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RESEARCH ARTICLE

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A physical classification of Killing magnetic fields in Thurston geometries

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In recent years, numerous studies have appeared that considered Killing vectors of three-dimensional Riemannian manifolds as magnetic fields, since these vector fields are divergenceless by definition. The existence of adivergenceless vector field modeled as a magnetic field does not imply that it is physically realizable. In this study, we propose a physical classification scheme based on the divergences of the integral that defines the energy of a Killing magnetic field. We consider all the Killing magnetic fields of Thurston geometries studied in the literature and classify them as either physical or nonphysical.

KEYWORDS
Killing magnetic fields, physical classification, Thurston geometries

MSC CLASSIFICATION
53Z05, 78A30

1 | INTRODUCTION

In differential geometry, the concept of Killing vectors is important in terms of determining the symmetries of a manifold. If K denotes a Killing vector and g the metric, the following equation must hold:

$$\mathcal{L}_K g = 0, \tag{1}$$

where $\mathcal{L}$ is the Lie derivative. In terms of metric-compatible covariant derivative, this equation can be written as follows in the index notation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{2}$$

The existence of a Killing vector is closely related to symmetries of a manifold. A Killing vector identifies a direction on a manifold where the metric remains constant. In physics, such as Einstein's general theory of relativity (see Carroll [1] for an introduction), Killing vectors are related to conserved quantities. In consideration of quantum effects on classical space-time structure (semiclassical physics), time-like Killing vector field is especially important, as it provides a way to define positive frequency solutions, hence a way to define *particles*.

As a visual illustration, we can imagine the two-sphere. It is invariant under rotations about x-, y-, and z-axes, and these correspond to three Killing vectors on $\mathbb{S}^2$. On the other hand, for the Euclidean plane $\mathbb{E}^2$, there are also three Killing vectors: one corresponding to rotation around the origin, and the other two corresponding to translations along the x- or y-axis. These two are also examples of what is called *maximally symmetric manifold*, characterized by the existence of $n(n+1)/2$ Killing vectors in a manifold of dimension n.

When we take the trace of Equation (2), we see that $\nabla_\mu K^\mu = 0$, which in turn means that the divergence of Killing vector field vanishes. This is also the defining property of magnetic fields in Maxwell equations, since there is no magnetic monopole, as far as it is known. Because Killing vectors can describe magnetic fields, there are many studies in the literature on various 3D Riemannian manifolds where Killing vector fields are interpreted as magnetic fields. Most of these studies are about magnetic curves in three-dimensional model spaces of Thurston geometry, which are $\mathbb{E}^3$ [2], $\mathbb{H}^3$ [3], $\mathbb{S}^3$ [4], $\mathbb{S}^2\times\mathbb{R}$ [5], $\mathbb{H}^2\times\mathbb{R}$ [6], $SL(2,\mathbb{R})$ [7], Nil_3 [8], Sol_3 [9].

Despite the existence of many studies on Killing magnetic fields on these manifolds, no study addresses their potential physical realizability as of now. While Killing magnetic fields satisfy the Maxwell equations mathematically, their potential physical existence requires fulfilling specific energy conditions that depend on the type of divergence (ultraviolet or infrared) of their respective energy densities.

The organization of the paper is as follows: In Section 2, we provide basic concepts such as energy density of Killing magnetic fields, and our classification scheme for them that is used in the rest of the article; in Section 3, we calculate electromagnetic energy for each Killing magnetic vector field in each manifold then classify these vector fields either as physical or as nonphysical, and finally, in Section 4, we conclude the paper.

2 | ENERGY DENSITY OF KILLING MAGNETIC FIELDS

In all of the cases, the electrical charge density is taken to be zero. Moreover, it is supposed that the electric field vanishes, and the magnetic field (as given by Killing vector fields) is time independent. Since electric field vanishes and the magnetic field is given by the Killing magnetic field K, we write the electromagnetic field strength as

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K^3 & -K^2 \\ 0 & -K^3 & 0 & K^1 \\ 0 & K^2 & -K^1 & 0 \end{pmatrix} \tag{3}$$

where $K = (K^1, K^2, K^3)$ whose components stand for the coordinates x_1, x_2, x_3. On the other hand, the electromagnetic energy-momentum tensor is given by Carroll [1] (where we mapped $\eta \to g$):

$$T^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{4}$$

Let $\mathcal{M}$ be one of the manifolds we consider. In order to calculate the energy density, we consider a manifold of topology $\mathbb{R}\times\mathcal{M}$. On this four-dimensional spacetime, we consider the metric as (we assume a mostly plus signature for metric):

$$g_4 = \begin{pmatrix} -1 & 0 \\ 0 & g_3 \end{pmatrix} \tag{5}$$

where g_3 is the metric on $\mathcal{M}$. In terms of matrix multiplication, we have

$$T^{\mu\nu} = \frac{1}{4}g^{-1}\mathrm{Tr}[g^{-1}Fg^{-1}F] - g^{-1}Fg^{-1}Fg^{-1} \tag{6}$$

where $F = F_{\mu\nu}$. On the other hand, we do not consider Einstein equations in $\mathbb{R}\times\mathcal{M}$. Our aim is to calculate the energy density of the electromagnetic field, and then by integrating it on $\mathcal{M}$, we will calculate the energy that is necessitated by the Killing magnetic vector. It will be crucial in our classification scheme. If u is the four-velocity of the observer, the energy density perceived by this observer is as follows:

$$\xi(u) = u_\mu u_\nu T^{\mu\nu}. \tag{7}$$

Using Equation (4), we can write $\xi(u)$ as (using $u\cdot u = -1$):

$$\xi(u) = \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + u_\mu u_\nu F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta}, \tag{8}$$

$$= \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + (u^{\mu}F_{\mu\alpha})(u^{\nu}F_{\nu\beta})g^{\alpha\beta}. \tag{9}$$

Since electric field is zero, the term $u^{\mu}F_{\mu\alpha}$ has no temporal component. Because the three-metric is Riemannian, the last term is greater than or equal to zero. Hence, we find a lower bound for electromagnetic energy, which is

$$\xi = \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}. \tag{10}$$

Because the first row and column of $F_{\mu\nu}$ and $F^{\mu\nu}$ are zero, we can write the lower bound as

$$\xi = \frac{1}{4}F^{ij}F_{ij}, \tag{11}$$

where the three-metric is used for contraction of indices. Note that the Latin indices run from 1 to 3, while the Greek indices run from 0 to 3. In essence, after a few calculations,[1] in the absence of electric field (which is the case in this study), we have

$$\xi(u) = \frac{K^2}{2|g|} + \frac{|\tilde{u} \times K|^2}{|g|}, \tag{12}$$

where $\tilde{u}$ is the spatial component of the four-velocity and $(\tilde{u} \times K)_i = \epsilon_{ijk}\tilde{u}^j K^k$ where ϵ is the Levi–Civita tensor. Let us mention two concepts regarding divergences of an integral:

- Infrared (IR) divergence of an integral is due to nonfiniteness of the volume of the manifold.
- Ultraviolet (UV) divergence of an integral is due to singularity of the integrand at some finite coordinate.

Here, we state our criterion for physicality of Killing magnetic fields (or of magnetic fields in general). If the energy as given by Equation (12) diverges due to IR divergence, this Killing magnetic vector field is nonphysical and UV divergence may or may not occur for physical Killing magnetic vector fields.

Theorem 1. *Let $\mathcal{M}$ be one of the three-dimensional manifolds of Thurston model geometries. (i) If $\mathcal{M}$ is compact, all the Killing magnetic fields are physical. (ii) If $\mathcal{M}$ is noncompact, all the Killing magnetic fields are nonphysical.*

Proof. The first part of the theorem is true, since the norm of Killing magnetic field (K) and the volume of the manifold are bounded on compact manifolds, due to extreme value theorem; hence, there is no IR divergence. For the second part of the theorem, we do explicit calculations in Section 3 and prove the theorem. □

Note that the Thurston model geometries cover a large class of three-dimensional Riemannian manifolds. It may be possible to extend Theorem 1 to include a more general result for all three-dimensional Riemannian manifolds, which is out of the scope of this study.

3 | ELECTROMAGNETIC ENERGY FOR EACH MANIFOLD

In each subsection of this section, we will provide the metric and Killing vector fields of each manifold. Then we will find the electromagnetic energy on manifolds for each case and classify Killing magnetic fields as either physical or nonphysical. For that purpose, we define the following function:

$$E(u) \equiv \int_{\mathcal{M}} \xi(u)\,\mathrm{d}V. \tag{13}$$

On the other hand, the lower bound on energy is given by

$$E \equiv \int_{\mathcal{M}} \xi\,\mathrm{d}V = \int_{\mathcal{M}} \frac{K^2}{2\sqrt{|g|}}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3. \tag{14}$$

[1] Using $F_{ij} = \epsilon_{ijk}K^k$ where ϵ_{ijk} is the Levi-Civita symbol and the speed dependent part of $\xi(u)$ is found using $(F_{ik}u^k)(F_{jl}u^l)g^{ij}$ and $F_{ij}u^j = \epsilon_{ijk}u^jK^k = (\tilde{u} \times K)_i/\sqrt{|g|}$.

TABLE 1 Classification of divergences of $\int \xi \, dV$ in terms of physicality.

UV divergence	IR divergence	Physical
×	×	✓
×	✓	×
✓	×	✓
✓	✓	×

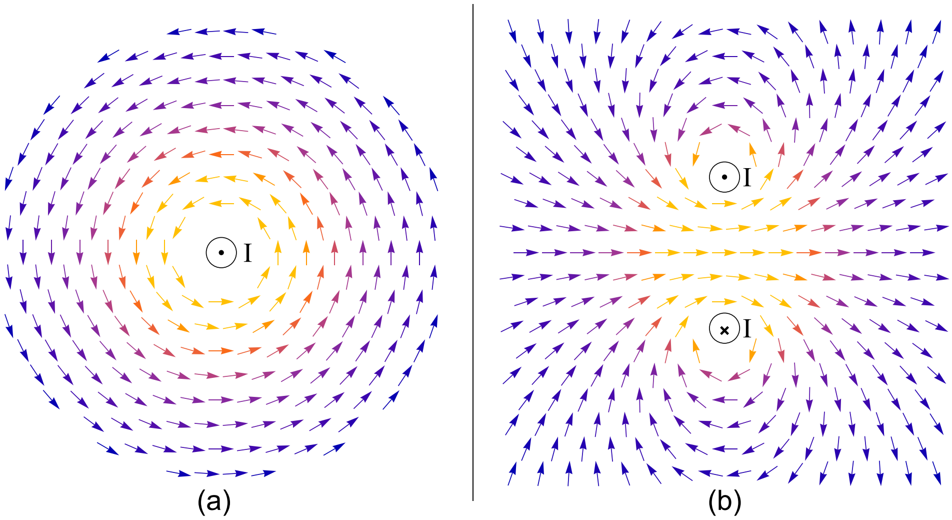

FIGURE 1 Two cases for current carrying wires illustrating the existence of UV and IR divergences. Magnetic field lines are on the x and y plane. (a) Current carrying wire in the $\hat{z}$ axis. (b) Current carrying wire loop in the y and z plane. Figures are drawn by Mathematica [10], and the numerical integration of the Biot–Savart law was done for the figure on the right hand side, again using this software. [Colour figure can be viewed at wileyonlinelibrary.com]

If this integral does not converge, due to IR divergence, we can safely say that the Killing magnetic field is nonphysical. If it has UV divergence but not IR divergence, we will classify those Killing magnetic field vectors as physical, considering that the corresponding UV divergence is due to *self-energy*. In Table 1, we classify the divergences.

In essence, our classification scheme is well illustrated in Figure 1 where we consider two setups for current carrying wires. In both cases, there are UV divergences coming from nearby regions of wires. However, the energy of the linear wire has an IR divergence (magnetic field is proportional to $1/\rho$ where $\rho = \sqrt{x^2 + y^2}$); hence, it would require an infinite amount of energy to realize this scenario which is nonphysical because it cannot be achieved by finite resources. Last but not least, in the case of circular wire, there are no IR divergences since this is a magnetic dipole and magnetic field goes like $1/r^3$ in the $r \to \infty$ limit.

3.1 | $\mathbb{E}^3$

Here, we provide basic information about the geometry of $\mathbb{E}^3$, the three Euclidean space. The metric is given by $ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates, and there are six Killing vectors:

$$K_1 = \partial_x, \tag{15}$$

$$K_2 = \partial_y, \tag{16}$$

$$K_3 = \partial_z, \tag{17}$$

$$K_4 = -z\partial_y + y\partial_z, \tag{18}$$

$$K_5 = z\partial_x - x\partial_z, \tag{19}$$

$$K_6 = -y\partial_x + x\partial_y. \tag{20}$$

In Druţă-Romaniuc and Munteanu [2], Killing magnetic curves have been investigated in the geometry of $\mathbb{E}^3$. We will make calculations for K_1 and K_4, and the results for others will follow from symmetry. For K_1, the lower bound of energy density is $\xi = 1/2$. Hence, when integrated over the manifold, the integral diverges due to IR divergence. So K_1 is non-physical, as well as K_2 and K_3. The lower bound of energy density for K_4 is $(y^2+z^2)/2$. When integrated on $\mathbb{R}^3$, the energy diverges due to IR divergence. Hence, we conclude that K_4, as well as K_5 and K_6, is not physical. To conclude, none of Killing magnetic field vectors are physical.

3.2 | $\mathbb{H}^3$

Killing magnetic fields on $\mathbb{H}^3$ are considered in Kelekçi et al. [3]. Here, we provide basic information about the geometry of $\mathbb{H}^3$, the three hyperbolic space. The metric is given by $\mathrm{d}s^2 = e^{-2\alpha z}\mathrm{d}x^2 + e^{-2\alpha z}\mathrm{d}y^2 + \mathrm{d}z^2$ for some constant α, and there are six Killing vectors [3]:

$$K_1 = \partial_x, \tag{21}$$

$$K_2 = \partial_y, \tag{22}$$

$$K_3 = y\partial_x - x\partial_y, \tag{23}$$

$$K_4 = \alpha x\partial_x + \alpha y\partial_y + \partial_z, \tag{24}$$

$$K_5 = \left[\frac{\alpha}{2}(x^2-y^2) - \frac{e^{2\alpha z}}{2\alpha}\right]\partial_x + \alpha xy\partial_y + x\partial_z, \tag{25}$$

$$K_6 = \alpha xy\partial_x + \left[\frac{\alpha}{2}(y^2-x^2) - \frac{e^{2\alpha z}}{2\alpha}\right]\partial_y + y\partial_z. \tag{26}$$

For each Killing magnetic field vector, we calculate $\xi\mathrm{d}V$. Note that in the coordinates we use, $x, y, z \in \mathbb{R}$. For K_1, we obtain $\xi\mathrm{d}V = 1/2(\mathrm{d}x\mathrm{d}y\mathrm{d}z)$, so the integral has IR divergence. Using symmetry arguments, we classify K_1 and K_2 as non-physical. For K_3, we have $\xi\mathrm{d}V = \frac{x^2+y^2}{2}\mathrm{d}x\mathrm{d}y\mathrm{d}z$ so it is non-physical as well, due to IR divergence. For K_4, $\xi\mathrm{d}V = \frac{e^{2\alpha z}+\alpha^2(x^2+y^2)}{2}\mathrm{d}x\mathrm{d}y\mathrm{d}z$, so this Killing magnetic field vector is non-physical as well. Lastly, we consider K_5. In this case, $\xi\mathrm{d}V = \frac{(e^{2\alpha z}+\alpha^2(x^2+y^2))^2}{8\alpha^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z$. The integral of this quantity diverges due to IR divergence, so K_5 (and by symmetry arguments, K_6) is not physical. To conclude, none of Killing magnetic field vectors are physical, where all of the energy calculations have IR divergence.

3.3 | $\mathbb{S}^3$

Line element for $\mathbb{S}^3$ in Hopf coordinates (θ, χ, ϕ) is given as [11]

$$ds^2 = d\theta^2 + \sin^2\theta d\chi^2 + \cos^2\theta d\phi^2 \tag{27}$$

where $\theta \in [0, \pi/2]$ and $\chi, \phi \in [0, 2\pi]$. Killing vector fields are obtained as

$$\begin{aligned}
K_1 &= \partial_\chi + \partial_\phi, \\
K_2 &= \cos(\chi+\phi)\partial_\theta - \cot\theta\sin(\chi+\phi)\partial_\chi + \tan\theta\sin(\chi+\phi)\partial_\phi, \\
K_3 &= \sin(\chi+\phi)\partial_\theta + \cot\theta\cos(\chi+\phi)\partial_\chi - \tan\theta\cos(\chi+\phi)\partial_\phi, \\
K_4 &= -\partial_\chi + \partial_\phi, \\
K_5 &= -\cos(\chi-\phi)\partial_\theta + \cot\theta\sin(\chi-\phi)\partial_\chi + \tan\theta\sin(\chi-\phi)\partial_\phi, \\
K_6 &= -\sin(\chi-\phi)\partial_\theta - \cot\theta\cos(\chi-\phi)\partial_\chi - \tan\theta\cos(\chi-\phi)\partial_\phi
\end{aligned} \tag{28}$$

For all of the six Killing magnetic fields, the lower bound for the energy density is found as $2/\sin^2(2\theta)$. The volume form is $\mathrm{d}V = \sin\theta\cos\theta\,\mathrm{d}\theta\mathrm{d}\chi\mathrm{d}\phi$. Hence, $\xi\mathrm{d}V = \frac{1}{\sin(2\theta)}\mathrm{d}\theta\mathrm{d}\chi\mathrm{d}\phi$. The θ integral from 0 to $\pi/2$ does not converge due to UV divergence, and there is no IR divergence in this case which is because the volume of $\mathbb{S}^3$ is finite. To conclude, all of Killing magnetic field vectors are physical.

3.4 | $\mathbb{S}^2 \times \mathbb{R}$

Killing magnetic fields on $\mathbb{S}^2 \times \mathbb{R}$ are considered in Munteanu and Nistor [5]. Here, we provide basic information about the geometry of $\mathbb{S}^2 \times \mathbb{R}$. The metric is given by $\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2$, and there are four Killing vectors:

$$K_1 = \partial_r, \tag{29}$$

$$K_2 = \partial_\phi, \tag{30}$$

$$K_3 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \tag{31}$$

$$K_4 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi. \tag{32}$$

The range of variables are: $r \in \mathbb{R}, \theta \in [0,\pi], \phi \in [0,2\pi]$. The volume form is $\mathrm{d}V = \sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$. Lower bounds for energy densities are respectively: $\csc^2\theta/2$, $1/2$, $(\csc^2\theta - \sin^2\phi)/2$ and $(\csc^2\theta - \cos^2\phi)/2$. Their integrals do not converge and all have IR divergences. To conclude, none of Killing magnetic field vectors are physical.

3.5 | $\mathbb{H}^2 \times \mathbb{R}$

Killing magnetic fields on $\mathbb{H}^2 \times \mathbb{R}$ are considered in Nistor [6]. Here, we provide basic information about the geometry of $\mathbb{H}^2 \times \mathbb{R}$. The metric is given by $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2 + \mathrm{d}z^2$ (where $y > 0$, and note that we did not modify the metric as it is given in Nistor [6] to make it similar to the form we used previously in Section 3.2 about the $\mathbb{H}^3$ geometry), and there are four Killing vectors:

$$K_1 = \partial_z, \tag{33}$$

$$K_2 = \partial_x, \tag{34}$$

$$K_3 = (x^2 - y^2 + 1)/2\partial_x + xy\partial_y, \tag{35}$$

$$K_4 = x\partial_x + y\partial_y. \tag{36}$$

The volume form for this manifold is $\mathrm{d}V = (\mathrm{d}x\mathrm{d}y\mathrm{d}z)/y^2$. Energy density lower bounds for these Killing magnetic vector fields are respectively: $y^4/2$, $y^2/2$, $y^2(1+x^2-2y+y^2)(1+x^2+2y+y^2)/8$ and $y^2(x^2+y^2)/2$. As it is easily seen, the integrals $\int \xi \mathrm{d}V$ do not converge, and all have IR divergences. To conclude, none of Killing magnetic field vectors are physical.

3.6 | $\mathbf{SL}(2,\mathbb{R})$

Killing magnetic fields on $\mathrm{SL}(2,\mathbb{R})$ are considered in Ref. [7]. Here, we provide basic information about the geometry of $\mathrm{SL}(2,\mathbb{R})$. The metric is given by $\mathrm{d}s^2 = (\mathrm{d}x/2y)^2 + (\mathrm{d}y/2y)^2 + (\mathrm{d}x/2y + \mathrm{d}\theta)^2$, and there are four Killing vectors [7]:

$$K_1 = \partial_x, \tag{37}$$

$$K_2 = \partial_\theta, \tag{38}$$

$$K_3 = x\partial_x + y\partial_y, \tag{39}$$

$$K_4 = (x^2 - y^2)/2\partial_x + xy\partial_y + y/2\partial_\theta, \tag{40}$$

where we added the $y/2\partial_\theta$ term to the fourth Killing vector. In the cited article it is missing, and without this term, it is not a Killing vector field. The range of coordinates are as follows [7]: $x \in \mathbb{R}, y \in \mathbb{R}^+, \theta \in [0,2\pi]$.

Volume form is $\mathrm{d}V = 1/(4y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}\theta$, and lower bounds of energy density for each Killing magnetic field vector are found as follows: $4y^2$, $8y^4$, $2y^2(2x^2 + y^2)$ and $(y(x^2 + y^2))^2$. Hence, the integrals $\int \xi \mathrm{d}V$ do not converge, and all have IR divergences. To conclude, none of Killing magnetic field vectors are physical.

3.7 | Nil_3

Here, we provide basic information about the geometry of Nil_3 space. The metric is given by $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\mathrm{d}y)^2$ [12]. For this manifold, there are four Killing vectors:

$$K_1 = \partial_y, \tag{41}$$

$$K_2 = \partial_z, \tag{42}$$

$$K_3 = \partial_x + y\partial_z, \tag{43}$$

$$K_4 = y\partial_x - x\partial_y + (y^2 - x^2)/2\partial_z. \tag{44}$$

The range of variables is given as $x, y, z \in \mathbb{R}$ [12]. The volume form is $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. For K_1, we have $\xi = (1 + x^2)/2$; hence, it is non-physical. For K_2, we find $\xi = 1/2$, and it is non-physical as well. For K_3, we obtain $\xi = (1 + y^2)/2$, and K_3 is not physical. Lastly, for K_4, we obtain $\xi = (x^2 + y^2)(4 + x^2 + y^2)/8$ whose integral does not converge. So K_4 is non-physical. All integrals have IR divergences. To conclude, none of Killing magnetic field vectors are physical.

3.8 | Sol_3

Killing magnetic fields on Sol_3 are considered in Erjavec and Inoguchi [9]. Here, we provide basic information about the geometry of Sol_3. The metric is given by $\mathrm{d}s^2 = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$, and there are three Killing vectors [9]:

$$K_1 = \partial_x, \tag{45}$$

$$K_2 = \partial_y, \tag{46}$$

$$K_3 = x\partial_x - y\partial_y - \partial_z. \tag{47}$$

For this manifold, the range of variables are $x, y, z \in \mathbb{R}$ [9]. The volume form is $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. For K_1, $\xi = e^{2z}/2$, and for K_2, $\xi = e^{-2z}/2$. The integrals of ξ diverges in both cases. Hence, K_1, K_2 are non-physical. On the other hand, for K_3, we have $\xi = (1 + e^{2z}x^2 + e^{-2z}y^2)/2$; hence, its integral is also not finite. All integrals have IR divergences. To conclude, none of Killing magnetic field vectors are physical.

4 | CONCLUSION

As is well known, a mathematical solution does not always imply that the solution is also physical. A vivid example from the theory of general relativity is Gödel's Universe [13]. What makes the Gödel's Universe non-physical is that it allows closed time-like paths that is paradoxical to say the least, and it supposes that the entire universe rotates, which cannot be done using finite resources.

In Einstein's general theory of gravity, black hole solutions were also considered to be non-physical; however, due to singularity theorems (e.g., see [14, 15]), it has been understood that black holes are inevitable in the universe. On the other hand, the findings uncovered by the Event Horizon Telescope (e.g. see [16, 17] for studies that have put tests) did not find any phenomena that is incompatible with the theory of general relativity (e.g. see [18, 19]) concerning black holes and the strong field regime.

A recent study [20] has ruled out many self-interacting vector field theories as nonphysical, using an effective metric approach. Investigations in this line of research, including ours, demonstrate the significance of physical classification of mathematical solutions.

The main reason for the non-physicality of Killing magnetic vector fields is that when the three-manifold $\mathcal{M}$ does not have a finite volume, the energy required by these vector fields are not finite. The only exception is $\mathbb{S}^3$ whose volume is finite. But even in this case, the θ integral of the energy lower bound diverges due to UV divergence. However, since we allow UV divergences in our criteria for a Killing magnetic field to be physical, the Killing vectors of $\mathbb{S}^3$ are considered as physical. All in all, for all of the Thurston geometries considered, the only Killing magnetic field vectors that are found to be physical are only for $\mathbb{S}^3$ and for the rest of the Thurston geometries they are non-physical.

AUTHOR CONTRIBUTIONS

Furkan Semih Dündar: Writing—review and editing; visualization; investigation; writing—original draft; conceptualization; methodology. **Özgür Kelekçi**: Writing—review and editing; investigation; writing—original draft; conceptualization; methodology. **Gülhan Ayar**: Writing—review and editing; investigation; writing—original draft; conceptualization; methodology.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interest.

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