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Charmonia decay constants from the QCD lattice and QCD sum rules

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Abstract

Using lattice QCD and QCD sum rules we compute the lowest state charmonia $J^{PC} = 0^{-+}$ (η_c), 1^{--} (J/ψ), and 1^{+-} (h_c) decay constants. For calculating the decay constant of J/ψ we use both the vector and tensor currents and compare the results. Lattice QCD results are obtained from the unquenched ($N_\mathrm{f} = 2$) simulations using twisted mass QCD at four lattice spacings and taking the continuum limit. In the QCD sum rule calculation we apply the moment sum rules. We also comment the phenomenological implications of calculated charmonia decay constants in $\eta_c \to \gamma\gamma$ decay, and $B \to X_{c\bar{c}}K$ decays.

Keywords: QCD lattice, QCD sum rules, charmonia form factors

1. Introduction

In the last ten years there have been numerous new charmonia states observed, mainly by BaBar and Belle experiments. This again raised interest to the charmonia systems and their properties as a place where one could try to understand the phenomena of the quark confinement and to test the validity of various quark models describing the spectrum of charmonium states including the orbital and radial excitations. In addition there are many well measured weak processes involving charmonia, like $B \to X_{c\bar{c}}K$ decays, where there is a need for the precise knowledge of the charmonia decay constants in order to describe the hadronic matrix elements and get some insight into the non-perturbative QCD dynamics of such hadronic two-body interactions.

The non-perturbative dynamics can be reached successfully by two methods. The first one is the method of *QCD sum rules* (QCDSR) which was in fact first tested on charmonium systems around 1980-ties [2, 3, 4]. The first measurements of $J^{PC} = 1^{--}$, its excitations and the electronic width were used to fix some of the QCDSR parameters. The second method is of the simulation of *QCD at the lattice*.

We are going to use both methods to calculate matrix elements of three charmonia states (η_c, J/ψ, h_c) and their four decay constants (f_{η_c}, $f_{J/\psi}$, $f^T_{J/\psi}$, f_{h_c}) defined as,

$$\langle 0|\bar{c}(0)\gamma_\mu\gamma_5 c(0)|\eta_c(p)\rangle = -if_{\eta_c}p_\mu \,,$$

$$\langle 0|\bar{c}(0)\gamma_\mu c(0)|J/\psi(p,\lambda)\rangle = f_{J/\psi}m_{J/\psi}e^\lambda_\mu \,,$$

$$\langle 0|\bar{c}(0)\sigma_{\mu\nu} c(0)|J/\psi(p,\lambda)\rangle = if^T_{J/\psi}(\mu)\left(e^\lambda_\mu p_\nu - e^\lambda_\nu p_\mu\right) \,,$$

$$\langle 0|\bar{c}(0)\sigma_{\mu\nu} c(0)|h_c(p,\lambda)\rangle = if_{h_c}(\mu)\varepsilon_{\mu\nu\alpha\beta}e^\alpha_\lambda p^\beta \,, \qquad (1)$$

where the μ-dependence of the couplings to the tensor current indicates the renormalization scale and scheme dependence.

Only $f_{J/\psi}$ can be directly extracted from experiment via

$$\Gamma(J/\psi \to e^+e^-) = \frac{4\pi\alpha_\mathrm{em}}{3m_{J/\psi}}\frac{4}{9}f^2_{J/\psi}. \qquad (2)$$

We will calculate above matrix elements using both non-perturbative methods of QCDSR and QCD lattice calculation and will compare the results. The phenomenological implications of the results will be discussed.

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2. Two-point QCD sum rules

QCD sum rules are based on the general idea of calculating a relevant quark- current correlation function and relating it to the hadronic parameters of interest via a dispersion relation. The starting point of the QCDSR calculation is a definition of the correlation function. For the calculation of charmonia systems we will employ following two-point functions:

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= i\int dx\, e^{iqx}\langle 0|\mathcal{T}\left[V_\mu^\dagger(x)V_\nu(0)\right]|0\rangle\,,\\
\Pi_P(q^2) &= i\int dx\, e^{iqx}\langle 0|\mathcal{T}\left[P^\dagger(x)P(0)\right]|0\rangle\,,\\
\Pi_{\mu\nu\rho\sigma}(q) &= i\int dx\, e^{iqx}\langle 0|\mathcal{T}\left[T_{\mu\nu}^\dagger(x)T_{\rho\sigma}(0)\right]|0\rangle\,, \qquad (3)
\end{aligned}
$$

where $V_\mu = \bar{c}\gamma_\mu c$, $P = 2m_c\, i\bar{c}\gamma_5 c$, and $T_{\mu\nu} = \bar{c}\sigma_{\mu\nu}c$, with $\sigma_{\mu\nu} = i/2\times[\gamma_\mu,\gamma_\nu]$. The Lorentz structure of the vector and tensor correlation functions can be written as:

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi_V(q^2)\,,\\
\Pi_{\mu\nu\rho\sigma}(q) &= P^-_{\mu\nu\rho\sigma}\Pi_-(q^2) + P^+_{\mu\nu\rho\sigma}\Pi_+(q^2)\,, \qquad (4)
\end{aligned}
$$

where the projectors

$$
\begin{aligned}
P^-_{\mu\nu\rho\sigma} &= g_{\mu\sigma}q_\nu q_\rho + g_{\nu\rho}q_\mu q_\sigma - g_{\mu\rho}q_\nu q_\sigma - g_{\nu\sigma}q_\mu q_\rho\,,\\
P^+_{\mu\nu\rho\sigma} &= q^2\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) - P^-_{\mu\nu\rho\sigma}\,, \qquad (5)
\end{aligned}
$$

separate the even and odd parity states, and therefore $\Pi_+(q^2)$ will be used to discuss the $h_c(1^{+-})$ channel and $\Pi_-(q^2)$ the ordinary $J/\psi(1^{--})$ state.

We follow the standard procedure for the derivation of QCDSR. In the Euclidean region of q momenta, $q^2 < 0$, we will perform a perturbative calculation in terms of quarks and gluons by applying the short-distance operator-product expansion (OPE) to the correlation function $\Pi(q)$. The correlation function is then expressed via a dispersion relation in terms of the spectral function ρ_{OPE}, representing the perturbative part, and the quark and gluon condensates, i.e $\langle qq\rangle$, $\langle GG\rangle$, representing the non-perturbative contributions. For charmonia, the leading non-perturbative contributions to the correlation functions are power corrections proportional to the gluon condensate, $\langle \frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{\mu\nu\, a}\rangle \equiv \langle\frac{\alpha_s}{\pi}G^2\rangle$. Therefore for each of the invariant functions $\Pi_i(q^2)$ $(i = P, V, +, -)$ we can write :

$$
\Pi_i(q^2) = \int_0^\infty \frac{\rho_i^{\rm pert}(s)}{s - q^2 ds} + C_i^{\rm G}(Q^2)\langle\frac{\alpha_s}{\pi}G^2\rangle\Big|_{Q^2=-q^2}\,, \qquad (6)
$$

with a suitable number of subtractions, where

$$
\rho_i^{\rm pert}(s) = \frac{{\rm Im}\Pi_i(s)}{\pi} = \rho_i^{(0)}(s) + \frac{\alpha_s}{\pi}\rho_i^{(1)}(s)\,, \qquad (7)
$$

and the Wilson coefficients $C_i^{\rm G}(Q^2) \propto 1/Q^{2n_i}$ ($n_i > 0$ depending on the operators used) are computed perturbatively, Fig.1 and 2. The explicit results can be found in Appendix of [1].

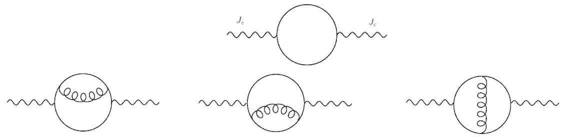

Figure 1: LO and NLO contributions to the correlation functions $\Pi_i(s)$ Only imaginary part is needed.

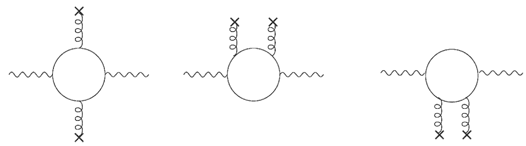

Figure 2: Gluon condensate contribution to the correlation functions $\Pi_i(s)$.

When studying charmonia it is convenient to use the so called moment sum rules [2, 3, 4]. The moments are defined as

$$
\mathcal{M}_n(Q_0^2) = \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi_i(q^2)\Big|_{q^2=-Q_0^2} \qquad (8)
$$

at some spacelike Q_0^2 far from the resonance region. In practice Q_0^2 is a parameter adjusted to improve the convergence of the integral. Since the mass of the charm quark $m_c \gg \Lambda_{\rm QCD}$ we use $Q_0^2 = 4m_c^2\xi$, and by replacing $s \to v^2 = 1 - 4m_c^2/s$, write the OPE part of the n^{th} moment as

$$
\begin{aligned}
\mathcal{M}_n^{{\rm OPE}\ i}(\xi) &= \mathcal{M}_n^{\rm pert.}(\xi) + \mathcal{M}_n^{\rm non-pert.}(\xi)\\
&= \frac{1}{(4m_c^2)^n}\int_0^1 \frac{2v(1-v^2)^{n-1}\rho_i(v)}{\left[1+\xi(1-v^2)\right]^{n+1}}\,dv\\
&\quad + \frac{1}{n!}\left(-\frac{d}{dQ^2}\right)^n C_i^{\rm G}(Q^2)\,\langle\frac{\alpha_s}{\pi}G^2\rangle\Big|_{Q^2=Q_0^2=4m_c^2\xi}\,. \qquad (9)
\end{aligned}
$$

On the other hand, in the physical (Minkowskian) region, $q^2 > 0$, we insert the complete sum over hadronic states starting from the ground state charmonia:

$$
\mathcal{M}_n^{{\rm phen.}\ i}(Q_0^2) = \sum_{k=0}^\infty \frac{|\langle 0|J^i(0)|H_k\rangle|^2}{\left(m_{H_k}^2 + Q_0^2\right)^{n+1}}\,, \qquad (10)
$$

where the sum runs over all possible single or multiparticle hadronic states, and J^i stands for a generic bilinear quark operator. Usually precisely measured quantities include only the ground states. We therefore take only one resonance to the hadronic side of the sum rules. All higher resonances and non-resonant states with the particle charmonium quantum numbers are then replaced by the hadronic spectral function $\rho_i^{\rm pert}(s)$ via a dispersion relation assuming the quark-hadron duality, starting from some effective threshold, $s_0^i > m^2_{H_0^{(i)}}$ of the order of the mass of the first excited charmonium resonance squared. In our case, using the definitions from (1), we have

$$\mathcal{M}_n^{\text{phen. }V}(Q_0^2) = \frac{f^2_{J/\psi}}{\left(m^2_{J/\psi}+Q_0^2\right)^{n+1}} + \int_{s_0^\psi}^\infty \frac{\rho_V^{\rm pert.}(s)ds}{\left(s+Q_0^2\right)^{n+1}},$$
$$\mathcal{M}_n^{\text{phen. }P}(Q_0^2) = \frac{\left(f_{\eta_c}m^2_{\eta_c}\right)^2}{\left(m^2_{\eta_c}+Q_0^2\right)^{n+1}} + 4m_c^2\int_{s_0^{\eta_c}}^\infty \frac{\rho_P^{\rm pert.}(s)ds}{\left(s+Q_0^2\right)^{n+1}},$$
$$\mathcal{M}_n^{\text{phen. }+}(Q_0^2) = \frac{f^2_{h_c}}{\left(m^2_{h_c}+Q_0^2\right)^{n+1}} + \int_{s_0^{h_c}}^\infty \frac{\rho_+^{\rm pert.}(s)ds}{\left(s+Q_0^2\right)^{n+1}},$$
$$\mathcal{M}_n^{\text{phen. }-}(Q_0^2) = \frac{[f^T_{J/\psi}(\mu)]^2}{\left(m^2_{J/\psi}+Q_0^2\right)^{n+1}} + \int_{s_0^{\psi^T}}^\infty \frac{\rho_-^{\rm pert.}(s)ds}{\left(s+Q_0^2\right)^{n+1}},$$

where the renormalization scale is chosen to be $\mu^2 = m_c^2 + Q_0^2$, with $m_c \equiv m_c^{\overline{\rm MS}}(m_c)$. By equating eqs.(9) and (11) and defining $\widetilde{\mathcal{M}}_n^i(\xi,s_0) = \frac{1}{(4m_c^2)^n}\int_0^{v[s_0^i]} \frac{2v(1-v^2)^{n-1}\rho_i^{\rm pert.}(v)}{\left[1+\xi(1-v^2)\right]^{n+1}}\,dv + \frac{1}{n!}\left(-\frac{d}{dQ^2}\right)^n C_i^{\rm G}(Q^2)\,\langle\frac{\alpha_s}{\pi}G^2\rangle\Big|_{Q^2=4m_c^2\xi}$, where $v[s_0] = \sqrt{1-4m_c^2/s_0}$, we arrive to the QCDSR for charmonia masses

$$m^2_{J/\psi} = -4m_c^2\xi + \frac{\widetilde{\mathcal{M}}_n^V(\xi,s_0^\psi)}{\widetilde{\mathcal{M}}_{n+1}^V(\xi,s_0^\psi)},$$
$$m^2_{\eta_c} = -4m_c^2\xi + \frac{\widetilde{\mathcal{M}}_n^P(\xi,s_0^{\eta_c})}{\widetilde{\mathcal{M}}_{n+1}^P(\xi,s_0^{\eta_c})},$$
$$m^2_{h_c} = -4m_c^2\xi + \frac{\widetilde{\mathcal{M}}_n^+(\xi,s_0^{h_c})}{\widetilde{\mathcal{M}}_{n+1}^+(\xi,s_0^{h_c})},$$
$$m^2_{J/\psi} = -4m_c^2\xi + \frac{\widetilde{\mathcal{M}}_n^-(\xi,s_0^\psi)}{\widetilde{\mathcal{M}}_{n+1}^-(\xi,s_0^\psi)}, \tag{11}$$

and decay constants:

$$f_{J/\psi} = \left(m^2_{J/\psi}+4m_c^2\xi\right)^{\frac{n+1}{2}}\left[\widetilde{\mathcal{M}}_n^V(\xi,s_0^\psi)\right]^{1/2},$$
$$\eta_c = \left(m^2_{\eta_c}+4m_c^2\xi\right)^{\frac{n+1}{2}}\left[\widetilde{\mathcal{M}}_n^P(\xi,s_0^{\eta_c})\right]^{1/2}\frac{2m_c}{m^2_{\eta_c}},$$
$$f_{h_c}(\mu_0) = \left(m^2_{h_c}+4m_c^2\xi\right)^{\frac{n+1}{2}}\left[\widetilde{\mathcal{M}}_n^+(\xi,s_0^{h_c})\right]^{1/2}\Big|_{\mu_0=m_c\sqrt{1+4\xi}},$$
$$f^T_{J/\psi}(\mu_0) = \left(m^2_{J/\psi}+4m_c^2\xi\right)^{\frac{n+1}{2}}\left[\widetilde{\mathcal{M}}_n^-(\xi,s_0^\psi)\right]^{1/2}. \tag{12}$$

We can see that the extraction of the masses is more precise since they are obtained from the ratios of moments.

2.1. Evaluation of Sum Rules

For the numerical evaluation of the QCDSR given in eq. (12) we take the masses of the three lowest lying states from [5]. The charm quark mass and the value of the gluon condensate are taken from ref. [6],

$$m_c^{\overline{\rm MS}}(m_c) = 1.275(15)\ \text{GeV},\qquad \langle\frac{\alpha_s}{\pi}G^2\rangle = 0.009(7)\ \text{GeV}^4, \tag{13}$$

that are found to be highly correlated (cf. Fig.5 in ref. [6]). We take that correlation into account and also vary the threshold parameter s_0 between the square of the mass of the lowest state and its first radial excitation. More specifically,

$$s_0^{\eta_c}\in[3.1^2,3.5^2]\ \text{GeV}^2,\quad s_0^\psi\in[3.3^2,3.65^2]\ \text{GeV}^2,\quad s_0^{h_c}\in[3.6^2,4.0^2]\ \text{GeV}^2. \tag{14}$$

With the sum rule parameters $[m_c^{\overline{\rm MS}}(m_c), \langle\frac{\alpha_s}{\pi}G^2\rangle, s_0^i]$ varied in the intervals indicated above, we then look for the moments n such that $\delta m^{\rm QCDSR}_{\eta_c,J/\psi}/m^{\rm exp.}_{\eta_c,J/\psi} \leq 1\%$ and $\delta m^{\rm QCDSR}_{h_c}/m^{\rm exp.}_{h_c} \leq 5\%$. Furthermore we impose the standard QCDSR requirements, namely that the next-to-leading order correction to the moments represents less than 30% with respect to the leading order term (this appears to be satisfied only with $\xi \neq 0$; we have performed check with $\xi = 1$ and $\xi = 2$ and included this variation in the error of the results), and that the contribution coming from the gluon condensate does not exceed 50% of the perturbative part. The above criteria are fulfilled and there is no large variation in the results for charmonia decay constants apart from the sum rules for h_c which show some instability.

We have carefully examined all parameter dependencies and found stability windows for QCDSR, see discussion in Sec. 2.1 of [1], and have obtained the following results:

$$f_{J/\psi} = (335\div 447)\ \text{MeV} = (401\pm 46)\ \text{MeV},$$
$$f_{\eta_c} = (270\div 348)\ \text{MeV} = (309\pm 39)\ \text{MeV},$$
$$f^T_{J/\psi}(2\ \text{GeV}) = (346\div 436)\ \text{MeV} = (391\pm 45)\ \text{MeV},$$
$$f_{h_c}(2\ \text{GeV}) = (140\div 184)\ \text{MeV} = (162\pm 22)\ \text{MeV} \tag{15}$$

Since the behavior of $f^T_{J/\psi}(\mu)$ with respect to the variation of the QCD sum rule parameters is very similar to that of $f_{J/\psi}$, we compute the ratio of the two,

$$R^T_{J/\psi} = \frac{f^T_{J/\psi}(\mu)}{f_{J/\psi}}, \tag{16}$$

By following the same criteria as for the decay constants themselves, and by using $\mu = 2$ GeV, we get

$$R^T_{J/\psi} = (0.965 \div 0.984) = 0.975 \pm 0.010\,. \qquad (17)$$

and see that the ratio $R^T_{J/\psi}$ is much more accurately estimated. We emphasize that this is the first determination of $f^T_{J/\psi}$ and $R^T_{J/\psi}$.

The results are illustrated in Fig.3., obtained by varying all of the QCD sum rule parameters: s_0, n, $m_c^{\overline{\rm MS}}(m_c)$, $\langle \frac{\alpha_s}{\pi} G^2 \rangle$, and for $\xi \in \{1, 2\}$. Some comments are in order: (i) for $f_{J/\psi}$ the calculated value is with 10% uncertainty, which is a typical accuracy of the sum rule computation of the hadronic decay constants [4]; (ii) for f_{η_c} we find somewhat lower value than usual. The main reason is that in the earlier sum rule estimates the computations were often done by using the pole charm quark mass, so that the approximation $2M_c \approx m_{\eta_c}$ was justified and in that way the resulting value for f_{η_c} was larger, which is not the case here where we consistently use the $m_c^{\overline{\rm MS}}$ mass; (iii) f_{h_c} shows very small stability region, for very small n, $n \in [1, 3]$ for $\xi = 2$, so the result has a larger error; (iv) the calculation represents the first QCDSR analysis of the h_c state[1].

3. Lattice QCD results

Now we compute the same quantities discussed above but by means of numerical simulations of QCD on the lattice. We will use the gauge field configurations generated by European Twisted Mass Collaboration (ETMC), in which the effect of $N_{\rm f} = 2$ dynamical ("*sea*") light quarks have been included by using the Wilson regularization of QCD on the lattice with the maximally twisted mass term, namely [8], with the action is written in the "physical basis" and not in the twisted one:

$$S = a^4 \sum_x \bar{\psi}(x) \left\{ \frac{1}{2} \sum_\mu \gamma_\mu \left(\nabla_\mu + \nabla^*_\mu \right) \right.$$
$$\left. - i\gamma_5 \tau^3 r \left[m_{\rm cr} - \frac{a}{2} \sum_\mu \nabla^*_\mu \nabla_\mu \right] + \mu_c \right\} \psi(x)\,, \quad (18)$$

where ∇_μ (∇^*_μ) stands for the forward (backward) covariant derivative, $m_{\rm cr}$ is the critical mass term tuned to restore the chiral symmetry of the massless action, otherwise broken by the Wilson term (also in the brackets), and μ_c is the bare charm quark mass. In the above action $\psi(x) = [c(x)\, c'(x)]^T$ is a doublet of the charm quark field and its replica. The factor $i\gamma_5\tau^3 r$ cures the pathology of the standard Wilson quark action by rotating the Wilson term to the imaginary axis which is why one can simulate with sea quark masses considerably closer to the chiral limit. The quark propagators $S_c(0, 0; \vec{x}, t)$ and $S'_c(0, 0; \vec{x}, t)$ are then obtained by inverting the above Wilson-Dirac operator with r and $-r$, respectively. In practice $r = 1$. Detailed information about the lattices used in this work are given in Tab.1 of [1].

Similarly to the previous section, hadron masses and decay constants are extracted from the study of the two-point correlation functions with operators chosen with desired quantum numbers, namely:

$$\begin{aligned} J^{PC} &= 0^{-+} \quad P = 2\mu_c\, \bar{c}\gamma_5 c'\,, \\ J^{PC} &= 1^{--} \quad V_i = Z_A\, \bar{c}\gamma_i c' \quad \text{or} \quad T_{0i} = Z_T(\mu)\, \bar{c}\sigma_{0i} c'\,, \\ J^{PC} &= 1^{+-} \quad T_{ij} = Z_T(\mu)\, \bar{c}\sigma_{ij} c' \quad i, j \in (1, 2, 3)\,, \end{aligned} \quad (19)$$

To extract masses and decay constants one studies the large time separation between the operators in the two-point correlation functions. For example for η_c,

$$\begin{aligned} C_P(t) &= \langle \sum_{\vec{x}} P(\vec{x}; t) P^\dagger(0; 0) \rangle \\ &= -4\mu_c^2 \sum_{\vec{x}} \langle {\rm Tr} \left[S_c(\vec{0}, 0; \vec{x}, t)\gamma_5 S'_c(\vec{x}, t; \vec{0}, 0)\gamma_5 \right] \rangle \\ &\xrightarrow{t \gg 0} \frac{\cosh[m_{\eta_c}(T/2 - t)]}{m_{\eta_c}} \left| \langle 0|P(0)|\eta_c(\vec{0})\rangle \right|^2 e^{-m_{\eta_c} T/2} \end{aligned} \quad (20)$$

and similarly for the other particles. In above T stands for the size of the periodic lattice in the time direction and

$$\langle 0|P|\eta_c(\vec{0})\rangle = f_{\eta_c} m^2_{\eta_c}\,, \qquad (21)$$

since charmonia are taken to be at rest.

In eq. (20) we assumed the local source operators, which are needed for extraction of the decay constants. In practice, however, we implement the Gaussian smearing procedure in order to increase the overlap between the interpolating operator and the lowest state coupling to a given operator. The smearing procedure and the parameters used in actual computations have been discussed in refs. [9, 10]. The matrix elements are then extracted by dividing the local-smeared and smeared-smeared correlation functions.

Hadron masses am_H ($H = \eta_c, J/\psi, h_c$) are determined from the fit to a constant on the plateau of the effective

[1] An attempt was made in ref. [7] in which was estimated $f_{h_c} = 490(60)$ MeV. By careful examination of the reported calculation we realized that there is a clear mistake made by calculating only the first part of $P^+_{\mu\nu\rho\sigma}$ in (5), and therefore the result of [7] does not correspond to any physical state.

mass $m_H^{\rm eff}(t)$ defined from

$$\frac{\cosh\left[m_H^{\rm eff}(t)\left(\frac{T}{2}-t\right)\right]}{\cosh\left[m_H^{\rm eff}(t)\left(\frac{T}{2}-t-1\right)\right]}=\frac{C_J(t)}{C_J(t+1)}\,, \tag{22}$$

with $J = P, V, T^{(+)}$ respectively. The results for the masses have already been presented in [9]. The novelty here is the result for the mass of J/ψ state obtained from the correlation function $C_T^{(1)}(t)$. It coincides with the one we obtain from the study of $C_V(t)$ except that the errors are about $2 \div 3$ times larger. Notice that only the mass of m_{η_c} is given in the lattice units while the other masses are obtained from the fit to a constant of the ratios

$$R_{J/\psi}(t)=\frac{m_{J/\psi}^{\rm eff}(t)}{m_{\eta_c}^{\rm eff}(t)}\,,\quad R_{h_c}(t)=\frac{m_{h_c}^{\rm eff}(t)}{m_{\eta_c}^{\rm eff}(t)}\,, \tag{23}$$

in which the statistical uncertainties cancel to a large extent. Once the masses are fixed, the matrix elements are extracted from the correlation functions as indicated in (20), and then related to the decay constants according to definitions (1). That is obtained for all our lattice ensembles, corresponding to four small lattice spacings and a number of light sea quarks masses.

To reach a physically interesting results we need to extrapolate our decay constants obtained at four lattice spacings to the continuum limit. We do the fit to the following form,

$$f_H = f_H^{\rm cont.}\left[1+b_H m_q + c_H\frac{a^2}{(0.086~{\rm fm})^2}\right]\,, \tag{24}$$

where the parameter b_H measures the dependence on the sea quark mass, denoted by $m_q \equiv m_q^{\overline{\rm MS}}(2~{\rm GeV})$, while the parameter c_H measures the dependence on the lattice spacing.

After accounting for all systematic uncertainties (see discussion in [1]) we obtain our final estimates:

$$\begin{aligned}&f_{\eta_c} = 387(7)(2)~{\rm MeV}\quad, f_{J/\psi} = 418(8)(5)~{\rm MeV}\,,\\ &f_{J/\psi}^T(2~{\rm GeV}) = 410(8)(6)~{\rm MeV}\,,\\ &f_{h_c}(2~{\rm GeV}) = 235(8)(5)~{\rm MeV}\,.\end{aligned} \tag{25}$$

Comments on the results: (i) $f_{J/\psi}$ and f_{η_c} have already been computed on the lattice in an unquenched setup but with the different lattice regularization by using the staggered quark action and by including $N_{\rm f} = 2 + 1$ dynamical light flavors in the continuum limit with the results: $f_{\eta_c} = 395(2)$ MeV, [11] and $f_{J/\psi} = 405(6)(2)$ MeV [12] which agrees very well with our results; (ii) our results indicate that the charmonium quantities (masses, decay constants and the form factors) do not depend on the light sea quark mass; (iii) we remark that the values for $f_{J/\psi}^T$ and f_{h_c}, as in QCDSR approach, are new results.

4. Comparison of the QCD lattice results and those obtained by QCDSR

By adopting the strategy of "one resonance + continuum" in the moment QCDSR analysis, we found that the values of the decay constants $f_{J/\psi}$ and $f_{J/\psi}^T$, as well as for $R_{J/\psi}^T = f_{J/\psi}^T(2~{\rm GeV})/f_{J/\psi}$, agree quite well with those obtained through the simulations of QCD on the lattice, in the continuum limit. On the other side the QCDSR results for the pseudoscalar meson decay constant f_{η_c} are lower than those obtained on the lattice. Similar holds true for f_{h_c}, decay constant of the recently observed $J^{PC} = 1^{+-}$ charmonium state. Adding more states to the hadronic side of the sum rules helps improving the stability of the sum rules, while the value of the decay constant remains practically unchanged. One reason for disagreement of the QCDSR estimate of f_{η_c} with that obtained on the lattice might be related to the fact that the non-perturbative contribution to the sum rules, proportional to the gluon condensate, has been fixed from the detailed analysis of the vector-vector correlation function. A possible explanation of that discrepancy is that the series of power corrections is truncated and that the higher order terms affect different correlation function differently, which is why f_{η_c} and f_{h_c} are not as well reproduced by the QCSR as it is the case with $f_{J/\psi}$ and $f_{J/\psi}^T$.

5. Phenomenological implications

Now when we have determined the decay constants of lowest-state charmonia we can consider their impact on the phenomenological considerations.

5.1. $\eta_c \to \gamma\gamma^{(*)}$

For a theoretical estimate of $\Gamma(\eta_c \to \gamma\gamma)$ the non-perturbative information is essential and is related to f_{η_c}.

The process $\eta_c \to \gamma\gamma^*$ is described by

$$\begin{aligned}\Gamma(\eta_c\to\gamma\gamma) &= \frac{4\pi\alpha_{\rm em}^2}{81}m_{\eta_c}^3|F_{\gamma\eta_c}(0)|^2\\ &= \frac{4\pi\alpha_{\rm em}^2}{81}m_{\eta_c}^3\left(\frac{f_{\eta_c}}{m_{\eta_c}^2(1+\delta)}\right)^2,\end{aligned} \tag{26}$$

where $F_{\gamma\eta_c}(0)$ is the form factor and the second expression accounts for the pole behavior of the form factor studied experimentally by BaBar Collaboration through

$d\sigma(e^+e^- \to e^+e^-\eta_c)/dQ^2$ ($Q^2 = -q^2 > 0$), a process driven by $\gamma\gamma^* \to \eta_c$. They found that the data are very well described by a single pole form, with the pole being at $m_{\rm pole} = 2.9(1)(1)$ GeV. Such a pole-like behavior was predicted by (quenched) QCD on the lattice [13], and is compatible with the vector meson dominance hypothesis [14]:

$$F^{\rm VMD}_{\eta_c\gamma}(0) = 2\frac{f_{J/\psi}}{m_{J/\psi}}\frac{2V^{J/\psi\to\eta_c}(0)}{m_{J/\psi}+m_{\eta_c}}\,, \qquad (27)$$

By using $V^{J/\psi\to\eta_c}(0) = 1.92(3)(2)$ computed from QCD lattice studies in [9], together with our result for $f_{J/\psi}$, we get $\Gamma(\eta_c \to \gamma\gamma) = 6.0(4)$ keV, which is in a fair agreement with the experimental value $\Gamma^{\rm exp}(\eta_c \to \gamma\gamma) = 5.0(4)$ keV, deduced from measured $\mathcal{B}(\eta_c \to \gamma\gamma) = (1.57 \pm 0.12)\times 10^{-4}$, and $\Gamma(\eta_c) = 32.0(9)$ MeV.

Usual expression for $\Gamma(\eta_c \to \gamma\gamma)$ based on factorization approximation [$\delta = 0$ in eq. (26)] leads to the result larger than the experimental value: $\Gamma^{\rm fact.}(\eta_c \to \gamma\gamma) = (6.64 \pm 0.27)$ keV. Taking our $f_{\eta_c} = 0.387(8)$ GeV and the experimental value for $\Gamma(\eta_c \to \gamma\gamma)$ allows us to deduce the value of $\delta = 0.15(5)$ GeV^2. That value is too large to be interpreted as $\sqrt{\langle \mathbf{k}_\perp^2\rangle}$, the mean transverse momentum of the c quark with respect to the momentum of η_c, which define $F_{\gamma\eta_c}(0)$ in the perturbative QCD approach [15] where $F_{\eta_c\gamma}(0) \simeq \frac{4f_{\eta_c}}{m^2_{\eta_c}+2\langle \mathbf{k}_\perp^2\rangle}$, since deduced $\sqrt{\langle \mathbf{k}_\perp^2\rangle} = 0.81(14)$ GeV. It is also too large to be identified as a parameter $b_{\eta_c} = 2m_c - m_{\eta_c} = 0.46(16)$ GeV in the heavy quark model of [16] where the pole-like behavior is simulating by replacing $2\langle \mathbf{k}_\perp^2\rangle \to b_{\eta_c}m_{\eta_c}$.

5.2. Non-leptonic B decays to charmonia

Last years there have been considerable progress in measurements of B decays into diverse charmonium final states. These decays governed by a color-suppressed $b \to c$ transition provide a valuable information on the factorization properties of B mesons. Usually the factorization of the four quark operator to a product of two currents is applied and for the case of B decays this takes the form

$$\begin{aligned}\langle \eta_c|\bar c\gamma_5 c|0\rangle\langle K|\bar d\Gamma_\mu b|B\rangle &\approx f_{\eta_c}\left[f_0^{B\to K}(m^2_{\eta_c})\right]\,,\\ \langle J/\psi|\bar c\gamma_\mu c|0\rangle\langle K|\bar d\Gamma_\mu b|B\rangle &\approx f_{J/\psi}\left[f_+^{B\to K}(m^2_{J/\psi})\right]\,. \end{aligned}\qquad (28)$$

In the framework of the generalized factorization of ref.[17], the ratio the branching ratios is given by

$$\frac{B(B\to\eta_c K)}{B(B\to J/\psi K)} \sim \left(\frac{f_{\eta_c}}{f_{J/\psi}}\right)^2\left(\frac{f_0^{B\to K}(m^2_{\eta_c})}{f_+^{B\to K}(m^2_{J/\psi})}\right)^2\,, \qquad (29)$$

where, for simplicity, we dropped a known kinematical factor. Applying our result $f_{\eta_c}/f_{J/\psi} = 0.926(6)$, we can compare the measured charged and neutral B-decay modes with eq. (29) and deduce,

$$\frac{f_+^{B\to K}(m^2_{J/\psi})}{f_0^{B\to K}(m^2_{\eta_c})} = 1.53(10)|_{B^\pm-{\rm mode}}\,,\ 1.56(13)|_{B^0-{\rm mode}}\,. \qquad (30)$$

These results are consistent with ≈ 1.44, as obtained from the QCDSR calculation near the light cone in ref. [18, 19]. They are also consistent with 1.51(3) obtained in the quenched lattice QCD study of ref. [20], but not as well with 1.37(2) recently obtained in the unquenched lattice study with non-relativistic QCD employed to treat the heavy quark [21]. However, one should keep in mind that nonfactorizable corrections in $B \to \eta_c K$ and in particular $B \to J/\psi K$ can be sizable, as shown by using the light-cone sum rules in [22].

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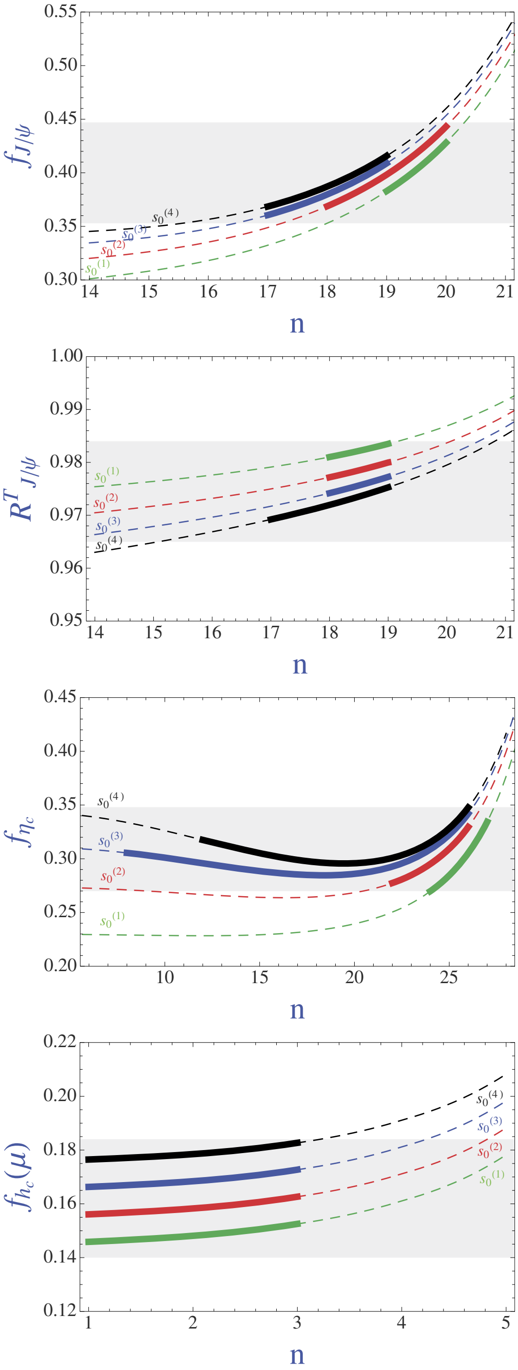

Figure 3: Couplings f_{η_c}, f_{h_c}(2 GeV), $f_{J/\psi}$ (in GeV), and the ratio $R^T_{J/\psi}$(2 GeV) computed by means of the moment sum rules. Thick lines correspond to the moments satisfying the requirement that $\delta m^{\rm QCDSR}_{\eta_c,J/\psi}/m^{\rm phys.}_{\eta_c,J/\psi} \leq 1\%$, and $\delta m^{\rm QCDSR}_{h_c}/m^{\rm phys.}_{h_c} \leq 5\%$. Illustration is provided for the central values of the charm quark mass and the gluon condensate, and for four equidistant values of the threshold parameter $s_0 \in [s_0^{(1)}, s_0^{(4)}]$. Shaded area display the range of values obtained after varying all the QCDSR parameters.

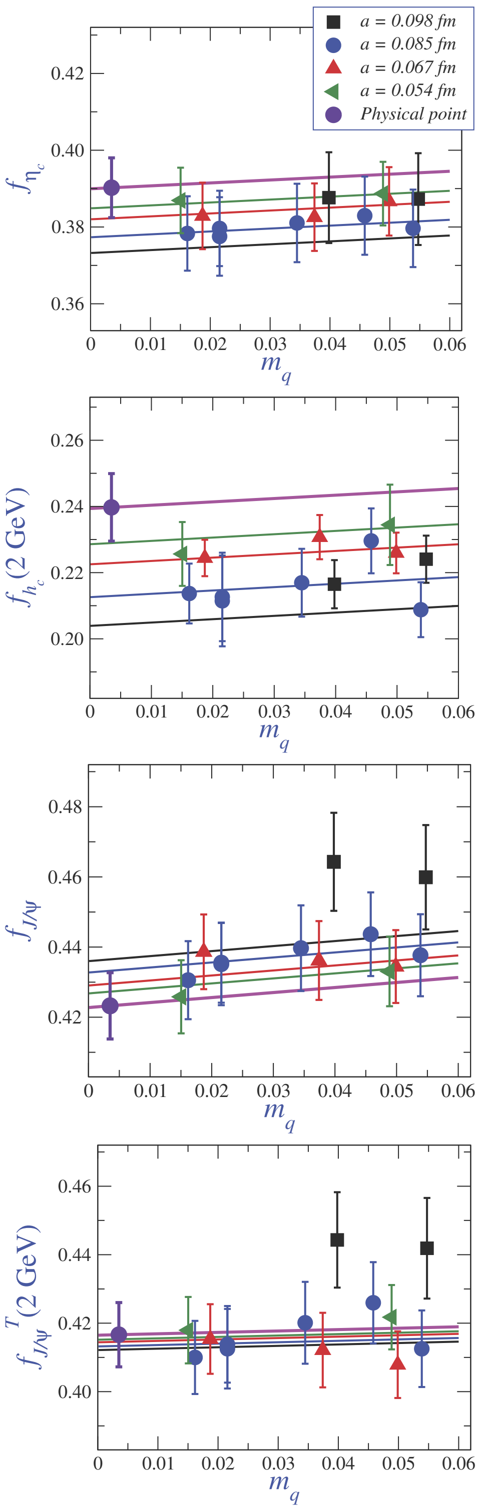

Figure 4: Dependence of f_{η_c}, f_{h_c}(2 GeV), $f_{J/\psi}$, and $f^T_{J/\psi}$(2 GeV) on the sea quark mass $m_q \equiv m_q^{\overline{\rm MS}}$(2 GeV) at each of our lattice spacings, as well as in the continuum limit. Separation among the curves, obtained from the simultaneous fit of our data to eq. (24), indicates the dependence on the finite lattice spacing. All quantities are displayed in physical units (in GeV).