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Particles in 3D Regge Gravity and BMS symmetry

Thesis MSc Physics in Particle and Astrophysics

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Abstract

Finding a quantum theory for gravity could be the most difficult problem in physics. The direct approach not leading to any solutions, many methodologies are explored to solve this complicated problem. One such method is quantum Regge calculus. In this thesis we consider the 3D gravitational action, specifically the Einstein-Hilbert action along with the Gibbson Hawking term and a massive point particle. We support the Hamilton-Jacobi and one-loop determinant results obtained in the discretized setting, by analyzing the gravitational action in the continuum. As claimed in this thesis both scenarios the one loop determinant display the massive BMS_3 character, showing the possibility of gravity being a 'Holography' theory similar to the entropy of a black hole. We further analyze the behaviour of metric perturbations in the continuum in the presence of a massive point particle in 3D gravity.

Contents

1 Why theory of Quantum gravity?

1.1 Quantum theory vs General relativity

Quantum theory and the standard model is universally accepted theory with many experimental tests such as determining the Lamb shift with high accuracy [28] and the casimir effect [29]. Other than a few ongoing discussions of the interpretational foundations such as to whether the quantum nature obeys the the copenhagen interpretation [30] or the pilot wave theory [6] (among others) the standard model is a well established theory leading to development of practical applications such as the quantum computers and quantum entangled communications [33]. The standard model accommodates three of the four known fundamental forces of the universe. Electromagnetic, weak and strong forces are well described in the standard model. The only interaction not included in the standard model is the gravitational field. But the gravitational interactions is well described by another theory, Einsteins general theory of relativity also called geometrodynamics. The fundamental equations of general relativity is presented in simple geometric terms and is a well proven theory by many experimental tests. One best example is the the decrease of the orbital period of the binary pulsar PSR 1913+16. This phenomenon is fully explained by the emission of gravitational waves as predicted by general relativity. There exists other phenomena that hints at a more fundamental theory than general relativity such as dark energy and dark matter. As stated by Claus Kiefer in his book"Quantum Gravity" [20] every theory with regards to the gravity must contain general relativity at certain limits. Also from the"singularity theorems" by Hawking and Penrose in 1996 it is clear that general relativity is not complete and breaks down under very general conditions because the existence of space-time singularities are unavoidable. These encountering space time singularities are predicted to be of quantum in nature. Hence paved way to unify the gravitational field into a quantum framework.

1.2 Why quantize gravity?

There are three main motivations to quantize gravity. Unification, the existence of singularities in cosmology and black holes and the concept of time. The reasons for unification are for one is to hoping to solve the divergence problem of quantum field theory although its being unsuccessful so far (as yet to prove any quantum gravity theory) canonical quantum gravity and string theory are possible candidates for a divergence free theory. Another reason for unification is, its possible to implement gravity into the standard model if there existed a universal coupling of gravity to all forms of energy. But the main reason for unification of gravity with quantum is the failure of constructing a semi-classical theory where gravity stays classical and other fields remaining quantum. A good example for the invalidation of a semi-classical theory is proven by Ford [16] for the inconsistent results in the emission of the gravitational radiation by quantum systems. The emission of gravitational radiation energy of a quantum system with a superposition of coherent states as predicted by linearized gravity in the semi-classical theory and quantization of the linear theory differ by macroscopic amounts. While for non-superposition coherent states both yield identical results. Another invalidation for the semi classical theory is that it does not predict the Casimir effect [20].

Breakdown of general relativity occurs at singularities involving the initial conditions near the 'big bang' and the final stages of a black hole evolution. Therefore it is required an encompassing theory which can describe these phenomena more concretely. Since these singularities reach Planck scales (and considering a historical analogy of quantum mechanics [20]) this theory is expected to be a quantum theory.

One would intuitively expect that time is of the same nature at the quantum scale and the macroscopic scale (especially since the semi classical approach has not been viable). But quantum theory and general relativity contain vastly different concepts for time also known as"problem of time". In quantum mechanics, time is considered as an universal and absolute element similar to classical physics (in quantum field theory Minkowiski space-time is an external absolute element). Time here mostly corresponds to the evolving entanglements and propagation of particles. Very similar how time is mostly involved in the movement and order of interaction of things (matter etc.) in Newtonian physics. But in general relativity time is relative in other words it depends on the observer. Also time is not an absolute but a dynamical element which is inextricably interwoven with space such that the nature of space (curvature) in where the observer lies affects the flow of time. Since time is as its very well proven to not to be an absolute element, we try to bring the intertwined nature of space-time to quantum theory. With these motivations physicists attempt to solve for the theory of quantum gravity. We will now look into some approaches made in order to do so.

1.3 Approaches to quantum theory of gravity

The main aim when constructing a quantum theory of gravity is to have a consistent theory that can be subjected to experimental tests. As discussed in the previous section the gravitational field is of quantum nature in the fundamental level, a true fundamental theory must have a rigid structure as to which it can predict values such as particle masses and coupling constants in the low energy regime. Since no experimental evidence for quantum gravity exists at the moment most approaches to quantum gravity focus on constructing a mathematically and conceptually consistent framework. As described by Isham [15] there is a distinction between the approaches to construct a theory. Namely, 'primary theory of quantum gravity' and a 'secondary theory'.

In the primary approach one applies heuristic quantization rules for a classical theory. Often being general relativity leading to 'quantum general relativity'. In the primary approach another distinction is the canonical and covariant approaches. The canonical approach adopts the split of 4 dimensional space-time to 3+1 space and time dimensions. Example that adoptes this is 'Loop quantum gravity'[27]. The covariant approach preserves the 4 dimensional covariance of space-time throughout the construction. Examples for this approach are 'Regge gravity'[4] and 'Causal dynamical triangulations'[22].

The the secondary theory approach begins from a fundamental quantum framework for every interaction and attempts to derive general relativity under certain conditions such as at high energy limits. The best example for this approach is 'string theory' [23]. A advantage of this approach as we start from a fundamental theory of all interactions we automatically have a 'Theory of Everything', but a shortcoming is that the starting point the fundamental theory itself is highly speculative.

This work is based on the results obtained in 3D Regge gravity mainly by the works of Alicia [10] followed by the continuum regime by Seth [1]. A detailed discussion about Regge gravity is in section 3. In the next section we will discuss a few concepts that will be used in a later sections 3 and 4. Section 4 contains original work conducted for the thesis.

2 Symmetry Group BMS

2.1 Asymptotic BMS symmetry

A system if under a set of transformations preserves it's invariance the system is said to be 'symmetric'. These set of transformations form a group and the main mathematical tool used to analyze symmetries is group theory. Here we look at 'asymptotic symmetries' which is also described as a generalization Poincaré symmetry in a weak gravitational field. This is a symmetry on which set of co-ordinate transformations of space-time, preserving the system's invariance at (almost) infinite distances in a gravitational field. In other words symmetries observed at (almost) infinitely away from a gravitational system. First we introduce the standard Poincaré group

$$Poincaré = Lorentz \ \rtimes \ Translations. \tag{1}$$

The notation $\rtimes$ is the semi-direct product of the Lorentz group of special relativity and the Translations group [25].A semi-direct product is defined as follows.

Consider two Lie groups A and B with elements $a, b \in A$ and $\alpha, \beta \in B$. Let σ be a function on : $A \times B$ such that $\sigma : A \times B \longrightarrow B : (a, \alpha)$ is a smooth action of A on B. Let σ_a be a automorphism of B such that $B : (a, \alpha) \longmapsto \sigma_a(\alpha)$. The semi-direct product of A and B with respect to σ consists of the group operation

$$(a, \alpha) \cdot (b, \beta) = (a \cdot b, \alpha \cdot \sigma_a(\beta)), \tag{2}$$

and the group is denoted by $A \rtimes B$ containing elements (a, α).

In a gravitational field of a asymptotically flat space-time it is found that a symmetry group much larger than of the Poincaré group exists. This group is the 'Bondi Metzner Sacha group' or BMS group and it is an infinite-dimensional extension of the Poincaré group. While is it similar to the Poincaré group, the space-time translations are replaced with a infinite-dimensional extension of a Abelian group called 'supertranslations' [25]

$$BMS = Lorentz \ \rtimes \ Supertranslations. \tag{3}$$

This makes the Poincaré group a subgroup of the BMS group. After recent work by Barnich and Troessaert [2] this is known as 'global BMS group' and the true physically relevant symmetry of four dimensional asymptotically flat space times is shown by the 'extended BMS group'. The extended BMS group is obtained by replacing Lorentz transformations by local conformal transformations (conformal transformation is a function that locally preserves angles, but not necessarily lengths). Local conformal transformations of celestial spheres (see figure below) are 'superrotations' and is considered as a infinite dimensional extension of Lorentz transformations,

$$BMS_{extended} = Superrotations \ \rtimes \ Supertranslations. \tag{4}$$

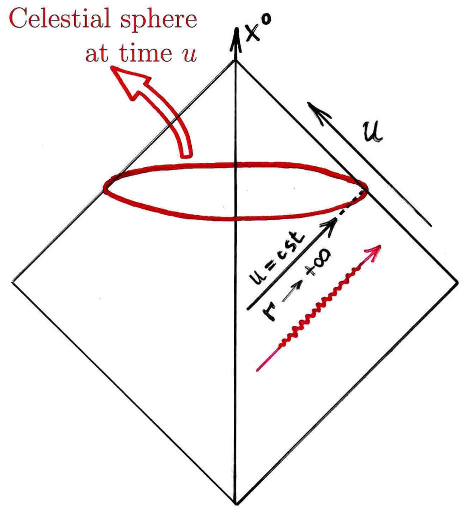

Figure 1: This represents the Penrose diagram of Minkowski space in three dimensions. The outgoing light ray (towards infinity) is shown by the wavy red arrow. In four dimensional space-time the red circle will represent a celestial sphere at time u. [25]

2.2 Massive BMS_3 character

It is needed to note that BMS group in four dimensions is poorly understood. The study of toy model namely BMS in three dimensions (BMS_3) is more well explored. BMS in three dimensions is quite sufficient to capture key features of BMS symmetry.

We will look into the characteristic of BMS particles in three dimensions. In Poincaré the irreducible unitary representations define 'particles'. They can be classifield by the mass and spin. Since the Poincaré group is a subgroup of BMS with this picture we can define a BMS particle as an irreducible, unitary representation of the BMS group [25]. A massive BMS_3 particle is defined as "whose supermomenta span a Virasoro coadjoint orbit that admits a generic constant representative". [25]

We are interested in the character (of irreducible representations of BMS group) for a BMS_3 massive particle and it is given by [25]

$$\chi[(f,\alpha)] = e^{is\theta} e^{i\alpha^0(m-c_2/24)} \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2}, \tag{5}$$

where f, α rotations and supertranslations in induced representations of the BMS_3 group. $q = exp(i\theta)$ and θ is a real angle, $\mathcal{M}$ is the rest BMS mass, c_2 is the central charge and α^0 denotes the zeroth Fourier mode of a supertranslation. $e^{is\theta}$ is the little group representation where s is the spin of particle. If we are considering a massive BMS_3 particle without spin the above expression reduces to

$$\chi[(f,\alpha)] = e^{i\alpha^0(m-c_2/24)} \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2}. \tag{6}$$

2.3 Holography and AdS/CFT

The concept of holography in quantum gravity is that all information of a gravitational phenomena taking place in a space-time manifold are stored in a lower-dimensional, "dual" theory [25]. The Bekenstein-Hawking entropy formula is a prime example for a holography theory [19]. It states that the the entropy of a black hole is directly proportional to the area of the horizon as opposed to the volume as one would generally expect.

Maldecena being one of the first in her work on 'string theory' illustrated the holography principle in the context of quantum gravity. It is now known as the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. According to AdS/CFT all gravitational observables' information of AdS (bulk space) are stored at spatial infinity (at the boundary) in terms of CFT. Two different theories contain the same physical information is the concept of duality.

2.4 Liouville's Theory

Liouville theory is a two dimensional unitary conformal field theory. The classical equations of motion of Liouville theory (or action) yields the Liouville equation. The Liouville equation describes for a surface of constant Gaussian curvature a non-linear partial differential equation with a conformal factor. This is used to study conformal manifolds. In the context of string theory Liouville theory is one of the simplest models to study non-trivial, non-compact backgrounds. [31, 21].

Starting from the three dimensional Einstein-Hilbert action with negative cosmological constant it has being shown that it can be reduced to a two dimensional boundary Liouville theory by [11]. According to Chern-Simons theory the Einstein action is equivalent to the $SL(2,R) \times SL(2,R)$ (gauge group) Chern-Simons action. $SL(2,R)$ is a special linear group with elements containing 2×2 real matrices and each matrix having determinant value of one. For a gauge field A_μ the Chern-Simons action is defined as [18]

$$S_{CS}[A] \equiv \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} Tr \left(A_\mu (\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3} A_\mu [A_\nu, A_\rho] \right). \tag{7}$$

Upon imposing the conditions of opposite chiralities on each $SL(2,R)$ factor in a space-time with a cylindrical boundary the Chern-Simons action reduces to the non-chiral $SL(2,R)$ Wess-Zumino-Novikov-Witten (WZW) model. WZW model can be regarded as the boundary theory for Chern-Simons theory. Next by imposing anti-de Sitter boundary conditions (i.e. space-time is asymptotically AdS), the constraints reduces the WZW model to the Liouville theory essentially reducing the 3D gravity to a 2D boundary model making it a holographic theory. It is also shown that by [4] Liouville like dual field theory can also be seen in finite boundary setting with flat space-time in vacuum with no particles using Regge calculus. As we see later we have shown that this could be extended for the case of having a massive BMS point particle.

For a boundary with background extrinsic curvature K_{AB} and background intrinsic metric h_{AB} we consider the scalar field taking the same form as used by Seth [1],

$$\rho \Delta \rho = \rho(2(K^{CD} - K h^{CD}) D_C D_D - \, ^2RK)\rho) \tag{8}$$

where ρ is the Liouville field and D_C is the covariant derivative with respect to the metric h_{AB}. The full lagrangian for this scalar field is given by [1]

$$\mathcal{L} = \sqrt{h}(\rho \Delta \rho - 2\rho \delta(^2R)). \tag{9}$$

Here $\delta(^2R)$ is the first order perturbation of the boundary Ricci scalar. Here it describes on how the scalar field (Liouville field) couples to the first order perturbation of the Ricci scalar. This lagrangian is the key component is proving that the effective action is of 'dual field theory' nature.

2.5 Why study 3D gravity?

It is well established by Einstein's theory of general relativity that this universe is well represented in four dimensional space-time. In Newtonian mechanics it is straight forward to translate from two spatial dimensional theories to three spatial dimensions. For example calculating (Newtonian) momentum of collisions of objects in two or three dimensions. But this is not the case in general relativity. We will discuss briefly one of the main reasons for this.

In any spacetime the Riemann (curvature) tensor can be written in terms of Ricci scalar R, Ricci tensor R_{im} and the Weyl tensor C_{iklm} and is given by [32]

$$\begin{aligned} C_{iklm} = R_{iklm} + \frac{1}{n-2}(R_{im}g_{kl} - R_{il}g_{km} + R_{kl}g_{im} - R_{km}g_{il}) \\ + \frac{1}{(n-1)(n-2)}R(g_{il}g_{km} - g_{im}g_{kl}) \end{aligned} \tag{10}$$

where n is the number of dimensions of the manifold. The Weyl tensor is traceless, conformally invariant and defined such that for any tensor contraction between integers it value becomes zero i.e. $C^i_{kim} = 0$. Therefore for any dimensions three or lower the Weyl tensor vanishes. This leads that for three dimensions the above equation reduces to

$$R_{iklm} = R_{il}g_{km} + R_{km}g_{il} - R_{im}g_{kl} - R_{kl}g_{im} - \frac{1}{2}(g_{il}g_{km} - g_{im}g_{kl})R. \tag{11}$$

Then with cosmological constant Λ any solutions for the Einstein field equations results in [9],

$$R_{im} = 2\Lambda g_{im}. \tag{12}$$

This means it has constant curvature, it is either locally flat for $\Lambda = 0$, de sitter and anti-de sitter for $\Lambda > 0$ and $\Lambda < 0$ respectively. This leads to the conclusion that three dimensional space-time has no local degrees of freedom (i.e. propagating degrees of freedom) which means no gravitational waves are emitted in a three dimensional setting [9]. Lack of local degrees of freedom greatly differentiates from a four dimensional setting (physical reality) which contains propagating degrees of freedom hence it is very difficult to translate the theory to higher dimensions. Another reason for it to differentiate is that three dimensional gravity lacks of a good Newtonian limit [9].

So why study three dimensional gravity? The reason is the surprise discovery by Banados, Teitelboim, and Zanelli (BTZ) of a black hole solution in three dimensional gravity (with $\Lambda < 0$) [8]. The BTZ black hole differed in some aspects to the Schwarzschild and Kerr black holes such as not containing a curvature singularity at the origin. But the BTZ black hole contains a event horizon, contains thermodynamic properties of a four dimensional black hole and it is shown that the BTZ black hole is formed by the gravitational collapse of matter [8]. This popularized the investigation into three dimensional gravity especially in the study of the nature of observable such as the "problem of time" as discussed previously. Some other reasons to study 3D gravity is that 3D gravity contains the same conceptual foundations of 4D gravity while being vastly simpler to analyze and calculate. Although it is quite unrealistic, 3D gravity is widely used as a toy model to investigate a theory of quantum gravity.

3 Introduction to 3D Regge Gravity

As mentioned previously our main motivational basis is the works by Seth and Alicia, in this section we will very briefly describe Alicia's work on 3D Regge gravity with point particle and introducing Regge calculus. As discussed in section 1.3 on the approaches for the construction of the theory the starting point for Regge calculus is the quantization of the path integral. A very popular method used in quantum mechanics and general relativity. In the case of quantum mechanics the probability amplitude for a propagator (in this case a propagator carrying the force of gravity known as 'graviton') to go from a postion x' at time t' to position x'' at time t'' is given by the path integral [20],

$$\langle x'', t'' | x', t' \rangle = \int Dx(t) e^{iS[x(t)]/\hbar} \tag{13}$$

Where $Dx(t)$ is the formal notation for the limiting process. But this holds for a propagator in external time and does not hold for quantum gravity since it is fundamentally timeless [20]. This path integral can be generalized to quantum field theory but it loses its physical interpretation. But it is valuable and plays a key role in gauge theories. As first formulated by Misner the quantum gravitational path integral takes the form [20]

$$Z[g] = \int Dg_{\mu\nu}(x) e^{iS[g_{\mu\nu}(x)]}, \tag{14}$$

where it is considered the summation over all metrics in a four dimensional manifold $\mathcal{M}$ and divided by (reducing) the diffeomorphism group Diff$\mathcal{M}$. This is highly complicated in terms of technical and conceptual stand point. To rectify this one usually performs a Wick rotation to the Euclidean regime. But it leads to numerous issues one especially being that the Euclidean gravitational action is not bounded from below. This leads to the path integral being divergent, this is known as the 'conformal factor problem'. Since there isn't a straight forward method to evaluate this integral approaches are made to evaluate the path integral by discretization and performing the continuum limit. There are several methods to perform this but here we will focus on 'Regge Calculus'.

In Regge calculus the central concept is to define smooth manifolds using basic topological principles using 'Euclidean simplexes' instead of co-ordinates. A simplex is the space time manifold fundamental building block. Consider a two dimensional wall, the simplexes would be the tiles covering the wall. Although in the case of Regge calculus these tiles need not be of the same size, they must all fit perfectly like peices of a puzzle and must be self-joining in a way covering the whole surface (manifold). A good example of discretized curved surface is the Atibaia's radio telescope in Brazil as shown in the figure below.

Figure 2: This image displays the discretization of a spherical surface using a set of juxtaposed plane polygons [12].

Intensely curved surfaces require more simplexes to cover it. This shows that the number of simplexes is directly related to the curvature. The curvature is measured at the vertexes. consider the image below,

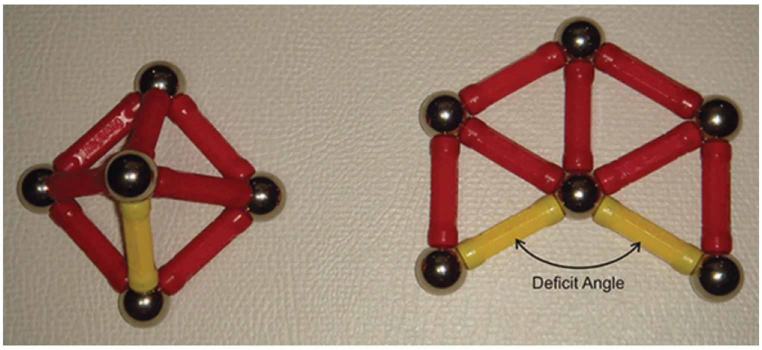

Figure 3: The curvature of the square pyramid (without base) shown on the left can be shown to be associated with the vertex as the faces of the pyramid can be flattened out on a 2D surface revealing a deficit angle [12].

in this scenario (ignoring the base of the pyramid for the sake of the argument), the sum of the dihedral angles θ_n (in the n triangles) around the top vertex of the pyramid (on the left) is below 2π in the flatten surface (to the right). This angular difference is defined as the deficit angle ε which measures the curvature of the pyramidal surface.

$$\varepsilon = 2\pi - \sum_n \theta_n \tag{15}$$

Under the same principle we can measure the extrinsic curvature of a boundary of a manifold. For boundary deficit angles ω we consider the difference with π instead of 2π

at the vertex [4].

$$\omega = \pi - \sum_{n} \theta_n \tag{16}$$

Here we see that the curvature is centered at vertexes and deficit angles quantify the curvature. In General Relativity, the Riemann tensor R^a_{bcd} quantifies the space-time curvature of continues manifolds. By studying the parallel transport of a vector along an infinitesimal closed loop in a discrete and continues manifold, a relationship can be obtained between deficit angles and the Riemann tensor [12] and in turn the Ricci scalar. In a three-dimensional manifold the lengths of the edges contains information of the metric (a very similar concept is adopted to discritize extrinsic curvature of a boundary).

In quantum Regge calculus the considered path integral is given by [10],

$$Z = \int D\mu(l) exp\left(-\frac{S_R}{\hbar}\right). \tag{17}$$

This implies integration of all edges of the triangulation with some measure $\mu(l)$. 3D gravity which is a topological theory as it describes the dynamics using only global topological variables, we expect 3D gravity to be discretization independent (invariant). This should imply that at least for the case of linearized theory the invariance should hold for the one loop determinant. The measure is given by [4]

$$D\mu(l) = \prod_{\sigma} \frac{1}{\sqrt{12\pi V_\sigma}} \prod_{e \in bulk} \frac{l_e dl_e}{\sqrt{8\pi G\hbar}} \prod_{e \in bdry} \sqrt{\frac{l_e}{\sqrt{8\pi G\hbar}}}\,. \tag{18}$$

When integrating out edge variables the form of the measure must not change in order to perserve the invariance of the measure. The integration process in terms of Regge calculus is interpreted as a local change in triangulation. There are two local changes (for 3D) of triangulation called Pachner moves. When integrating out edges it has been explicitly shown that the form of the measure does not change [13]. A small introduction of the Pachner moves is followed in the next section.

3.1 Pachner moves in 3D

In this section we will very briefly introduce the pachner moves for three-dimensions. Pachner moves are local changes of triangulation of a given manifold. We will first look at the 3-2 Pachner move.

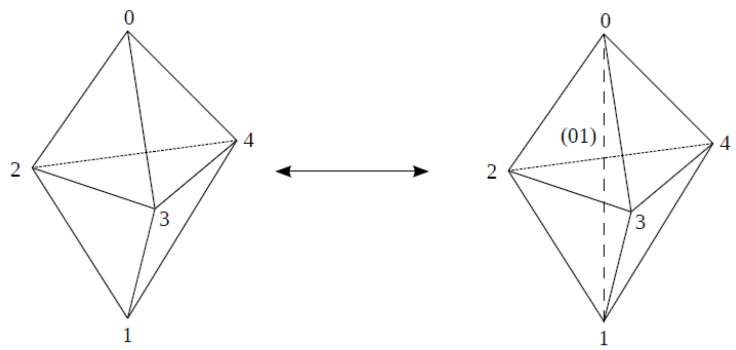

Figure 4: This figure shows the 2 states of the 3-2 pachner move. The left configuration consists of two tetrahedra and the right consists of three. The dashed line represents a bulk edge. [13].

In the configuration to the right of the figure above all three tetrahedra (0123), (0124) and (0134) have a common edge which is the only bulk edge (01). By removing this bulk edge or integrating out this edge the configuration reduces to the configuration as shown to the left of the figure above. This configuration consists of two tetrahedra (0234) and (1234) sharing the triangle (234). As the name suggests the 3-2 move is the transition from the three tetrahedra configuration to the 2 tetrahedra configuration while retaining its structure. [14, 5]

Now we will discuss the 4-1 pachner move.

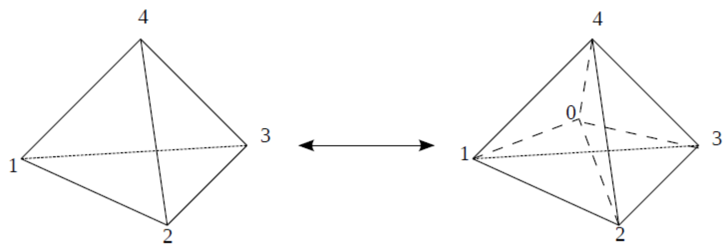

Figure 5: The two configurations of the 4-1 move. The dashed lines represents the internal bulk edges. The left and right configurations consists of one and four tetrahedra respectively. [13].

Similar tot he 3-2 move but as we can clearly see the as in the figure to the right the four tetrahedra shares a common vertex (0) instead of an edge. But integrating or removing all edges connected to this vertex the configuration reduces to a single tetrahedra (1234). These are the ways one can integrate out edge variables while preserving the same form of the measure in the path integral for quantum Regge calculus in 3D.

3.2 Regge gravity for a Point Particle

We will now introduce the Regge action. A term is added to the Regge action in vacuum which adds the affect of deformation of the triangulation due to a massive particle in it's rest frame, located at the center of a torus [10].

$$S_p = -\frac{1}{8\pi G} \sum_{e \subset T^o} l_e \epsilon_e - \frac{1}{8\pi G} \sum_{e \subset \partial T} l_e \omega_e + \sum_{e \subset WL} 8\pi G M l_e \tag{19}$$

with

$$\epsilon_e = 2\pi - \sum_{e \subset \sigma} \theta_e^\sigma, \tag{20}$$

$$\omega_e = \pi - \sum_{e \subset \sigma} \theta_e^\sigma. \tag{21}$$

Here T^o and ∂T represent the bulk and boundary edges respectively. The first sum is over all bulk edges of triangulation T and the deficit angles measured around the edge e. The angle θ_e^σ denotes for the tetrahedron σ the interior dihedral angle measured at the edge e. The second sum corresponds to the Gibbons-Hawking-York and involves the boundary angles on a given edge e measuring the extrinsic curvature. The last term is the discretized worldline of a massive point particle in it's rest frame. The equations of motion with respect to length variables yields,

$$\epsilon_e = 0 \;\; e \not\subset WL \tag{22}$$

$$\epsilon_e = 8\pi GM \;\; e \subset WL \tag{23}$$

The path integral cannot be computed analytically due to the complexity of the functions of the length variables and having to consider the range of integration due to Regge calculus having need to satisfy the generalized triangle inequalities. This issue is circumvented by considering linearized Regge calculus.

This is done by first choosing a background structure or solution. According to Rickles [26] a theory contains dynamic structures and background structures. The dynamic structure represents the physical degrees of freedom of a theory such as the electromagnetic field in Maxwell's theory and the metric in general relativity. The background structures are the values (parameters) put 'by hand' such as quantities involving the shape of space-time (like triangulation), topology, fields and the metric (not in general relativity). A good example of this is general relativity is a background independent theory as the metric is a solution of the Einstein's field equations while in quantum field theory we assign Minkowski space-time for the theory making it background dependent. Although ideally we would want background independence in our theories (making it a more fundamental theory) this is a very difficult condition to maintain (another reason as to why gravity have not yet unified with other forces). Here the background structure considered is the triangulation having fixed edge lengths $l_e^{(0)}$ already satisfy the triangle inequalities and integration are performed for the length perturbations λ_e [4].

$$l_e = l_e^{(0)} + \lambda_e \tag{24}$$

we consider Regge action up to the second order in the perturbation variables λ_e as the hessian produces the interesting properties need to study the path integral,

$$S = S^{(0)}\Big|_{l_e = l_e^{(o)}} + \frac{\partial S}{\partial l_e}\Big|_{l_e = l_e^{(o)}} \lambda_e + \frac{\partial^2 S}{2\partial l_e \partial l_{e'}}\Big|_{l_e = l_e^{(o)}} \lambda_e \lambda_{e'} \; . \tag{25}$$

The zeroth order and the first order term of 25 vanishes for the bulk edges, because of the Schläfli identity ($\sum_{e \in m} l_e \delta\theta_e^m = 0$ where θ_e^m is the interior dihedral angle at the edge e in the tertrahedron m [4])and specifically we take the background solution to be (locally) flat, that is $\omega_h^{(bulk)} = 0$.

3.3 The Triangulation

We will now look at the triangulation configuration. The space-time topology is of a solid torus. The cylindrical height of the torus is denoted as β and it contains (divided into) N_T cylinders of height T each. Each cylinder is divided by N_A radial lines of length R with boundary edge length A. Each of the prisms are triangulated into three tetrahedra as shown in the figures below.

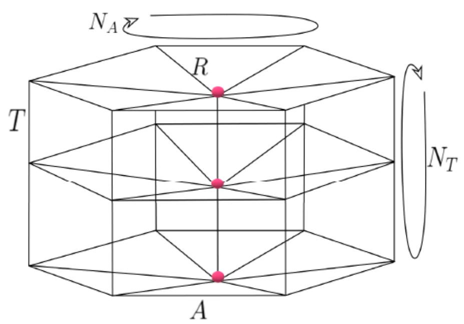

Figure 6: Chosen background triangulation of torus. The red dot represents the point particle. [10]

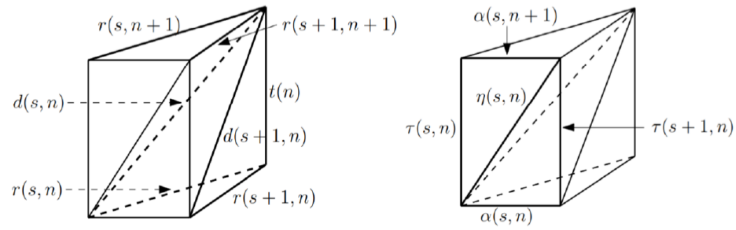

Figure 7: Subdivision of each prism into three tetrahedra. The left and right figures shows the bulk and boundary fluctuations of its lengths respectively. [4]

It is useful to define the following quantity for the relation between the background lengths A and R [10],

$$x = \frac{A^2}{2R^2} = 1 - cos\left(\frac{2\pi\mu}{N_A}\right) \tag{26}$$

with $\mu = 1 - 4GM$. By imposing flatness condition to all of the triangles, the boundary deficit angle along the time axis (which spans along the center of the torus where the particle lies as shown in figure 6) θ_b becomes [10],

$$\theta_b = \pi - \frac{2\pi\mu}{N_A} \ . \tag{27}$$

We denote the vector for the length fluctuation of the triangulation as follows [10],

$$\lambda(s,n) = (t(n), r(s,n), d(s,n), \tau(s,n), \alpha(s,n), \eta(s,n)) \ . \tag{28}$$

The partition function (17) in this triangulation considering the linearized action takes the following form [10],

$$Z = \int \prod_n dt(n) \prod_{(s,n)} dr(s,n)dd(s,n)d\tau(s,n)d\alpha(s,n)d\eta(s,n)\mu(l)exp\left(-\frac{S_p^{LR}}{\hbar}\right) \ . \tag{29}$$

As discussed earlier integrating out bulk edges preserves the form of the measure. First we evaluate the bulk edge fluctuations (i.e integrating out), for this we rewrite (29) as [10],

$$Z = \int d\tau(s,n)d\alpha(s,n)d\eta(s,n)\mu(l)Z(\tau(s,n), \alpha(s,n), \eta(s,n)) \tag{30}$$

where

$$Z(\tau(s,n), \alpha(s,n), \eta(s,n)) = \int \prod_n dt(n) \prod_{(s,n)} dr(s,n)dd(s,n)\mu(l)exp\left(-\frac{S_p^{LR}}{\hbar}\right) \ . \tag{31}$$

Here $\mu(l)$ is the density of the measure $D\mu(l)$, for this triangulation it gives the following form [10],

$$\mu(l) = \frac{R^{N_A N_T} T^{N_T} (R^2+T^2)^{N_A N_T}}{(8\pi G\hat{h})^{N_T(2N_A+1)/2}} \sqrt{\frac{A^{N_A N_T} T^{N_A N_T} (A^2+T^2)^{(}N_A N_T/2)}{(8\pi G\hat{h})^{3N_A N_T/2}}} \frac{1}{(12\pi V_\sigma)^{3N_A N_T/2}} \tag{32}$$

where the tetrahedron background volume is given by $V_\sigma = (ATR\sqrt{4-2x})/2$.

By considering the linearized action for the Regge action we get the first order and zeroth order terms for the bulk edges as zero. The Hamilton jacobi functional also known as effective boundary action is “the on-shell action, for 3D linearized gravity, for a large class of boundaries”[1] gives the boundary terms (edges) for the first and zeroth order of the linearized action [10].

$$S_{(p)}^{(0)} = -\frac{\beta}{4G}\mu \tag{33}$$

$$S_{(p)}^{(1)} = -\frac{1}{8\pi G}\frac{2\pi\mu}{N_A}\sum_{s,n}\tau(s,n) \tag{34}$$

The second order terms of the linearized action of the bulk and the boundary edges form the Hessian. The Hessian is defined as the square matrix containg all second order partial derivatives of all variables of a scalar function or scalar field [3]. For Hessian for one prism $H^{pr}_{ee'}$ we obtain a matrix of the form [10],

$$H^{pr}_{ee'} = \sum_{\sigma\in pr}\sum_{\sigma\supset e,e'}\frac{\partial\theta^{\sigma}_{e'}}{\partial l_e} = \frac{L_e L_{e'}}{6V_\sigma}M^{pr}_{ee'}(x) \tag{35}$$

where $M_{ee'}$ is dimensionless and the only background length dependence appears is as function of the ratio x. We replace fluctuation variables λ_e with rescaled variables [10]

$$\hat{\lambda}_e := \frac{L_e}{\sqrt{6V_\sigma}}\lambda_e \; . \tag{36}$$

Now when considering the hessian for the full triangulation since we choose a solid twisted torus it is suitable to Fourier transform the fluctuation variables in temporal and angular directions [10],

$$\hat{\lambda}(k,n) = \frac{1}{\sqrt{N_A}}\sum_{s} e^{-i\frac{2\pi}{N_A}k.s}\hat{\lambda}(s,n) \tag{37}$$

$$\hat{\lambda}(k,\nu) = \frac{1}{\sqrt{N_T}}\sum_{n} e^{-i\frac{2\pi}{N_T}(\nu-\frac{\gamma}{2\pi}k).n}\hat{\lambda}(k,n) \tag{38}$$

where $k \in \{0,1,2,...,N_A-1\}$ and $\nu \in \{0,1,2,...,N_T-1\}$. The parameter γ is called the twist angle and it measures the angular rotation made before identifying $t \sim t+\beta$. It is given by [10]

$$\gamma = 2\pi\frac{N_\gamma}{N_A}, \tag{39}$$

where N_γ is the number of prisms its rotated ($N_\gamma = 0,1,....,N_A-1$). It is also convenient to define $\upsilon = \nu - \frac{\gamma}{2\pi}k$. By implementing the above transformed variables we can write the second order term of the linearized Regge action as follows [10],

$$S_P^{(2)} = \frac{1}{16\pi G}\sum_{k,\nu}(\hat{\lambda}(k,\nu))^t.\tilde{M}(k,\nu).(\hat{\lambda}(k,\nu)) \tag{40}$$

with

$$(\hat{\lambda}(k,\nu))^t = (\hat{t}(\nu),\hat{r}(k,\nu),\hat{d}(k,\nu),\hat{\tau}(k,\nu),\hat{\alpha}(k,\nu),\hat{\eta}(k,\nu)) \; . \tag{41}$$

The Hessian matrix $\tilde{M}(k,\nu)$ takes the same form for both vacuum and particle cases, only difference being the function ratio x and it given by the matrix [10],

$$\begin{pmatrix} 0 & -2x\sqrt{N_A}\delta_{k,0} & 0 & 0 & \sqrt{N_A}\delta_{k,0} & 0 \\ -2x\sqrt{N_A}\delta_{k,0} & \Delta_k & 2x(1-\omega_\upsilon)-\delta_k & (\omega_k-1+2x)\omega_\upsilon & \omega_k\omega_\upsilon-1 & \omega_\upsilon-\omega_k\omega_\upsilon \\ 0 & 2x(1-\frac{1}{\omega_\upsilon})-\Delta_k & \Delta_k & \frac{1}{\omega_k}-1 & \frac{1}{\omega_\upsilon}-1 & \omega_k-1 \\ 0 & \frac{1}{\omega_\upsilon}(\frac{1}{\omega_k}-1+2x) & \omega_k-1 & 1 & \frac{\omega_k}{2}-\frac{1}{2\omega_\upsilon} & -\omega_k \\ \sqrt{N_A}\delta_{k,0} & \frac{1}{\omega_k\omega_\upsilon}-1 & \omega_\upsilon-\frac{1}{\omega_k} & \frac{1}{2\omega_k}-\frac{\omega_\upsilon}{2} & 1 & -\frac{\omega_\upsilon}{2}-\frac{1}{2} \\ 0 & \frac{1}{\omega_\upsilon}-\frac{1}{\omega_\upsilon\omega_k} & \frac{1}{\omega_k}-1 & -\frac{1}{\omega_k} & -\frac{1}{2\omega_\upsilon}-\frac{1}{2} & 1 \end{pmatrix} \tag{42}$$

where

$$\omega_k = e^{i\frac{2\pi}{N_A}k}\ ,\ \ \Delta_k = 2-\omega_k-\omega_k^{-1}\ , \tag{43}$$

$$\omega_\upsilon = e^{i\frac{2\pi}{N_A}\upsilon}\ ,\ \ \Delta_\upsilon = 2-\omega_\upsilon-\omega_\upsilon^{-1}\ . \tag{44}$$

As mentioned previously we integrate the bulk variables of the Hessian. We do this by first identifying the null vectors of the Hessian. The null vectors represent the symmetries of the system. Also when integrating bulk variables we consider only the null vectors which contains only bulk terms. These represent the gauge modes of the system which we will discuss on how they contribute to the partition function in the next section.

3.4 Gauge Modes

To find the null vectors of the gauge modes we analyze the bulk part of the hessian matrix. First we consider the k=0 case, this matrix is given by [10]

$$\tilde{M}_{bulk}(k=0,\nu) = \begin{pmatrix} 0 & -2x\sqrt{N_A} & 0 \\ -2x\sqrt{N_A} & 0 & 2x(1-\omega_\upsilon) \\ 0 & 2x(1-\omega_\upsilon^{-1}) & 0 \end{pmatrix} \tag{45}$$

The matrix has one null vector [10]

$$(n_t)(k=0,\nu) = (\frac{1-\omega_\upsilon}{\sqrt{N_A}}, 0, -\omega_\upsilon, 0, 0, 0). \tag{46}$$

The next step is to integrate out the $\hat{r}(0,\nu)$ and $\hat{d}(0,\nu)$ variables. We negate the negative sign (by considering it as positive) of one of the eigenvalues arising due to the conformal factor mode problem. This gives the eigenvalue [10]

$$\begin{aligned} \int d\hat{r}(0,\nu)d\hat{d}(0,\nu)d\hat{r}(0,-\nu)d\hat{d}(0,-\nu)\times \\ exp\left(-\frac{1}{2}(\hat{t}(\nu),\hat{r}(0,\nu),\hat{d}(0,\nu)).\tilde{M}_{bulk}(k=0,\nu).(\hat{t}(-\nu),\hat{r}(0,-\nu),\hat{d}(0,-\nu))^t\right) \\ = \frac{4\pi^2}{4x^2\Delta_\nu}\ . \end{aligned} \tag{47}$$

Taking the product over the ν modes the square root value of (47) becomes the contribution to the partition function. A corresponding measure needs to be removed due to the overlapping contribution of the gauge orbits describing the invariance of the Regge action along the t vector. For each vertex $p(n)$ along t the removed measure factor is given by [10]

$$\frac{1}{2\pi}\prod_{a=1,..,n}\frac{1}{\sqrt{8\pi G\hbar}}dx^a. \tag{48}$$

Where x^a are Cartesian coordinates of the vertex $p(n)$. Since there exist only null gauge mode in the t vector, after translating the variables to $t(\nu)$ we obtain the measure term corresponding to gauge orbits as [10],

$$\frac{1}{2\pi}\frac{1}{(8\pi G\hbar)^{\frac{1}{2}}}\prod_{\nu}\frac{1}{\sqrt{\Delta_\nu}}dt(\nu) \tag{49}$$

From the integration over the modes ($k = 0, \nu$) and removing (dividing) the measure over the gauge orbits we obtain a factor [10]

$$(8\pi G\hbar)^{\frac{3}{2}N_T}\left(\frac{2\pi}{2x}\right)^{N_T}\frac{(6V_\sigma)^{\frac{3}{2}N_T}}{R^{2N_T}T^{N_T}}\ . \tag{50}$$

3.5 Physical modes

All modes with $k \neq 0$ do not have null vectors, therefore they correspond to the physical modes of the theory. To evaluate the physical modes the full hessian is considered in order to integrate the bulk variables. We proceed by integrating $\hat{d}$ variables. In order to accomplish this we first solve for $\frac{\delta S}{\delta \hat{d}(k,\nu)} = 0$ and $\frac{\delta S}{\delta \hat{d}(-k,-\nu)} = 0$ for $\hat{d}(k,\nu)$ and $\hat{d}(-k,-\nu)$. Then substituting these values to the matrix (42), the resulting matrix $\tilde{M}_r$ is used to compute the gaussian integrals and summing over all the modes ($k \neq 0, \nu$). Following this procedure we obtain the following factor [10],

$$\begin{aligned}
&\prod_{k=1}^{N_A-1}\prod_{\nu=1}^{N_T-1}\int d\hat{d}(k,\nu)d\hat{d}(-k,-\nu)exp\big(-\frac{1}{2\times 8\pi G\hbar}(\hat{d}(k,\nu))\cdot\Delta_k\cdot(\hat{d}(-k,-\nu))\\
&\quad=(8\pi G\hbar)^{\frac{N_T(N_A-1)}{2}}\prod_{k=1}^{N_A-1}\prod_{\nu=1}^{N_T-1}\frac{(2\pi)^{1/2}}{\sqrt{\Delta_k}}\\
&\quad=(8\pi G\hbar)^{\frac{N_T(N_A-1)}{2}}(2\pi)^{\frac{N_T(N_A-1)}{2}}N_A^{-N_T}.
\end{aligned} \tag{51}$$

3.6 One loop determinant

The one loop determinant is the determinant of the Hessian consisting of only the bulk variables. For $k > 0$ the Hessian is [10]

$$\tilde{M}_{bulk}(k\neq 0,\nu)=\begin{pmatrix}\Delta_k & -2x(1-\omega_\upsilon)\\ -2x(1-\omega_\upsilon^{-1})-\Delta_k & \Delta_k\end{pmatrix}, \tag{52}$$

integrating out the $\hat{d}$ variables of the partition function relating to the Hessian of the above matrix leads to a scalar value [10],

$$2x\Delta_\upsilon\left(1-\frac{2x}{\Delta_k}\right)\ . \tag{53}$$

Now we take the product of all the modes ($k \neq 0, \nu$) we get [10],

$$
\begin{aligned}
&\prod_{k=1}^{N_A-1} \prod_{\nu=0}^{N_T-1} 2x\Delta_\upsilon \left(1 - \frac{2x}{\Delta_k}\right) \\
&= 2x^{N_T(N_A-1)} \left(\prod_{k=1}^{N_A-1} \left(1-\frac{2x}{\Delta_k}\right)^{N_T} \right) \left(\prod_{\nu=0}^{N_T-1} \prod_{k=1}^{N_A-1} \Delta_\upsilon \right) \\
&= (2x)^{N_T(N_A-1)} \left(\frac{\left(\frac{1}{1-x+\sqrt{(x-2)x}}; e^{\frac{i2\pi i}{N_A}}\right)_{N_A} \left(-\frac{1}{1-x+\sqrt{(x-2)x}}; e^{\frac{i2\pi i}{N_A}}\right)_{N_A}}{2\left(\left(e^{\frac{i2\pi i}{N_A}}; e^{\frac{i2\pi i}{N_A}}\right)_{N_A-1}\right)^2} \right)^{N_T} \\
&\times \left(\prod_{k=1}^{N_A-1} (2 - 2cos(\gamma k)) \right)
\end{aligned}
\tag{54}
$$

where $(.;.)_N$ is the q-Pochhammer symbol [10].

Now equation (31) is evaluated through the expansion of the Regge action to the second order in the bulk and boundary fluctuations. The calculations results in the form for (31) as [10],

$$
Z(\tau(s,n), \alpha(s,n), \eta(s,n)) = e^{-\frac{1}{\hbar}S_R^{(0)}} D e^{-F(\tau(s,n),\alpha(s,n),\eta(s,n))} \tag{55}
$$

where D is the one loop determinant and F is the boundary part of the Hamilton-Jacobi functional. We will evaluate F by integrating the boundary fluctuations.

The initial measure for the path integral is [10],

$$
\begin{aligned}
&\prod_\sigma \frac{1}{\sqrt{12\pi V_\sigma}} \prod_{e\in bulk} \frac{L_e d\lambda_e}{\sqrt{8\pi G\hbar}} \prod_{e\in bdry} \sqrt{\frac{L_e}{\sqrt{8\pi G\hbar}}} \\
&= \prod_{e\in bdry} \sqrt{\frac{L_e}{\sqrt{8\pi G\hbar}}} \frac{(6V_\sigma)^{N_A N_T + N_T/2}}{(12\pi V_\sigma)^{\frac{3}{2}N_A N_T}} \frac{1}{(8\pi G)^{N_A N_T + N_T/2}} \prod_{e\in bulk} d\hat{\lambda}_e
\end{aligned}
\tag{56}
$$

Now evaluating all the factors contributing to the path integral from the integration over the bulk variables. First integration over the gauge modes ($k = 0, \nu$) we receive a factor as in expression (47). Next integration over the $\hat{d}$ variables for all modes ($k \neq 0$) yields a factor as shown in (51). Finally for the integration of $\hat{r}$ variables for all modes ($k \neq 0$) using the expression (54) [taking the square root reciprocal of the expression and adding the constants] leads to the following factor with the integration over the $\hat{r}$ modes [10]

$$
(8\pi G\hbar)^{\frac{N_T(N_A-1)}{2}} (2\pi)^{\frac{N_T(N_A-1)}{2}} (2x)^{-\frac{N_T(N_A-1)}{2}} f(x, N_A)^{-\frac{N_T}{2}} \left[\prod_{k=1}^{N_A-1} (2 - 2cos(\gamma k)) \right]^{-1/2} \tag{57}
$$

Taking all those into account we finally receive an expression for D [10]

$$
\begin{aligned}
D = &2^{-N_T} (2\pi)^{-\frac{N_T N_A}{2}} (\frac{R}{A})^{N_T(N_A-1)} (ART)^{-\frac{N_T N_A}{2}} (A^2 T)^{N_T} \left(4 - \frac{A^2}{R^2}\right)^{-N_T(\frac{N_A}{4}+1)} \\
&\times \left(\frac{N_A^2}{2^{N_A-2}} f(x, N_A) \right)^{-\frac{N_T}{2}} \left[\prod_{e\in bdry} \sqrt{\frac{L_e}{\sqrt{8\pi G\hbar}}} \right] \left[\prod_{k=1}^{(N_A-1)/2} \frac{1}{|1-q^k|^2} \right],
\end{aligned}
\tag{58}
$$

where $q = e^{i\gamma}$. Now, using this result and the zeroth and first order parts of the action, we can write the resulting partition function

$$Z(\tau(s,n),\alpha(s,n),\eta(s,n)) = exp\left(\frac{\beta}{\hbar 4G}\mu\right)\tilde{D}\prod_{k=1}^{\infty}\frac{1}{|1-q^k|^2}e^{F(\tau(s,n),\alpha(s,n),\eta(s,n))} \tag{59}$$

As we can see here the one loop correction of the partition function shows character of a massive BMS_3 particle. By making the choice that $\alpha^0 = i\beta$ the following relations are obtained [10]

$$m = \frac{M}{\hbar}\ , \quad c_2 = \frac{6}{G\hbar}. \tag{60}$$

where m is the BMS_3 mass, M the geometrical mass and c_2 the central charge. Compared to the massless vacuum case as computed in [4] we see here an extra physical Fourier mode. This of course corresponds to the existence of the BMS_3 particle and how it existence makes the radial component not a gauge mode.

In order to verify the discretized calculations (since discretization is essentially an approximation) we must consider continuum limit of the discrete results and compare it with results obtained directly from the continuum theory. In order to do this length variables are matched to metric fluctuations.

The relation between the (boundary) length fluctuations λ_b and the metric fluctuations in the boundary δh_{ab} are determined by where $h_{ab} = diag(T^2, A^2)$ and basis vectors are (e^a_τ, e^a_α) [10],

$$\begin{aligned}
(h_{ab}+\delta h_{ab})e^a_\tau e^b_\tau =& h_{\tau\tau}+\delta h_{\tau\tau} = T^2 + 2T\tau + O(\tau^2)\\
(h_{ab}+\delta h_{ab})e^a_\alpha e^b_\alpha =& h_{\alpha\alpha}+\delta h_{\alpha\alpha} = A^2 + 2A\alpha + O(\alpha^2)\\
(h_{ab}+\delta h_{ab})(e^a_\tau + e^a_\alpha)(e^b_\tau + e^b_\alpha) =& h_{\alpha\alpha}+h_{\tau\tau}+2h_{\tau\alpha}+\delta h_{\alpha\alpha}+\delta h_{\tau\tau}+\delta 2h_{\tau\alpha}\\
=& A^2 + T^2 + 2\sqrt{A^2+T^2}\eta + O(\eta^2)
\end{aligned} \tag{61}$$

Taking into account the rescaling the following relation was obtained [10]:

$$\begin{pmatrix}\delta h_{\tau\tau}\\ \delta h_{\alpha\alpha}\\ \delta h_{\tau\alpha}\end{pmatrix} = \sqrt{6V_\sigma}\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}\hat{\tau}\\ \hat{\alpha}\\ \hat{\eta}\end{pmatrix} + O(\lambda^2) \tag{62}$$

We can calculate $S^{(2)}_{HJ}$ in terms of metric perturbation [10],

$$8\pi G S^{(2)}_{HJ} = \frac{1}{2}\sum_{k,\nu}(\delta h(k,\nu))^t\cdot\tilde{M}^h_b(k,\nu)\cdot(\delta h(-k,-\nu)) \tag{63}$$

We implement the continuum limit by setting for $\epsilon \ll 1$ [10]

$$A = \varepsilon A_0\ , \quad T = \varepsilon T_0. \tag{64}$$

In order to compare discreet results to the continuum as in [4] the action is evaluated on perturbations of the metric induced by infinitesimal diffeomorphisms. These infinitesimal diffeomorphisms in the discrete are given by vertex displacements outwards to the boundary (radial $n^h_{b,r}$) and time $n^h_{b,\tau}$ and angular $n^h_{b,\alpha}$ directions along the boundary. These vectors describing of metric perturbations (vertex displacements) are given by δh [10],

$$(n^h_{b,\tau})^t(k,\nu) = -(2(1-\omega_\nu), 0, \omega_\nu(1-\omega_k))X_\tau(k,\nu) \tag{65}$$

$$(n^h_{b,\alpha})^t(k,\nu) = (0, 2(1-\omega_k), \omega_k(1-\omega_\nu))X_\alpha(k,\nu) \tag{66}$$

$$(n^h_{b,r})^t(k,\nu) = Asin(\frac{\pi}{N_A})(0, 2(1+\omega_k), \omega_k\omega_\nu(1-\omega_\nu^{-1}))X_\alpha(k,\nu) \tag{67}$$

Here X_τ, X_α give the distance between old and new vertex positions. In the continuum limit $X_\tau = \varepsilon X^0_\tau(k,\nu)$, $X_r = \varepsilon X^0_r$ and $X_\alpha = \varepsilon X^0_\alpha(k,\nu)$. The above equations results in the continuum results in [10],

$$(n^h_{b,\tau})^t(k,\nu) = \varepsilon^2(2i\hat{\upsilon}, 0, i\hat{k})X^0_\tau(k,\nu) + O(\varepsilon^3) \tag{68}$$

$$(n^h_{b,\alpha})^t(k,\nu) = \varepsilon^2(0, 2i\hat{k}, i\hat{\upsilon})X^0_\alpha(k,\nu) + O(\varepsilon^3) \tag{69}$$

$$(n^h_{b,r})^t(k,\nu) = \varepsilon^2\frac{A_0^2}{R}(0,2,0)X^0_r(k,\nu) + O(\varepsilon^3) \tag{70}$$

These vertex displacements evaluated to the Hamilton Jacobi functional (63) is given by [10],

$$\begin{aligned} 8\pi GS^{(2)}[(n^h_{b,\tau})] =& \frac{1}{2}\sum_{k,\nu}(n^h_{b,\tau}(k,\nu))^t \cdot \tilde{M}^h_b(k,\nu)\cdot(n^h_{b,\tau}(-k,-\nu)) \\ =& \frac{1}{2}\sum_{k,\nu}\varepsilon^2\frac{A_0}{RT_0^3}\hat{\upsilon}^2X^0_\tau(k,\nu)X^0_\tau(-k,-\nu) + O(\varepsilon^3) \end{aligned} \tag{71}$$

$$\begin{aligned} 8\pi GS^{(2)}[(n^h_{b,\alpha})] =& \frac{1}{2}\sum_{k,\nu}(n^h_{b,\alpha}(k,\nu))^t \cdot \tilde{M}^h_b(k,\nu)\cdot(n^h_{b,\alpha}(-k,-\nu)) \\ =& \frac{1}{2}\sum_{k,\nu}\varepsilon^2\frac{1}{A_0RT_0}\hat{\upsilon}^2X^0_\alpha(k,\nu)X^0_\tau(-k,-\nu) + O(\varepsilon^3) \end{aligned} \tag{72}$$

$$\begin{aligned} 8\pi GS^{(2)}[(n^h_{b,r})] =& \frac{1}{2}\sum_{k,\nu}(n^h_{b,r}(k,\nu))^t \cdot \tilde{M}^h_b(k,\nu)\cdot(n^h_{b,r}(-k,-\nu)) \\ =& -\frac{1}{2}\sum_{k,\nu}\varepsilon^2\frac{A_0}{RT_0}\hat{\upsilon}^2X^0_r(k,\nu)X^0_r(-k,-\nu) + O(\varepsilon^3) \end{aligned} \tag{73}$$

This shows that after considering the continuum limit the second order part of the action does not depend on the mass. But this is not a inconsistant theory as the mass appears in the action as the zeroth order term but it is invariant in the continuum limit. Here we see that the difference of $8\pi GS^{(2)}[(n^h_{b,r})]$ between the massless case is that there exists an extra mode ($k = 1$) in the expression (which resulted in because of the additional physical degrees of freedom).

Direct calculation of the $k = 1$ mode of the above second order action contribution does not yield a valid result without assigning a function for X. In order to analyze what this extra term entails we attempt to derive the second order action contribution due to perturbation in the boundary of a foliation of a solid torus using first order metric perturbations without discretization (in the continuum). We will use similar methodology as used in [1] to arrive at the second order action in the continuum!

4 The Hamilton–Jacobi Functional for 3D Gravity with Point Particle

4.1 Metric in Gaussian coordinates

In sections 4.1, 4.2 and 4.4 we will introduce some background concepts and sections 4.3, 4.5 and 4.6 the original contributions for this thesis is presented. It should be emphasized that this work is conducted by closely following Seth's work on [1]. Its is proven in Seth's work the massless BMS_3 character can be obtained in the case of flat space-time in vacuum and we can analyze (73) in non-discreet terms (in terms of diffeomorphic inducing vectors). So one would expect following a similar procedure we should expect the massive BMS character is a case of a massive point particle on flat space-time in a vacuum. Here we have shown that (not exactly calculated) we could expect the massive BMS_3 character.

In this section we will choose our metric coordinate system and address some properties derived from the metric. We need to create a similar environment as used in the case of Regge gravity as described in the previous section. Just as we considered fixed lengths for background solutions in Regge gravity here we choose background solutions to have homogeneous curvature $R_{abcd} = \Lambda(g_{ac}g_{bd} - g_{ad}g_{bc})$. Also we need to implement infinitesimal diffeomorphisms in the boundary of a torus. It is convenient to use Gaussian coordinates for the background solution [1]

$$g_{ab}dx^a dx^b = dr^2 + h_{AB}dy^A dy^B. \tag{74}$$

Some assumptions are made to implement smoothness and convenience in order to introduce boundary diffeomorphisms for the manifold $\mathcal{M}$. Firstly, $r = 0$ defines a point or a one-dimensional sub-manifold of $\mathcal{M}$. Also a boundary $\partial\mathcal{M}$ of the manifold is a surface spanned by a fixed $r = r_b$ radial coordinate. Here $a, b, ...$ denotes space–time indices and $A, B, ... = 1, 2$ denotes "spatial" indices for the constant r surfaces. With the background solution introduced we will now consider metric perturbations for it [1]

$$g^{full}_{ab} = g_{ab} + \gamma_{ab}. \tag{75}$$

Here γ_{ab} contains components $\gamma_{\perp\perp}$, $\gamma_{\perp A}$ and γ_{AB} which are the metric perturbations defined on the boundary, $\perp$ denotes the index for the radial coordinate. It is further assumed that the background boundary curvature is homogeneous (i.e. $\partial_A {}^2R = 0$) and also the background boundary has a non–vanishing extrinsic curvature. ${}^2R_{AB} = \frac{1}{2}\,{}^2Rh_{AB}$ can be used to determine the Ricci tensor for a two dimensional boundary metric. The Christoffel symbols with the extrinsic curvature tensor given by $K_{AB} = \frac{1}{2}\partial_\perp h_{AB}$ for a Gaussian metric are [1]

$$\Gamma^a_{\perp\perp} = 0\ , \quad \Gamma^\perp_{\perp B} = 0\ , \quad \Gamma^\perp_{AB} = -K_{AB} \tag{76}$$

$$\Gamma^A_{\perp B} = K^A_B\ , \quad \Gamma^A_{BC} =^2 \Gamma^A_{BC}. \tag{77}$$

With this we can determine the relations between space–time covariant derivatives and spatial covariant derivatives and for a vector ξ this is given by [1],

$$\nabla_A\xi_B = D_A\xi_B + K_{AB}\xi_\perp, \tag{78}$$

$$\nabla_A\xi_\perp = D_A(\xi_\perp) + K^B_A\xi_B \tag{79}$$

The symbol ∇ is denoted for the covariant derivative with respect to metric g, and D is denoted for the covariant derivative with respect to h. Since $\xi_\perp$ is a vector perpendicular to the boundary surface it is treated as a spatial scalar, implying $D_A\xi_\perp = \partial_A\xi_\perp$. As it is used later we will mention here the Gauss–Codazzi relations for a surface embedded into a $3D$ vacuum solution [1]

$$K^2 - K_{AB}K^{AB} = {}^2R - 2\Lambda\ , \quad D_A K^A_B - D_B K = 0. \tag{80}$$

4.2 A basis for the boundary metric perturbations

Since we are attempting to replicate the results obtained in the discrete we need to compute the Hamilton Jacobi functional for linearized gravity but in the continuum. A space-time with a homogeneous intrinsic curvature and non-zero extrinsic curvature is considered. As shown by Seth in [1] the metric perturbations becomes the on-shell solutions for the Hamilton Jacobi functional if the functional is expressed in terms of diffeomorphism generating vectors. We need this condition in order to replicate equation (73) in the continuum setting. But unlike the massless case as computed in [1] as we see later the Hamilton Jacobi does take a much more complicated form when paired with a massive test particle. We introduce vector components $\xi^\perp$ and ξ^A parametrized to γ_{AB} [1],

$$\begin{aligned} \gamma_{AB} = [L_\xi g]_{AB} =& \nabla_A\xi_B + \nabla_B\xi_A \\ =& 2\xi^\perp K_{AB} + [L_{\xi^{||}}h]_{AB} \end{aligned} \tag{81}$$

This parametrization is possible because the solutions of the equations of motion for 3D gravity is diffeomorphism equivalent to a homogeneously curved space-time. We will assume that the transformation from $(\xi^\perp, \xi^1, \xi^2)$ to $(\gamma_{11}, \gamma_{22}, \gamma_{12})$ is invertible. This means the extrinsic curvature tensor K_{AB} is non-vanishing. With this we can determine the vector components $\xi^\perp$ and ξ^A [1]

$$\begin{aligned} \Delta\xi^\perp =& \Pi^{AB}\gamma_{AB} \\ D^A_B\xi^B =& 2(K^{BC} - Kh^{BC})\delta\,{}^2\Gamma^A_{BC} \end{aligned} \tag{82}$$

where

$$\begin{aligned} \Delta =& 2(K^{CD} - Kh^{CD})D_C D_D - {}^2RK, \\ D^A{}_B =& 2(K^{BC} - Kh^{CD})D_C D_D h^A{}_B - {}^2RK^A{}_B, \\ \Pi^{AB} =& D^A D^B - h^{AB}D_C D^C - \frac{1}{2}\,{}^2Rh^{AB}, \\ \delta\,{}^2\Gamma^A_{BC} =& \frac{1}{2}h^{AD}(D_B\gamma_{AC} + D_C\gamma_{BA} - D_A\gamma_{BC}). \end{aligned} \tag{83}$$

From here the operators Δ and $D^A{}_B$ is inverted. Seth suggests a relationship between $\Pi^{AB}\gamma_{AB}$ and the first variation of the boundary Ricci scalar $\delta({}^2R)$ given by [1],

$$\begin{aligned} \Pi^{AB}\gamma_{AB} =& (D^A D^B - h^{AB}D_C D^C)\gamma_{AB} - \frac{1}{2}\,{}^2Rh^{AB}\gamma_{AB} \\ =& (D^A D^B - h^{AB}D_C D^C)\gamma_{AB} - \frac{1}{2}\,{}^2R^{AB}h^{AB}\gamma_{AB} \\ =& \delta({}^2R) \end{aligned} \tag{84}$$

with ${}^2R_{AB} = \frac{1}{2}\,{}^2Rh_{AB}$ for two dimensional metrics. This relationship arises due to $\xi^\perp$ being invariant under linearized boundary tangential diffeomorphisms, that leads to Π^{AB} vanishing on perturbations induced by tangential diffeomorphisms.

The lapse $\gamma_{\perp\perp}$ and shift $\gamma_{\perp A}$ of the metric perturbations can be written as functions of the generating vector fields $(\xi^\perp, \xi^A)$ given by [1],

$$
\begin{aligned}
\gamma_{\perp\perp} =& 2\partial_\perp \xi^\perp, \\
\gamma_{\perp A} =& \nabla_\perp \xi_A + \nabla_A \xi_r \\
=& \partial_\perp (h_{AB}\xi^B) - \Gamma^B_{\perp A}\xi_B + \partial_A (g_{\perp\perp}\xi^\perp) - \Gamma^B_{A\perp}\xi_B \\
=& D_A \xi^\perp + h_{AB}\partial_\perp \xi^B.
\end{aligned}
\tag{85}
$$

4.3 Hamilton jacobi functional first order

In this section we will compute the Hamilton–Jacobi functional. To replicate the discrete setting we consider 3D linearized gravity, with Euclidean signature and vanishing cosmological constant on a manifold $\mathcal{M}$ with smooth boundary $\partial\mathcal{M}$ coupled with a point particle. We are interested in studying how a massive point particle couples to Regge gravity. We start by analysing the classical continuum solution to this problem in order to get geometrical information that help us in the discrete setting. We will work in a 3D Euclidean torus since we want to compare its one-loop partition function with a massive BMS_3 character.

The (Euclidean) Einstein–Hilbert action, with Gibbons-Hawking-York boundary term and a massive point particle in vacuum is given by [10],

$$
S_p = -\frac{1}{16\pi G}\left(\int d^3x\sqrt{g}R - 2\int d^2y\sqrt{h}K\right) + M\int d\tau\sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}. \tag{86}
$$

where G is the newton's constant. The Einstein-Hilbert's action, the Gibbons-Hawking-York boundary term and worldline of a point particle with rest mass $\mathcal{M}$ are the first, second and third terms of the action respectively. Next is to solve for Einstein's Equations [10]

$$
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{87}
$$

for an energy-momentum tensor of the particle is decribed by $(t,\bar{r}) = (M,\bar{0})$ with $\bar{r} = (r,\theta)$ [10],

$$
T_{00} = M\delta^{(2)}(\bar{r}),\;\; T_{i0} = 0 = T_{ij}. \tag{88}
$$

The solution yields [10]

$$
ds^2 = (1-4GM)^2 dt^2 + (1-4GM)^2 dr^2 + r^2 d\theta^2, \tag{89}
$$

rescaling t and r by $(1-4GM)$, and θ results in the following metric [10]

$$
ds^2 = dt^2 + dr^2 + r^2 d\theta^2, \tag{90}
$$

with $0 \leqslant \theta \leqslant 2\pi(1-4GM)$, which describes a cone. $8\pi GM$ is the deficit angle at he center see figure 4.3. The identification $t+\beta \sim t$ is considered which determines a torus.

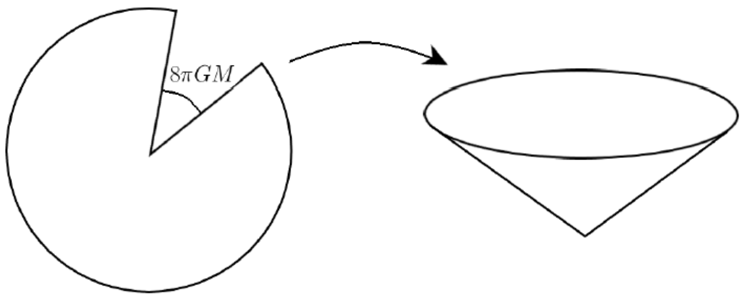

Figure 8: Conical spatial geometry caused by the presence of a point particle [10].

The extrinsic curvature tensor, K_{AB} is associated to the foliation by surfaces of constant radius. Using Gaussian coordinates the extrinsic curvature tensor is defined by $K_{AB} = \frac{1}{2}\partial_\perp h_{AB}$ [1]. The equations of motion demand $R_{ab} = 0$ for vanishing cosmological constant and thus $R = 0$. This leads to a classical background solution [10]

$$S_p = \frac{1}{8\pi G}\int d^2y\sqrt{h}K + M\int d\tau\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}. \tag{91}$$

We will now calculate the first variation of the gravitational action. Also it should be emphasized that the calculations in this work are mainly focus on the particle segment of the action. As for the first order and second order variations of Einstein-Hilbert action and the Gibbson-Hawking boundary term the result is exactly the same as [1]. Here we consider a test particle case hence and only consider the variation with respect to the metric and not the degrees of freedom of the particle. Any variation with respect to the metric on the particle mass and its (rest) coordinates will be nullified.

$$\begin{aligned}\delta S =& -\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(\frac{1}{2}Rg^{ab} - R^{ab}\right)\delta g_{ab} + \frac{1}{16\pi G}\int d^2y\sqrt{h}(Kh^{AB} - K^{AB})\delta g_{AB} \\ &+ M\int d\tau\left(\frac{1}{2\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\dot{x}^C\dot{x}^D\delta g_{CD}\right)\end{aligned} \tag{92}$$

is used to determine the first order on-shell action, the (background) equations of motions and the momentum conjugated to the metric $\pi^{AB} = \sqrt{h}(K^{AB} - Kh^{AB})$. Using the parametrization $\delta g_{ab} = \gamma_{ab} = L_\xi g_{ab} = \gamma_{AB}$ since only h_{AB} varies for the boundary metric fluctuations, the first order of the on-shell action evaluates to,

$$
\begin{aligned}
S^{(1)}_{HJ} =& \frac{1}{16\pi G}\int d^2y\sqrt{h}(Kh^{AB}-K^{AB})(\nabla_A\xi_B+\nabla_B\xi_A)\\
&+M\int d\tau\left(\frac{\dot{x}^C\dot{x}^D}{2\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\right)(\nabla_C\xi_D+\nabla_D\xi_C)\\
=&\frac{1}{16\pi G}\int d^2y\sqrt{h}(Kh^{AB}-K^{AB})(D_A\xi_B+K_{AB}\xi^\perp)\\
&+M\int d\tau\left(\frac{\dot{x}^C\dot{x}^D}{2\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\right)(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})\\
=&\frac{1}{16\pi G}\int d^2y\sqrt{h}((-D^BK+D_AK^{AB})\xi_B+(K^2-K_{AB}K^{AB})\xi^\perp)\\
&+M\int d\tau\left(\frac{\dot{x}^C\dot{x}^D}{2\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\right)(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})\\
=&\frac{1}{16\pi G}\int d^2y\sqrt{h}({}^2R)\\
&+M\int d\tau\left(\frac{\dot{x}^C\dot{x}^D}{2\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\right)(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})
\end{aligned} \tag{93}
$$

where we have used the Gauss–Codazzi relations. The bulk and boundary terms of the gravitational action are not uniquely determined. This can be redefined using integration by parts (see 113). These terms are chosen such that the bulk term vanishes on-shell. This leads to the second order on-shell action being only contributed by the boundary term and the particle term in the equation above.

We now calculate the second order of the Hamilton- Jacobi functional. As calculated in [1] the first sum of the first order Hamilton-Jacobi equation remain the same. Hence,

$$
\begin{aligned}
S^{(2)}_{HJ} =& \frac{1}{32\pi G}\int d^2y\sqrt{h}(\xi^\perp\Delta\xi^\perp-\xi^AD_{AB}\xi^B)\\
&+M\int d\tau\delta\left[\left(\frac{\dot{x}^C\dot{x}^D}{2\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\right)(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})\right]\\
=&S^{(2)}_{HJ^V}+S^{(2)}_{HJ^M}
\end{aligned} \tag{94}
$$

Where $S^{(2)}_{HJ^V}$ is the second order Hamilton Jacobi for the vacuum and $S^{(2)}_{HJ^M}$ for the second sum of the equation which involved to the point particle. We will now calculate $S^{(2)}_{HJ^M}$.

$$
\begin{aligned}
S^{(2)}_{HJ^M} =& \frac{M}{2}\int d\tau(-\frac{1}{2})\frac{\dot{x}^E\dot{x}^F\delta g_{EF}}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}\dot{x}^C\dot{x}^D(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})+\\
&\frac{M}{2}\int d\tau\frac{\dot{x}^C\dot{x}^D}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\delta(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})
\end{aligned} \tag{95}
$$

By definition $\delta\xi^\perp=\delta\xi^A=0$. Therefore,

$$
\begin{aligned}
S^{(2)}_{HJ^M} =& \frac{M}{2}\int d\tau(-\frac{1}{2})\frac{\dot{x}^E\dot{x}^F(2\xi^\perp K_{EF}+L_{\xi^{||}}h_{EF})}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}\dot{x}^C\dot{x}^D(2\xi^\perp K_{CD}+L_{\xi^{||}}h_{CD})+\\
&\frac{M}{2}\int d\tau\frac{\dot{x}^C\dot{x}^D}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(2\xi^\perp(\delta K_{CD})+L_{\xi^{||}}(\delta h_{CD}))
\end{aligned} \tag{96}
$$

We know $\delta h_{CD} = 2\xi^{\perp} K_{CD} + L_{\xi^{||}} h_{CD}$

$$S^{(2)}_{HJ^M} = -\frac{M}{4}\int d\tau \frac{\dot{x}^E \dot{x}^F (2\xi^{\perp} K_{EF} + L_{\xi^{||}} h_{EF})}{(\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu})^{3/2}} \dot{x}^C \dot{x}^D (2\xi^{\perp} K_{CD} + L_{\xi^{||}} h_{CD}) + \frac{M}{2}\int d\tau \frac{\dot{x}^C \dot{x}^D}{\sqrt{\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu}}} (2\xi^{\perp}(\delta K_{CD}) + L_{\xi^{||}}(2\xi^{\perp} K_{CD}) + L_{\xi^{||}} L_{\xi^{||}} h_{CD}) \tag{97}$$

According to [1] it is considered $\pi^{AB} L_{\xi^{||}} h_{AB}$ is modulo a total divergence and $D_A \pi^{AB}$ is a vanishing momentum constraint. Also $\pi^{AB} L_{\xi^{||}} h_{AB}$ vanishes. Similarly as π^{AB} is the momentum constraint for the case of vacuum, for point particle we will consider $\dot{x}^A \dot{x}^B$ the momentum constraint (I would like to note that the basis of this consideration is not entirely clear as we try to achieve our desired results using minimal number of assumptions). Let $\dot{x}^A \dot{x}^B = \dot{X}^{AB}$ then $\dot{X}^{AB} L_{\xi^{||}} h_{AB}$ is modulo a total divergence and $D_A X^{AB} = 0$. $\simeq$ indicates calculations are performed considering modulo total divergences.

We have for the term,

$$\begin{aligned} &\dot{x}^C \dot{x}^D L_{\xi^{||}} (2\xi^{\perp} K_{CD}) \\ &= \dot{X}^{CD} L_{\xi^{||}} (2\xi^{\perp} K_{CD}) \\ &= L_{\xi^{||}} (\dot{X}^{CD} 2\xi^{\perp} K_{CD}) - 2\xi^{\perp} L_{\xi^{||}} (\dot{X}^{CD} K_{CD}) + 2\xi^{\perp} \dot{X}^{CD} L_{\xi^{||}} (K_{CD}) \\ &\simeq -2\xi^{\perp} L_{\xi^{||}} (\dot{X}^{CD} K_{CD}) + 2\xi^{\perp} \dot{X}^{CD} L_{\xi^{||}} (K_{CD}) \end{aligned} \tag{98}$$

where the first term on the RHS is dropped because the total derivative of a Lie derivative of a scalar density is zero. This results in

$$\begin{aligned} S^{(2)}_{HJ^M} \simeq & -\frac{M}{4}\int \frac{d\tau \dot{X}^{CD} \dot{X}^{EF}}{(\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu})^{3/2}} (4\xi^{\perp} K_{CD} K_{EF} \xi^{\perp} + \\ & 2\xi^{\perp} K_{CD} L_{\xi^{||}} h_{EF} + 2\xi^{\perp} K_{EF} L_{\xi^{||}} h_{CD} + L_{\xi^{||}} h_{CD} L_{\xi^{||}} h_{EF}) + \\ & \frac{M}{2}\int \frac{d\tau}{\sqrt{\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu}}} (2\dot{X}^{CD} \xi^{\perp} \delta K_{CD} - 2\xi^{\perp} L_{\xi^{||}} (\dot{X}^{CD} K_{CD}) + \\ & 2\xi^{\perp} \dot{X}^{CD} L_{\xi^{||}} (K_{CD}) + \dot{X}^{CD} L_{\xi^{||}} L_{\xi^{||}} h_{CD}) \\ \simeq & -\frac{M}{4}\int \frac{d\tau \dot{X}^{CD} \dot{X}^{EF}}{(\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu})^{3/2}} [4\xi^{\perp} K_{CD} K_{EF} \xi^{\perp} + 8\xi^{\perp} K_{CD} (D_E \xi_F) + (D_C \xi_D)(D_E D_F)] \\ & \frac{M}{2}\int \frac{d\tau}{\sqrt{\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu}}} (2\dot{X}^{CD} \xi^{\perp} \delta K_{CD} - 2\xi^{\perp} L_{\xi^{||}} (\dot{X}^{CD} K_{CD}) + \\ & 2\xi^{\perp} \dot{X}^{CD} L_{\xi^{||}} (K_{CD}) + \dot{X}^{CD} L_{\xi^{||}} L_{\xi^{||}} h_{CD}) \end{aligned} \tag{99}$$

by running similar methodology as [1] we look at the term

$$\begin{aligned} &\dot{X}^{AB} L_{\xi^{||}} L_{\xi^{||}} h_{AB} \\ &= -2\dot{X}^{AB} (\xi_C D_A D_B \xi^C - \xi^C D_C D_A \xi_B + \xi^C D_A D_C \xi_B) + \\ &\quad [2\dot{X}^{AB} (D_A(\xi_C D_B \xi^C) + D_A(\xi^C D_C \xi_B))] \\ &\simeq -2\xi^C \dot{X}^{AB} D_A D_B \xi^C - 2\dot{X}^{AB}\, {}^2R_{ABCD} \xi^C \xi^D \end{aligned} \tag{100}$$

The term in square bracket was dropped, due to it being a total divergence of a momentum constraint $D_A \dot{X}^{AB} = 0$. Now ${}^2R_{ABCD} = \frac{1}{2}\, {}^2R(h_{AB} h_{CD} - h_{AD} h_{CB})$ yields,

$$
\begin{aligned}
&\dot{X}^{AB} L_{\xi^{||}} L_{\xi^{||}} h_{AB} \\
&\quad \simeq -2\xi^C \dot{X}^{AB} D_A D_B \xi_C - 2\dot{X}^{AB}\frac{1}{2}{}^2R(h_{AB}h_{CD} - h_{AD}h_{CB})\xi^C\xi^D \\
&\quad \simeq -2\xi^C \dot{X}^{AB} D_A D_B \xi_C - \dot{X}^{AB}\,{}^2R(h_{AB}\xi^C\xi_C - \xi_A\xi_B)
\end{aligned} \tag{101}
$$

We will now calculate the term δK_{AB},

$$
\begin{aligned}
\delta K_{AB} =&\delta\left(\frac{1}{2}\partial_\perp h_{AB}\right) \\
=&\frac{1}{2}\xi^p\partial_p(\delta h_{AB}) \\
=&\frac{1}{2}\xi^\perp\partial_\perp(2\xi^\perp K_{AB} + L_{\xi^{||}}h_{AB}) \\
=&\frac{1}{2}\xi^\perp\partial_\perp(D_A\xi_B + D_B\xi_A) + \xi^\perp\partial_\perp(\xi^\perp K_{AB}) \\
=&\frac{1}{2}\xi^\perp\partial_\perp(D_A\xi_B + D_B\xi_A) + \xi^\perp\partial_\perp(\xi^\perp)K_{AB} + \xi^\perp\xi^\perp\partial_\perp(K_{AB})
\end{aligned} \tag{102}
$$

With these terms we substitute to $S^{(2)}_{HJ^M}$

$$
\begin{aligned}
S^{(2)}_{HJ^M} \simeq& -\frac{M}{4}\int\frac{d\tau\dot{X}^{CD}\dot{X}^{EF}}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}[4\xi^\perp K_{CD}K_{EF}\xi^\perp + 8\xi^\perp K_{CD}(D_E\xi_F) + (D_C\xi_D)(D_E D_F)] + \\
&\frac{M}{2}\int\frac{d\tau}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\dot{X}^{CD}\xi^\perp\xi^\perp\partial_\perp(D_C\xi_D + D_D\xi_C) + \xi^\perp(2\xi^\perp\partial_\perp\xi^\perp)K_{CD} + \\
&2\xi^\perp\xi^\perp\xi^\perp\partial_\perp K_{CD} - 2\xi^\perp L_{\xi^{||}}(\dot{X}^{CD}K_{CD}) + 2\xi^\perp\dot{X}^{CD}L_{\xi^{||}}(K_{CD}) + \\
&\dot{X}^{CD}(-2\xi^C\dot{X}^{AB}D_A D_B\xi_C - \dot{X}^{AB}\,{}^2R(h_{AB}\xi^C\xi_C - \xi_A\xi_B)))
\end{aligned} \tag{103}
$$

Here we see that for both S_{HJ^V} and S_{HJ^M} the Hamiltonian-Jacobi can be expressed in terms of the diffeomorphism generating vector fields. But unlike the Hamiltonian-Jacobi for the massless 3D gravity (S_{HJ^V}) when a test particle is introduced the form of the Hamiltonian-Jacobi is much more complicated. In a massless segment ($S^{(2)}_{HJ^V}$) case we can see that the boundary normal component $\xi^\perp$ and the boundary tangential components ξ^A of the diffeomorphism generating vector field decouple. But clearly as shown in equation (103) this does not hold in our case.

In the next section we will discuss as shown by Seth [1] that the length of the geodesics which are normal to the boundary will be given by $\xi^\perp$ and that the geodesic length is a viable boundary field to consider as part of the dual boundary field theory. Ideally we would analyze each and every part of $S^{(2)}_{HJ^M}$ but for this work we will pay attention to the segments of $S^{(2)}_{HJ^M}$ with quadratic terms of $\xi^\perp$ (our main focus being to analyze (73) we expect quadratic terms of $\xi^\perp$ will help us in order do to so),

$$
\begin{aligned}
S^{(2)}_{M^\perp} \equiv& -\frac{M}{4}\int\frac{d\tau\dot{X}^{CD}\dot{X}^{EF}}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}[4\xi^\perp K_{CD}K_{EF}\xi^\perp] + \\
&\frac{M}{2}\int\frac{d\tau\dot{X}^{CD}}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\xi^\perp\xi^\perp\partial_\perp(D_C\xi_D + D_D\xi_C) + \xi^\perp(2\xi^\perp\partial_\perp\xi^\perp)K_{CD})
\end{aligned} \tag{104}
$$

as mentioned $\dot{X^{CD}}$ is considered at the center of the torus.

$$\dot{X}^{CD} = \dot{x}^C \dot{x}^D \delta(r) = \delta^{CD}\delta^{C1}\delta(r) \tag{105}$$

$\dot{x}^C$ takes the form,

$$\dot{x}^C = (\dot{x}_t, \dot{x}_\theta, \dot{x}_r) = (\dot{x}_A, \dot{x}_\perp) = (1, 0, 0) \tag{106}$$

With this $\dot{x}^C\dot{x}^D\delta(r) = \delta^{CD}\delta^{C1}\delta(r)$. $S^{(2)}_{M^\perp}$ reduces to,

$$\begin{aligned}
S^{(2)}_{M^\perp} =& - M\int \frac{d\tau\delta(r)}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}\delta^{CD}\delta^{EF}\delta^{C1}\delta^{E1}[\xi^\perp K_{CD}K_{EF}\xi^\perp]+ \\
&\frac{M}{2}\int \frac{d\tau\delta(r)}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}\delta^{CD}\delta^{C1}(\xi^\perp\xi^\perp\partial_\perp(2D_C\xi_D) + 2\xi^\perp(\xi^\perp\partial_\perp\xi^\perp)K_{CD}) \\
S^{(2)}_{M^\perp} =& - M\int \frac{d\tau\delta(r)}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}\xi^\perp K^2_{11}\xi^\perp+ \\
&M\int \frac{d\tau\delta(r)}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\xi^\perp\xi^\perp\partial_\perp(D_1\xi_1) + \xi^\perp(\xi^\perp\partial_\perp\xi^\perp)K_{11})
\end{aligned} \tag{107}$$

We see here a much more simplified form of the Hamilton-Jacobi for the test particle segment of the action. Here we can see decoupled terms of $\xi^\perp$ and ξ^A components along with a coupled term. As to what this entails will be discussed in the conclusions chapter of this thesis. The complete new Hamilton Jacobi will take the form,

$$\begin{aligned}
S^{(2)}_{HJ} =& S^{(2)}_{HJ^V} + S^{(2)}_{M^\perp} \\
=& \frac{1}{32\pi G}\int d^2y\sqrt{h}(\xi^\perp\Delta\xi^\perp - \xi^A D_{AB}\xi^B) - \\
&M\int \frac{d\tau\delta(r)}{(\dot{x}^\mu\dot{x}^\nu g_{\mu\nu})^{3/2}}\xi^\perp K^2_{11}\xi^\perp+ \\
&M\int \frac{d\tau\delta(r)}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\xi^\perp\xi^\perp\partial_\perp(D_1\xi_1) + \xi^\perp(\xi^\perp\partial_\perp\xi^\perp)K_{11})
\end{aligned} \tag{108}$$

4.4 Geodesic Length as a boundary field

Here we attempt to define a local field theory defined on the boundary $\partial\mathcal{M}$, whose Hamilton-Jacobi functional is equivalent (not exactly equal but agrees with under certain conditions) to the Hamilton-Jacobi functional of gravity. This is known as 'dual boundary field theory'. As shown in [1] geodesic length from a boundary point to the central bulk point at $r = 0$ is considered as a field defined on the boundary itself. As we will see this will act as our Liouville field defined on the boundary. The main reasons for using geodesic length as a boundary field are that it is an geometric observable in the gravitational theory, and it can describe the shape of the boundary. Another motivation to use geodesic lengths from boundary to a central axis while considering it as boundary field variables as shown by [13], although in a discrete setting it is shown that one can easily integrate out all variables except these boundary field variables. Another property that we will use in our computations. Finally in [7] it is argued that choosing geodesic lengths as boundary variables gives rise to boundary degrees of freedom (as boundary breaks diffeomorphsims).

We consider the same Gaussian metric defined earlier (74) this makes the tangent vector to the geodesic orthogonal to the boundary. This makes the geodesic length invariant under boundary diffeomorphisms and evaluated on the boundary is proportional to $\xi^{\perp}$. This will lead the geodesic length being related to the first order variation of the Ricci scalar. This is important as we see in the Liouville Lagrangian (9) we have a scalar field term coupled with the first order variation of the Ricci scalar. As stated by Seth this allows to 'guess' a candidate for the dual field theory [1], and it should meet the following two conditions.

- It should reproduce the equation of motion for this geodesic length
- It should reproduce the boundary diffeomorphism invariant part of the gravitational Hamilton–Jacobi functional as the on shell solution.

The geodesic length l for the full metric g_{ab}^{full} given by to the first order metric perturbations [1],

$$l = \frac{1}{2}\int_{r_1}^{r_2} dr\gamma_{\perp\perp}(r) \tag{109}$$

where r_1 and r_2 denotes to two end points of the geodesic (in which later we would consider it a point on the boundary and the center bulk point). Considering Gaussian coordinates and using (81) we can find that,

$$\gamma_{\perp\perp} = (L_\xi g)_{\perp\perp} = 2\partial_\perp \xi^\perp \tag{110}$$

and hence,

$$l = \xi^\perp(r_2) - \xi^\perp(r_1). \tag{111}$$

We can express $\xi^{\perp}$ as a functional of the boundary metric using (82) and (83) and it results in,

$$\xi^\perp = \frac{1}{\Delta}\Pi^{AB}\gamma_{AB} = \frac{1}{\Delta}\delta({}^2R). \tag{112}$$

It should be noted that it is assumed that $r_{in} = 0$ describes a zero-dimensional locus (can be considered as a central axis or a point).

We will now look into the derivation of the effective action for the geodesic length observable from the gravitation action. Geodesic lengths are our dynamical variables therefore expect those variables parametrizing geodesic lenghts, we integrate all other degrees of freedom to obtain the effective action. The resulting effective action will be the same as obtained in [1], but the difference will be the non-vanishing modes (similar to as in the Hamilton-Jacobi in the discreet Regge gravity). It is difficult to perform these integrations directly using the gravitational action therefore we will consider the following second order action with a lagrangian multiplier term as utilized by Seth [1],

$$\begin{aligned} -8\pi G S^{(2)} =& \frac{1}{4}\int_{\mathcal{M}} d^3x\sqrt{g}\gamma_{ab}(V^{abcd}\gamma_{cd} + \frac{1}{2}G^{abcdef}\nabla_c\nabla_d\gamma_{ef}) + \\ & \frac{1}{4}\int_{\partial\mathcal{M}} d^2y\sqrt{h}\gamma_{ab}((B_1)^{abcd}\gamma_{cd} + (B_2)^{abcde}\nabla_c\gamma_{de}) + \\ & \frac{1}{4}\int_{(\partial\mathcal{M})_{out}} d^2y\lambda(y)(\rho(y) - l[\gamma_{\perp\perp}]) \end{aligned} \tag{113}$$

where

$$V^{abcd} = \frac{1}{2}\left[\frac{1}{2}(R-2\Lambda)(g^{ab}g^{cd} - 2g^{ac}g^{bd}) - R^{ab}g^{cd} - g^a bR^{cd} + 2(g^{ac}R^{bd} + g^{bc}R^{ad})\right] \tag{114}$$

$$G^{abefcd} = g^{ab}g^{ec}g^{fd} + g^{ac}g^{bd}g^{ef} + g^{ae}g^{bf}g^{cd} - g^{ab}g^{ef}g^{cd} - g^{af}g^{bd}g^{ec} - g^{ac}g^{bf}g^{ed} \tag{115}$$

$$B_1^{abcd} = \frac{1}{2}(Kh^{ab} - K^{ab})g^{cd} - h^{ac}h^{bd}K - h^{ab}K^{cd} + h^{ac}K^{bd} + h^{bc}K^{ad} \tag{116}$$

$$B_2^{abecd} = \frac{1}{2}((h^{ae}h^{bd} - h^{ab}h^{ed})n^c + (h^{ac}h^{be} - h^{ab}h^{ce})n^d - (h^{ac}h^{bd} - h^{ab}h^{cd})n^e. \tag{117}$$

The first two terms of (113) represent the bulk and boundary terms of the gravitational action respectively and these are not uniquely determined. It can take different forms by using integration by parts and it is chosen such that the bulk term vanishes on shell as mentioned earlier. The λ is a first order variable and it is the scalar density with respect to the boundary metric. ρ is a scalar defined on the boundary. The lagrangian multiplier term is introduced such that the equations of motion for λ evaluated on (background) solutions yields the geodesic length [1]

$$l = \frac{1}{2}\int_{r_{in}}^{r_{out}} dr\ \gamma_{\perp\perp}. \tag{118}$$

This interprets two scenarios. One is the existence of an outer and inner boundary. In this case geodesics span from a point on the outer boundary to the inner boundary. The other scenario is the existence of only an outer boundary. Here geodesics span from the boundary to a point in the bulk at $r = 0$.

In order to calculate the solve for metric (perturbative) components and the lagrangian multiplier we consider the equations of motion by varying the action (113) with respect to the metric components this gives [1]

$$\hat{G}^{ab} := V^{abcd}\gamma_{cd} + \frac{1}{2}G^{abcdef}\nabla_c\nabla_d\gamma_{ef} = \frac{1}{4}\frac{\lambda(y)}{\sqrt{h}}\delta^a_\perp\delta^b_\perp, \tag{119}$$

where since Gaussian coordinates are used we can write $\sqrt{g} = \sqrt{h}$. This does hold the contracted Bianchi identities. With this we have the necessary tools to solve for the three metric components $\gamma_{\perp\perp}$ and $\gamma_{\perp A}$ in term of γ_{AB} and λ. In the next section we will derive the effective action for a torus boundary embedded into flat space.

4.5 Twisted Thermal Flat Space with finite Boundary

We will now consider a background solution that directly and simplest to work with in Gaussian coordinates. We will choose the twisted or spinning thermal flat space which is also worked by Seth in the massless case. The reason for choosing this is to replicate the results as mentioned in section 3 by Alicia [10] and compare the one loop determinant that will be obtained in both discretized and in the continuum case. The metric of the thermal spinning flat space is [1]

$$ds^2 = dr^2 + dt^2 + r^2 d\theta^2. \tag{120}$$

The periodic identification for this metric is $(r, t, \theta) \sim (r, t + \beta, \theta + \gamma)$ and the additional identification for the angular variable $\theta \sim \theta + 2\pi(1 - 4GM)$.

As described in the case of Regge gravity we need to consider a solid torus and it is given when considered the space-time of range $0 \leq r \leq r_{out}$ [1] and the height of the cylinder is β with a twisting angle γ. For the twisted flat space we calculate the boundary extrinsic (background) curvature, differential operator Δ and D_{AB},

$$\begin{aligned} K_{AB} &= \frac{1}{2}\partial_\perp h_{AB} = r\delta^\theta_A\delta^\theta_B \\ K_{11} &= r\delta^\theta_t\delta^\theta_t = 0 \end{aligned} \tag{121}$$

$$
\begin{aligned}
\Delta =&2(K^{CD}-Kh^{CD})D_C D_C\\
=&2(rh^{\theta C}h^{D\theta}-r\delta^\theta_A\delta^\theta_B h^{AB}h^{CD})D_C D_D\\
=&2\left(\frac{1}{r^3}\partial_\theta\partial_\theta - rh^{\theta\theta}\left(\partial_t\partial_t+\frac{\partial_\theta\partial_\theta}{r^2}\right)\right)\\
=&-2r^{-1}\partial_t^2
\end{aligned}
\tag{122}
$$

$$
\begin{aligned}
D_{AB} =&2(K^{CD}-Kh^{CD})D_C D_D h_{AB}\\
=&2(rh^{\theta C}h^{D\theta}-Kh^{CD})D_C D_D h_{AB}\\
=&2\left(\frac{1}{r^3}\partial_\theta\partial_\theta\right)-2(r\delta^\theta_A\delta^\theta_B h^{AB}h^{CD})D_C D_D h_{AB}\\
=&\frac{2}{r^3}\partial_\theta^2-\frac{2}{r}\left(\partial_t^2+\frac{\partial_\theta^2}{r^3}\right)h_{AB}\\
=&-\frac{2}{r}h_{AB}\partial_t^2
\end{aligned}
\tag{123}
$$

and the boundary intrinsic (background) curvature is vanishing ${}^2R=0$. Substituting these values to 107 we can further simplify $S^{(2)}_{M^\perp}$,

$$
S^{(2)}_{M^\perp}=M\int\frac{d\tau\delta(r)}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\xi^\perp\xi^\perp\partial_\perp(\partial_t\xi_t))
\tag{124}
$$

With analogous to the Fourier transformation in [1],

$$
\gamma_{ab}(r,k'_t,k'_{\theta'})=\frac{1}{\sqrt{2\pi\beta}}\int_{\beta/2}^{\beta/2}dt\int_{-\pi}^{\pi}d\theta'\gamma_{ab}(r,t,\theta')e^{-i\theta' k'_{\theta'}}e^{-i\frac{2\pi t}{\beta}(k'_t-\frac{\gamma}{2\pi}k'_{\theta'})}
\tag{125}
$$

we transform $\theta=\theta'(1-4GM)$ and let $\mu=1-4MG$

$$
\gamma_{ab}(r,k'_t,k'_\theta)=\frac{1}{\sqrt{2\pi\beta}}\int_{\beta/2}^{\beta/2}dt\int_{-\pi\mu}^{\pi\mu}d\theta\gamma_{ab}(r,t,\theta)e^{-i(\frac{\theta}{\mu})k'_\theta}e^{-i\frac{2\pi t}{\beta}(k'_t-\frac{\gamma}{2\pi}k'_\theta)}
\tag{126}
$$

where we will use the abbreviation $k_t=\frac{2\pi}{\beta}(k'_t-\frac{\gamma}{2\pi}k_{\theta'})$ and $k_\theta=(\frac{k'_\theta}{\mu})$ and $k'_{\theta'},k'_t\in\mathbb{Z}$. The fourier inverse transformation is as follows

$$
\gamma_{ab}(r,t,\theta)=\frac{1}{\sqrt{2\pi\beta}}\sum_{k'_t,k'_\theta}\gamma_{ab}(r,k_t,k_\theta)e^{i\theta k_\theta}e^{itk_t}.
\tag{127}
$$

By Fourier transformation the equations of motions of (119) can be used to now solve for the lapse and shift components $\gamma_{\perp\perp}$ and $\gamma_{\perp A}$ of the metric perturbations [1].

$$
\begin{aligned}
\gamma_{\perp\perp}=&2\partial_\perp\left(\frac{1}{2r}\left(\gamma_{\theta\theta}+\frac{k_\theta^2}{k_t^2}\gamma_{tt}-2\frac{k_\theta}{k_t}\gamma_{\theta t}\right)\right)\\
=&2\partial_\perp\xi^\perp
\end{aligned}
\tag{128}
$$

$$
\begin{aligned}
\gamma_{\perp\theta}=&ik_\theta\frac{1}{2r}\left(\gamma_{\theta\theta}+\frac{k_\theta^2}{k_t^2}\gamma_{tt}-2\frac{k_\theta}{k_t}\gamma_{\theta t}\right)+r^2\partial_\perp\left(\frac{i}{r^2}\left(\frac{k_\theta}{2k_t^2}\gamma_{tt}-\frac{1}{k_t}\gamma_{\theta t}\right)\right)-ik_\theta\lambda\frac{1}{4k_t^2}\\
=&ik_\theta\xi^\perp+r^2\partial_\perp\xi^\theta-ik_\theta\lambda\frac{1}{4k_t^2}
\end{aligned}
\tag{129}
$$

$$\gamma_{\perp t} = ik_t \frac{1}{2r}\left(\gamma_{\theta\theta} + \frac{k_\theta^2}{k_t^2}\gamma_{tt} - 2\frac{k_\theta}{k_t}\gamma_{\theta t}\right) + \partial_\perp\left(-\frac{i}{2k_t}\gamma_{tt}\right) - ik_t\lambda\frac{1}{4k_t^2} \tag{130}$$
$$= ik_t\xi^\perp + \partial_\perp\xi^t - ik_t\lambda\frac{1}{4k_t^2}$$

By comparing these equations with (85) it can be deduced that [1]

$$\xi^\perp = \frac{1}{2r}\left(\gamma_{\theta\theta} + \frac{k_\theta^2}{k_t^2}\gamma_{tt} - 2\frac{k_\theta}{k_t}\gamma_{\theta t}\right), \tag{131}$$

$$\xi^\theta = \frac{i}{r^2}\left(\frac{k_\theta}{2k_t^2}\gamma_{tt} - \frac{1}{k_t}\gamma_{\theta t}\right) \tag{132}$$

and

$$\xi^t = -\frac{i}{2k_t}\gamma_{tt}. \tag{133}$$

The λ dependence of $\xi^\perp$ is implemented by rescaling $\xi^\perp$ [1]

$$\hat{\xi}^\perp = \xi^\perp - \frac{1}{2\Delta}\frac{\lambda}{\sqrt{h}} = \xi^\perp - \frac{1}{4k_t^2}\lambda \tag{134}$$

coming from the variation of the Lagrange multiplier as mentioned earlier we also have [1],

$$\rho = \frac{1}{2}\int_{r_1}^{r_2} dr\gamma_{\perp\perp} = \hat{\xi}^\perp(r_2) - \hat{\xi}^\perp(r_1). \tag{135}$$

We see that since the λ term in $\xi^\perp$ is r independent for non vanishing radius (r_1) the scalar field ρ does not depend on λ. In the case of non-vanishing radius λ is not solvable and will be a free parameter, the resulting action becomes the gravitational Hamilton–Jacobi functional plus the Lagrange multiplier term which does not solve our problem. To overcome this problem we follow the procedure as conducted by Seth and that is to only consider having an outer boundary and implement $r = 0$ into the bulk manifold $\mathcal{M}$. When implementing this condition we have to consider smoothness conditions for the metric perturbations at $r = 0$. With these conditions imposed $\xi^\perp$ will be λ dependent. This mechanism allows to solve for λ and compute an effective action for the geodesics lengths. This action will help us to predict a possible gravitational dual boundary field theory.

4.6 Implementing smoothness conditions for the metric at $r = 0$

With analogous to [1] we will impose the smoothness condition at $r = 0$. Smoothness conditions are implemented by Taylor expanding the metric perturbations around the origin. In cylindrical coordinates the metric perturbations become r [1],

$$\gamma_{ab} = a_{ab}^{(0)} + a_{ab}^{(1)}r + a_{ab}^{(2)}r^2 + O(r^3)\ for\ ab = \perp\perp, tt, \perp t; \tag{136}$$
$$\gamma_{ab} = a_{ab}^{(1)}r + a_{ab}^{(2)}r^2 + O(r^3)\ for\ ab = \perp\theta, \theta t;$$
$$\gamma_{\theta\theta} = a_{\theta\theta}^{(2)}r^2 + O(r^3).$$

We will see that we now unlike the massless case we only need to consider two separate cases, one is $|k_\theta| \geq 1/\mu$ and $k_\theta = 0$. For the case of $|k_\theta| \geq 1/\mu$ we Taylor expand all metric perturbations in the solutions for the lapse and shift variables (128, 129, 130) in r and we are able to obtain relations for the expansion coefficients $a_{ab}^{(n)}$. Imposing the conditions that $a_{ab}^{(n)} = 0$ for $n < 0$ and that $a_{a\theta}^{(0)} = 0$ as well as $a_{\theta\theta}^{(1)}$ then the following conclusions obtained was the same as in [1]:

$$\frac{k_\theta^2}{k_t^2} a_{tt}^{(0)} = 0, \tag{137}$$

$$\left(1 - \frac{1}{k_\theta^2}\right)\left(\frac{k_\theta^2}{k_t^2} a_{tt}^{(1)} - 2\frac{k_\theta}{k_t} a_{\theta t}^{(1)}\right) = \frac{\lambda}{2k_t^2}. \tag{138}$$

We have $a_{tt}^{(0)} = 0$ for $k_\theta \neq 0$, this also implies $a_{r\theta}^{(-1)}$ and $a_{rt}^{(-1)}$ vanishes. From the second equation we see that for $k_\theta = 0$ needs special treatment. With this the value of $\xi^\perp$ at $r = 0$ can be calculated,

$$\begin{aligned}
\xi^\perp(0) =& \lim_{r \to 0} \frac{1}{2r}\left(\gamma_{\theta\theta} + \frac{k_\theta^2}{k_t^2}\gamma_{tt} - 2\frac{k_\theta}{k_t}\gamma_{\theta t}\right) \\
=& \frac{1}{2}\left(\frac{k_\theta^2}{k_t^2} a_{tt}^{(1)} - 2\frac{k_\theta}{k_t} a_{\theta t}^{(1)}\right) \\
=& \frac{1}{4}\frac{k_\theta^2}{(k_\theta^2 - 1)}\frac{\lambda}{k_t^2}.
\end{aligned} \tag{139}$$

Next we look at the equation of motion imposed by the Lagrange multiplier [1],

$$\rho = \frac{1}{2}\int_{r_1}^{r_2} dr \gamma_{\perp\perp} = \xi^\perp(r_2) - \xi^\perp(r_1), \tag{140}$$

where

$$\xi^\perp(r_{out}) = \frac{1}{2r_{out}}\left(\gamma_{\theta\theta}(r_{out}) + \frac{k_\theta^2}{k_t^2}\gamma_{tt}(r_{out}) - 2\frac{k_\theta}{k_t}\gamma_{\theta t}(r_{out})\right) \tag{141}$$

with this the solution for the lagrange multiplier is obtained ($r_2 = r_{out}, r_1 = 0$),

$$\lambda = 4k_t^2\left(1 - \frac{1}{k_\theta^2}\right)\left(\xi^\perp(r_{out}) - \rho\right) \tag{142}$$

4.7 The Dual Boundary Field Action

We will now derive the Hamilton-Jacobi of a possible dual boundary field theory. Now we will substitute solutions (128),(129) and (128) to the Lagrangian multiplier term 113. Evaluation of the bulk term will be,

$$\begin{aligned}
-8\pi G S_{bulk}^{(2)} =& \frac{1}{4}\int_{\mathcal{M}} d^3x \sqrt{g}\gamma_{ab}\hat{G}^{ab} \\
=& \frac{1}{16}\int_{\mathcal{M}} d^2y dr \gamma_{\perp\perp}(r, y)\lambda(y) \\
=& \frac{1}{8}\int d^2y \lambda(y)(\xi^r(r_{out}, y) - \xi^\perp(0, y)),
\end{aligned} \tag{143}$$

The boundary term contains two parts. The Hamilton-Jacobi ($S_{HJ}^{(2)}$) and the term containing lagrangian multiplier. The first is the calculated term (108) and the λ dependent term is the boundary integral over $\lambda\xi^\perp$ [1] (and considering we only have an outer boundary),

$$-8\pi G S^{(2)}_{bdry} = -8\pi G S^{(2)}_{HJ}(r_{out}) - \frac{1}{8}\int d^2y \lambda(y)(\xi^r(r_{out}, y)), \tag{144}$$

To obtain the on-shell gravitation action we of course add (143) and (144) expressions. With this we obtain,

$$\begin{aligned} -8\pi G S^{(2)}_{\lambda} &= -8\pi G S^{(2)}_{bulk} - 8\pi G S^{(2)}_{bdry} \\ &= -8\pi G S^{(2)}_{HJ}(r_{out}) - \frac{1}{8}\int d^2y \lambda(y)(\xi^r(0, y)) \end{aligned} \tag{145}$$

We substitute the results for λ and $\xi^{\perp}(0, y)$ to the above expression

$$\begin{aligned} -8\pi G S^{(2)}_{\lambda}|_{solu} &= -8\pi G S^{(2)}_{HJ}(r_{out}) - \frac{1}{8}\int d^2y \left[4k_t^2\left(1-\frac{1}{k_\theta^2}\right)(\xi^{\perp}(r_{out})-\rho)\right]\left[\frac{1}{4}\frac{k_\theta^2}{k_\theta^2-1}\left(\frac{\lambda}{k_t^2}\right)\right] \\ &= -8\pi G S^{(2)}_{HJ}(r_{out}) - \frac{1}{2}\int d^2y \left[(\xi^{\perp}(r_{out})-\rho)k_t^2\left(1-\frac{1}{k_\theta^2}\right)(\xi^{\perp}(r_{out})-\rho)\right] \end{aligned} \tag{146}$$

We now fourier inverse transform,

$$\begin{aligned} -8\pi G S^{(2)}_{\lambda}|_{solu} &= -8\pi G S^{(2)}_{HJ}(r_{out}) + \frac{1}{2}\int d^2y \left[(\xi^{\perp}(r_{out})-\rho)\partial_t^2\left(1+\frac{1}{\partial_\theta^2}\right)(\xi^{\perp}(r_{out})-\rho)\right] \\ &= -8\pi G S^{(2)}_{HJ}(r_{out}) + \frac{1}{2}\int d^2y \xi^{\perp}(r_{out})\partial_t^2\left(1+\frac{1}{\partial_\theta^2}\right)\xi^{\perp}(r_{out}) + \\ &\frac{1}{2}\int d^2y\left[\rho\partial_t^2\left(1+\frac{1}{\partial_\theta^2}\right)\rho - 2\rho\partial_t^2\left(1+\frac{1}{\partial_\theta^2}\right)\xi^{\perp}(r_{out})\right]. \end{aligned} \tag{147}$$

The Hamilton-Jacobi functional is given by (for twisted thermal flat space with finite boundary using (121, 122, 123))

$$\begin{aligned} S^{(2)}_{HJ} &= S^{(2)}_{HJ^V} + S^{(2)}_{M^{\perp}} \\ &= -\frac{1}{16\pi G}\int d^2y(\xi^{\perp}\partial_t^2\xi^{\perp} - \xi^A h_{AB}\partial_t^2\xi^B) + M\int \frac{d\tau\delta(r)}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\xi^{\perp}\xi^{\perp}\partial_{\perp}(\partial_t\xi_t)) \end{aligned} \tag{148}$$

and with $\xi^{\perp} = \Delta^{-1}\delta\ (^2R) = -2^{-1}r\partial_t^{-2}\delta\ (^2R)$ we can write

$$\begin{aligned} -8\pi G S^{(2)}_{\lambda}|_{solu} = &-\frac{1}{4}\int d^2y\sqrt{h}\left(\rho\Delta\left(1+\frac{1}{\partial_\theta^2}\right)\rho - 2\rho\left(1+\frac{1}{\partial_\theta^2}\right)\delta\ (^2R)\right) + \\ &\frac{1}{4}\int d^2y\sqrt{h}\left(\xi^{\perp}\Delta\frac{1}{\partial_\theta^2}\xi^{\perp} - \xi^A D_{AB}\xi^B\right) + \\ &M\int \frac{d\tau\delta(r)}{\sqrt{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}}(\xi^{\perp}\xi^{\perp}\partial_{\perp}(\partial_t\xi_t)) \end{aligned} \tag{149}$$

This does define an action for the boundary field ρ, We see that it it very much the same as the action in the vacuum case but with an additional term representing the particle. By analyzing the Hamilton-Jacobi expression we can 'guess' a candidate for the dual field action and it is predicted as follows,

$$8\pi G S'_{\rho} := -\frac{1}{4}\int d^2y\sqrt{h}\left(\rho\Delta\left(1+\frac{1}{\partial_\theta^2}\right)\rho - 2\rho\left(1+\frac{1}{\partial_\theta^2}\right)\delta\ (^2R)\right). \tag{150}$$

We see that the expression for the dual field action S'_ρ is the same as in the massless scenario [1]. We will discuss the results and conclude them in the next chapter.

5 Conclusion and Discussion

As mentioned previously the candidate for the postulated dual field action needs to reproduce the geodesic length as a solution for the equations of motion and also should reproduce the diffeomorphism invariant part of the gravitational-Hamilton Jacobi functional. As shown in [1] the proposed Louville like action (9) does solve equation of motion such that $\rho = \xi^\perp$ and produces the gravitational gravitational-Hamilton Jacobi functional (massless case) as the on-shell action [1].

$$S_\rho|_{sol} = \frac{1}{4}\int d^2y\sqrt{h}\left(\xi^\perp \Delta \xi^\perp\right) \tag{151}$$

But this postulated action (9) does not display the gauge symmetries when calculated the one-loop correction for the gravitational path integral. Hence the dual field action (150) was found to be the better candidate for the dual field action. This does produce (151) but multiplied with the non-local operator $(1 + \partial^{-2})$ (can be compensated by adding the gravitational action with ∂^{-2} inserted [1]).

But why was the postulated candidate for the case with a mass is the same as the massless-case? Looking at the gravitational Hamiltonian-function with particle (108) derived for our scenario we see that boundary diffeomorphism invariant part includes the same term as the massless case and some additional terms involving the mass. But these terms do couple with the extrinsic curvature of the boundary metric. But as shown in (121) for the spinning thermal flat space (solid torus) this quantity vanishes leaving only the same boundary diffeomorphism invariant part as in the massless case. Also as shown in (149) there are no addition or modifications involving the boundary field ρ. Therefore our predicted dual field action is the same as the massless case. But for different boundaries this may not be the case, this is will require further research.

The dual action S'_ρ differs from the Louville action (9) by the insert of $(1+\partial_\theta^{-2})$. This additional term has consequences. In the massless case the geodesic length vanishes for modes $k_\theta = \pm 1$ and is ill defined for $k_\theta = 0$. But in this case for $k'_\theta = \pm 1$ the geodesic length does not vanish but it is still ill defined for $k'_\theta = 0$. Exactly as in [1] $k_\theta = 0$ will be a gauge mode and $a^{(0)}_{tt}$ is considered as a gauge parameter. Which is also confirmed in the Regge calculus setting [10] (also see section 3.4). We will now look at the $k'_\theta = \pm 1$ modes,

The above equation (138) we find that for $k'_\theta = \pm 1$

$$\left(1-\mu^2\right)\left(\frac{1}{\mu^2 k_t^2}a^{(1)}_{tt} \pm 2\frac{1}{\mu k_t}a^{(1)}_{\theta t}\right) = \frac{\lambda}{2k_t^2} \tag{152}$$

we know that $\mu = 1 - 4GM$

$$(8GM(1-2GM))\left(\frac{1}{(1-4GM)^2 k_t^2}a^{(1)}_{tt} \pm 2\frac{1}{(1-4GM)k_t}a^{(1)}_{\theta t}\right) = \frac{\lambda}{2k_t^2}. \tag{153}$$

We see here that unlike for the massless case as described in [1] for $k_\theta = \pm 1$ the value for $\lambda \neq 0$. This means that $\xi^\perp(0)$ will be λ dependent and will not be considered as a gauge parameter (like in the massless case) but a physical parameter. This claim is also supported in the Regge calculus setting (see section 3.5 or [10]) where we found two extra physical modes (compared to the massless case) corresponding to two Fourier modes. It should be noted that for a specific mass value $M = 1/2G$ the value for λ becomes zero (Further study is required as to determine what does that entail).

Now we look into the determining the one loop determinant for the postulated S'_ρ (150). We already know the one loop determinant has been calculated in the discreet setting (see section 3.6 or [10], one loop determinant has being calculated in the continuum for massive particles but in AdS (anti-de sitter) setting [17] giving similar results.) Since the dual action (150) obtained is the same as in [1] we can safely assume (predict) the one loop determinant in our case. A simple lattice regularization for the Hessian is adopted to calculate the one loop determinant, it is given by $k_t^2(1-k_\theta^{-2})$ [1]. The Fourier modes when calculated produce the following expressions [1],

$$k_\theta^2 \to \left(2 - 2cos\left(\frac{2\pi}{N_\theta}\right)\right)^{-1}\left(2 - 2cos\left(\frac{2\pi}{N_\theta}\kappa_\theta\right)\right) \tag{154}$$

$$k_t^2 \to \frac{N_t^2}{\beta^2}\left(2 - 2cos\left(\frac{2\pi}{N_t}(\kappa_t - \frac{\gamma}{2\pi}\kappa_\theta),\right)\right) \tag{155}$$

where $\kappa = 0, ..., N_\theta - 1$ and $\kappa_t = 0, ...N_t - 1$. With this choice unlike in the massless flat gravity case we have that $(1 - k_\theta^{-2}) \neq 0$ for $\kappa_\theta = \pm 1$. Therefore in this case our dual action can be defined for $|k_\theta| \geq 1$ so then we can consider ignoring some inessential constant we can predict that the one loop determinant will take the following form as in [1],

$$\prod_{\kappa_\theta=1}^{N_\theta-1}\prod_{\kappa_t=0}^{N_t-1}\frac{1}{\sqrt{k_t^2(1-k_\theta^{-2})}} \sim \prod_{\kappa_\theta=1}^{(N_\theta-1)/2}\frac{1}{|1-q^{\kappa_\theta}|^2} \tag{156}$$

where $q = exp(i\gamma)$. As we can see we can reproduce the one loop determinant of the gravitational theory with massive particle displaying the massive BMS_3 character as shown in [10, 24]. The results obtained do not contradict with the solutions obtained for the 3D flat gravity (massless case) as we substitute $M = 0$ all results such as $S_\lambda^{(2)}$ will result in all expressions in the flat 3D case. This confirms the interpretation of S'_ρ as a dual action for 3D gravity in vacuum with a massive point particle. Now we address a a few aspects of this thesis that needs to be studied further.

Firstly, our original objective to analyze (73) in the continuum setting. Although it is not explicitly analyzed in this thesis we can predict that the extra term arising from the $k = 1$ mode in (73) could refer to the extra (mass) terms obtained in the Hamilton-Jacobi functional for gravity $S_M^\perp$ (but 73 being mass independent could cause contradictions in that case the integral of the affine parameter τ could play a significant role). Another reason for studying specifically the quadratic $\xi^\perp$ terms in the Hamilton-Jacobi because $X_r^0(k,\nu)$ in (73) is directly related to $\xi^\perp$. In order to properly analyze and compare $S_H J^{(2)}$ needs to be discretized woth analogus to the procedure as shown in [4]. Another aspect to reconsider is the that we considered the smoothness at $r = 0$ to obtain solutions for the Lagrangian multiplier term. If we consider the canonical defect (at the center) as mentioned (section 4.3) we may have to re-evaluate our approach to obtain solutions for the Lagrangian multiplier term. Finally, looking at equation (153) we see that we need to pay special attention for when $M = 1/4G$.

This work can be expanded much further by looking into more examples of manifolds, here we have studied a torus boundary embedded into flat space but we could also consider hyperbolic AdS space or a spherical boundary embedded into flat space. Also it is needed to study a more general approach such as adding spin to the particle, or considering a particle with finite volume. I hope this thesis has contributed into understanding more about solving different aspects of issues arise when studying a theory for quantum gravity.

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