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THE APPLICATION OF LIE ALGEBRA TECHNIQUES TO BEAM TRANSPORT DESIGN*

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INTRODUCTION

The techniques described here can be used for beam line design in quite general circumstances. We are introducing them for a final focus system design in linear colliders because this is where we have applied them, and to provide a specific context for our discussion. Other optical systems may require modifications or extensions to the methods we introduce here. We have kept the formalism and mathematics to a bare minimum, hoping to simplify the presentation and clarify its connection with other methods. The territory we sketch is the tip of the iceberg of Lie algebraic methods. In the last section, we describe briefly a broader context, though the interested reader will need to consult the extensive literature on this subject.[1] As far as we know, the particulars we present here are original; however, the essence of the method comes from Alex Dragt and collaborators.

1. HAMILTONIANS, KICKS, AND POISSON BRACKETS

Hamiltonian Reminders

The elegant and powerful formulation of classical mechanics given by Hamilton is summed up in pairs of first-order differential equations.

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_{\rm x}} \quad , \qquad \frac{dp_x}{dt} = -\frac{\partial H}{\partial {\rm x}} \tag{1.1}$$

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The state of motion of a particle is described by giving a position and a momentum, which can be identified with a point in a $2n$ dimensional space of these variables, for n degrees of freedom. The velocity of this point in this space is prescribed by one function defined in this space, the Hamiltonian, as indicated above. Thus one function determines completely the ensuing motion once initial conditions are specified.

The Hamiltonian function for the motion of particles in magnetic optical elements can be derived, after introduction of the appropriate coordinates and approximations, from the Hamiltonian for motion in a general electromagnetic field. This procedure has been described in many places.[2] The principal elements in our beam line consist of static transverse magnetic fields. If one chooses the distance s measured along the design orbit as the timelike variable, then within uniform elements with no dipole field the Hamiltonian can be transformed to

$$H = \left[p^2 - p_x^2 - p_y^2\right]^{1/2} - \frac{e}{c}A_s\left(x, y\right) \tag{1.2}$$

where p_x and p_y are the transverse components of the momentum, p is the total momentum of the particle, e is the electronic charge, c is the speed of light and A_s is the magnetic vector potential in the s direction.

For p much greater than p_x or p_y the square root can be expanded in powers of $(p_x^2 + p_y^2)/p^2$ as follows:

$$H = p\left\{1 + \frac{1}{2}\left(\frac{p_x^2 + p_y^2}{p^2}\right) - \frac{1}{8}\left(\frac{p_x^2 + p_y^2}{p^2}\right)^2 + \ldots\right\} - \frac{e}{c}A_s \quad . \tag{1.3}$$

For particles of constant p the first term can be dropped and the transverse momentum variables changed from p_x and p_y to $p_x/p \equiv x'(\approx dx/ds)$ and $p_y/p \equiv y'$. The latter is a scale transformation for which Hamilton's equations remain intact if H is scaled by p. The new H is

$$H = \frac{1}{2}\left(x'^2 + y'^2\right) - \frac{1}{8}\left(x'^2 + y'^2\right)^2 + \ldots - \frac{e}{pc}A_s \quad . \tag{1.4}$$

The scale transformation we performed here is not appropriate if the total energy is needed as a dynamical variable for the third degree of freedom. However, since there is no time dependence present, we can legitimately perform this transformation and pursue the track followed by standard optical system methods.

It is usual to write $p = (1+\delta)p_0$ where p_0 is the design momentum. We prefer to introduce a variable $\bar{\delta}$ defined by $\bar{\delta} = \delta/(1+\delta)$ arriving finally at

$$H = \frac{1}{2}\left(x'^2 + y'^2\right) - \frac{1}{8}\left(x'^2 + y'^2\right)^2 + \ldots - \left(1-\bar{\delta}\right)\frac{e}{p_0 c}A_s\left(x,y\right) \quad . \tag{1.5}$$

In our application the transverse emittance is very small, and the higher-order terms in x' and y' become miniscule. In our case $1/8\ (x'^2 + y'^2)^2$ is 100 times smaller than other typical aberrations in the beam line of the same order. Henceforth we will drop this and higher-order terms in x' and y'. Under these conditions, the Hamiltonian for a quadrupole magnet is given to high accuracy by[6]

$$H(x,x',y,y') = \frac{1}{2}(x'^2 + y'^2) + \frac{1}{2}K_Q(1-\bar{\delta})(x^2 - y^2) \quad . \tag{1.6}$$

Furthermore, in our situation, the Hamiltonians for a sextupole, a skew quadrupole, and a skew sextupole are given to high accuracy by

$$H(x,x',y,y') = \frac{1}{2}(x'^2 + y'^2) + \frac{1}{3!}K_S(1-\bar{\delta})(x^3 - 3xy^2)$$

$$H(x,x',y,y') = \frac{1}{2}(x'^2 + y'^2) - \frac{1}{2}K'_Q(1-\bar{\delta})(2xy) \tag{1.7}$$

$$H(x,x',y,y') = \frac{1}{2}(x'^2 + y'^2) - \frac{1}{3!}K'_S(1-\bar{\delta})(3xy^2 - y^3) \quad .$$

The Hamiltonian functions completely determine the dynamical situation. Accordingly for a normal or skew quad $|K_Q| = |K'_Q| = (B_T/a)\ [1/(p_0/q)]$ where B_T is the pole tip field and a is the radius of the aperture. For a normal or skew sextupole $|K_s| = |K'_s| = (2B_T/a^2)\ [1/(p_0/q)]$. In general, if $2n$ is the number of poles, $|K_n| = |K'_n| = (n-1)\ !\ (B_T/a^{n-1})\ [1/(p_0/q)]$.

2. POISSON BRACKETS (PBs)

Hamilton's equations of motion describe the change of the position and momentum of a particle, but we often would like to know the change of other functions defined on the particle state space. For example, consider the function[9]

$$U\left(x,x'\right) = \frac{\partial^2 H\left(x,x'\right)}{\partial x^2} \quad . \tag{2.1}$$

We may wish to know dU/ds as the particle moves along its trajectory in state space determined by Hamilton's equations. This derivative is given by the Poisson bracket (PB), as follows:

$$\begin{aligned}\frac{dU}{ds} &= \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial x'}\frac{dx'}{ds} \\ &= \frac{\partial U}{\partial x}\frac{\partial H}{\partial x'} - \frac{\partial U}{\partial x'}\frac{\partial H}{\partial x} \\ &\equiv [U, H] = [-H, U] \quad .\end{aligned} \tag{2.2}$$

Physicists often use the symbol x and the function $\mathbf{x}(x, x') = x$ interchangeably. With this as a given, Hamilton's equations of motion can be written as:

$$\begin{aligned}\frac{dx}{ds} &= [-H, x] \qquad \frac{dx'}{ds} = [-H, x'] \\ \frac{dy}{ds} &= [-H, y] \qquad \frac{dy'}{ds} = [-H, y'] \quad .\end{aligned} \tag{2.3}$$

The Linear Design Hamiltonian

To proceed with this problem—which can become quite complex in the presence of many elements and many small effects such as errors in placement and strength—it is advantageous to separate out a problem we can handle analytically: the linear problem. The linear problem includes the quadrupole magnets, the free space between elements, and the dipoles used to change the course of direction of the beam line. These elements for a particle of design energy ($\delta = 0$) are what we refer to as the design beam line, and the Hamiltonians associated with this motion we call the linear design Hamiltonian. Everything else is the "rest." This includes nonlinear elements such as sextupoles, the terms in the Hamiltonian that depend on particle energy, deviations of element parameters from design values (such as quad position or strength) and fringing fields. The "rest" includes all and any departures of the real system as it exists, or might exist, from the linear design Hamiltonian.

This is a useful, conceptual distinction as well as a valuable mathematical distinction, and this paper is about how to handle the various and sundry departures from ideal linear motion. In quantum mechanics what we do here is known as using an "interaction representation." It is nothing other than the standard device of beginning with a known and soluble Hamiltonian and then obtaining the departure from

this motion when a small perturbative Hamiltonian is present. Perturbations cause an increase in the spot size at the focal point of the system and create the aberrations we seek to minimize.

Kicks

We now consider a small slice of the beam line of width Δs at $s = s_1$, and look at the Hamiltonian in this slice:

$$H = H_{lin} + H_{rest} \quad . \tag{2.4}$$

Except for fringing fields or higher-order kinematic terms as discussed in Section 1, H_{rest} has no x' or y' dependence. In this case H_{rest} gives the particle a slight change in slope while traversing the slice, but no change in position. (The change in position is in H_{lin} in the form of the $x'^2 + y'^2$ term.) Let us denote H_{rest} (at $s = s_1$) $= V_1(x_1, y_1)$. The change in the momentum is given by an "impulse." Starting with Hamilton's equation:

$$\begin{aligned} \Delta x'(at \;\; s = s_1) &= \left[-V_1(x_1, y_1), x_1'\right] \Delta s \\ &= -\frac{\partial V_1(x_1, y_1)}{\partial x} \Delta s \\ &= \text{“Force (at } \; s = s_1)\text{”} \Delta s = \text{“Impulse”} \quad . \end{aligned} \tag{2.5}$$

We note the particle position at the final focus point $s = s_f$ by x^* or y^*. Its change due to the impulse $\Delta x'$ is

$$\begin{aligned} \Delta x^* &= \Delta x'(at \;\; s = s_1) \;\; \frac{\partial x^*}{\partial x_1'} \\ &= -\frac{\partial V_1(x_1, y_1)}{\partial x_1} \cdot \frac{\partial x^*}{\partial x_1'} \Delta s \\ &= \left[-V_1(x_1, y_1), x^*\right] \Delta s \quad . \end{aligned} \tag{2.6}$$

Note that $\partial x^* / \partial x_1'$, the change in x* due to a change in x_1', is just the (1,2) element of the linear transfer matrix between point $s = s_1$ and $s = s_f$. It is also the linear Green's function.

If H_{rest} had contained some x' or y' dependence, say, from a kinematic term or a fringe field, Equation (2.6) is still valid with H_{rest} replacing V:

$$\Delta x^* = \left[-H_{rest}(x_1, x_1', y_1, y_1'), x^*\right] \Delta s \quad . \tag{2.7}$$

This follows from

$$\begin{aligned} [-H_{rest}, x^*]\,\Delta s &= \left(-\frac{\partial H}{\partial x_1}\frac{\partial x^*}{\partial x_1'} + \frac{\partial H}{\partial x_1'}\frac{\partial x^*}{\partial x_1}\right)\Delta s \\ &= \Delta x_1' \frac{\partial x^*}{\partial x_1'} + \Delta x_1 \frac{\partial x^*}{\partial x_1} = \Delta x^* \quad . \end{aligned} \tag{2.8}$$

Hence there is no need to limit our discussion to kicks. However, we will so limit ourselves in the interest of conceptual simplicity because all our examples are indeed kicks.

Global Variables

It follows that, to first-order in H_{rest}, the change in the particle position at s_f is given by a sum of Equation (2.7) over all slices in the beam line where H_{rest} is unequal to zero, namely,

$$\Delta x^* = \sum_i \left[-H_{rest}(x_i, x_i', y_i, y_i'), x^*\right] \Delta s_i \quad . \tag{2.9}$$

From this expression we see that it makes sense to define a first-order (in kick strength) Hamiltonian for the beam line by

$$H_{NL}^1 = \sum_i H_{rest}(x_i, x_i', y_i, y_i')\Delta s_i \quad . \tag{2.10}$$

Within thick elements, this sum, in the limit of small Δs, becomes an integral. We can carry out these integrals, since within each element we know the equations of motion and can express x and x' in terms of x_c and x_c', the coordinates at the center of the element. We will carry out these integrals for quads and sextupoles in the section "Thick Element Sums." After carrying out these integrals, what remains is a discrete sum of terms in which the x_i and x_i' are the coordinates at the centers of beam line elements. To carry out these sums we must express the coordinates x_i and x_i' in some global way, say, in terms of the particle coordinates at some fixed s. For this purpose we express all coordinates in terms of the particle coordinates at the final focus plane, $s = s_f$. x_i and x_i' are linear functions of these coordinates, of course.

It is important to note that the sum we are calculating is predominantly the sum of polynomials of order 3. The chromatic aberration term in the quad is $\bar{\delta}(x^2 - y^2)$, the sextupole is a third-order polynomial, and a strength error of a quad would be a $\Delta K_Q(x^2 - y^2)$, again third-order in "deviation" variables. Thus the sum of such terms is a third-order polynomial in several variables. Fourth-order terms could occur in this sum if there were octupoles present in the beam line.

To first-order in kick strengthens the quantities

$$\Delta x^* = \left[-H^1_{NL}, x^*\right] = \frac{\partial H^1_{NL}}{\partial x^{*\prime}}$$
$$\Delta y^* = \left[-H^1_{NL}, y^*\right] = \frac{\partial H^1_{NL}}{\partial y^{*\prime}} \quad . \tag{2.11}$$

Thus an optimal design would arrange the coefficients of all monomials in H^1_{NL} which contain either $x^{*\prime}$ or $y^{*\prime}$ to be zero or negligibly small. Note that all of these terms are independent. There is no redundancy or interdependence. Each coefficient of a monomial with a distinct dependence on the variables $x^*, y^*, x^{*\prime}, y^{*\prime}$, and $\bar{\delta}$ represents an independent aberration. This feature is an important advantage of this formulation. The number of all such third-order terms is, of course, quite small, and only a handful are finally important in our application.

Equation (2.11) establishes that a term in H^1_{NL} of n^{th} order will become a term of order $n-1$ in the expression for Δx^* or Δy^*, so in discussing the order of a term we will specify H-order for the order in H_{NL} and optical order for order in the expression for the coordinates, the latter corresponding to the usual use of order in optics.

3. KICKS ON KICKS

Kicks on Kicks

A kick at the beginning of the beam line will affect the particle coordinates throughout the remainder of the line, and thus alter the magnitude of subsequent kicks. The change in a kick can be determined by writing down the first terms of a Taylor series expansion of the kick about the linear value of the particle position there. Using the first kick (transferred to the position of the second kick) to determine the deviation of the coordinates from their linear value, one finds the following intuitively satisfying result for two kicks to second-order in the kick strengths:

$$\Delta x^* = \left[-V_1(x_1, y_1), x^*\right] \Delta s_1 + \left[-V_2(x_2, y_2), x^*\right] \Delta s_2$$
$$+ \left[-V_1(x_1, y_1), \left[-V_2(x_2, y_2), x^*\right]\right] \Delta s_1 \Delta s_2 \quad . \tag{3.1}$$

In other words, just as $[-V_1, x^*]$ *give the change in* x^* *due to* V_1, *so* $[-V_1, [-V_2, x^*]]$ *gives the change in* $[-V_2, x^*]$ *due to* V_1.

The Total Beam Line Hamiltonian

Consider all V_i in the beam line which are of third H-order (second optical order). The additional PB, representing effects of the kicks on one another, give terms of third optical order. Hence all second optical-order kicks occur only to first-order in the kick strength. The PB of such terms with x^* or y^* must be zero, or quite small, so that these terms do not enlarge the beamsize. In other words,

$$\left[- \sum_{V_i \text{ 3rd order}} V_i \Delta s_i,\ x^* \right] = 0 \quad . \tag{3.2}$$

Under these conditions

$$\sum_{j<i} [V_j \Delta s_j, [V_i \Delta s_i, x^*]] = 1/2 \left[\sum_{j<i} [V_j \Delta s_j, V_i \Delta s_i], x^* \right] \quad . \tag{3.3}$$

This can be proven by writing out the expression

$$\left[\sum_{all\ j} V_j \Delta s_j, \left[\sum_{all\ i} V_i \Delta s_i, x^* \right] \right] = 0 \tag{3.4}$$

which is zero by virtue of Equation (3.2). We also use the fact that for kicks $[V_i, [V_i, x^*]] = 0$, which follows from the fact that $[V_i, x^*]$ is a function of x_i, $(\partial x^*/\partial x_i'$ is a number).

This technical fact is of great use because it enables us to find a beam line Hamiltonian containing all first- and second-order terms in kick strengths. It is just

$$H^2_{NL} = H^1_{NL} - 1/2 \sum_{j<i} [V_j \Delta s_j, V_i \Delta s_i] \tag{3.5}$$

where the sum is assumed to be over V_i of third-order. H^1_{NL} can contain higher than third-order terms. The ability to find a beam line Hamiltonian is actually valid in general, and does not depend on the restricted assumptions we have used here. This fact is perhaps the heart of the usefulness of the Lie Algebra approach, and is formalized in what is known as the Cambell–Baker–Hausdorff theorem. See Section 10 below.

Again, we have the satisfactory result that each coefficient represents a unique identifiable aberration. The sources of terms are readily identified, and correction strategies suggest themselves quite naturally. *The problem of beam line design is thus transformed to a problem in calculating a low-order polynomial, and arranging that the $x^{*\prime}$- and $y^{*\prime}$-dependent monomials have small coefficients. There is no need to calculate kicks, and follow the details of their interplay. All the relevant information is contained in the potentials, V_i. Their Poisson brackets with one another contain the relevant information on their interplay.*

4. THICK ELEMENT SUMS

Chromatic Aberration from Quads

Within quadrupole elements

$$H_{rest} = -\frac{1}{2}K_Q\bar{\delta}(x^2 - y^2) \quad . \tag{4.1}$$

If $K_Q > 0$ then

$$\begin{aligned} x(s) &= x_c\cos(ks) + x_c'\frac{\sin(ks)}{k} \\ y(s) &= y_c\cosh(ks) + y_c'\frac{\sinh(ks)}{k} \end{aligned} \tag{4.2}$$

where

$$k = \sqrt{|K_Q|}$$

and s is measured from the quad center. For $K_Q < 0$, interchange cos with cosh and sin with sinh.

For an element of length L, the sum over thin slices of the quad becomes the integral

$$\int_{-L/2}^{L/2} H_{rest}(x(s), y(s))ds =$$

$$-\frac{|K_Q|L\bar{\delta}}{2}\left\{\frac{1}{2}\left[1 + \frac{\sin(\phi)}{\phi}\right]x_c^2 + \frac{1}{2}\left[1 - \frac{\sin(\phi)}{\phi}\right]\frac{x_c'^2}{|K_Q|}\right. \tag{4.3}$$

$$\left. -\frac{1}{2}\left[1 + \frac{\sinh(\phi)}{\phi}\right]y_c^2 - \frac{1}{2}\left[\frac{\sinh(\phi)}{\phi} - 1\right]\frac{y_c'^2}{|K_Q|}\right\}$$

where

$$\phi = kL \quad .$$

For all but the final quads in our example, it is an excellent approximation to set $1/2[1 + \sin(\phi)/\phi] = 1/2[1 + \sinh(\phi)/\phi] = 1$ and the remaining bracketed expressions equal to 0, obtaining

$$\int_{-L/2}^{L/2} H_{rest}(x(s), y(s))ds = -\frac{K_Q L\bar{\delta}}{2}(x_c^2 - y_c^2) \quad . \tag{4.4}$$

Table 1 shows the bracketed expressions for the last quads in the SLAC Final Focus Test Beam (FFTB) line.

We could now evaluate the double integral for the second-order term in the thick quadrupoles, but since it is a negligible term we omit its calculation here.

Thick Sextupoles

For sextupoles

$$H_{rest} = \frac{K_S}{3!}\left(x^3 - 3xy^2\right) \tag{4.5}$$

and

$$\begin{aligned} x &= x_c + sx_c' \\ y &= y_c + sy_c' \quad . \end{aligned} \tag{4.6}$$

The first-order thin slice sums become the integrals

$$\begin{aligned} \int_{-L/2}^{L/2} x^3 ds &= L\left[x_c^3 + \frac{1}{4}x_c x_c'^2 L^2\right] \\ \int_{-L/2}^{L/2} xy^2 ds &= L\left[x_c y_c^2 + \frac{1}{12}x_c y_c'^2 L^2 + \frac{1}{6}y_c x_c' y_c' L^2\right] \quad . \end{aligned} \tag{4.7}$$

The terms with L^2 are smaller than the leading term by a factor of 10^6 for our FFTB.

The double sum in Equation (3.5) within the sextupole becomes the integral

$$\int_{-L/2}^{L/2} ds \int_{s}^{L/2} ds' \left[H_{rest}(s), H_{rest}(s')\right] = \frac{K_S^2 L^3}{48} \left(x_c^2 + y_c^2\right)^2 \quad . \tag{4.8}$$

We have now carried out all the necessary thick element sums. The complications introduced by thick elements are thus fully represented by expressions which involve only the linear coordinates of the particle at the center of the element.

5. DIPOLE EFFECTS

Dipole Insertions

Suppose an additional small dipole element is placed in the beam line. (By small we mean that the effect of such an element is well represented by a kick at its center. This definition includes all dipoles in our example, the FFTB line.) A dipole kick in the horizontal plane changes x' by a constant amount, independent of the particle coordinate, and leaves x, y, and y' unchanged. Furthermore, suppose that in the calculation of H_{NL}^1 we have expressed all coordinates in the sum by the linear coordinates at the end of the beam line, which we will refer to as the IP. We may properly account for the effect of the dipole insertion on the nonlinear Hamiltonian by (a) dividing the sum into two parts, those elements occurring before the kick and those elements occurring after the kick, (b) finding the change in the IP coordinates that are caused by the dipole kick, and (c) using these new coordinates when evaluating the sum over elements occurring after the dipole insertion. The new coordinates differ by a constant from the original coordinates. If the kick is large enough that the delta-dependent aspect of the kick is important, the "constant" may contain delta (momentum) dependence. In either case, the transverse coordinate in the sum is displaced. When the resulting binomial expressions are expanded [e.g., $(x + \Delta x)^n$], terms of lower order in the transverse variables occur, an effect often referred to as feeddown. For example, a chromatic aberration term ($\bar{\delta} x^{*\prime 2}$) feeds down a dispersion term ($2\Delta x^{*\prime} \bar{\delta} x^{*\prime}$).

Quadrupole Displacement

The discussion of the previous paragraph applies directly to the problem of quadrupole displacement, since a displaced quadrupole can be thought of as a dipole field superimposed on an (undisplaced) quadrupole field. It is usually an excellent approximation to replace the dipole field by a kick at the center of the quad. The chromatic effects of the quad are then represented by a term at the same location, the interplay between dipole and quadrupole usually being negligible.

Dispersion

Often in a beam line there are intentional bends to redirect the design orbit of the beam line. For such elements the design momentum dipole field is in H^1_{lin}, but there is an additional part to the Hamiltonian:

$$H_{rest} = -\bar{\delta} K_D x \quad . \tag{5.1}$$

We will not put this in H^1_{NL}, since it is not of third-order, but instead consider simply that the particle has received a kick proportional to $\bar{\delta}$ in traversing this element. This could be handled as we have sketched above, but there is another more common and convenient alternative. For each element in the beam line, every preceding bend causes a shift proportional to $\bar{\delta}$ in the tranverse coordinate at that element. The sum of all such shifts is referred to as the "dispersion" at that element. The shift in x is often denoted by $\eta_x \bar{\delta}$. η_x is referred to as the horizontal dispersion function. In other words, at every point in the beam line there is a dispersion function, $\eta_x(s)$, such that if we replace each occurrence of $x(s)$ and $x'(s)$ by $x(s) + \eta_x(s)\bar{\delta}$ and $x'(s) + \eta'_x(s)\bar{\delta}$, respectively, we obtain the correct expression for H^1_{NL}. Within a quad or free space, the η function obeys the same equation of motion as the linear coordinates. Hence in the "Thick Element Sums" we may replace x_c and x'_c by $x_c + \eta_{xc}\bar{\delta}$ and $x'_c + \eta'_{xc}\bar{\delta}$, respectively. In this way we completely account for the dispersive effects of the dipole bends.

6. APPLICATION TO THE FFTB EXAMPLE

FFTB Lattice Description

In Figure 1 we show a block diagram of a typical final focus system where one degree of freedom is more strongly focused than the other, as is likely for the next generation of linear colliders. The system consists of five sections: (i) beta preparation, (ii) x chromatic correction, (iii) beta exchange, (iv) y chromatic correction, and (v) final transformer.[3] The final transformer in this example ends with a very strong doublet of quadrupoles, referred to as the final doublet. The beam is broad with approximately parallel particle trajectories in the final doublet, then rotates 90° in phase space traveling to the IP, where the beam is narrow and divergent. The final doublet is the principal focusing element, and one could say that the whole system is a preparation for this doublet, including precompensation for the aberrations produced there.

Figure 2 shows the optical functions: the x and y beam envelopes and the horizontal dispersion function. The solution to the linear equations of motion can be written in the form:

$$x(s) = \sqrt{2\beta_x(s) J_x} \cos\left(\theta_x + \phi_x(s)\right) \tag{6.1}$$

where β_x and ϕ_x depend on s and are related through the equation

$$\phi_x(s) = \int_{s_o}^{s} \frac{d\hat{s}}{\beta_x(\hat{s})} \quad . \tag{6.2}$$

J_x and θ_x are the two constants needed to specify the solution of a second-order linear differential equation, and are chosen so that they correspond to the standard choice of action-angle coordinates. θ_x is the phase angle in the plane $s = s_0$, and ϕ_x is called the phase advance through the lattice.

The choice of normalization in the equation for ϕ_x implies that at points where $d\beta/ds = 0, \langle x' \rangle_{rms} = \langle x \rangle_{rms}/\beta$.

Figure 2 also designates the five sections of the system we referred to above. Things to note are:

(a) The phase advance across the chromatic correction sections and beta exchange section is π, and across the final transformer is very close to $3\pi/2$.

(b) For each chromatic correction section all optical functions at the beginning are equal to their values at the ends of the section. In fact, the transfer matrix of the chromatic correction sections is $-I$ (minus the identity matrix.)

(c) $\beta_y > \beta_x$ in the y-correction section and $\beta_x > \beta_y$ in the x-correction section.

(d) All bend dipoles are approximately π phase shift from one another and the IP.

Since we wish to express our coordinates in the H^1_{NL} sums in terms of the IP coordinates we will choose $s_0 = s_f$. This will imply that $\phi_x(s)$ is always negative. Hence often we refer to $\bar{\phi}_x \equiv -\phi_x$.

Figure 3 shows a plot of the phase functions through the lattice. The functions are very close to step functions. This follows from Equation (6.2) relating the beta function to the phase function: where the β function is small, the phase advances very rapidly; where it is large, the phase advances slowly. The step function behavior of the phase functions reflects the "mountain and valley" behavior of the beta functions. This mountainous behavior is, of course, intentional. If the trajectories are very parallel in the final doublet, then after $\pi/2$ phase rotation the beam will be very narrow. Thus the beta functions should be large at the final quads. However, large β functions here imply large chromatic aberrations, which will be corrected by sextupole pairs in the chromatic correction sections. To have these sextupoles be effective, it is advantageous to have large β functions at their position.

As we shall now see, the step function behavior of the phase functions greatly simplifies the analysis of this optical system.

7. THIRD H-ORDER ABERRATIONS

Third H-Order Chromaticity

We have found the chromatic term for the final quads:

$$-\frac{1}{2}K_Q\bar{\delta}\left(x_c^2 - y_c^2\right) \quad . \tag{7.1}$$

Both the x and y contributions to the H^1_{NL} polynomial need to be compensated. Expressed in terms of IP coordinates, and using the fact that $\bar{\phi}_{x_c} = \bar{\phi}_{y_c} = \pi/2$,

$$x_c = \sqrt{2\beta_x J_x}\sin(\theta_x) = -\sqrt{\beta_x\beta_x^*}x^{*\prime} \quad . \tag{7.2}$$

It is clear that we would be better off using $\bar{x} = x^*/\sqrt{\beta_x^*}$ and $\bar{p}_x = \sqrt{\beta_x^*}x^{*\prime}$ as common coordinates, removing β_x^* from such expressions. The transformation from * to bar coordinates is symplectic. The transformation matrix, which is

$$\begin{bmatrix} 1/\sqrt{\beta^*} & 0 \\ 0 & \sqrt{\beta^*} \end{bmatrix} \tag{7.3}$$

has determinant unity. In terms of those coordinates

$$x_c = -\sqrt{\beta_x}\bar{p}_x \qquad y_c = -\sqrt{\beta_y}\bar{p}_y \quad , \tag{7.4}$$

and H^1_{NL} contain the monomials $-1/2K_Q\beta_{xQ}\bar{\delta}\bar{p}_x^2$ and $+1/2K_Q\beta_{yQ}\bar{\delta}\bar{p}_y^2$.

The $\bar{p}_y$ term, if left uncompensated, would blow up the beam 20 times larger than the design spot size. It is compensated so that the contribution of the chromatic term would give a spot that is one-fifth the design spot; in other words, the net result is smaller by a factor of 100 than the quad contribution. This is an important number, which we call the chromatic correction ratio (CCR).

Sextupoles

To compensate the final quad chromaticity, the polynomial H^1_{NL} must get contributions elsewhere which are of the same form but opposite in sign. This comes from the sextupoles. Suppose at a sextupole we have nonzero dispersion. Then H^1_{NL} will get a contribution, from the sextupole Hamiltonian, equal to

$$\frac{1}{3!}K_S\left[(x_c + \eta_x\bar{\delta})^3 - 3(x_c + \eta_x\bar{\delta})y_c^2\right] \quad . \tag{7.5}$$

If we place this sextupole $\pi/2 + n\pi$ from the IP, then Equation (7.4) multiplied by $(-1)^n$ is valid and the last term contributes the desired monomial. Of course there is

an unwanted new term, $x_c y_c^2$, that we get in the bargain. This can be compensated if there is another sextupole with opposite sign x_c but same sign η_x. This can be achieved by placing it a phase difference of π from the first. Thus we see the need for sextupole pairs.[3]

The first term above, which has only x dependence, when expanded, contains x_c^3, $\bar{\delta} x_c^2$, $\bar{\delta}^2 x_c$, and $\bar{\delta}^3$. The last term has no dependence on transverse coordinates and can't contribute to the spot size. The first and third term are odd in x_c and so they cancel out for a sextupole pair, if η_x is the same at the two sextupoles, leaving the second term which is the desired term.

Figure 4 is a graph showing the magnitude of all contributions to these two chromatic terms. For tracking purposes the sextupoles are divided into four pieces. Their contribution is shown as four separate terms. The sum for all quads, and the sum of quad plus sextupoles is also shown. The net sum for the $\bar{p}_y$ term is +50, smaller than the final quad contribution by a factor of 100 as required.

In Figure 5 we keep track of the $\bar{p}_y$-dependent monomials in H_{NL}^1 and H_{NL}^2 we will be discussing. We show the main chromatic term $\delta \bar{p}_y^2$ and its main sources, the final quads and the sextupole pairs. The wiggly line is meant to indicate that this aberration is traveling through the system toward the IP and can give rise to higher-order terms. The termination of the line indicates that the aberration has been corrected and is no longer present in the system. Thus the geometric effect $\bar{p}_x \bar{p}_y^2$ travels between sextupole pairs. The other lines show higher-order aberrations which are generated by these aberrations.

8. FOURTH H-ORDER ABERRATIONS

Fourth H-Order Chromaticity

A third-order chromatic error slightly alters subsequent chromatic errors, and we have argued that this alteration is given by the Poisson bracket $-1/2[V_1(x_1, y_1), V_2(x_2, y_2)]$ where V_1 and V_2 are third H-order chromatic corrections. Writing out this expression in terms of partial derivatives it is seen to be equivalent to the expression

$$-1/2\,[V_1(x_1,y_1), V_2(x_2,y_2)] = -1/2\left\{\frac{\partial V_1}{\partial x_1}\frac{\partial V_2}{\partial x_2}\,[x_1,x_2] + \frac{\partial V_1}{\partial y_1}\frac{\partial V_2}{\partial y_2}\,[y_1,y_2]\right\} \quad . \tag{8.1}$$

As mentioned earlier, $[x_1, x_2] = \partial x_2/\partial x_1'$ is the (1,2) element of the transfer matrix between elements 1 and 2. Lets look at two quads in the final transformer with $\bar{\phi}_y \approx \pi/2$, and concentrate on the y degree of freedom. For such quads

$$\frac{\partial V}{\partial y} = -K_Q \bar{\delta} y_c = K_Q \sqrt{\beta_Q}\; \bar{\delta} \bar{p}_y \tag{8.2}$$

and if the quads are separated by a drift then

$$[y_1, y_2] = s_2 - s_1 \quad , \tag{8.3}$$

hence the y-dependent part of the above PB for two consecutive quads equals

$$-\frac{1}{2} K_{Q1} K_{Q2} \sqrt{\beta_{yQ_1} \beta_{yQ_2}} \, (s_2 - s_1) \bar{\delta}^2 \bar{p}_y^2 \quad . \tag{8.4}$$

This is a fourth H-order aberration or a third-order optical aberration. The transverse coordinate dependence of $\bar{p}_y^2$ tells us that it is a quadlike aberration, like a change in focal length. The $\bar{\delta}^2$ indicates it is second-order in the energy deviation variable. Hence this is a higher-order chromatic aberration. The large values of β_y at these quads suggest that this term, though smaller than the third-order term, may not be negligible, and this is indeed the case.

Suppose we take one quad in the final transformer and one quad in the y-correction section. Here, a more convenient expression for $[y_1, y_2]$ is the standard expression for the R_{12} element of the transfer matrix:

$$\begin{aligned} [y_1, y_2] &= \sqrt{\beta_{yQ_1} \beta_{yQ_2}} \, \sin(\phi_{y_2} - \phi_{y_1}) \\ &= -\sqrt{\beta_{yQ_1} \beta_{yQ_2}} \, \epsilon_{12} \end{aligned} \tag{8.5}$$

where since $\phi_{y_2} - \phi_{y_1}$ is close to π we have chosen to represent

$$\phi_{y_2} - \phi_{y_1} = \pi + \epsilon_{12} \quad . \tag{8.6}$$

The value of the PB is

$$\frac{1}{2} K_{Q1} K_{Q2} \beta_{yQ1} \beta_{yQ2} \epsilon_{12} \bar{\delta}^2 \bar{p}_y^2 \quad . \tag{8.7}$$

Here, it is clear that the magnitude and sign of the term depends on the magnitude and sign of ϵ_{12}. This suggests that if we slide the phase of the y-correction section a bit, we may be able to correct this higher-order chromatic aberration.[4] Indeed this is possible, and the design of the lattice reflects this fact.

Thick Sextupoles

In "Thick Element Sums" above, we calculated the PB integral for thick sextupoles. This fourth H-order term is the strongest uncorrected aberration in the present FFTB lattice design. It could be corrected by octupoles if desired, though one needs to worry whether additional aberrations might be introduced, especially if

the correction requires strong octupoles beating against each other, like the sextupole pairs we have already introduced. However, this is not the case, as we show now. Expanding the expression in Equation (4.8) we get a term proportional to

$$x_c^4 + 2x_c^2 y_c^2 + y_c^4 \quad . \tag{8.8}$$

To evaluate this at the y-correction quads $x_c + \eta_{xc}\bar{\delta}$ must be inserted for the local x_c. Doing this, and adding the two sextupole terms together, the terms which are odd in x_c or y_c will drop out, and we are left with the following y-dependent terms:

$$4x_c^2 y_c^2 + 4\eta_{xc}^2 \bar{\delta}^2 y_c^2 + 2y_c^4 \quad . \tag{8.9}$$

At the position of the sextupoles y_c and x_c are $3\pi/2$ out of phase from the IP, thus the middle term here has the same variable dependence as the fourth-order chromatic correction we calculated in the previous section. Therefore the middle term can be compensated by the method described in the preceding section. The remaining terms have no $\bar{\delta}$ dependence and can be corrected with octupoles located in a dispersion-free region.

Octupoles

An octupole placed in the beam line after the bend dipoles in the final transformer where there is no dispersion, will have the form

$$V_O = \frac{K_O}{4!}\left(x^4 - 6x^2y^2 + y^4\right) \quad . \tag{8.10}$$

Concentrating on the y-dependent terms, we see they are exactly the ones we need to correct for the geometric aberrations of the thick sextupoles [see Equation (4.8.)]. To do this we must use two octupoles, one where $\beta_x \approx \beta_y$ to correct the x^2y^2 term, and one where $\beta_y \gg \beta_x$ to correct the y^4 without altering the x^2y^2-correction. The octupole strengths are small, and so the x^4 does not appreciably increase the beam width which is, for our case, much larger than the beam height.

Chromogeometric Aberrations

The cancellation of the geometric terms created by introduction of the sextupole pairs requires that the optical matrix between them be $-I$ in both the x and y degree of freedom. This can be achieved for the design momentum particle, but for particles with $\bar{\delta}$ unequal to zero, the transformation will not be exactly $-I$. If we calculate

the sum of the chromatic terms between the sextupole pairs, and express the result in terms of the coordinates at the center of the $-I$ section, the result will look like

$$\bar{\delta}\left\{ax_c^2 + bx'^2 + cy_c^2 + dy_c'^2\right\} \quad . \tag{8.11}$$

The possible xx' and yy' terms vanish because of the forward-backward symmetry of this section.

Here, we are interested in how these chromatic terms alter the second sextupole's geometric term, so that it does not exactly cancel the first sextupole. The alteration for the y degree of freedom is given by the PB:

$$\left[-\bar{p}_x\bar{p}_y^2, \bar{\delta}\left(c\beta_{y_c}\bar{y}^2 + \frac{d\bar{p}_y^2}{\beta_{y_c}}\right)\right] = 4c\beta_{y_c}\bar{\delta}\bar{p}_x\bar{p}_y\bar{y} \quad . \tag{8.12}$$

This is clearly a fourth H-order term. It is quite small since β_{yc} at the center of the $-I$ section is very small. However, it gives rise to a very interesting effect to which we now turn.

9. FIFTH H-ORDER ABERRATIONS

Fifth H-Order Chromogeometric

The fourth H-order chromogeometric aberration we have just calculated occurs before the final quads, hence we must ask if this alters the chromatic correction of the final quad which we know to be quite strong. The fact that this aberration has a $\bar{y}$ dependence means that it will have a nonzero PB with the final quad term. The dependence of this term is given by

$$\left[\bar{\delta}\bar{p}_x\bar{p}_y\bar{y}, \bar{\delta}\bar{p}_y^2\right] = 2\bar{\delta}^2\bar{p}_x\bar{p}_y^2 \quad . \tag{9.1}$$

Interestingly, in our case this term is larger than the fourth-order chromogeometric term that induces it. It is comparable to the thick sextupole aberration which we have corrected by the octupole insertions.[5] Also interesting is the fact that this aberration now has no $\bar{y}$ dependence, only $\bar{p}_x$ and $\bar{p}_y$. Since most of the lattice has the phase of these variables, a compensation using decapoles is possible.

Decapoles

The Hamiltonian for a decapole is

$$H_0 = \frac{K_0}{5!}\left(x^5 - 10x^3y^2 + 5xy^4\right) \quad . \tag{9.2}$$

A pair of opposite sign decapoles in the y chromatic corrector located alongside the sextupoles would yield the following y-dependent terms after substitution of $x+\eta\bar{\delta}$ for x:

$$2\frac{K_0}{5!}\Big\{-10x^3y^2-30\eta^2\bar{\delta}^2xy^2+5xy^4\Big\} \quad . \tag{9.3}$$

The middle term is the correction term we seek. The first and last terms may be cancelled by decapoles placed near the final quad, similar to our placement of octupoles to cancel the unwanted octupole terms of the thick sextupoles.

10. EXPONENTIAL LIE OPERATORS

The Exponential Operator as Taylor Series

We now wish to show how our methods relate to the usual Lie method presentations. The heart of the method is the definition and use of the Lie exponential operator. It is the object which in the full, nonlinear treatment of beam lines replaces the transfer matrix so familiar in linear systems. What we seek is an object that gives the output coordinates of a beam line element in terms of the input coordinates. The familiar Taylor series gives the output coordinates in terms of the input coordinates and all the derivatives at the input plane:

$$x_2 = x(s_2) = x(s_1+L) = \sum \frac{L^n}{n!}\left.\frac{d^nx}{ds^n}\right|_{s=s_1} \quad . \tag{10.1}$$

At this point we do two things:

(a) We need all the derivatives at the input points. However, since we know the Hamiltonian for the element, all of the derivatives can be expressed in terms of the initial position and slope by using PBs.

$$\frac{d^nx}{ds^n} = [-H,[-H,\ldots,[-H,x]]\ldots] \equiv : -H :^n x \quad . \tag{10.2}$$

(b) We use the suggestive notation

$$\sum \frac{L^n}{n!}\frac{d^n}{ds^n} = \exp\left(L\frac{d}{ds}\right) \tag{10.3}$$

or

$$\sum \frac{L^n}{n!} : -H :^n = \exp(L : -H :) \quad . \tag{10.4}$$

We have the result that

$$x(s_2) = \exp(L\frac{d}{ds})x(s)\Big|_{s=s_1}$$
$$= \exp\left(L : -H :\right) x\Big|_{x=x_1, x'=x'_1} \tag{10.5}$$

and

$$x'(s_2) = \exp(L\frac{d}{ds})x'(s)\Big|_{s=s_1}$$
$$= \exp\left(L : -H :\right) x'\Big|_{x=x_1, x'=x'_1} . \tag{10.6}$$

Concatenation for Beam Lines

Next we must find the expression for two successive beam line elements. In the linear theory this is given by the matrix multiplication. In the nonlinear theory we have the result

$$x_3 = \exp\left(L_1 : -H_1 :\right) \exp\left(L_2 : -H_2 :\right) x\Big|_{x=x_1, x'=x'_1} . \tag{10.7}$$

This may be seen by noting

$$x_3 = \exp\left(L_2 : -H_2 :\right) x|_{x=x_2, x'=x'2} \equiv f(x_2, x'_2)$$
$$= f\left(\exp\left(L_1 : -H_1 :\right) x, \exp\left(L_1 : -H :\right) x'\right)\Big|_{x=x, x'=x'_1} \tag{10.8}$$
$$= \exp\left(L_1 : -H_1 :\right) f\left(x, x'\right)\Big|_{x=x_1, x'=x'_1} .$$

The last step is the crucial one, and follows from the fact that for $f(s) = f(g(s), h(s))$

$$f(s+L) = f\left(g\left(s+L\right), h\left(s+L\right)\right) \quad . \tag{10.9}$$

Similarity Transformation

There is a very intuitive similarity transformation rule for these operators

$$\exp(:A:)\exp(:B:)\exp(-:A:) = \exp(:C:) \tag{10.10}$$

with

$$C = \exp(:A:)\,B \quad . \tag{10.11}$$

In other words, to find the transformed operator, transform the generator.

Interaction Representation

The beam line is written down as a product of the exponential operator elements. Some of these elements will be purely linear, such as the drifts between magnets. Some will be purely nonlinear, such as very thin sextupoles. Most elements are a combination. For these we consider the element as sliced into many thin elements and the operator for each slice written as a product of the nonlinear term times the linear term.

Next, we move all the linear elements to one end of the beam line, by using the appropriate similarity transformation. In the process we must transform the coordinates in the remaining nonlinear terms. What we get for the beam line is then one linear term times a nonlinear term which is the product of operators of the form

$$\exp\left(\Delta s_i : -H_{rest}\left(x_i, x_i'\right) :\right) \quad . \tag{10.12}$$

where x_i is the local linear coordinate.

The CBH Theorem

It remains to find the beam line Hamiltonian for this product. To do that we need the Cambell–Baker–Hausdorf theorem.

$$\exp(:A:)\exp(:B:) = \exp(:C:) \tag{10.13}$$

where

$$C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A-B,[A,B]] + \ldots . \tag{10.14}$$

If we apply this theorem many times in succession so as to combine all the factors in the beam line, we get one grand exponent

$$C = \sum_i A_i + \frac{1}{2}\sum_{j<i}[A_j, A_i] + \ldots . \tag{10.15}$$

This should now begin to look familiar. Compare Equation (3.5) which can be written in the form

$$-H_{NL}^2 = \sum_i -V_i \Delta s_i + \frac{1}{2} \sum_{j<i} [-V_j \Delta s_j, -V_i \Delta s_i] \quad . \tag{10.16}$$

Computational Methods

To carry out the above program for actual beam lines, with all their important complications, involves manipulating low-order polynomials in several variables. Recently, M. Berz has created a tool which does this. Polynomials can be defined and most standard operations can be performed on them, including taking PBs. G. Roy and M. Berz are presently providing a MAD[6] lattice input for COSY *Infinity.*[7] G. Roy and the author are implementing the procedures described in this article. Implementations with slightly different objectives, for circular machines, were performed previously.[8]

ACKNOWLEDGEMENTS

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4. This phase adjustment argument was described to me by K. Brown, private communication (October 1989).
5. The importance of this aberration was pointed out by K. Oide, private communication.
6. C. Iselin, MAD Reference Manual, CERN (1985).
7. Cosy Infinity is a shell for the polynomial manipulation tool of M. Berz. See M. Berz, *Proceedings of the Third International Conference on Charged Particle Optics,* Toulouse, France, April 24–27, 1990.
8. É. Forest, M. Berz, and J. Irwin, "Normal Form Methods for Complicated Periodic Systems," *Particle Accelerators* **24**, 91 (1989).
9. Often, as here, we indicate dependence on x and x', with dependence on y and y' understood. The required definition for the Poisson bracket for 2 degrees of freedom follows directly from the considerations presented here.

Table 1: Quadrupole chromatic correction coefficients.

Coeff. of	x_c^2	$\frac{x_c'^2}{\|K\|}$	y_c^2	$\frac{y_c'^2}{\|K\|}$
QC1	1.1158	0.1158	0.8988	0.1012
QX1	1.0063	0.0063	0.9938	0.0062
QC2	0.9233	0.0767	1.0845	0.0845
QC3	1.0042	0.0042	0.9959	0.0041
QC4	0.9989	0.0011	1.0011	0.0011
QC5	1.0068	0.0068	0.9933	0.0067
QN1	0.9919	0.0081	1.0082	0.0082
QM3	1.0043	0.0043	0.9957	0.0043

FIGURE CAPTIONS

1. A block diagram of a typical final focus system.
2. Optical functions for the final focus test beam (FFTB) facility under construction at the Stanford Linear Accelerator Center (SLAC).
3. The phase functions for the FFTB.
4. Coefficients of the chromaticity terms in the FFTB line Hamiltonian.
5. Important y-dependent aberration terms in the FFTB line Hamiltonian.

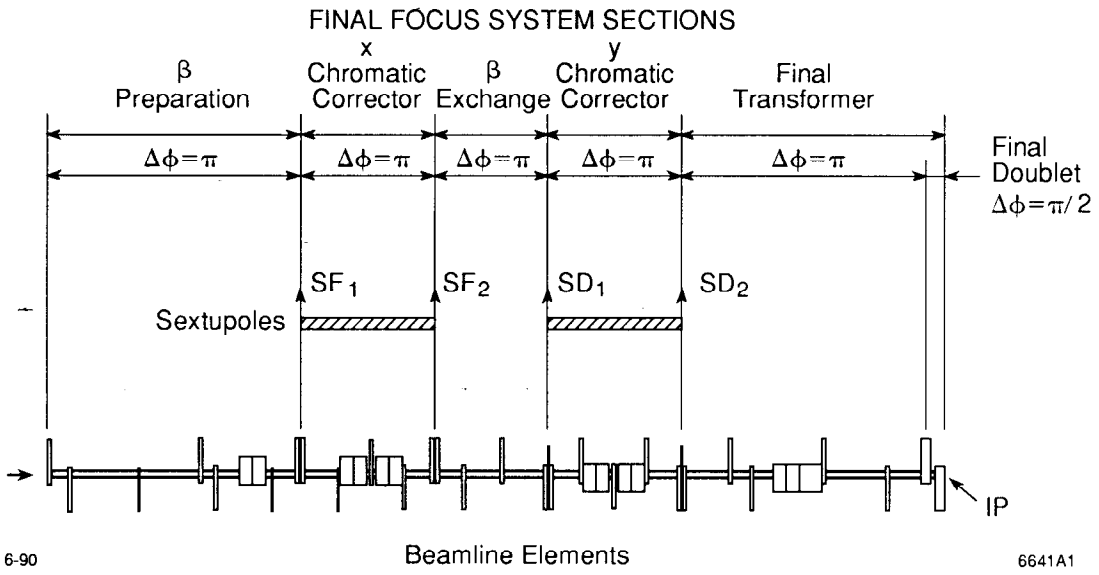

FINAL FOCUS SYSTEM SECTIONS
β Preparation
x Chromatic Corrector
β Exchange
y Chromatic Corrector
Final Transformer
Δϕ=π
Δϕ=π
Δϕ=π
Δϕ=π
Δϕ=π
Final Doublet Δϕ=π/2
SF1
SF2
SD1
SD2
Sextupoles
IP
6-90
Beamline Elements
6641A1

Fig. 1

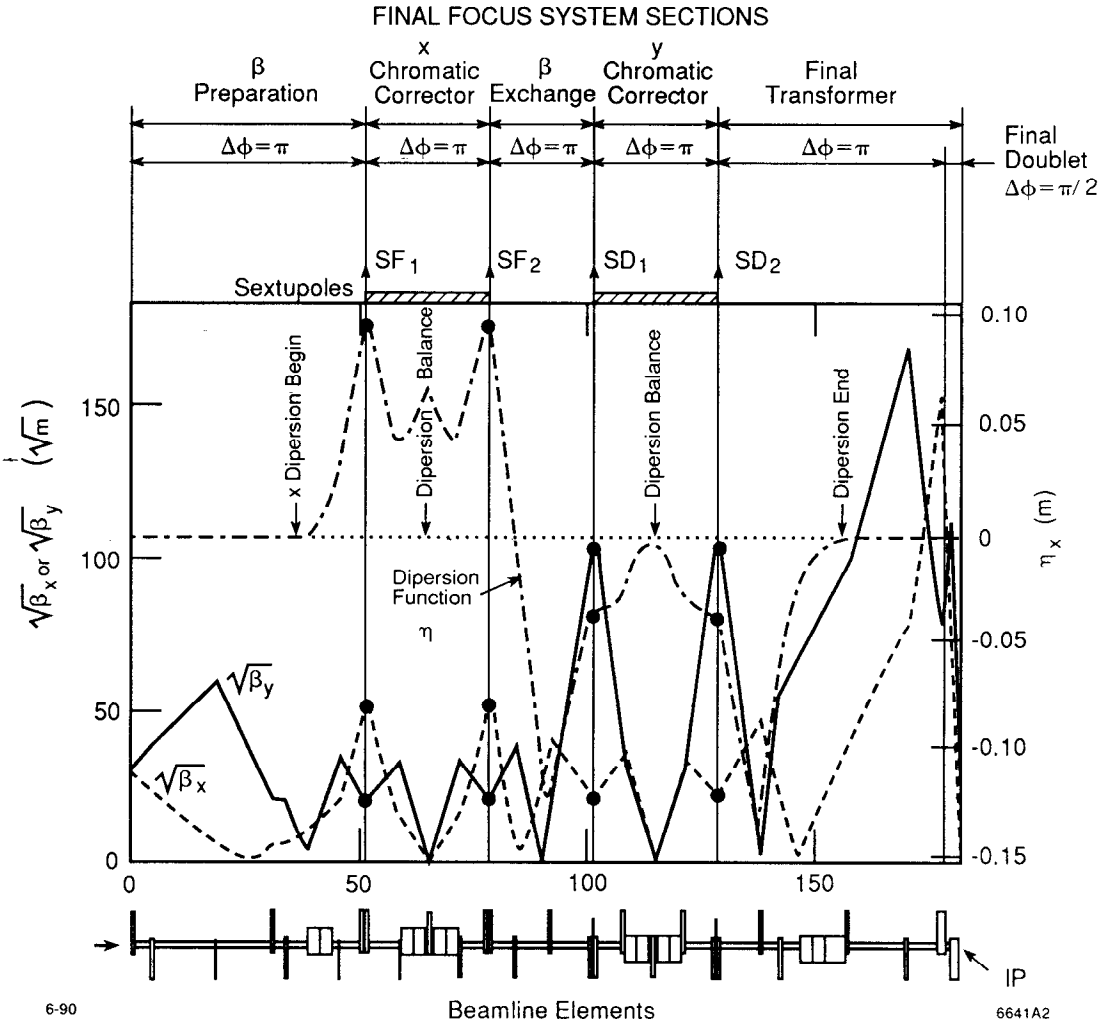

FINAL FOCUS SYSTEM SECTIONS
β Preparation
x Chromatic Corrector
β Exchange
y Chromatic Corrector
Final Transformer
Δφ=π
Final Doublet
Δφ=π/2
SF1
SF2
SD1
SD2
Sextupoles
x Dipersion Begin
Dipersion Balance
Dipersion Balance
Dipersion End
Dipersion Function
η
√βy
√βx
√βx or √βy (√m)
ηx (m)
0.10
0.05
0
-0.05
-0.10
-0.15
150
100
50
0
0
50
100
150
IP
Beamline Elements
6-90
6641A2

Fig. 2

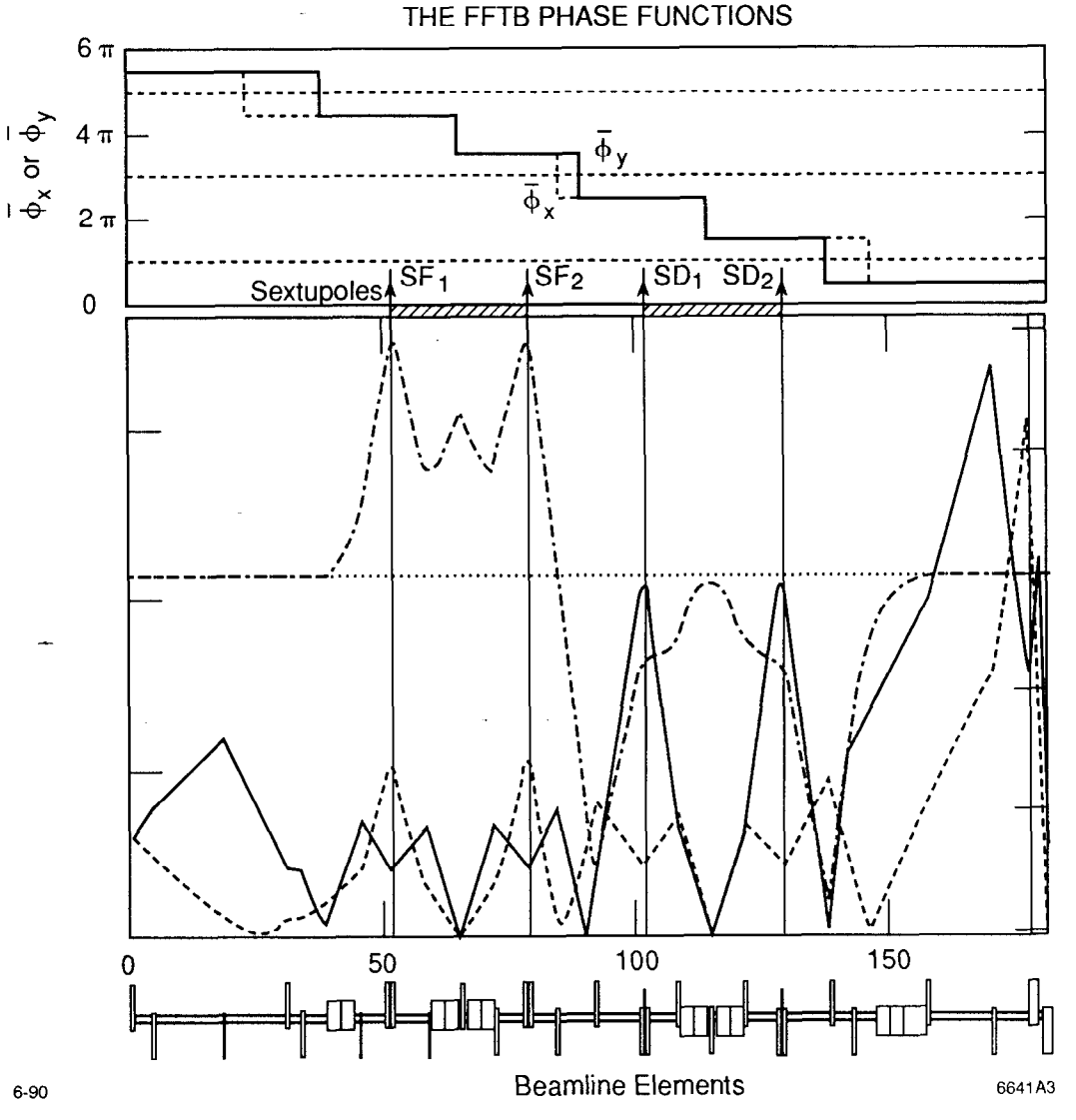

THE FFTB PHASE FUNCTIONS
6 π
4 π
2 π
0
$\bar{\phi}_x$ or $\bar{\phi}_y$
$\bar{\phi}_y$
$\bar{\phi}_x$
Sextupoles
SF1
SF2
SD1
SD2
0
50
100
150
Beamline Elements
6-90
6641A3

Fig. 3

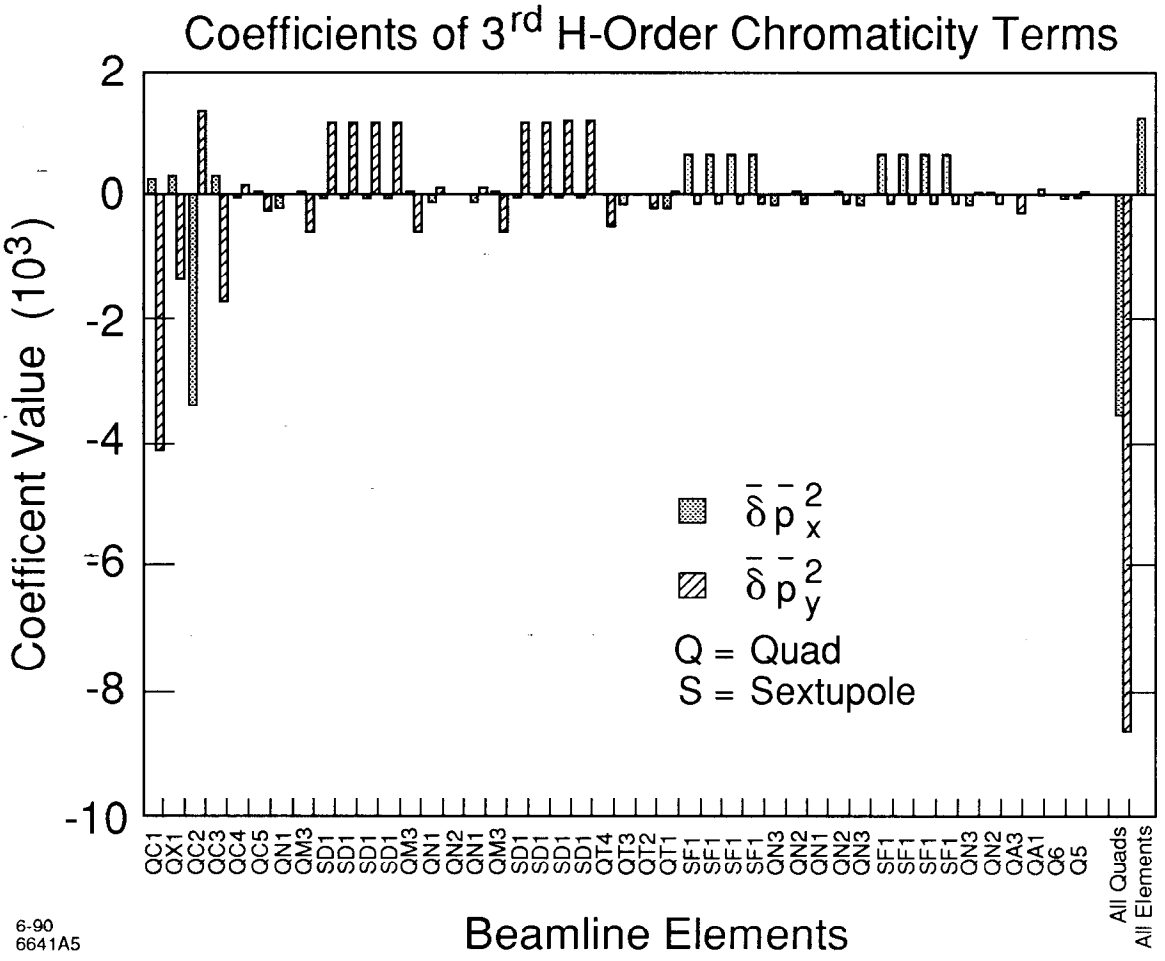
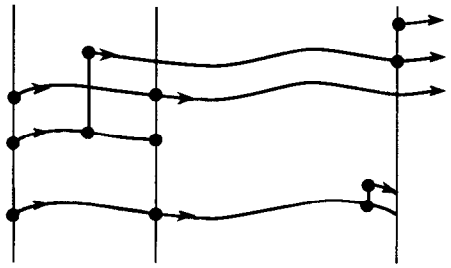
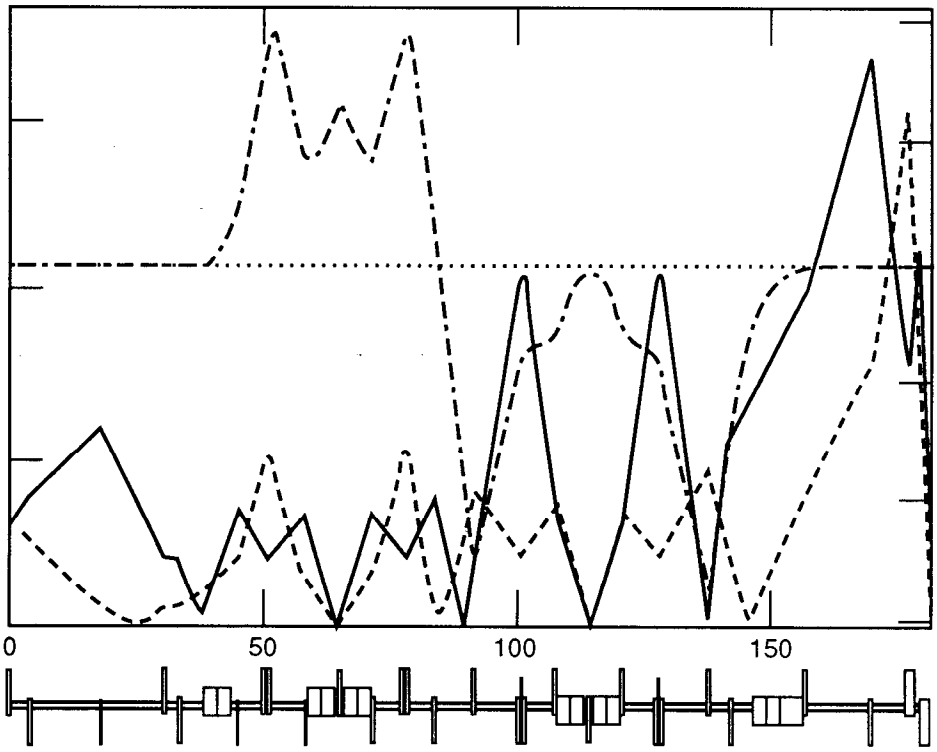

Coefficients of 3rd H-Order Chromaticity Terms
Coefficent Value (10^3)
2
0
-2
-4
-6
-8
-10
$\bar{\delta}\bar{p}_x^2$
$\bar{\delta}\bar{p}_y^2$
Q = Quad
S = Sextupole
QC1 QX1 QC2 QC3 QC4 QC5 QN1 QM3 SD1 SD1 SD1 SD1 QM3 QN1 QN2 QN1 QM3 SD1 SD1 SD1 SD1 QT4 QT3 QT2 QT1 SF1 SF1 SF1 SF1 QN3 QN2 QN1 QN2 QN3 SF1 SF1 SF1 SF1 QN3 QN2 QA3 QA1 Q6 Q5
All Quads
All Elements
Beamline Elements
6-90
6641A5

Fig. 4

IMPORTANT Y-DEPENDENT ABERRATIONS

Aberation Name	H-Order	Functional Form
Chromo-geom	5 th	$\bar{\delta}^2 \bar{p}_x \bar{p}_y^2$
	4 th	$\bar{\delta} \bar{p}_x y \bar{p}_y$
Geometric	3 rd	$\bar{p}^4$ & $\bar{p}^2 \bar{p}^2$
	4 th	$\bar{p}_x \bar{p}_y^2$
Chromatic	4 th	$\bar{\delta}^2 \bar{p}_y^2$
	3 rd	$\bar{\delta} \bar{p}_y^2$





6-90

Beamline Elements

6641A4

Fig. 5