

## Review

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# Canonical Construction of Invariant Differential Operators: A Review

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**Abstract:** In the present paper, we review the progress of the project of the classification and construction of invariant differential operators for non-compact, semisimple Lie groups. Our starting point is the class of algebras which we called earlier ‘conformal Lie algebras’ (CLA), which have very similar properties to the conformal algebras of Minkowski space-time, though our aim is to go beyond this class in a natural way. For this purpose, we introduced recently the new notion of a *parabolic relation* between two non-compact, semi-simple Lie algebras  $\mathcal{G}$  and  $\mathcal{G}'$  that have the same complexification and possess maximal parabolic subalgebras with the same complexification.

**Keywords:** invariant differential operators; non-compact semisimple Lie groups; conformal Lie algebras

## 1. Introduction and Preliminaries

*Invariant differential operators* play a very important role in the description of physical symmetries—starting from the early occurrences in the Maxwell, d’Allembert and Dirac, equations to the latest applications of (super)differential operators in conformal field theory, supergravity and string theory (for reviews, cf., e.g., [1,2]). Thus, it is important for applications in physics to systematically study such operators. For more relevant references cf., e.g., [3–73], and others throughout the text. Especially, we would like to point out the book [74] which contains a section devoted to groups of conformal transformations of curved spacetime.

In a recent paper [75], we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the *parabolic subgroups and subalgebras* from which the necessary representations are induced. Thus, we have set the stage for the study of different non-compact groups. Up to 2016, relevant references may be found in our monograph [76] and also in [77–111].

Our canonical construction is applicable also to *quantum groups, super groups, to (super-)Virasoro and Kac-Moody algebras*, see our monographs: [112–114].

### Preliminaries

Let  $G$  be a semi-simple, non-compact Lie group, and  $K$  a maximal compact subgroup of  $G$ . Then, we have an *Iwasawa decomposition*  $G = KA_0N_0$ , where  $A_0$  is an Abelian simply connected vector subgroup of  $G$  and  $N_0$  is a nilpotent simply connected subgroup of  $G$  preserved by the action of  $A_0$ . Furthermore, let  $M_0$  be the centralizer of  $A_0$  in  $K$ . Then, the subgroup  $P_0 = M_0A_0N_0$  is a *minimal parabolic subgroup* of  $G$ . A *parabolic subgroup*  $P = M'A'N'$  is any subgroup of  $G$  which contains a minimal parabolic subgroup.

Furthermore let  $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$  denote the Lie algebras of  $G, K, P, M, A, N$ , resp.

For our purposes, we shall be restrict *maximal parabolic subgroups*  $P = MAN$ , i.e.,  $\text{rank } A = 1$ , resp., to *maximal parabolic subalgebras*  $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$  with  $\dim \mathcal{A} = 1$ .

Let  $\nu$  be a (non-unitary) character of  $A$ ,  $\nu \in \mathcal{A}^*$ , parameterized by a real number  $d$ , called the *conformal weight* or energy.

Furthermore, let  $\mu$  fix a discrete series representation  $D^\mu$  of  $M$  on the Hilbert space  $V_\mu$ , or the finite-dimensional (non-unitary) representation of  $M$  with the same Casimirs.



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We call the induced representation  $\chi = \text{Ind}_P^G(\mu \otimes \nu \otimes 1)$  an *elementary representation* of  $G$  [24]. (These are called *generalized principal series representations* (or *limits thereof*) in [115].) Their spaces of functions are

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g)\} \quad (1)$$

where  $a = \exp(H) \in A'$ ,  $H \in \mathcal{A}'$ ,  $m \in M'$ ,  $n \in N'$ . The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \quad (2)$$

- An important ingredient in our considerations are the *highest/lowest-weight representations* of  $\mathcal{G}^\mathbb{C}$ . These can be realized as (factor-modules of) Verma modules  $V^\Lambda$  over  $\mathcal{G}^\mathbb{C}$ , where  $\Lambda \in (\mathcal{H}^\mathbb{C})^*$ ,  $\mathcal{H}^\mathbb{C}$  is a Cartan subalgebra of  $\mathcal{G}^\mathbb{C}$  and weight  $\Lambda = \Lambda(\chi)$  is determined uniquely from  $\chi$  [76].

Actually, since our ERs may be induced from finite-dimensional representations of  $\mathcal{M}$  (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules*  $\tilde{V}^\Lambda$  such that the role of the highest/lowest-weight vector  $v_0$  is taken by the (finite-dimensional) space  $V_\mu v_0$ . For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight  $d$ . Relatedly, for the intertwining differential operators, only the reducibility with regard to non-compact roots is essential.

- Another main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [76]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The explicit parameterization of the multiplets and of their ERs is important in understanding of the situation. The notion of multiplets was introduced in [116] and applied to representations of  $SO_o(p, q)$  and  $SU(2, 2)$ , resp., induced from their minimal parabolic subalgebras. Then it was applied to the conformal superalgebra [117], to infinite-dimensional (super)algebras [113] and to quantum groups [112]. (For other applications, we refer to [114].)

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consist of the pair  $(\beta, m)$ , where  $\beta$  is a (non-compact) positive root of  $\mathcal{G}^\mathbb{C}$ ,  $m \in \mathbb{N}$ , such that the *BGG Verma module reducibility condition* (for highest-weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee \equiv 2\beta/(\beta, \beta) \quad (3)$$

where  $\rho$  is half the sum of the positive roots of  $\mathcal{G}^\mathbb{C}$ . When the above holds, then the Verma module with shifted weight  $V^{\Lambda - m\beta}$  (or  $\tilde{V}^{\Lambda - m\beta}$  for GVM and  $\beta$  non-compact) is embedded in the Verma module  $V^\Lambda$  (or  $\tilde{V}^\Lambda$ ). This embedding is realized by a singular vector  $v_s$  determined by a polynomial  $\mathcal{P}_{m, \beta}(\mathcal{G}^-)$  in the universal enveloping algebra  $(U(\mathcal{G}_-)) v_0$ , and  $\mathcal{G}^-$  is the subalgebra of  $\mathcal{G}^\mathbb{C}$  generated by the negative root generators [118]. More explicitly, [76,119],  $v_{m, \beta}^s = \mathcal{P}_{m, \beta} v_0$  (or  $v_{m, \beta}^s = \mathcal{P}_{m, \beta} V_\mu v_0$  for GVMs). Then, there exists [76,119] an **intertwining differential operator**

$$\mathcal{D}_{m, \beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \quad (4)$$

given explicitly by:

$$\mathcal{D}_{m, \beta} = \mathcal{P}_{m, \beta}(\widehat{\mathcal{G}^-}) \quad (5)$$

where  $\widehat{\mathcal{G}^-}$  denotes the *right action* on the functions  $\mathcal{F}$ .

In most of these situations, the invariant operator  $\mathcal{D}_{m,\beta}$  has a non-trivial invariant kernel in which a subrepresentation of  $\mathcal{G}$  is realized. Thus, studying the equations with trivial RHS is also very important:

$$\mathcal{D}_{m,\beta} f = 0, \quad f \in \mathcal{C}_{\chi(\Lambda)}, \quad (6)$$

For example, in many physical applications, in the case of first-order differential operators, i.e., for  $m = m_\beta = 1$ , these equations are called *conservation laws*, and the elements  $f \in \ker \mathcal{D}_{m,\beta}$  are called *conserved currents*.

The above construction also works for the *subsingular vectors*  $v_{ssv}$  of Verma modules. Such vectors are also expressed by a polynomial  $\mathcal{P}_{ssv}(\mathcal{G}^-)$  in the universal enveloping algebra:  $v_{ssv}^s = \mathcal{P}_{ssv}(\mathcal{G}^-) v_0$ , cf. [120]. Thus, there exists a *conditionally invariant differential operator* given explicitly by  $\mathcal{D}_{ssv} = \mathcal{P}_{ssv}(\widehat{\mathcal{G}^-})$ , and a *conditionally invariant differential equation*; for many more details, see [121]. (Note that these operators (equations) are not of the first order.)

In our exposition below, we shall use the so-called Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee), \quad i = 1, \dots, n, \quad (7)$$

where  $\Lambda = \Lambda(\chi)$ ,  $\rho$  is half the sum of the positive roots of  $\mathcal{G}^\mathbb{C}$ .

We shall use also the so-called Harish–Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta), \quad (8)$$

where  $\beta$  is any positive root of  $\mathcal{G}^\mathbb{C}$ . These parameters are redundant, since they are expressed in terms of the Dynkin labels; however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish–Chandra parameters have their origin in the BGG reducibility condition (3).)

Finally, we shall introduce the notion of ‘parabolically related non-compact semisimple Lie algebras’ [122]. This notion is not part of our procedure for constructing invariant differential operators, but just a tool to extend the construction from one Lie algebra to another.

**Definition 1.** Let  $\mathcal{G}, \mathcal{G}'$  be two non-compact semi-simple Lie algebras with the same complexification  $\mathcal{G}^\mathbb{C} \cong \mathcal{G}'^\mathbb{C}$ . We call them **parabolically related** if they have parabolic subalgebras  $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ,  $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$ , such that  $\mathcal{M}^\mathbb{C} \cong \mathcal{M}'^\mathbb{C}$  ( $\Rightarrow \mathcal{P}^\mathbb{C} \cong \mathcal{P}'^\mathbb{C}$ ).  $\diamond$

Certainly, there are many such parabolic relationships for any given algebra  $\mathcal{G}$ . Furthermore, two algebras  $\mathcal{G}, \mathcal{G}'$  may be parabolically related via different parabolic subalgebras.

The paper is organized as follows. In Section 2, we consider the case of the pseudo-orthogonal algebras  $so(p, q)$  which are parabolically related to the conformal algebra  $so(n, 2)$  for  $p + q = n + 2$ . In Section 3, we consider the CLA  $su(n, n)$  and the parabolically related  $sl(2n, \mathbb{R})$ , and for  $n = 2k$ :  $su^*(4k)$ . In Section 4, we consider the CLA  $sp(n)$  and—for  $n = 2r$ —the parabolically related  $sp(r, r)$ . In Section 5, we consider the algebras  $so^*(2n)$  (which are CLA when  $n$  is even) and the parabolically related algebras. In Section 6, we consider the CLA  $E_{7(-25)}$  and the parabolically related  $E_{7(7)}$ . In Section 7, we consider the hermitian symmetric case  $E_{6(-14)}$  and the parabolically related  $E_{6(6)}$  and  $E_{6(2)}$ . In Section 8, we consider the algebra  $F_4$  and its real forms  $F'_4$  and  $F''_4$ . In Section 9, we consider the algebra  $G_{2(2)}$ .

We would also like to list some more recent relevant references [123–229].

## 2. Conformal Algebras $so(n, 2)$ and Parabolically Related Algebras

The most widely used algebras are the *conformal algebras*  $so(n, 2)$  in  $n$ -dimensional Minkowski space-time. In that case, there is a maximal *Bruhat decomposition* [230] that has direct physical meaning:

$$so(n, 2) = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}, \quad (9)$$

$$\mathcal{M} = so(n-1, 1), \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n$$

where  $so(n-1, 1)$  is the *Lorentz algebra* of  $n$ -dimensional Minkowski space-time, the sub-algebra  $\mathcal{A} = so(1, 1)$  represents the *dilatations* and the conjugated subalgebras  $\mathcal{N}, \tilde{\mathcal{N}}$  are the algebras of *translations* and *special conformal transformations*, both being isomorphic to  $n$ -dimensional Minkowski space-time.

Another physically important feature is that the algebras  $so(n, 2)$  have discrete series representations. We recall that by the Harish-Chandra criterion [231], these are groups where the following holds:

$$\text{rank } G = \text{rank } K,$$

where  $K$  is the *maximal compact subgroup* of the non-compact group  $G$ .

Furthermore, the algebras  $so(n, 2)$  belong to the class of *Hermitian symmetric spaces*. The practical criterion is that in these cases, the *maximal compact subalgebra*  $\mathcal{K}$  is of the form:

$$\mathcal{K} = so(2) \oplus \mathcal{K}' \quad (10)$$

The Lie algebras from this class are as follows:

$$so(n, 2), \quad sp(n, R), \quad su(m, n), \quad so^*(2n), \quad E_{6(-14)}, \quad E_{7(-25)} \quad (11)$$

These groups/algebras have *highest/lowest-weight representations*, and relatedly, *holomorphic* discrete series representations.

We label the signature of the ERs of  $\mathcal{G}$  as follows:

$$\begin{aligned} \chi = \{ & n_1, \dots, n_{\tilde{h}}; c \}, \quad n_j \in \mathbb{Z}/2, \quad c = d - \frac{n}{2}, \quad \tilde{h} \equiv [\frac{n}{2}], \\ & |n_1| < n_2 < \dots < n_{\tilde{h}}, \quad n \text{ even}, \\ & 0 < n_1 < n_2 < \dots < n_{\tilde{h}}, \quad n \text{ odd}, \end{aligned} \quad (12)$$

where the last entry of  $\chi$  labels the characters of  $\mathcal{A}$ , and the first  $\tilde{h}$  entries are labels of the finite-dimensional nonunitary irreps of  $\mathcal{M} \cong so(n-1, 1)$ .

The reason to use the parameter  $c$  instead of  $d$  is that the parametrization of the ERs in the multiplets is given in a simple intuitive way (cf. [232]):

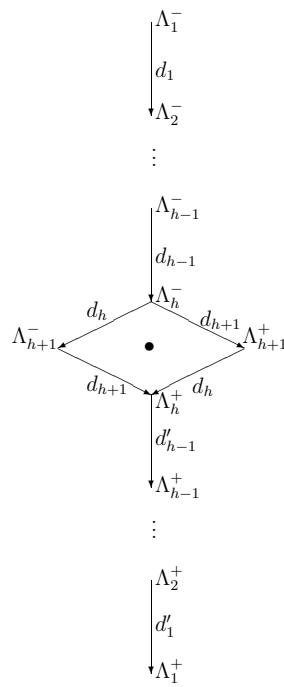
$$\begin{aligned} \chi_1^\pm &= \{ \epsilon n_1, \dots, n_{\tilde{h}}; \pm n_{\tilde{h}+1} \}, \quad n_{\tilde{h}} < n_{\tilde{h}+1}, \\ \chi_2^\pm &= \{ \epsilon n_1, \dots, n_{\tilde{h}-1}, n_{\tilde{h}+1}; \pm n_{\tilde{h}} \} \\ \chi_3^\pm &= \{ \epsilon n_1, \dots, n_{\tilde{h}-2}, n_{\tilde{h}}, n_{\tilde{h}+1}; \pm n_{\tilde{h}-1} \} \\ &\dots \\ \chi_{\tilde{h}}^\pm &= \{ \epsilon n_1, n_3, \dots, n_{\tilde{h}}, n_{\tilde{h}+1}; \pm n_2 \} \\ \chi_{\tilde{h}+1}^\pm &= \{ \epsilon n_2, \dots, n_{\tilde{h}}, n_{\tilde{h}+1}; \pm n_1 \} \\ \epsilon &= \begin{cases} \pm, & n \text{ even} \\ 1, & n \text{ odd} \end{cases} \end{aligned} \quad (13)$$

Furthermore, we denote by  $\tilde{\mathcal{C}}_i^\pm$  the representation space with signature  $\chi_i^\pm$ . The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^\mathbb{C}, \mathcal{H}^\mathbb{C})| / |W(\mathcal{M}^\mathbb{C}, \mathcal{H}_m^\mathbb{C})| = 2(1 + \tilde{h}) \quad (14)$$

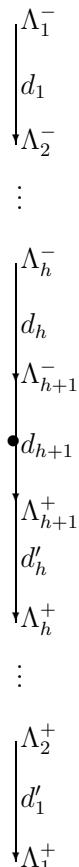
where  $\mathcal{H}^\mathbb{C}, \mathcal{H}_m^\mathbb{C}$  are Cartan subalgebras of  $\mathcal{G}^\mathbb{C}, \mathcal{M}^\mathbb{C}$ , resp. This formula is valid for the main multiplets of all conformal Lie algebras.

Now, in Figure 1, we show the general *even* case  $so(p, q)$ ,  $p + q = 2h + 2$ -even [76,232].



**Figure 1.** Diagram for the cases  $so(p, q)$ ,  $p + q = 2h + 2$ , even, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

In Figure 2, we show the general odd case  $so(p, q)$ ,  $p + q = 2h + 3$ -odd [76,232].



**Figure 2.** The cases  $so(p, q)$ ,  $p + q = 2h + 3$ , showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

The ERs in the multiplet are related by *intertwining integral and differential operators*. The *integral operators* were introduced by Knapp and Stein [233]. They correspond to elements of the restricted Weyl group of  $\mathcal{G}$ . These operators intertwine the pairs  $\tilde{\mathcal{C}}_i^\pm$

$$G_i^\pm : \tilde{\mathcal{C}}_i^\mp \longrightarrow \tilde{\mathcal{C}}_i^\pm, \quad i = 1, \dots, 1 + \tilde{h} \quad (15)$$

The *intertwining differential operators* correspond to non-compact positive roots of the root system of  $so(n+2, \mathbb{C})$ , cf. [76]. (In the current context, compact roots of  $so(n+2, \mathbb{C})$  are those that are roots also of the subalgebra  $so(n, \mathbb{C})$ , the rest of the roots are non-compact.) The degrees of these intertwining differential operators are given just by the differences of the  $c$  entries [76]:

$$\begin{aligned} \deg d_i &= \deg d'_i = n_{\tilde{h}+2-i} - n_{\tilde{h}+1-i}, & i = 1, \dots, \tilde{h}, \quad \forall n \\ \deg d_{\tilde{h}+1} &= n_2 + n_1, \quad n \text{ even} \end{aligned} \quad (16)$$

where  $d'_h$  is omitted from the first line for  $(p+q)$  even.

Matters are arranged so that in every multiplet only the ER with signature  $\chi_1^-$  contains a *finite-dimensional nonunitary subrepresentation* in a subspace  $\mathcal{E}$ . The latter corresponds to the finite-dimensional unitary irrep of  $so(n+2)$  with signature  $\{n_1, \dots, n_{\tilde{h}}, n_{\tilde{h}+1}\}$ . The subspace  $\mathcal{E}$  is annihilated by the operator  $G_1^+$ , and is the image of the operator  $G_1^-$ .

- *Interlude:*

We mention one more special feature of  $so(n, 2)$ , namely that the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^\mathbb{C} = so(n, \mathbb{C}) \oplus so(2, \mathbb{C}) \cong so(n-1, 1)^\mathbb{C} \oplus so(1, 1)^\mathbb{C} = \mathcal{M}^\mathbb{C} \oplus \mathcal{A}^\mathbb{C}. \quad (17)$$

The coincidence of the complexification of the semi-simple subalgebras

$$\mathcal{K}'^\mathbb{C} = \mathcal{M}^\mathbb{C} \quad (18)$$

means that the sets of finite-dimensional (nonunitary) representations of  $\mathcal{M}$  are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of  $\mathcal{K}'$ . The latter leads to the fact that the corresponding induced representations are representations of finite  $\mathcal{K}$ -type [234].

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of  $so(n, 2)$ . This subclass consists of

$$so(n, 2), \ sp(n, \mathbb{R}), \ su(n, n), \ so^*(4n), \ E_{7(-25)} \quad (19)$$

with the corresponding analogs of Minkowski space-time  $V$  being

$$\mathbb{R}^{n-1, 1}, \ Sym(n, \mathbb{R}), \ Herm(n, \mathbb{C}), \ Herm(n, \mathbb{Q}), \ Herm(3, \mathbb{O}) \quad (20)$$

In view of applications to physics, we proposed to call these algebras '*conformal Lie algebras*' (or groups).

We summarize the algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling (18) in Table 1 below. Also, some non-CLAs are included.

There,  $sl(n, \mathbb{C})_{\mathbb{R}}$  denotes  $sl(n, \mathbb{C})$  as a real Lie algebra (thus,  $(sl(n, \mathbb{C})_{\mathbb{R}})^\mathbb{C} = sl(n, \mathbb{C}) \oplus sl(n, \mathbb{C})$ );  $e_6$  denotes the compact real form of  $E_6$ ; and we have imposed restrictions to avoid coincidences or degeneracies due to well-known isomorphisms:  $so(1, 2) \cong sp(1, \mathbb{R}) \cong su(1, 1)$ ,  $so(2, 2) \cong so(1, 2) \oplus so(1, 2)$ ,  $su(2, 2) \cong so(4, 2)$ ,  $sp(2, \mathbb{R}) \cong so(3, 2)$ ,  $sp(1, 1) \cong so(4, 1)$ ,  $so^*(4) \cong so(3) \oplus so(2, 1)$ ,  $so^*(8) \cong so(6, 2)$ .

**Table 1.** Table of conformal Lie algebras (CLA)  $\mathcal{G}$  with  $\mathcal{M}$ -factor fulfilling (18) and the corresponding parabolically related algebras  $\mathcal{G}'$ ; we display also some non-CLA cases.

$\mathcal{G}$	$\mathcal{K}$	$\mathcal{M}$ $\dim V$	$\mathcal{G}'$	$\mathcal{M}'$
$so(n, 2)$ $n \geq 3$	$so(n) \oplus so(2)$	$so(n-1, 1)$ $n$	$so(p, q),$ $p+q = n+2$	$so(p-1, q-1)$
$su(n, n)$ $n \geq 3$	$u(n) \oplus su(n)$	$sl(n, \mathbb{C})_{\mathbb{R}}$	$sl(2n, \mathbb{R})$	$sl(n, \mathbb{R}) \oplus$ $sl(n, \mathbb{R})$
			$n^2$	$su^*(2n), n = 2k$
				$su^*(2k) \oplus$ $su^*(2k)$
$sp(n, \mathbb{R})$ $\text{rank } n \geq 3$	$u(n)$	$sl(n, \mathbb{R})$	$sp(r, r), n = 2r$	$su^*(2r), n = 2r$
			$n(n+1)/2$	
$so^*(4n)$ $n \geq 3$	$u(2n)$	$su^*(2n)$	$so(2n, 2n)$	$sl(2n, \mathbb{R})$
			$n(2n-1)$	
$E_{7(-25)}$	$e_6 \oplus so(2)$	$E_{6(-26)}$ $27$	$E_{7(7)}$	$E_{6(6)}$
below not CLA				
$so^*(10)$	$u(5)$	$su(3, 1) \oplus su(2)$	$so(5, 5)$	$sl(4, \mathbb{R}) \oplus$ $sl(2, \mathbb{R})$
			$13$	
$E_{6(-14)}$	$so(10) \oplus so(2)$	$su(5, 1)$ $21$	$E_{6(6)}$ $E_{6(2)}$	$sl(6, \mathbb{R})$ $su(3, 3)$
$F'_4$	$sp(3) \oplus su(2)$	$sl(3, \mathbb{R}) \oplus$ $sl(2, \mathbb{R})$ $20$		
$F''_4$	$so(9)$	$so(7)$ $15$		
$G_{2(2)}$	$su(2) \oplus su(2)$	0 min. $sl(2, \mathbb{R})$ max. 6 min. 5 max.		

Although the diagram in Figure 1 is valid for arbitrary  $so(p, q)$  (even  $p+q > 5$ ) due to the parabolic relatedness, the contents are very different. (The same remark holds for the diagram in Figure 2 valid for  $so(p, q)$  for odd  $p+q \leq 5$ .) We comment only on the ER with signature  $\chi_1^+$ . In all cases, it contains a UIR of  $so(p, q)$  realized on an invariant subspace  $\mathcal{D}$  of the ER  $\chi_1^+$ . That subspace is annihilated by the operator  $G_1^-$ , and is the image of the operator  $G_1^+$ . (Other ERs contain more UIRs.)

If  $pq \in 2\mathbb{N}$ , the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if  $q = 2$ , the invariant subspace  $\mathcal{D}$  is the direct sum of two subspaces  $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$ , in which a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp., are realized. Note that the corresponding *lowest-weight GVM* is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate *highest-weight GVM* is infinitesimally equivalent to the anti-holomorphic discrete series.

Note that  $\deg d_i, \deg d'_i$  are Harish-Chandra parameters corresponding to the non-compact positive roots of  $so(n+2, \mathbb{C})$ . From these, only  $\deg d_1$  corresponds to a simple root; i.e., it is a Dynkin label.

Above, we considered  $so(n, 2)$  for  $n > 2$ . The case  $n = 2$  is reduced to  $n = 1$  since  $so(2, 2) \cong so(1, 2) \oplus so(1, 2)$ . The case  $so(1, 2)$  is special and must be treated separately. But, in fact, it is contained in what we presented already. In that case, the multiplets contain only *two* ERs which may be depicted by the *top pair*  $\chi_1^\pm$  in the pictures that we presented. And they have the properties that we described for  $so(n, 2)$  with  $n > 2$ . The case  $so(1, 2)$  was given already in 1946-7 independently by Gel'fand et al. [235] and Bargmann [236].

### 3. The Lie Algebra $su(n, n)$ and Parabolically Related Algebras

Let  $\mathcal{G} = su(n, n)$ ,  $n \geq 2$ . The maximal compact subgroup is  $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$ , while  $\mathcal{M} = sl(n, \mathbb{C})_{\mathbb{R}}$ . The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = \binom{2n}{n}$$

The signature of the ERs of  $\mathcal{G}$  is

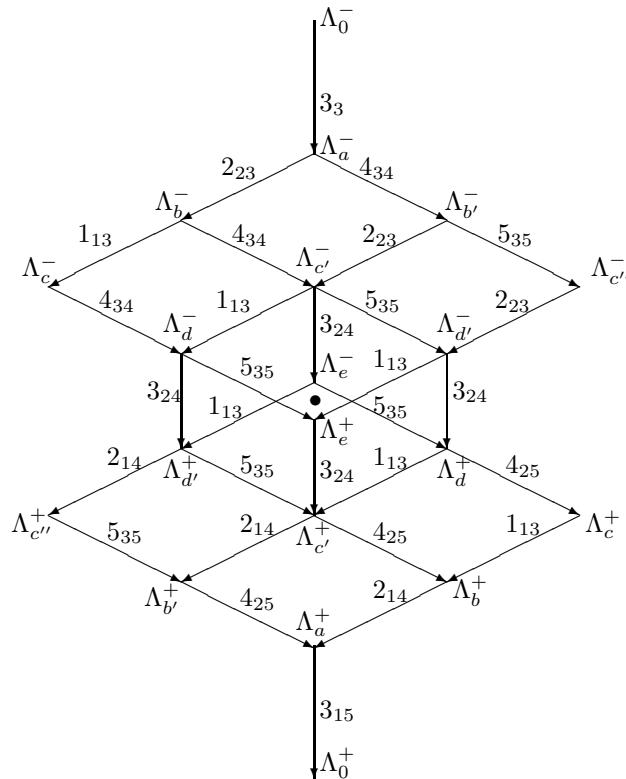
$$\chi = \{n_1, \dots, n_{n-1}, n_{n+1}, \dots, n_{2n-1}; c\}, \quad n_j \in \mathbb{N}, \quad c = d - n$$

The Knapp–Stein restricted Weyl reflection is given by

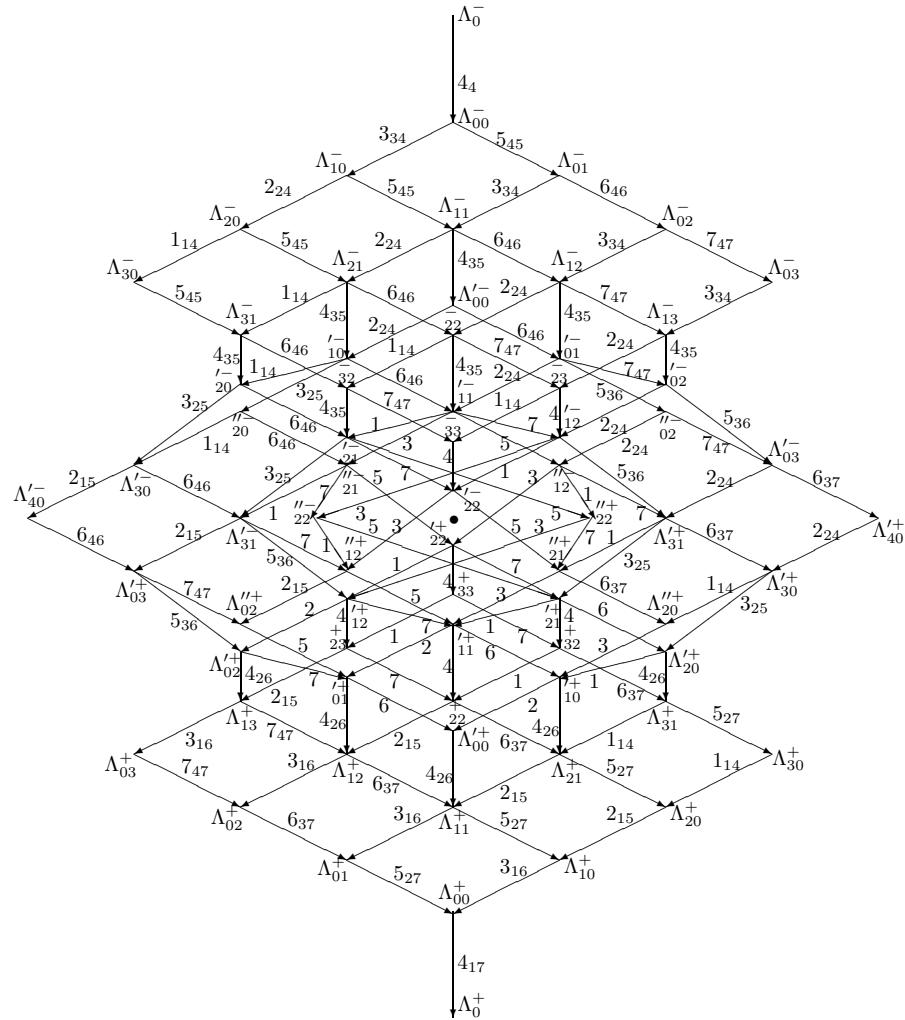
$$G_{KS} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}, \quad \chi' = \{(n_1, \dots, n_{n-1}, n_{n+1}, \dots, n_{2n-1})^*; -c\}$$

$$(n_1, \dots, n_{n-1}, n_{n+1}, \dots, n_{2n-1})^* \doteq (n_{n+1}, \dots, n_{2n-1}, n_1, \dots, n_{n-1})$$

Below, in Figures 3 and 4, we give the diagrams for  $su(n, n)$  for  $n = 3, 4$  [237]. (The case  $n = 2$  is already considered since  $su(2, 2) \cong so(4, 2)$ .) These are also diagrams for the parabolically related  $sl(2n, \mathbb{R})$ , and for  $n = 2k$ , these are also diagrams for the parabolically related  $su^*(4k)$  [122].



We use the following conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol  $i_{j\dots k}$  encoding the root  $\beta_{j\dots k}$  and the number  $m_{\beta_{j\dots k}}$  which is involved in the BGG criterion.



**Figure 4.** Pseudo-unitary symmetry in 16-dimensional space. By parabolic relation the  $su(4, 4)$  diagram above is valid also for  $sl(8, R)$  and  $su^*(8)$ .

#### 4. The Lie Algebras $sp(n, \mathbb{R})$ and $sp(\frac{n}{2}, \frac{n}{2})$ ( $n$ -even)

Let  $n \geq 2$ . Let  $\mathcal{G} = sp(n, \mathbb{R})$  be the split real form of  $sp(n, \mathbb{C}) = \mathcal{G}^{\mathbb{C}}$ . The maximal compact subgroup is  $\mathcal{K} \cong u(1) \oplus su(n)$ , while  $\mathcal{M} = sl(n, \mathbb{R})$ . The number of ERs in the corresponding multiplets is

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = 2^n$$

The signature of the ERs of  $\mathcal{G}$  is

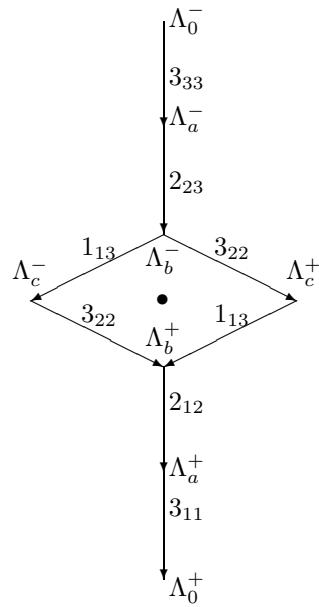
$$\chi = \{n_1, \dots, n_{n-1}; c\}, \quad n_j \in \mathbb{N},$$

The Knapp–Stein Weyl reflection acts as follows:

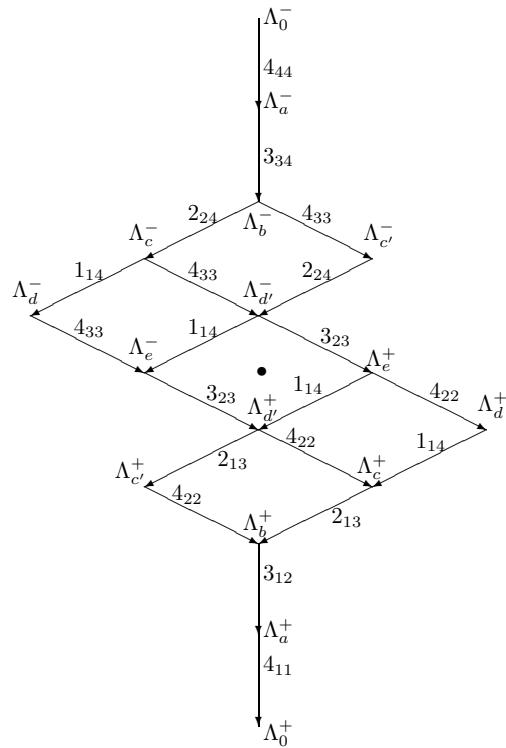
$$\begin{aligned} G_{KS} : \mathcal{C}_{\chi} &\longrightarrow \mathcal{C}_{\chi'}, \chi' = \{(n_1, \dots, n_{n-1})^*; -c\}, \\ (n_1, \dots, n_{n-1})^* &\doteq (n_{n-1}, \dots, n_1) \end{aligned}$$

Below, in Figures 5–8, we give pictorially the multiplets for  $sp(n, \mathbb{R})$  for  $n = 3, 4, 5, 6$  [238]. (The case  $n = 2$  is already considered since  $sp(2, \mathbb{R}) \cong so(3, 2)$ .) For  $n = 2r$ , these are also

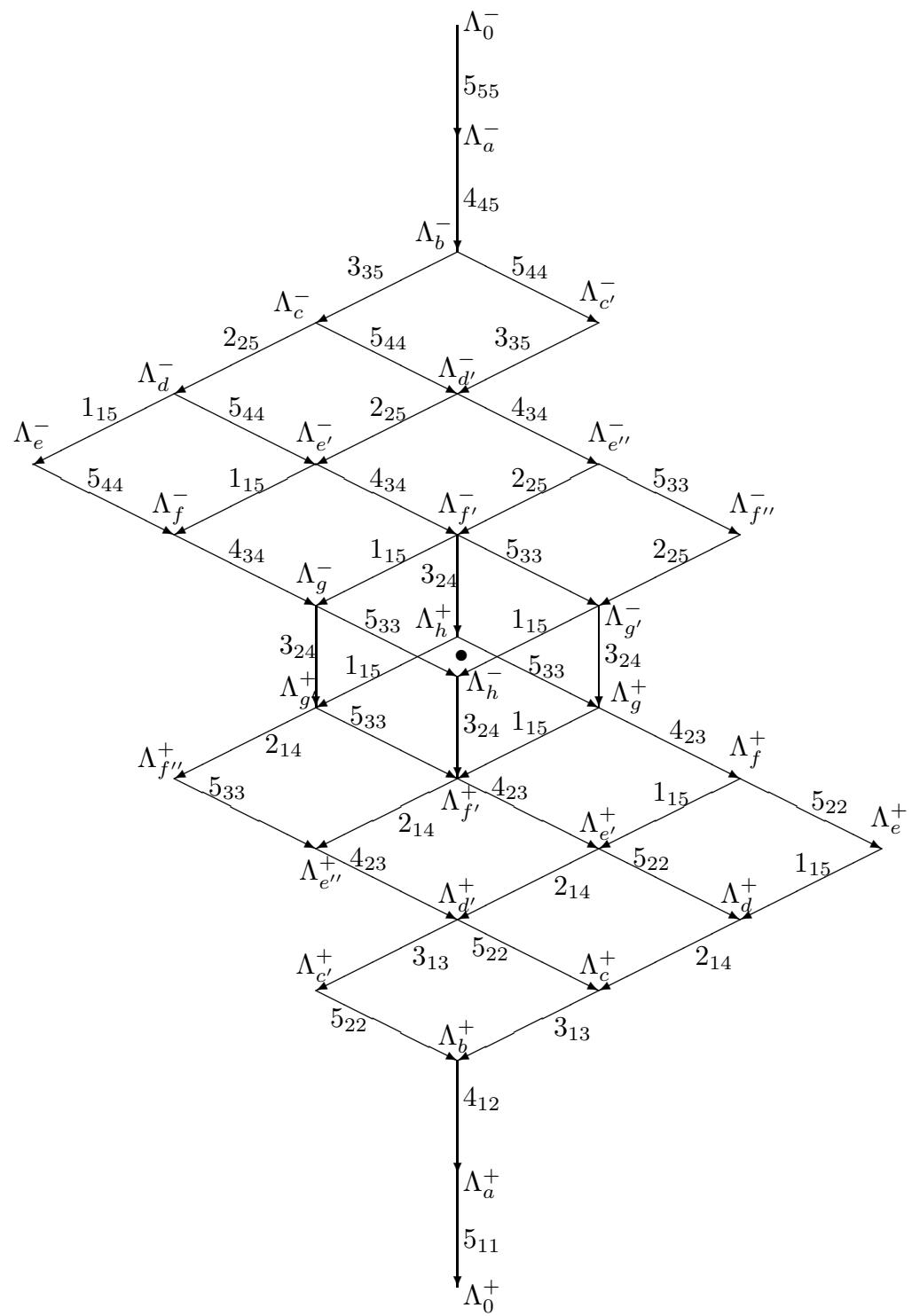
multiplets for  $sp(r, r)$ ,  $r = 1, 2, 3$  [122]. (The case  $n = 2, r = 1$  is already considered due to  $sp(1, 1) \cong so(4, 1)$  and the parabolic relation between  $so(3, 2)$  and  $so(4, 1)$ .)



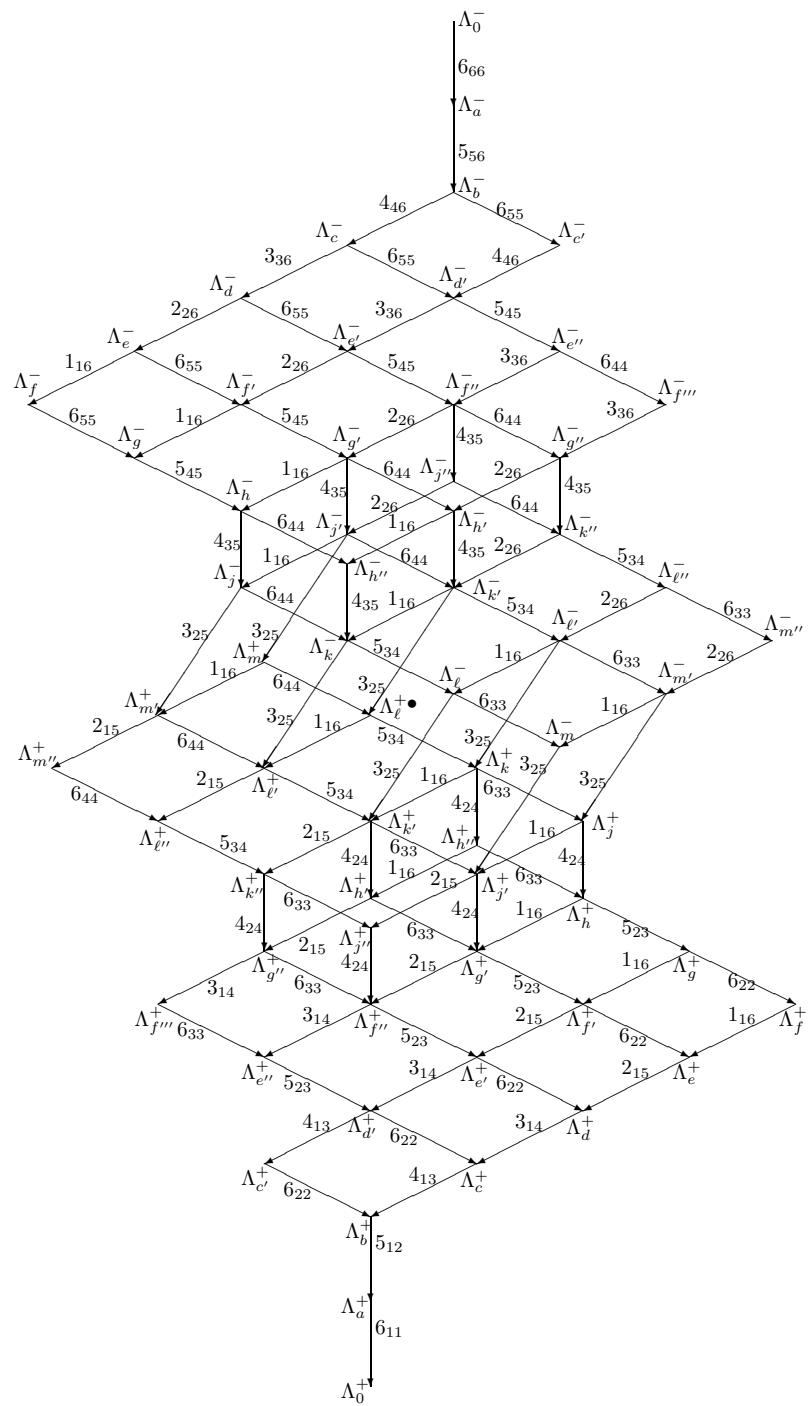
**Figure 5.** Main multiplets for  $Sp(3, \mathbb{R})$ .



**Figure 6.** Main multiplets for  $sp(4, \mathbb{R})$  and  $sp(2, 2)$ .



**Figure 7.** Main multiplets for  $Sp(5, \mathbb{R})$ .



**Figure 8.** Main multiplets for  $Sp(6, \mathbb{R})$  and  $sp(3, 3)$ .

## 5. The Lie Algebra $so^*(2n)$

The Lie algebra  $\mathcal{G} = so^*(2n)$  is given by

$$\begin{aligned}
so^*(2n) &\doteq \{X \in so(2n, \mathbb{C}) : J_n CX = X J_n C\} \\
&= \{X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in gl(n, \mathbb{C}), \\
&\quad {}^t a = -a, \quad b^\dagger = b\}.
\end{aligned}$$

$$\dim_R \mathcal{G} = n(2n-1), \operatorname{rank} \mathcal{G} = n.$$

The maximal compact subalgebra is  $\mathcal{K} \cong u(n)$ . Thus,  $\mathcal{G} = so^*(2n)$  has discrete series representations and highest/lowest-weight representations. The split rank is  $r \equiv [n/2]$ .

The maximal parabolic subalgebras have  $\mathcal{M}$ -factors as follows [75]:

$$\begin{aligned} \mathcal{M}_j^{\max} &= so^*(2n-4j) \oplus su^*(2j), \\ j &= 1, \dots, r. \end{aligned} \quad (21)$$

### 5.1. Case of $so^*(12)$

For even  $n = 2r$ , we may choose a *maximal* parabolic  $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$  such that  $\mathcal{A} \cong so(1, 1)$ ,  $\mathcal{M} = \mathcal{M}_r^{\max} = su^*(n)$ . We note also that

$$\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(n, \mathbb{C}) \cong \mathcal{A}^{\mathbb{C}} \oplus \mathcal{M}^{\mathbb{C}}$$

Thus, with this choice we utilize the property which distinguishes the class of 'conformal Lie algebras' to which class the algebras  $so^*(4r)$  belong.

Furthermore, we restrict ourselves to  $\mathcal{G} = so^*(12)$ .

We label the signature of the ERs of  $\mathcal{G}$  as follows:

$$\begin{aligned} \chi &= \{n_1, n_2, n_3, n_4, n_5; c\}, \\ n_j &\in \mathbb{Z}_+, \quad c = d - \frac{15}{2} \end{aligned}$$

where the last entry of  $\chi$  labels the characters of  $\mathcal{A}$ , and the first five entries are labels of the finite-dimensional (nonunitary) irreps of  $\mathcal{M} = su^*(6)$  when all  $n_j > 0$  or limits of the latter when some  $n_j = 0$ .

Below, we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, n_2, n_3, n_4, n_5)^* \doteq (n_5, n_4, n_3, n_2, n_1).$$

The ERs in the multiplet are related also by intertwining integral operators introduced in [KnSt]. These operators are defined for any ER, the general action being:

$$\begin{aligned} G_{KS} : \mathcal{C}_{\chi} &\longrightarrow \mathcal{C}_{\chi'}, \\ \chi &= \{n_1, \dots, n_5; c\}, \\ \chi' &= \{(n_1, \dots, n_5)^*; -c\}. \end{aligned}$$

Furthermore, we give the correspondence between the signatures  $\chi$  and the highest weight  $\Lambda$ . The connection is through the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \alpha_i^{\vee}) = (\Lambda + \rho, \alpha_i), \quad i = 1, \dots, 5,$$

where  $\Lambda = \Lambda(\chi)$ ,  $\rho$  is half the sum of the positive roots of  $\mathcal{G}^{\mathbb{C}}$ . The explicit connection is

$$\begin{aligned} n_i &= m_i, \\ c &= -\frac{1}{2}(m_1 + 2m_2 + 3m_3 + 4m_4 + \\ &\quad + 2m_5 + 3m_6) \end{aligned}$$

Finally, we recall that according to [75], the above considerations are applicable also for the algebra  $so(6, 6)$  with parabolic  $\mathcal{M}$ -factor  $sl(6, \mathbb{R})$ .

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of  $so^*(12)$ ; i.e., they are labeled by the six positive Dynkin labels  $m_i \in \mathbb{N}$ .

The number of ERs/GVMs in the corresponding multiplets is [75]

$$\begin{aligned} |W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| &= \\ &= |W(so(12, \mathbb{C}))| / |W(sl(6, \mathbb{C}))| = 32 \end{aligned}$$

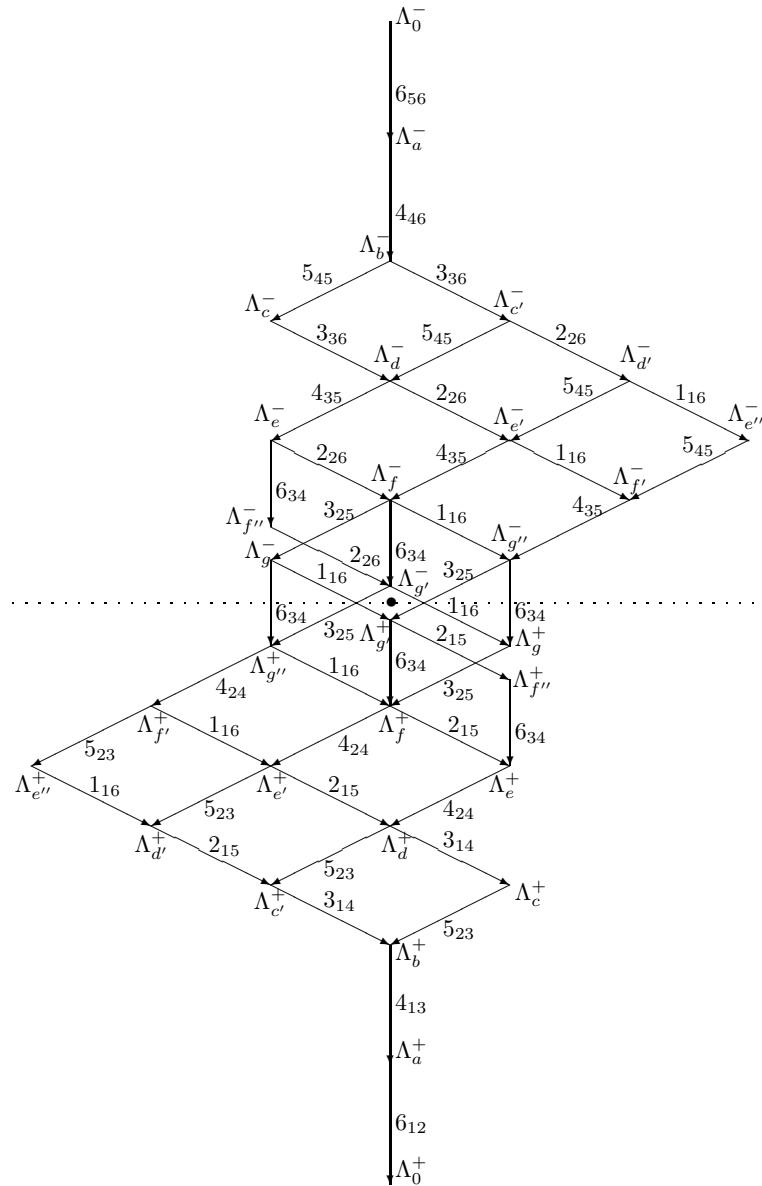
where  $\mathcal{H}$  is a Cartan subalgebra of both  $\mathcal{G}$  and  $\mathcal{K}$ .

They are given explicitly in the Figure 9 below. The pairs  $\Lambda^\pm$  are symmetric with regard to the bullet in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators

The statements made for the ER with signature  $\chi_0^-$  in the previous cases also remain valid here. Also, the conjugate ER  $\chi_0^+$  contains a unitary discrete series subrepresentation split in highest/lowest-weight representations.

All the above is valid also for the algebra  $so(6, 6)$ , cf. [75]; however, the latter algebra does not have highest/lowest-weight representations.

We present only the main multiplets. The reduced multiplets may be seen in [92].



**Figure 9.**  $SO^*(12)$  main multiplets.

### 5.2. Case of $so^*(8)$

This case was already considered for the choice of generic maximal parabolic subalgebra of  $so(6, 2) \cong so^*(8)$ , but here, we shall consider a Heisenberg maximal parabolic subalgebra.

We recall that  $\mathcal{G} = so^*(8) \cong so(6, 2)$  has minimal parabolic:

$$\mathcal{M}_0 = so(3) \oplus so(3) \quad (22)$$

The Satake–Dynkin diagram of  $\mathcal{G}$  is



where, by standard convention, the black dots represent the  $so(3)$  subalgebras of  $\mathcal{M}_0$ .

We shall use the Heisenberg maximal parabolic (21) with  $\mathcal{M}$ -subalgebra:

$$\mathcal{M} = so^*(4) \oplus so(3) \cong so(2, 1) \oplus so(3) \oplus so(3) \quad (24)$$

The Satake–Dynkin diagram of  $\mathcal{M}$  is a subdiagram of (23):



From the above, it follows that the  $\mathcal{M}$ -compact roots of  $\mathcal{G}^{\mathbb{C}}$  are (given in terms of the simple roots):

$$\alpha_{12} = \gamma_1, \alpha_{34} = \gamma_3, \beta_{34} = \gamma_4 \quad (26)$$

By definition, the above are the positive roots of  $\mathcal{G}^{\mathbb{C}}$ .

The positive  $\mathcal{M}$ -noncompact roots of  $\mathcal{G}^{\mathbb{C}}$  in terms of the simple roots are:

$$\gamma_{12} = \gamma_1 + \gamma_2, \gamma_{13} = \gamma_1 + \gamma_2 + \gamma_3, \gamma_{23} = \gamma_2 + \gamma_3, \quad (27a)$$

$$\begin{aligned} \beta_{12} &= \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4, \beta_{13} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \\ \beta_{14} &= \gamma_1 + \gamma_2 + \gamma_4, \beta_{23} = \gamma_2 + \gamma_3 + \gamma_4, \beta_{24} = \gamma_2 + \gamma_4 \end{aligned} \quad (27b)$$

where, for convenience, we use the notation  $\gamma_{ij} \equiv \alpha_{i,j+1}$

To characterize the Verma modules, we shall first use the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^{\vee}), \quad i = 1, \dots, 4, \quad (28)$$

where  $\rho$  is half the sum of the positive roots of  $\mathcal{G}^{\mathbb{C}}$ . Thus, we shall use:

$$\chi_{\Lambda} = \{m_1, m_2, m_3, m_4\} \quad (29)$$

Note that when all  $m_i \in \mathbb{N}$ , then  $\chi_{\Lambda}$  characterizes the finite-dimensional irreps of  $\mathcal{G}^{\mathbb{C}}$  and its real forms, in particular,  $so^*(8)$ . Furthermore,  $m_1, m_3, m_4 \in \mathbb{N}$  characterizes the finite-dimensional irreps of the  $\mathcal{M}$  subalgebra.

For the  $\mathcal{M}$ -noncompact roots of  $\mathcal{G}^{\mathbb{C}}$ , we shall also use the Harish–Chandra parameters:

$$m_{ij} = (\Lambda + \rho, \gamma_{ij}^{\vee}), \quad (30a)$$

$$\hat{m}_{ij} = (\Lambda + \rho, \beta_{ij}^{\vee}) \quad (30b)$$

and explicitly in terms of the Dynkin labels (compare (27)):

$$\begin{aligned} \chi_{HC} = & \{ m_{12} = m_1 + m_2, m_{13} = m_1 + m_2 + m_3, \\ & m_{23} = m_2 + m_3, \end{aligned} \quad (31a)$$

$$\begin{aligned} & \hat{m}_{12} = m_1 + 2m_2 + m_3 + m_4, \\ & \hat{m}_{13} = m_1 + m_2 + m_3 + m_4, \\ & \hat{m}_{14} = m_1 + m_2 + m_4, \\ & \hat{m}_{23} = m_2 + m_3 + m_4, \hat{m}_{24} = m_2 + m_4 \} \end{aligned} \quad (31b)$$

### Main multiplets of $\text{SO}^*(8)$

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of  $so^*(8)$ ; i.e., they are labeled by the four positive Dynkin labels  $m_i \in \mathbb{N}$ .

We take  $\chi_0 = \chi_{HC}$ . It has one embedded Verma module with HW  $\Lambda_a = \Lambda_0 - m_2\gamma_2$ . The number of ERs/GVMs in a main multiplet is 24. We give the whole multiplet as follows:

$$\begin{aligned} \chi_0 &= \{m_1, m_2, m_3, m_4\} \\ \chi_a &= \{m_{12}, -m_2, m_{23}, m_{2,4}\}, \quad \Lambda_a = \Lambda_0 - m_2\gamma_2 \\ \chi_b &= \{m_2, -m_{12}, m_{13}, m_{12,4}\}, \quad \Lambda_b = \Lambda_a - m_1\gamma_{12} \\ \chi_c &= \{m_{13}, -m_{23}, m_2, m_{24}\}, \quad \Lambda_c = \Lambda_a - m_3\gamma_{23} \\ \chi_d &= \{m_{12,4}, -m_{2,4}, m_{24}, m_2\}, \quad \Lambda_d = \Lambda_a - m_4\beta_{24} \\ \chi_e &= \{m_{23}, -m_{13}, m_{12}, m_{14}\}, \quad \Lambda_e = \Lambda_c - m_1\gamma_{12} \\ \chi_f &= \{m_{2,4}, -m_{12,4}, m_{14}, m_{12}\}, \quad \Lambda_f = \Lambda_b - m_4\beta_{24} \end{aligned} \quad (32)$$

$$\begin{aligned} \chi_g &= \{m_{14}, -m_{24}, m_{2,4}, m_{23}\}, \quad \Lambda_g = \Lambda_c - m_4\beta_{24} = \Lambda_d - m_3\gamma_{23} \\ \chi_h &= \{m_{24}, -m_{14}, m_{12,4}, m_{13}\}, \quad \Lambda_h = \Lambda_e - m_4\beta_{24} \\ \chi_i &= \{m_3, -m_{13}, m_1, m_{14,2}\}, \quad \Lambda_i = \Lambda_e - m_2\gamma_{13} \\ \chi_j &= \{m_4, -m_{12,4}, m_{14,2}, m_1\}, \quad \Lambda_j = \Lambda_f - m_2\beta_{14} \\ \chi_k &= \{m_{14,2}, -m_{24}, m_4, m_3\}, \quad \Lambda_k = \Lambda_g - m_2\beta_{23} \end{aligned}$$

$$\begin{aligned} \chi_k^+ &= \{m_{14,2}, -m_{14}, m_4, m_3\}, \quad \Lambda_k^+ = \Lambda_k - m_1\beta_{12} \\ \chi_j^+ &= \{m_4, -m_{14}, m_{14,2}, m_1\}, \quad \Lambda_j^+ = \Lambda_j - m_3\beta_{12} \\ \chi_i^+ &= \{m_3, -m_{14}, m_1, m_{14,2}\}, \quad \Lambda_i^+ = \Lambda_i - m_4\beta_{12} \\ \chi_h^+ &= \{m_{24}, -m_{14,2}, m_{12,4}, m_{13}\}, \quad \Lambda_h^+ = \Lambda_h - m_2\beta_{12} \\ \chi_e^+ &= \{m_{23}, -m_{14,2}, m_{12}, m_{14}\}, \quad \Lambda_e^+ = \Lambda_i^+ - m_2\beta_{24} \\ \chi_f^+ &= \{m_{2,4}, -m_{14,2}, m_{14}, m_{12}\}, \quad \Lambda_f^+ = \Lambda_h^+ - m_3\beta_{14} \end{aligned} \quad (33)$$

$$\begin{aligned} \chi_g^+ &= \{m_{14}, -m_{14,2}, m_{2,4}, m_{23}\}, \quad \Lambda_g^+ = \Lambda_h^+ - m_1\beta_{23} = \Lambda_k^+ - m_2\gamma_{12} \\ \chi_d^+ &= \{m_{12,4}, -m_{14,2}, m_{24}, m_2\}, \quad \Lambda_d^+ = \Lambda_f^+ - m_1\beta_{23} \\ \chi_c^+ &= \{m_{13}, -m_{14,2}, m_2, m_{24}\}, \quad \Lambda_c^+ = \Lambda_g^+ - m_4\gamma_{13} \\ \chi_b^+ &= \{m_2, -m_{14,2}, m_{13}, m_{12,4}\}, \quad \Lambda_b^+ = \Lambda_e^+ - m_3\beta_{14} = \Lambda_f^+ - m_4\gamma_{13} \\ \chi_a^+ &= \{m_{12}, -m_{14,2}, m_{23}, m_{2,4}\}, \quad \Lambda_a^+ = \Lambda_b^+ - m_1\beta_{23} \\ \chi_0^+ &= \{m_1, -m_{14}, m_3, m_4\}, \quad \Lambda_0^+ = \Lambda_a^+ - m_2\beta_{13} \end{aligned}$$

We shall label the signature of the ERs of  $\mathcal{G}$  also as follows:

$$\chi = [n_1, n_2, n_3; c], \quad c = -\frac{1}{2}m_{14,2}, \quad n_1 = m_1, n_2 = m_3, n_3 = m_4, \quad (34)$$

where the last entry labels the characters of  $\mathcal{A}$ , and the first three entries of  $\chi$  are labels of the finite-dimensional irreps of  $\mathcal{M}$  when all  $n_j > 0$  or limits of the latter when some  $n_j = 0$ . Note that  $m_{14,2} = m_1 + 2m_2 + m_3 + m_4$  is the Harish–Chandra parameter for the highest root  $\beta_{12}$ .

Use of this labeling signatures may be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= [m_1, m_3, m_4; \pm \frac{1}{2}m_{14,2}] \\
\chi_a^\pm &= [m_{12}, m_{23}, m_{2,4}; \pm \frac{1}{2}m_{14}], \\
\chi_b^\pm &= [m_2, m_{13}, m_{12,4}; \pm \frac{1}{2}m_{24}], \\
\chi_c^\pm &= [m_{13}, m_2, m_{24}; \pm \frac{1}{2}m_{12,4}], \\
\chi_d^\pm &= [m_{12,4}, m_{24}, m_2; \pm \frac{1}{2}m_{13}], \\
\chi_e^\pm &= [m_{23}, m_{12}, m_{14}; \pm \frac{1}{2}m_{2,4}], \\
\chi_f^\pm &= [m_{2,4}, m_{14}, m_{12}; \pm \frac{1}{2}m_{23}], \\
\chi_g^\pm &= [m_{14}, m_{2,4}, m_{23}; \pm \frac{1}{2}m_{12}], \\
\chi_h^\pm &= [m_{24}, m_{12,4}, m_{13}; \pm \frac{1}{2}m_2], \\
\chi_i^\pm &= [m_3, m_1, m_{14,2}; \pm \frac{1}{2}m_4], \\
\chi_j^\pm &= [m_4, m_{14,2}, m_1; \pm \frac{1}{2}m_3], \\
\chi_k^\pm &= [m_{14,2}, m_4, m_3; \pm \frac{1}{2}m_1]
\end{aligned} \tag{35}$$

The ERs in the multiplet are also related by intertwining integral operators introduced in [233]. These operators are defined for any ER, the general action being:

$$\begin{aligned}
G_{KS} : \mathcal{C}_\chi &\longrightarrow \mathcal{C}_{\chi'}, \\
\chi = [n_1, n_2, n_3; c], \quad \chi' = [n_1, n_2, n_3; -c].
\end{aligned} \tag{36}$$

The main multiplets are given explicitly in Figure 10. The pairs  $\chi^\pm$  are symmetric with regard to the dashed line in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators  $G_{KS} : \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm}$ .

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature  $\chi_0^-$  contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace  $\mathcal{E}$ . The latter corresponds to the finite-dimensional irrep of  $so^*(8)$  with signature  $[m_1, \dots, m_4]$ . The subspace  $\mathcal{E}$  is annihilated by the operator  $G^+$ , and is the image of the operator  $G^-$ . The subspace  $\mathcal{E}$  is annihilated also by the intertwining differential operator acting from  $\chi_0^-$  to  $\chi_a^+$ . When all  $m_i = 1$ , then  $\dim \mathcal{E} = 1$ , and in that case,  $\mathcal{E}$  is also the trivial one-dimensional UIR of the whole algebra  $\mathcal{G}$ . Furthermore, in that case, the conformal weight is zero:  $d = \frac{5}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + 2m_2 + m_3 + m_4)|_{m_i=1} = 0$ .

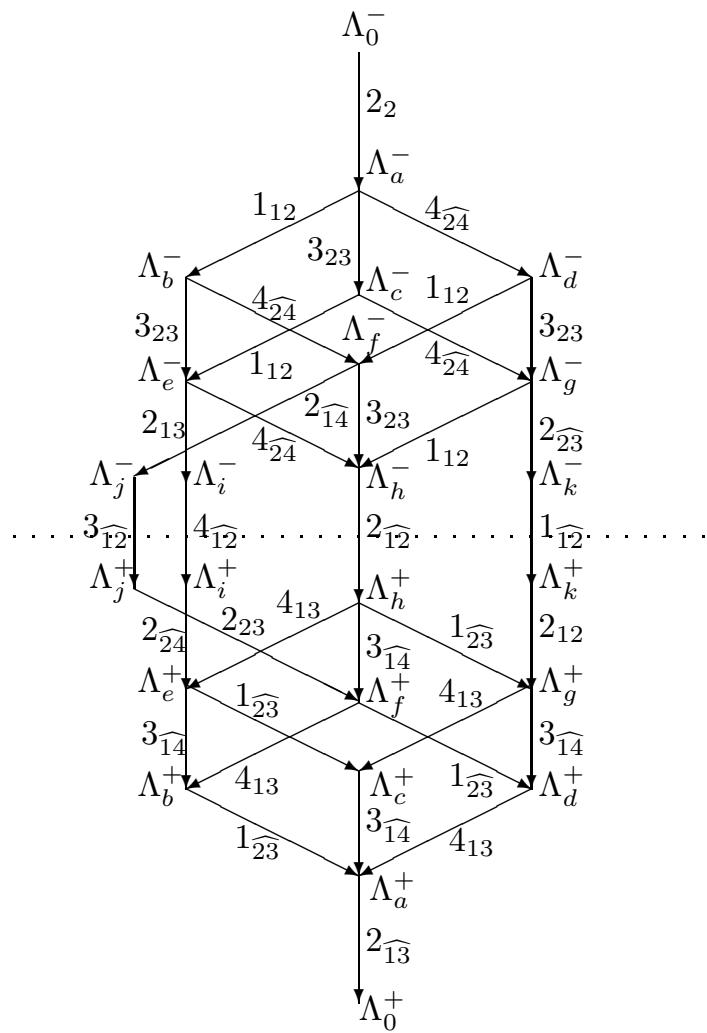
In the conjugate ER  $\chi_0^+$ , there is a unitary discrete series subrepresentation in an infinite-dimensional subspace  $\mathcal{D}$  with conformal weight  $d = \frac{5}{2} + c = \frac{7}{2} + \frac{1}{2}(m_1 + 2m_2 + m_3 + m_4)$ . It is annihilated by the operator  $G^-$ , and is the image of the operator  $G^+$ .

Thus, for  $so^*(8)$  the ER with signature  $\chi_0^+$  contains both a holomorphic discrete series representation and a conjugate anti-holomorphic discrete series representation. The direct sum of the holomorphic and the anti-holomorphic representations spaces form the invariant subspace  $\mathcal{D}$  mentioned above. Note that the corresponding lowest-weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest-weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

In Figure 10, we use the notation:  $\Lambda^\pm = \Lambda(\chi^\pm)$ . Each intertwining differential operator is represented by an arrow accompanied either by a symbol  $i_{jk}$  encoding the root  $\gamma_{jk}$  and the number  $m_{\gamma_{jk}}$  which is involved in the BGG criterion, or a symbol  $i_{\hat{j}k}$  encoding the root  $\beta_{jk}$  and the number  $m_{\beta_{jk}}$  from BGG.

Finally, we recall that according to [122], the above considerations are applicable also for the algebra  $so(p, q)$  (with  $p + q = 8$ ,  $p \geq q \geq 3$ ) with maximal Heisenberg parabolic subalgebra:  $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$ ,  $\mathcal{M}' = so(p-2, q-2) \oplus sl(2, \mathbb{R})$ .

We present only the main multiplets. The reduced multiplets may be seen in [239].



**Figure 10.** Main multiplets for  $SO^*(8)$  using induction from maximal Heisenberg parabolic.

### 5.3. Case of $so^*(10)$

The case  $so^*(10)$  is also special. It is not of the class of ‘conformal Lie algebras’ but belongs to the wider class of Hermitian symmetric spaces as described in the Introduction.

The maximal parabolic subalgebras of  $so^*(2n)$  have  $\mathcal{M}$ -factors as follows [75]:

$$\mathcal{M}_j^{\max} = \mathfrak{so}^*(2n-4j) \oplus \mathfrak{su}^*(2j), \quad j = 1, \dots, r. \quad (37)$$

The  $\mathcal{N}^\pm$  factors in the maximal parabolic subalgebras have dimensions  $\dim(\mathcal{N}_j^\pm)^{\max} = j(4n - 6j - 1)$ .

The case  $j = 1$  is special. In this case, we have a maximal Heisenberg parabolic with  $\mathcal{M}$ -factor

$$\mathcal{M}_{\text{Heisenberg}}^{\max} = \text{so}^*(2n-4) \oplus \text{su}(2) \quad (38)$$

which we use in this section.

Furthermore, we restrict ourselves to our case of study  $\mathcal{G} = so^*(10)$  with minimal parabolic:

$$\mathcal{M}_0 = so(2) \oplus so(3) \oplus so(3) \quad (39)$$

The Satake–Dynkin diagram of  $\mathcal{G}$  is:

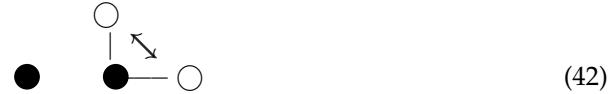


where, by standard convention, the black dots represent the  $so(3)$  subalgebras of  $\mathcal{M}_0$  and the left-right arrow represents the  $so(2)$  subalgebra of  $\mathcal{M}_0$ .

We shall use the Heisenberg maximal parabolic (37) with  $\mathcal{M}$ -subalgebra:

$$\mathcal{M} = so^*(6) \oplus so(3) \cong su(3,1) \oplus su(2) \quad (41)$$

The Satake–Dynkin diagram of  $\mathcal{M}$  is a subdiagram of (40):



where the single black dot represents the  $so(3)$  subalgebra, while the connected part of the diagram represents the  $su(3,1)$  subalgebra.

From the above follows that the  $\mathcal{M}$ -compact roots of  $\mathcal{G}^{\mathbb{C}}$  are (given in terms of the simple roots):

$$\alpha_{12} = \gamma_1, \quad (43a)$$

$$\alpha_{34} = \gamma_3, \alpha_{45} = \gamma_4, \beta_{45} = \gamma_5, \quad (43b)$$

$$\alpha_{35} = \gamma_3 + \gamma_4, \beta_{34} = \gamma_3 + \gamma_4 + \gamma_5, \beta_{35} = \gamma_3 + \gamma_5$$

By definition, the above are the positive roots of  $\mathcal{M}^{\mathbb{C}}$ , namely:  $su(2)^{\mathbb{C}}$  (43a) and  $su(3,1)^{\mathbb{C}} = sl(4, \mathbb{C})$  (43b).

The positive  $\mathcal{M}$ -noncompact roots of  $\mathcal{G}^{\mathbb{C}}$  in terms of the simple roots are

$$\begin{aligned} \gamma_{12} &= \gamma_1 + \gamma_2, \gamma_{13} = \gamma_1 + \gamma_2 + \gamma_3, \gamma_{14} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \\ \gamma_{23} &= \gamma_2 + \gamma_3, \gamma_{24} = \gamma_2 + \gamma_3 + \gamma_4, \end{aligned} \quad (44a)$$

$$\beta_{12} = \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5, \beta_{13} = \gamma_1 + \gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5,$$

$$\beta_{14} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5, \beta_{15} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_5,$$

$$\beta_{23} = \gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5, \beta_{24} = \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,$$

$$\beta_{25} = \gamma_2 + \gamma_3 + \gamma_5 \quad (44b)$$

where, for convenience, we use the notation  $\gamma_{ij} \equiv \alpha_{i,j+1}$

To characterize the Verma modules, we shall use first the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^{\vee}), \quad i = 1, \dots, 5, \quad (45)$$

where  $\rho$  is half the sum of the positive roots of  $\mathcal{G}^{\mathbb{C}}$ . Thus, we shall use:

$$\chi_{\Lambda} = \{m_1, m_2, m_3, m_4, m_5\} \quad (46)$$

Note that when all  $m_i \in \mathbb{N}$ , then  $\chi_{\Lambda}$  characterizes the finite-dimensional irreps of  $\mathcal{G}^{\mathbb{C}}$  and its real forms, in particular,  $so^*(10)$ . Furthermore,  $m_1 \in \mathbb{N}$  characterizes the finite-dimensional irreps of the  $su(2)$  subalgebra, while the set of positive integers  $\{m_3, m_4, m_5\}$  characterizes the finite-dimensional irreps of  $su(3,1)$ .

For the  $\mathcal{M}$ -noncompact roots of  $\mathcal{G}^{\mathbb{C}}$ , we shall also use the Harish–Chandra parameters:

$$m_{ij} = (\Lambda + \rho, \gamma_{ij}^{\vee}), \quad (47a)$$

$$\hat{m}_{ij} = (\Lambda + \rho, \beta_{ij}^{\vee}) \quad (47b)$$

and explicitly in terms of the Dynkin labels (compare (44)):

$$\begin{aligned} \chi_{HC} = & \{m_{12} = m_1 + m_2, m_{13} = m_1 + m_2 + m_3, \\ & m_{14} = m_1 + m_2 + m_3 + m_4, m_2, \\ & m_{23} = m_2 + m_3, m_{24} = m_2 + m_3 + m_4, \end{aligned} \quad (48a)$$

$$\begin{aligned} & \hat{m}_{12} = m_1 + 2m_2 + 2m_3 + m_4 + m_5, \\ & \hat{m}_{13} = m_1 + m_2 + 2m_3 + m_4 + m_5, \\ & \hat{m}_{14} = m_1 + m_2 + m_3 + m_4 + m_5, \\ & \hat{m}_{15} = m_1 + m_2 + m_3 + m_5, \\ & \hat{m}_{23} = m_2 + 2m_3 + m_4 + m_5, \\ & \hat{m}_{24} = m_2 + m_3 + m_4 + m_5, \\ & \hat{m}_{25} = m_2 + m_3 + m_5 \} \end{aligned} \quad (48b)$$

### Main multiplets of $\text{SO}^*(10)$

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of  $so^*(10)$ ; i.e., they are labeled by the five positive Dynkin labels  $m_i \in \mathbb{N}$ .

We take  $\chi_0 = \chi_{HC}$ . It has one embedded Verma module with HW  $\Lambda_a = \Lambda_0 - m_2\gamma_2$ . The number of ERs/GVMs in a main multiplet is 40. We give the whole multiplet as follows:

$$\begin{aligned} \chi_0 &= \{m_1, m_2, m_3, m_4, m_5\} \\ \chi_a &= \{m_{12}, -m_2, m_{23}, m_4, m_5\}, \quad \Lambda_a = \Lambda_0 - m_2\gamma_2 \\ \chi_b &= \{m_2, -m_{12}, m_{13}, m_4, m_5\}, \quad \Lambda_b = \Lambda_a - m_1\gamma_{12} \\ \chi_c &= \{m_{13}, -m_{23}, m_2, m_{34}, m_{3,5}\}, \quad \Lambda_c = \Lambda_a - m_3\gamma_{23} \\ \chi_d &= \{m_{23}, -m_{13}, m_{12}, m_{34}, m_{3,5}\}, \quad \Lambda_d = \Lambda_b - m_3\gamma_{23} = \Lambda_c - m_1\gamma_{12} \\ \chi_e &= \{m_{14}, -m_{24}, m_2, m_3, m_{35}\}, \quad \Lambda_e = \Lambda_c - m_4\gamma_{24} \\ \chi_f &= \{m_{13,5}, -m_{23,5}, m_2, m_{35}, m_3\}, \quad \Lambda_f = \Lambda_c - m_5\beta_{25} \\ \chi_g &= \{m_3, -m_{13}, m_1, m_{24}, m_{23,5}\}, \quad \Lambda_g = \Lambda_d - m_2\gamma_{13} \\ \chi_h &= \{m_{24}, -m_{14}, m_{12}, m_3, m_{35}\}, \quad \Lambda_h = \Lambda_d - m_4\gamma_{24} \\ \chi_i &= \{m_{23,5}, -m_{13,5}, m_{12}, m_{35}, m_3\}, \quad \Lambda_i = \Lambda_d - m_5\beta_{25} = \Lambda_f - m_1\gamma_{13} \\ \chi_j &= \{m_{15}, -m_{25}, m_2, m_{3,5}, m_{34}\}, \quad \Lambda_j = \Lambda_e - m_5\beta_{25} \\ \chi_k &= \{m_{34}, -m_{14}, m_1, m_{23}, m_{25}\}, \quad \Lambda_k = \Lambda_g - m_4\gamma_{24} = \Lambda_h - m_2\gamma_{13} \\ \chi_l &= \{m_{3,5}, -m_{13,5}, m_1, m_{25}, m_{23}\}, \quad \Lambda_l = \Lambda_g - m_5\beta_{25} \\ \chi_m &= \{m_{25}, -m_{15}, m_{12}, m_{3,5}, m_{34}\}, \quad \Lambda_m = \Lambda_h - m_5\beta_{25} \\ \chi_n &= \{m_{15,3}, -m_{25,3}, m_{23}, m_5, m_4\}, \quad \Lambda_n = \Lambda_j - m_3\beta_{24} \\ \chi_p &= \{m_4, -m_{14}, m_1, m_2, m_{25,3}\}, \quad \Lambda_p = \Lambda_k - m_3\gamma_{14} \\ \chi_q &= \{m_{35}, -m_{15}, m_1, m_{23,5}, m_{24}\}, \quad \Lambda_q = \Lambda_k - m_5\beta_{25} \\ \chi_r &= \{m_5, -m_{13,5}, m_1, m_{25,3}, m_2\}, \quad \Lambda_r = \Lambda_l - m_3\beta_{15} \\ \chi_s &= \{m_{25,3}, -m_{15,3}, m_{13}, m_5, m_4\}, \quad \Lambda_s = \Lambda_m - m_3\beta_{24} \\ \chi_t &= \{m_{15,23}, -m_{25,3}, m_3, m_5, m_4\}, \quad \Lambda_t = \Lambda_n - m_2\beta_{23} \end{aligned} \quad (49)$$

$$\begin{aligned}
\chi_p^+ &= \{m_4, -m_{15}, m_1, m_2, m_{25,3}\}, \quad \Lambda_p^+ = \Lambda_p - m_5\beta_{12} \\
\chi_q^+ &= \{m_{35}, -m_{15,3}, m_1, m_{23,5}, m_{24}\}, \quad \Lambda_q^+ = \Lambda_q - m_3\beta_{12} \\
\chi_r^+ &= \{m_5, -m_{15}, m_1, m_{25,3}, m_2\}, \quad \Lambda_r^+ = \Lambda_r - m_4\beta_{12} \\
\chi_s^+ &= \{m_{25,3}, -m_{15,23}, m_{13}, m_5, m_4\}, \quad \Lambda_s^+ = \Lambda_s - m_2\beta_{12} \\
\chi_t^+ &= \{m_{15,23}, -m_{15,3}, m_3, m_5, m_4\}, \quad \Lambda_t^+ = \Lambda_t - m_1\beta_{12} \\
\chi_k^+ &= \{m_{34}, -m_{15,3}, m_1, m_{23}, m_{25}\}, \quad \Lambda_k^+ = \Lambda_p^+ - m_3\beta_{25} \\
\chi_l^+ &= \{m_{3,5}, -m_{15,3}, m_1, m_{25}, m_{23}\}, \quad \Lambda_l^+ = \Lambda_q^+ - m_4\beta_{15} = \Lambda_r^+ - m_3\gamma_{24} \\
\chi_m^+ &= \{m_{25}, -m_{15,23}, m_{12}, m_{3,5}, m_{34}\}, \quad \Lambda_m^+ = \Lambda_q^+ - m_2\beta_{24} = \Lambda_s^+ - m_3\gamma_{13} \\
\chi_n^+ &= \{m_{15,3}, -m_{15,23}, m_{23}, m_5, m_4\}, \quad \Lambda_n^+ = \Lambda_s^+ - m_1\beta_{23} = \Lambda_t^+ - m_2\gamma_{12} \\
\chi_h^+ &= \{m_{24}, -m_{15,23}, m_{12}, m_3, m_{35}\}, \quad \Lambda_h^+ = \Lambda_k^+ - m_2\beta_{24} = \Lambda_m^+ - m_5\gamma_{14} \\
\chi_g^+ &= \{m_3, -m_{15,3}, m_1, m_{24}, m_{23,5}\}, \quad \Lambda_g^+ = \Lambda_l^+ - m_5\gamma_{14} \\
\chi_i^+ &= \{m_{23,5}, -m_{15,23}, m_{12}, m_{35}, m_3\}, \quad \Lambda_i^+ = \Lambda_l^+ - m_2\beta_{24} = \Lambda_m^+ - m_4\beta_{15} \quad (50) \\
\chi_j^+ &= \{m_{15}, -m_{15,23}, m_2, m_{3,5}, m_{34}\}, \quad \Lambda_j^+ = \Lambda_m^+ - m_1\beta_{23} = \Lambda_n^+ - m_3\gamma_{13} \\
\chi_e^+ &= \{m_{14}, -m_{15,23}, m_2, m_3, m_{35}\}, \quad \Lambda_e^+ = \Lambda_h^+ - m_1\beta_{23} \\
\chi_d^+ &= \{m_{23}, -m_{15,23}, m_{12}, m_{34}, m_{3,5}\}, \quad \Lambda_d^+ = \Lambda_g^+ - m_2\beta_{24} = \Lambda_i^+ - m_5\gamma_{14} \\
\chi_f^+ &= \{m_{13,5}, -m_{15,23}, m_2, m_{35}, m_3\}, \quad \Lambda_f^+ = \Lambda_i^+ - m_1\beta_{23} = \Lambda_j^+ - m_4\beta_{15} \\
\chi_\delta^+ &= \{m_{14}, -m_{15,23}, m_2, m_3, m_{35}\}, \quad \Lambda_\delta^+ = \Lambda_j^+ - m_5\gamma_{14} = \chi_e^+ \quad !!! \\
\chi_b^+ &= \{m_2, -m_{15,23}, m_{13}, m_4, m_5\}, \quad \Lambda_b^+ = \Lambda_\beta^+ - m_3\beta_{14} \\
\chi_c^+ &= \{m_{13}, -m_{15,23}, m_2, m_{34}, m_{3,5}\}, \quad \Lambda_c^+ = \Lambda_d^+ - m_1\beta_{23} \\
\chi_v^+ &= \{m_{13}, -m_{15,23}, m_2, m_{34}, m_{3,5}\}, \quad \Lambda_v^+ = \Lambda_f^+ - m_5\gamma_{14} = \chi_c^+ \quad !!! \\
\chi_a^+ &= \{m_{12}, -m_{15,23}, m_{23}, m_4, m_5\}, \quad \Lambda_a^+ = \Lambda_b^+ - m_1\beta_{23} = \Lambda_c^+ - m_3\beta_{14} \\
\chi_0^+ &= \{m_1, -m_{15,3}, m_3, m_4, m_5\}, \quad \Lambda_0^+ = \Lambda_a^+ - m_2\beta_{13}
\end{aligned}$$

We shall label the signature of the ERs of  $\mathcal{G}$  also as follows:

$$\chi = [n; c; n_1, n_2, n_3], \quad n \in \mathbb{N}, \quad c = -\frac{1}{2}m_{15,23}, \quad n_j = m_{j+2} \in \mathbb{Z}_+, \quad (51)$$

where the first entry  $n = m_1$  labels the finite-dimensional irreps of  $su(2)$ , the second entry labels the characters of  $\mathcal{A}$ , the last three entries of  $\chi$  are labels of the finite-dimensional (nonunitary) irreps of  $\mathcal{M} = su(3, 1)$  when all  $n_j > 0$  or limits of the latter when some  $n_j = 0$ . Note that  $m_{15,23} = m_1 + 2m_2 + 2m_3 + m_4 + m_5$  is the Harish-Chandra parameter for the highest root  $\beta_{12}$ .

Using this labeling, signatures may be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= [m_1; m_3, m_4, m_5; \pm \frac{1}{2}m_{15,23}] \\
\chi_a^\pm &= [m_{12}; m_{23}, m_4, m_5; \pm \frac{1}{2}m_{15,3}] \\
\chi_b^\pm &= [m_2; m_{13}, m_4, m_5; \pm \frac{1}{2}m_{25,3}] \\
\chi_c^\pm &= [m_{13}; m_2, m_{34}, m_{3,5}; \pm \frac{1}{2}m_{15}] \\
\chi_d^\pm &= [m_{23}; m_{12}, m_{34}, m_{3,5}; \pm \frac{1}{2}m_{25}] \\
\chi_e^\pm &= [m_{14}; m_2, m_3, m_{35}; \pm \frac{1}{2}m_{13,5}] \\
\chi_f^\pm &= [m_{13,5}; m_2, m_{35}, m_3; \pm \frac{1}{2}m_{14}]
\end{aligned}$$

$$\begin{aligned}
\chi_g^\pm &= [m_3; m_1, m_{24}, m_{23,5}; \pm \frac{1}{2}m_{35}] \\
\chi_h^\pm &= [m_{24}; m_{12}, m_3, m_{35}; \pm \frac{1}{2}m_{23,5}] \\
\chi_i^\pm &= [m_{23,5}; m_{12}, m_{35}, m_3; \pm \frac{1}{2}m_{24}] \\
\chi_j^\pm &= [m_{15}; m_2, m_{3,5}, m_{34}; \pm \frac{1}{2}m_{13}] \\
\chi_k^\pm &= [m_{34}; m_1, m_{23}, m_{25}; \pm \frac{1}{2}m_{3,5}] \\
\chi_l^\pm &= [m_{3,5}; m_1, m_{25}, m_{23}; \pm \frac{1}{2}m_{34}] \\
\chi_m^\pm &= [m_{25}; m_{12}, m_{3,5}, m_{34}; \pm \frac{1}{2}m_{23}] \\
\chi_n^\pm &= [m_{15,3}; m_{23}, m_5, m_4; \pm \frac{1}{2}m_{12}] \\
\chi_p^\pm &= [m_4; m_1, m_2, m_{25,3}; \pm \frac{1}{2}m_5] \\
\chi_q^\pm &= [m_{35}; m_1, m_{23,5}, m_{24}; \pm \frac{1}{2}m_3] \\
\chi_r^\pm &= [m_5; m_1, m_{25,3}, m_2; \pm \frac{1}{2}m_4] \\
\chi_s^\pm &= [m_{25,3}; m_{13}, m_5, m_4; \pm \frac{1}{2}m_2] \\
\chi_t^\pm &= [m_{15,23}; m_3, m_5, m_4; \pm \frac{1}{2}m_1]
\end{aligned}$$

The ERs in the multiplet are also related by intertwining integral operators introduced in [233]. These operators are defined for any ER, the general action being:

$$\begin{aligned}
G_{KS} : \mathcal{C}_\chi &\longrightarrow \mathcal{C}_{\chi'}, \\
\chi = [n; n_1, n_2, n_3; c], \quad \chi' = [n; n_1, n_2, n_3; -c].
\end{aligned} \tag{52}$$

The main multiplets are given explicitly in Figure 11. The pairs  $\chi^\pm$  are symmetric with regard to the bullet in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators:  $G_{KS} : \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm}$ .

Some comments are in order.

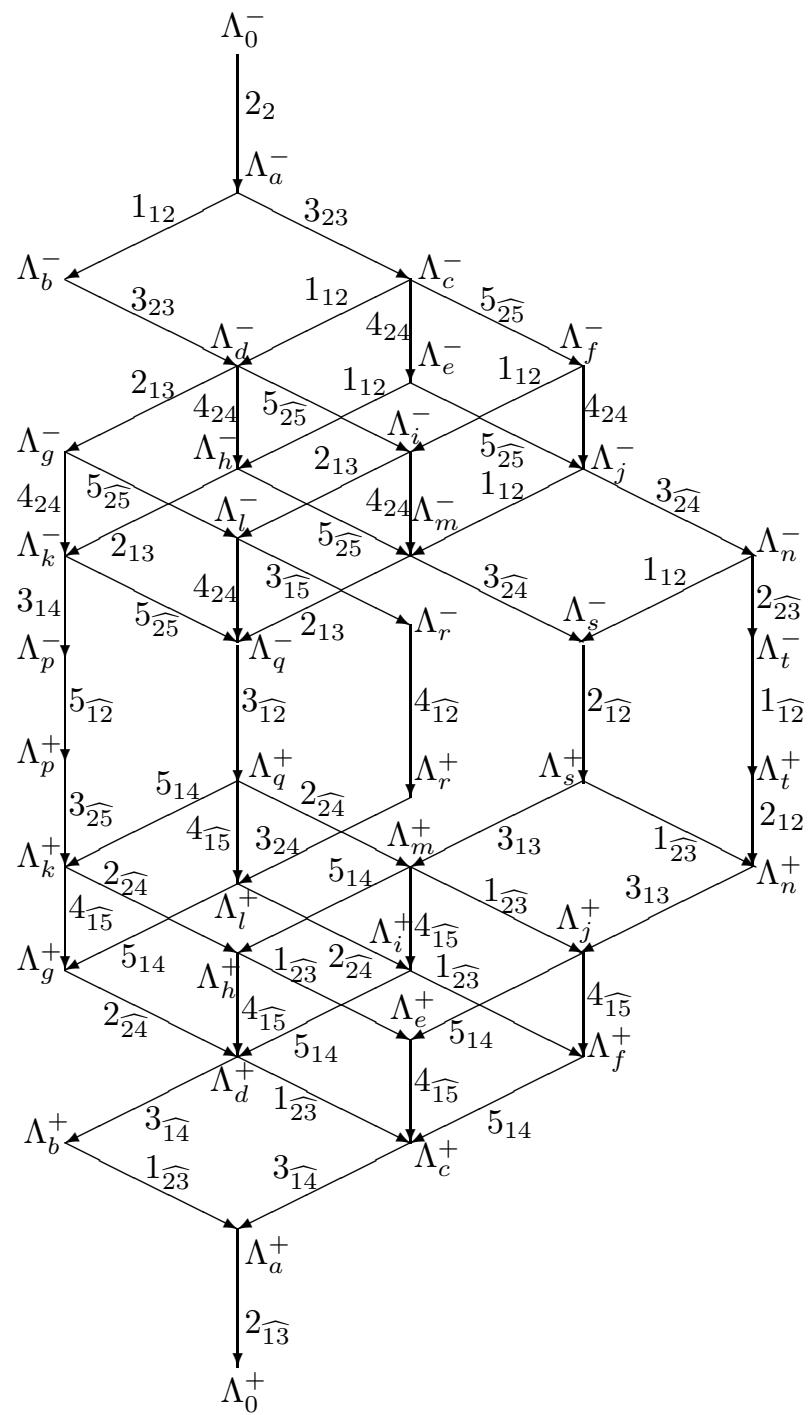
Matters are arranged so that in every multiplet only the ER with signature  $\chi_0^-$  contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace  $\mathcal{E}$ . The latter corresponds to the finite-dimensional irrep of  $so^*(10)$  with signature  $\{m_1, \dots, m_5\}$ . The subspace  $\mathcal{E}$  is annihilated by the operator  $G^+$ , and is the image of the operator  $G^-$ . The subspace  $\mathcal{E}$  is also annihilated by the intertwining differential operator acting from  $\chi_0^-$  to  $\chi_a^-$ . When all  $m_i = 1$  then  $\dim \mathcal{E} = 1$ , and in that case,  $\mathcal{E}$  is also the trivial one-dimensional UIR of the whole algebra  $\mathcal{G}$ . Furthermore, in that case, the conformal weight is zero:  $d = \frac{7}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + 2m_2 + 2m_3 + m_4 + m_5)|_{m_i=1} = 0$ .

In the conjugate ER  $\chi_0^+$ , there is a unitary discrete series subrepresentation in an infinite-dimensional subspace  $\mathcal{D}$ . It is annihilated by the operator  $G^-$ , and is the image of the operator  $G^+$ .

Thus, for  $so^*(10)$  the ER with signature  $\chi_0^+$  contains both a holomorphic discrete series representation and a conjugate anti-holomorphic discrete series representation. The direct sum of the holomorphic and the antiholomorphic representations spaces form the invariant subspace  $\mathcal{D}$  mentioned above. Note that the corresponding lowest-weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Finally, we recall that according to [122], the above considerations are applicable also for the algebra  $so(p, q)$  (with  $p + q = 10$ ,  $p \geq q \geq 2$ ) with maximal Heisenberg parabolic subalgebra:  $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$ ,  $\mathcal{M}' = so(p - 2, q - 2) \oplus sl(2, \mathbb{R})$ .

We present only the main multiplets. The reduced multiplets may be seen in [240].

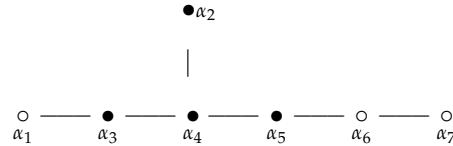


**Figure 11.** Main multiplets for  $SO^*(10)$  using induction from maximal Heisenberg parabolic.

## 6. The Lie Algebras $E_{7(-25)}$ and $E_{7(7)}$

Let  $\mathcal{G} = E_{7(-25)}$ . The maximal compact subgroup is  $\mathcal{K} \cong e_6 \oplus so(2)$ , while  $\mathcal{M} \cong E_{6(-6)}$ .

The Satake diagram [241] is:



The signatures of the ERs of  $\mathcal{G}$  are:

$$\chi = \{n_1, \dots, n_6; c\}, \quad n_j \in \mathbb{N}.$$

expressed through the Dynkin labels:

$$n_i = m_i, \quad c = -\frac{1}{2}(m_{\tilde{\alpha}} + m_7) = -\frac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7)$$

The same signatures can be used for the parabolically related exceptional Lie algebra  $E_{7(7)}$  (with  $\mathcal{M}$ -factor  $E_{6(6)}$ ).

The noncompact roots of the complex algebra  $E_7$  are:

$$\begin{aligned} & \alpha_7, \alpha_{17}, \dots, \alpha_{67}, \\ & \alpha_{1,37}, \alpha_{2,47}, \alpha_{17,4}, \alpha_{27,4}, \\ & \alpha_{17,34}, \alpha_{17,35}, \alpha_{17,36}, \alpha_{17,45}, \alpha_{17,46}, \\ & \alpha_{27,45}, \alpha_{27,46}, \\ & \alpha_{17,25,4}, \alpha_{17,26,4}, \alpha_{17,35,4}, \alpha_{17,36,4}, \\ & \alpha_{17,26,45}, \alpha_{17,36,45}, \\ & \alpha_{17,26,35,4}, \alpha_{17,26,45,4}, \\ & \alpha_{17,16,35,4} = \tilde{\alpha}, \end{aligned}$$

given through the simple roots  $\alpha_i$ :

$$\begin{aligned} \alpha_{ij} &= \alpha_i + \alpha_{i+1} + \dots + \alpha_j, \quad i < j, \\ \alpha_{ij,k} &= \alpha_{k,ij} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j + \alpha_k, \quad i < j, \quad \text{etc.} \end{aligned}$$

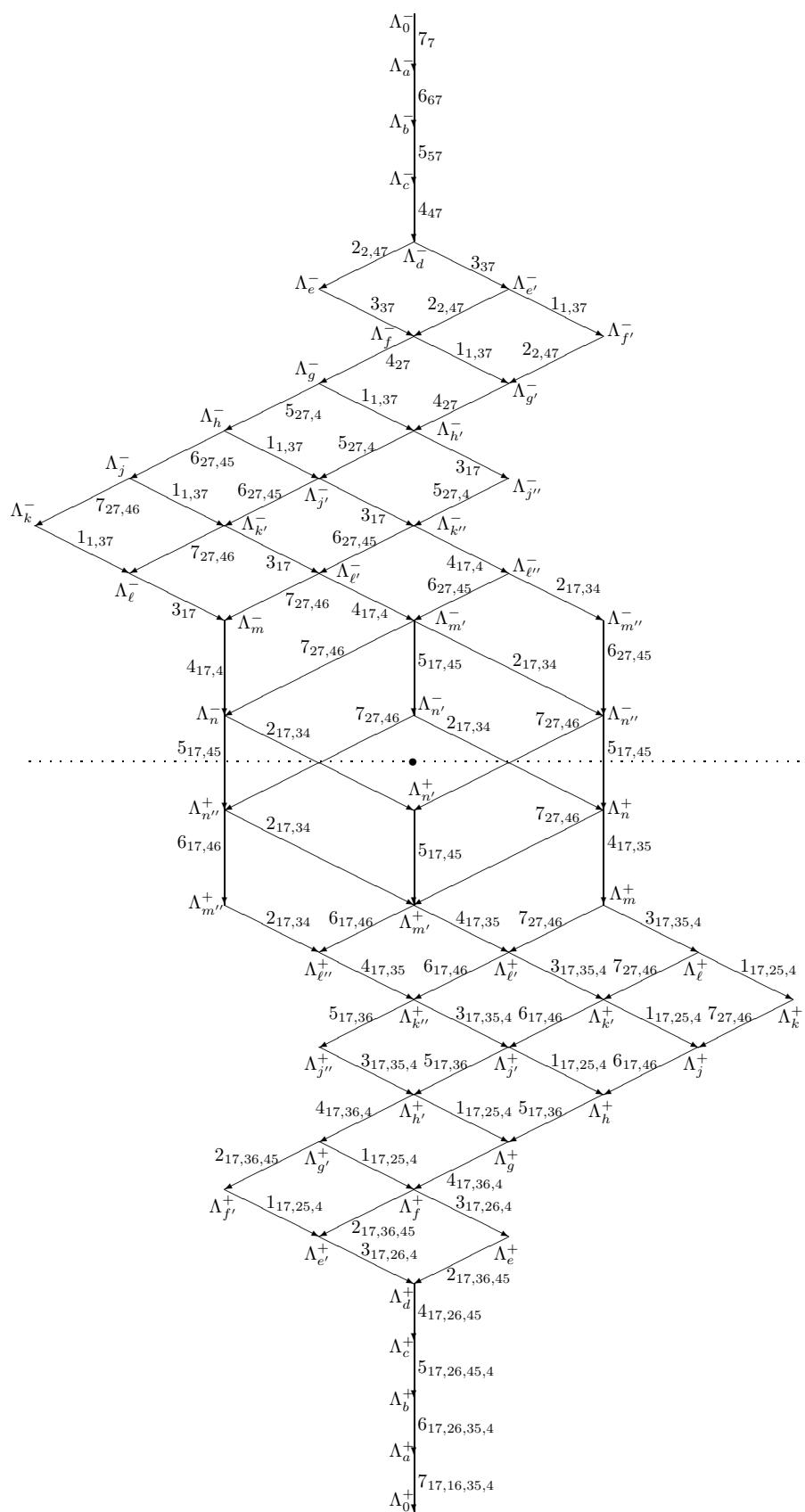
The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of  $E_7$ ; i.e., they will be labeled by the seven positive Dynkin labels  $m_i \in \mathbb{N}$ .

The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| = 56$$

The multiplets are given in Figure 12 [122,242].

The Knapp–Stein operators  $G_{\chi}^{\pm}$  act pictorially as reflections with regard to the bullet intertwining each  $\mathcal{T}_{\chi}^{-}$  member with the corresponding  $\mathcal{T}_{\chi}^{+}$  member.

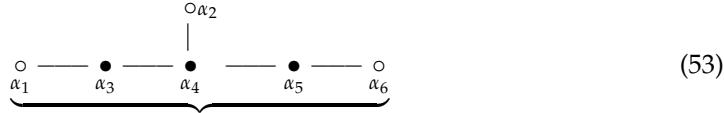


**Figure 12.** Main type for  $E_{7(-25)}$ .

## 7. The Lie Algebras $E_{6(-14)}$ , $E_{6(6)}$ and $E_{6(2)}$

Let  $\mathcal{G} = E_{6(-14)}$ . The maximal compact subalgebra is  $\mathcal{K} \cong so(10) \oplus so(2)$ , while  $\mathcal{M} \cong su(5, 1)$ .

The Satake diagram [241] is:



The signature of the ERs of  $\mathcal{G}$  is:

$$\chi = \{n_1, n_3, n_4, n_5, n_6; c\}, \quad c = d - \frac{11}{2}. \quad (54)$$

expressed through the Dynkin labels as:

$$n_i = m_i, \quad -c = \frac{1}{2}m_{\tilde{\alpha}} = \frac{1}{2}(m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6) \quad (55)$$

The same signatures can be used for the parabolically related exceptional Lie algebras  $E_{6(6)}$  and  $E_{6(2)}$  with  $\mathcal{M}$ -factors  $sl(6, \mathbb{R})$  and  $su(3, 3)$ , resp.

Furthermore, we need the noncompact roots of the complex algebra  $E_6$ :

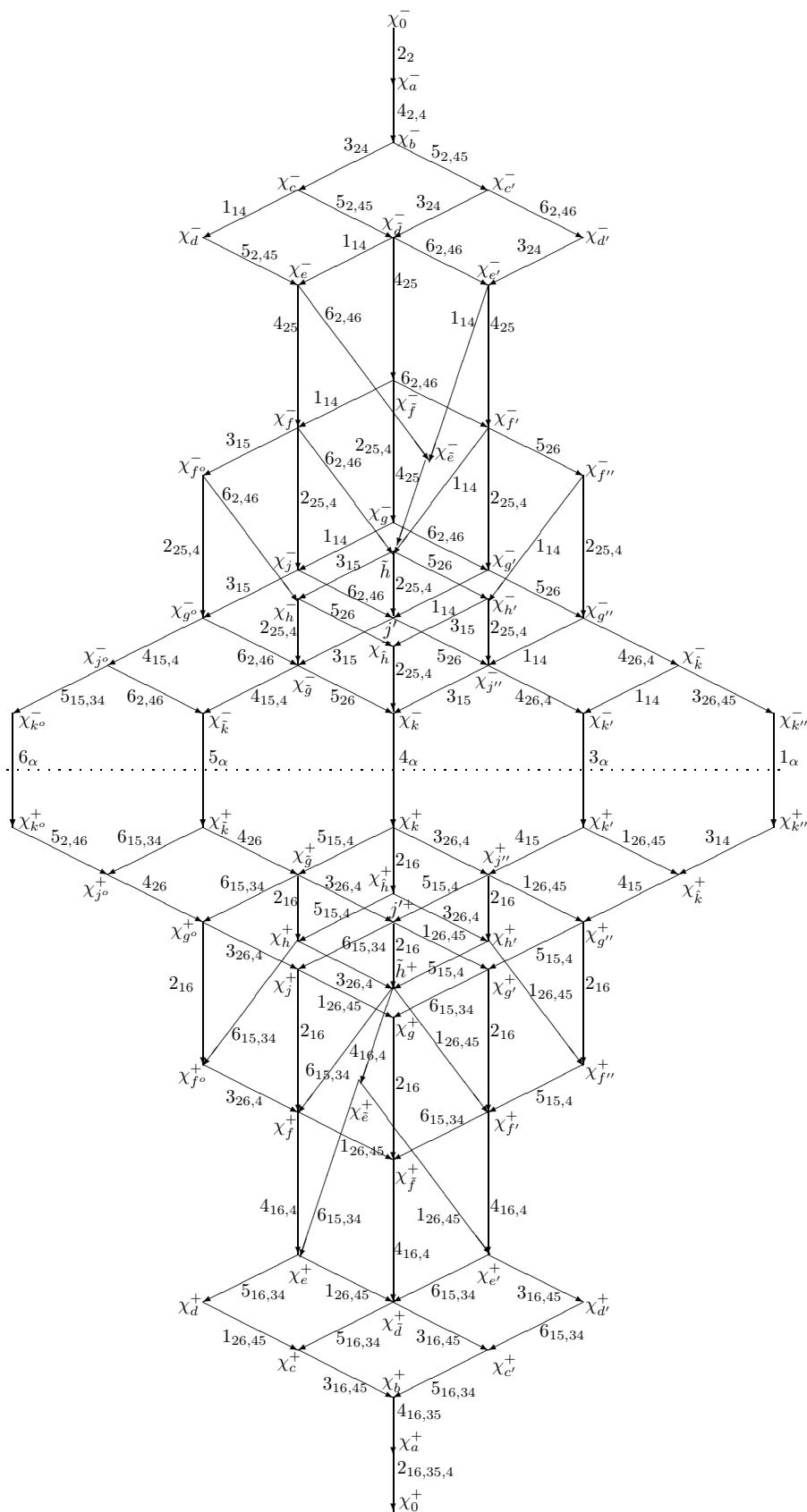
$$\begin{aligned} & \alpha_2, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{24}, \alpha_{25}, \alpha_{26} \\ & \alpha_{2,4}, \alpha_{2,45}, \alpha_{2,46}, \alpha_{25,4}, \alpha_{15,4}, \alpha_{26,4} \\ & \alpha_{16,4}, \alpha_{15,34}, \alpha_{26,45}, \alpha_{16,34}, \alpha_{16,45} \\ & \alpha_{16,35}, \alpha_{16,35,4}, \alpha_{16,25,4} = \tilde{\alpha} \end{aligned} \quad (56)$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of  $\mathcal{G}$ ; i.e., they will be labeled by the six positive Dynkin labels  $m_i \in \mathbb{N}$ .

Since these algebras do not belong to the class of conformal Lie algebras (CLA), the number of ERs/GVMs in the multiplet is not given by formula (14). It turns out that each such multiplet contains 70 ERs/GVMs—see Figure 13 [122,243]. Another difference with the CLA class is that, pictorially, the Knapp–Stein operators  $G_{\chi}^{\pm}$  act as reflections with regard to the dotted line separating the  $\mathcal{T}_{\chi}^{-}$  members from the  $\mathcal{T}_{\chi}^{+}$  members (and not as reflections with regard to a central dot (bullet) as in the CLA cases).

Note that there are five cases when the embeddings correspond to the highest root  $\tilde{\alpha} : V^{\Lambda^-} \longrightarrow V^{\Lambda^+}, \Lambda^+ = \Lambda^- - m_{\tilde{\alpha}} \tilde{\alpha}$ . In these five cases, the weights are denoted as:  $\Lambda_{k''}^{\pm}, \Lambda_{k'}^{\pm}, \Lambda_{\tilde{k}}^{\pm}, \Lambda_k^{\pm}, \Lambda_{k^o}^{\pm}$ ; then,  $m_{\tilde{\alpha}} = m_1, m_3, m_4, m_5, m_6$ , resp. Thus, their action coincides with the action of the Knapp–Stein operators  $G_{\chi}^{+}$  which, in the above five cases, degenerate to differential operators as we discussed for  $so(3, 2)$ .

Note that the figure has the standard  $E_6$  symmetry, namely, conjugation exchanging indices  $1 \longleftrightarrow 6, 3 \longleftrightarrow 5$ .



**Figure 13.** Main Type for  $E_{6(-24)}$ .

## 8. The Lie Algebra $F_4$

The complex Lie algebra  $F_4$  has two real forms denoted by  $F'_4$  and  $F''_4$ .

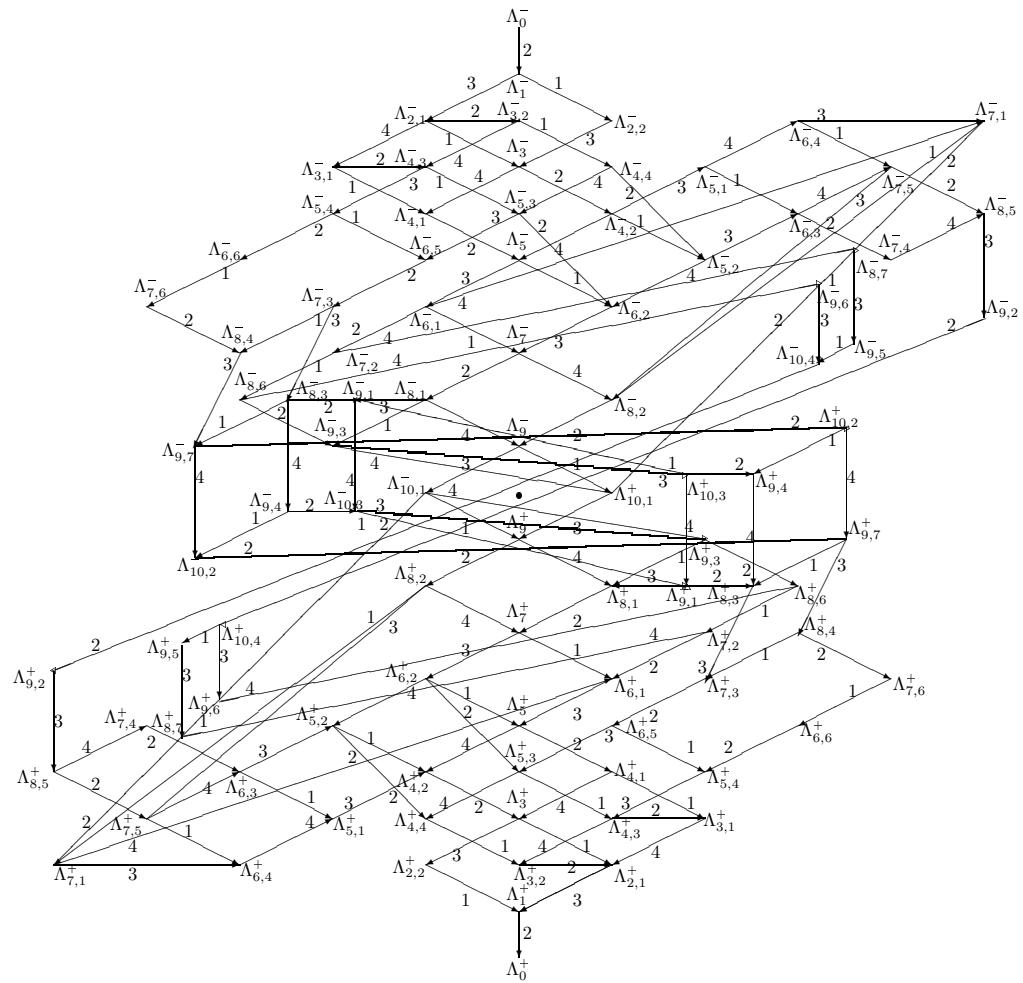
The first  $F'_4$  is the split real form (denoted also as  $F_{4(4)}$ ). It has discrete series representations since  $\text{rank } F'_4 = \text{rank } \mathcal{K} = 4$ , where  $\mathcal{K} = \text{sp}(3) \oplus \text{su}(2)$  is the maximal compact subalgebra.

The real form  $F'_4$  has several parabolic subalgebras. We shall consider a maximal (also called Heisenberg) parabolic subalgebra:

$$\begin{aligned}\mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}, \\ \mathcal{M} &= \text{sl}(3, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}), \\ \dim \mathcal{A} &= 1, \quad \dim \mathcal{N} = 20\end{aligned}$$

Note that in what follows we shall use the case when the  $\text{sl}(3, \mathbb{R})$  subalgebra is formed by the two short roots of  $F_4$ , and the  $\text{sl}(2, \mathbb{R})$  subalgebra is formed by a long root of  $F_4$ . The other (equivalent in our considerations) possibility is to flip the short and the long roots.

The embedding diagram is given in Figure 14.



**Figure 14.** Multiplets for the real split form  $F'_4$  using maximal parabolic with  $\mathcal{M} = \text{sl}(3, \mathbb{R})_{\text{shortroots}} \oplus \text{sl}(2, \mathbb{R})_{\text{longroots}}$ .

The other (split rank one) real form of  $F_4$  is denoted as  $F''_4$ , sometimes as  $F_{4(-20)}$ . This real form also has discrete series representations since  $\mathcal{K} = \text{so}(9)$ . The minimal (also maximal) parabolic  $\mathcal{P}$  and the corresponding Bruhat decomposition are:

$$\begin{aligned}\mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}, \quad \mathcal{M} = \text{so}(7) \\ \mathcal{G} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^+ \oplus \mathcal{N}^-\end{aligned}$$

Each main multiplet contains 24 GVMs (ERs) an777d is given in Figure 15.

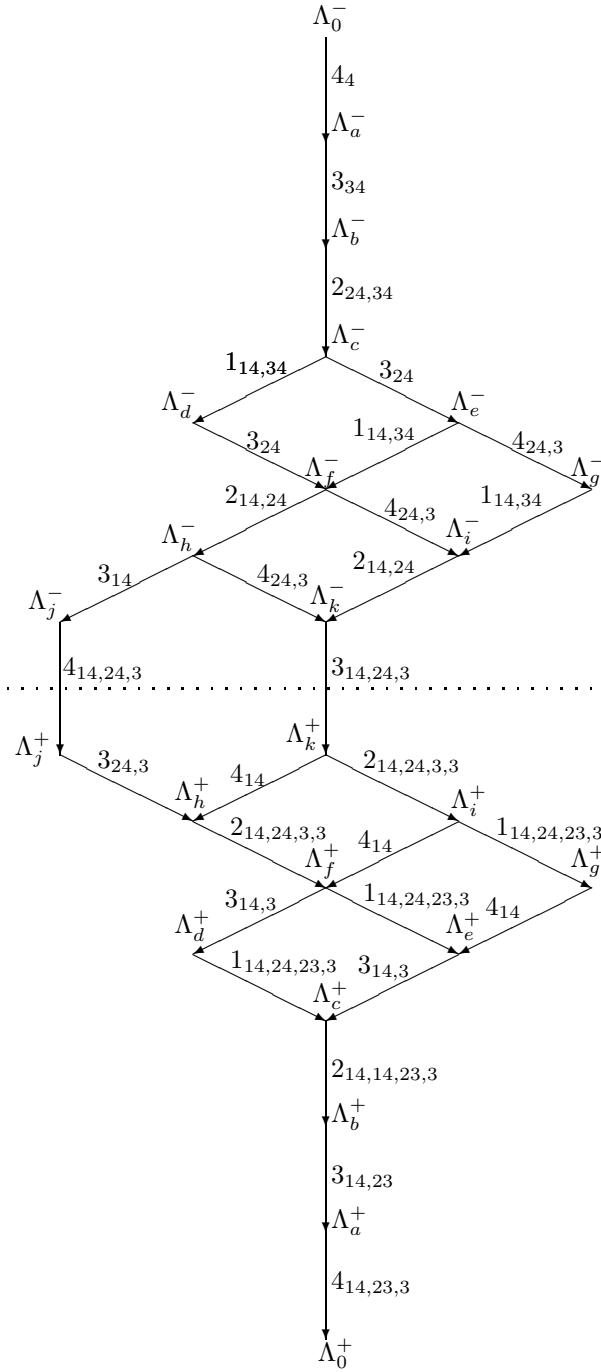


Figure 15. Main multiplets for  $F_4''$ .

### 9. The Case of Lie Algebra $G_{2(2)}$

Let  $\mathcal{G}^{\mathbb{C}} = G_2$ , with Cartan matrix:  $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ , simple roots  $\alpha_1, \alpha_2$  with products:  $(\alpha_1, \alpha_1) = 3(\alpha_2, \alpha_2) = -2(\alpha_1, \alpha_2)$ . We choose  $(\alpha_2, \alpha_2) = 2$ ; then,  $(\alpha_1, \alpha_1) = 6, (\alpha_1, \alpha_2) = -3$ . As we know,  $G_2$  is 14-dimensional. The positive roots are:

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\} \quad (57)$$

We shall use the orthonormal basis  $\varepsilon_1, \varepsilon_2, \varepsilon_3$ . In its terms for the simple roots, we may choose:

$$\alpha_1 = \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3 \quad (58)$$

With the chosen normalization, the roots  $\alpha_1, \alpha_1 + 3\alpha_2 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3, 2\alpha_1 + 3\alpha_2 = 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3$  have a length of 6, while  $\alpha_2, \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_2, \alpha_1 + 2\alpha_2 = \varepsilon_1 - \varepsilon_3$  have a length of 2. Another way to write the roots in general is  $\beta = (a_1, a_2, a_3)$  under the condition  $a_1 + a_2 + a_3 = 0$ . Then,

$$\begin{aligned}\alpha_1 &= (1, -2, 1), \quad \alpha_1 + 3\alpha_2 = (1, 1, -2), \quad 2\alpha_1 + 3\alpha_2 = (2, -1, -1), \\ \alpha_2 &= (0, 1, -1), \quad \alpha_1 + \alpha_2 = (1, -1, 0), \quad \alpha_1 + 2\alpha_2 = (1, 0, -1)\end{aligned}\quad (59)$$

The dual roots are:  $\alpha_1^\vee = a_1/3, \alpha_2^\vee = a_2, (\alpha_1 + \alpha_2)^\vee = \alpha_1 + \alpha_2 = 3\alpha_1^\vee + \alpha_2^\vee, (\alpha_1 + 2\alpha_2)^\vee = \alpha_1 + 2\alpha_2 = 3\alpha_1^\vee + 2\alpha_2^\vee, (\alpha_1 + 3\alpha_2)^\vee = (\alpha_1 + 3\alpha_2)/3 = \alpha_1^\vee + \alpha_2^\vee, (2\alpha_1 + 3\alpha_2)^\vee = (2\alpha_1 + 3\alpha_2)/3 = 2\alpha_1^\vee + \alpha_2^\vee$ .

The Weyl group of  $G_2$  is the dihedral group of order 12. This follows from the fact that  $(s_1 s_2)^6 = 1$ , where  $s_1, s_2$  are the two simple reflections.

The algebra  $G_2$  has one non-compact real form:  $\mathcal{G} = G_{2(2)}$ , which is naturally split. Its maximal compact subalgebra is  $\mathcal{K} = su(2) \oplus su(2)$ . Thus,  $\mathcal{G} = G_{2(2)}$  has discrete series representations. The complimentary space  $\mathcal{Q}$  is eight-dimensional. The positive root system of  $\mathcal{K}^\mathbb{C}$  consists of  $\alpha_2 = (0, 1, -1), 2\alpha_1 + 3\alpha_2 = (2, -1, -1)$  (chosen to be orthogonal to each other).

The minimal parabolic of  $\mathcal{G}$  is:

$$\begin{aligned}\mathcal{P}_0 &= \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 \\ \mathcal{M}_0 &= 0, \quad \dim \mathcal{A}_0 = 2, \quad \dim \mathcal{N}_0 = 6\end{aligned}\quad (60)$$

There are two isomorphic maximal parabolic subalgebras of  $\mathcal{G}$  which are of Heisenberg type:

$$\begin{aligned}\mathcal{P}_k &= \mathcal{M}_k \oplus \mathcal{A}_k \oplus \mathcal{N}_k, \quad k = 1, 2, \\ \mathcal{M}_k &= sl(2, \mathbb{R})_k, \quad \dim \mathcal{A}_k = 1, \quad \dim \mathcal{N}_k = 5\end{aligned}\quad (61)$$

where  $sl(2, \mathbb{R})_k$  inherits from  $\mathcal{G}^\mathbb{C}$  the simple root  $\alpha_k$  ( $k = 1, 2$ ). Equivalently, the  $\mathcal{M}_k$ -compact root of  $\mathcal{G}^\mathbb{C}$  is  $\alpha_k$  ( $k = 1, 2$ ). In each case, the remaining five positive roots of  $\mathcal{G}^\mathbb{C}$  are  $\mathcal{M}_k$ -noncompact.

The positive roots of  $\mathcal{G}^\mathbb{C}$  in terms of the simple roots will be denoted as :

$$\gamma_1 = \alpha_1, \quad \gamma_{13} = \alpha_1 + 3\alpha_2, \quad \gamma_{23} = 2\alpha_1 + 3\alpha_2 \quad (62a)$$

$$\gamma_2 = \alpha_2, \quad \gamma_{11} = \alpha_1 + \alpha_2, \quad \gamma_{12} = \alpha_1 + 2\alpha_2 \quad (62b)$$

(where, as above, in (62a) are the long roots, in (62b) are the short roots).

To characterize the Verma modules, we shall use first the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^\vee), \quad i = 1, 2, \quad (63)$$

where  $\rho$  is half the sum of the positive roots of  $\mathcal{G}^\mathbb{C}$ . Thus, we shall use:

$$\chi_\Lambda = \{m_1, m_2\} \quad (64)$$

Note that when all  $m_i \in \mathbb{N}$ , then  $\chi_\Lambda$  characterizes the finite-dimensional irreps of  $\mathcal{G}^\mathbb{C}$  and its real forms, in particular,  $\mathcal{G}$ . Furthermore,  $m_k \in \mathbb{N}$  characterizes the finite-dimensional irreps of the  $\mathcal{M}_k$  subalgebra.

We shall use also the Harish-Chandra parameters:

$$m_{ij} = (\Lambda + \rho, \gamma_{ij}^\vee), \quad (65)$$

and explicitly in terms of the Dynkin labels:

$$\chi_{HC} = \{m_1, m_{13} = m_1 + m_2, \quad m_{23} = 2m_1 + m_2, \quad (66a)$$

$$m_2, m_{11} = 3m_1 + m_2, \quad m_{12} = 3m_1 + 2m_2\} \quad (66b)$$

### 9.1. Induction from Minimal Parabolic Main Multiplets

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of  $G_2$ ; i.e., they are labeled by the two positive Dynkin labels  $m_i \in \mathbb{N}$ . When we induce from the minimal parabolic, the main multiplets of  $\mathcal{G}$  are the same as for the complexified Lie algebra  $\mathcal{G}^{\mathbb{C}}$ .

We take  $\chi_0 = \chi_{HC}$ . It has two embedded Verma modules with HW  $\Lambda_1 = \Lambda_0 - m_1\gamma_1$ , and  $\Lambda_2 = \Lambda_0 - m_2\gamma_2$ . The number of ERs/GVMs in a main multiplet is  $12 = |W(\mathcal{G}^{\mathbb{C}})|$ . We give the whole multiplet as follows:

$$\begin{aligned}
 \chi_0 &= \{m_1, m_2; -\frac{1}{2}(2m_1 + m_2)\} \\
 \chi_1 &= \{-m_1, 3m_1 + m_2; -\frac{1}{2}(m_1 + m_2)\}, \quad \Lambda_1 = \Lambda_0 - m_1\gamma_1 \\
 \chi_2 &= \{m_1 + m_2, -m_2; -\frac{1}{2}(2m_1 + m_2)\}, \quad \Lambda_2 = \Lambda_0 - m_2\gamma_2 \\
 \chi_{12} &= \{-m_1 - m_2, 3m_1 + 2m_2; -\frac{1}{2}m_1\}, \quad \Lambda_{12} = \Lambda_1 - m_2\gamma_{11} \\
 \chi_{21} &= \{2m_1 + m_2, -3m_1 - m_2; -\frac{1}{2}(m_1 + m_2)\}, \quad \Lambda_{21} = \Lambda_2 - m_1\gamma_{13} \\
 \chi_{121} &= \{-2m_1 - m_2, 3m_1 + 2m_2; \frac{1}{2}m_1\}, \quad \Lambda_{121} = \Lambda_{12} - m_1\gamma_{23} \\
 \chi_{212} &= \{2m_1 + m_2, -3m_1 - 2m_2; -\frac{1}{2}m_1\}, \quad \Lambda_{212} = \Lambda_{21} - m_2\gamma_{12} \\
 \chi_{1212} &= \{-2m_1 - m_2, 3m_1 + m_2; \frac{1}{2}(m_1 + m_2)\}, \quad \Lambda_{1212} = \Lambda_{121} - m_2\gamma_{12} \\
 \chi_{2121} &= \{m_1 + m_2, -3m_1 - 2m_2; \frac{1}{2}m_1\}, \quad \Lambda_{2121} = \Lambda_{212} - m_1\gamma_{23} \\
 \chi_{12121} &= \{-m_1 - m_2, m_2; \frac{1}{2}(2m_1 + m_2)\}, \quad \Lambda_{12121} = \Lambda_{1212} - m_1\gamma_{13} \\
 \chi_{21212} &= \{m_1, -3m_1 - m_2; \frac{1}{2}(m_1 + m_2)\}, \quad \Lambda_{21212} = \Lambda_{2121} - m_2\gamma_{11} \\
 \chi_{121212} &= \{-m_1, -m_2; \frac{1}{2}(2m_1 + m_2)\} = \chi_{212121}, \\
 \Lambda_{121212} &= \Lambda_{12121} - m_2\gamma_2 = \Lambda_{21212} - m_1\gamma_1 \\
 \Lambda_{21} &= \Lambda_1 - (3m_1 + m_2)\gamma_2 \\
 \Lambda_{212} &= \Lambda_{12} - (3m_1 + 2m_2)\gamma_2 \\
 \Lambda_{2121} &= \Lambda_{121} - (3m_1 + 2m_2)\gamma_2 \\
 \Lambda_{21212} &= \Lambda_{1212} - (3m_1 + m_2)\gamma_2 \\
 \Lambda_{12} &= \Lambda_2 - (m_1 + m_2)\gamma_1 \\
 \Lambda_{121} &= \Lambda_{21} - (2m_1 + m_2)\gamma_1 \\
 \Lambda_{1212} &= \Lambda_{212} - (2m_1 + m_2)\gamma_1 \\
 \Lambda_{12121} &= \Lambda_{2121} - (m_1 + m_2)\gamma_1
 \end{aligned} \tag{67}$$

where we have included as third entry also the parameter  $c = -\frac{1}{2}(2m_1 + m_2)$ , related to the Harish-Chandra parameter of the highest root (recalling that  $m_{\gamma_{23}} = 2m_1 + m_2$ ). It is also related to the conformal weight  $d = \frac{3}{2} + c$ .

The ER  $\chi_{121212}$  contains discrete series representation according to the Harish-Chandra criterion [3] (all HC parameters are negative).

These labeling signatures may be given in the following pair-wise manner:

$$\begin{aligned}
 \chi_0^{\pm} &= \{\mp m_1, \mp m_2; \pm \frac{1}{2}(2m_1 + m_2)\} \\
 \chi_1^{\pm} &= \{\pm m_1, \mp (3m_1 + m_2); \pm \frac{1}{2}(m_1 + m_2)\}, \\
 \chi_2^{\pm} &= \{\mp (m_1 + m_2), \pm m_2; \pm \frac{1}{2}(2m_1 + m_2)\}, \\
 \chi_{12}^{\pm} &= \{\pm (m_1 + m_2), \mp (3m_1 + 2m_2); \pm \frac{1}{2}m_1\} \\
 \chi_{21}^{\pm} &= \{\mp (2m_1 + m_2), \pm (3m_1 + m_2); \pm \frac{1}{2}(m_1 + m_2)\} \\
 \chi_{121}^{\pm} &= \{\pm (2m_1 + m_2), \mp (3m_1 + 2m_2); \mp \frac{1}{2}m_1\},
 \end{aligned} \tag{68}$$

where  $\chi_{...}^- = \chi_{...}$  from (67),  $\chi_0^+ = \chi_{121212}$ ,  $\chi_1^+ = \chi_{21212}$ ,  $\chi_2^+ = \chi_{12121}$ ,  $\chi_{12}^+ = \chi_{2121}$ ,  $\chi_{21}^+ = \chi_{1212}$ ,  $\chi_{121}^+ = \chi_{212}$ .

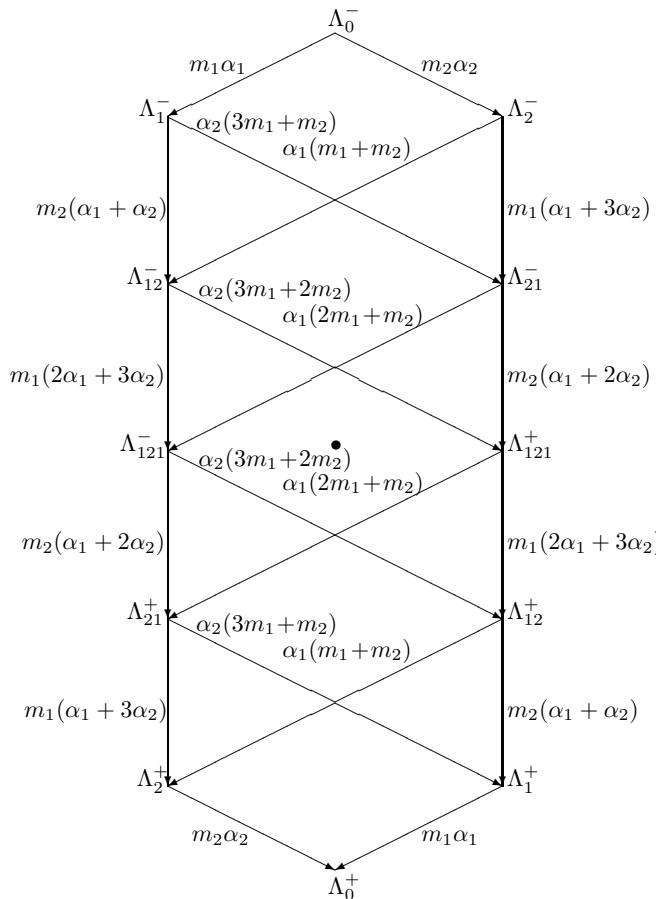
The ERs in the multiplet are also related by intertwining integral operators introduced in [233]. These operators are defined for any ER, the general action in our situation being:

$$G_{KS} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}, \quad \chi = [n_1, n_2; c], \quad \chi' = [-n_1, -n_2; -c]. \quad (69)$$

This action is consistent with the parameterization in (68).

The main multiplets are given explicitly in Figure 16. The pairs  $\chi^\pm$  are symmetric with regard to the bullet in the middle of the picture—this symbolizes the Weyl symmetry realized by the Knapp–Stein operators:

$$G_{KS} : \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm}.$$



**Figure 16.** Main multiplets for  $G_{2(2)}$  using induction from the minimal parabolic.

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature  $\chi_0^-$  contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace  $\mathcal{E}$ . The latter corresponds to the finite-dimensional irrep of  $G_{2(2)}$  with signature  $[m_1, m_2]$ . The subspace  $\mathcal{E}$  is annihilated by the operator  $G^+$ , and is the image of the operator  $G^-$ . When all  $m_i = 1$ , then  $\dim \mathcal{E} = 1$ , and in that case,  $\mathcal{E}$  is also the trivial one-dimensional UIR of the whole algebra  $\mathcal{G}$ . Furthermore, in that case, the conformal weight is zero:  $d = \frac{3}{2} + c = \frac{3}{2} - \frac{1}{2}(2m_1 + m_2)_{|m_i=1} = 0$ .

In the conjugate ER  $\chi_0^+$  there is a unitary discrete series subrepresentation in an infinite-dimensional subspace  $\mathcal{D}$  with conformal weight  $d = \frac{3}{2} + c = \frac{7}{2} + \frac{1}{2}(2m_1 + m_2)$ . It is annihilated by the operator  $G^-$ , and is the image of the operator  $G^+$ .

## 9.2. Induction from Maximal Parabolics

### 9.2.1. Main Multiplets When Inducing from $\mathcal{P}_1$

When inducing from the maximal parabolic  $\mathcal{P}_1 = \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{N}_1$  there is one  $\mathcal{M}_1$ -compact root, namely,  $\alpha_1$ . We take again the Verma module with  $\Lambda_{HC} = \Lambda_0^{1-}$ . We take  $\chi_0^{1-} = \chi_{HC}$ . The GVM  $\Lambda_0^{1-}$  has one embedded GVM with HW  $\Lambda_2^{1-} = \Lambda_0^{1-} - m_2\gamma_2$ ,  $m_2 \in \mathbb{N}$ . Altogether, the main multiplet in this case includes the same number of ERs/GVMs as in (32), so we use the same notation only adding super index 1, namely

$$\begin{aligned}\chi_0^{1\pm} &= \{\mp m_1, \mp m_2; \pm \frac{1}{2}(2m_1 + m_2)\} \\ \chi_1^{1\pm} &= \{\pm m_1, \mp(3m_1 + m_2); \pm \frac{1}{2}(m_1 + m_2)\}, \\ \chi_2^{1\pm} &= \{\mp(m_1 + m_2), \pm m_2; \pm \frac{1}{2}(2m_1 + m_2)\}, \\ \chi_{12}^{1\pm} &= \{\pm(m_1 + m_2), \mp(3m_1 + 2m_2); \pm \frac{1}{2}m_1\} \\ \chi_{21}^{1\pm} &= \{\mp(2m_1 + m_2), \pm(3m_1 + m_2); \pm \frac{1}{2}(m_1 + m_2)\} \\ \chi_{121}^{1\pm} &= \{\pm(2m_1 + m_2), \mp(3m_1 + 2m_2); \mp \frac{1}{2}m_1\},\end{aligned}\tag{70}$$

In addition, in order to avoid coincidence with (35) we must impose in (70) the conditions:  $m_1 \notin \mathbb{N}$ ,  $m_1 \notin \mathbb{N}/2$ ,  $m_1 \notin \mathbb{N}/3$ .

What is peculiar is that the ERs/GVMs of the main multiplet (70) actually consist of three submultiplets with intertwining diagrams as follows:

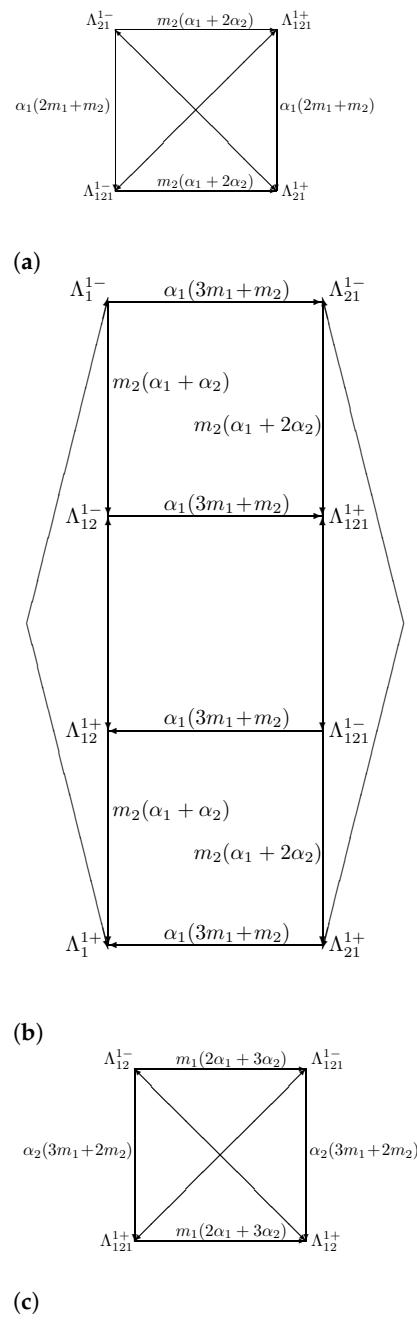
$$\begin{array}{ccc} \Lambda_0^{1-} & \xrightarrow{\mathcal{D}_{\gamma_2}^{m_2}} & \Lambda_2^{1-} \\ \downarrow & & \downarrow \text{ subtype (A}_1\text{)} \end{array}\tag{71a}$$

$$\begin{array}{ccc} \Lambda_0^{1+} & \xleftarrow{\mathcal{D}_{\gamma_2}^{m_2}} & \Lambda_2^{1+} \\ \downarrow & & \downarrow \text{ subtype (B}_1\text{)} \end{array}\tag{71b}$$

$$\begin{array}{ccc} \Lambda_1^{1-} & \xrightarrow{\mathcal{D}_{\gamma_{11}}^{m_2}} & \Lambda_{12}^{1-} \\ \downarrow & & \downarrow \text{ subtype (C}_1\text{)} \\ \Lambda_1^{1+} & \xleftarrow{\mathcal{D}_{\gamma_{11}}^{m_2}} & \Lambda_{12}^{1+} \\ \Lambda_{21}^{1-} & \xrightarrow{\mathcal{D}_{\gamma_{12}}^{m_2}} & \Lambda_{121}^{1+} \\ \downarrow & & \downarrow \text{ subtype (C}_1\text{)} \\ \Lambda_{21}^{1+} & \xleftarrow{\mathcal{D}_{\gamma_{12}}^{m_2}} & \Lambda_{121}^{1-} \end{array}\tag{71c}$$

Next, we relax in (70) one of the conditions, namely, we allow  $m_1 \in \mathbb{N}/2$ , still keeping  $m_1 \notin \mathbb{N}$ ,  $m_1 \notin \mathbb{N}/3$ . This changes the diagram of subtype (C<sub>1</sub>), (71c), as given in Figure 17a below.

Next, we relax in (70) another condition, namely, we allow  $m_1 \in \mathbb{N}/3$ , still keeping  $m_1 \notin \mathbb{N}$ ,  $m_1 \notin \mathbb{N}/2$ . This changes the diagrams of subtypes (B<sub>1</sub>) and (C<sub>1</sub>) combining them as given in Figure 17b below.



**Figure 17.** (a) Submultiplets type ( $C_1$ ) for  $G_{2(2)}$  using induction from the maximal parabolic  $P_1$  for  $m_2 \in \mathbb{N}$ ,  $m_1 \notin \mathbb{N}$ ,  $m_1 \in \mathbb{N}/2$ ,  $m_1 \notin \mathbb{N}/3$ ; the (anti)diagonal arrows represent the KS operators. (b) Submultiplets type ( $B_1$ )+ ( $C_1$ ) for  $G_{2(2)}$  using induction from the maximal parabolic  $P_1$  for  $m_2 \in \mathbb{N}$ ,  $m_1 \notin \mathbb{N}$ ,  $m_1 \notin \mathbb{N}/2$ ,  $m_1 \in \mathbb{N}/3$ ; the up-down arrows represent four pairs of KS operators. (c) Submultiplets type ( $C_2$ ) for  $G_{2(2)}$  using induction from the maximal parabolic  $P_2$  for  $m_1 \in \mathbb{N}$ ,  $m_2 \notin \mathbb{N}/2$ ; the (anti)diagonal arrows represent the KS operators.

### 9.2.2. Main Multiplets When Induction from $P_2$

This case is partly dual to the previous one. When inducing from the maximal parabolic  $P_2 = \mathcal{M}_2 \oplus \mathcal{A}_2 \oplus \mathcal{N}_2$ , there is one  $\mathcal{M}_2$ -compact root, namely,  $\alpha_2$ . We take again the Verma module with  $\Lambda_{HC} = \Lambda_0^{2-}$ . We take  $\chi_0^{2-} = \chi_{HC}$ . The GVM  $\Lambda_0^{2-}$  has one embedded GVM with HW  $\Lambda_1^{2-} = \Lambda_0^{2-} - m_1\gamma_1$ ,  $m_1 \in \mathbb{N}$ . Altogether, the main multiplet in this case includes the same number of ERs/GVMs as in (32), so we use the same notation only adding super index 2, namely

$$\begin{aligned}
\chi_0^{2\pm} &= \{\mp m_1, \mp m_2; \pm \frac{1}{2}(2m_1 + m_2)\} \\
\chi_1^{2\pm} &= \{\pm m_1, \mp(3m_1 + m_2); \pm \frac{1}{2}(m_1 + m_2)\}, \\
\chi_2^{2\pm} &= \{\mp(m_1 + m_2), \pm m_2; \pm \frac{1}{2}(2m_1 + m_2)\}, \\
\chi_{12}^{2\pm} &= \{\pm(m_1 + m_2), \mp(3m_1 + 2m_2); \pm \frac{1}{2}m_1\} \\
\chi_{21}^{2\pm} &= \{\mp(2m_1 + m_2), \pm(3m_1 + m_2); \pm \frac{1}{2}(m_1 + m_2)\} \\
\chi_{121}^{2\pm} &= \{\pm(2m_1 + m_2), \mp(3m_1 + 2m_2); \mp \frac{1}{2}m_1\},
\end{aligned} \tag{72}$$

In addition, in order to avoid coincidence with (35) we must impose in (70) the conditions  $m_2 \notin \mathbb{N}$ ,  $m_2 \notin \mathbb{N}/2$ .

Similarly to the  $\mathcal{P}_1$  case, the ERs/GVMs of the main multiplet (72) actually consist of three submultiplets with intertwining diagrams as follows:

$$\begin{array}{ccc}
\Lambda_0^{2-} & \xrightarrow{\mathcal{D}_{\gamma_1}^{m_1}} & \Lambda_1^{2-} \\
\downarrow & & \downarrow \text{ subtype (A}_2\text{)}
\end{array} \tag{73a}$$

$$\begin{array}{ccc}
\Lambda_0^{2+} & \xleftarrow{\mathcal{D}_{\gamma_1}^{m_1}} & \Lambda_1^{2+} \\
\Lambda_2^{2-} & \xrightarrow{\mathcal{D}_{\gamma_{13}}^{m_1}} & \Lambda_{21}^{2-} \\
\downarrow & & \downarrow \text{ subtype (B}_2\text{)}
\end{array} \tag{73b}$$

$$\begin{array}{ccc}
\Lambda_2^{2+} & \xleftarrow{\mathcal{D}_{\gamma_{13}}^{m_1}} & \Lambda_{21}^{2+} \\
\Lambda_{12}^{2-} & \xrightarrow{\mathcal{D}_{\gamma_{23}}^{m_1}} & \Lambda_{121}^{2-} \\
\downarrow & & \downarrow \text{ subtype (C}_2\text{)}
\end{array} \tag{73c}$$

Next, we relax in (70) one of the conditions, namely, we allow  $m_2 \in \mathbb{N}/2$ , still keeping  $m_2 \notin \mathbb{N}$ . This changes the diagram of subtype (C<sub>2</sub>), (73c), as given in Figure 17c.

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**Conflicts of Interest:** The author declares no conflict of interest.

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