

Current algebra, a $U(1)$ gauge theory and the Wess–Zumino–Witten model

Spenta R. Wadia*

International Centre for Theoretical Sciences (ICTS-TIFR), Tata Institute of Fundamental Research, Shivakote, Bangalore 560089, India

*E-mail: spenta.wadia@icts.res.in

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In this note we realise current algebra with anomalous terms in terms of a $U(1)$ gauge theory, in the space of maps M from S^1 into a compact Lie group corresponding to the current algebra. The Wilson loop around a closed curve in M is shown to be the Wess–Zumino–Witten term. This discussion enables a simple understanding of the non-Abelian anomaly in the Schrödinger picture.
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1. Reminiscences of the Chicago years

Tohru Eguchi and I arrived in Chicago at approximately the same time, in the fall of 1978. He came from SLAC to join as an Assistant Professor, and I joined as a postdoc having graduated from City University of New York under the guidance of Bunji Sakita. Eguchi was already well known for the Eguchi–Hanson instanton of Euclidean self-dual Einstein equations [1]. I know that my appointment at the University of Chicago was recommended by Eguchi, who recognised my work with Tamiaki Yoneya on the gauge fixing ambiguity in non-Abelian gauge theories [2], which predates the work of V. N. Gribov [3] by more than half a year.

Around that time one of the exciting problems was to understand and demonstrate quark confinement in a non-Abelian gauge theory. Ken Wilson [4] had demonstrated confinement in the strong coupling expansion. On the other hand, non-Abelian gauge theories are asymptotically free, and it would be important to demonstrate quark confinement, or equivalently the area law for the Wilson loop in the continuum theory. In a parallel development, Gerard 't Hooft [5] had demonstrated the connection between non-Abelian gauge theories and string theory in the $1/N$ expansion. Under some plausible assumptions, Nambu [6] and Gervais and Neveu [7] had derived an approximate string-like equation for the Wilson loop expectation value, with the string tension proportional to the vacuum condensate $\langle F_{\mu\nu} F^{\mu\nu} \rangle$, with the hope that such an equation would yield an area law as a solution and explain quark confinement.

My efforts to understand the gauge fixing problem during my PhD days had convinced me that non-perturbative gauge theories must be defined on a lattice. Eguchi and I set out to put Nambu's program on a firm footing, but it was almost impossible to make progress except by making plausible assumptions. We did make physically motivated but unproven assumptions and derived a lattice version of Nambu's equation. Tohru was having a strongly worded back and forth with the referee, who had many objections. I was not aware of all this correspondence until Nambu walked over to my

office and said that he had had a call from the referee about my paper with Eguchi and his objections, even though he would recommend it for publication in *PRL* [8]. The referee was Ken Wilson.

I then started working on deriving the loop equations as Dyson–Schwinger equations for Wilson loops on the lattice in the context of the $1/N$ expansion, using the properties of the path integral measure which I was familiar with from Ken Wilson’s paper [4]. Eguchi had independently thought about doing this, and we started discussing the derivation. Here we had a disagreement about the factorisation of a single trace intersecting loop, which was much in the spirit of Nambu’s original derivation and our own previous work. He invited me to be a co-author but I did not agree as I was by then certain that the approximation that produces the string tension was not justified [9]. This was also pointed out to Eguchi in a letter by David Gross. I continued to work on deriving the exact loop equations on the lattice using the factorisation property of gauge-invariant operators [10], which was by then realised by Makeenko and Migdal in their paper on a proposed loop equation in the continuum theory [11]. Their loop equations have regularisation ambiguities associated with string splitting terms and the definition of the loop derivative operator. However, the factorisation property of gauge-invariant operators to leading order in $1/N$ was an important insight in obtaining the correct set of closed-loop equations. Unfortunately, these equations are too difficult to solve except in simple cases, e.g. the one-matrix model [10]. During this period Tohru also produced an excellent and influential *Physics Report* article with Peter Gilkey and Andrew Hanson on “Gravitation, Gauge Theories and Differential Geometry” [12].

The disagreement that arose while working on the loop equations was in the nature of a conflict between intuition based on what to expect, and precise derivation, abated by an implicitly hierarchical relationship that I only slowly learnt to resist. See Ref. [13].

The years in Chicago were difficult for both of us for different reasons. In retrospect, I think that there were too many hidden undercurrents (unknown to me) that contributed to a difficult atmosphere, but we eventually left our differences behind and emerged stronger. Barring this, cheer, shine, and insights were brought in by people like Yoichiro Nambu, Steve Shenker, Leo Kadanoff, Peter Freund, and S. Chandrasekhar, and by Sumit Das, Nino Bralic, and Mark Sweeny who were students at that time. Long-term visitors like Michael Atiyah, Don Weingarten, and John Kogut contributed too.

In 1980 Tohru returned to Japan and joined the University of Tokyo, and I returned to India in 1982 and joined the Tata Institute of Fundamental Research in Bombay. He visited the Tata Institute in 1984 to lecture at a Winter School, and then during Strings 2001 at the Tata Institute in Mumbai, and Strings 2015 at ICTS-TIFR in Bangalore. I also met him on several occasions during my many visits to Japan. I was glad to see him during Strings 2018 in Okinawa, unfortunately for the last time.

During the decades following his return to Japan he greatly contributed to physics there, as is evidenced by some outstanding physicists he mentored. They include Hiroshi Ooguri, Tadashi Takayanagi, and Yuji Tachikawa.

In the following I will present an elementary result pertaining to a simple reinterpretation of the Wess–Zumino–Witten (WZW) model and current algebra in $1 + 1$ dimensions.

2. Current algebra, the WZW term, and an associated $U(1)$ gauge theory

In the presence of chiral couplings, current conservation laws are anomalous. At low energies this is reflected by the presence of the WZW term in the effective action [14]. From a microscopic point of view one can derive this term by computing the phase of the determinant of the Dirac operator in the presence of chiral couplings. The infinitesimal gauge transformation of the phase is equal to the

chiral anomaly term. Integrating this equation along a closed curve in the configuration space of the slowly varying chiral field gives the WZW term [15,16]. The presence of this term is also reflected in Schwinger terms in the corresponding current algebra. For the purposes of this note we take the current algebra to be the starting point and restrict to two spacetime dimensions. The gauge group is $SU(2)$ for the purposes of illustration, but the discussion easily generalises to $SU(N)$.

We discuss here a generalisation of an idea that was worked out in the case of the collective coordinate dynamics of the baryon solution of the chiral model in $3 + 1$ dimensions [17]. Our aim is to recast the WZW model as the dynamical system of an infinite-dimensional rotor defined by the local group element $U(x)$, minimally coupled to an Abelian gauge field defined on the configuration space M of maps from a circle S_1 of unit radius to the group manifold: $x \rightarrow U(x) \in G$.

2.1. Current algebra and an associated Abelian gauge theory

Consider the currents $L_0^\alpha(x, t)$ and $L_1^\alpha(x, t)$ corresponding to the compact Lie group $SU(2)$. We will also employ the parametrization $U = \exp(it_\alpha \theta^\alpha)$, where t_α are the generators of G , $[t_\alpha, t_\beta] = if_{\alpha\beta\gamma} t_\gamma$. The equal-time algebra of currents is

$$[L_0^\alpha(x), L_0^\beta(y)] = i\lambda^2 f_{\alpha\beta\gamma} L_0^\gamma + i\alpha f_{\alpha\beta\gamma} L_1^\gamma, \quad (1a)$$

$$[L_0^\alpha(x), L_1^\beta(y)] = i\lambda^2 f_{\alpha\beta\gamma} L_1^\gamma + \frac{\lambda^4}{2} \delta_{\alpha\beta} \partial_x \delta(x-y), \quad (1b)$$

where α and λ^2 are couplings related by $4\pi\alpha\lambda^{-2} = k$, an integer. (At the conformal invariant point, $\alpha = 1$).

Our main point is at the current algebra, Eqs. (1a) and (1b) can be realised in terms of a $U(1)$ fibre bundle picture. To see this, consider the current operators

$$l_0^\alpha(x) = \pi_\alpha - A_\alpha, \quad \pi_\alpha(x) = \text{tr} \left(t_\alpha U(x) \frac{\partial}{\partial U(x)} \right), \quad (2a)$$

$$l_1^\alpha(x) = \text{tr}(t_\alpha \partial_x U(x) U^{-1}(x)). \quad (2b)$$

In Eq. (2a), $A_\alpha \equiv A_\alpha(U(x))$ is a $U(1)$ connection at the point $U(x)$ in M . Introducing the notation $\delta_{\alpha x} U = [\pi_\alpha(x), U]$, it is easy to check that

$$\begin{aligned} \delta_{\alpha x} U(y) &= t_\alpha U(y) \delta(x-y), \\ \delta_{\alpha x} U^{-1}(y) &= -U(y) t_\alpha \delta(x-y), \\ [\delta_{\alpha x}, \delta_{\beta y}] &= -i f_{\alpha\beta\gamma} \delta_{\gamma x} \delta(x-y). \end{aligned} \quad (3)$$

The gauge transformation of $A_\alpha(U(x))$ is defined by

$$A_\alpha \rightarrow A_\alpha + \delta_{\alpha x} \phi, \quad (4)$$

where $\phi = \phi(U(x))$ is a function on M . The gauge-invariant field strength is given by

$$F_{\alpha x, \beta y} = i \delta_{\alpha x} A_{\beta y} - i \delta_{\beta y} A_{\alpha x} - f_{\alpha\beta\gamma} A_{\gamma x} \delta(x-y). \quad (5)$$

With these definitions it is easy to see that

$$[l_0^\alpha(x), l_0^\beta(x)] = -i f_{\alpha\beta\gamma} l_0^\gamma(x) \delta(x-y) + i F_{\alpha x, \beta y}, \quad (6a)$$

$$[l_0^\alpha(x), l_1^\beta(y)] = i f_{\alpha\beta\gamma} l_1^\gamma(x) \delta(x-y) + i f_{\alpha\beta\gamma} l_1^\gamma(x) \delta(x-y). \tag{6b}$$

l_0^α generates gauge transformations, but its algebra has an anomalous piece proportional to a specific value of the field strength F .

A representation of the current algebra in Eqs. (1a) and (1b) is given by

$$L_0^\alpha(x) = \lambda^2 l_0^\alpha(x), \quad L_1^\alpha(x) = \lambda^2 l_1^\alpha(x), \tag{7a}$$

$$F_{\alpha x, \beta y} = \left(\frac{\alpha}{\lambda^2}\right) f_{\alpha\beta\gamma} l_1^\gamma \delta(x-y) = \frac{k}{4\pi} f_{\alpha\beta\gamma} l_1^\gamma \delta(x-y). \tag{7b}$$

The above gives us a specific value for the field strength F relevant to the current algebra.

2.2. The Wilson line and the WZW term

To construct the Wilson line corresponding to the $U(1)$ gauge field, we note the following definitions and identities. The vierbein on the group manifold is defined by

$$\Sigma_\alpha(\delta\theta_\alpha e_a^\alpha) = \text{tr}(t_a \delta U U^{-1}). \tag{8}$$

Introducing the redefined connection

$$A_a = e_a^\alpha A_\alpha, \tag{9}$$

one can prove using the Maurer–Cartan equation that

$$e_a^\alpha e_b^\beta F_{\alpha\beta} = F_{ab}, \tag{10}$$

where $F_{ab} = \left(\frac{\partial}{\partial\theta^a} A_b - \frac{\partial}{\partial\theta^b} A_a\right)$. For the specific two-form field strength in Eq. (7b), which is well defined, the gauge potential in Eq. (9) has a Dirac-like singularity, like monopole potentials in electromagnetism. However, we will not try to construct the potential in this note.

The Wilson line is given by $e^{i\Gamma}$, and the phase Γ is given by

$$\Gamma = \int_{-1}^1 dt \int_0^{2\pi} dx A_\alpha(U(x, t)) l_t^\alpha(U(x, t)). \tag{11}$$

In the above, $t \rightarrow U(x, t)$ is a closed curve in the space of maps M with $U(x, 1) = U(x, -1)$, so that the domain of (x, t) is the two-sphere S^2 , and $l_t^\alpha = \text{tr}\left(t_\alpha \frac{\partial}{\partial t} U(x, t) U^{-1}(x, t)\right)$ is the tangent or “velocity” vector. Using the definition in Eq. (8) and the parametrization $U = \exp(it_a \theta^a)$, the phase Γ can be written as

$$\Gamma = \int_{-1}^1 dt \int_0^{2\pi} dx A_a(\theta(x, t)) \frac{d\theta^a(x, t)}{dt}. \tag{12}$$

Using Stokes’ theorem, we get a formula for Γ in terms of the two-form field strength:

$$\Gamma = \int_0^1 ds \int_{-1}^1 dt \int_0^{2\pi} dx F_{ab}(\theta(x, t, s)) \frac{d\theta^a}{dt} \frac{d\theta^b}{ds}, \tag{13}$$

where $\theta^a(x, t; s = 0) = 0$ and $\theta^a(x, t; s = 1) = \theta^a(x, t)$. Thus, $(s, x, t) \rightarrow \theta^a$ is a disc in $SU(2) \cong S^3$, with boundary S_2 . Using Eq. (9), we have

$$\Gamma = \int_0^1 ds \int_{-1}^1 dt \int_0^{2\pi} dx F_{\alpha,\beta} l_t^\alpha l_s^\beta, \tag{14}$$

where $l_s^\beta = \text{tr}(t_\beta \frac{\partial}{\partial s} U U^{-1})$. From the current algebra we know the expression in Eq. (7b) for the field strength. Substituting into Eq. (13), we get

$$\Gamma = \frac{k}{4\pi} \int_0^1 ds \int_{-1}^1 dt \int_0^{2\pi} dx f_{\alpha\beta\gamma} l_x^\alpha l_t^\beta l_s^\gamma, \tag{15}$$

which can be written in the standard form

$$\Gamma = \frac{k}{24\pi} \int_D d^3 x \varepsilon^{ijk} \text{tr}(\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1}). \tag{16}$$

This is the familiar form of the Wess–Zumino–Witten term written in terms of a closed three-form in *spacetime*; D is a disc in S_3 with boundary S_2 . In what we have described it arose as the phase of a $U(1)$ connection (one-form) on the space of maps M , or equivalently after using Stokes’ theorem in terms of a two-form on M . In the Schrödinger representation of the quantum field theory, the wave function carries a phase which is given precisely by Eq. (11), except that $t \rightarrow U(x, t)$ is an open curve with end point $U(x, T)$:

$$\Psi(\{U(x)\}, T) \sim \exp\left(i \int_{-1}^T dt \int_0^{2\pi} dx A_a(\theta(x, t)) \frac{d\theta^a(x, t)}{dt}\right). \tag{17}$$

The change of the wave function as one traverses a closed curve is given by

$$\Delta \Psi \sim \exp(i \Gamma). \tag{18}$$

This means that the wave function carries a projective representation of the non-Abelian current algebra that is encoded in the topologically non-trivial $U(1)$ gauge field.

3. Conclusion

In summary, we have shown that the WZW phase factor arises naturally as the phase of the Wilson line of a non-trivial $U(1)$ gauge field in the space of maps from S^1 into the group. It is then natural to assign the integer k as the quantized charge of a “magnetic monopole” of an Abelian gauge theory. This discussion can be carried over to the $SU(N)$ current algebra, and can also be extended to $3 + 1$ dimensions.

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