AN OVERVIEW OF EXACT SOLUTIONS OF EINSTEIN'S EQUATIONS

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Abstract

In this note an overview of reviews on exact solutions for stationary axisymmetric fields and anisotropic cosmological models has been presented. Senovilla class of singularity free space time is reviewed.

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A final observation bearing on the attitude to physical problems that we have maintained in our studies of the mathematical theories of black holes and of colliding waves: in the theory of spacetimes with two Killing fields one timelike and one spacelike or both spacelike- the basic governing equations are one or more Laplace's equations, the Ernst equations, and the X- and Y-equations. The simplest solutions of the Laplace and the Ernst equations have provided all the fundamental solutions describing black holes and colliding waves. But the simplest solution of the X- and Y- equations have been left out. At long last it has found its place: it provides the first nontrivial binary black hole solution with supporting strings.

— S. CHANDRASEKHAR.

1 Introduction

Einstein's field equations

$$G^{ij} = -8\pi T^{ij} \tag{1.1}$$

where T^{ij} represents the energy momentum tensor of the matter and field, are second order non linear partial differential equations. There are basically three approaches to study them : exact solutions, approximation schemes and numerical computation. These approaches are listed in order of decreasing aesthetic appeal. The exact solutions as explained below, are necessarily very special and constitute a measure of zero in the space of all solutions. However, exact solutions play an important role in generating tests for numerical codes [Centrella (1986)] and also for providing checks on the validity of the approximation schemes. They can also provide models for important, though oversimplified, physical situations e.g. the Schwarzchild, and Kerr black holes and the Friedmann cosmological model on which almost all of the relativistic astrophysics and astrophysical cosmology is based. Besides, they can also provide counter-examples to conjectures [Misner (1963)]; for a discussion of these aspects one may refer to [MacCallum (1984, 1985 a,b)].

Due to the nonlinearity of the Einstein's equations it is impossible to solve these equations in their full generality. The usual practice is to assume, at the outset, some symmetry e.g. Cylindrical, Planar, Axial or the ones described in terms of Killing vectors. In addition to this, one has to resort to some specific energy momentum tensor. The most commonly used energy momentum tensors are that characterising dust, perfect fluid, electromagnetic field, imperfect fluid and radiation field. At this point it is to be noted that the solutions where energy momentum tensor is generalized by way of introducing, say null radiation, electromagnetic field, heat conducting fluid etc., and which do not leave any of its trace in the metric coefficients are not important unless a physical interpretation of decomposing the energy momentum tensor and an explanation of how ambiguities in the decomposition are to be removed is provided. [MacCallum (1987), Krasinski (1993)].

Exact solutions is one of the topics which attract many scientists working in General Relativity. About 10% of the research papers at the International conferences on the subject and 40% on the average at the Indian conferences are devoted to this topic. Hence, it is very difficult to present an overview of the subject in such a short duration unless some selection rules and procedures are adopted. Fortunately, in the literature, good review articles are available from time to time on this topic and it would be easier to paint a 'broad-brush' picture of the state of affairs by looking at these reviews. I would present an overview of these reviews with the year 1974 as the cut off. Besides, we have to narrow down our interests to (i) stationary axi-symmetric fields and (ii) inhomogeneous cosmologies. This restriction, is guided mainly by the fact that the physically relevant solutions fall into these two subtopics but reflects my personal interests and prejudices as well.

2 Stationary Axisymmetric Fields

Kinnersley (1974) describes briefly all known vacuum solutions including electrovacuum and their inter-relationships. He discusses space times admitting two commuting 2-forming Killing vectors containing (i) Tomimatsu-Sato solution (ii) stationary axially symmetric solutions and various transformation theorems for generating solutions from other known solutions. Here, the algebraically special solutions with non-diverging rays and with non rotating rays have also been discussed.

Starting from the early seventies and for about a decade there has been active interest and considerable progress in obtaining new axisymmetric stationary solutions of Einstein's field equations. The main motivation was the close resemblance of Einstein's equations, for this case, with the non-linear differential equations e.g. Sine-Gordon or Korteweg-de Vries (KdV) equations arising in other branches of Physics. Consequently, extensive use of Backlund transformations and Soliton techniques has been made. The various topics e.g. Hoensaelers, Kinnersley, Xanthopolous (H K X) transformations, Kinnersley-Chitre(K C) transformation, Geroch group have been discussed thoroughly and every aspect has been considered in detail at the Seminar with the title "Solutions of Einstein's Equation : Techniques and Results" edited by Hoensealers and Dietz (1994). For the benefit of the reader list of topics covered there are included as Appendix I. For an earlier review of the topic one may refer to Kramer and Stephani (1980).

2.1 Anisotropic Cosmology

The well known Friedmann-Lemaitre Robertson-Walker (FLRW) model, represents Universe in which all space points and all directions at any space point are equivalent. If one assumes the matter to be represented by dust the Universe turns out to be homogeneous. The isotropy on a large scale is confirmed by experiments but the issue whether the Universe was homogeneous and isotropic already at its early stages and has it still these properties in very distant regions is so far unresolved. Besides, there are mathematical arguments in favour of inhomogeneous models [Tavakole and Ellis (1988)]. It is to be pointed that even at the very early stage of Relativistic Cosmology several physicists were able to see the importance and necessity of investigating inhomogeneous cosmological models [Dingle (1933), Tolman (1934)]. For a detailed discussion of these aspects one may refer to a recent review by Krasinski (1993) where one may find topics e.g. memorable statements about the cosmological principle, why one should consider inhomogeneous models of the universe. In view of the above there has always been active interest in inhomogeneous cosmological models and in the literature several good review articles e.g. MacCallum (1984), Jantzen (1987), Kramer and stephani (1980) are available for a recent review one may refer to Krasinski [1993, 1994].

2.1.1 Senovilla Class of Solution

Senovilla (1990) has discovered an important solution representing a perfect fluid distribution with cylindrical symmetry and obeying an equation of state $\rho = 3p$. This solution represents the distribution for $-\infty < t < \infty$ having the curvature as well as matter invariants regular and smooth everywhere. The fluid flow lines have inhomogeneous expansion, nonvanishing shear and anisotropic acceleration. Until now the conventional and prevalent view of Cosmology was that of FLRW which represents spherically symmetric isotropic and homogeneous perfect fluid distribution with vanishing shear and homogeneous acceleration. The anisotropic and inhomogeneous models obtained by earlier workers viz. Wainwright and Goode (1980), Feinstein and Senovilla (1980), Davidson (1991), possess space - like Big Bang singularity. Consequently, it was considered that this would represent general singularity structure in other models as well. This experience was strongly aided by the singularity theorems, according to which if one adheres to Einstein's theory of General Relativity and assumes physically reasonable conditions of positivity of energy, causality and regularity etc, the initial singularity ($t = 0$) is inescapable. Chinea, Fernandez-Jambrina and Senovilla (1992) have carried out the analysis of geodesics and have found that the Senovilla spacetime is geodesically complete. Senovilla and his coworkers have made a thorough scrutiny to unravel how this solution could avoid the powerful singularity theorem. Senovilla argues that the general reason presented by Hawking and Ellis (1973) in establishing the singularity theorem makes use of the assumption of geodesic motion of the cosmological matter which is clearly not supported by any theoretical reasoning. He further finds that the singularity free nature of his solution is in accordance with Raychaudhuri's work [1955] according to which the presence of acceleration or rotation may prevent the existence of a universal singularity in our past [Senovilla (1995)]. The other reason is attributed to the shear of the fluid flow- lines which is nonvanishing and to the nonexistence of compact surfaces. This is interpreted to mean that nowhere in the spacetime gravity becomes strong enough to focus geodesics in a small compact region for the trapping of the all particles including photons to take place [Dadhich, Tikekar and Patel (1993)]. Recently Ruiz and Senovilla (1992) have obtained a general class of inhomogeneous perfect fluid solution which contains singular solutions due to Wainwright and Goode (1980), Feinstein and Senovilla (1989) and the solution being discussed presently. It is easier to understand Senovilla's solution if we express FLRW solution in cylindrical coordinates (τ, r, ϕ, z) as

$$ds^2 = T^2[-d\tau^2 + dr^2 + (1 + M \sum^2) d\phi^2 + \sum^2 dz^2] \quad (2.1)$$

where T is an arbitrary function of time and M is an arbitrary constant ; its sign is related the usual curvature index in FLRW model. $\sum$ is a function of r given by

$$\sum'^2 = 1 + M \sum^2 \quad (2.2)$$

where a prime indicates derivative with respect to r. The density ρ and the pressure p of the fluid are given by

$$\rho = \frac{3}{T^2}\left(\frac{\dot{T}^2}{T^2} - M\right) \quad , \quad p = -\frac{1}{T^2}\left(\frac{2\dot{T}}{T} - \frac{\dot{T}^2}{T^2} - M\right) \quad (2.3)$$

where an overhead dot represents derivative with respect to time. The Senovilla solution is given by

$$\begin{aligned} ds^2 &= T^{2(1+n)} \sum^{2n(n-1)} \left[-d\tau^2 + dr^2\right] \\ &+ T^{2(1+n)} \sum^{2n} \sum'^2 d\phi^2 \\ &+ T^{2(1-n)} \sum^2 (1-n) dz^2 \end{aligned} \quad (2.4)$$

where $\sum$ is a function of r satisfying

$$\sum'^2 = M \sum^2 + 1 - nK \sum^{2(1-2n)} \quad (2.5)$$

and M, n and k are arbitrary constants. There arises two cases :

$$(i) \quad n = 0 \quad \text{and} \quad T \quad \text{arbitrary} \quad (2.6)$$

$$(ii) \quad n = 0 \quad \text{and} \quad T^2 = \left\{ \begin{array}{lll} A \cos h(2\sqrt{M}C) + B \sin h(2\sqrt{\tau}) & , & M > 0 \\ A\tau + B & , & M = 0 \\ A \cos h(2 - \sqrt{-M}\tau) + B \sin h(2\sqrt{-M}\tau) & , & M < 0 \end{array} \right\} \quad (2.7)$$

The expansion θ, acceleration a_i and shear σ_{ij} have their nonvanishing components as given by

$$\left. \begin{aligned} \theta &= (n+3) \sum^{n(1-n)} \frac{\dot{T}}{T^{n+2}} \\ a_1 &= n(n-1) \sum^{n(1-n)} \frac{\sum}{T^{n+1} \sum} \\ \sigma_{11} &= \sigma_{22} = \sigma_{33} = \frac{2n}{3} \sum^{n(1-n)} \frac{\dot{T}}{T^{n+2}} \end{aligned} \right\} \quad (2.8)$$

The anisotropy of the fluid motion, the scale factor $R(\tau)$ and deceleration parameter q are given by

$$\left. \begin{aligned} \frac{\sigma}{\theta} &= \frac{2}{\sqrt{3}}\left[1 - \frac{3}{n+3}\right] \\ R(\tau) &= T^{(n+3)/3} \\ q &= 2 - \frac{n}{n+3}\left(1 - \frac{T\dot{T}}{T^2}\right) \end{aligned} \right\} \quad (2.9)$$

The solution reduces to FLRW solution for $n = 0$. The physical scenario described by this solution is as follows. The distribution if ; we assume it initially to contract, collapses to $r = 0$ and then reverses its motion and expands for ever. This model may be compatible with observational fact that the present Universe is to be homogeneous and expanding provided one prefers to have the dynamics of the distribution as given by

$$T(\tau) \sim \begin{cases} T_{SF} & \text{around } \tau = 0 \text{ with any } n \\ T_{FLRW} & \text{for } C = \bar{\tau} \text{ with } n = 0 \end{cases} \tag{2.10}$$

where T_{SF} represents the time scale for singularity free Senovilla solution valid up to $\bar{\tau}$, the instant of bounce and T_{FLRW} is the time scale as given by FLRW solution. However, there are many conceptual issues to be resolved. For a discussion of these one may refer to Senovilla (1995). Some recent works attempting to establish the uniqueness of Senovilla class of cylindrically symmetric solution among separable, irrotational space times are going on and encouraging results are reported [Dadhich, Patel, Govinder and Leach (1994), Dadhich, and Patel (1995).]

2.1.2 Spherically Symmetric Non-vanishing Shear Models

I have remarked above that the existence of the shear of the four velocity is attributed to be a reason for resulting in a singularity free solution. For spherically symmetric perfect fluid distribution with vanishing shear there is no dearth of exact solutions. [Srivastava (1987,1992)] But the investigations with non-vanishing shear are rare and may be counted. The solutions for a stiff matter [Wesson (1978), Vaidya (1968)], and for equation of state $\rho = \rho(p)$, under the assumption of separability of metric coefficientists [Vandenberg and Wils (1985)] are available. The additional assumptions to yield a solution are (i) existence of self similarity [Collins and Lang (1987)], (ii) The four velocity begin perpendicular to surface = constant [Vaidya 1968)]. Biech and Das (1990) have shown null coordinates to be useful for obtaining solutions of this class. This investigation also presents reviews of the earlier works. Recently an interesting result has emerged out of the earlier works of Mc Vittie and Wiltshire (1977) ; the solution is found to have shear. [Bonnor and Knutsen (1993), and Knutsen (1995)]. This investigation employs noncomoving coordinates. It is interesting to remark that investigations for solutions with shear has been a topic of interest in the very early days. [Narlikar (1936), Narlikar and Moghe (1935)].

2.1.3 Shearfree Models

In contrast to the above there is sufficient reason to study the perfect fluid distributions with vanishing shear. As has been discussed by Collins (1986) shear free solutions would retain the feature of isotropy of local motions but redshifts need not be isotropic. It is shown following Ellis (1971) that relative recessional motion of the neighbouring galaxies is isotropic iff $\sigma_{ij} = 0$ whereas for the relative red shift to be isotropic one would need in additon vanishing of $\dot{U}_i$. Further the isotropy of the transverse motion of the neighbouring galaxies would require $\sigma_{ij} = w_{ij} = 0$. Thus $\sigma_{ij} = 0$ is a common feature of all these aspects. For a review of shear free perfect fluid solutions one may refer to Barnes (1983). Besides, there is another strong reason to study shear free models. It is because of following conjecture due to Collins [1986].

$$\sigma_{ij} = 0 \quad \Rightarrow \quad w\theta = 0 \tag{2.11}$$

This conjecture is based on the observation that for shear free motions;it must have either $w = 0$ or $\theta = 0$. The conjecture has been proved for (i) and parallel to each other [White and Collins (1984)] and (ii) the vanishing magnetic part of the Weyl tensor [White (1986)]. The importance of the study is in obtaining a counter-example to the conjecture because in Newtonian theory it is fairly readily shown that for because in Newtonian theory it is fairly readily shown that for self gravitating shear free fluids neither vorticity nor expansion vanish. The intuitive picture in general relativity would therefore, be illusive if conjecture holds. At this point I would like to quote Chandrasekher.

"On the relativistic theory, the frequencies of oscillation of the non-radial models (as we have shown) depend only on the distribution of the energy density and the pressure in the static configuration and the equation of state only to the extent of its adiabatic exponent. If this is a true representation of the physical situation then it must be valid in the Newtonian theory as well : the true nature of an object cannot change with modes and manner of one's perception".

— S. CHANDRASEKHAR (1991).

3 Commentary

Kinnersely (1974) while presenting his talk on "Recent progress in exact solutions" at the GR7 International Conference has remarked "the study of exact solutions has acquired a rather low reputation in the past, for which there are several explanations. Most of known exact solutions describe situations which are frankly unphysical and these do have a tendency to distract attention from useful ones. But the situation is also partially the fault of us who work in the field. We toss in null currents, macroscopic neutrino fields and tachyons for the sake of greater "generality"; We seem to take delight at the invention of confusing anti-intuitive notion ; and when all is done we leave our newborn wobbling on its vierbien without any visible means of interpretation".

More or less similar views have been expressed by later workers. There is another aspect of the problem. We have too many solutions rather than too few solutions. Simple minded attempts to derive a new solution from natural assumptions are likely to result in yet another discovery. As a side effect, some simple results are published again and again while important fine developments remain unnoticed for a long time [Krasinski (1993)]. For example, the investigation of Kustaanheimo and Kvist (1948) pertaining to shear free perfect fluids could only be noticed after the publication of book by Kramer et al. (1980). To avoid such repetition and also for our own sake we should not stop just after obtaining a solution but should look for invariant characterisation. At present the literature is full of materials where these aspects are clearly discussed e.g. MacCallum (1987), Krasinski (1994), Kramer et al. (1980).

In fact even in the early seventies it was being felt that number of exact solutions was growing; consequently to any serious minded research worker in the field it was necessary to have a stock - taking of known solutions in order to avoid duplication. The notable investigations among these are Kramer, Neugebauer and Stephani (1972) and Kinnersley (1974). Kramer et. al started in 1975 an exhaustive survey of the literature on exact solutions of Einstein's field equations and have published their results in the form of a book [Kramer et. al (1980)]. It is devoted to the classification and construction of exact solutions. They have discussed the classification schemes for exact solutions viz. (i) algebraic classification of conformal curvature tensor (Petrov types), (ii) algebraic classification of the Ricci tensor (Plebanski types) and the physical characterisation of the energy momentum tensor, (iii) the existence and structure of the preferred vector fields and (iv) the group of symmetry admitted by the metric. However, they have based the classification of the known solutions primarily on (i) symmetry group and (ii) Petrov types and subdivided these to include other main classification schemes. Besides, these studies have remarkably brought to our notice many old works to name a few, Wyman (1946), Narlikar (1947), Kustaanheimo and Qvist (1948) which were otherwise almost out of sight but are more rigorous and deep as compared to the later works which rederive their results.

3.1 What should we be doing?

An uptodate account of exact solutions has been presented by MacCallum at ICGC 87 and the material covered there under the titles what solutions do we know? Can we find physically interesting solutions?, Can we put realistic physics into our solution? Can we interpret the solutions we find? Do we find interesting mathematics? is relevant even today and the reader may refer to it

[MacCallum (1987)]. However, I would conclude my comments, which are of course, not completely independent.

- Tackle whole class of solutions at once.
- Analyse the symmetry structure of field equations viz. Lie, Painle've analysis. A good recent reference to Painle've analysis of which I am aware may be the lecture notes (unpublished) of Leach, Govinder (1994).
- Explore new solutions for the well identified field e.g. solution having two Killing vector fields, source for Kerrmetric, anisotropic inhomogeneous cosmologies, radiating star, colliding plane waves, etc.
- Apply more efforts towards physical interpretation of the known solutions rather than to increase the number of solutions.
- Have more physical motivation in obtaining exact solution than the mathematical skill. I would explain this point by an example from one of my work [Srinivastava (1987)] in appendix II.
- Obtain the invariant characterisation of the solution e.g. kinematical parameters, classification of Weyl tensor, and of principal null eigen vectors after one has obtained it.

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APPENDIX – I

Some topics in the Proceedings, Seminar on Solutions of Einstein's field Equations: Techniques and results

Sl.No.	Title	Page	Authors
1.	Backlund transformations in general relativity	1-25	Kramer, D. and Neugehauer, G.
2.	Vector Backlund transformation and associated superposition principle	55-67	Chinea, F.J.
3.	H K X transformations : An introduction	68-84	Hoensaelers, C.
4.	The Geroch group is a Banch Lie group	113-127	Schmodt, B.G.
5.	On the homogeneous Hilbert problem for effecting Kinnersley-Chitre transformations	128-175	Hauser, I.
6.	Non-iterative methods for constructing exact solutions of Einstein equations	186-198	Guo, D.S.
7.	Inverse scattering, differential geometry, Einstein Maxwell solutions and one soliton Backlund transformations	199-234	Gurses, M.
8.	N Kerr particles	311-320	Yamazaki, M.
9.	Algebraically special, shearfree diverging and twisting vacuum and Einstein-Maxwell fields	321-333	Stephani, H.
10.	Exact solutions in Cosmology	334-366	MacCallum, M.A.H.

APPENDIX – II

The purpose of this appendix is to show, with the help of an example, how physical motivation renders a solution not to be new which otherwise had appeared as new. The solution refers to spherically symmetric perfect fluid distribution executing shearfree motion [Srivastava (1987a)].

We start with the metric ansatz

$$dS^2 = \left(\frac{1-\Phi}{1+\Phi}\right)^2 dt^2 - \left[\frac{(1+\Phi)^2 S}{V}\right]^2 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \qquad (II.1)$$

where $V = V(r,t), S = S(t)$ and $\Phi = \Phi(r,t)$ are arbitrary functions of their arguments. An analysis of field equations reveals that

$$\Phi = \frac{1}{C}\left[\int \left(\frac{C}{S}\right) S dt + U(r)\right] \qquad (II.2)$$

$$C = bS[1-2as]^{-1/2} \qquad (II.3)$$

$$U'' - \frac{U'}{r} + \frac{2U'V'}{V} = \begin{cases} -\frac{1}{2}V\tilde{F} & , \quad a = 0 \\ \frac{a}{2}UV\tilde{F} & , \quad a \neq 0 \end{cases} \qquad (II.4)$$

$$\frac{V''}{V} - \frac{V'}{rV} = \frac{a}{2}V\tilde{F} \qquad (II.5)$$

where a and b are arbitrary constants and $U(r)$ and $\tilde{F}(r)$ are arbitrary functions of r. It is to be noted for later reference that $\tilde{F}$ is related to Weyl conformal curvature invariant as

$$\psi_2 = -\frac{1}{3}\tilde{F}\left[\frac{S}{V}(1+\Phi)\right]^{-3} \qquad (II.6)$$

Hereafter the analysis gets divided into two classes.

$$Case I \ : \ a = 0 \ , \quad Case II \ : \ a \neq 0 \qquad (II.7)$$

In the first case the functions U, V and $\tilde{F}$ are determined explicitly and the solution becomes the one obtained earlier by Glass and Mashhoon (1976). Whereas in the second case one has to solve equations :

$$U'V^2 = q^4(a^2U^2 - b^2) \ , q = const. \qquad (II.8)$$

$$\frac{4a^2b^2U'^2}{(a^2U^2-b^2)^3} - \frac{2U'''}{U'} + \left(\frac{U'}{U}\right)^2 = \frac{1}{r^2} \qquad (II.9)$$

These equations can be solved in principle but it had not been possible to find $U(r)$ explicitly. At this stage I thought that the two cases ought to be new precisely because of the way they appear in (II.4). However, before the solution for the case (ii) could be claimed to be new its invariant classification is needed. For this, one way is to evaluate $\tilde{F}(r)$ for the two cases. Fortunately, following the analysis made by Stephani (1983) I could evaluate the function $\tilde{F}(r)$ for the second case and met happy surprise to see that (II.9) is a perfect third order differential equation.

$$\left(\frac{a^2U^2-b^2}{U_x}\right)_{xxx} = 0, (\ )_x = \frac{d}{dx}(\ ) \ , x = r^2 \qquad (II.10)$$

The function $\tilde{F}(r)$ for the case (ii) has the same functional form as in the case (i) hence the two cases should be the same. This means that by a suitable parameterisation equations (II.4)-(II.5) should reduce to corresponding ones with $a = 0$. The required parameterisation is found to be

$$\left.\begin{array}{lll} U & \rightarrow u & = \frac{aU-b}{aU+b} \\ V & \rightarrow v & = \frac{-2ab^2V}{(aU+b)^2} \end{array}\right\} \qquad (II.11)$$

which in no way is an obvious parameterisation. This illustrates the point.

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