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Hadronic Isospin Helicity and the Consequent $SU(4)$ Gauge Theory

Eckart Marsch [1,*] and Yasuhito Narita [2]

1 Institute for Experimental and Applied Physics, Christian Albrechts University at Kiel, Leibnizstraße 11, 24118 Kiel, Germany
2 Institut für Theoretische Physik, Technische Universität Braunschweig, Mendelssohnstraße 3, 38106 Braunschweig, Germany; y.narita@tu-braunschweig.de
* Correspondence: marsch@physik.uni-kiel.de

Abstract: A new approach to the Dirac equation and the associated hadronic symmetries is proposed. In this approach, we linearize the second Casimir operator of the Lorentz Group, which is defined by the energy–momentum four-vector and the fermion spin, thereby using the spinor-helicity representation instead of the three-vector representation of the particle momentum and spin vector. We then expand the so-obtained standard Dirac equation by employing an inner abstract "hadronic" isospin, initially describing a $SU(2)$ fermion doublet. Application of the spin-helicity representation of that isospin leads to the occurrence of a quadruplet of inner states, revealing the $SU(4)$ symmetry via the isospin helicity operator. This further leads to two independent fermion state spaces, specifically, singlet and triplet states, which we interpret as $U(1)$ symmetry of the leptons and $SU(3)$ symmetry of the three quarks, respectively. These results indicate the genuinely very different physical nature of the strong $SU(4)$ symmetry in comparison to the chiral $SU(2)$ symmetry. While our approach does not require the a priori concept of grand unification, such a notion arises naturally from the formulation with the isospin helicity. We then apply the powerful procedures developed for the electroweak interactions in the SM, in order to break the $SU(4)$ symmetry by means of the Higgs mechanism involving a scalar Higgs field as an $SU(4)$ quadruplet. Its finite vacuum creates the masses of the three vector bosons involved, which can change the three quarks into a lepton and vice versa. Finally, we consider a toy model for calculation of the strong coupling constant of a Yukawa potential.

Keywords: Lorentz group symmetry; expanded Dirac equation; isospin helicity; $SU(4)$ electro-strong symmetry

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1. Introduction

$SU(N)$ symmetries, Lie groups of the special unitary type, play an important role in describing physical symmetries, in particular in the standard model (SM) of elementary particle physics [1–3], such as $SU(2)$ for the electroweak interaction for the leptons and $SU(3)$ for the strong interaction among quarks. The task of finding a way to include the associated gauge bosons in the quantum mechanical framework of modern field theory was first demonstrated in the seminal work by Yang and Mills [4]. While the mathematical procedure of describing the fermions and their interactions through the $SU(N)$ gauge bosons by means of the covariant derivative is well established nowadays, to date, the physical origin of these important symmetries remains unexplained from the theoretical point of view. Rather, the physical origin of the gauge symmetries is of empirical nature. $SU(2)$ was chosen to describe the effects of parity violation on the assumption that the weak interactions flipping the isospin only involve the left-chiral Weyl field [5] of massless fermions. $SU(3)$ was introduced ad hoc by Gell-Mann [6] and also independently by Zweig [7], in order to bring some schematics into a variety of hadrons found in the early days of particle accelerators.

The fundamental empirical hadron symmetry $SU(3)$ is derived here anew, whereby the Dirac equation is discussed on the basis of the usual spinor representation of the Lorentz group (LG). This is achieved by linearization of the Pauli–Lubański (or the second Casimir) operator of the Lorentz Algebra (LA) in terms of the energy–momentum four-vector. We then add to this physical picture an abstract internal isospin $\mathbf{I}$ of $1/2$, which is not connected with the LG, but intended to cope with the hadronic particle symmetry of the two basic empirical fermion types of leptons and quarks. Isospin was originally suggested by Heisenberg [8] to explain the nuclear force in the early days of nuclear physics, when the heavy fermions (protons and neutrons) seemed to be the elementary building blocks of nuclei.

Application of the spinor helicity formalism (which means mathematically going from three-vector to two-component Pauli spinors) yields four orthogonal singlet and triplet fermion states for spin $1/2$. Thus, the spinor-helicity formalism applied to isospin causes a splitting into two unequal subspaces. We interpret the singlet state as the origin of $U(1)$ symmetry (one axis) corresponding to the charged lepton, and the triplet state as the origin of the $SO(3)$ symmetry group corresponding to the three quark colors (three axes). These basic symmetries can be combined into a larger group of $SU(4)$, which has been suggested in a merely ad hoc manner half a century ago by Pati and Salam [9,10] to unify the fermion interactions. For more recent discussions, references, and experimental material with respect to the leptoquarks involved, see, for example, the paper by Blumhofer and Lampe [11] and the reviews of Tanabashi and Rolli [12] and Tanabashi et al. [13].

Moreover, we apply the symmetry breaking mechanism of the SM here, as is used for the breaking of chiral symmetry in the weak interactions. This new approach enhances the complexity, while yielding completeness of the unified $SU(4)$ theory as compared to the Standard Model (SM), which employs only quantum chromodynamics (QCD). The spinless scalar Klein–Gordon equation can be accordingly generalized to include isospin as well, thus changing this field into an isospin quadruplet associated with the "strong Higgs" field. This quantum field (QF) can finally be coupled to the spinor fermion QF via the SM methods developed for the $SU(2)$ symmetry breaking. Three of the gauge bosons involved obtain masses through the Higgs mechanism, whereas the quarks and lepton remain unchanged and still have a common mass m.

The way we obtain these results suggests the very different physical nature of the hadronic $SU(4)$ as compared to the chiral $SU(2)$ symmetry. This new route does not require, but implies, the idea of unification at the outset; however, the notion arises naturally from isospin helicity. We then also apply the powerful procedures developed in the SM for the electroweak interactions in order to break $SU(4)$ symmetry. The finite Higgs vacuum creates the masses of the three vector bosons, which can change the three quarks into a lepton and vice versa. We finally discuss their masses. In Appendix A, we consider a simple toy model for the calculation of the coupling constant related to a Yukawa [14] model potential.

Here, we present a model of electro-strong interactions in the spirit of casting somewhat more light on the genuine nature of the $U(1)$ and $SU(3)$ symmetries, with the help of the mathematical and physical principles of Lorentz invariance of the two associated Casimir operators of the Lorentz group (LG) and their connections with the notion of isospin. The paper extends and deepens the recent theoretical approaches by Marsch and Narita [15–20]. Following the physical reasoning and mathematical procedure of the electroweak unification of the SM, we can combine gauging of the strong hypercharge field with gauging of the $SU(4)$-related fields. Thus, we retain strong electromagnetism and obtain a kind of "strong" charge-exchange reactions, the strengths of which depend on the two coupling constants g and g'. Three new types of vector gauge boson fields are involved, which are named here $V_\mu^\pm$ and carry the strong charge. The present theory is constructed in analogy to the Glashow–Weinberg–Salam (GWS) theory [5,21] of the SM quantum electroweak dynamics, but it is chirally symmetric and essentially relates electric charge to isospin helicity and to the strong hypercharge introduced in analogy to the weak hypercharge. In such a theory, all particles carrying the strong charge interact by charge exchange reactions mediated by massive vector bosons. Of course, only the quarks finally take part in QCD after $SU(4)$ symmetry breaking.

2. Spin and Casimir Operators of the LG, the Mass Squared and Pauli–Lubański Operator

2.1. Spin

The spin $\mathbf{S}$ is a fundamental physical quantity in particle quantum mechanics. It may be considered as inner property of the particle, and represents a kind of intrinsic rotation operator described by the three-vector $\mathbf{S}$ having non-commuting components. But it has nothing to do with the Lorentz invariant propagation of any particle per se. Spin obeys the commutator relation

$$\mathbf{S} \times \mathbf{S} = \mathrm{i}\mathbf{S}. \tag{1}$$

For its physics and mathematics see, for example, the modern textbook of Weinberg [22]. Tomonaga [23] wrote an enlightening book about the story of spin.

2.2. The Role of Spin in Defining the Generators of the Lorentz Group

Yet, it turns out that spin also plays an important role in space-time symmetries. It defines the generators of the rotation group in the three-dimensional Euclidean space, for $SO(3)$ in the adjoint representation and for $SU(2)$ in the fundamental spinor representation. Moreover, the two vector generators, rotation $\mathbf{J}$ and boost $\mathbf{K}$ of the Lorentz Group [24] in Minkowski space-time, can be combined to define the two spins

$$\mathbf{S}_{\pm} = \frac{1}{2}(\mathbf{J} \pm \mathrm{i}\mathbf{K}), \quad \mathbf{S}_{\pm}^2 = \frac{3}{4}1_4, \tag{2}$$

which expresses the basic property of chirality of the LG. The chiral spins form two subalgebras of the Lorentz algebra, commute with each other $[\mathbf{S}_{\pm}, \mathbf{S}_{\mp}] = 0$, and obey the spin commutator (1). In the fundamental spinor representation of the LG, one simply has $\mathbf{S}_{+} = 1/2\sigma$, and $\mathbf{S}_{-} = 0$, or $\mathbf{S}_{-} = 1/2\sigma$, and $\mathbf{S}_{+} = 0$ in terms of the Pauli matrices. For the four-vector representation of the LG, Marsch and Narita [17] have shown that the right- and left-chiral spin can be expressed as $\mathbf{S}_{\pm} = 1/2\mathbf{\Sigma}_{\pm}$ in terms of the larger 4×4 matrices

$$\Sigma_{\pm\,\mathrm{x}} = \begin{pmatrix} 0 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \; \Sigma_{\pm\,\mathrm{y}} = \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \pm 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix},$$
$$\Sigma_{\pm\,\mathrm{z}} = \begin{pmatrix} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

They have been used to define an extended Dirac equation [17,19], which also encompasses the up and down components of the chiral fermion doublet associated with the $SU(2)$ symmetry group.

2.3. Casimir Operators of the LG and Dirac Equation

Wigner [25] and Bargman and Wigner [26], in their influential work, emphasized the important role of the two Casimir operators of the Poincaré group and Lorentz group (LG) [24,27]. Let us start with the first Casimir operator of the LG, which is the squared four momentum of a particle and equals to its mass squared in the case of finite mass. The second is the first times the spin squared, $\mathbf{S}^2$, a rotational invariant.

We will not consider orbital angular momentum here, but just the intrinsic spin of a particle. We stay in Fourier space for the four-momentum $p^{\mu} = (E, \mathbf{p})$, and obtain

$$C_1 = p^{\mu} p_{\mu} = m^2, \tag{4}$$

$$C_2 = m^2\mathbf{S}^2 = m^2 s(s+1)1_{2s+1}. \tag{5}$$

On the basis of Equation (5), we describe here a lucid method to derive the Pauli–Lubański [28,29] operator. We make use of the fact that, for the three-vector momentum $\mathbf{p}$ (the components of which commute) and any spin $\mathbf{S}$ we obtain from (4) and (5), another equation involving the spin explicitly is as follows :

$$\begin{aligned}(E^2 - \mathbf{p}^2)\mathbf{S}^2 &= \mathbf{S}^2E^2 - ((\mathbf{S}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) + (\mathbf{S}\times\mathbf{p})\cdot(\mathbf{S}\times\mathbf{p})) \\ &= (\mathbf{S}E + \mathrm{i}\mathbf{S}\times\mathbf{p})^2 - (\mathbf{S}\cdot\mathbf{p})^2 = -W^\mu W_\mu,\end{aligned} \tag{6}$$

which is nothing but the covariant form for the negative square of the Pauli–Lubański operator, reading

$$W^\mu = (\mathbf{S}\cdot\mathbf{p}, \mathbf{S}E + \mathrm{i}\mathbf{S}\times\mathbf{p}) \tag{7}$$

in its four-vector form. Note that we made use of the identity $\mathbf{S}\cdot(\mathbf{S}\times\mathbf{p}) + (\mathbf{S}\times\mathbf{p})\cdot\mathbf{S} = 0$. Therefore, Equation (6) is Lorentz invariant, and is just the second Casimir operator C_2 of the Lorentz group. For spin $s = 1/2$, the square of cross-product in (6) is equal to the square of scalar product multiplied by a factor of 2, which gives $(\sigma\times\mathbf{p})\cdot(\sigma\times\mathbf{p}) = 2(\sigma\cdot\mathbf{p})^2$. Note that this relation is only valid for the Pauli matrices, and not for any other matrices of higher spins. When we exploit that relation, we obtain from (6) the equation

$$(E^2 - m^2)1_2 = (\sigma\cdot\mathbf{p})^2, \tag{8}$$

from which the Dirac equation in the Weyl basis readily follows by operating on Pauli bi-spinors, and by subsequent factorization of the resulting equation in terms of first-order operators. Consequently, the Casimir operator C_2 can be written as the square of the $\gamma^\mu p_\mu$. We use the natural units, $c = \hbar = 1$. Acting with $p_\mu = \mathrm{i}\partial_\mu$ on a four-component spinor field $\psi(x)$ yields the famous Dirac [30] equation

$$\gamma^\mu \mathrm{i}\partial_\mu\psi = m\psi. \tag{9}$$

This equation describes a fermion with mass m and spin 1/2 by the four 4×4 Dirac gamma matrices [1,3] obeying the Clifford algebra,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}1_4, \tag{10}$$

with the Minkowski metric $g^{\mu\nu}$. The Dirac equation can be extended [17,19,20] by including the chiral spins as defined in (2) and (3).

2.4. Spinor-Helicity Formalism

Returning to the Casimir operators, the general Equation (6) is valid for any spin. The key question, then, is "can it, while being a second-order (in E and $\mathbf{p}$) algebraic equation in Fourier space, also be factorized to the first order in a mathematically convenient and appealing form like that used in deriving the Dirac equation?" The algebraic way to achieve this goal is to use the relation $(\sigma\cdot\mathbf{p})^2 = \mathbf{p}^2 1_2$. This formula has as an interesting property, in that any three-vector scalar product can be replaced by matrices, a procedure that is called spinor-helicity formalism [3].

However, if one applies this formalism to an arbitrary intrinsic spin $\mathbf{S}$, then its non-commutativity yields more complicated results. First, note that with the Pauli matrices we obtain, for any three vectors $\mathbf{a}$ and $\mathbf{b}$ which do not commute, the special result

$$(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})1_2 + \mathrm{i}\sigma\cdot(\mathbf{a}\times\mathbf{b}). \tag{11}$$

Application of this relation to the three-vector operator $\mathbf{S}$ yields

$$1_2\mathbf{S}^2 = (\sigma\cdot\mathbf{S})((\sigma\cdot\mathbf{S}) + 1_{2(2s+1)}) = 1_{2(2s+1)}(s(s+1)). \tag{12}$$

Here, the dot symbol again denotes a scalar product of the three vectors, but in the case of matrices, it also implies Kronecker multiplication (symbol $\otimes$ left off here) with the Pauli matrices. This relation was first obtained in 1936 by Dirac [31] in his early attempt to derive a relativistic wave equation for any spin. At this point, it is natural to define the novel spin helicity [19] as follows:

$$H(s) = (\boldsymbol{\sigma} \cdot \mathbf{S}) = \boldsymbol{\sigma} \cdot \mathbf{S} = \sigma_{\mathrm{x}} \otimes S_{\mathrm{x}} + \sigma_{\mathrm{y}} \otimes S_{\mathrm{y}} + \sigma_{\mathrm{z}} \otimes S_{\mathrm{z}}. \tag{13}$$

We can thus rewrite Equation (8) in matrix form, and then expand C_2 to encompass the kinetic helicity, $\boldsymbol{\sigma} \cdot \mathbf{p}$, as well as the spin helicity $\boldsymbol{\sigma} \cdot \mathbf{S}$. This modified matrix equation finally reads

$$(1_2 E^2 - (\boldsymbol{\sigma} \cdot \mathbf{p})^2)(\boldsymbol{\sigma} \cdot \mathbf{S})(\boldsymbol{\sigma} \cdot \mathbf{S} + 1_{2(2\mathrm{s}+1)}) = m^2 s(s+1) 1_{2(2\mathrm{s}+1)}. \tag{14}$$

The key advantage here is the fact that there are no three-vector scalar products of either $\mathbf{p}$ or $\mathbf{S}$ with itself any more; we introduced matrix multiplications and two degrees of freedom associated with the Pauli matrices. We emphasize that the above equation connects the Lorentz group (LG) with the intrinsic spin group by Kronecker multiplication. This multiplication procedure is in compliance with the strict requirements of the Coleman–Mandula theorem [32].

3. Physics of the Spin Helicity Operator

Here, we first derive the basis vectors of eigenfunctions of the spin helicity $H(s)$ operator [19,20]. There are $2(2s+1)$ orthogonal basis vectors, i.e., twice as many as the case for the original spin $\mathbf{S}$ with quantum number s. Therefore, the dimension of the space associated with $H(s)$ is doubled in comparison with that of the genuine matrices related to the spin vector $\mathbf{S}$. The corresponding eigenvector equation is

$$H(s)\phi_j(s) = (\boldsymbol{\sigma} \cdot \mathbf{S})\phi_j(s) = s\phi_j(s). \tag{15}$$

Here, the index j has $2(2s+1)$ possible values, corresponding to to the dimension of the spin helicity representation. For the Kronecker product with the Pauli matrices, the helicity matrices are 4×4 matrices for $s = 1/2$. According to its definition, we explicitly have

$$H(s) = \begin{pmatrix} S_{\mathrm{z}} & S_{\mathrm{x}} - \mathrm{i}S_{\mathrm{y}} \\ S_{\mathrm{x}} + \mathrm{i}S_{\mathrm{y}} & -S_{\mathrm{z}} \end{pmatrix}. \tag{16}$$

We only quote this matrix operator for the spin quantum number $s = 1/2$, for which we obtain the real matrix

$$H(\frac{1}{2}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

The four eigenvectors form, respectively, an orthogonal set for the four-dimensional configuration space of the spin helicity. The related eigenvalues are $1/2$ for the first three and $-3/2$ for the fourth one. These eigenvectors read

$$\begin{gathered} \phi_1 = (1,0,0,0)^\dagger, \;\; \phi_2 = \tfrac{1}{\sqrt{2}}(0,1,1,0)^\dagger, \\ \phi_3 = (0,0,0,1)^\dagger, \;\; \phi_4 = \tfrac{1}{\sqrt{2}}(0,1,-1,0)^\dagger. \end{gathered} \tag{18}$$

We can then diagonalize the $H(s)$ matrix by unitary transformation using the above eigenvectors. Thus, one obtains the diagonal matrix in the following form:

$$H(\frac{1}{2}) = \frac{1}{2}\mathrm{diag}[1,1,1,-3]. \tag{19}$$

Apparently, the spin helicity space of dimension 4 factorizes into two distinct, asymmetric spaces of dimensions, 3 and 1, for spin 1/2. This asymmetry originates in the basic Equation (12), in which the spin helicity enters with a linear and quadratic term, owing to the fact that the three components of the spin **S** do not commute.

Inspection of the symmetry group $SU(4)$, describing the general rotations in the four-dimensional complex space, reveals that the spin helicity matrix is identical (aside from a normalization constant) with the fifteenth matrix λ^{15} of the $SU(4)$ group. This is a strong hint to the possible importance of this group to describe the hadronic interactions. It is well known that the group $SU(3)$ is a subgroup of $SU(4)$ and fundamental for QCD in the SM. So, the suggestive interpretation of the two subspaces is that $SU(3)$ is connected to the three colored quarks, whereas the remaining subspace with $U(1)$ symmetry is connected with the single lepton. Returning again to (14), we obtain for the spin 1/2 the result

$$(1_2 E - \boldsymbol{\sigma} \cdot \mathbf{p})(1_2 E + \boldsymbol{\sigma} \cdot \mathbf{p}) \otimes (H(\frac{1}{2})(H(\frac{1}{2}) + 1_4) = m^2 \frac{3}{4} 1_8, \tag{20}$$

whereby we expressed the spin helicity operator in terms of its eigenfunctions according to Equation (19). We use it here to introduce two normalized diagonal matrices, which obey $H_0 H_1 = H_1 H_0 = 1_4$, and read as follows:

$$H_0 = \mathrm{diag}[1,1,1,-3], \quad H_1 = \mathrm{diag}[1,1,1,-1/3]. \tag{21}$$

With their help, and by introduction of the four-momentum differential operator $p_\mu = \mathrm{i}\partial_\mu = \mathrm{i}(\partial/\partial t, \partial/\partial \mathbf{x})$, we can write (20) as a second-order wave equation with differential operators acting on a two-component Pauli spinor $\varphi(x)$. This equation reads

$$\left(\mathrm{i}(1_2 \frac{\partial}{\partial t} + \boldsymbol{\sigma} \cdot \frac{\partial}{\partial \mathbf{x}}) \otimes H_0\right)\left(\mathrm{i}(1_2 \frac{\partial}{\partial t} - \boldsymbol{\sigma} \cdot \frac{\partial}{\partial \mathbf{x}}) \otimes H_1\right) \varphi \otimes \phi = m^2 \varphi \otimes \phi. \tag{22}$$

Following the mathematical procedure presented in the papers by Marsch and Narita [18,20], we can readily linearize the above wave equation and finally obtain an expanded Dirac equation in the Weyl basis. It involves, of course, the Dirac fermion bi-spinor $\psi^\dagger = (\varphi_1^\dagger, \varphi_2^\dagger)$, describing a spin one-half particle and antiparticle, as well as the four-component state vector ϕ of the $SU(4)$ group. We obtain

$$\gamma^\mu \mathrm{i}\partial_\mu \Psi = m\Psi. \tag{23}$$

We have the new expanded spinor $\Psi = \psi \otimes \phi$, whereby the first three components of $\phi^\dagger = (\chi_{q1}, \chi_{q2}, \chi_{q3}, \chi_l)$ belong to the $SU(3)$ subgroup of quarks, and the fourth to the $U(1)$ of the lepton, and the chis just denote complex numbers.

4. Hadronic Isospin and $SU(4)$

4.1. The Hadronic Isospin I

We recapitulate that there is an empirical duality between quarks and leptons, although, as derived in the previous section, the dimensions of the relevant state spaces are different and determined by spin helicity. According to the early intuitive ideas of Heisenberg [8] concerning isospin, he just considered the proton and neutron as a fundamental and stable fermion isospin doublet, and assumed them to have equal masses to preserve the employed $SU(2)$ symmetry. Yet, here we will not disregard the composite structure of the hadrons, but consider instead the quarks and leptons themselves as the basic particle degrees of freedom.

Concerning nomenclature, we use the traditional symbol **I** for the abstract hadronic or nuclear isospin, which is defined just like the spin **S** and based on the Pauli matrices as well, and so define

$$\mathbf{I} = \frac{1}{2}(\sigma_\mathrm{x}, \sigma_\mathrm{y}, \sigma_\mathrm{z}), \quad \mathbf{I} \times \mathbf{I} = \mathrm{i}\mathbf{I}. \tag{24}$$

As compared to the kinetic helicity of the momentum $\mathbf{p}$, $\boldsymbol{\sigma} \cdot \mathbf{p}$, which has two values corresponding to parallel and antiparallel orientation of the momentum and spin of the fermion, the isospin helicity $\boldsymbol{\sigma} \cdot \mathbf{I}$ reveals four orientations, related with three quarks and one lepton. Yet, we know from the SM that leptons do not take part in the strong interactions. How can one obtain this result from the previous considerations? Like in the weak interaction connected with the chirality of the fermions and symmetry group $SU(2)$, the mechanism to separate the two fermion species is by breaking the $SU(4)$ symmetry. This is the topic of the next subsections.

4.2. Hypercharge Operator and Helicity Operator

The symmetry group $SU(4)$ describes the possible rotations around the four axes of the complex Euclidian space $\mathcal{C}^4$ of four dimensions. We determined it above as the space of the spin helicity, which has four real orthogonal eigenfunctions spanning that space. We refer to the literature for the detailed representation matrices of $SU(4)$ [16].

What matters here is, firstly, the so-called hypercharge operator named Y, and secondly, the operator $\tilde{H}$, which we already identified as the spin helicity operator. The two associated real and diagonal matrices can then be written as follows:

$$Y = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \tilde{H} = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{25}$$

They reflect the fact that the original isospin duality leads via the helicity to three quarks, which share their available space equally, and thus each has a hypercharge of $1/3$, and then to only a single lepton having a hypercharge of 1. The zero trace of $\tilde{H}$ guarantees strong charge neutrality of the quadruplet. As diagonal matrices these two operators commute with each other. Furthermore, Y and $\tilde{H}$ also commute with the eight matrices of the subgroup $SU(3)$, yet not with the matrices λ^a with a running from 9 to 14. Both matrices have been normalized such that $\mathrm{Tr}Y^2 = 1/2$ and $\mathrm{Tr}\tilde{H}^2 = 1/2$. The same standard normalization is applied to all the other $SU(4)$ group matrices as well.

4.3. Electro-Strong Symmetry Breaking

In this subsection, we partly follow the physical reasoning presented in [16], and also employ the mathematical procedures known from electroweak unification and $SU(2)$ symmetry breaking in the SM [3]. We describe the hadronic dynamics by means of the isospin helicity and the consequent $SU(4)$ symmetry, which is then broken by a gauge-field rotation. We employ similar techniques than in the Glashow–Weinberg–Salam theory in order to break that symmetry. The general unitary phase transformation operator U, which acts on the state vector ϕ and yields $\phi_{\mathrm{U}} = U\phi$, leaves the scalar product invariant, $\phi^\dagger \phi = \phi_{\mathrm{U}}^\dagger \phi_{\mathrm{U}}$. The phase operator U can be cast into an instructive form, which includes the hypercharge and helicity operators (25) and the generators of $SU(4)$. They are represented by the fifteen 4×4 matrices λ^a, given for example in [16], and describe the possible rotations in the four-dimensional complex state space spanned by the four basis vectors given in (18). Thus, we obtain the unitary phase operator

$$U = \exp\left(\mathrm{i}(g'\beta Y + g\alpha^{15}\tilde{H} + g\sum_{a=1}^{14} \alpha^a \lambda^a)\right). \tag{26}$$

A gauge-field theory using the isospin-related $SU(4)$ symmetry in this way is, to our best knowledge, a novel approach in the hadronic sector of elementary particle physics. We use the conventional notation such that g' stands for the coupling constant for the hypercharge operator Y, and g the coupling constant for the $SU(4)$ gauge symmetry. The related scalar angle β and the 15 angles α^a (a runs from 1 to 15) are all real. In a local gauge symmetry, they depend of course upon the space-time coordinate $x = (t, \mathbf{x}) = x^\mu$, which we may omit here to ease the notation. We just need to consider the two diagonal-matrix operators Y and the isospin helicity $\tilde{H}$ that is identical to λ^{15}. Both can now be mixed by a rotation, such that we obtain the new linear combinations as follows:

$$\begin{pmatrix} \beta \\ \alpha^{15} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega' \\ \omega \end{pmatrix}. \tag{27}$$

Here, θ is the rotation or mixing angle that still is to be determined. Inserting the new coordinates into the operators appearing in the phase (26) of the operator U, we obtain for the argument in the exponential function

$$g'\beta Y + g\alpha^{15}\tilde{H} = \omega' Q + \omega R. \tag{28}$$

with the new matrix operators Q and R and gauge rotation angles ω and ω', the gradients of which determine the new gauge fields. We obtain the electro-strong field as $A_\mu = \partial_\mu \omega'(x)$, and the other gauge field $B_\mu = \partial_\mu \omega(x)$, which is also a mixture of the hypercharge operator Y and the original $SU(4)$ helicity operator λ^{15}. B_μ is the analogue of the field Z_μ in the electroweak section of the SM. By their constructions, the charge operators are both represented by diagonal 4×4 matrices, which read

$$Q = \frac{1}{\sqrt{24}} \begin{pmatrix} (g'\cos\theta + g\sin\theta)1_3 & 0 \\ 0 & 3(g'\cos\theta - g\sin\theta) \end{pmatrix}, \tag{29}$$

and, similarly,

$$R = \frac{1}{\sqrt{24}} \begin{pmatrix} (-g'\sin\theta + g\cos\theta)1_3 & 0 \\ 0 & -3(g'\sin\theta + g\cos\theta) \end{pmatrix}. \tag{30}$$

Here, 1_3 is the 3×3 unit matrix and Q and R are of block-diagonal form. Close inspection of (29) and (30) shows that, if $g'\cos\theta = g\sin\theta = q$ for a particular angle $\theta_{gg'}$, these matrices become more transparent and Q contains merely integers. The above condition implies that $\tan\theta_{gg'} = g'/g$, which is defined in analogy to the so called Weinberg–Glashow angle [5] of the SM. Once it is fixed by the ratio of the coupling constants, the entity Q turns out to be the operator of the "electro-strong" charge of the quarks, with the charge unit $q = gg'/\sqrt{g^2 + (g')^2}$. The quark charge operator Q couples to the electro-strong gauge field A_μ. Thus, one obtains the new operators

$$Q = \frac{q}{\sqrt{6}} \begin{pmatrix} 1_3 & 0 \\ 0 & 0 \end{pmatrix}, \tag{31}$$

$$R = \frac{q}{\sqrt{6}} \begin{pmatrix} 1_3(-\frac{g'}{g} + \frac{g}{g'}) & 0 \\ 0 & -3(\frac{g'}{g} + \frac{g}{g'}) \end{pmatrix}. \tag{32}$$

Of course, these two operators commute, since they are represented by diagonal matrices. So, we are dealing here with Q with a hadronic multiplet of three charged quarks and a lepton of charge zero, which therefore does not take part in the remaining $SU(3)$ strong interaction.

With the redefinition of the gauge fields by a simple rotation, we have separated the quarks from lepton, yet not entirely, because the coupling between the two species is still included in the six generators λ^a, with a running from 9 to 14. They correspond to what have been called "leptoquarks" in the literature [33–36], which are the corresponding vector bosons exchanging the charges of the hadronic fermions, i.e., transforming quarks into leptons and vice versa. We appropriately rename Q as Q_q. However, there is also still the

R matrix stemming from λ^{15}. For equal coupling constants, $g = g'$, q turns out to be just $g/\sqrt{2}$. Moreover, R then takes the simple form

$$R = q\sqrt{6}\begin{pmatrix} 0_3 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33}$$

Thus, we are dealing here with a lepton of charge -1, the electron, and the upper part of the diagonal being zero that corresponds to a decoupling of the quarks from the gauge field B_μ. According to Equation (31), for Q_q, the quark charge is $1/3$. Then, what is the nature of R? It is just the strong charge operator acting on the electron that is coupled to the gauge field B_μ associated with the $U(1)$ symmetry. We, therefore, also rename R in the special form (33) as Q_l.

4.4. Covariant Derivative after Symmetry Breaking

The general covariant derivative, as obtained after the $SU(4)$ symmetry breaking, then reads

$$D_\mu = \partial_\mu - \mathrm{i}(Q_q A_\mu(x) + Q_l B_\mu(x) + \sum_{a=1}^{14} \partial_\mu \alpha^a(x)\lambda^a). \tag{34}$$

The first eight λ^a generators correspond to the $SU(3)$ subgroup of QCD, and the remaining three to the "leptoquarks". From now on, instead of the superscript a, we use a simple subscript n for counting of the fields from 9 to 14, and can then define the following six convenient linear combinations of the lambdas:

$$\begin{aligned} \lambda^{\pm}_{9,10} &= \tfrac{1}{2}(\lambda_9 \pm \mathrm{i}\lambda_{10}), \\ \lambda^{\pm}_{11,12} &= \tfrac{1}{2}(\lambda_{11} \pm \mathrm{i}\lambda_{12}), \\ \lambda^{\pm}_{13,14} &= \tfrac{1}{2}(\lambda_{13} \pm \mathrm{i}\lambda_{14}). \end{aligned} \tag{35}$$

They are all real, and contain a single one, but are otherwise zeros. By definition, they are not hermitian, yet obey $(\lambda^{\pm})^\dagger = \lambda^{\mp}$. Therefore, we quote only three of them here. They connect quarks with the lepton. The plus and minus signs indicate opposite rotations in state space around the relevant axes. To ease the notation, we introduce $\Lambda^{\pm}_1 = \lambda^{\pm}_{9,10}$, $\Lambda^{\pm}_2 = \lambda^{\pm}_{11,12}$, and $\Lambda^{\pm}_3 = \lambda^{\pm}_{13,14}$. Then, $(\Lambda^{\pm}_n)^\dagger = \Lambda^{\mp}_n$ for $n = 1,2,3$. These three matrices read

$$2\Lambda^{+}_1 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, 2\Lambda^{+}_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, 2\Lambda^{+}_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}. \tag{36}$$

The summed up matrices obey the relation

$$\sum_{n=1}^{3}(\Lambda^{+}_n\Lambda^{-}_n + \Lambda^{-}_n\Lambda^{+}_n) = \frac{1}{4}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3 \end{pmatrix}, \tag{37}$$

which is aside from a factor the previous Y operator (25). Moreover, we find that $(\Lambda^{\pm}_n)^2 = 0$, and $\Lambda^{\pm}_n\Lambda^{\pm}_m = 0$ for n unequal m. Thus, any product of the lambdas for equal superscript vanishes. The expectation value of (37) evaluated with the simple field vector $\phi^\dagger = (0,0,0,1)$ delivers the value 3. Correspondingly, we redefine the associated gauge fields as follows:

$$\begin{aligned}(V_1^{\pm})_\mu(x) &= \partial_\mu(\alpha^9 \mp \mathrm{i}\alpha^{10}),\\ (V_2^{\pm})_\mu(x) &= \partial_\mu(\alpha^{11} \mp \mathrm{i}\alpha^{12}),\\ (V_3^{\pm})_\mu(x) &= \partial_\mu(\alpha^{13} \mp \mathrm{i}\alpha^{14}),\\ U_\mu^a(x) &= \partial_\mu \alpha^a(x),\end{aligned} \tag{38}$$

where a now runs only from 1 to 8. In the end, the covariant derivative consisting of hermitian operators reads

$$\mathrm{i}D_\mu = \mathrm{i}\partial_\mu + Q_q A_\mu(x) + Q_l B_\mu(x) + g\sum_{a=1}^{8} U_\mu^a(x)\lambda^a \tag{39}$$

$$+g\sum_{n=1}^{3}((V_n^+)_\mu(x)\Lambda_n^- + (V_n^-)_\mu(x)\Lambda_n^+). \tag{40}$$

This way, we have clearly separated the notations for the $U(1)$ and $SU(3)$ quantities from the remaining $SU(4)$ ones, which provide the connections between quarks and leptons.

5. The Higgs Mechanism Applied to Hadronic Interactions

5.1. The Hadronic Higgs

In the above described way, we can, consistently with the empirical SM phenomenology, describe how the hadronic fermions with the single mass m transform under the fundamental representations of the $SU(2)$ isospin group, yielding the $SU(4)$ symmetry, and assemble them in a multiplet carrying the strong charge. The hadronic fermion is described by its spinor field Ψ obeying the Dirac Equation (23). We consider an additional scalar Higgs-type particle field Φ with unspecified assumed mass M_{H}. Then, the particle part of the Lagrangian for the combined boson and fermion sectors reads

$$\mathcal{L}_{\mathrm{P}} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - M_{\mathrm{H}}^2\Phi^\dagger\Phi + \bar{\Psi}\gamma^\mu \mathrm{i}D_\mu\Psi - m\bar{\Psi}\Psi, \tag{41}$$

with the covariant derivative given in (39). As usual, $\bar{\Psi} = (\gamma_0\Psi)^\dagger$. For the sake of completeness, D_μ is repeated and further discussed below. The covariant derivative may be split appropriately as

$$D_\mu = \partial_\mu - \mathrm{i}\Delta_\mu, \tag{42}$$

with the abbreviation for the gauge-field interaction term Δ_μ in (39). For the following, we require the validity of the Lorentz condition for all gauge fields, i.e., that $\partial^\mu A_\mu = \partial^\mu B_\mu = \partial^\mu U_\mu = \partial^\mu V_\mu^{\pm} = 0$. Moreover, the interaction term is hermitian, i.e., $(\Delta_\mu)^\dagger = \Delta_\mu$.

It is instructive to multiply, by use of the decomposition (42), the squared kinetic term out, as it appears in the boson sector of the Lagrangian $\mathcal{L}_{\mathrm{P}}$. The result is

$$\begin{aligned}(D^\mu\Phi)^\dagger(D_\mu\Phi) &= (\partial^\mu\Phi)^\dagger(\partial_\mu\Phi)\\ -(\Phi\Delta^\mu)^\dagger(\mathrm{i}\partial_\mu\Phi) &- (\mathrm{i}\partial_\mu\Phi)^\dagger(\Delta^\mu\Phi) + \Phi^\dagger\Delta^\mu\Delta_\mu\Phi.\end{aligned} \tag{43}$$

The first term is the kinetic term of the free scalar Higgs boson field, the second two terms correspond to the coupling of the Higgs boson current density with the vector boson gauge fields, and the third describes the interaction between all the gauge fields. It also contains terms (quadratic in the gauge fields), which can give rise to masses of the gauge bosons by means of finite expectation values in the Higgs field.

Of course, the scalar Higgs field has to be a quadruplet consistent with $SU(4)$ symmetry. It should be uncharged with respect to the strong interaction, and therefore we chose the state vector of the simplest form:

$$\Phi_{\mathrm{v}}^\dagger = \mathrm{v}(0,0,0,1). \tag{44}$$

Here, v is the vacuum expectation value of the strong Higgs field, which is expected to be nonzero in accordance with the known SM assumptions as applied to the weak interactions.

Trivially, $\partial^\mu \Phi_{\rm v} = 0$. This is used in evaluating the interaction term of the vector boson fields in the Higgs vacuum which, after (43), gives the subsequent contribution to the Lagrangian:

$$\mathcal{L}_M = \Phi_{\rm v}^\dagger \Delta^\mu \Delta_\mu \Phi_{\rm v}. \tag{45}$$

This enormous simplification is due to the fact that all kinetic derivative terms in (43) vanish in the constant Higgs vacuum v.

To calculate $\mathcal{L}_M$ requires us to consider the expectation values of the various matrices appearing in (45). Trivially, $\lambda^a \Phi_{\rm v} = 0$ for $a = (1, \ldots, 8)$ running through the elements of the $SU(3)$ subgroup. Also, $Q\Phi_{\rm v} = 0$. Moreover, $(\lambda^a)^2 \Phi_{\rm v} = \Phi_{\rm v}$ for $a = (9, \ldots, 14)$. Furthermore, $\Phi_{\rm v}^\dagger (\lambda^a)^2 \Phi_{\rm v} = 0$ for $a = (9, \ldots, 14)$ and, finally, we obtain $\Phi_{\rm v}^\dagger (\lambda^a \lambda^b) \Phi_{\rm v} = 0$ for the three index pairs $(a, b) = (9, 10), (11, 12), (13, 14)$. The lambda matrices and related calculations can also be found, for example, in the previous paper by Marsch and Narita [16]. We also make use of the fact that matrices λ^9 and λ^{10}, and λ^{11} and λ^{12}, as well as λ^{13} and λ^{14}, anticommute. We further use the relation (37). In the ,end we obtain the result

$$\mathcal{L}_M = B^\mu(x) B_\mu(x) (\Phi_{\rm v}^\dagger R^2 \Phi_{\rm v}) + \tag{46}$$

$$g^2 \Phi_{\rm v}^\dagger \left(\sum_{n=1}^{3} ((V_n^+)^\mu(x) \Lambda_n^- + (V_n^-)^\mu(x) \Lambda_n^+) \right)^2 \Phi_{\rm v}. \tag{47}$$

The squared expression involving the three V vector bosons is to be understood as being summed up over the Greek relativistic indices as well. Evaluation of the expectation values of the lambda matrices in the Higgs vacuum leads to the final result for the mass Lagrangian:

$$\mathcal{L}_M = M_{\rm R}^2 B^\mu B_\mu + M_{\rm V}^2 \sum_{n=1}^{3} ((V_n^+)^\mu (V_n^-)_\mu). \tag{48}$$

The squared masses of the three vector bosons $V_{1,2,3}^\pm$ turn out to be the same, and arise from expectation values of the $SU(4)$ generators in the Higgs vacuum. The squared mass of the vector boson of the R field is given by $M_{\rm R}^2 = (\Phi_{\rm v}^\dagger R^2 \Phi_{\rm v})$. The mass squared of the V bosons are all equal and given by $M_{\rm V}^2 = g^2 {\rm v}^2/4$. In conclusion, all vector bosons that change the strong charge have become massive by the famous Higgs mechanism, involving here a scalar quadruplet vacuum field with a strong charge of -1, like that of the electron.

5.2. Discussion of the Vector Boson Masses

In order to confine the nuclear forces to the size of the nucleus (about 1.5 fm), the vector gauge bosons should, in fact, be massive, unlike the photon mediating electromagnetism through the vector potential A_μ that reaches out to infinity. In the SM, it is the Higgs mechanism [1–3,37] that gives the weak gauge bosons masses by breaking chiral symmetry, whereby the related Higgs field also has a non-vanishing vacuum expectation value. In the present case, the above vector boson masses are given by

$$M_{\rm A} = 0, \quad M_{\rm R} = {\rm v}\sqrt{\frac{3}{2}(g^2 + (g')^2)}, \quad M_{\rm V} = \frac{1}{2}{\rm v}g. \tag{49}$$

Consequently, the masses are entirely determined by the two coupling constants. Apparently, the R boson mass is greater than the V mass. For $g = g'$, we obtain $M_{\rm R} = 2\sqrt{3} M_{\rm V}$, i.e., the R boson mass then is 3.46 times larger than the V boson mass. These results mean that the field B_μ acting solely on the lepton becomes massive, whereas the electro-strong field A_μ acting on the three quarks remains massless, and thus its quanta propagate like photons at the speed of light. In our opinion, there is no good physical reason to assume that g and g' differ. So, we shall assume from now on that they are equal. Then, the vector boson of the gauge field B_μ is heavy, with the mass $M_{\rm R} = \sqrt{3}{\rm v}g$.

In the SM, the coupling constants are well known experimentally from QED and QCD, and are applied to interpret the particle accelerator and scattering results [38,39]. The electromagnetic fine structure constant is defined as $\alpha_e = e^2/(4\pi)$, and has the value $\alpha_e = 1/137 = 0.0073$. The "strong fine-structure" constant is defined as $\alpha_s = g^2/(4\pi)$, and is experimentally known to be running between about 0.1 at 200 GeV, and attains the value of about 0.5 at 1 GeV. In a simple toy model in Appendix A, we show that only a coupling constant greater unity may lead to bound states in a Yukawa potential [14].

How great then is the mass of the vector bosons that transfer the electro-strong charge from the quarks to the lepton and vice versa? This depends on the strong Higgs vacuum v, which remains a purely theoretical concept at this point. However, for an estimation, we may assume that v about equals the value of the electroweak Higgs, which gives v = (247–251) GeV [3]. Let us further assume then that the mass is equal to that of the electroweak W, which is $M_{\rm W} = 80.4\,{\rm GeV}$. Then, we can simply infer after (49) that the strong coupling constant is $g = g_w$, with $g_w = e/\sin\theta_{\rm W} = 0.64e_0$ and $e = 0.303e_0$, whereby e_0 is the elementary electrostatic charge unit. The Weinberg angle is $\sin\theta = 0.472$ [3], as determined experimentally. Under our above assumptions, the vector boson associated with the gauge field B_μ is rather heavy, and its mass $M_{\rm R}$ is in energy units equal to 278.18 GeV, which is well in reach for the CERN accelerator.

In conclusion, we obtain the rounded value $g = 0.64e_0$ for the strong coupling constant, and thus, under the above assumption, the electro-strong charge is equal to that of the electroweak interaction. From the measured value of $\alpha_s = 0.12$ at 80 GeV [3], one obtains $g = \sqrt{4\pi\alpha_s} = 1.2e_0$, which is about two times the above-estimated value. The V bosons in the present model exchange charges of value $\pm\frac{4}{3}$ between the quarks and the lepton.

Under the equal mass assumption, the strong (V) and weak (W) charge-changing vector bosons could not be easily experimentally distinguished, because one would have to determine, in addition to the charge, whether they carry the color charge or weak charge. It should be stressed, though, that the equal mass assumption does not seem to be that unreasonable. Marsch and Narita [16] have already applied it with some success to the electroweak $SU(2)$ symmetry breaking within their $SU(8)$ unification model.

6. Results and Conclusions

The main results of this paper are the following. The starting point of our derivations was the second Casimir operator C_2 of the Lorentz group, which links in a multiplicative way the Lorentz invariant squared four-momentum with the intrinsic spin squared (a rotational invariant) of any massive particle. For the fermion spin of one-half, the standard Dirac equation can readily be derived from C_2. When using the vector representation of the Lorentz group, with the chiral spin operators (2) and (3) acting on complex four-component Minkowski spinors [17], one can extend the Dirac equation to include also the chiral symmetry involving the $SU(2)$ group. This subject was dealt with extensively in references [19,20].

In order to also obtain the $SU(3)$ symmetry group of the strong interactions as in the SM, we introduced the hadronic isospin, the helicity of which leads naturally to the unifying $SU(4)$ symmetry group that contains $SU(3)$ as a subgroup. Application of the usual methods of the SM to break the $SU(4)$ symmetry by means of the (what we called for the present purpose hadronic) Higgs mechanism then yields the Quantum Chromodynamics (QCD) for the three quarks and the $U(1)$ symmetry group for the resulting single lepton. In addition, one obtains three "leptoquarks", i.e., massive vector bosons the masses of which we estimated roughly to be of the order of the W of the SM electroweak interactions. They can, by charge exchange, transform quarks into leptons and vice versa. As compared to previous unification models proposed in the cited literature, the present model makes a minimum number of assumptions, is quite transparent, and is based on the new concept of isospin helicity which, for the isospin 1/2, hints directly to the importance of $SU(4)$ for unification.

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Appendix A. On the Strong Coupling Constant

Here, we shall present another derivation of the coupling constant g for the hadron interaction. In the main text, we just assumed that it may be given by the QCD coupling constant at the energy of 1 GeV, which yields the "strong fine-structure" constant $\alpha_s = 0.12$, and consequently a strong coupling constant of $g = 1.23e_0$. Here, we consider a simple Yukawa-type shielded but attractive potential $U(r)$, with a shielding length λ_M determined by the mass M of the involved boson and with the coupling constant G. The potential reads

$$U(r) = -\frac{G^2 \exp(-r/\lambda_M)}{4\pi\epsilon_0 r}. \tag{A1}$$

Here, ϵ_0 is the vacuum permittivity or dielectric constant. In adequate units for quantum field theory, the potential has to be divided by $\hbar c$. For electromagnetism, the fine structure constant has been measured, and it is of course known. We define the constant α_G in analogy to the fine-structure constant α_e for electrodynamics with $g = e_0$. In electrostatic cgs units, we have $4\pi\epsilon_0 = 1$, and thus $\alpha_e = e_0^2/(\hbar c)$. In the natural units as used in high-energy-particle (hep) physics $c = \hbar = \epsilon_0 = 1$, we obtain $\alpha_e = e_0^2/(4\pi)$.

For the hadronic interaction mediated by an extremely short-lived vector boson considered here, the coupling constant has not directly been measured, but it must be assumed in reasonable accord with indirect empirical evidence, like obtained from hadron scattering [38]. However, we may instead consider the simple energy argument, saying that G may be determined by equating the Yukawa potential at the Compton wave length of the boson $\lambda_M = \hbar/(Mc)$ with its energy at rest. Thus, we obtain with $x = r/\lambda_M$, and the toy potential

$$U(x) = -\alpha_G \frac{\exp(-x)}{x} Mc^2. \tag{A2}$$

The total, kinetic and potential energy of a relativistic particle with momentum $p = c\hbar/r$ moving in such an attractive potential is given by

$$\frac{E(x)}{Mc^2} = -\alpha_G \frac{\exp(-x)}{x} + \frac{1}{x} = \frac{1}{x} f(x), \tag{A3}$$

with the function $f(x) = 1 - \alpha_G \exp(-x)$. To obtain a bound state at some x, the value of $f(x)$ must be negative, which means that the potential term dominates. But, for $x > 0$, the exponential term becomes smaller, and thus α_G must be larger than unity to obtain a bound state at a finite distance from the singular point at $x = 0$. It is intuitively clear that to excite the trapped particle to a higher energy state, it must absorb an energy amount of the order of Mc^2, i.e., the energy of the boson providing the potential.

The extremum of the energy is attained at the location x_0, where $E'(x_0) = dE(x_0)/dx_0 = 0$. As the coupling constant is positive, the resulting maximum of $E(x_0)$ is determined by the condition

$$\alpha_G(x_0) = \frac{\exp(x_0)}{1 + x_0}. \tag{A4}$$

See the graphical representation in Figure A1. Then, a relativistic particle can be trapped in the potential well at $x < x_0$. For example, $x_0 = 0$ for $\alpha_G(x_0) = 1$, or $x_0 = 2$ for $\alpha_G(x_0) = 2.46$. The relation (A4) turns out to be independent of the boson mass M. Then, one can solve for the coupling constant, giving for example for $x_0 = 2$ the value $G = 5.56e_0$. In that case, the hadronic coupling constant is about four times that of the strong one and eighteen times that of the electromagnetic one! That appears to be an unrealistic result, owing to the simplicity of the assumed model. However, it illustrates that a short-range potential must be rather deep to bind a relativistic particle.

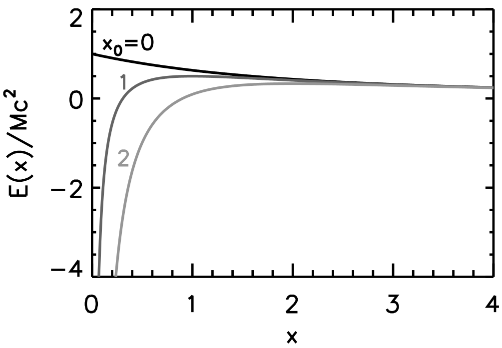

Figure A1. Graphical representation of the function $E(x)/Mc^2$ in Equation (A3) for $x_0 = 0$ (in black), $x_0 = 1$ (in dark gray), and $x_0 = 2$ (in light gray).

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