce a set of boundary conditions and $1/r$ fall–off rates on the fields based on our prior knowledge of the well–established physical principles of non–general relativistic circsustances. Therefore, there are well–posed notions of energy, momentum and angular momentum associated to the disturbances on these fields (such as electromagnetic waves). However, when one is interested in the gravitational interaction, taking into account the non–linear character of the Einstein's field equations, the metric tensor field, which would provide a well–defined meaning to "boundary conditions" and "fall–offs," is itself a dynamical variable. Thus, in the study of isolated systems emitting gravitational radiation, one is lead to introduce the notion of an *asymptotically flat spacetime*, for which the physical metric tensor field $\mathbf{g}_{\mu\nu}$ approaches a Minkowski metric $\eta_{\mu\nu}$ as $1/r$ (where r denotes the luminosity distance) whenever one recedes from the sources along the null geodesics. The asymptotic symmetry group, as it turns out, is an infinite–dimensional generalisation of the Poincaré group, the reason of which can be intuitively understood as follows.

Let us suppose that one is provided with an inertial coordinate system $(t, \vec{x})$, with respect to which the Minkowski metric tensor field $\eta_{\mu\nu}$, for which the family of isometry transformations are realised as the action of the Poincaré group $\mathscr{P}$, can be written in the canonical form, $\eta = -dt \otimes dt + d\vec{x} \cdot d\vec{x}$. Then, consider a coordinate transformation:

$$t \mapsto t' = t + f(\theta, \phi), \quad \vec{x} \mapsto \vec{x}' = \vec{x} \quad \left(f \in \mathscr{L}^2\left(\mathbb{S}^2; \mathbb{R}\right)\right), \tag{1.1}$$

which is allowed by the property of diffeomorphism invariance enjoyed by GR. The transformation $(t, \vec{x}) \mapsto (t', \vec{x}')$, which is an angle–dependent translation parametrised by the (square–integrable in the sense of Lebesgue) function $f(\theta, \phi)$ on the standard 2–sphere, $\mathbb{S}^2$, mapping the original Minkowski metric $\eta_{\mu\nu}$ to another Minkowski metric $\eta'_{\mu\nu}$, the latter of which is associated to the Poincaré isometry group $\mathscr{P}'$. Therefore, the asymptotic symmetry group of the physical metric tensor field $\mathbf{g}_{\mu\nu}$ under consideration must include both the Poincaré subgroups $\mathscr{P}$ and $\mathscr{P}'$. Thence, the asymptotic BMS group can be intuitively conceived as the disjoint union of the each Poincaré group associated to any Minkowski metric that can be obtained by an angle–dependent translation (also referred to as a *supertranslation*). So, on the following two paragraphs, we shall provide a rigorous mathematical definition of the notions of asymptotically flat spacetimes and the definition of the BMS group as a semi–direct product between the homogeneous Lorentz group $\mathscr{L}$ and the supertranslations.

Let $(\mathscr{M}, \mathbf{g})$ be an asymptotically flat. By definition, there exists a geodesically complete Lorentzian manifold[2] $(\hat{\mathscr{M}}, \hat{\mathbf{g}})$ and a conformal embedding $\psi : \mathscr{M} \longrightarrow \hat{\mathscr{M}}$, with conformal factor $\Omega \in \mathscr{C}^{\omega}$, such that $\psi^{*}(\hat{\mathbf{g}}) = \Omega^{2}\mathbf{g}$. The conformal boundary $\mathscr{I} := \partial(\psi(\mathscr{M}))$ is a smooth submanifold of $\hat{\mathscr{M}}$, which inherits via the inclusion map $\iota_{\mathscr{I}} : \mathscr{I} \longrightarrow \hat{\mathscr{M}}$ a degenerate fundamental form $\mathbf{q} := \iota^{*}_{\mathscr{I}}(\hat{\mathbf{g}})$ such that $(\mathscr{I}, \mathbf{q})$ is a null hypersurface isomorphic to $\mathbb{R} \times \mathbf{S}^{n-2}$. The conformal boundary can be written as the disjoint union $\mathscr{I} = \mathscr{I}^{+} \sqcup \mathscr{I}^{-}$, where $\mathscr{I}^{+}$ and $\mathscr{I}^{-}$ are (respectively) the future and past null infinity, containing the future and past endpoints of every null geodesic of the spacetime $(\mathscr{M}, \mathbf{g})$. The conformal factor can be analytically continued to $\hat{\mathscr{M}}$ so that $\Omega\big|_{\mathscr{I}} = 0$ and $\hat{\nabla}_{a}\Omega\big|_{\mathscr{I}} \neq 0$, where $\hat{\nabla}_{a}$ is the covariant derivative operator associated to the Riemann–Levi-Civita connection induced by the metric tensor $\hat{\mathbf{g}}$ on $\hat{\mathscr{M}}$. The condition of asymptotic flatness is realised by further assumption that $\hat{\nabla}_{a}\hat{\nabla}_{b}\Omega + \frac{1}{2}\hat{\mathbf{g}}_{ab}\hat{\mathbf{g}}^{cd}\hat{\nabla}_{c}\hat{\nabla}_{d}\Omega = 0$ on $\mathscr{I}$.

Now, concerning the group of asymptotic isometries acting on null infinity $\mathscr{I}$, let us conceive the Abelian group of supertranslations $\mathscr{T} := \mathscr{L}^{2}(\mathbb{S}^{2}; \mathbf{R})$ as the Lebesgue space of square–integrable real–valued functions defined on the standard 2–sphere $\mathbb{S}^{2}$ and equipped with the linear structure induced by pointwise addition. The (right) action $g \in SL(2; \mathbf{C}) \mapsto \sigma_{g} \in \mathbb{CP}^{1}$ of the 2×2 unimodular group (consisting of the 2×2 complex matrices with unit determinant), $SL(2; \mathbf{C})$, on the Riemann sphere (the complex projective line), $\mathbb{CP}^{1}$, is determined by the assignment:

$$z \in \mathbb{CP}^{1}, g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C}) \mapsto z \cdot g := \sigma_{g}(z) = \frac{az + c}{bz + d} \in \mathbb{CP}^{1}. \tag{1.2}$$

In order to state the definition of the BMS group as a semi–direct product, let us introduce the group homomorphism $\varphi : SL(2; \mathbf{C}) \longrightarrow Aut(\mathscr{T})$, which maps from the group of unimodular 2×2 complex matrices into the module constituted of the automorphism transformations of the group of supertranslations, $Aut(\mathscr{T})$, and which is given by the rule:

$$[\varphi(g) f](z, \bar{z}) := K_{g}(z, \bar{z}) f(z \cdot g, \bar{z} \cdot g), \tag{1.3}$$

whenever $f \in \mathscr{T}$ and $g \in SL(2; \mathbf{C})$. On the above equation, one recognises $K_{g}(z, \bar{z})$ as the conformal factor associated to the BMS transformation g, which is defined by the following expression:

$$K_{g}(z, \bar{z}) := \frac{1}{1 + z\bar{z}}\left(|az + c|^{2} + |bz + d|^{2}\right). \tag{1.4}$$

Therefore, let us employ the mathematical framework determined by the algebraic structures

[2]Also supposed to be connected, simply connected, locally convex.

introduced above to define the Bondi–Metzner–Sachs (BMS) group as the semi–direct product:

$$BMS \coloneqq \mathcal{T} \rtimes_{\varphi} SL\left(2;\mathbf{C}\right), \tag{1.5}$$

the law of multiplication of which is explicitly given by the following associative binary operation:

$$\left(f_1, g_1\right) \circ \left(f_2, g_2\right) \coloneqq \left(f_1 + \varphi\left(g_1\right) f_2, g_1 g_2\right). \tag{1.6}$$

Now, let us briefly outline the geometric significance of the BMS symmetry group. We start from the observation that the degeneracy of the first fundamental form $\mathbf{q}_{ab}$ on $\mathscr{I}$ (which, we recall, was defined as the pull–back of the metric tensor field $\hat{\mathbf{g}}_{\mu\nu}$ from $\hat{\mathscr{M}}$ to $\mathscr{I}$) is degenerate, thus implying that $\mathscr{I}$ is a null Lorentzian hypersurface embedded on $\hat{\mathscr{M}}$. Hence, there exists an infinite family $\{D\}$ of torsion–free, metrically compatible connections on $\mathscr{I}$, each member of which is defined intrinsically on the pseudo–Riemannian structure of null infinity, such that[3] $D_c \mathbf{q}_{ab} \equiv 0$. Moreover, there exists a natural equivalence class $\{(\mathbf{q}_{ab}, n^a)\}$ canonically defined on $\mathscr{I}$ composed of ordered pairs $(\mathbf{q}_{ab}, n^a)$ where $\mathbf{q}_{ab}$ is a degenerate fundamental form on $\mathscr{I}$ and $n^a \coloneqq \hat{\mathbf{g}}^{ab} \nabla_b \Omega$ is a normal vector field to the hypersurface $\mathscr{I} \subset \hat{\mathscr{M}}$; each conformal transformation maps $(\mathbf{q}_{ab} n^a) \mapsto (\omega^2 \mathbf{q}_{ab}, \omega^{-1} n^a)$, whenever the Lie derivative of the conformal factor vanishes along the flow generated by the vector field n^a, that is to say, $\mathscr{L}_n \omega \equiv 0$. From this, and recalling that the standard 2–sphere $\mathbb{S}^2$ admits a unique conformal structure, the class of asymptotically flat spacetimes is provided with a universal structure, constituted of the families $\{D\}$ and $\{(\mathbf{q}_{ab}, n^a)\}$ defined above. The BMS asymptotic symmetry group associated to such a class of spacetimes is therefore the subgroup of the group of diffeomorphisms $\mathscr{D}iff\left(\mathscr{I}\right)$ preserving this universal structure.

The meaning of the BMS group from a general relativistic point of view should be clarified by the following observations. Let us recall that we introduced $(\mathscr{M}, \mathbf{g}_{\mu\nu})$ as an asymptotically flat spacetime conformally completed by the embedding $\psi : \mathscr{M} \longrightarrow \hat{\mathscr{M}}$ into the unphysical ambient pseudo–Riemannian manifold $(\hat{\mathscr{M}}, \hat{\mathbf{g}}_{\mu\nu})$, with Bondi conformal factor Ω, such that the corresponding metric tensor fields are conformally related by $\mathbf{g}_{\mu\nu} = \left(\Omega \circ \psi\right)^2 \psi^* \left(\hat{\mathbf{g}}_{\mu\nu}\right)$, where ψ^* denotes the pull–back from the tensor bundle of $\hat{\mathscr{M}}$ to the tensor bundle of $\mathscr{M}$. Furthermore, we discovered that the null–hypersurface $\mathscr{I} \coloneqq \partial\left(\psi\left(\mathscr{M}\right)\right)$, endowed with the degenerate first fundamental form $\mathbf{q}_{ab} \coloneqq \psi^*\left(\hat{\mathbf{g}}_{\mu\nu}\right)_{ab}$, provides a mathematical model to the Penrose conception of *null infinity*, which is naturally endowed with the universal structure determined by the family $\{(\mathbf{q}_{ab}, n^a)\}$ discussed above. Let us recall, from our knowledge of particle physics, that the set of flat connections in Yang–Mills theory determines the *classical vacua* of the theory; similarly, let us refer to the family of connections $\{\mathring{D}\}$, intrinsically defined on the pseudo–Riemannian structure $(\mathscr{I}, \mathbf{q}_{ab})$, as the *classical*

[3]On what follows, we denote by $\equiv$ the equality on the restriction to the null infinity; that is, given a tensor field $\mathbf{T}^{a\ldots b}_{c\ldots d}$ on $\hat{\mathscr{M}}$, we write $\mathbf{T}^{a\ldots b}_{c\ldots d} \equiv 0$ *iff* $\mathbf{T}^{a\ldots b}_{c\ldots d}\big|_{\mathscr{I}} = 0$.

gravitational vacua, whenever the tensor field $*\mathbf{K}^{ab}$ which is uniquely defined by the relation:

$$\overset{\circ}{D}_{[a}\,\mathbf{G}_{b]}{}^{c} =: \frac{1}{4}\varepsilon_{abm} * \mathbf{K}^{mc}, \tag{1.7}$$

where $\mathbf{G}_{a}{}^{b} := \mathbf{R}_{a}{}^{b} - (1/2)\,R\delta_{a}^{b}$ is the Einstein tensor, vanishes identically, $\star\mathbf{K}^{ab} = 0$. It follows from the definitions that the four–dimensional subgroup of BMS corresponding to the translations acts trivially on $\{\overset{\circ}{D}\}$, contrarily to the infinite–dimensional normal subgroup consisting of the super-translations. Thus, there are as many classical gravitational vacua as supertranslations, and thus, the enhancement of the Poincaré group $\mathscr{P}$ to the BMS group can be understood as a manifestation of the degeneracy of the gravitational connections.

2 A first look on the Celestial Wavefunctions

Let us consider the four–dimensional Minkowski spacetime $\mathbb{R}^3_1 := \langle\mathbb{R}^4, \eta\rangle$, endowed with a global inertial coordinate system $X := (X^\mu : 0 \le \mu \le 3)$, in terms of which the Minkowski metric tensor field reads[4] $\eta = -\eta_{\mu\nu} dX^\mu \otimes dX^\nu$. The Minkowski spacetime is naturally equipped with the structure of a vector space, for which the Minkowski inner product is given by $\langle V, W\rangle := \eta_{\mu\nu}V^\mu W^\nu = -V^0W^0 + \vec{V}\cdot\vec{W}$, for each pair $V, W \in \mathbb{R}^3_1$ of worldvectors. Furthermore, the Laplace–Beltrami differential operator, induced by the Riemannian structure determined by the Lorentzian metric tensor field η, will be denoted by $\Box_X := \eta^{\mu\nu}\nabla_\mu\nabla_\nu = (\partial/\partial X^\mu)(\partial/\partial X_\mu)$, where[5] ∇_μ denotes the covariant derivative operator defined by the Riemann–Levi-Civita connection associated to the Lorentzian structure defined on $\mathbb{R}^3_1$.

In order to introduce the notion of a conformal primary wavefunction, also referred to simply as a *celestial wavefunction*, it will be necessary first to review some elementary concepts on hyperbolic geometry. Let us define the *standard three–dimensional hyperbolic space*, $\mathbf{H}_3$, as the Lorentzian hypersurface:

$$\mathbf{H}_3 := \left\{u \in \mathbb{R}^3_1 : \langle u, u\rangle = -1\right\}. \tag{2.1}$$

Let $u := (y, z, \bar{z}) : \mathbf{H}_3 \longrightarrow \mathbb{R}\times\mathbb{C}$ be a Poincaré patch on the 3–hyperboloid $\mathbf{H}_3$, in terms of which the first fundamental form $\mathbf{h}$, inherited from the extrinsic Lorentzian geometry, reads:

$$\mathbf{h} = \frac{1}{y^2}\left[dy\otimes dy + \frac{1}{2}\left(dz\otimes d\bar{z} + d\bar{z}\otimes dz\right)\right]. \tag{2.2}$$

The isometry group of $\mathbf{H}_3$ is the unimodular group of 2×2 complex matrices, $SL(2;\mathbb{C})$, whose

[4]Here, $\eta_{\mu\nu} := \mathrm{diag}\,[-1, +1, +1, +1]$ denotes the canonical form of the Minkowski metric.

[5]The subscript X in $\Box_X$ should be understood as implying that each derivative operator differentiates with respect to the coordinate–functions belonging to the chart $\langle\mathbb{R}^4, X\rangle$.

action $(y, z, \bar{z}) \mapsto (y', z', \bar{z}')$ is given by the following transformation laws:

$$y \mapsto y' = \frac{y}{|cz+d|^2 + |c|^2 y^2}, \tag{2.3}$$

$$z \mapsto z' = \frac{(az+b)\left(\bar{c}\bar{z}+\bar{d}\right) + a\bar{c}y^2}{|cz+d|^2 + |c|^2 y^2}, \tag{2.4}$$

$$\bar{z} \mapsto \bar{z}' = \frac{\left(\bar{a}\bar{z}+\bar{b}\right)(cz+d) + \bar{a}cy^2}{|cz+d|^2 + |c|^2 y^2}, \tag{2.5}$$

for each:

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2; \mathbb{C}). \tag{2.6}$$

In this manner, $\mathbf{H}_3$ can be canonically embedded in the energy–momentum space, which is dual to the four–dimensional Minkowski spacetime under a Fourier transformation, with embedding $\hat{p}^\mu : \mathbf{H}_3 \longrightarrow \mathbb{R}^3_1$ given by:

$$\hat{p}^\mu (y, z, \bar{z}) = \left(\frac{1+y^2+|z|^2}{2y}, \frac{\mathrm{Re}(z)}{y}, \frac{\mathrm{Im}(z)}{y}, \frac{1-y^2-|z|^2}{2y} \right), \quad z = \frac{\hat{p}^1 + i\hat{p}^2}{\hat{p}^0 + \hat{p}^3} \in \mathbb{C}, \tag{2.7}$$

We are finally prepared to introduce the celestial wavefunctions. First, on what follows, let us denote by $\mathscr{H}$ the Hilbert space whose members are the scalar wavefunctions defined on the Minkowski spacetime, such that the Klein–Gordon operator (parametrised by the mass $m > 0$) is the linear differential map acting on $\mathscr{H}$ by $\Box_X - m^2$, whose kernel (when projected, of course, to the subspace obeying appropriate boundary conditions) provides a model for the physical state space. Second, with the objective of introducing the notion of a scalar conformal primary wavefunction as the integral representation of a convolution of plane waves, for which the kernel is the Witten bulk–to–boundary propagator, let us remember that, in quantum field theory, the scattering amplitudes of fields belonging to the four–dimensional Minkowski spacetime $\mathbb{R}^3_1$ are expressed (following the standard textbook expositions) in terms of asymptotic plane wave solutions to the free field equations.

Thus, the invariance of the corresponding scattering amplitudes, under transformations induced by the elements of the non–homogeneous component of the Lorentz group, becomes manifest, by virtue of the fact that plane waves develops phases which cancel each other from the constraint imposed by the law of energy–momentum conservation. Nonetheless, the invariance under the $SL(2; \mathbb{C})$ representation of the Lorentz group is more intricate, since the plane waves transform into one another in a non–trivial manner. Hence, we should describe a new basis in terms of which the scattering amplitudes can be written in a manifestly $SL(2; \mathbb{C})$–covariant form, such that the resulting expressions are the familiar ones from the study of $2d$ CFT. For, in the latter case, the $SL(2;\mathbb{C})$ group acts as global conformal transformations. (The investigation of the $SL(2;\mathbb{C})$ invariance of scattering amplitudes started with Dirac[6]. Our motivation, based on holographic considerations, is the following. When gravity is coupled to our bulk theory, the $SL(2;\mathbb{C})$–covariance of the theory is assumed to be enhanced to the full local conformal group, namely, the Virasoro group.)

A scalar conformal primary wavefunction $\phi_{\Delta,m} \in \mathscr{H}$, $X \in \mathbb{R}^3_1 \mapsto \phi_{\Delta,m}(X; z, \bar{z})$, with conformal dimension Δ and mass m, is a solution to the Klein–Gordon equation,

$$\left(\Box_X - m^2\right)\phi_{\Delta,m}\left(X^\mu; w, \bar{w}\right) = 0, \tag{2.8}$$

such that, under a homogeneous Lorentz transformation, realised as the action of the unimodular group of complex 2×2 matrices, $SL(2;\mathbb{C})$, on the celestial sphere at null infinity, transforms covariantly as a conformal quasi–primary operator[7],

$$\phi_{\Delta,m}\left(\Lambda(g)^\mu{}_\rho X^\rho; \frac{aw+b}{cw+d}, \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right) = |cw+d|^{2\Delta}\phi_{\Delta,m}\left(X^\mu; w, \bar{w}\right), \tag{2.9}$$

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2;\mathbb{C}) \leftrightarrow \Lambda(g) \in SO(1,3).$$

Let us, then, introduce the next constituent of the wavefunction for the scalar conformal quasi–primary fields on the Minkowski spacetime, namely, the Witten bulk–to–boundary propagator:

$$W_\Delta\left(y, z, \bar{z} \middle| w, \bar{w}\right) := \left(\frac{y}{y^2 + |z-w|^2}\right)^\Delta. \tag{2.10}$$

It follows by a simple computation that, for each $g \in SL(2;\mathbb{C})$ corresponding to the homogeneous Lorentz transformation $X^\mu \mapsto \Lambda^\mu{}_\rho X^\rho$ ($\Lambda \in SO(1,3)$), the Witten propagator transforms covariantly:

$$W'_\Delta\left(y', z', \bar{z}' \middle| w', \bar{w}'\right) = \left(|cw+d|^\Delta\right)^2 W_\Delta\left(y, z, \bar{z} \middle| w, \bar{w}\right), \quad w' = w \cdot g, \bar{w}' = \overline{w \cdot g}, \tag{2.11}$$

where y', z' and $\bar{z}'$ are given by Eqs. (2.3, 2.4, 2.5).

Therefore, the scalar conformal primary wavefunction $\phi_{\Delta,m}(X^\mu; z, \bar{z})$ can be regarded as a convolution of plane wavefunctions, for which the Witten propagator is the kernel of the integral

[6] Dirac, Paul AM. "Wave equations in conformal space." Annals of Mathematics (1936): 429-442.

[7] Here, we denote by $\Lambda(g) \in SO(1,3)$ the homogeneous Lorentz transformation associated to the global conformal transformation induced by the unimodular matrix $g \in SL(2;\mathbb{C})$.

representation, in such a manner that:

$$\phi^{\pm}_{\Delta,m}\left(X^{\mu};w,\bar{w}\right)=\int_{0}^{\infty}\frac{dy}{y^{3}}\int dzd\bar{z}\;\;W_{\Delta}\left(y,z,\bar{z}\middle|w,\bar{w}\right)e^{\pm im\,\eta_{\mu\nu}X^{\mu}\hat{p}^{\nu}(y,z,\bar{z})}\,, \tag{2.12}$$

where $\hat{p}^{\mu}\left(y,z,\bar{z}\right)$ is the parametrisation given by Eq. (2.7) of the embedding $\hat{p}:\mathscr{H}_{3}\longrightarrow\mathbb{R}^{(1,3)}$, and moreover, the superscript plus (minus, resp.) indicates an incoming (outgoing, resp.) particle state. In order to verify that Eq. (2.12) indeed corresponds to a scalar conformal quasi–primary wavefunction, it is sufficient to note that, by virtue of the fact that each plane wave $e^{\pm im\,\eta_{\mu\nu}X^{\mu}\hat{p}^{\nu}(y,z,\bar{z})}$ verifies the Klein–Gordon equation with mass m, one concludes that each $\phi^{\pm}_{\Delta,m}\left(X^{\mu};w,\bar{w}\right)$ is also a solution to the Klein–Gordon equation; and finally, the covariance of Eq. (2.12) under the $SL\left(2;\mathbb{C}\right)$ realisation of the homogeneous Lorentz group follows from the covariance of the Witten propagator $W_{\Delta}\left(y,z,\bar{z};w,\bar{w}\right)$, as stated in Eq. (2.11).

Our next task will be to consider the covariance of the scattering amplitudes, and we shall illustrate our method with the example of the tree–level 3–point function for the scalar conformal quasi–primary wavefunctions.

3 A Primer on Celestial Amplitudes[8]

The covariance of the conformal primary wavefunctions with respect to the $SL\left(2;\mathbb{C}\right)$ transformations implies the covariance of any scattering amplitudes which can be constructed from the former. Thus, let us denote by p^{μ}_{j} $(1\leq j\leq n)$ the on–shell momenta corresponding to n massive scalars, whose respective masses are m_{j} $(1\leq j\leq n)$. Let us consider a Lorentz invariance n–point scattering amplitude in the energy–momentum representation $\mathcal{M}\left(p^{\mu_1}_{1},...,p^{\mu_n}_{n}\right)$, including the energy–momentum conservation δ–function, $\delta^{(4)}\left(\sum_{j=1}^{n}p^{\mu}_{j}\right)$. Therefore, the conformal primary scattering amplitudes, denoted by:

$$\hat{\mathscr{A}}_{\Delta_{1},...,\Delta_{n}}\left(w_{i},\bar{w}_{i}\right)\coloneqq\hat{\mathscr{A}}_{\Delta_{1},...,\Delta_{n}}\left(w_{1},\bar{w}_{1},...,w_{n},\bar{w}_{n}\right), \tag{3.1}$$

admits the following integral representation:

$$\hat{\mathscr{A}}_{\Delta_{1},...,\Delta_{n}}\left(w_{i},\bar{w}_{i}\right)=\prod_{i=1}^{n}\left(\int_{0}^{\infty}\frac{dy_{i}}{y_{i}^{3}}\int dz_{i}d\bar{z}_{i}\right)W_{\Delta}\left(y_{i},z_{i},\bar{z}_{i}\middle|w_{i},\bar{w}_{i}\right)\mathcal{M}\left(m_{j}\hat{p}^{\mu}_{j}\right), \tag{3.2}$$

where each $\hat{p}^{\mu}_{j}\coloneqq\hat{p}^{\mu}\left(y_{j},z_{j},\bar{z}_{j}\right)$ $(1\leq j\leq n)$ is determined by the parametrisation whose form was defined in Eq. (2.7). Thence, it follows by our construction that the n–point scattering amplitudes

[8]This section follows closely Ref.([3]).

transforms in an $SL(2;\mathbb{C})$–covariant manner, so that:

$$\hat{\mathcal{A}}_{\Delta_1,\ldots,\Delta_n}\left(w_j\cdot g,\overline{w_j\cdot g}\right)=\left(\prod_{\ell=1}^{n}\left|cw_\ell+d\right|^{2\times\Delta_\ell}\right)\hat{\mathcal{A}}_{\Delta_1,\ldots,\Delta_n}\left(w_i,\bar{w}_i\right). \tag{3.3}$$

Let us, now, consider the tree–level 3–point scattering amplitude $\hat{\mathcal{A}}\left(w_i,\bar{w}_i\right)$ of the conformal primary wavefunction $\phi^{\pm}_{\Delta,m}\left(X^\mu;w_i,\bar{w}_i\right)$, whose dynamics is governed by interaction term which is assumed to be added (adiabatically) to the total Lagrangian density:

$$\mathcal{L}_{\text{Int}}=\frac{\lambda}{3!}\phi_1\phi_2\phi_3. \tag{3.4}$$

It follows from the Feynman rules that the first–order contribution to the perturbative expansion in the coupling constant λ of the 3–point scattering amplitude is given by:

$$\mathcal{A}\left(p_1,p_2,p_3\right)=\left(2\pi\right)^4\left(i\lambda\right)\times\delta^{(4)}\left(-p_1+p_2+p_3\right). \tag{3.5}$$

Hence, the amplitude for the scattering of 3 particles, the first of which is represented by an incoming state and the latter two as an outgoing state, is determined by the equation:

$$\hat{\mathcal{A}}\left(w_i,\bar{w}_i\right)=i\lambda\int d^4X\,\phi^{-}_{\Delta_1,m_1}\left(X^\mu;w_1,\bar{w}_1\right)\prod_{i=2}^{2}\phi^{+}_{\Delta_i,m_i}\left(X^\mu;w_i,\bar{w}_i\right). \tag{3.6}$$

From the $SL(2;\mathbb{C})$ covariance, the holomorphic form of the amplitude is fixed to be such that:

$$\hat{\mathcal{A}}\left(w_i,\bar{w}_i\right)\propto\frac{\lambda}{\left|w_1-w_2\right|^{\Delta_1+\Delta_2-\Delta_3}\left|w_2-w_3\right|^{\Delta_2+\Delta_3-\Delta_1}\left|w_3-w_1\right|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{3.7}$$

Moreover, the above analytic form is the one expected for the correlation function of a two–dimensional CFT. Our next work, then, is to compute the proportionality factor and show that the latter is completely determined as a function of the masses and conformal dimensions, Δ_j, from the integral representation with the kernel given by the Witten propagator, as in Eq. (3.2).

Since our aim is pedagogic, it will be more illuminating to consider the limiting case in which the mass of the decaying particle is equal to $2\left(1+\varepsilon\right)m$, where m is the mass of the outgoing particles and $0<\varepsilon$ is a small parameter. So, after computing the X^μ–integral, one deduces the following formula for the scalar 3–point function:

$$\hat{\mathcal{A}}\left(w_i,\bar{w}_i\right)=\frac{\left(2\pi\right)^4 i\lambda}{m^4}\prod_{i=1}^{3}\left(\int_0^\infty\frac{dy_i}{y_i^3}\int dz_i d\bar{z}_i\right)\prod_{i=i}^{3}K_{\Delta_i}\left(y_i,z_i,\bar{z}_i\middle|w_i,\bar{w}_i\right)\delta^{(4)}\left(-2\left(1+\varepsilon\right)\hat{p}_1+\hat{p}_2+\hat{p}_3\right). \tag{3.8}$$

The last integral can be reduced to the transition amplitude corresponding to the tree–level 3–point Witten diagram in the hyperbolic space $\mathscr{H}_3$ by taking the limit $\varepsilon \to 0^+$. The leading order term thus becomes:

$$\hat{\mathscr{A}}\left(w_i, \bar{w}_i\right) = \frac{\kappa \mathscr{C}\left(\Delta_1, \Delta_2, \Delta_3\right)}{\left|w_1 - w_2\right|^{\Delta_1+\Delta_2-\Delta_3}\left|w_2 - w_3\right|^{\Delta_2+\Delta_3-\Delta_1}\left|w_3 - w_1\right|^{\Delta_3+\Delta_1-\Delta_2}} + \mathscr{O}\left(\varepsilon\right), \tag{3.9}$$

where:

$$\kappa \coloneqq \frac{i 2^{9/2}\pi^6\lambda}{m^4}\sqrt{\varepsilon}, \tag{3.10}$$

and:

$$\mathscr{C}\left(\Delta_1, \Delta_2, \Delta_3\right) \coloneqq \frac{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_1-\Delta_2}{2}\right)}{\Gamma\left(\Delta_1\right)\Gamma\left(\Delta_2\right)\Gamma\left(\Delta_3\right)}.$$

To summarise, we have succeeded in showing that the near extremal massive 3–point scattering amplitude takes the form of the 3–point correlation function of scalar primaries with conformal dimensions Δ_i, belonging to a two–dimensional CFT, which illustrates the general philosophy of the celestial holography research programme. Further progress may utilise tools inspired from recent insights in pure mathematics, twistor geometry, the spinor–helicity formalism or the revival of the conformal bootstrap program.

4 Symplectic structure of radiative modes[9]

Let us consider the problem of defining a symplectic structure of the radiative modes at null infinity and let us proceed in such a manner as to generalise the conformal primary wavefunctions discussed above to an arbitrary number of spatial dimensions.

Let $\mathscr{M}_{d+2} \coloneqq \left(\mathbb{R}^{d+2}, \eta_{MN}\right)$ be the $(d+2)$–dimensional Minkowski spacetime, equipped with a global Lorentz coordinate system $X^M : \mathscr{M}_{d+2} \longrightarrow \mathbb{R}^{d+2}$, in terms of which the Minkowski metric tensor field assumes the canonical form $\eta_{MN} = \mathrm{diag}\left[-1, +1, ..., +1\right]$. The Minkowski space $\mathscr{M}_{d+2}$ is naturally endowed with a linear structure which is isomorphic to $\mathbb{R}^{d+1}_1 \coloneqq \left(\mathbb{R}^{d+2}, \langle\cdot,\cdot\rangle\right)$, for which the Minkowski inner product is given by $\langle V, W\rangle \coloneqq \eta_{MN}V^M W^N = -V^0W^0 + \mathbf{V}\cdot\mathbf{W}$ for each pair of $(d+2)$–dimensional worldvectors $V^M = (V^0, \mathbf{V})$ and $W^M = (W^0, \mathbf{W})$. Let the Penrose conformal completion of the Minkowski spacetime be determined by the embedding $\psi : \mathscr{M}_{d+2} \longrightarrow \tilde{\mathscr{M}}_{d+2}$ into the unphysical ambient Lorentzian space $(\tilde{\mathscr{M}}_{d+2}, \tilde{\mathbf{g}}_{MN})$, such that $\eta_{MN} = \left(\Omega \circ \psi\right)^2 \hat{\mathbf{g}}_{MN}$, where $\Omega : \tilde{\mathscr{M}}_{d+2} \longrightarrow \mathbb{R}_+$ is the Bondi conformal factor; the null hypersurface $\mathscr{I} \coloneqq \partial\left(\psi\left(\mathscr{M}_{d+2}\right)\right)$, endowed with the degenerate first fundamental form $\mathbf{q}_{AB} \coloneqq \psi^*\left(\hat{\mathbf{g}}_{MN}\right)_{AB}$, defines the null infinity associated to the Penrose compactification of the Minkowski spacetime; let us denote by $\hat{\nabla}_A$ the covariant

[9] In this section, let us employ capital letters $A, B, ...$ from the beginning of the latin alphabet to index tensor components living on $\mathscr{I}$, and capital letters $M, N...$ from the middle of the latin alphabet to index tensor components corresponding to fields on the $(n+2)$–dimensional spacetime manifold $\mathscr{M}_{d+2}$.

derivative operator, acting on the sections of the tensor bundle[10] $\mathbf{T}(\mathscr{I}) := \sqcup_p \mathbf{T}_p(\mathscr{I})$, which is induced by the unique torsion–free connection on $\mathscr{I}$ satisfying $\hat{\nabla}_C \mathbf{q}_{AB} \equiv 0$. In order to introduce the space of regular (also referred to as "tempered") scalar fields on $\mathscr{I}$, let $\left(\hat{\mathbf{q}}_{AB}, \hat{n}^A\right)$ be a conformal frame on $\mathscr{I}$, which means that there exists a differentiable function $\omega : \mathscr{I} \longrightarrow \mathbb{R}_+$ such that $\hat{\mathbf{q}}_{AB} = \omega^2 q_{AB}$ and $\hat{n}^A = \omega^{-1} \mathbf{q}^{AB} \hat{\nabla}_B \Omega$. Moreover, let (u, θ, ϕ) be a coordinate system on $\mathscr{I}$ for which $\mathscr{L}_{\hat{n}} u = 1$ and (θ, ϕ) is a chart on the 2–sphere of the generators of the BMS group on $\mathscr{I}$; let us also suppose that u is globally defined on $\mathscr{I}$. Thus, the norm $\|\cdot\|$ on the module of complex–valued twice differentiable functions, $\mathcal{F}' := \mathscr{C}^2(\mathscr{I})$, is such that, for every $\phi \in \mathcal{F}'$ and some fixed $\varepsilon > 0$,

$$\|\phi\| := \sum_{|\alpha|=0}^{\infty} \frac{1}{2^{|\alpha|}} \left(\sup_{u \in \mathbb{R}} \left[\left(1 + u^2\right)^{\frac{1}{2}|\alpha| + \varepsilon} |\partial^\alpha \phi| \right] \right) \left(1 + \sup_{u \in \mathbb{R}} \left[\left(1 + u^2\right)^{\frac{1}{2}|\alpha| + \varepsilon} |\partial^\alpha \phi| \right] \right)^{-1}. \tag{4.1}$$

Thus, we define $\mathcal{F} := \{\phi \in \mathcal{F}' : \|\phi\| < \infty\}$ to be the Banach space of complex–valued regular scalar fields on $\mathscr{I}$ endowed with the topology induced by the norm $\|\cdot\|$. Lastly, in order to conclude our preliminary remarks, let us denote the Laplace–Beltrami differential operator written with respect to the inertial coordinates X^M by $\Box_X = \eta^{MN} \nabla_M \nabla_N = \left(\partial / \partial X^M\right)\left(\partial / \partial X_M\right)$, where $\partial / \partial X_M := \eta^{MN} \partial / \partial X^N$; therefore, the Klein–Gordon operator with mass m acts on $\mathcal{F}$ by $\phi\left(X^M\right) \mapsto \left(\Box_X - m^2\right) \phi\left(X^M\right)$. Finally, letting $\Pi : \mathcal{F} \longrightarrow \mathcal{F}$ be the projection (namely, $\Pi \circ \Pi = \Pi$) onto the subspace of regular scalar fields obeying physically appropriate boundary conditions, the Hilbert state space can be identified with $\mathscr{H} = \Pi\left(\ker\left(\Box_X - m^2\right)\right)$. Note that, on the one hand, the norm $\|\cdot\|$ defined in Eq. (4.1) depends on the choice of conformal frame $\left(\hat{\mathbf{q}}_{AB}, \hat{n}^A\right)$ and coordinate system (u, θ, φ) on $\mathscr{I}$, while that, on the other hand, the topology of the Fréchet space $\mathcal{F}$, induced by the norm $\|\cdot\|$, is uniquely defined by the universal structure of the Penrose null infinity.

There exists a symplectic structure on the space of scalar fields on the null infinity of any asymptotically flat spacetime; in particular, one can define a symplectic phase space of a scalar field theory on the asymptotic structure associated to the Penrose conformal compactification of the Minkowski spacetime. To begin with, let us remember that for any Cauchy surface Σ on $\mathscr{M}_{d+2}$ with inclusion map $\iota_\Sigma : \Sigma \longrightarrow \mathscr{M}_{d+2}$, there exists a symplectic structure naturally defined on the Fréchet space $\mathcal{F}$ which is provided by the anti–symmetric bilinear form $\Omega_\Sigma : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{R}$ such that, to each pair $f, g \in \mathcal{F}$,

$$\Omega_\Sigma(f, g) := \int_\Sigma \left(f \mathscr{L}_n g - g \mathscr{L}_n f\right) \iota_\Sigma^*(\varepsilon), \tag{4.2}$$

[10]Here, for every differentiable submanifold Σ of $\mathscr{M}$, with inclusion map $\iota : \Sigma \longrightarrow \mathscr{M}$, let us denote by $\mathbf{T}_p(\Sigma) \subset \mathbf{T}_{\iota(p)}(\mathscr{M})$ for every point $p \in \Sigma$ the module constituted of all the multilinear transformations $T_p(\Sigma) \longrightarrow \mathbb{R}$ acting on the tangent space at p, endowed with pointwise addition and scalar multiplication.

where $\varepsilon \in \bigwedge^{d+2} \mathscr{M}_{d+2}$ is the canonical element of volume differential $(d+2)$–form, which in terms of the global inertial coordinates introduced above reads $\varepsilon = \frac{1}{(d+2)!}\varepsilon_{M_1 \ldots M_{d+2}} dX^{M_1} \wedge \ldots \wedge dX^{M_{d+2}}$. The requirement that each scalar field on $\mathscr{F}$ obeys the Klein–Gordon equation, one deduces that the symplectic form Ω_Σ is independent of the choice of the Cauchy surface Σ. Furthermore, letting $\hat{n}^M$ be the normal vector field on Σ, the symplectic form can be rewritten in the following manner:

$$\Omega_\Sigma (f, g) \coloneqq \int_\Sigma (f \hat{\nabla}_M g - g \hat{\nabla}_M f) dS^M. \tag{4.3}$$

One is therefore lead to introduce the symplectic structure $\Gamma \coloneqq (\mathscr{F}, \hat{\Omega})$ on the Fréchet space of regular scalar fields living on $\mathscr{I}$ for which the symplectic tensor is given by:

$$\hat{\Omega} (\phi_1, \phi_2) \coloneqq \int_{\mathscr{I}} (\phi_1 \mathcal{L}_{\hat{n}} \phi_2 - \phi_2 \mathcal{L}_{\hat{n}} \phi_1) \, \varepsilon_{A_1 \ldots A_{d+1}} dS^{A_1 \ldots A_{d+1}}, \tag{4.4}$$

where $\varepsilon_{A_1 \ldots A_{d+1}}$ is the Levi–Civita totally anti–symmetric tensor field density induced on null infinity with respect to the pull–back of the inclusion map $\iota_{\mathscr{I}} : \mathscr{I} \longrightarrow \mathscr{M}_{d+2}$, in such a manner that (employing indiscriminately the abstract index notation) $\varepsilon_{A_1 \ldots A_{d+1}} = \left[\iota^*_{\mathscr{I}} \left(\varepsilon_{M_1 \ldots M_{d+2}}\right)\right]_{A_1 \ldots A_{d+1}}$.

Let $\hat{\xi}^M$ be the generator of a BMS transformation with flow $\lambda \in \mathbb{R} \mapsto \psi_\lambda \in \mathrm{Diff}\,(\mathscr{M}_{d+2})$, the latter of which being completely determined by the ordinary differential equation:

$$\frac{d}{d\lambda} \left[X^M \circ \psi_\lambda (p)\right]_{\lambda=0} = \hat{\xi}^M_p \in T_p (\mathscr{M}_{d+2}), \tag{4.5}$$

for every spacetime event $p \in \mathscr{M}_{d+2}$, together with the initial–value $\psi_0 (p) = p$. Thus, there exists a family scalar fields $k(\lambda) : \mathscr{I} \longrightarrow \mathbb{R}$, each of which is preserved under the flow ψ_λ, $(d\psi^*_\lambda (k(\lambda)) / d\lambda)_{\lambda=0} = \mathcal{L}_{\hat{\xi}} k(\lambda) = 0$, such that:

$$\mathcal{L}_{\hat{\xi}} \hat{\mathbf{q}}_{AB} = 2k(\lambda) \, \hat{\mathbf{q}}_{AB}, \tag{4.6}$$

$$\mathcal{L}_{\hat{\xi}} \hat{n}^A = -k(\lambda) \, \hat{n}^A. \tag{4.7}$$

Let us denote the image of a tensor $\mathbf{T}^{M_1 \ldots M_r}_{N_1 \ldots N_s} (p) \in T_p (\mathscr{M}_{d+2})$, $p \in \mathscr{M}_{d+2}$, under the flow $\psi_\lambda : \mathscr{M}_{d+2} \longrightarrow \mathscr{M}_{d+2}$, by the push–forward $(\psi_\lambda)_* \left(\mathbf{T}^{M_1 \ldots M_r}_{N_1 \ldots N_s}\right) \in T_{\psi_\lambda (p)} \mathscr{M}_{d+2}$. Thus,

$$(\psi_\lambda)_* (\hat{\mathbf{q}}_{AB}) = [\omega (\lambda)]^2 \, \hat{\mathbf{q}}_{AB}, \tag{4.8}$$

$$(\psi_\lambda)_* \left(\hat{n}^A\right) = [\omega (\lambda)]^{-1} \, \hat{n}^A, \tag{4.9}$$

for a one–parameter family of scalar fields, $(\omega (\lambda))_{\lambda \in \mathbb{R}}$, such that $[(d/d\lambda) \circ \omega (\lambda)]_{\lambda=0} = k(\lambda)$. As a consequence, the action induced by $(\psi_\lambda)_*$ on the Fréchet space of regular scalar fields on $\mathscr{I}$, $\mathscr{F}$, is determined by the transformation law:

$$\left(\left(\psi_\lambda\right)_* f\right)(\lambda) = \omega(\lambda)\left(f \circ \psi_\lambda\right). \tag{4.10}$$

Therefrom, $(\psi_\lambda)_*(f) \in \mathscr{F}$ whenever $f \in \mathscr{F}$; the mapping $\mathbb{R} \longrightarrow \mathscr{F}$, $\lambda \mapsto (\psi_\lambda)_*(f)$ is a continuous curve on the Fréchet space passing through $(\psi_0)_*(f) = f$; and $\lim_{\lambda\to 0^+}\left((\psi_\lambda)_*(f) - f\right)/\lambda = \mathscr{L}_{\hat{\xi}} f$. Consequently, $\lambda \mapsto (\psi_\lambda)_*$ is a one–parameter family of automorphisms on $\mathscr{F}$ generated by the linear transformation $T = \mathscr{L}_{\hat{\xi}}$. Moreover, the action $(\psi_\lambda)_*$ is symplectic, since:

$$\hat{\Omega}\left((\psi_\lambda)_*(f), (\psi_\lambda)_*(g)\right) = \int_{\mathscr{I}} \left[(\psi_\lambda)_*(f)\,\hat{\nabla}_M (\psi_\lambda)_*(g) - (\psi_\lambda)_*(f)\,\hat{\nabla}_M (\psi_\lambda)_*(g)\right] \hat{n}^M \iota^*_{\mathscr{I}}(\varepsilon) \tag{4.11}$$

$$= \int_{\mathscr{I}} \left(f\hat{\nabla}_M g - g\hat{\nabla}_M f\right) \hat{n}^M \iota^*_{\mathscr{I}}(\varepsilon) \tag{4.12}$$

$$= \hat{\Omega}(f,g), \tag{4.13}$$

for each pair $f, g \in \mathscr{F}$.

With the aim of providing a rigorous discussion of the structure of the symplectic geometry associated to the radiative modes of the regular scalar fields on null infinity, let us recall some mathematical facts regarding the linear semi–group theory of Hamiltonian systems. Let $\mathscr{E}$ be a Fréchet manifold endowed with a symplectic form $\tilde{\omega} : \mathscr{E} \times \mathscr{E} \longrightarrow \mathbb{R}$. The structure $(\mathscr{E}, \tilde{\omega})$ determines a differential 2–form $\tilde{\Omega} \in \bigwedge^2 \mathscr{E}$ in the following manner; since there exists a natural identification between the total space $\mathscr{E}$ and every tangent space $T_p(\mathscr{E})$, for each $p \in \mathscr{E}$, one can define $\tilde{\Omega}(p) : T_p(\mathscr{E}) \times T_p(\mathscr{E}) \longrightarrow \mathbb{R}$ by $\tilde{\Omega}(p)(v,w) := \tilde{\omega}(v,w)$. Observe that $\mathbf{d}\tilde{\Omega} \equiv 0$ because the bilinear map $\tilde{\Omega}(p)$ is constant when regarded as a function of p. Furthermore, denoting the Fréchet directional derivative operator along $\mathbf{X} \in \mathscr{E}$ by $\mathscr{D}_{\mathbf{X}}$, let us consider a Fréchet automorphism $S : \mathscr{E} \longrightarrow \mathscr{E}$ enjoying the property $\mathscr{D}_{\mathbf{X}} S = S$; then, one deduces:

$$\left(S^*(\tilde{\Omega})\right)_{\mathbf{X}}(v,w) = \tilde{\Omega}_{S(\mathbf{X})}\left(S(v), S(w)\right) = \tilde{\omega}\left(S(v), S(w)\right), \tag{4.14}$$

for every pair $(v,w) \in \mathscr{E} \times \mathscr{E}$. Therefore, a necessary and sufficient condition in order for the transformation S to be symplectic, namely, $S^*(\tilde{\Omega}) = \tilde{\Omega}$, is that the bilinear form $\tilde{\omega}$ is invariant under the pull–back by S. Concerning the one–parameter semi–groups of canonical transformations mapping $\mathscr{E}$ onto itself, let us suppose that $(\varphi_t)_{t\in\mathscr{I}}$ $(0 \in \mathscr{I} \subseteq \mathbb{R})$ is a semi–group generated by the linear differential operator $\mathbf{T}$, such that $\varphi_t = e^{t\mathbf{T}}$; the domain $\mathscr{D}(\mathbf{T}) \subseteq \mathscr{E}$ is a dense linear subspace and one may regard $\mathbf{T}$ as a (linear) vector field whenever the canonical identification $\mathbf{T}(p) \in \mathscr{E} \simeq T_p(\mathscr{E})$ is throughly adopted. Thus, the Chernoff–Marsden theorem (the details of which can be found in Ref.([4])) asserts that the following properties are equivalent: (i) the generator $\mathbf{T}$ is a locally Hamiltonian vector field, $\mathbf{T} \neg \tilde{\Omega} = 0$; (ii) the linear transformation $\mathbf{T}$ is anti–symmetric with respect to the symplectic form $\tilde{\omega}$, that is to say, $\tilde{\omega}(\mathbf{T}(v), w) = -\tilde{\omega}(v, \mathbf{T}(w))$,whenever $v, w \in \mathscr{D}(\mathbf{T})$; (iii) $\mathbf{T}$ is globally Hamiltonian for which the corresponding energy–function is defined by the mapping $v \mapsto \mathscr{H}(v) := (1/2)\,\tilde{\omega}(\mathbf{T}(v), v)$; and, (iv) the semi–group of transformations $(\varphi_t)_{t\in\mathcal{I}}$ is symplectic. Moreover, whenever one of these properties holds, the law of conservation of energy follows, the precise statement of which reads $\mathscr{H} \circ \varphi_t = \mathscr{H}$ on $\mathscr{D}(\mathbf{T})$ for each $t \in \mathcal{I}$.

We are finally prepared to introduce the symplectic phase space of radiative modes for regular scalar fields associated to the Penrose completion of Minkowski spacetime $\mathscr{M}_{d+2}$. Let $U \subseteq \mathscr{I}$ be an open, locally convex and connected neighbourhood contained in null infinity endowed with a compact closure; let $\left(\hat{\mathbf{q}}_{AB}, \hat{n}^A\right)$ be a conformal frame and open a chart (u, θ, φ), such that u is globally defined on $\mathscr{I}$ and (θ, φ) are the standard angular coordinates on the d–sphere generated by the action of the BMS group. Introduce the following inner product on $\mathscr{C}^2(U;\mathbb{R})$,

$$\langle f, g\rangle_U = \int_U \left(\left|\frac{\partial f}{\partial u}\frac{\partial g}{\partial u}\right| + \left|\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\right| + \left|\frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial \varphi}\right|\right) \iota_U^*(\varepsilon), \tag{4.15}$$

where $\iota_U : U \longrightarrow \tilde{\mathscr{M}}_{d+2}$ is the inclusion map into the conformal completion and $\iota_U^*(\varepsilon)$ denotes the pull–back of the volume element to U. Let $\mathcal{F}_U$ be the Fréchet space consisting of all functions $f \in \mathscr{C}^2(U;\mathbb{R})$ such that $\langle f, f\rangle_U < \infty$, and given a BMS generator $\hat{\xi}^M$ whose infinitesimal action on $\mathscr{I}$ is determined by the Lie derivative operator $\mathbf{T} = \mathcal{L}_{\hat{\xi}}$, let $\mathscr{D}(\mathbf{T})$ be the dense subspace of $\mathcal{F}_U$ constituted of every scalar field $f \in \mathcal{F}_U$ such that[11] $\mathbf{T}(f) \hat{=} 0$ on the boundary ∂U. Furthermore, denote by Θ_U the characteristic function of the subset U, so that $\Theta_U(p) = 1$ whenever $p \in U$ and $\Theta_U(p) = 0$ if otherwise. We then define the symplectic tensor $\hat{\Omega}_U : \mathcal{F}_U \times \mathcal{F}_U \longrightarrow \mathbb{R}$ to be the continuous, weakly non–degenerate bilinear mapping determined by the assignment:

$$\hat{\Omega}_U(f, g) := \int_U (f\mathcal{L}_{\hat{n}}g - g\mathcal{L}_{\hat{n}}f)\, \iota_U^*(\varepsilon),$$

for every pair $f, g \in \mathcal{F}_U$. Therefore, as a consequence of the fact that $h \in \mathcal{F} \mapsto \Theta_U h \in \mathcal{F}_U$ is a surjection, whenever $(f, g) \in \mathcal{F}_U \times \mathcal{F}_U$, one has that $\hat{\Omega}_U(\Theta_U f, \Theta_U g) = \hat{\Omega}(f, g)$; hence,

$$\hat{\Omega}_U = \Theta_U^* \hat{\Omega}. \tag{4.16}$$

The linear differential operator $\mathbf{T} = \mathcal{L}_{\hat{\xi}}$, acting on the dense subspace $\mathcal{F}_U$ of the Fréchet space $\mathcal{F}$, induced by the vector field $\hat{\xi}^M$ whose flow constitute a BMS transformation on $\mathscr{I}$, is the generator of a canonical transformation on the symplectic Fréchet manifold $(\mathcal{F}_U, \hat{\Omega}_U)$. In fact, letting $f, g \in \mathscr{C}^2(\mathscr{I};\mathbb{R})$ such that $\langle f, f\rangle_U < \infty$ and $\langle g, g\rangle_U < \infty$, under the assumption that

[11] The symbol $\hat{=}$ should be understood as implying an equality restricted to the subset under consideration, for which, in our discussion above, means $A \hat{=} B$ if and only if $(A - B)\big|_U = 0$.

$\mathscr{L}_{\hat{\xi}}\left(g\mathscr{L}_{\hat{n}}f\right)=0$, one derives[12]:

$$\hat{\Omega}_U\left(\mathbf{T}\left(f\right),g\right)=\int_U\left(\mathscr{L}_{\hat{\xi}}f\mathscr{L}_{\hat{n}}g-g\mathscr{L}_{\hat{n}}\mathscr{L}_{\hat{\xi}}f\right) \tag{4.17}$$

$$=\int_U\left(\mathscr{L}_{\hat{\xi}}f\mathscr{L}_{\hat{n}}g-\mathscr{L}_{\hat{\xi}}\left(g\mathscr{L}_{\hat{n}}f\right)+\mathscr{L}_{\hat{\xi}}g\mathscr{L}_{\hat{n}}f\right) \tag{4.18}$$

$$=-\hat{\Omega}_U\left(f,\mathbf{T}\left(g\right)\right). \tag{4.19}$$

Hence, the generating function of the canonical transformation induced by the action of $\hat{\xi}$ is given by the energy–function:

$$\mathscr{H}_U^{\hat{\xi}}\left(f\right)=\frac{1}{2}\hat{\Omega}_U\left(\mathbf{T}f,f\right)=\int_U\mathscr{L}_{\hat{\xi}}f\mathscr{L}_{\hat{n}}f. \tag{4.20}$$

Let us note that $f\mapsto\mathscr{H}_U^{\hat{\xi}}\left(f\right)$ is continuous on $\mathscr{F}_U$ and $\mathscr{D}\left(\mathbf{T}\right)$ is a dense subspace of $\mathscr{F}$; therefore, the analytic extension $\mathscr{H}^{\hat{\xi}}$ to the Fréchet space $\mathscr{F}$ exists and is unique, and our result follows by the Chernoff–Marsden theorem. **QED.**

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[12]For the sake of notational simplicity, since there is no risk of ambiguites, from now on, we shall omit the volume element $\iota_U^*\left(\varepsilon\right)$ induced by the pull–back of the volume element ε belonging to the Penrose conformal completion,