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Gravitational waves and stellar structure in effective quantum gravity

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Abstract

Questions about the origin and structure of the universe have been asked for a long time. Our current understanding of the universe is that it is dominated by dark matter and dark energy. Most of the matter and energy in the universe are invisible, and their origin is still a mystery. We aspire to discover the origin of these dark components in our universe and answer the questions.

Gravity plays an important role in the evolution of the universe. The general theory of relativity (GR), as proposed by Einstein in 1915, is a theory of gravity which describes gravitation in terms of curved geometry of spacetime. GR has established itself as the most successful theory of gravity to date through experimental and observational tests. GR is, however, not considered to be a complete theory today. One of the serious theoretical problems is the occurrence of singularity under general conditions, which brings about mathematical failure. Furthermore, modification of GR is considered necessary in order to make it renormalizable in quantization. In addition, in the context of cosmological observations, it is anticipated that the mystery of accelerated expansion (dark energy) may be solved by modifying GR. As implied above, after 100 years of the proposal of GR, the search for a more correct theory of gravity is strongly motivated from various theoretical and observational aspects.

The purpose of this thesis is to improve our understanding of gravity. We use two classes of extended gravity theories in this thesis: generic higher-curvature gravity (HCG) and generic linear massive gravity (MG). HCG is a theory that adds higher-order terms of spacetime curvature as a correction at low energies to the Einstein-Hilbert action. Some of its variants are known to be renormalizable and/or appear in the low-energy limit of superstring theory. Thus, it is one of the candidates for quantum gravity at low energies. When expanded around a flat background, the linear dynamical degrees of freedom (DOFs) in this theory are identified as massless spin-2, massive spin-2, and massive spin-0 modes. MG is a theory that adds mass to the spin-2 graviton of GR, and it is pointed out that it can be related to the late accelerated expansion of the universe. Generic linear MG loosens the Fierz-Pauli tuning for the Lorentz-invariant mass term so the spin-0 ghost is allowed to propagate. These extra DOFs create physical effects that are not present in GR.

Recently, human beings have acquired a direct and very powerful tool for testing theories of gravity – the observation of gravitational waves (GWs). GWs are a ripple of spacetime, which are modally decomposed into physical DOFs called polarizations. Since the contents of polarization modes differ depending on the theory of gravity, it is possible to test the correctness of a theory by observing the polarizations. Although the present-day GW observations have not succeeded in determining the polarizations due to the smaller number of available detectors than necessary and insufficient observational accuracy, it is expected to be achieved in the near future. In this thesis, we

identify the polarizations of GWs in generic HCG and generic linear MG. In generic HCG, we have shown that (i) the massless spin-2 is the ordinary graviton with 2 tensor-type (helicity-2) polarisations, (ii) the massive spin-2 breaks down into 2 tensor-type (helicity-2), 2 vector-type (helicity-1) and 1 scalar-type (helicity-0) polarizations, (iii) the massive spin-0 provides 1 scalar-type (helicity-0) polarization. Therefore, GWs in generic HCG exhibit 6 massive polarizations on top of the ordinary 2 massless ones. The generic linear MG with generic mass terms of non-Fierz-Pauli type consists of spin-2 and spin-0 modes, the former breaking down into 2 tensor (helicity-2), 2 vector (helicity-1), and 1 scalar (helicity-0) components, while the latter just corresponding to a scalar. We also find convenient representations of the scalar-polarization modes connected directly to the theory parameters of HCG. They are utilised to determine those parameters by GW-polarization observations.

Non-relativistic stars such as white, brown, and red dwarfs can be powerful probes of gravitational interactions. In general, modifications to GR would change the hydrostatic equilibrium condition in stars, thus having a large impact on the stellar radius and mass. In this thesis, we study the structure of static spherical stars made up of a nonrelativistic polytropic fluid in generic HCG and generic linear MG. We first formulate the modified Lane–Emden (LE) equation for the stellar profile function, finding it boils down to a sixth-order differential equation in the generic case. In special cases, it reduces to a fourth-order equation reflecting the number of additional massive gravitons arising in each theory. Moreover, the existence of massive gravitons renders the nature of the boundary-value problem unlike the standard LE: some of the boundary conditions can no longer be formulated in terms of physical conditions at the stellar centre alone, but some demands at the stellar surface necessarily come into play. We present a practical scheme for constructing solutions to such a problem and demonstrate how it works in the cases of the polytropic index $n = 0$ and 1, where analytical solutions to the modified LE equations exist. As physical outcomes, we clarify how the stellar radius, mass, and Yukawa charges depend on the theory parameters and how these observables are mutually related.

In conclusion, if GW polarizations become observable in the future, we may confirm whether or not HCG or MG provides a theory beyond GR. Observations of stellar radii will be helpful to examine the constraints based on GW observations.

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Preface

Organization of this thesis: In Chapter 1, we will provide an overview of theories of gravity, including general relativity, higher-curvature gravity (HCG), and massive gravity (MG). We will explain why we require theories beyond GR and introduce gravitational waves (GWs) and non-relativistic stars as tools for testing gravity. In Chapter 2, we will study the number of physical degrees of freedom (DOFs) in MG and HCG. We will identify GW polarization modes and show how the physical DOFs propagate in Minkowski spacetime. We will subsequently discuss the methodology for determining theory parameters from GW observations with laser interferometers and pulsar timing arrays, such as LIGO and NANOGrav. In Chapter 3, we will study non-relativistic stellar structure in MG and HCG. We will derive a master equation that provides the stellar profile function and find exact solutions. We will then show how to determine the theory parameters through stellar observations. In Chapter 4, we will conclude this thesis.

Notations: Throughout the thesis, we will work with natural units with $c = 1$. Greek indices of tensors such as $\mu, \nu, \cdots$ are of space-time, while Latin ones such as $i, j, \cdots$ are spatial. We employ the Einstein summation convention, such as

$$\begin{aligned} A_\mu B^\mu &= \sum_{\mu=0}^{3} A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3 \,, \\ a_i\, b^i &= \sum_{i=1}^{3} a_i\, b^i = a_1 b^1 + a_2 b^2 + a_3 b^3 \,. \end{aligned} \tag{1}$$

We introduce background coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ in which Minkowski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)\,. \tag{2}$$

The partial differentiation is

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_i)\ . \tag{3}$$

$\Box \equiv \eta^{\mu\nu}\, \partial_\mu \partial_\nu$ is the d'Alembertian and $\triangle \equiv \delta^{ij}\, \partial_i \partial_j$ the Laplacian. We denote the curved spacetime metric by $g_{\mu\nu}(x)$ and its determinant by g. The Christoffel symbol is

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\,. \tag{4}$$

The Riemann tensor is defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\alpha\rho} \Gamma^\alpha{}_{\nu\sigma} - \Gamma^\mu{}_{\alpha\sigma} \Gamma^\alpha{}_{\nu\rho}\,. \tag{5}$$

The Ricci tensor is $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, and the Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$. The energy-momentum tensor for the matter field is defined from the variation of the matter action S_M under a change of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, according to

$$\delta S_M = \frac{1}{2} \int \mathrm{d}^4 x \, \sqrt{-g} \, T^{\mu\nu} \delta g_{\mu\nu} \, . \tag{6}$$

Parentheses around tensor indices denote symmetrisation such as

$$T_{(\mu\nu)} \equiv \frac{1}{2} \left(T_{\mu\nu} + T_{\nu\mu} \right) , \tag{7}$$

while square brackets denote antisymmetrisation such as

$$T_{[\mu\nu]} \equiv \frac{1}{2} \left(T_{\mu\nu} - T_{\nu\mu} \right) . \tag{8}$$

Our conventions of the Fourier transform for a function of time and its inverse are

$$\begin{aligned} \tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\omega t} \, f(t) \, , \\ f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \mathrm{e}^{-\mathrm{i}\omega t} \, \tilde{f}(\omega) \, . \end{aligned} \tag{9}$$

Declaration

This thesis includes materials presented in the following three papers co-authored by myself and my colleagues:

Tomoya Tachinami, Shinpei Tonosaki, and Yuuiti Sendouda, "Gravitational-wave polarizations in generic linear massive gravity and generic higher-curvature gravity", Physical Review D **103**, 104037 (2021),

Shinpei Tonosaki, Tomoya Tachinami, and Yuuiti Sendouda, "Non-relativistic stellar structure in higher-curvature gravity: Systematic construction of solutions to the modified Lane–Emden equations", Physical Review D **108**, 024037 (2023),

and

Tomoya Tachinami and Yuuiti Sendouda, "Non-relativistic stellar structure in the Fierz–Pauli theory and generic linear massive gravity", Physical Review D **110**, 064013 (2024).

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Chapter 1

Introduction

1.1 General theory of relativity

The general theory of relativity (GR), which describes gravity in the language of spacetime geometry, was proposed by Einstein in 1915. This is the most successful theory of gravity to date having passed various observational tests. The predictions of GR are incredibly consistent with observations from the sub-millimetre scale to the solar-system scale. Observational evidence includes Mercury's perihelion shift, light bending, and Shapiro time delay caused by the Sun. In 2015, gravitational waves (GWs) propagating as dynamical fluctuations in spacetime were directly observed [A$^+$16a]. This has given us one more piece of evidence supporting GR.

However, GR is not the only theory of gravity, and there are many alternatives that extend and modify GR. The history of the modification of gravity goes back to Weyl in 1919 [Wey19], just after the proposal of GR. Modifying gravity is considered to have two main motivations: to overcome the cosmological problems in GR and to construct a quantum theory of gravity.

The present standard cosmological model is the Big Bang model originally proposed by Gamow in 1946, which states that the universe began explosively as a high-temperature and high-density state and, through the expansion, the temperature and density decreased to the values in the present universe [Gam46, ABG48]. The Big Bang model was developed on the basis of general relativity as a theory capable of treating the whole universe. It also assumes the cosmological principle that the universe is not uneven and has no special direction, i.e., the metric of spacetime and distribution of matter in the universe are uniform and isotropic. One of the predictions of the Big Bang model is that there be the Plank-distributed cosmic microwave background (CMB) radiation in the present universe. This is because a universe that began in a hot state should be filled with thermally distributed radiation. In fact, CMB was discovered by Penzias and Wilson in 1965 [PW65], and the Big Bang model became the present standard cosmological model. However, the Big Bang model left several problems. In particular, the horizon problem and the flatness problem are significant unsolved problems.

From the beginning of the decelerated universe until a certain time, only matter in a finite region of the universe can causally interact with a given point, and the edge of this causal region is called the particle horizon. In an expanding universe, the horizon continues to expand with time, so the size of the horizon becomes smaller if we go back to the early universe. The CMB is observed as a thermal radiation of 2.73 Kelvins coming isotropically from all directions on the celestial sphere [M+94], but the horizon of the cosmic recombination period, when the CMB was last scattered, is only about 2 degrees on the present celestial sphere. This means that all regions that have been causally independent of each other before the recombination of the universe should have different temperatures. It is unnatural for regions of the universe that have never been causally connected to each other to be in the same state, and this is called the horizon problem.

Observations show that the present universe is extremely flat in space [A+16b]. The natural initial value of curvature at the birth of the universe at the Planck temperature is considered to be about (Planck length)$^{-2}$ = $10^{66}\,\mathrm{cm}^{-2}$. However, if we go back to the Planck temperature from the presently observed value of $10^{-56}\,\mathrm{cm}^{-2}$ or less, we would find its initial value around $10^{8}\,\mathrm{cm}^{-2}$ or less. Comparing the two, the flatness problem is the need to set an initial condition that is at least 58 orders of magnitude smaller than the natural value.

The inflation theory is expected to solve these problems of the Big Bang model. It was proposed around 1980 by Sato [Sat81] and Guth [Gut81]. The simplest mechanism for inflation to occur is an exponential expansion due to the energy of a scalar field (inflaton). Apart from the introduction of scalar fields, Starobinsky has already pointed out in 1980 that an exponential expansion of the universe occurs by introducing a higher order curvature term into the action [Sta80]. Inflation theory is an essential part of the modern standard cosmological model, but its realization mechanism is still unknown.

Observations of Type Ia supernovae (SN Ia) tell us that the current universe is undergoing accelerated expansion. It was achieved by two independent research teams, High-redshift Supernova Search Team (HSST) [R+98] and supernova Cosmology Project (SCP) [P+99], in 1998. This fact has been confirmed by many subsequent observations (LSS, BAO, and CMB). The unknown energy component that causes this late accelerated expansion of the universe is called dark energy, which, assuming a positive energy density, has a negative pressure and is, therefore, completely distinct from ordinary matter (baryons and radiation). The equation of state of dark energy is expressed as $w = P/\rho$, where P is pressure and ρ is energy density. The theoretical value of w depends on the model; in the case of a cosmological constant, $w = -1$, which is constant over time, but in many other models, the value of w changes over time. Since the cosmological constant can cause this accelerated expansion within the framework of general relativity, which is the standard theory of gravity and is consistent with observations, many people believe that the cosmological constant is the true identity of dark energy. Considering the possibility that the origin of dark energy is not a cosmological constant, we can assume that its equation of state is of the form

$$w(a) = w_0 + (1-a)w_a \tag{1.1}$$

as a function of the scale factor, where $a = 1$ is the current value. Then, the constraints are given by

$$w_0 = -0.957 \pm 0.080\,, \quad w_a = -0.29^{+0.32}_{-0.26}\,. \tag{1.2}$$

In fact, this shows that considering the cosmological constant ($w_0 = -1$ and $w_a = 0$) to be dark energy is consistent with observation [A$^+$20]. In addition, the ΛCDM model, which is the standard model of cosmology, assumes a cosmological constant as the origin of dark energy, and it is estimated that dark energy accounts for approximately 70 % of the total energy in the current universe.

As mentioned above, adding a cosmological constant to the framework of general relativity is consistent with observations and seems natural. However, the observed dark energy density is extremely small as $\rho_\Lambda = 10^{-47}\,\mathrm{GeV}^4$, and it is unclear how such a fine-tuning is achieved. On the other hand, from the viewpoint of particle physics, the cosmological constant is interpreted as the energy density of the vacuum, and its value is estimated to be $\rho_{vac} = 10^{74}\,\mathrm{GeV}^4$, which is approximately 10^{121} times larger than the observed value. It remains a big mystery. If the cosmological constant is causing the late accelerated expansion of the universe, we must find a mechanism that allows us to obtain a value of dark energy density that agrees with the observations (the cosmological constant problem) [Wei89]. Because of these problems, it is quite possible that dark energy is not a cosmological constant. Furthermore, it has recently been pointed out that in observations of the Hubble constant, the values obtained from low-redshift surveys and those obtained from CMB observations are significantly different. This discrepancy may suggest an origin of dark energy other than the cosmological constant.

Modifying gravity is also considered as a bottom-up approach to constructing the quantum gravity. In this context, higher-curvature gravity is an old idea as an effective field theory model for more fundamental string or quantum gravity theories. For example, in the 1960s and later, attempts were made to remove the quantum divergence by modifying the action. In 1962, Utiyama and DeWitt showed that renormalizability at one loop requires that the Einstein-Hilbert action be supplemented by higher order curvature terms [UD62], and in 1977, Stelle showed that the higher order action is indeed renormalizable [Ste77]. We introduce these higher-curvature models in the next section.

Recently, Chern-Simons gravity [JP03] is also considered, as reviewed in [AY09]. It can arise in various anomaly cancellation schemes in the standard model of particle physics, in cancelling the Green-Schwarz anomaly in string theory, or in effective field theories of inflation [Wei08]. Einstein-Gauss-Bonnet gravity is also taken into account in the above motivations. It has been reviewed in [MS07], for instance.

1.2 Higher-curvature gravity

In this thesis, we consider gravity theories whose Lagrangian can be written in a generic form

$$L = \frac{1}{2\kappa}\, f(R^{\mu}{}_{\nu\rho\sigma}, g_{\mu\nu})\,, \tag{1.3}$$

where $g_{\mu\nu}$ is the space-time metric, $R^{\mu}{}_{\nu\rho\sigma}$ the Riemann tensor and κ the bare gravitational constant. The scalar function f is almost generic but we here conservatively assume that, when Taylor expanded around $R^{\mu}{}_{\nu\rho\sigma} = 0$, it only gives positive powers of the Riemann tensor so that Minkowski spacetime is a solution of the full theory. Studies of such models date back to Weyl [Wey19], who suggested

$$f = C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma}\,, \tag{1.4}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl curvature tensor defined by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{R}{3} g_{\mu[\rho}g_{\sigma]\nu}\,. \tag{1.5}$$

Relatively modern motivations also come from the developments in string theories, e.g., [GW86]. Einstein's general relativity is defined by the linear function $f = R$, while the presence of any higher-order terms characterizes how the theory differs from GR.

Higher-curvature gravity (HCG) generically exhibits more dynamical degrees of freedom (DOFs) than GR does. For instance, the theory with

$$f = R + \beta\, R^2\,, \tag{1.6}$$

can be shown to be equivalent to a scalar-tensor theory [SF10], which can be generalized to the case of a generic function f of the Ricci scalar, called $f(R)$ gravity. As a cosmological application, the $R + \beta\, R^2$ model was utilized by Starobinsky to realize inflation [Sta80]. A generic class of $f(R)$ gravity has provided candidates for the dark energy [DFT10] , although such theories with negative powers of curvature are out of our scope in this thesis. Also, it was shown by Stelle [Ste78] that in the theory with

$$f = R - \alpha\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} + \beta\, R^2\,, \tag{1.7}$$

there arises another massive spin-2 particle on top of a zero-mass graviton, which was utilized to render the quantum theory renormalizable [Ste77]. Afterwards, a general Hamiltonian analysis of f(Riemann) gravity, keeping f undetermined, was done in [DSSY10]. Recently conducted research on quadratic-curvature gravity (QCG) includes the study of QCG-specific black hole solutions [LPPS15] .

We would like to mention the problem that arises in HCG. Some modified gravity theories produce ghost fields that violate theoretical consistency. A ghost field is a DOF that gives rise to a Hamiltonian non-bounded underneath, and when the ghost field interacts with a normal field bounded underneath, they transfer energy to each other, and the fields become unstable. In particular, ghost fields arising from theories that include higher-order derivatives in the Lagrangian are called Ostrogradsky ghosts, and in HCG, it is known that ghost fields emerge when the action includes higher-order terms of the Riemann or Ricci tensors [Sal18]. Thus, sometimes HCG may not be considered as a consistent theory. However, it can happen that HCG avoids instability because the ghost field can sometimes be separated from the normal fields, at least within the linear approximations. The basis for determining the presence or absence of ghosts is Ostrogradsky's theorem reviewed in [Woo07] and [Sal18].

We begin with discussing perturbative degrees of freedom in this HCG in the Minkowski background. If the Lagrangian f consists only of terms that have smooth behavior around $R_{\mu\nu\rho\sigma} = 0$, its expansion in curvature tensors up to the quadratic order can be arranged as

$$f = \chi\, R - \alpha\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} + \beta\, R^2 + \gamma\, \left(R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} - 4\, R_{\mu\nu}\, R^{\mu\nu} + R^2\right) + \mathcal{O}(R_{\mu\nu\rho\sigma})^3\,, \tag{1.8}$$

where χ, α, β and γ are constants. The combination in the parentheses, the so-called Gauss–Bonnet invariant, is topological in four dimensions and can be discarded in the action integral. Then we find that the generic higher-curvature action expanded up to the second order in the metric perturbation $h_{\mu\nu}$

$$S_{\mathrm{HCG}}[h_{\mu\nu}] = \frac{1}{2\kappa} \int \mathrm{d}^4x \left(-\frac{\chi}{2}\, {}^{(1)}G_{\mu\nu}\, h^{\mu\nu} - \alpha\, {}^{(1)}C_{\mu\nu\rho\sigma}\, {}^{(1)}C^{\mu\nu\rho\sigma} + \beta\, {}^{(1)}R^2\right)\,, \tag{1.9}$$

where ${}^{(1)}G_{\mu\nu}$ is the linearized Einstein tensor

$${}^{(1)}G_{\mu\nu} \equiv -\frac{1}{2}\, \Box h_{\mu\nu} + \partial_{(\mu}\partial^\lambda h_{\nu)\lambda} - \frac{1}{2}\, \partial_\mu\partial_\nu h + \frac{1}{2}\, \eta_{\mu\nu}\, \left(\Box h - \partial^\rho\partial^\sigma h_{\rho\sigma}\right)\,, \tag{1.10}$$

${}^{(1)}C_{\mu\nu\rho\sigma}$ and ${}^{(1)}R$ are the linear perturbations of the Weyl tensor and Ricci scalar, respectively, whose expressions are presented in Appendix A. When $\chi = 0$, the theory cannot be seen as GR with corrections and, moreover, as we will see in Appendix A, there arise instabilities in the tensor and scalar parts. So, hereafter, we assume $\chi \neq 0$.

1.3 Massive gravitons

As first shown by Stelle [Ste78] in Minkowski, there is an equivalence of the action (1.9) to GR "minus" massive gravity. In this formalism, the original metric perturbation is given by

$$h_{\mu\nu} = \phi_{\mu\nu} + \tilde{\phi}_{\mu\nu}\,. \tag{1.11}$$

Then, the quadratic action (1.9) can be arranged into the form

$$\begin{aligned} S_{\mathrm{HCG}}[\phi_{\mu\nu}, \tilde{\phi}_{\mu\nu}] &= \chi\, S_{\mathrm{GR}}[\phi_{\mu\nu}] - \chi\, S_{\mathrm{MG}}[\tilde{\phi}_{\mu\nu}] \\ &= \frac{\chi}{4\kappa} \int \mathrm{d}^4x \left[-{}^{(1)}G_{\mu\nu}[\phi]\, \phi^{\mu\nu} + {}^{(1)}G_{\mu\nu}[\tilde{\phi}]\, \tilde{\phi}^{\mu\nu} + \frac{m^2}{2}\, \left(\tilde{\phi}_{\mu\nu}\, \tilde{\phi}^{\mu\nu} - (1-\epsilon)\, \tilde{\phi}^2\right)\right]\,, \end{aligned} \tag{1.12}$$

where we introduced $m^2 \equiv \chi/(2\alpha)$ and $\epsilon = 9\beta/(2\alpha+12\beta)$. For the discussion extended to Einstein manifolds, see [NNZ+]. Clearly, we need to assume $\alpha \neq 0$. The case with $\alpha = 0$ can be treated in a similar manner by means of a conformal transformation, after which the theory takes the form of a scalar-tensor theory; See [MBM19] for an example.

As seen from the structure of the action (1.12), the dynamical contents in this theory are $\phi_{\mu\nu}$ as a massless spin-2 field and $\tilde{\phi}_{\mu\nu}$ as a mixture of a massive spin-2 and a spin-0 fields. It is worth mentioning that, while $\phi_{\mu\nu}$ field has the same gauge symmetry as GR, $\tilde{\phi}_{\mu\nu}$ is not subject to gauge transformations.

Massive gravity (MG), the idea to give a mass to the graviton, has been studied based on several motivations. One is to offer a candidate for the cosmic acceleration mechanism. MG gives rise to a large-distance modification of gravity because a massive graviton only propagates over a finite distance. Another is a theoretical interest because the theory of massive graviton is an extended version of GR which is the theory of massless spin-2 graviton.

When deviation from the flat space-time, $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$, is small, the second-order perturbed action of a generic linear theory for massive gravitons is

$$\begin{aligned} S_{\rm MG}[h_{\mu\nu}] &= S_{\rm GR}[h_{\mu\nu}] - \frac{m^2}{8\kappa} \int \mathrm{d}^4x \, \left[h_{\mu\nu}\, h^{\mu\nu} - (1-\epsilon)\, h^2\right] \\ &= \frac{1}{4\kappa} \int \mathrm{d}^4x \left[-{}^{(1)}G_{\mu\nu}\, h^{\mu\nu} - \frac{m^2}{2} \left(h_{\mu\nu}\, h^{\mu\nu} - (1-\epsilon)\, h^2\right) \right] , \end{aligned} \tag{1.13}$$

where $S_{\rm GR}$ is thesecond-order GR action

$$S_{\rm GR}[h_{\mu\nu}] = -\frac{1}{4\kappa} \int \mathrm{d}^4x \, {}^{(1)}G_{\mu\nu}\, h^{\mu\nu} \, , \tag{1.14}$$

m corresponds to the mass of the spin-2 graviton and ϵ is a nondimensional parameter. This is the most generic extension of linear general relativity that incorporates Lorentz-invariant mass terms. For $\epsilon = 0$, the action reduces to that of the Fierz–Pauli theory [FP39], where the graviton is pure spin-2, otherwise a spin-0 “ghost” graviton emerges, as we shall confirm below.

A crucial observation associated with the FP theory with a matter field was made by van Dam and Veltman and by Zakharov in 1970 [vDV70, Zak70]. They added a minimally coupled matter source

$$S_{\rm int} = \frac{1}{2} \int \mathrm{d}^4x \, h_{\mu\nu}\, T^{\mu\nu} \tag{1.15}$$

to the gravitational action (1.13) with $\epsilon = 0$. Their discovery was that the prediction for the bending angle of light due to a massive body in the massless limit of the FP theory differs from GR by a factor of 3/4, a phenomenon called the van Dam–Veltman–Zakharov (vDVZ) discontinuity. Rather than rejecting the FP theory, however, the discovery has stimulated inspections of “screening” mechanisms in MG that can cancel the discrepancy and make predictions on the solar-system scale consistent with GR. One of the most notable proposals was made by Vainshtein [Vai72], who considered a non-linear kinetic self-interaction and found recovery of consistency with GR within a certain radius. See [Hin12] for historical overviews and recent developments.

1.4 Scalar-tensor theories

There are many models of modified gravity theory, but the scalar-tensor theory has been attracting particular attention as a promising candidate to explain the late accelerated expansion of the universe without requiring a cosmological constant. Applications of various modified gravity theories to cosmology are comprehensively reviewed in [CFPS12].

A simplest model of the scalar-tensor theory is the Brans–Dicke (BD) theory with a scalar potential

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g} \left[\varphi R - \frac{w}{\varphi} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \right] . \tag{1.16}$$

Dirac suggested that gravitational coupling may not be constant, at least in a cosmological context (Dirac's giant number hypothesis) [Dir37], and Jordan developed this idea [Jor59]. Later, Brans and Dicke proposed the action (1.16) without scalar potential $V(\varphi)$ [BD61]. While philosophical motives predominated in the past, modern motives include more physical ones, such as dark energy or implications from string theory. The BD theory, sometimes called massive dilaton gravity as proposed by [O'H72], with a scalar potential with $w = 0$ is equivalent to the $f(R)$ gravity, where the Lagrangian is an arbitrary function of the Ricci scalar. Therefore, BD theory is often investigated in conjunction with $f(R)$ gravity theory. The literature reviewing $f(R)$ gravity and BD theories and their applications to cosmology includes [DFT10, SF10, CDL11]. In particular, a theory with $w = 0$ and $V(\varphi) = \frac{1}{4\beta}(\varphi - 1)^2$ is equivalent to $f = R + \beta R^2$ gravity (β is a constant). This is known as the Starobinsky model [Sta80] and it is a promising inflation model to explain the accelerated expansion (inflation) of the early universe.

One of the features of BD theory is that gravitational coupling is not constant. According to [Wil14], the current value of gravitational coupling is

$$G_{\text{today}} = \frac{4 + 2w}{3 + 2w} \frac{1}{\phi_0} , \tag{1.17}$$

where ϕ_0 is a constant such that the potential takes the minimum value $V(\phi_0) = 0$. The original BD theory has a constraint $w > 4 \times 10^4$ from Cassini's observation [Wil14], but this is not the case if a potential exists. The gravitational constant in GR is about $G_{\text{today}} = 6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$. This is an overwhelmingly low precision compared to other fundamental constants because the gravitational interaction is very weak, and there is no negative mass, so shielding is impossible.

1.5 Gravitational waves

Since the first observation of gravitational-waves (GWs) in 2015, the development of gravitational wave astronomy has been remarkable [A+16a]. GWs have a physical degree of freedom (DOF) called polarization, which represents the modes of oscillation of GWs, and in general relativity, there are two polarizations called plus (+) mode and cross (×) mode. At present, the type of polarization has not been identified, but it is expected to be achieved in the near future, considering the recent remarkable improvements in observation accuracy. The successful observation of polarization is expected to provide evidence for the correctness of general relativity or to suggest some kind of its correction.

It was argued in [ELL+73b, ELL73a] that GWs in a certain generic class of gravity theories can have maximally six polarizations. Although there has been no contradiction with the hypothesis of only two polarizations in the GW experiments [A+18] including the observed orbital decay rate of a neutron star binary PSR B1913+16 [TW82], the ongoing progress in the construction of the worldwide GW-observatory network has motivated the developments of various methods for detecting those beyond-Einstein polarization modes [HN13, IWMP15, IPW17, TNM+18]. In practice, however, a complete decomposition into the possible six polarizations and determination of each amplitude in an observed GW signal cannot be done with the limited number of detectors that we currently have, so there have been only weak constraints on the existence of non-GR polarizations; An example is the constraint on the vector-type polarizations obtained by a method to detect scalar and vector polarizations with four interferometers of LIGO (Hanford and Livingston), Virgo, and KAGRA developed in [HEIA18, HEI+19, HEI+20]. In this regard, there are still open and wide theoretical possibilities to explore.

A representative example of extended gravity theories is the linear massive gravity (MG), in which gravitational waves acquire masses. Adding a generic combination of possible two mass terms to GR breaks its four linear gauge symmetries and gives rise to six DOFs in total. A special class of MG introduced by Fierz and Pauli (FP) [FP39] with a single mass parameter m is known to avoid the appearance of the spin-0 mode that would have a negative kinetic term, see e.g. [Hin12] for a review.

If GWs are massive, their group velocity $c_{\rm g}$ deviates from the speed of light c due to the modification of the dispersion relation. The multi-messenger analysis of the GW event GW170817 [A+17b] has put a tight constraint on the deviation, $|c_{\rm g} - c|/c \lesssim \mathcal{O}(10^{-15})$ [A+17a], which can be interpreted as an upper limit on the mass.

1.5.1 Geodesic deviation

GWs have the property of inducing a change in the spatial distance between two distant points. The principle of GW observation using laser interferometers and pulsar timing arrays is to measure the spatial separation of a proper distance. As long as we are considering a single geodesic line, we cannot measure the curvature of spacetime, but if we consider two geodesic lines slightly apart, we can determine the curvature of space-time by observing changes in the distance between them. Therefore, we consider a geodesic congruence $x^\mu(\lambda\,,\sigma)$, which is a collection of geodesic curves that exist continuously in a finite region of space and time. λ is an affine parameter of the geodesic lines, and σ is a parameter that distinguishes each trajectory. Two vectors

$$k^\mu = \frac{\partial x^\mu}{\partial \lambda}\,, \quad Z^\mu = \frac{\partial x^\mu}{\partial \sigma} \tag{1.18}$$

locally characterise the geodesic bundle, where k^μ and Z^μ represent the tangent vector of a geodesic line and the vector connecting adjacent geodesic lines that are infinitesimally apart, respectively. Because geodesic bundles cover a finite space-time volume, these vectors can be considered as fields that exist continuously in spacetime. The variation of the vector k^μ along the geodesic line is the geodesic equation

$$k^\nu \nabla_\nu k^\mu = \frac{\mathrm{D}k^\mu}{\partial \lambda} = 0\,. \tag{1.19}$$

D is a differential operator that gives the change due to parallel translation of a vector over an infinitesimal distance δx^μ defined as

$$\begin{aligned} \mathrm{D}A^i &:= A^\mu(x+\delta x) - \tilde{A}^\mu(x+\delta x) \\ &= A^\mu(x+\delta x) - A^i(x) + \Gamma^\mu{}_{\nu\lambda} A^\nu \delta x^\lambda \\ &= (\partial_\lambda A^\mu + \Gamma^\mu{}_{\nu\lambda} A^\nu)\delta x^\lambda\,, \end{aligned} \tag{1.20}$$

where we defined the infinitesimal translation as

$$\tilde{A}^\mu(x+\delta x) \equiv A^\mu(x) - \Gamma^\mu{}_{\nu\lambda}(x) A^\nu(x) \delta x^\lambda\,. \tag{1.21}$$

We want to know how the vector Z^μ , called the Jacobi field, changes along the geodesic line. So, we take the differential of Z^μ in the k^μ direction to get

$$\begin{aligned} k^\nu \nabla_\nu Z^\mu &= \frac{\mathrm{D}Z^\mu}{\partial \lambda} \\ &= Z^\nu \nabla_\nu k^\mu\,. \end{aligned} \tag{1.22}$$

Considering the further differential in the k^μ direction, we have

$$\begin{aligned} \frac{\mathrm{D}^2 Z^\mu}{\partial \lambda^2} &= k^\alpha \nabla_\alpha k^\nu \nabla_\nu Z^\mu \\ &= -R^\mu{}_{\alpha\nu\beta} k^\alpha k^\beta Z^\nu\,. \end{aligned} \tag{1.23}$$

This is the geodesic deviation equation, which describes the change in distance between adjacent geodesics due to space-time curvature, that is, a tidal phenomenon due to gravity.

When considering a timelike geodesic bundle, we can take the proper time τ along the geodesic as an affine parameter without loss of generality. Then, we can impose $k_\mu k^\nu = -1$ and the geodesic deviation equation becomes

$$\frac{\mathrm{D}^2\zeta^\mu}{\partial\tau^2} = -R^\mu{}_{\alpha\nu\beta}u^\alpha u^\beta \zeta^\nu \tag{1.24}$$

for $k^\mu = u^\mu$, $Z^\mu = \zeta^\mu$, $\lambda = \tau$. If we can know u^μ and $R^\mu{}_{\alpha\nu\beta}$ on each geodesic, we can find the time evolution of ζ^μ by solving a set of ordinary differential equations along that geodesic. Furthermore, introducing local Cartesian coordinates around the point where $\zeta^\mu = (0\,, \zeta^i)$, the Christoffel symbol can be zeroed at any point and we get

$$\frac{\mathrm{d}^2\zeta^i}{\mathrm{d}^2\tau} = -R^i{}_{\alpha j\beta}u^\alpha u^\beta \zeta^j \,. \tag{1.25}$$

When gravitational waves are present, the Riemann tensor is no longer zero, and the distance between each geodesic curve changes. Since its order is $\mathcal{O}(h)$, we can set $u^\mu \simeq (1\,, 0\,, 0\,, 0)$. From $u^0 \simeq \frac{\mathrm{d}t}{\mathrm{d}\tau} \simeq 1$, we use the relation $\mathrm{d}\tau \simeq \mathrm{d}t$ and get the geodesic deviation equation up to the first order in h as

$$\ddot{\zeta}^i = -{}^{(1)}R^i{}_{0j0}\,\zeta^j \,, \tag{1.26}$$

where the dot denotes the derivative with respect to $x^0 = t$ and ${}^{(1)}R^i{}_{0j0}$ is the linear Riemann tensor

$${}^{(1)}R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij} + \partial_{(i}\dot{h}_{j)0} - \frac{1}{2}\partial_i\partial_j h_{00} \,. \tag{1.27}$$

Thus, the separation of two test bodies would fluctuate in response to the GWs embodied by $h_{\mu\nu}$, and conversely, by tracking their movements, one could decode the dynamical contents of the gravitational theory.

Since the Riemann tensor is invariant under gauge transformation (A.8), the Riemann tensor can be written in terms of gauge invariant variables (see Appendix A) as

$${}^{(1)}R_{i0j0} = -\ddot{H}_{ij} - \partial_{(i}\dot{\Sigma}_{j)} + \partial_i\partial_j\Psi - \delta_{ij}\,\ddot{\Phi} \,. \tag{1.28}$$

It is worth stressing that, despite their apparent advantage, each gauge-invariant variable does not necessarily correspond to a single dynamical DOF in a given gravity theory. Instead, we generally expect that the gauge-invariant variables become linear combinations of independent DOFs. In particular, a mixture of different spins occurs in the scalar part, as we will see in the cases of massive gravity and higher-curvature gravity.

1.5.2 Polarization basis

The connection between the irreducible decomposition of the Riemann tensor (1.28) and observables in GW experiments is made explicit by considering a wave solution propagating in a fixed direction. As we already noted, we should keep in mind that each part can contain multiple DOFs.

Suppose a tensor variable $T_{ij}(z-c_T t)$ propagates in the z direction with velocity c_T. A conventional orthogonal basis that is compatible with the conditions $T_i^{\ i} = \partial_i T^{ij} = 0$ is the $+$ and $\times$ polarizations

$$e^{+}_{ij} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \quad e^{\times}_{ij} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{1.29}$$

Using the basis tensor, we can decompose the tensor variable as

$$T_{ij} = T_{+} e^{+}_{ij} + T_{\times} e^{\times}_{ij} \,, \tag{1.30}$$

where the polarization components are defined as

$$T_{\lambda} = \frac{1}{2} e^{ij}_{\lambda} T_{ij} \,, \tag{1.31}$$

for $\lambda = + \,, \times$. Explicitly, $T_{+} = T_{xx} = -T_{yy}$ and $T_{\times} = T_{xy} = T_{yx}$.

For a vector variable propagating in the z direction with velocity c_V ,$V_i(z - c_V t)$, a polarization basis compatible with the condition $\partial_i V^i = 0$ is

$$e^{x}_{ij} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \,, \quad e^{y}_{ij} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \,. \tag{1.32}$$

Using these, we can decompose the vector part of the symmetric tensor as

$$\partial_{(i} V_{j)} = \frac{1}{2} V'_x e^{x}_{ij} + \frac{1}{2} V'_y e^{y}_{ij} \,, \tag{1.33}$$

where the prime $'$ denotes derivative with respect to z .

As for scalar-type polarizations, natural transverse ("breathing") and longitudinal polarization bases are

$$e^{\mathrm{B}}_{ij} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \quad e^{\mathrm{L}}_{ij} \equiv \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{1.34}$$

For a scalar variable propagating in the z direction with velocity c_S , $S(z - c_S t)$,

$$(\partial_i \partial_j - \triangle \delta_{ij}) S = e^{\mathrm{B}}_{ij} S'' \,, \partial_i \partial_j S = e^{\mathrm{L}}_{ij} S'' \,. \tag{1.35}$$

Sometimes, it is also useful to introduce other scalar bases instead of B and L , such as

$$e^{\mathrm{T}}_{ij} \equiv \sqrt{\frac{2}{3}}\, e^{\mathrm{B}}_{ij} + \frac{1}{\sqrt{3}}\, e^{\mathrm{L}}_{ij} = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \quad e^{\mathbb{T}}_{ij} \equiv \frac{1}{\sqrt{3}}\, e^{\mathrm{B}}_{ij} - \sqrt{\frac{2}{3}}\, e^{\mathrm{L}}_{ij} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \,. \tag{1.36}$$

These basis tensors satisfy an orthonormal condition $e^{\alpha}_{ij}\, e^{\beta}_{ij} = 2\delta_{\alpha\beta}$ for $\alpha, \beta = +, \times, x, y, \mathrm{B}, \mathrm{L}$. Seen as a spatial symmetric tensor, the Riemann tensor ${}^{(1)}R_{0i0j}$ can be decomposed with the above introduced polarization basis. It is understood from (1.26) that the amplitude of small oscillation of the distance between two bodies ζ^i due to the α polarization component of GW is proportional to $A_\alpha \equiv \mathcal{A}\, e^{ij}_{\alpha}\, {}^{(1)}R_{0i0j}$, where $\mathcal{A}$ is a constant independent of the type of polarization.

1.5.3 Linearized Einstein field equation

The action of GR expanded around a flat background to the quadratic order is

$$S_{\mathrm{GR}}[h_{\mu\nu}] = -\frac{1}{4\kappa} \int \mathrm{d}^4x\, {}^{(1)}G_{\mu\nu}\, h^{\mu\nu}\,, \tag{1.37}$$

where ${}^{(1)}G_{\mu\nu}$ is the linearized Einstein tensor

$${}^{(1)}G_{\mu\nu} \equiv -\frac{1}{2}\,\Box h_{\mu\nu} + \partial_{(\mu}\partial^{\lambda}h_{\nu)\lambda} - \frac{1}{2}\,\partial_\mu\partial_\nu h + \frac{1}{2}\,\eta_{\mu\nu}\,\left(\Box h - \partial^\rho\partial^\sigma h_{\rho\sigma}\right)\,. \tag{1.38}$$

The equation of motion (EOM) for $h_{\mu\nu}$ in vacuum is obtained as

$${}^{(1)}G_{\mu\nu} = 0\,. \tag{1.39}$$

The GR action (1.37) and EOM (1.39) are invariant under the gauge transformation (A.14). We can choose the transverse-traceless (TT) gauge $h_{\mu\nu} \to h^{\mathrm{TT}}_{\mu\nu}$ such that

$$h^{\mathrm{TT}}_{00} = 0\,, \quad h^{\mathrm{TT}}_{0i} = 0\,, \quad \delta^{ij}\, h^{\mathrm{TT}}_{ij} = 0\,, \quad \partial^i h^{\mathrm{TT}}_{ij} = 0\,. \tag{1.40}$$

Then the EOM (1.39) reduces to a massless Klein–Gordon-type equation for h^{TT}_{ij}:

$$\Box h^{\mathrm{TT}}_{ij} = 0\,. \tag{1.41}$$

We can take a plane-wave solution $h^{\mathrm{TT}}_{ij} \propto \mathrm{e}^{\mathrm{i}\,\omega\,(z-t)}$ and obtain the linear Riemann tensor (1.27) as

$${}^{(1)}R_{i0j0} = -\frac{1}{2}\,\ddot{h}^{\mathrm{TT}}_{ij} = \frac{1}{2}\,\omega^2\,\left(h^{\mathrm{TT}}_{+}\, e^{+}_{ij} + h^{\mathrm{TT}}_{\times}\, e^{\times}_{ij}\right)\,. \tag{1.42}$$

The same conclusion can be drawn in the gauge-invariant formulation, in which the GR action (1.37) is rewritten in terms of the gauge-invariant variables as

$$S_{\mathrm{GR}}[H_{ij}, \Sigma_i, \Phi, \Psi] = \frac{1}{2\kappa} \int \mathrm{d}^4x\, \left[H^{ij}\,\Box H_{ij} + \frac{1}{2}\,\partial^j\Sigma^i\,\partial_j\Sigma_i - 6\Phi\,\Box\Phi - 4\Phi\,\triangle(\Psi - \Phi)\right]\,. \tag{1.43}$$

The EOMs in a vacuum are

$$\Box H_{ij} = 0\,, \quad \triangle\Sigma_i = 0\,, \quad 3\ddot{\Phi} - \triangle\Psi - \triangle\Phi = 0\,, \quad \triangle\Phi = 0\,. \tag{1.44}$$

These imply $\Sigma_i = 0$ and $\Phi = \Psi = 0$ thanks to the assumed asymptotic flatness. Therefore, for a plane-wave solution $H_{ij} \propto \mathrm{e}^{\mathrm{i}\,\omega\,(z-t)}$, the Riemann tensor (1.28) is

$${}^{(1)}R_{i0j0} = -\ddot{H}_{ij} = \omega^2\,\left(H_+\, e^{+}_{ij} + H_\times\, e^{\times}_{ij}\right)\,. \tag{1.45}$$

1.6 Nonrelativistic stars

This section mostly follows [KWW12]. The polytropic relation $P \propto \rho^{1+\frac{1}{n}}$ is often used as a simple model for the equation of state (EOS) of stars. $n = 3$ can handle stars that meet the Eddington standard model (stars like the Sun) and stars that consist of a completely degenerate relativistic electron gas (white dwarfs or brown dwarfs consisting of relativistic electrons). $n = 1.5$ describes a star consisting of completely degenerate non-relativistic electron gas (white dwarfs or brown dwarfs consisting of non-relativistic electrons) [Sak15]. In these cases, it is necessary to rely entirely on numerical calculations.

1.6.1 Coordinates

Most astronomical objects in the universe are composed of fluids, which are general terms for gases and liquids. Since these generally consist of a large number of particles, we focus on macroscopic quantities and define physical quantities as average values. For example, let N be the number of particles contained in the volume dV, and let M be the mass of one particle. Then, the average number of particles per unit volume (particle number density) is

$$n = \frac{N}{dV}\,, \tag{1.46}$$

and the mass per unit volume (mass density) is

$$\rho = \frac{MN}{dV} = nM\,. \tag{1.47}$$

Other macroscopic physical quantities that describe the behavior of fluids include thermodynamic quantities such as pressure and temperature and the average velocity and force of a fluid element. To understand the structure of astronomical objects, it is necessary to express these physical quantities as functions of space and time coordinates. Although there is a wide variety of fluids, including those with viscosity and electromagnetic force, we will only consider completely compressible fluids.

There are two ways to describe the behaviour of a fluid: the Eulerian description and the Lagrangian description. This thesis primarily employs the Eulerian description, but we can convert the variables to the Lagrangian description.

The Eulerian description is a method for examining the change in state at each point in space over time. For example, let us consider a spherically symmetric star. It is natural to introduce a distance r, with the stellar centre considered as the origin of the spatial coordinates, and R representing its surface. It is necessary to consider solely the radial direction since it is a spherically symmetric star. If stars evolve over time, it is imperative to introduce a time coordinate t. Therefore, to characterise a spherically symmetric star, we may select two independent variables: the distance r and the time t. For example, mass density is expressed as $\rho = \rho(r\,, t)$. We define a mass $m(r\,,t)$ contained in a sphere of radius r at the time t. Subsequently, the mass m varies with respect to the distance r and the time t as expressed by

$$dm = \frac{\partial m}{\partial r} dr + \frac{\partial m}{\partial t} dt\,. \tag{1.48}$$

The first term on the right-hand side is the enclosed mass

$$\frac{\partial m}{\partial r} dr = 4\pi\, r^2 \rho\, dr \tag{1.49}$$

in the thin spherical shell dr with the constant time t. The second term on the right-hand side is the spherical symmetric mass flow

$$\frac{\partial m}{\partial t} dt = -4\pi\, r^2 \rho\, v\, dt \tag{1.50}$$

out of the sphere of constant radius r, with a radial velocity v directed outward during the time interval dt. Differentiating (1.49) with respect to t and (1.50) with respect to r and combining two results gives

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v)\,. \tag{1.51}$$

This is the continuity equation of hydrodynamics for the spherical symmetry case.

The continuity equation in the general case can be derived by considering the following. Let us consider a rectangular parallelepiped with each side $dx\,, dy\,, dz$ in a fluid with density ρ. The mass of fluid dm contained within this region is $dm = \rho dV$. The difference between the mass flowing in and the mass flowing out through the boundary S of this infinitesimal volume dV is the difference in the mass of the fluid in dV. When the volume is fixed, it is a change in density. In the following, we fix the volume and study the variation of density in dV. Let ρv_x be the mass flux flowing into dV from an infinitesimal area $dydz$, then the mass flowing out of the same sized infinitesimal area $dydz$ at a distance of dx is $\rho v_x + \frac{\partial}{\partial x}(\rho v_x) dx$, so the time variation of mass in dV in the direction x is

$$\frac{\partial}{\partial t}(\rho dV) = -\frac{\partial}{\partial x}(\rho v_x) dx dy dz\,. \tag{1.52}$$

Thus, the time variation of mass density in dV is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v_x)\,. \tag{1.53}$$

Similarly, considering the y and z directions, we have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y) - \frac{\partial}{\partial z}(\rho v_z) = -\nabla \cdot (\rho \mathbf{v})\,. \tag{1.54}$$

This is the well-known continuity equation of hydrodynamics.

The Lagrangian description is a method for tracking the motion of a fluid, where an observer follows individual fluid parcels as they move through space and time. As before, let us consider a spherically symmetric star. There is no possibility of replacing the time coordinate t with a new one. On the other hand, the spatial coordinate does not need to be the distance r, and it is possible to take the mass element m as an independent variable of the new spatial coordinate. The spatial coordinates of a given mass element do not vary with time. The origin of the mass element is the stellar centre $m = 0$, and $m = M$ is the stellar surface. The Lagrangian description has the advantage when the stellar radius changes dramatically over time, while the mass only changes by several times. For example, the mass density is expressed as $\rho = \rho(m\,, t)$, and the radial distance is also expressed as $r = r(m\,, t)$. Lagrangian description and Eulerian description are related by

$$\begin{aligned} \frac{\partial}{\partial m} &= \frac{\partial r}{\partial m}\frac{\partial}{\partial r}\,, \\ \left(\frac{\partial}{\partial t}\right)_m &= \left(\frac{\partial r}{\partial t}\right)_m \frac{\partial}{\partial r} + \left(\frac{\partial}{\partial t}\right)_r\,. \end{aligned} \tag{1.55}$$

Using the formula

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi\, r^2 \rho} \tag{1.56}$$

to the first equation, we obtain

$$\frac{\partial}{\partial m} = \frac{1}{4\pi\, r^2 \rho}\frac{\partial}{\partial r}\,. \tag{1.57}$$

The second equation represents the time variation of the substantial hydrostatic equilibrium. In the case of Eulerian description, the handling is complicated by the existence of the first term, so Lagrangian description is also convenient in this respect.

1.6.2 Hydrostatic equilibrium state

The mechanical equilibrium in a star is called the hydrostatic equilibrium state. For a static spherically symmetric star, the only forces are due to gravity and pressure gradients. In this thesis, we do not consider stellar rotation, magnetic field, or close companions. Let us consider a thin spherical mass shell with a thickness dr at a radius r inside the star. The mass of a unit area of the shell is $\rho\, dr$, and the weight of the shell is $-g\rho\, dr$. g is the gravitational acceleration at a distance r from the stellar centre defined as

$$g = \frac{\mathrm{d}\phi}{\mathrm{d}r}\,, \tag{1.58}$$

where ϕ is the gravitational potential.

We assume that $P_e = P + dP$ with $dP < 0$ for the pressure outside the spherical shell and $P_i = P$ for the pressure inside the spherical shell, the difference in pressure acting per unit area on the spherical shell is given by

$$P_i - P_e = -dP = -\frac{\mathrm{d}P}{\mathrm{d}r}\,dr\,. \tag{1.59}$$

In the hydrostatic equilibrium state, the sum of gravity and pressure gradient is zero, leading

$$g\rho + \frac{\mathrm{d}P}{\mathrm{d}r} = 0\,. \tag{1.60}$$

Therefore, we obtain the hydrostatic equilibrium condition

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\phi}{\mathrm{d}r}\,. \tag{1.61}$$

Next, we consider two timescales that characterise the gravitational phenomena of stars and explain why we can assume that the hydrostatic equilibrium state exists for many stars. Here, we employ Lagrangian description. We consider a spherically symmetric star and its thin mass shell dm at a distance r from the stellar centre. The pressure gradient per unit area is

$$f_P = -\frac{\partial P}{\partial m}\,dm\,, \tag{1.62}$$

and the gravitational force acting per unit area on a thin spherical shell is

$$f_g = -\frac{g}{4\pi r^2}\,dm\,. \tag{1.63}$$

If the two forces are not equal, the hydrostatic equilibrium is broken, and the mass element is accelerated. The equation of motion of the mass element is

$$\frac{dm}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = f_P + f_g\,, \tag{1.64}$$

and substituting f_P and f_g, it becomes

$$\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{\partial P}{\partial m} - \frac{g}{4\pi r^2}\,. \tag{1.65}$$

Since $\partial P/\partial m < 0$, the pressure gradient term is on the outside, and gravity is responsible for the acceleration towards the inside. If the left-hand side is zero, that is, all mass elements are stationary or moving at a constant velocity in the radial direction, then the equation is strictly hydrostatic equilibrium. Furthermore, if the right-hand sides cancel each other out and are close to zero, then the left-hand side becomes unimportant, so that hydrostatic equilibrium is a good approximation. Using this fact, we can understand that hydrostatic equilibrium is established to a very good approximation in many stars. If we assume the pressure suddenly disappears in (1.65), the mass element falls free. As a characteristic timescale, we define the free fall time $\tau_{\rm ff}$ over the distance R after the pressure disappears by

$$\left|\frac{\partial^2 r}{\partial t^2}\right| = \frac{R}{\tau_{\rm ff}^2}\,, \tag{1.66}$$

and we get

$$\tau_{\rm ff} \approx \left(\frac{R}{g}\right)^{\frac{1}{2}}\,. \tag{1.67}$$

Similarly, if gravity suddenly disappears in (1.65), the mass element explodes. As a characteristic timescale, we define the time $\tau_{\rm expl}$ that is required for the sound speed to propagate from the stellar centre to the surface R after gravity disappears

$$\left|\frac{\partial^2 r}{\partial t^2}\right| = \frac{R}{\tau_{\rm expl}^2}\,. \tag{1.68}$$

Using the equation of motion (1.65), we obtain

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} = -\frac{1}{\rho}\frac{\partial P}{\partial r}\,. \tag{1.69}$$

Therefore, the explosion time is

$$\tau_{\rm expl} \approx R\left(\frac{\rho}{P}\right)^{\frac{1}{2}}\,, \tag{1.70}$$

where $(P/\rho)^{1/2}$ is the order of the average sound speed. If a state close to hydrostatic equilibrium is achieved, $\tau_{\rm ff} \approx \tau_{\rm expl}$. The timescale at which a star becomes dynamically stable is called the hydrostatic timescale $\tau_{\rm hydr}$.

For example, if we assume general relativity as the theory of gravity, the hydrostatic timescale $\tau_{\rm hydr}$ is about

$$\tau_{\rm hydr} \approx \left(\frac{R^3}{GM}\right)^{\frac{1}{2}} \approx \frac{1}{2}\,(G\bar{\rho})^{-\frac{1}{2}}\,, \tag{1.71}$$

where we used $g \approx GM/R^2$ and $\bar{\rho}$ is the average density. In the case of the Sun, it is $\tau_{\rm hydr} \approx 27\,$min., which is extremely short compared to the lifespan of the Sun. Therefore, most phases during the lifetime of a star are thought to change slowly enough to be approximated by hydrostatic equilibrium.

The gravitational potential ϕ is governed by the Poisson equation

$$\Delta\phi = 4\pi\, G\, \rho\,. \tag{1.72}$$

The solution is

$$\phi = -G\int_r^\infty {\rm d}r' \frac{m(r')}{r'^2} \tag{1.73}$$

with

$$m(r) \equiv 4\pi \int_0^r \mathrm{d}r' r'^2 \rho(r') \,. \tag{1.74}$$

Outside the star, the potential becomes

$$\phi(r \geq R) = -\frac{GM}{r} \,, \tag{1.75}$$

since $M \equiv m(r \geq R)$. ϕ is expanded as

$$\phi = \phi(0) + \frac{2\pi\, G}{3}\,\rho_c\, r^2 + \frac{\pi\, G}{3}\rho_c'\, r^3 + \frac{\pi\, G}{10}\rho_c''\, r^4 + \mathcal{O}(r^5) \tag{1.76}$$

around $r = 0$.

Substituting the gravitational potential in GR into the hydrostatic equilibrium condition

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\phi}{\mathrm{d}r} \tag{1.77}$$

yields

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\, m}{r^2} \tag{1.78}$$

In Lagrangian description, it is represented as

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{G\, m}{4\pi\, r^4} \,. \tag{1.79}$$

Let us roughly estimate the pressure and temperature at the stellar centre. For (1.79), replace the right-hand side with the average pressure gradient $(P_0 - P_c)/M$ where $P_0 = 0$ and P_c are the pressure at the stellar surface and center respectively, and replace m and r in the left-hand side with rough mean values represented as $M/2$ and $R/2$, then we get

$$P_c \approx \frac{2G\, M^2}{\pi\, R^4} \,. \tag{1.80}$$

Because the average density of a spherically symmetric star is

$$\bar{\rho} = \frac{3M}{4\pi\, R^3} \,, \tag{1.81}$$

and the equation of state of the ideal gas is (E.19), we obtain the temperature at the stellar center

$$T_c = \frac{\mu}{\mathscr{R}}\frac{P_c}{\rho_c} \approx \frac{8}{3}\frac{\mu}{\mathscr{R}}\frac{GM}{R}\frac{\bar{\rho}}{\rho_c} \,, \tag{1.82}$$

where μ is the dimensionless mean molecular weight and $\mathscr{R} = 8.315 \times 10^7 \mathrm{erg\, K^{-1}\, g^{-1}}$ is the gas constant. Generally, the density increases from the surface to the center, so we have $\bar{\rho}/\rho < 1$. Hence, the temperature becomes

$$T_c \leq \frac{8}{3}\frac{G\,\mu}{\mathscr{R}}\frac{M}{R} \,. \tag{1.83}$$

For the Sun with the radius $R_\odot = 6.96 \times 10^{10}\,\mathrm{cm}$ and the mass $M_\odot = 1.989 \times 10^{33}\,\mathrm{g}$, we can roughly estimate as

$$P_c \approx 7 \times 10^{15}\,\mathrm{dyn/cm^2}\,, \quad T_c \leq 3 \times 10^7\,\mathrm{K}\,. \tag{1.84}$$

The numerical culculation result is $P_c = 2.4 \times 10^{17}\,\mathrm{dyn/cm^2}\,, T_c = 1.6 \times 10^7\,\mathrm{K}$.

1.6.3 Virial theorem

We explain the Virial theorem, which is useful for investigating the global stability of stars. Multiplying $4\pi r^3$ on both side of (1.79) and integrate over dm, we obtain

$$\int_0^M \mathrm{d}m\, 4\pi\, r^3 \frac{\mathrm{d}P}{\mathrm{d}m} = -\int_0^M \mathrm{d}m\, \frac{Gm}{r}\,. \tag{1.85}$$

The left-hand side can be integrated as

$$\int_0^M \mathrm{d}m\, 4\pi\, r^3 \frac{\mathrm{d}P}{\mathrm{d}m} = \left[4\pi\, r^3\, P\right]_0^M - 3\int_0^M \mathrm{d}m\, \frac{P}{\rho}\,, \tag{1.86}$$

and the first term vanishes because the pressure is zero at the stellar surface $P(R) = 0$. Thus, we have

$$3\int_0^M \mathrm{d}m\, \frac{P}{\rho} = \int_0^M \mathrm{d}m\, \frac{Gm}{r}\,. \tag{1.87}$$

The right-hand side can be interpreted as the gravitational energy

$$\Omega \equiv -\int_0^M \mathrm{d}m\, \frac{Gm(r)}{r}\,. \tag{1.88}$$

To understand the left-hand side, let us consider an ideal gas. Using Mayer's relation

$$c_v(\gamma - 1) = \frac{R}{\mu} \tag{1.89}$$

to the equation of state of an ideal gas

$$P = \frac{R}{\mu}\rho T\,, \tag{1.90}$$

we get

$$\frac{P}{\rho} = c_v(\gamma - 1)T = (\gamma - 1)u\,, \tag{1.91}$$

where $c_v \equiv C_V/(n\mu)$ is the specific heat at constant volume, c_p/c_v is replaced by γ, and u is the internal energy per unit mass of the perfect gas. Thus, the left-hand side can be evaluated as

$$3\int_0^M \mathrm{d}m\, \frac{P}{\rho} = 3(\gamma - 1)\int_0^M \mathrm{d}m\, u = 3(\gamma - 1)U\,, \tag{1.92}$$

where we defined the total internal energy of the star

$$U \equiv \int_0^M \mathrm{d}m\, u\,. \tag{1.93}$$

Therefore, the integrated hydrostatic equilibrium condition (1.85) is expressed as

$$\Omega + 3(\gamma - 1)U = 0\,, \tag{1.94}$$

and is called the Virial theorem.

The total energy

$$E = \Omega + U \tag{1.95}$$

of the star is the sum of its gravitational energy Ω and its internal energy U. Thus, we have

$$E = -(3\gamma - 4)U = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega\,. \tag{1.96}$$

If γ is larger than 4/3, the total energy of the star is negative $E < 0$, and the star is gravitationally bound. On the other hand, if γ is smaller than 4/3, the stellar total energy is positive $E > 0$, and the star is not gravitationally bound and is unstable.

As the star loses energy with luminosity L, its internal energy increases, and its gravitational energy decreases, since

$$-L \equiv \frac{\mathrm{d}E}{\mathrm{d}t} = -(3\gamma - 4)\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{3\gamma - 4}{3(\gamma - 1)}\frac{\mathrm{d}\Omega}{\mathrm{d}t} < 0\,. \tag{1.97}$$

A decrease in gravitational energy means the star is more deeply bound, and an increase in internal energy means an increase in temperature. Therefore, when a star loses energy, its temperature increases, and thus, stars are said to have negative specific heat.

The timescale in which an astronomical object loses its thermal energy via radiation, called Kelvin–Helmholtz timescale, can be defined as

$$\tau_{\mathrm{KH}} = \frac{|\Omega|}{L} \approx \frac{U}{L}\,, \tag{1.98}$$

since L is on the order of $|\mathrm{d}\Omega/\mathrm{d}t|$. The gravitational energy is roughly estimated as

$$|\Omega| \approx \frac{G\bar{m}^2}{\bar{r}} \approx \frac{GM^2}{2R}\,, \tag{1.99}$$

where $\bar{m}$ and $\bar{r}$ denote the mean values and we have replaced them by $M/2$ and $R/2$. Thus, we obtain

$$\tau_{\mathrm{KH}} \approx \frac{GM^2}{2RL}\,. \tag{1.100}$$

In the case of the Sun, given the luminosity of the Sun $L_\odot = 3.827 \times 10^{33}$ erg/s, the Kelvin–Helmholtz timescale is estimated as $\tau_{\mathrm{KH}} \approx 1.6 \times 10^7$ years. However, stars like the Sun have lifetimes much longer than the Kelvin–Helmholtz timescale because they glow with the energy of nuclear reactions. The actual lifetime of the Sun is said to be about 10^9 years based on nuclear physics.

1.6.4 Lane–Emden equation

The hydrostatic equilibrium condition

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\phi}{\mathrm{d}r} \tag{1.101}$$

depends on density and pressure, not explicitly on temperature. Generally, when density (or pressure) is rewritten using the equation of state $\rho = \rho(P, T)$, it depends explicitly on temperature. However, the equation of state of stars, such as the adiabatic state of an ideal gas or a completely degenerate electron gas, often does not depend on temperature. Therefore, we adopt the polytropic relation (a relationship in which pressure is proportional to the power of density) as the stellar equation of state.

Operating $\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2$ to the hydrostatic equilibrium condition

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right), \tag{1.102}$$

and using the Poisson equation

$$\Delta\phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = 4\pi G\rho \tag{1.103}$$

for a spherically symmetric star, we have

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho\,. \tag{1.104}$$

Adopting the polytropic relation

$$P = K\rho^{1+\frac{1}{n}}, \tag{1.105}$$

we obtain the equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(K\rho^{1+\frac{1}{n}}\right)\right] = -4\pi G\rho \tag{1.106}$$

in terms of density only. Therefore, by giving boundary conditions at the centre and solving this differential equation, we determine the internal structure of the star. However, this equation requires boundary conditions to be given and resolved for each star with a different mass or radius. Hence, by performing the following variable transformation, we can make it dimensionless and make it into a convenient form.

Introducing a parameter

$$\ell \equiv \sqrt{\frac{(n+1)P_c}{4\pi G\rho_c^2}} \tag{1.107}$$

with the dimension of length, where $\rho_c \equiv \rho(0)\,, P_c \equiv P(0) = K\rho_c^{1+\frac{1}{n}}$, and define the dimensionless radius, mass density, and pressure as

$$\xi \equiv \frac{r}{\ell}\,, \quad \rho = \rho_c \theta^n\,, \quad P = P_c \theta^{n+1}\,. \tag{1.108}$$

The formula (1.106) is then transformed into a dimensionless form

$$\Delta_\xi \theta + \theta^n = 0\,. \tag{1.109}$$

This is the Lane–Emden (LE) equation.

Let us consider the boundary conditions. Since the LE equation is second-order in differential equations, two independent conditions are required to solve it. The dimensionless density θ is normalised at the stellar center $\theta_c \equiv \theta(0) = 1$ since the real central density is ρ_c. The hydrostatic equilibrium condition can be written as

$$\theta' = -\frac{1}{4\pi G\,\rho_c\,\ell}\frac{\mathrm{d}\phi}{\mathrm{d}r} \tag{1.110}$$

in terms of the dimensionless variables. Using the expanded gravitational potential (1.76), we find

$$\theta'_c \equiv \theta'(0) = -\frac{1}{4\pi G\,\rho_c\,\ell}\lim_{r\to 0}\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0\,. \tag{1.111}$$

This means that at the center of the star, the pressure gradient is zero, and therefore the gravitational acceleration is zero. Also, the higher-order differential coefficients are

$$\begin{aligned}
\theta''_c &\equiv \theta''(0) = -\frac{1}{4\pi G\,\rho_c}\lim_{r\to 0}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = -\frac{1}{3}\,,\\
\theta^{(3)}_c &\equiv \theta^{(3)}(0) = -\frac{\ell}{4\pi G\,\rho_c}\lim_{r\to 0}\frac{\mathrm{d}^3\phi}{\mathrm{d}r^3} = 0\,,\\
\theta^{(4)}_c &\equiv \theta^{(4)}(0) = -\frac{\ell^2}{4\pi G\,\rho_c}\lim_{r\to 0}\frac{\mathrm{d}^4\phi}{\mathrm{d}r^4} = \frac{n}{5}\,.
\end{aligned} \tag{1.112}$$

The behavior of θ at the center of the star is

$$\begin{aligned}
\theta &= \theta_c + \theta'_c\,\xi + \frac{1}{2}\theta''_c\,\xi^2 + \frac{1}{6}\theta^{(3)}_c\,\xi^3 + \frac{1}{24}\theta^{(4)}_c\,\xi^4 + \mathcal{O}(\xi^5)\\
&= 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 + \mathcal{O}(\xi^5)\,.
\end{aligned} \tag{1.113}$$

Since the stellar radius is $R = \ell_n \xi_R$, we get

$$R = \sqrt{\frac{(n+1)K}{4\pi\,G}}\,\rho_c^{\frac{1-n}{2n}}\,\xi_R\,. \tag{1.114}$$

It is worth noting that when $n = 1$, the radius of the star does not depend on the central density ρ_c. Therefore, for a star with $n = 1$, the polytropic constant K (stellar composition) can be found if the radius of the star is observed.

The stellar mass is given by

$$M = 4\pi\ell^3\rho_c\omega_R$$
$$= 4\pi\left(\frac{(n+1)K}{4\pi\, G}\right)^{\frac{3}{2}} \rho_c^{\frac{3-n}{2n}}\,\omega_R\,, \tag{1.115}$$

where

$$\omega_R \equiv \int_0^{\xi_R} \mathrm{d}\xi\,\xi^2\theta(\xi)^n = -\xi_R^2\,\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\bigg|_{\xi_R}\,, \tag{1.116}$$

and where $\xi_R \equiv R/\ell$. We used the LE equation for the second equality. From this, we can see that M does not depend on the central density ρ_c when $n = 3$. Therefore, for a star with $n = 3$, the polytropic constant K (stellar composition) can be found if the stellar mass is observed.

Eliminating ρ_c from the formula of R and M, we find

$$\frac{\xi_R^{n-3}}{\omega_R^{n-1}} = \frac{1}{4\pi}\left[\frac{(n+1)K}{G}\right]^n \frac{R^{n-3}}{M^{n-1}}\,. \tag{1.117}$$

For a general polytropic index n, the polytropic constant K can be found from this formula.

Consider the mass of the star with $n = 3$, for which

$$M = \frac{4}{\sqrt{\pi}\,G^{\frac{3}{2}}}\,K^{\frac{3}{2}}\,\omega_R\,. \tag{1.118}$$

The Eddington standard model satisfies the polytropic relation with $n = 3$

$$P = K\,\rho^{\frac{3}{4}} \tag{1.119}$$

with

$$K = \left(\frac{3\,\mathscr{R}^4}{a\,\mu^4}\right)^{\frac{1}{3}} \left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{3}}\,, \tag{1.120}$$

where a is the radiation density constant $a = 7.564\times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}$, $\beta \equiv P_{gas}/P$, and P is the total pressure consisting of the gas pressure P_{gas} and radiation pressure P_{rad}. Substituting this, we get

$$M = M_{\mathrm{Edd}}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}}\,, \tag{1.121}$$

where

$$M_{\mathrm{Edd}} = \frac{4}{\sqrt{\pi}\,G^{\frac{3}{2}}}\,\omega_R\left(\frac{3}{a}\right)^{\frac{1}{2}}\left(\frac{\mathscr{R}}{\mu}\right)^2 \tag{1.122}$$

is called Eddington mass.

The exact solutions to the LE equation are known for $n = 0\,, 1\,,$ and 5. In particular, when $n = 0$ and 1, the LE equation is linear in θ, so one would expect it to be solvable analytically.

When $n = 0$, the LE equation is

$$\Delta_\xi \theta = -1\,. \tag{1.123}$$

This can be easily integrated to give

$$\theta = -\frac{\xi^2}{6} - \frac{A_1}{\xi} + A_2\,. \tag{1.124}$$

By imposing the boundary conditions, we get

$$\theta = 1 - \frac{\xi^2}{6}\,. \tag{1.125}$$

The dimensionless radius of the star is ξ_R such that $\theta(\xi_R) = 0$, and in this case $\xi_R = \sqrt{6}$.

When $n = 1$, the LE equation is

$$\Delta_\xi \theta + \theta = 0\,, \tag{1.126}$$

and rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(\xi\theta) = -\xi\theta\,. \tag{1.127}$$

The general solution to this equation is given by

$$\theta = B_1 \frac{\sin\xi}{\xi} + B_2 \frac{\cos\xi}{\xi}\,. \tag{1.128}$$

Imposing the boundary conditions, we obtain a suitable solution

$$\theta = \frac{\sin\xi}{\xi}\,. \tag{1.129}$$

If the gas temperature is constant, that is, it is isothermal, it corresponds to a polytrope of $n = \infty$. In this case, the equation of state is

$$P = K\rho\,, \tag{1.130}$$

and substituting this into (1.104), we obtain

$$\frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho\right) = -4\pi\, G\,\rho\,. \tag{1.131}$$

Define

$$\ell \equiv \sqrt{\frac{K}{4\pi\, G\rho_c}}\,, \quad \xi \equiv \frac{r}{\ell}\,, \quad \rho = \rho_c \mathrm{e}^{-\psi}\,. \tag{1.132}$$

as dimensionless variables, where $\rho_c \equiv \rho(0)$. Using these, we obtain the isothermal Lane–Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) = \mathrm{e}^{-\psi}\,. \tag{1.133}$$

The boundary conditions are

$$\psi_c = 0\,, \quad \psi'_c = 0\,. \tag{1.134}$$

The solution to this generally needs to be found numerically.

Chapter 2

Degrees of freedom and gravitational waves

In this chapter, we study the GW polarizations and show how the dynamical degrees of freedom (DOFs) propagate in Minkowski spacetime. In Section 2.1, we formulate the polarizations in linear MG with generic mass terms of non-Fierz–Pauli type. We show all the independent variables obey Klein–Gordon-type equations and identify the polarization modes. In Section 2.2, we analyze the linear perturbations of generic HCG. In the analysis of HCG, we employ two distinct methods; One takes full advantage of the partial equivalence between the generic HCG and MG at the linear level, whereas the other relies upon a gauge-invariant formalism. We confirm that the two results agree. In Section 2.3, we discuss methods to determine the theory parameters by GW-polarization measurements. It is worth stressing that our method for determining the theory parameters does not require measuring the propagation speeds, whether absolute or relative or the details of the waveforms of the GWs.

2.1 Polarization modes from massive gravitons

We first consider GW polarizations in generic linear MG. In order to treat the dynamical DOFs efficiently, we use the tensor (T), vector (V), and scalar (S) variables defined as in (A.9). The second-order perturbed action of a generic linear theory for non-self-interacting massive gravitons (1.13) is decomposed as

$$S_{\rm MG}[h_{\mu\nu}] = S^{\rm (T)}_{\rm MG}[H_{ij}] + S^{\rm (V)}_{\rm MG}[B_i, E_i] + S^{\rm (S)}_{\rm MG}[A, B, C, E] \tag{2.1}$$

with

$$S^{(\mathrm{T})}_{\mathrm{MG}}[H_{ij}] = \frac{1}{2\kappa} \int \mathrm{d}^4x \, \left[H_{ij} \, \Box H^{ij} - m^2 \, H_{ij} \, H^{ij} \right] , \tag{2.2}$$

$$S^{(\mathrm{V})}_{\mathrm{MG}}[B_i, E_i] = \frac{1}{2\kappa} \int \mathrm{d}^4x \, \left[-\frac{1}{2} \, (B_i + \dot{E}_i) \, \triangle (B^i + \dot{E}^i) + \frac{m^2}{2} \, \left(B_i \, B^i + E_i \, \triangle E^i \right) \right] , \tag{2.3}$$

$$\begin{aligned} S^{(\mathrm{S})}_{\mathrm{MG}}[A, B, C, E] = \frac{1}{2\kappa} \int \mathrm{d}^4x \, \Big[& -6C \, \Box C - 4C \, \triangle (A - \dot{B} - \ddot{E} - C) & (2.4) \\ & - \frac{m^2}{2} \left(2\epsilon \, A^2 + B \, \triangle B + 6 \, (3\epsilon - 2) \, C^2 + 2\epsilon \, E \, \triangle^2 E \right. \\ & \left. + 4 \, (3\epsilon - 2) \, C \, \triangle E + 4 \, (\epsilon - 1) \, A \, \triangle E + 12 \, (\epsilon - 1) \, A \, C \right) \Big] . \end{aligned} \tag{2.5}$$

Contrary to GR, these vector and scalar actions cannot be solely expressed with the gauge-invariant variables due to the lack of the gauge symmetries. Thus, in order to calculate the Riemann tensor (1.28), we have to manipulate the original variables.

The equation of motion (EOM) for the tensor variable H_{ij} is obtained from (2.2) as

$$\Box H_{ij} - m^2 \, H_{ij} = 0 \, . \tag{2.6}$$

The EOM for the gauge-invariant variable Σ_i does not directly derive, but the EOMs for B_i and E_i from (2.3) are

$$-\triangle (B_i + \dot{E}_i) + m^2 \, B_i = 0 \, , \quad \dot{B}_i + \ddot{E}_i + m^2 \, E_i = 0 \, . \tag{2.7}$$

Combining these, we find that $\Sigma_i \equiv B_i + \dot{E}_i$ obeys

$$\Box \Sigma_i - m^2 \, \Sigma_i = 0 \, . \tag{2.8}$$

Σ_i embodies all the dynamical vector-type DOFs as B_i and E_i are dependent upon Σ_i via the nondynamical relations

$$B_i = m^{-2} \, \triangle \Sigma_i \, , \quad E_i = -m^{-2} \, \dot{\Sigma}_i \tag{2.9}$$

from (2.7). The most complicated is to find the governing equations for the gauge-invariant scalars Ψ and Φ. The scalar EOMs from variations of (2.5) with respect to A, B, C, E are, respectively,

$$\begin{aligned} & 2\triangle C + m^2 \, (\epsilon A + (\epsilon - 1) \, \triangle E + 3 \, (\epsilon - 1) \, C) = 0 \, , \\ & \triangle \left[4\dot{C} + m^2 \, B \right] = 0 \, , \\ & 6\Box C + 2\triangle (A - \dot{B} - \ddot{E}) - 4\triangle C + m^2 \, (3 \, (3\epsilon - 2) \, C + (3\epsilon - 2) \, \triangle E + 3 \, (\epsilon - 1) \, A) = 0 \, , \\ & \triangle \left[-2\ddot{C} + m^2 \, (\epsilon \triangle E + (3\epsilon - 2) \, C + (\epsilon - 1) \, A) \right] = 0 \, . \end{aligned} \tag{2.10}$$

Gathering these equations, it is found that the following two combinations

$$W \equiv A - \dot{B} - \ddot{E} - C\,, \quad h \equiv 2A + 6C + 2\triangle E \tag{2.11}$$

satisfy Klein–Gordon equations

$$\Box W - m^2\, W = 0\,, \quad \epsilon\,\Box h - \frac{3-4\epsilon}{2}\, m^2\, h = 0\,. \tag{2.12}$$

Observe that $W = \Psi - \Phi$ and $h = \eta^{\mu\nu}\, h_{\mu\nu}$. The second equation implies that, if the Fierz–Pauli tuning $\epsilon = 0$ is realised, then the four-dimensional trace h is constrained to vanish. We assume $\epsilon \neq 0$ and define $m_0^2 \equiv \frac{3-4\epsilon}{2\epsilon}\, m^2$ as the mass of h. These W and h are the only dynamical scalar-type DOFs. Indeed, the set of equations (2.10) can be solved for A, B, C, E in terms of W and h as

$$\begin{aligned} A &= \frac{2}{3m^4}\,\triangle^2 W + \left(\frac{1-\epsilon}{2} - \frac{\epsilon}{3m^2}\,\triangle\right)\, h\,, \\ B &= \frac{4}{3m^4}\,\triangle\dot{W} - \frac{2\epsilon}{3m^2}\,\dot{h}\,, \\ C &= -\frac{1}{3m^2}\,\triangle W + \frac{\epsilon}{6}\, h\,, \\ E &= \frac{1}{m^2}\, W - \frac{2}{3m^4}\,\triangle W + \frac{\epsilon}{3m^2}\, h\,. \end{aligned} \tag{2.13}$$

Moreover, the gauge-invariant variables are expressed as

$$\Psi = A - \dot{B} - \ddot{E} = W - \frac{1}{3m^2}\,\triangle W + \frac{\epsilon}{6}\, h\,, \quad \Phi = C = -\frac{1}{3m^2}\,\triangle W + \frac{\epsilon}{6}\, h\,, \tag{2.14}$$

where we have used (2.12) to eliminate $\ddot{W}$ and $\ddot{h}$. Using these relationships, the scalar part of the linear Riemann tensor is written as

$$\begin{aligned} {}^{(1)}R^{(\mathrm{S})}_{i0j0} &\equiv \partial_i\partial_j\Psi - \delta_{ij}\,\ddot{\Phi} \\ &= 3\partial_i\partial_j W - \frac{1}{3m^2}\,\triangle\left(\partial_i\partial_j W - \delta_{ij}\,\ddot{W}\right) + \frac{\epsilon}{6}\,\left(\partial_i\partial_j h - \delta_{ij}\,\ddot{h}\right)\,. \end{aligned} \tag{2.15}$$

Having found that the variables H_{ij} , Σ_i , W and h all obey Klein–Gordon-type equations, we are allowed to consider plane-wave solutions propagating along the z direction,

$$H_{ij} \propto \mathrm{e}^{\mathrm{i}\,(k_H z - \omega_H t)}\,, \quad \Sigma_i \propto \mathrm{e}^{\mathrm{i}\,(k_\Sigma z - \omega_\Sigma t)}\,, \quad W \propto \mathrm{e}^{\mathrm{i}\,(k_W z - \omega_W t)}\,, \quad h \propto \mathrm{e}^{\mathrm{i}\,(k_h z - \omega_h t)} \tag{2.16}$$

with

$$k_I \equiv \sqrt{\omega_I^2 - m^2} \quad (I = H, \Sigma, W)\,, \quad k_h \equiv \sqrt{\omega_h^2 - m_0^2}\,. \tag{2.17}$$

Then the Riemann tensor is calculated as

$$\begin{aligned} {}^{(1)}R_{i0j0} &= \omega_H^2\,(H_+\, e^+_{ij} + H_\times\, e^\times_{ij}) - \frac{1}{2}\,\sqrt{\omega_\Sigma^2 - m^2}\,\omega_\Sigma\,\left(\Sigma_x\, e^x_{ij} + \Sigma_y\, e^y_{ij}\right) \\ &\quad - \frac{1}{3m^2}\,\triangle W\,\left(\omega_W^2\, e^{\mathrm{B}}_{ij} - \sqrt{2}\, m^2\, e^{\mathrm{L}}_{ij}\right) + \frac{\epsilon}{6}\, h\,\left(\omega_h^2\, e^{\mathrm{B}}_{ij} + \frac{m_0^2}{\sqrt{2}}\, e^{\mathrm{L}}_{ij}\right)\,. \end{aligned} \tag{2.18}$$

This expression tells us that the six separate variables provide different polarizations. In particular, the information carried by the two scalar variables is distinctive. W, the helicity-0 mode of the spin-2 graviton, can be split into the transverse, or "breathing" (B), and longitudinal (L) polarizations based on its different dependences on the frequency: the former is proportional to ω_W^2 while the latter is m^2. Similarly, the spin-0 graviton h, which only exists if $\epsilon \neq 0$, can be decomposed into the transverse and longitudinal polarizations. Thus, if the amplitudes of each polarisation are separately measured in future gravitational-wave experiments, the longitudinal modes will provide a direct measure of the masses of the spin-2 and spin-0 gravitons. We will come back to this issue in Section 2.3.

Finally, let us mention the effectively massless case $m^2 \ll \omega_I^2$. Resembling the discontinuity in the bending angle of light [Iwa70, vDV70, Zak70], taking the continuous limit does not recover the set of polarizations expected in GR, as the vector and the transverse scalar polarizations would remain.

2.2 GWs in generic higher-curvature gravity

In the studies of linear perturbations of HCG, what is significant would be the equivalence between the quadratic curvature gravity and massive bigravity at the linear level, which was recently extended to arbitrary background with an Einstein metric (see Appendix B). This means that the analyses of the extra massive DOFs in HCG can be done in parallel to MG. On the other hand, we show that a dedicated analysis based on a gauge-invariant formalism is also useful and that the two results agree.

2.2.1 Massive-bigravity approach

As we mentioned in Section 1.2, there is an equivalence of the action (1.9) to GR "minus" massive gravity,

$$\begin{aligned} S[\phi_{\mu\nu}, \tilde{\phi}_{\mu\nu}] &= \chi\, S_{\mathrm{GR}}[\phi_{\mu\nu}] - \chi\, S_{\mathrm{MG}}[\tilde{\phi}_{\mu\nu}] \\ &= \frac{\chi}{4\kappa} \int \mathrm{d}^4x \left[-{}^{(1)}G_{\mu\nu}[\phi]\, \phi^{\mu\nu} + {}^{(1)}G_{\mu\nu}[\tilde{\phi}]\, \tilde{\phi}^{\mu\nu} + \frac{m^2}{2} \left(\tilde{\phi}_{\mu\nu}\, \tilde{\phi}^{\mu\nu} - (1-\epsilon)\, \tilde{\phi}^2 \right) \right] , \end{aligned} \tag{2.19}$$

where we introduced $m^2 \equiv \chi/(2\alpha)$ and $\epsilon = 9\beta/(2\alpha + 12\beta)$, see Appendix B for the derivation extended to Einstein manifolds [NNZ$^+$]. Clearly, we need to assume $\alpha \neq 0$. The case with $\alpha = 0$ can be treated in a similar manner by means of a conformal transformation, after which the theory takes a form of a scalar-tensor theory; See [MBM19] for an example.

In this formalism, the original metric perturbation is given by

$$h_{\mu\nu} = \phi_{\mu\nu} + \tilde{\phi}_{\mu\nu}\,, \tag{2.20}$$

and, as seen from the structure of the action (2.19), the dynamical contents in this theory are $\phi_{\mu\nu}$ as a massless spin-2 field and $\tilde{\phi}_{\mu\nu}$ as a mixture of a massive spin-2 and a spin-0 fields. It is worth mentioning that, while $\phi_{\mu\nu}$ field has the same gauge symmetry as GR, $\tilde{\phi}_{\mu\nu}$ is not subject to gauge transformations.

For the massless spin-2, we can choose the TT gauge as in GR treated in Sec. 1.5.3. The only dynamical DOF is $\phi_{ij}^{\rm TT} = 2H_{ij}$ with its EOM being $\Box H_{ij} = 0$. For the massive spin-2, the analysis is completely parallel to the case of massive gravity treated in Sec. 2.1. We decompose $\tilde{\phi}_{\mu\nu}$ as

$$\tilde{\phi}_{00} = -2\tilde{A}\,, \quad \tilde{\phi}_{0i} = -\partial_i \tilde{B} - \tilde{B}_i\,, \quad \tilde{\phi}_{ij} = 2\tilde{C}\,\delta_{ij} + 2\partial_i\partial_j\tilde{E} + 2\partial_{(i}\tilde{E}_{j)} + 2\tilde{H}_{ij} \tag{2.21}$$

and define two scalar variables

$$\tilde{W} \equiv \tilde{A} - \dot{\tilde{B}} - \ddot{\tilde{E}} - \tilde{C}\,, \quad \tilde{\phi} \equiv \tilde{\phi}_\mu{}^\mu = 2\tilde{A} + 6\tilde{C} + 2\triangle\tilde{E}\,. \tag{2.22}$$

Then we find the EOMs for the dynamical variables

$$\left(\Box - m^2\right)\,\tilde{H}_{ij} = 0\,, \quad \left(\Box - m^2\right)\,\tilde{\Sigma}_i = 0\,, \quad \left(\Box - m^2\right)\,\tilde{W} = 0\,, \quad \left(\frac{6\beta}{\chi}\Box - 1\right)\,\tilde{\phi} = 0\,. \tag{2.23}$$

In this case, the Fierz–Pauli tuning $\epsilon = 0$ is realized and the spin-0 mode $\tilde{\phi}$ is required to vanish when $\beta = 0$, or else it acquires a finite mass $m_0^2 \equiv \chi/(6\beta)$. Considering plane wave solutions

$$\begin{aligned} &H_{ij} \propto \mathrm{e}^{\mathrm{i}\,\omega_H\,(z-t)}\,, \\ &\tilde{H}_{ij} \propto \mathrm{e}^{\mathrm{i}\,(k_{\tilde{H}}\,z - \omega_{\tilde{H}}\,t)}\,, \quad \tilde{\Sigma}_i \propto \mathrm{e}^{\mathrm{i}\,(k_{\tilde{\Sigma}}\,z - \omega_{\tilde{\Sigma}}\,t)}\,, \quad \tilde{W} \propto \mathrm{e}^{\mathrm{i}\,(k_{\tilde{W}}\,z - \omega_{\tilde{W}}\,t)} \\ &\quad\tilde{\phi} \propto \mathrm{e}^{\mathrm{i}\,(k_{\tilde{\phi}}\,z - \omega_{\tilde{\phi}}\,t)} \end{aligned} \tag{2.24}$$

with dispersion relations

$$k_I = \begin{cases} \sqrt{\omega_I^2 - m^2} & (I = \tilde{H}, \tilde{\Sigma}, \tilde{W}) \\ \sqrt{\omega_I^2 - m_0^2} & (I = \tilde{\phi}) \end{cases}\,, \tag{2.25}$$

we get the following expression for the Riemann tensor:

$$\begin{aligned} {}^{(1)}R_{i0j0} = &\sum_{\lambda=+,\times} \left[\omega_H^2\,H_\lambda + \omega_{\tilde{H}}^2\,\tilde{H}_\lambda\right]\,e_{ij}^\lambda - \frac{1}{2}\sqrt{\omega_{\tilde{\Sigma}}^2 - m^2}\,\omega_{\tilde{\Sigma}}\sum_{p=x,y}\tilde{\Sigma}_p\,e_{ij}^p \\ &- \frac{1}{3m^2}\triangle\tilde{W}\,\left(\omega_{\tilde{W}}^2\,e_{ij}^{\rm B} - \sqrt{2}\,m^2\,e_{ij}^{\rm L}\right) + \frac{\epsilon}{6}\,\tilde{\phi}\,\left(\omega_{\tilde{\phi}}^2\,e_{ij}^{\rm B} + \frac{m_0^2}{\sqrt{2}}\,e_{ij}^{\rm L}\right)\,. \end{aligned} \tag{2.26}$$

To summarize, we have established that the gravitational-wave polarizations in generic HCG with $\alpha \neq 0$ is a sum of those in GR and in MG, with a difference that the mass parameters m and m_0 in this case are given by the expansion coefficients χ, α and β inherent in the nonlinear Lagrangian f.

2.2.2 Gauge-invariant approach

The action (1.9) can be rewritten in terms of the gauge-invariant variables as

$$S_{\mathrm{HCG}}[h_{\mu\nu}] = S^{(\mathrm{T})}_{\mathrm{HCG}}[H_{ij}] + S^{(\mathrm{V})}_{\mathrm{HCG}}[\Sigma_i] + S^{(\mathrm{S})}_{\mathrm{HCG}}[\Phi, \Psi]\,, \tag{2.27}$$

where each part is

$$S^{(\mathrm{T})}_{\mathrm{HCG}}[H_{ij}] = \frac{1}{2\kappa}\int \mathrm{d}^4x\ \left[H_{ij}\,\Box H^{ij} - 2\alpha\,\Box H_{ij}\,\Box H^{ij}\right]\,, \tag{2.28}$$

$$S^{(\mathrm{V})}_{\mathrm{HCG}}[\Sigma_i] = \frac{1}{2\kappa}\int \mathrm{d}^4x\ \left[\frac{1}{2}\,\partial_j\Sigma_i\,\partial^j\Sigma^i - \alpha\,\left(\partial_j\dot{\Sigma}_i\,\partial^j\dot{\Sigma}^i - \triangle\Sigma_i\,\triangle\Sigma^i\right)\right]\,, \tag{2.29}$$

$$S^{(\mathrm{S})}_{\mathrm{HCG}}[\Phi, \Psi] = \frac{1}{2\kappa}\int \mathrm{d}^4x\ \left[-6\Phi\,\Box\Phi - 4\Phi\,\triangle(\Psi-\Phi) - \frac{4}{3}\,\alpha\,(\triangle\Psi - \triangle\Phi)^2 + \beta\,{}^{(1)}R^2\right] \tag{2.30}$$

with ${}^{(1)}R = -6\Box\Phi - 2\triangle(\Psi-\Phi)$. As seen in the above expressions, when $\alpha = 0$, the tensor and vector parts reduce to the ones in GR. Conversely, the tensor and vector parts are modified in comparison to GR by the presence of the Weyl-squared term but unaffected by the Ricci-squared term in the expansion of the Lagrangian (1.8). On the other hand, the scalar part is affected by both the Weyl-squared and Ricci-squared terms. In the following, we proceed to the analyses for each part.

First, the tensor part is only affected by the presence of the Weyl term, as seen in (2.28). The tensor EOM is

$$\Box H_{ij} - 2\alpha\,\Box^2 H_{ij} = 0\,. \tag{2.31}$$

When $\alpha = 0$, the tensor eom reduces to that of GR, and the same result for the polarizations is obtained. When $\alpha \neq 0$, we find that the above EOM admits two independent solutions $H_{ij} = \phi_{ij}$ and $H_{ij} = \tilde{\phi}_{ij}$ which respectively satisfy

$$\Box\phi_{ij} = 0\,, \quad \Box\tilde{\phi}_{ij} - m^2\,\tilde{\phi}_{ij} = 0\,, \tag{2.32}$$

where, as before, $m^2 = \frac{1}{2\alpha}$. Hence the general solution for H_{ij} is

$$H_{ij} = \phi_{ij} + \tilde{\phi}_{ij}\,. \tag{2.33}$$

Considering plane-wave solutions propagating in the z direction,

$$\phi_{ij} \propto \mathrm{e}^{\mathrm{i}\,\omega_\phi\,(z-t)}\,, \quad \tilde{\phi}_{ij} \propto \mathrm{e}^{\mathrm{i}\,(k_{\tilde{\phi}}\,z - \omega_{\tilde{\phi}}\,t)}\,, \tag{2.34}$$

where $k_{\tilde{\phi}} = \sqrt{\omega^2_{\tilde{\phi}} - m^2}$, and substituting these into (1.28), we obtain the tensor part of the Riemann tensor as

$${}^{(1)}R^{(\mathrm{T})}_{i0j0} = \omega^2_\phi\,\left(\phi_+\,e^+_{ij} + \phi_\times\,e^\times_{ij}\right) + \omega^2_{\tilde{\phi}}\,\left(\tilde{\phi}_+\,e^+_{ij} + \tilde{\phi}_\times\,e^\times_{ij}\right)\,. \tag{2.35}$$

Next, as for the vector part, the EOM from the action (2.29) is

$$(1 - 2\alpha\,\Box)\;\triangle\Sigma_i = 0\,. \tag{2.36}$$

When $\alpha = 0\,,\ \Sigma_i = 0$ as expected. When $\alpha \neq 0\,,$ the EOM admits a plane-wave solution

$$\Sigma_i \propto \mathrm{e}^{\mathrm{i}\,(k_\Sigma\, z - \omega_\Sigma\, t)} \tag{2.37}$$

with $k_\Sigma = \sqrt{\omega_\Sigma^2 - m^2}\,$. For this, the vector part of the Riemann tensor (1.28) is

$${}^{(1)}R^{(\mathrm{V})}_{i0j0} = -\frac{1}{2}\sqrt{\omega_\Sigma^2 - m^2}\,\omega_\Sigma\,\left(\Sigma_x\, e^x_{ij} + \Sigma_y\, e^y_{ij}\right)\,. \tag{2.38}$$

Finally, we analyse the scalar part. As for scalar, counting of the number of DOFs is a nontrivial task since the action (2.30) contains second-order time derivatives non-linearly in the Ricci-squared term. Let us first introduce a variable $\Theta = \frac{2}{3}\triangle(\Phi - \Psi)$ to simplify the scalar action as

$$S^{(\mathrm{S})}_{\mathrm{HCG}}[\Phi,\Theta] = \frac{1}{2\kappa}\int \mathrm{d}^4x\;\left[-6\Phi\,\Box\Phi + 6\Phi\,\Theta - 3\alpha\,\Theta^2 + \beta\,{}^{(1)}R^2\right]\,. \tag{2.39}$$

To have an equivalent action with only derivatives lower than or equal to the second order, we replace the Ricci scalar ${}^{(1)}R = -6\Box\Phi + 3\Theta$ with an auxiliary variable Ξ introducing a Lagrange multiplier λ as

$$S^{(\mathrm{S})}_{\mathrm{HCG}}[\Phi,\Theta,\Xi,\lambda] = \frac{1}{2\kappa}\int \mathrm{d}^4x\;\left[-6\Phi\,\Box\Phi + 6\Phi\,\Theta - 3\alpha\,\Theta^2 + \beta\,\Xi^2 + \lambda\,({}^{(1)}R - \Xi)\right]\,. \tag{2.40}$$

The variation of the above action with respect to Ξ gives a constraint $\lambda = 2\beta\,\Xi$, which can be used to eliminate λ as

$$S^{(\mathrm{S})}_{\mathrm{HCG}}[\Phi,\Theta,\Xi] = \frac{1}{2\kappa}\int \mathrm{d}^4x\;\left[-6\Phi\,\Box\Phi + 6\Phi\,\Theta - 3\alpha\,\Theta^2 - \beta\,\Xi^2 - 12\beta\,\Xi\,\Box\Phi + 6\beta\,\Xi\,\Theta\right]\,. \tag{2.41}$$

Now, after some manipulation, we obtain three independent equations from the variations of the above action with respect to each variable,

$$\Theta - 2\alpha\,\Box\Theta = 0\,, \tag{2.42}$$
$$\bar{\beta}\,\Xi - 6\beta^2\,\Box\Xi = 0\,, \tag{2.43}$$
$$\Phi = \alpha\,\Theta - \beta\,\Xi\,. \tag{2.44}$$

The EOMs (2.42) and (2.43) imply that Θ and Ξ are independent DOFs with masses $m^2 = 1/2\alpha$ and $m_0^2 = 1/6\beta$, respectively. The algebraic constraint (2.44) indicates that when $\alpha \neq 0$ ($\beta \neq 0$), Θ (Ξ) can be eliminated. When both α and β are nonzero, we can solve (2.42) and (2.43) for Θ and Ξ and obtain Φ via (2.44). We then have the other gauge-invariant variable

$$\Psi = \alpha\,\Theta - \frac{3}{2}\,\triangle^{-1}\Theta - \beta\,\Xi\,. \tag{2.45}$$

Assuming plane-wave solutions

$$\Theta \propto \mathrm{e}^{\mathrm{i}\,(k_\Theta\, z - \omega_\Theta\, t)}\,, \quad \Xi \propto \mathrm{e}^{\mathrm{i}\,(k_\Xi\, z - \omega_\Xi\, t)} \tag{2.46}$$

with $k_\Theta = \sqrt{\omega_\Theta^2 - m^2}$ and $k_\Xi = \sqrt{\omega_\Xi^2 - m_0^2}$, we arrive at the expression for the scalar part of the linear Riemann tensor (1.28)

$$^{(1)}R^{(\mathrm{S})}_{i0j0} = \alpha\,\Theta\,\left(\omega_\Theta^2\, e^{\mathrm{B}}_{ij} - \sqrt{2}\, m^2\, e^{\mathrm{L}}_{ij}\right) - \beta\,\Xi\,\left(\omega_\Xi^2\, e^{\mathrm{B}}_{ij} + \frac{m_0^2}{\sqrt{2}}\, e^{\mathrm{L}}_{ij}\right)\,. \tag{2.47}$$

Our final task here is to investigate special cases with $\alpha = 0$ or $\beta = 0$. When $\alpha = 0$, the EOM (2.42) reduces to a constraint $\Theta = 0$, which implies $\Psi = \Phi$. Equation (2.44) reduces to $\Phi = -\beta\,\Xi$, so the variable Φ represents the spin-0 DOF and obeys the eom

$$\Box\Phi - m_0^2\,\Phi = 0\,. \tag{2.48}$$

Therefore, recalling $\Psi = \Phi$, we find the scalar-type polarization in (1.28) in this case as

$$^{(1)}R^{(\mathrm{S})}_{i0j0} = \Phi\,\left(\omega_\Phi^2\, e^{\mathrm{B}}_{ij} + \sqrt{2}\, m_0^2\, e^{\mathrm{L}}_{ij}\right)\,. \tag{2.49}$$

Also, the vector and tensor perturbations give the same result as in GR. This agrees with the result of Moretti *et al.* [MBM19].

When $\beta = 0$, the constraint (2.44) reduces to $\Phi = \bar{\alpha}\,\Theta$, which implies that

$$\Psi = \Phi - \frac{3}{2\bar{\alpha}}\,\triangle^{-1}\Phi \tag{2.50}$$

and the EOM for Φ is

$$\Box\Phi - m^2\,\Phi = 0\,. \tag{2.51}$$

It is clear that Φ, in this case, is the helicity-0 component of the massive spin-2. Recalling its relation to Ψ as given by (2.50), the scalar-type polarization in (1.28) is calculated as

$$^{(1)}R^{(\mathrm{S})}_{i0j0} = \Phi\,\left(\omega_\Phi^2\, e^{\mathrm{B}}_{ij} - \frac{m^2}{\sqrt{2}}\, e^{\mathrm{L}}_{ij}\right)\,. \tag{2.52}$$

2.3 Determining theory parameters by observations

In this section, we provide a brief discussion of how one could determine the theory parameters, namely α and β in HCG. For brevity, below, we collectively call the theory parameters "masses." We consider interferometers and pulsar timing arrays (PTAs) as GW measurement instruments.

To discuss the ability of GW detectors to determine the masses via polarization measurements, it is necessary to take into account the detector's response to each polarization in a GW propagating from the direction (θ, ϕ) with the polarization angle ψ, which is represented by the antenna pattern functions $F_\alpha(\theta,\phi,\psi)$ ($\alpha = +, \times, x, y, \mathrm{B}, \mathrm{L}$) summarized in Appendix D. The whole signal takes the form

$$S(t) = \sum_\alpha F_\alpha(\theta,\phi,\psi)\, A_\alpha(t)\,, \tag{2.53}$$

where A_α is the waveform of each polarization being proportional to the coefficient of the polarization basis e^α_{ij} appearing in the Riemann tensor.

We note that different spin components have different velocities, so it would be plausible to treat each spin component separately if one is interested in the case of a short-duration source like a burst from a black-hole merger. Thus, we first concentrate on the spin-0 GW alone, denoting its velocity as v. A spin-0 GW represented as a monochromatic plane-wave with frequency ω gives a signal of the form

$$S(t) = F_\mathrm{B}\, A_\mathrm{B}(t) + F_\mathrm{L}\, A_\mathrm{L}(t)\,. \tag{2.54}$$

From (2.47) we see that there is a relationship between the waveforms

$$\frac{A_\mathrm{L}(t)}{A_\mathrm{B}(t)} = \frac{m_0^2}{\omega^2}\,. \tag{2.55}$$

Suppose that two detectors respond to a single spin-0 GW. In this case, the signals to be detected by the two detectors are

$$\begin{cases} S^1(t) = F^1_\mathrm{B}\, A_\mathrm{B}(t) + F^1_\mathrm{L}\, A_\mathrm{L}(t)\,, \\ S^2(t+\Delta t) = F^2_\mathrm{B}\, A_\mathrm{B}(t) + F^2_\mathrm{L}\, A_\mathrm{L}(t)\,, \end{cases} \tag{2.56}$$

where $\Delta t = L/v$ is the time delay between the arrivals at the two detectors separated by L along the propagation direction. If the coefficient matrix

$$\mathcal{F} \equiv \begin{pmatrix} F^1_\mathrm{B} & F^1_\mathrm{L} \\ F^2_\mathrm{B} & F^2_\mathrm{L} \end{pmatrix} \tag{2.57}$$

is invertible, it is possible to solve for A_B and A_L as

$$\begin{pmatrix} A_\mathrm{B}(t) \\ A_\mathrm{L}(t) \end{pmatrix} = \mathcal{F}^{-1} \begin{pmatrix} S^1(t) \\ S^2(t+\Delta t) \end{pmatrix}\,. \tag{2.58}$$

From this, the ratio of the waveforms $\mathcal{R} \equiv A_\mathrm{L}/A_\mathrm{B}$ is obtained in terms of observable signals and, therefore, the spin-0 mass can be determined as

$$m_0^2 = \omega^2\, \mathcal{R}\,. \tag{2.59}$$

The above argument relies upon the invertibility of the matrix $\mathcal{F}$. Unfortunately, LIGO-like interferometers have degenerate antenna pattern functions

$$F_\mathrm{B} = -\frac{1}{2}\sin^2\theta\,\cos 2\phi\,, \quad F_\mathrm{L} = \frac{1}{\sqrt{2}}\sin^2\theta\,\cos 2\phi\,, \tag{2.60}$$

so it is not possible to use the above method. On the other hand, PTAs have antenna functions [YS13, LJP+10, QBK21]

$$F_{\rm B} = \frac{1}{2}\,\frac{\sin^2\theta}{1+v\,\cos\theta}\,, \quad F_{\rm L} = \frac{1}{\sqrt{2}}\,\frac{\cos^2\theta}{1+v\,\cos\theta}\,, \tag{2.61}$$

which are nondegenerate if the two detectors (pulsars) have different orientations with respect to the GW, $\theta_1 \neq \theta_2$, so they would allow the determination of the spin-0 mass. A straightforward extension to a setup with more detectors (pulsars) enables us to decompose a spin-2 GW into polarization modes and, in principle, determine the spin-2 mass m as well. Such multidetector measurements would also be needed to analyse GW backgrounds.

In an ideal case in which we know, or can predict, the spectral form of the spin-0 GWs $A_\alpha(\omega, t)$, i.e., its frequency dependence, we might be able to determine the mass even with a single detector. Consider two measurements of a GW at two different frequencies ω_1 and ω_2

$$\begin{cases} S(\omega_1) = F_{\rm B}\,A_{\rm B}(\omega_1) + F_{\rm L}\,A_{\rm L}(\omega_1)\,, \\ S(\omega_2) = F_{\rm B}\,A_{\rm B}(\omega_2) + F_{\rm L}\,A_{\rm L}(\omega_2)\,, \end{cases} \tag{2.62}$$

where we omitted t. Substituting the relationship $A_{\rm L}(\omega)/A_{\rm B}(\omega) = m_0^2/\omega^2$ and solving for m_0, we obtain

$$m_0^2 = \frac{F_{\rm B}}{F_{\rm L}}\,\frac{A_{\rm B}(\omega_1)/A_{\rm B}(\omega_2) - S(\omega_1)/S(\omega_2)}{\omega_2^{-2}\,S(\omega_1)/S(\omega_2) - \omega_1^{-2}\,A_{\rm B}(\omega_1)/A_{\rm B}(\omega_2)} \tag{2.63}$$

Since the ratio $A_{\rm B}(\omega_1)/A_{\rm B}(\omega_2)$ is assumed to be known, m_0^2 can be measured. This method can also be extended to determine the spin-2 mass m.

We emphasise the merit of our methods that the determination of the mass parameter m_0 (or β) can be carried out without measuring the velocity of the spin-0 GW, whether absolute or relative to other signals, or analysing the details of the waveforms. Naturally, this advantage is also enjoyed in the relevant analyses of the spin-2 part.

Chapter 3

Effects of graviton masses on astrophysical objects

In this chapter, we study the structure of static spherical stars composed of non-relativistic matter. In Section 3.1, we show the effects of the massive gravitons on the stellar structure. Adopting a polytropic equation of state, we construct master differential equations, the modified Lane–Emden (LE) equations, for the stellar profile function. In Section 3.2, we study the case of HCG. The method for MG can be partially applied since the massive gravitons also exist in HCG.

3.1 Non-relativistic stellar structure in massive gravity

In this section, we study the structure of non-relativistic polytropic stars in the Fierz–Pauli (FP) theory and generic linear massive gravity (MG) theories. Our aim is to study the effects of the graviton mass m and the "non-Fierz–Pauli" parameter ϵ incorporated in the MG action (1.13) on the stellar structures. The spin-2 graviton is the only content in the FP theory with $\epsilon = 0$, while the spin-0 ghost graviton is also included in generic non-FP MG with $\epsilon \neq 0$.

3.1.1 Massive gravitational potentials

We begin with summarising the EOMs for the scalar modes in generic linear MG, which will be relevant to static spherical configurations. Scalar-type metric perturbations are characterised by four variables as

$$h_{00}^{(\mathrm{S})} = -2A\,, \quad h_{0i}^{(\mathrm{S})} = -\partial_i B\,, \quad h_{ij}^{(\mathrm{S})} = 2\delta_{ij}\, C + 2\partial_i \partial_j E\,. \tag{3.1}$$

It is well known that one can choose two linearly independent combinations of the above four variables that are invariant under a small gauge transformation, see Appendix A for a summary of linear gauge transformations and invariant variables. In particular,

$$\Psi \equiv A - \dot{B} - \ddot{E}\,, \quad \Phi \equiv C \tag{3.2}$$

correspond to the gravitational potentials in the Newtonian gauge. When the action (1.13) does not possess the mass term, the scalar sector can be solely written in terms of the gauge-invariant variables because the massless theory inherits the linearized version of the gauge symmetry of GR. In the massive case, on the other hand, one cannot take such advantage because the mass term breaks the symmetry. Indeed, the action (1.13) is written in terms of the metric perturbations, A, B, C, and E, as

$$\begin{aligned} S^{(\mathrm{S})}_{\mathrm{MG}} = \frac{1}{16\pi G}\int \mathrm{d}^4x \Big[& -6C\,\Box C - 4C\,\triangle(A - \dot{B} - \ddot{E} - C) \\ & -\frac{m^2}{2}\left(2\epsilon\,A^2 + B\,\triangle B + 6(3\epsilon-2)\,C^2 + 2\epsilon\,E\,\triangle^2 E\right. \\ & \left. +4(3\epsilon-2)\,C\,\triangle E + 4(\epsilon-1)\,A\,\triangle E + 12(\epsilon-1)\,A\,C\right)\Big]\,, \end{aligned} \tag{3.3}$$

up to surface terms. In a similar fashion, we define the scalar components of the perturbative energy-momentum tensor $T_{\mu\nu}$ as

$$T^{(\mathrm{S})}_{00} = \rho\,, \quad T^{(\mathrm{S})}_{0i} = -\partial_i v\,, \quad T^{(\mathrm{S})}_{ij} = P\,\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\,\delta_{ij}\,\triangle\right)\sigma\,, \tag{3.4}$$

so the scalar sector of the minimally coupled matter action is

$$\begin{aligned} S^{(\mathrm{S})}_{\mathrm{int}} &= \frac{1}{2}\int \mathrm{d}^4x\, h^{(S)}_{\mu\nu} T^{(S)\mu\nu} \\ &= \int \mathrm{d}^4x \left[-A\,\rho + B\,\triangle v + 3C\,P + E\,\triangle\left(P + \frac{2}{3}\triangle\,\sigma\right)\right]\,, \end{aligned} \tag{3.5}$$

where surface terms have been discarded. We assume the conservation law $\partial_\mu T^{(\mathrm{S})\mu\nu} = 0$ holds, which settles down in the decomposed form:

$$\dot{\rho} + \triangle v = 0\,, \quad \dot{v} + P + \frac{2}{3}\,\triangle\sigma = 0\,. \tag{3.6}$$

Varying the action $S^{(\mathrm{S})} = S^{(\mathrm{S})}_{\mathrm{MG}} + S^{(\mathrm{S})}_{\mathrm{int}}$ with respect to $A\,,B\,,C\,,E$ and using the conservation law, we obtain the four EOMs

$$\begin{aligned} 2\triangle C + m^2\left[\epsilon\,A + (\epsilon-1)\,\triangle E + 3(\epsilon-1)\,C\right] &= -8\pi\,G\,\rho\,, \\ \triangle(4\dot{C} + m^2\,B) &= -16\pi\,G\,\dot{\rho}\,, \\ 6\Box C + 2\triangle(A - \dot{B} - \ddot{E} - 2C) + m^2\left[3(3\epsilon-2)\,C + (3\epsilon-2)\,\triangle E + 3(\epsilon-1)\,A\right] &= 24\pi\,G\,P\,, \\ \triangle\{-2\ddot{C} + m^2\left[\epsilon\,\triangle E + (3\epsilon-2)\,C + (\epsilon-1)\,A\right]\} &= 8\pi\,G\,\ddot{p}\,. \end{aligned} \tag{3.7}$$

The derivation of an equivalent set of equations can be found in Ref. [JMM13].

We rigorously proved in Section 2.1 that, in the vacuum case, the scalar-type dynamical content in this theory consists of the helicity-0 component of the spin-2 graviton and the spin-0 graviton, which may be defined as

$$\phi_2 \equiv \frac{1}{2}\left(A - \dot{B} - \ddot{E} - C\right), \quad \phi_0 \equiv -A + \frac{\dot{B}}{2} - 2C\,, \tag{3.8}$$

respectively. In fact, even in the presence of the matter sources, the EOMs (3.7) are neatly recast into the following two sourced Klein–Gordon-type equations for ϕ_2 and ϕ_0 ,

$$(\Box - m_2^2)\,\phi_2 = 4\pi\,G\,(\rho + \dot{v} - \Box\sigma)\,, \quad (\Box - m_0^2)\,\phi_0 = 4\pi\,G\,(\rho - 3P)\,, \tag{3.9}$$

where their masses are defined as

$$m_2^2 \equiv m^2\,, \quad m_0^2 \equiv \frac{3-4\epsilon}{2\epsilon}\,m^2\,, \tag{3.10}$$

respectively. By solving ϕ_2 and ϕ_0 , we can reconstruct the original metric variables A , B , C , and E as

$$\begin{aligned}
A &= \frac{4}{3m_2^4}\,\triangle^2\phi_2 - \left(\frac{1-\epsilon}{\epsilon} - \frac{2}{3m_2^2}\,\triangle\right)\phi_0 + \frac{8\pi\,G}{3m_2^2}\left(-3\rho - \frac{2}{m_2^2}\,\triangle\rho + \frac{2}{m_2^2}\,\triangle^2\sigma\right),\\
B &= \frac{8}{3m_2^4}\,\triangle\dot{\phi}_2 + \frac{4}{3m_2^2}\,\dot{\phi}_0 + \frac{16\pi\,G}{3m_2^2}\left(3v - \frac{2}{m_2^2}\,\dot{\rho} + \frac{2}{m_2^2}\,\triangle\dot{\sigma}\right),\\
C &= -\frac{2}{3m_2^2}\,\triangle\phi_2 - \frac{1}{2}\,\phi_0 + \frac{8\pi\,G}{3m_2^2}\,(\rho - \triangle\sigma)\,,\\
E &= \frac{2}{m_2^2}\left(1 - \frac{2}{3m_2^2}\triangle\right)\phi_2 - \frac{2}{3m_2^2}\,\phi_0 + \frac{8\pi\,G}{3m_2^2}\left(3\sigma - \frac{2}{m_2^2}\,\triangle\sigma + \frac{2}{m_2^2}\rho\right).
\end{aligned} \tag{3.11}$$

Therefore, the gauge-invariant variables are

$$\begin{aligned}
\Psi &= A - \dot{B} - \ddot{E} = 2\left(1 - \frac{1}{3m_2^2}\triangle\right)\phi_2 - \frac{1}{3}\,\phi_0 + \frac{8\pi\,G}{3m_2^2}\,(\rho - \triangle\sigma)\,,\\
\Phi &= C = -\frac{2}{3m_2^2}\,\triangle\phi_2 - \frac{1}{3}\,\phi_0 + \frac{8\pi\,G}{3m_2^2}\,(\rho - \triangle\sigma)\,.
\end{aligned} \tag{3.12}$$

In order to prevent ϕ_0 from being tachyonic, the non-FP parameter ϵ is required to be within the range $0 < \epsilon \leq 3/4$ so that $0 \leq m_0^2 < \infty$. We will shortly confirm that both ϕ_2 and ϕ_0 produce a Yukawa-type potential in the static case and, when non-tachyonic, ϕ_2 is attractive in the remote distances whereas ϕ_0 is repulsive. On the observational ground, the attractive force mediated by the spin-2 graviton must dominate, so we further restrict the range of ϵ within $0 < \epsilon \leq 1/2$ so that $m_2^2 \leq m_0^2$. On the other hand, m_2^2 must be so tiny that the speed of gravitational waves is sufficiently close to the speed of light in light of the observation of GW170817 [A+17a]. Notice that, as long as $\epsilon \neq 0$, vanishing of the spin-2 mass implies simultaneous vanishing of the spin-0 mass.

From now on, we specialise to static configurations with non-relativistic matter satisfying $\rho \gg |P|$. Then, our variables are related to the metric perturbations as

$$\phi_2 = \frac{A - C}{2}, \quad \phi_0 = -A - 2C \tag{3.13}$$

and they obey the Helmholtz-type equations

$$(\triangle - m_2^2)\,\phi_2 = 4\pi\, G\, \rho\,, \quad (\triangle - m_0^2)\,\phi_0 = 4\pi\, G\, \rho\,. \tag{3.14}$$

The gauge-invariant variables (3.2) can now be written in terms of ϕ_2 and ϕ_0 as

$$\Psi = A = \frac{4}{3}\,\phi_2 - \frac{1}{3}\,\phi_0\,, \quad \Phi = C = -\frac{2}{3}\,\phi_2 - \frac{1}{3}\,\phi_0\,. \tag{3.15}$$

For notational convenience, we introduce

$$\alpha_2 = \frac{4}{3}, \quad \alpha_0 = -\frac{1}{3}, \quad \beta_2 = -\frac{2}{3}, \quad \beta_0 = -\frac{1}{3} \tag{3.16}$$

and write the gauge-invariant potentials as

$$\Psi = \sum_{s=2,0} \alpha_s\, \phi_s\,, \quad \Phi = \sum_{s=2,0} \beta_s\, \phi_s\,. \tag{3.17}$$

It will be useful to notice that $\sum_s \alpha_s = \sum_s \beta_s = 1$.

Hereafter, we presume that the system is spherically symmetric. Since asymptotic flatness requires $\lim_{|\vec{x}|\to\infty} \Psi = \lim_{|\vec{x}|\to\infty} \Phi = 0$, ϕ_s must satisfy $\lim_{|\vec{x}|\to\infty} \phi_s = 0$. The general solution to the Helmholtz equation (3.14) with two integration constants, C_1 and C_2, is

$$\phi_s = -\frac{G}{r}\, \left(\sigma_s(r)\, \cosh m_s\, r - \chi_s(r)\, \sinh m_s\, r + C_1\, \sinh m_s\, r + C_2\, \cosh m_s\, r\right) \tag{3.18}$$

with the two functions σ_s and χ_s with the dimensions of mass being

$$\begin{aligned} \sigma_s(r) &\equiv \frac{4\pi}{m_s} \int_0^r \mathrm{d}r'\, r'\, \rho(r')\, \sinh m_s\, r'\,, \\ \chi_s(r) &\equiv \frac{4\pi}{m_s} \int_0^r \mathrm{d}r'\, r'\, \rho(r')\, \cosh m_s\, r'\,. \end{aligned} \tag{3.19}$$

For a star with the finite radius R, likewise, these quantities have a constant value outside the star:

$$\sigma_s(r \geq R) = \sigma_s(R) \equiv \Sigma_s\,, \quad \chi_s(r \geq R) = \chi_s(R) \; \equiv X_s\,. \tag{3.20}$$

In this case, regularity at $r = 0$ requires $C_2 = 0$, while asymptotic flatness fixes the other constant as

$$C_1 = X_s - \Sigma_s = \frac{4\pi}{m_s} \int_0^R \mathrm{d}r\, r\, \rho(r)\, \mathrm{e}^{-m_s r} \equiv I_s\,. \tag{3.21}$$

Thus, the spin-dependent massive potential is determined as

$$\phi_s = -\frac{G}{r}\left(\sigma_s(r)\, \cosh m_s\, r + (I_s - \chi_s(r))\, \sinh m_s\, r\right)\,. \tag{3.22}$$

Outside the star, ϕ_s reduces to a Yukawa-type potential

$$\phi_s(r \geq R) = -\frac{G\,\Sigma_s\, \mathrm{e}^{-m_s r}}{r}\,, \tag{3.23}$$

because $\sigma_s(r \geq R) = \Sigma_s$ and $I_s - \chi_s(r \geq R) = -\Sigma_s\,.$ We see that Σ_s plays a role of a Yukawa charge.

Let us gain some insights into the massless limit. The limiting value of the enclosed charge $\sigma_s(r)$ is nothing but the enclosed mass,

$$\lim_{m_s \to 0} \sigma_s(r) = 4\pi \int_0^r \mathrm{d}r'\, r'^2\, \rho(r') \equiv m(r)\,, \tag{3.24}$$

which proves that the massive potential (3.22) reproduces the Newtonian potential in this limit:

$$\lim_{m_s \to 0} \phi_s(r) = \int_\infty^r \frac{G\, m(r')}{r'^2}\, \mathrm{d}r'\,. \tag{3.25}$$

A remarkable consequence is that the massless limit of the gauge-invariant potentials external to an object with mass $M \equiv m(R)$ recovers the Newtonian potential,

$$\lim_{m \to 0} \Psi = -\lim_{m \to 0} \Phi = -\frac{GM}{r}\,, \tag{3.26}$$

thereby proving the absence of the vDVZ discontinuity in generic non-FP MG.

Notice that the above mentioned recovery of the Newtonian potential is a consequence of the compensating contribution from the spin-0 mode $\phi_0\,.$ The contribution to the gravitational potential from the spin-2 mode ϕ_2 alone exceeds the Newtonian potential by a factor of 1/3 as implied by the coefficient $\alpha_2 = 4/3$ in (3.15). That excess is exactly cancelled by ϕ_0 with the negative coefficient $\alpha_0 = -1/3$, where ϕ_0 is understood to provide a repulsive potential representing its ghost nature.

3.1.2 Stellar structure equation in massive gravity – Modified Lane–Emden equation

Given the gauge-invariant potential Ψ, the hydrostatic equilibrium condition in the Newtonian gauge reads

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,. \tag{3.27}$$

In order to obtain a differential equation for ρ, one supplies an equation of state (EOS) $P = P(\rho)$ and operates $\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2$ on the both sides to find

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -\triangle\Psi\,. \tag{3.28}$$

For now, we assume both m_2^2 and m_0^2 are bounded. Then we can safely use the Helmholtz equations (3.14) for ϕ_s's to obtain

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -\sum_s \alpha_s\left(m_s^2\,\phi_s + 4\pi G\rho\right). \tag{3.29}$$

Hereafter we adopt the polytropic relation $P = K\,\rho^{1+\frac{1}{n}}$ as the EOS for the stellar matter, where K is a constant and n is another constant called the polytropic index. Quantities at the stellar centre will be denoted by the subscript "c," such as $\rho_\mathrm{c} \equiv \rho(r=0)$ and $P_\mathrm{c} \equiv P(r=0) = K\,\rho_\mathrm{c}^{1+\frac{1}{n}}$. The stellar mass density and pressure can be expressed by a single profile function θ as

$$\rho = \rho_\mathrm{c}\,\theta^n\,,\quad P = P_\mathrm{c}\,\theta^{n+1}\,, \tag{3.30}$$

which is normalized as $\theta_\mathrm{c} = \theta(r=0) = 1$. We introduce a length scale

$$\ell \equiv \sqrt{\frac{(n+1)\,P_\mathrm{c}}{4\pi\,G\,\rho_\mathrm{c}^2}}\,, \tag{3.31}$$

note that ℓ depends on the polytropic index n. Then, the non-dimensional radial coordinate ξ and graviton mass parameters μ_s are defined as

$$\xi \equiv \frac{r}{\ell}\,,\quad \mu_s \equiv m_s\,\ell\,. \tag{3.32}$$

Following the standard derivation of the Lane–Emden equation in GR, we operate the non-dimensionalized laplacian operator $\triangle_\xi \equiv \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\right) = \ell^2\,\triangle$ on the hydrostatic equilibrium condition (3.29) to obtain

$$\triangle_\xi\theta + \sum_s \alpha_s\left(\theta^n + \frac{\mu_s^2\,\phi_s}{4\pi G\rho_\mathrm{c}\ell^2}\right) = 0\,. \tag{3.33}$$

Unlike GR, however, this is still an integro-differential equation for ρ since ϕ_s involve integrals of ρ in a non-trivial manner as given by (3.22). Since ϕ_s is a formal integral of the Helmholtz equation (3.14), it reduces in turn to the source term $4\pi\,G\,\rho$ by an operation of the Helmholtz operator $\triangle - m_s^2$. Thus, operating $(\triangle_\xi - \mu_2^2)\,(\triangle_\xi - \mu_0^2)$ and using (3.14), we obtain a sixth-order differential equation for θ:

$$\triangle_\xi\,\left[(\triangle_\xi - \mu_2^2)\,(\triangle_\xi - \mu_0^2)\,\theta + \alpha_2\,\left(\triangle_\xi - \mu_0^2\right)\,\theta^n + \alpha_0\,\left(\triangle_\xi - \mu_2^2\right)\,\theta^n\right] = 0\,. \tag{3.34}$$

This is our master equation for the profile function θ in generic linear MG, which is an extension of the original second-order Lane–Emden equation in GR. The stellar structure can be constructed upon integration of (3.34) with suitable boundary conditions. The fact that this is sixth order in differentiation is a consequence of the fact that there exist three physical degrees of freedom, one from the massive spin-2 graviton, one from the massive spin-0 and one from the matter. A word of caution is that the overall laplacian operator on the left-hand side of (3.34) cannot be dropped as in the case of field equations (3.7) since θ and its derivatives must satisfy non-trivial boundary conditions at a finite radius $r = R$, as we shall shortly see.

In terms of the non-dimensional variables, the stellar mass M and charge Σ_s are expressed as

$$M = 4\pi\,\ell^3\,\rho_{\rm c}\int_0^{\xi_R}{\rm d}\xi\,\xi^2\,\theta(\xi)^n\,, \quad \Sigma_s = \frac{4\pi\,\ell^3\,\rho_{\rm c}}{\mu_s}\int_0^{\xi_R}{\rm d}\xi\,\xi\sinh\mu_s\xi\,\theta(\xi)^n\,, \tag{3.35}$$

where $\xi_R \equiv R/\ell$ and is the first positive zero of θ. Unlike GR, M cannot be expressed in terms of derivatives at the stellar surface.

Now, we move on to discuss boundary conditions for the profile function θ. Since the master equation (3.34) is sixth order in differentiation, there is a need for six independent conditions, a priori. We shall demonstrate all such conditions can be derived as the requirements for the compatibility with the hydrostatic equilibrium condition (3.27). Here, we take the same strategy as in GR in the sense that we aim to impose all the conditions on the values of θ and its derivatives at the stellar center. It will turn out, however, that this is made only partially successful by the massive nature of the extra gravitational potential.

Let us start by re-expressing (3.27) in terms of θ as

$$\theta' = -\frac{\ell^{-1}}{4\pi\,G\,\rho_{\rm c}}\,\frac{{\rm d}\Psi}{{\rm d}r}\,, \tag{3.36}$$

where and hereafter, the prime denotes derivative with respect to ξ. The derivatives at the stellar center $\theta_{\rm c}^{(n)} \equiv \frac{{\rm d}^n\theta}{{\rm d}\xi^n}(0)$ are required to be consistent with the behavior of the potential $\Psi = \sum_s \alpha_s\,\phi_s$ evaluated there. Indeed, one finds the expansion of the potential (3.22) to a sufficient order as

$$\begin{aligned} G^{-1}\,\phi_s = {} & -m_s\,I_s + \left(\frac{2\pi}{3}\,\rho(0) - \frac{m_s^3\,I_s}{6}\right) r^2 + \frac{\pi}{3}\,\frac{{\rm d}\rho}{{\rm d}r}(0)\,r^3 \\ & + \left(\frac{\pi}{10}\,\frac{{\rm d}^2\rho}{{\rm d}r^2}(0) + \frac{\pi\,m_s^2\,\rho(0)}{30} - \frac{m_s^5\,I_s}{120}\right) r^4 + \mathcal{O}(r^5)\,. \end{aligned} \tag{3.37}$$

From above, it can be shown that the radial acceleration at the stellar center vanishes, $\lim_{r\to 0} -\frac{{\rm d}\Psi}{{\rm d}r} = 0$, which implies $\theta'_{\rm c} = 0$ via (3.36). Hence, the following two boundary conditions have so far been obtained:

$$\theta_{\rm c} = 1\,, \quad \theta'_{\rm c} = 0\,. \tag{3.38}$$

These two conditions just suffice in the case of the second-order LE equation in GR. By contrast, four more boundary conditions are required in order to solve the full sixth-order differential equation (3.34). They are found from derivatives of (3.36) as

$$\begin{aligned}
\theta_{\mathrm{c}}'' &= -\frac{1}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} = \sum_s \alpha_s \left(-\frac{1}{3} + \frac{1}{3}\, \mu_s^2\, \iota_s \right) , \\
\theta_{\mathrm{c}}''' &= -\frac{\ell}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^3\Psi}{\mathrm{d}r^3} = 0 \, , \\
\theta_{\mathrm{c}}^{(4)} &= -\frac{\ell^2}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^4\Psi}{\mathrm{d}r^4} = \sum_s \alpha_s \left(-\frac{3}{5}\, n\, \theta_{\mathrm{c}}'' - \frac{1}{5}\, \mu_s^2 + \frac{1}{5}\, \mu_s^4\, \iota_s \right) , \\
\theta_{\mathrm{c}}^{(5)} &= -\frac{\ell^3}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^5\Psi}{\mathrm{d}r^5} = 0 \, ,
\end{aligned} \tag{3.39}$$

where we have non-dimensionalized the constants I_s (3.21) appearing in the potentials ϕ_s such that

$$\iota_s \equiv \frac{\mu_s\, I_s}{4\pi\, \ell^3\, \rho_{\mathrm{c}}} = \int_0^{\xi_R} \mathrm{d}\xi\, \xi\, \theta(\xi)^n\, \mathrm{e}^{-\mu_s \xi} \tag{3.40}$$

and iteratively applied the conditions arising from lower derivatives to those from higher ones. It is observed that θ_{c}'' and $\theta_{\mathrm{c}}^{(4)}$ are now related to the stellar global quantities ι_s, which was moreover introduced to guarantee flatness at spatial infinity. The value of ι_s is as yet undetermined until the profile function θ is solved. Therefore, these expressions of the boundary values should be considered merely formal. Namely, the problem is not formulated as a simple initial value problem as in GR, but we will have to perform some matching procedure between the boundary values at the stellar center and the integrals of the solution over the whole domain. Note that the stellar radius ξ_R is simultaneously determined by this procedure.

It might be useful to provide an integro-differential equation equivalent to the master equation (3.34). Integrating (3.34) twice, we have

$$(\triangle_\xi - \mu_2^2)\, (\triangle_\xi - \mu_0^2)\, \theta + \alpha_2\, \left(\triangle_\xi - \mu_0^2\right)\, \theta^n + \alpha_0\, \left(\triangle_\xi - \mu_2^2\right)\, \theta^n = \mu_2^2\, \mu_0^2\, \left(1 - \alpha_2\, \iota_2 - \alpha_0\, \iota_0\right) , \tag{3.41}$$

where we have already utilised the boundary conditions (3.39).

When the mass parameter μ_0^2 goes to infinity, i.e., the non-FP parameter ϵ goes to zero, the massive gravity theory reduces to the FP theory. A little care has to be taken when one wants to consider such a limit because we have assumed finiteness of the graviton masses in the derivation of the full equation. Actually, in the massive limit, the spin-0 Yukawa potential itself must be absent from the beginning. A secure way to go is to begin by setting $\alpha_0 = 0$ in (3.33), which has the effect of turning off the spin-0 Yukawa potential ϕ_0. Then, operating $(\triangle_\xi - \mu_2^2)$ only, the correct master equation for the FP theory is found to be

$$\triangle_\xi \left[(\triangle_\xi - \mu_2^2)\, \theta + \frac{4}{3}\, \theta^n \right] = 0 \, , \tag{3.42}$$

where we have restored α_2 to $4/3$. The reason for this being the fourth order is that the helicity-0 component of the massive spin-2 graviton, which is absent in GR, is still at work in addition to the matter DOF. It is interesting to note that this can also be obtained by taking a formal limit $\mu_0^2 \to \infty$ in the master equation (3.34) for a generic theory.

We need four boundary conditions in total, two of which are the same as the Lane–Emden conditions (3.38). The remaining two are obtained by setting $\alpha_0 = 0$ in (3.39) as

$$\begin{aligned} \theta_{\mathrm{c}}'' &= -\frac{1}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} = -\frac{4}{9}(1-\mu_2^2\, \iota_2)\,, \\ \theta_{\mathrm{c}}''' &= -\frac{\ell}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^3\Psi}{\mathrm{d}r^3} = 0\,. \end{aligned} \tag{3.43}$$

It is useful here to consider an integro-differential equation equivalent to the fourth-order LE equation (3.42),

$$(\triangle_\xi - \mu_2^2)\, \theta + \frac{4}{3}\theta^n = \frac{4}{3}\, \mu_2^2\, \iota_2 - \mu_2^2\,. \tag{3.44}$$

From this, one can deduce that, when the graviton mass parameter μ_2 goes to zero, the master equation reduces to

$$\triangle_\xi \theta + \frac{4}{3}\, \theta^n = 0 \tag{3.45}$$

with the boundary conditions $\theta_{\mathrm{c}} = 1\,, \theta_{\mathrm{c}}' = 0$. This resembles the ordinary Lane–Emden equation in GR but has a different numerical coefficient, implying that its solutions do not recover those of GR although the mass parameter μ_2 vanishes. This discontinuous behavior is reminiscent of the vDVZ discontinuity.

Historically, since Vainshtein [Vai72], this kind of discontinuity in the FP theory has been believed to be cured by a screening mechanism inherent in the nonlinear completion of the theory, where gravity within some radius r_{V} is expected to mimic GR. It would be of interest here to compare the stellar radius R, and the screening radius r_{V} taking its generic expression $r_{\mathrm{V}} \equiv (r_{\mathrm{S}}/m^p)^{1/(p+1)}$, where r_{S} is the Schwarzschild radius of a stellar object, m is the graviton mass, and p is a theory-dependent parameter. $p = 4$ for the original Vainshtein model [Vai72] and $p = 2$ for the de Rham–Gabadadze–Tolley (dRGT) theory [dRGT11], see, e.g., [BD13] for generalisations. Then, the ratio of the screening radius to the stellar radius reads $r_{\mathrm{V}}/R = (r_{\mathrm{S}}/R)^{1/(p+1)}/(m\, R)^{p/(p+1)}$. Although the ratio r_{S}/R in the numerator should be reasonably small for any nonrelativistic stars, the product $m\, R$ in the denominator is requested to be far more smaller once one demands the inverse-square force exerted by the object reaches astronomical distances, which are generically greater than R by tremendous orders of magnitude. Hence, one can generically expect that $r_{\mathrm{V}}/R \gg 1$ and, therefore, some screening effect should work within non-relativistic stellar objects.

Last but not least, we consider the massless limit with a non-vanishing non-FP parameter ϵ. When $\epsilon \neq 0$, the doubly massless limit can be taken, where both μ_2 and

μ_0 go to zero. Then, our master equation (3.34) becomes

$$\triangle_\xi^2 \left(\triangle_\xi \theta + \theta^n\right) = 0\,. \tag{3.46}$$

This equation can be integrated four times and imposing the suitable boundary conditions (3.39) yields

$$\triangle_\xi \theta + \theta^n = 0\,. \tag{3.47}$$

This is the same as the standard Lane–Emden equation in GR. Thus, we can expect that the solutions to the master equation (3.34) will recover the corresponding solutions of the standard LE equation in GR. This proves the absence of the discontinuity in the generic, non-FP MG.

We will present analytical solutions to the FP and non-FP master equations for the polytropic indices $n = 0$ and 1. All the differential equations treated in this section will be a linear homogeneous equation for θ sharing the common form

$$f(\triangle_\xi)\,\theta = 0\,, \tag{3.48}$$

where the characteristic polynomial $f(X)$ is degree 2 in X for the FP theory or 3 for the non-FP theories. The general solution is characterised by the set of the roots for the characteristic equation $f(X) = 0$. It always has $X_0 = 0$ as a root, as is obvious from Eqs. (3.34) and (3.42). For each root X_i, one finds a fundamental solution by solving $\triangle\theta_i = X_i\,\theta_i$,

$$\theta_i = \begin{cases} A_i + \dfrac{B_i}{\xi} & (X_i = 0) \\ A_i\,\dfrac{\sinh\sqrt{X_i}\xi}{\xi} + B_i\,\dfrac{\cosh\sqrt{X_i}\xi}{\xi} & (X_i > 0) \\ A_i\,\dfrac{\sin\sqrt{-X_i}\xi}{\xi} + B_i\,\dfrac{\cos\sqrt{-X_i}\xi}{\xi} & (X_i < 0) \end{cases}\,, \tag{3.49}$$

where A_i and B_i are arbitrary real constants. When there is no degeneracy, the general solution is just a sum of the fundamental solutions θ_i. If there are degeneracies, on the other hand, special but straightforward mathematical treatments such as variation of constants will be necessary. Later, we will take care of an example of a degenerate situation.

3.1.3 Case I: Fierz–Pauli theory

We first present the exact solutions to the Fierz–Pauli master equation (3.42) in the cases of the polytropic indices $n = 0$ and 1. In this case, only the spin-2 graviton exists.

For $n = 0$, the fourth-order equation (3.42) reduces to a homogeneous linear equation

$$\left(\triangle_\xi - \mu_2^2\right)\triangle_\xi \theta = 0\,. \tag{3.50}$$

The general solution can be found as

$$\theta = A_1 + \frac{A_2}{\xi} + A_3 \frac{\sinh \mu_2 \xi}{\mu_2 \xi} + A_4 \frac{\cosh \mu_2 \xi}{\mu_2 \xi} \,, \tag{3.51}$$

where A_1 , A_2 , A_3 , and A_4 are constants of integration. The set of the boundary conditions (3.43) determines the constants as

$$A_1 = 1 - A_3 \,, \quad A_3 = -\frac{4}{3} \frac{(1 + \mu_2 \xi_R) \, \mathrm{e}^{-\mu_2 \xi_R}}{\mu_2^2} \,, \quad A_2 = A_4 = 0 \,. \tag{3.52}$$

Therefore, we get the exact solution

$$\theta = 1 - \frac{4}{3} \frac{(1 + \mu_2 \xi_R) \, \mathrm{e}^{-\mu_2 \xi_R}}{\mu_2^2} \frac{\sinh \mu_2 \xi - \mu_2 \xi}{\mu_2 \xi} \,. \tag{3.53}$$

As anticipated, the solution is parameterized by the stellar radius ξ_R , the value of which must be determined by numerically solving $\theta(\xi_R) = 0$. The analytical expression for the gravitational potential Ψ can be found as

$$\Psi = \begin{cases} -\dfrac{4}{3} \dfrac{G\Sigma_2}{r} \dfrac{m_2 r - (1 + m_2 R) \mathrm{e}^{-m_2 R} \sinh m_2 r}{m_2 R \cosh m_2 R - \sinh m_2 R} & (r \leq R) \\ -\dfrac{4}{3} \dfrac{G\Sigma_2}{r} \, \mathrm{e}^{-m_2 r} & (r \geq R) \end{cases} \,. \tag{3.54}$$

Figure 3.1 shows the profile function θ for several values of μ_2 and compares them with GR. In the massless limit $\mu_2 \to 0$, the solution reduces to $\theta = 1 - 2\xi^2/9$, which has a different coefficient from the Lane–Emden solution in GR. As a consequence, the stellar radius shrinks from the GR value of $\sqrt{6}$ to $3/\sqrt{2}$. On the other hand, in the massive limit $\mu_2 \to \infty$, the solution has an infinite radius because gravity no longer works. The profile best resembles that of GR when μ_2 has some finite value around 0.4, but such a large value is physically unreasonable.

The expressions for the stellar mass M and the Yukawa charge Σ_2 can be obtained by substituting the solution to (3.35):

$$M = \frac{4\pi \ell^3 \rho_{\mathrm{c}} \, \xi_R^3}{3} \,, \quad \Sigma_2 = 4\pi \ell^3 \rho_{\mathrm{c}} \frac{\mu_2 \xi_R \cosh \mu_2 \xi_R - \sinh \mu_2 \xi_R}{\mu_2^3} \,. \tag{3.55}$$

Figure 3.2 shows the dependence of the stellar radius R/ℓ (blue) , the mass $M/(4\pi\ell^3\rho_{\mathrm{c}})$ (red) and the charge $\Sigma_2/(4\pi\ell^3\rho_{\mathrm{c}})$ (yellow), each appropriately normalised, on the graviton mass parameter μ_2 . As we explained earlier, when the graviton mass parameter μ_2 goes to zero, the charge Σ_2 and the mass M have the same limiting value. One also confirms that the massless limit of R and M does not converge to their corresponding GR values indicated by the star symbols. This discontinuous behavior is analogous to what is predicted for the bending of light. Hence, one could conclude that, in order for the graviton in the FP theory to acquire a tiny mass, some screening mechanism that

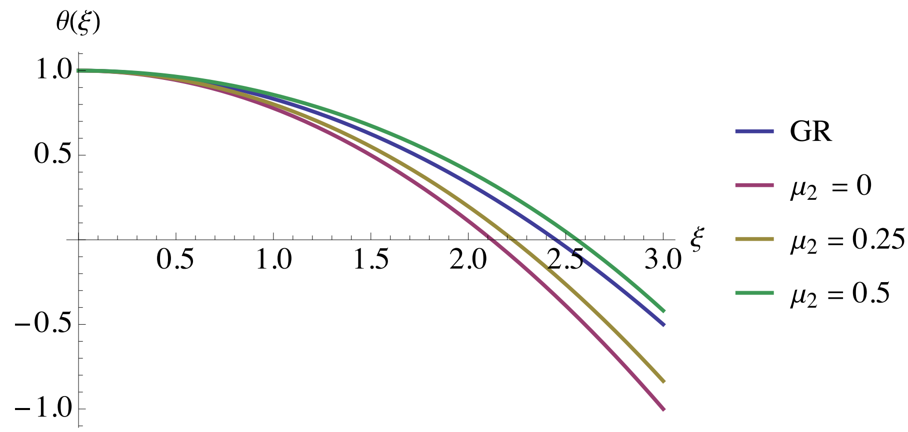

Figure 3.1: The profile function θ for the polytropic index $n = 0$ in the Fierz–Pauli theory and GR.

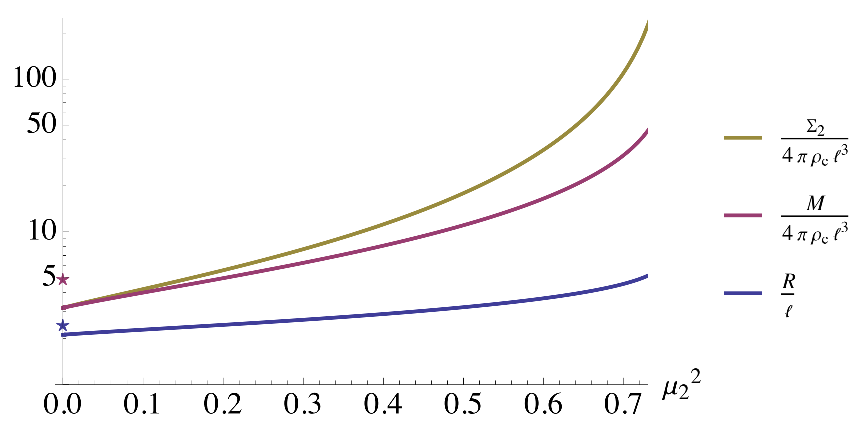

Figure 3.2: The dependence of the stellar radius R/ℓ (blue), the mass $M/(4\pi\ell^3\rho_c)$ (red) and the charge $\Sigma_2/(4\pi\ell^3\rho_c)$ (yellow), each appropriately normalised, on the graviton mass μ_2^2 for the polytropic index $n = 0$. The star symbols indicate the GR values of R (bottom, blue) and M (top, red).

helps the recovery of GR has to be invoked even inside stars. On the other hand, as the mass of the graviton increases, each quantity diverges.

For the polytropic index $n = 1$, the fourth-order equation (3.42) reduces to a linear homogeneous equation

$$\left[\triangle_\xi - \left(\mu_2^2 - \frac{4}{3}\right)\right] \triangle_\xi \theta = 0\,. \tag{3.56}$$

We assume $\mu_2^2 < 4/3$ since the graviton mass should be small enough to be compatible with the gravitational-wave experiments. The general solution is obtained as

$$\theta = B_1 + \frac{B_2}{\xi} + B_3 \frac{\sin\lambda\xi}{\lambda\xi} + B_4 \frac{\cos\lambda\xi}{\lambda\xi} \tag{3.57}$$

with $\lambda = \sqrt{4/3 - \mu_2^2}$ and B_1, B_2, B_3, and B_4 are constants of integration. The boundary conditions (3.43) determine the constants as

$$B_1 = 1 - B_3\,, \quad B_3 = \frac{4}{3\lambda^2}(1 - \mu_2^2 \iota_2)\,, \quad B_2 = B_4 = 0\,. \tag{3.58}$$

Therefore, the exact solution is

$$\theta = 1 + B_3 \, \frac{\sin\lambda\xi - \lambda\xi}{\lambda\xi} \tag{3.59}$$

with

$$B_3 = -\frac{4\lambda(1 + \mu_2\xi_R)}{3\mu_2^2\,(\lambda\,\cos\lambda\xi_R + \mu_2\,\sin\lambda\xi_R) - 4\lambda\,(1 + \mu_2\xi_R)}\,. \tag{3.60}$$

An expression for the interior gravitational potential can also be found, but we do not show it because the result is too complicated and not illuminating.

Figure 3.3 shows the profile functions in the Fierz–Pauli theory and GR for $n = 1$. Although the functional shapes are different from those for $n = 0$, the trends with respect to the change in the mass parameter are similar. When $\mu_2 \to 0$, the solution reduces to $\theta = \sqrt{3}/(2\xi)\,\sin 2\xi/\sqrt{3}$, so the stellar radius ξ_R shrinks from the GR value of π to $\sqrt{3}\,\pi/2$.

Once ξ_R is determined, the stellar mass M and charge Σ_2 can be evaluated via their expressions for $n = 1$:

$$\begin{aligned}
M &= \frac{4\pi\ell^3\rho_c}{3}\left[\xi_R^3 + \frac{3B_3}{\lambda^2}\left(\sin\lambda\xi_R - \lambda\xi_R\,\cos\lambda\xi_R - \frac{1}{3}\,\lambda^3\,\xi_R^3\right)\right], \\
\Sigma_2 &= \frac{4\pi\ell^3\rho_c}{\mu_2^3}\Bigg[\mu_2\xi_R\,\cosh\mu_2\xi_R - \sinh\mu_2\xi_R \\
&\quad + B_3\,\lambda\left\{\sinh\mu_2\xi_R\left(1 - \frac{3}{4}\,\mu_2^2\,\cos\lambda\xi_R\right) - \mu_2\,\cosh\mu_2\xi_R\left(\xi_R + \frac{3}{4}\,\mu_2^2\,\cos\lambda\xi_R\right)\right\}\Bigg].
\end{aligned} \tag{3.61}$$

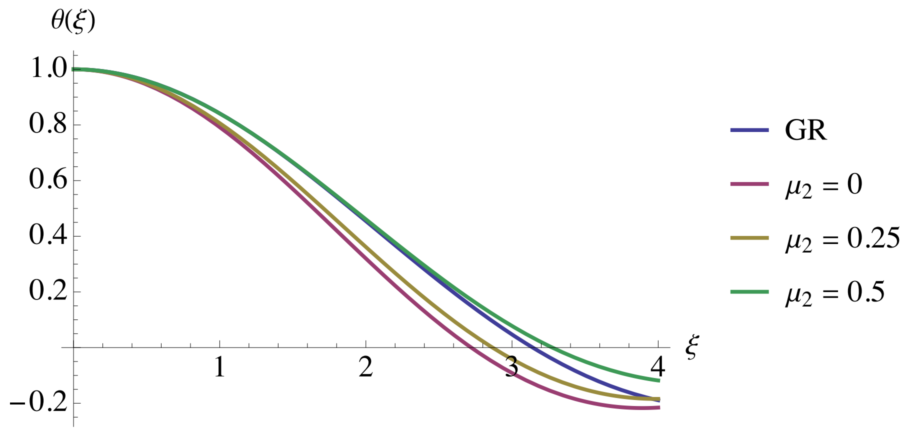

Figure 3.3: The profile function θ for the polytropic index $n = 1$ in the Fierz–Pauli theory and GR.

3.1.4 Case II: Non-Fierz–Pauli generic theories

Now, we would like to tackle the full master equation (3.34) in generic linear MG. The existence of the two massive gravitons, spin-2 and -0, renders the analysis considerably messy, but most features of the solutions will be reasonably understood as collective contributions from these gravitons. As in the FP case, we find exact solutions for the polytropic indices $n = 0, 1$.

In the case of $n = 0$, the master equation (3.34) reduces to

$$(\triangle_\xi - \mu_2^2)\,(\triangle_\xi - \mu_0^2)\,\triangle_\xi \theta = 0\,. \tag{3.62}$$

When $\mu_2 \neq \mu_0$, the general solution with six arbitrary constants is found to be

$$\theta = C_1 + \frac{C_2}{\xi} + C_3\,\frac{\sinh \mu_2 \xi}{\mu_2 \xi} + C_4\,\frac{\cosh \mu_2 \xi}{\mu_2 \xi} + C_5\,\frac{\sinh \mu_0 \xi}{\mu_0 \xi} + C_6\,\frac{\cosh \mu_0 \xi}{\mu_0 \xi}\,, \tag{3.63}$$

where C_1–C_6 are constants of integration. The boundary conditions (3.39) determine the constants as

$$C_1 = 1 - C_3 - C_5\,, \quad C_3 = -\frac{4}{3}(\mu_2^{-2} - \iota_2)\,, \quad C_5 = \frac{1}{3}(\mu_0^{-2} - \iota_0)\,, \quad C_2 = C_4 = C_6 = 0\,. \tag{3.64}$$

Therefore, the exact solution is

$$\theta = 1 - \frac{4}{3}\,\frac{(1+\mu_2\xi_R)\mathrm{e}^{-\mu_2\xi_R}}{\mu_2^2}\,\frac{\sinh \mu_2\xi - \mu_2\xi}{\mu_2\xi} + \frac{1}{3}\,\frac{(1+\mu_0\xi_R)\mathrm{e}^{-\mu_0\xi_R}}{\mu_0^2}\,\frac{\sinh \mu_0\xi - \mu_0\xi}{\mu_0\xi}\,. \tag{3.65}$$

When $\mu_2 = \mu_0$, the characteristic roots degenerate, and the general solution then is

$$\theta = \tilde{C}_1 + \frac{\tilde{C}_2}{\xi} + \tilde{C}_3\,\frac{\sinh \mu_2\xi}{\mu_2\xi} + \tilde{C}_4\,\frac{\cosh \mu_2\xi}{\mu_2\xi} + \tilde{C}_5\,\sinh \mu_2\xi + \tilde{C}_6\,\cosh \mu_2\xi\,. \tag{3.66}$$

A calculation leads to the solution

$$\theta = 1 - \frac{(1+\mu_2\xi_R)\mathrm{e}^{-\mu_2\xi_R}}{\mu_2^2}\,\frac{\sinh\mu_2\xi - \mu_2\xi}{\mu_2\xi}\,, \tag{3.67}$$

which happens to be identical with the formal $\mu_0 \to \mu_2$ limit of the non-degenerate solution (3.65). The analytical expression for the gravitational potential is found as

$$\Psi = \begin{cases} -\dfrac{4}{3}\,\dfrac{G\Sigma_2}{r}\,\dfrac{m_2 r - (1+m_2R)\,\mathrm{e}^{-m_2R}\,\sinh m_2 r}{m_2R\,\cosh m_2R - \sinh m_2R} \\ \quad +\dfrac{1}{3}\,\dfrac{G\Sigma_0}{r}\,\dfrac{m_0 r - (1+m_0R)\,\mathrm{e}^{-m_0R}\,\sinh m_0 r}{m_0R\,\cosh m_0R - \sinh m_0R} & (r \le R) \\ -\dfrac{4}{3}\,\dfrac{G\Sigma_2}{r}\,\mathrm{e}^{-m_2r} + \dfrac{1}{3}\,\dfrac{G\Sigma_0}{r}\,\mathrm{e}^{-m_0r} & (r \ge R) \end{cases} . \tag{3.68}$$

Typical solutions are shown in Fig. 3.4 together with the $n = 0$ LE solution. As is obvious from the expression (3.65), the spin-2 graviton mediates an attractive force while the spin-0 is repulsive. When there is a hierarchy between the graviton masses, the graviton with lighter mass dominates. It is worth mentioning that the stellar structure is maintained even in the presence of the ghost spin-0 graviton as long as $\mu_0 \gtrsim \mu_2$. In the case where the graviton masses are comparable, $\mu_2 \approx \mu_0$, the spin-2 graviton exceeds because its magnitude is four-fold stronger than the spin-0.

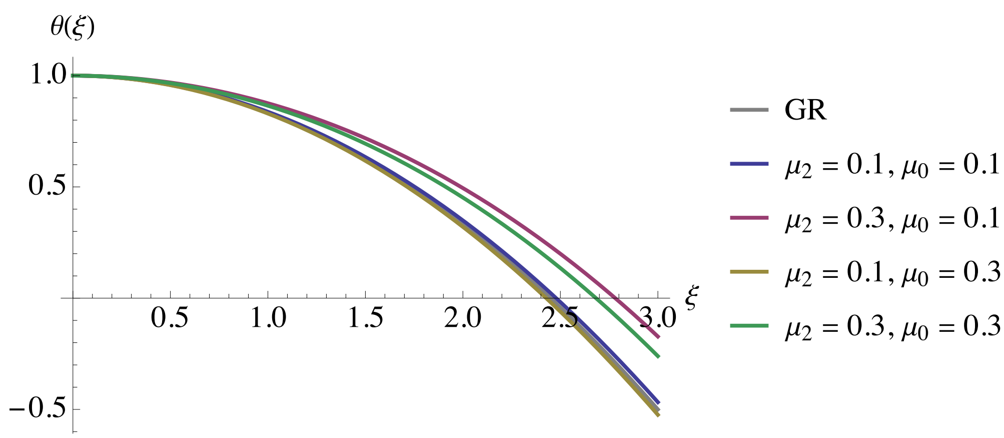

Figure 3.4: Typical profile functions θ in generic non-FP linear MG for the polytropic index $n = 0$ compared with the LE solution in GR.

Let us mention some interesting limits. In the limit $\mu_0 \to \infty$, i.e., $\epsilon \to 0$, the solution reduces to the one in the FP theory obtained previously. In another special case where the spin-2 graviton has zero mass, $\mu_2 \to 0$, the spin-0 mass is simultaneously zero, $\mu_0 \to 0$, by definition. The massless limit of the solution (3.65) recovers the $n = 0$ LE solution $\theta_{\mathrm{LE},0} = 1 - \xi^2/6$, so no discontinuity appears when the spin-0 graviton is included.

Once ξ_R is determined, the stellar mass M and charge Σ_s can be evaluated via their expressions, which are essentially the same as the one for the FP theory (3.55) ($\mu_2 \to \mu_0$ for Σ_0) since $n = 0$. Figure 3.5 shows the dependences of the stellar radius R/ℓ (blue) and the mass $M/(4\pi\ell^3\rho_c)$ (red) on the spin-2 graviton mass μ_2^2 for several values of the non-FP parameter ϵ. A larger value of ϵ leads to a larger gradient. Here, the Yukawa charges Σ_s are omitted, but their behavior is similar to the stellar mass M as seen in Fig. 3.2 for the FP case. In the presence of the spin-0 graviton, the values of R and M in the massless limit $\mu_2 \to 0$ both converge to the values of GR, which is in sharp contrast with the FP case (grey) studied in the previous subsection.

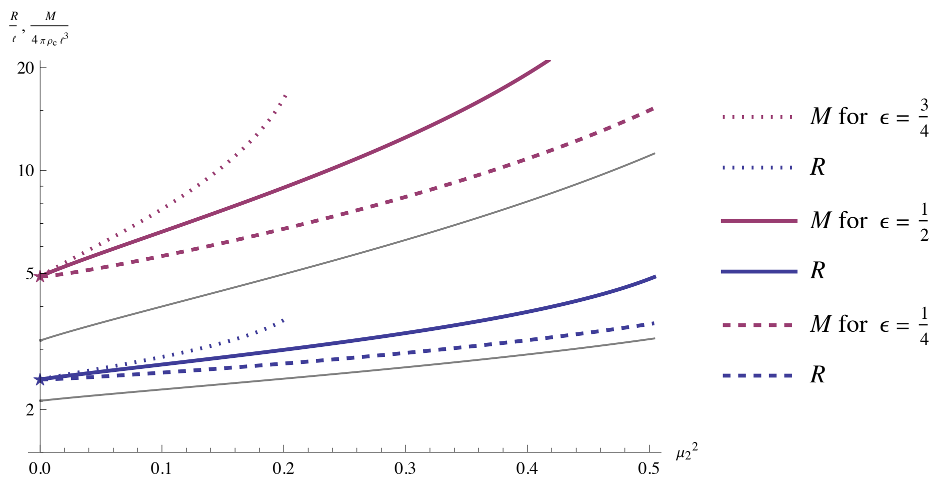

Figure 3.5: Dependences of the stellar radius R/ℓ (blue) and the mass $M/(4\pi\ell^3\rho_c)$ (red), each appropriate normalized, on the graviton mass μ_2^2 for several values of the non-FP parameter ϵ for the polytropic index $n = 0$. The grey lines correspond to the FP case with $\epsilon = 0$. The star symbols indicate the GR values of R (bottom) and M (top).

Figure 3.6 shows contours of the ratio of the stellar radius to the value of GR, $R/R_{\rm LE}$, in the $(\mu_2^2\,, \mu_0^2)$ plane. On the curve with a ratio $R/R_{\rm LE} = 1$, the MG solution has the same radius as GR as a result of competition between the attractive spin-2 and repulsive spin-0. This diagram has a potential usage in constraining the graviton mass parameters. We do not go into the detailed analysis here, but it is likely that the regions far from the line with $R/R_{\rm LE} = 1$, for instance, those with values smaller than 0.9 or greater than 1.1, may be rejected since the discrepancy from GR could be too large. In particular, when $\mu_2^2 \ll 1$, the spin-0 mass would be constrained as $\mu_0^2 \lesssim \mathcal{O}(1)$.

In the case of $n = 1$, the master equation (3.34) becomes

$$(\triangle_\xi - X_1)\,(\triangle_\xi - X_2)\,\triangle_\xi\theta = 0\,, \tag{3.69}$$

where X_i's are roots of the characteristic equation

$$f(x) = x^2 - (\mu_2^2 + \mu_0^2 - 1)\,x + \frac{1}{3}(\mu_2^2 - 4\mu_0^2 + 3\mu_2^2\,\mu_0^2) = 0 \tag{3.70}$$

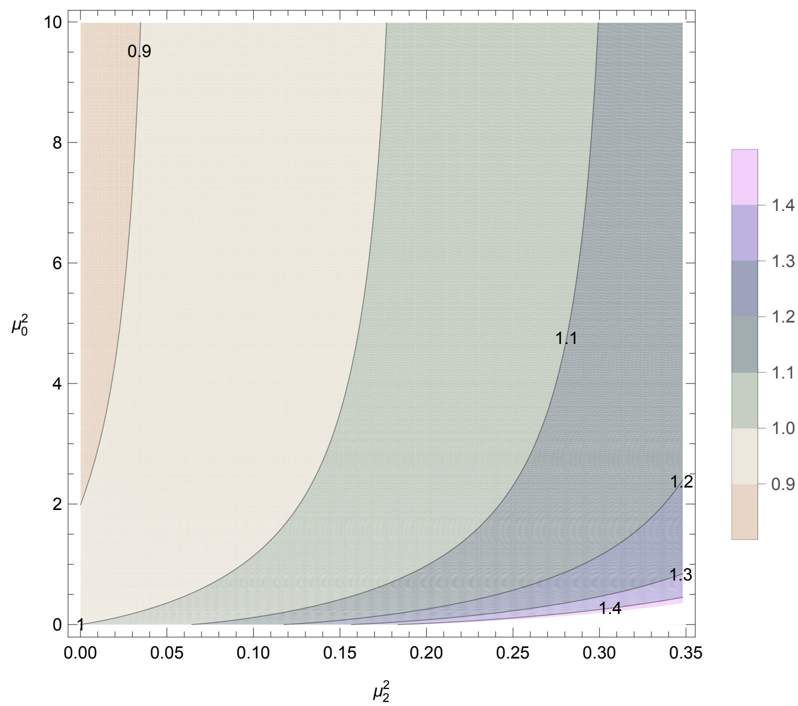

Figure 3.6: Contours of the ratio of the stellar radius to the value of GR, $R/R_{\rm LE}$, in the $(\mu_2^2\,,\mu_0^2)$ plane for the polytropic index $n = 0$.

with its discriminant being

$$D = \mu_2^4 + \mu_0^4 - \frac{10}{3}\mu_2^2 + \frac{10}{3}\mu_0^2 - 2\mu_2^2\,\mu_0^2 + 1\,. \tag{3.71}$$

In the case when $\mu_2^2 \ll 1$, which is of most importance on the observational grounds, the above equation has two real roots. A further restriction $\mu_2^2 \leq \mu_0^2$ forces the two real roots X_1 and X_2 to have opposite signs. The positive and negative roots then are $\lambda_+^2 = (\mu_2^2 + \mu_0^2 - 1 + \sqrt{D})/2$ and $-\lambda_-^2 = (\mu_2^2 + \mu_0^2 - 1 - \sqrt{D})/2$, respectively. The general solution is

$$\theta = D_1 + \frac{D_2}{\xi} + D_3\,\frac{\sinh\lambda_+\xi}{\lambda_+\xi} + D_4\,\frac{\cosh\lambda_+\xi}{\lambda_+\xi} + D_5\,\frac{\sin\lambda_-\xi}{\lambda_-\xi} + D_6\,\frac{\cos\lambda_-\xi}{\lambda_-\xi}\,, \tag{3.72}$$

where D_i's are arbitrary constants of integration. The set of boundary conditions (3.39) determines the constants as

$$\begin{aligned}
&D_1 = 1 - D_3 - D_5\,, \quad D_2 = D_4 = D_6 = 0\,,\\
&D_3 = \frac{1}{\lambda_+^2\,(\lambda_+^2 + \lambda_-^2)}\left[1 - \lambda_-^2 - \frac{4}{3}\,\mu_2^2\,\left(1 + \iota_2\,(1 - \lambda_-^2 - \mu_2^2)\right) + \frac{1}{3}\,\mu_0^2\,\left(1 + \iota_0\,(1 - \lambda_-^2 - \mu_0^2)\right)\right],\\
&D_5 = \frac{1}{\lambda_-^2\,(\lambda_+^2 + \lambda_-^2)}\left[1 + \lambda_+^2 - \frac{4}{3}\,\mu_2^2\,\left(1 + \iota_2\,(1 + \lambda_+^2 - \mu_2^2)\right) + \frac{1}{3}\,\mu_0^2\,\left(1 + \iota_0\,(1 + \lambda_+^2 - \mu_0^2)\right)\right].
\end{aligned} \tag{3.73}$$

Therefore, the exact solution is

$$\theta = 1 + D_3\,\frac{\sinh\lambda_+\xi - \lambda_+\xi}{\lambda_+\xi} + D_5\,\frac{\sin\lambda_-\xi - \lambda_-\xi}{\lambda_-\xi}\,. \tag{3.74}$$

Figure 3.7 shows the form of the profile functions θ for several combinations of μ_2 and μ_0. The trends with respect to the changes of the values of μ_2 and μ_0 are almost in parallel to the case with $n = 0$.

Here, we focus on the limiting case with the vanishing spin-2 mass, i.e., $\mu_2^2 \to 0$. Figure 3.8 shows the dependence of the stellar radius R/ℓ on the spin-0 graviton mass μ_0^2. The first thing to note is that, as is expected, the simultaneous massless limit $\mu_0^2 \to 0$ recovers the GR value of π indicated by the star symbol in the figure. As μ_0 increases, the repulsive force weakens, so the stellar radius shrinks. The grey dashed line in Fig. 3.8 indicates the 90 % value of the normalized radius in the case of GR, i.e., $0.9 \times \pi \approx 2.83$, which we shall regard as a tentative lower bound. If one demands the radius should be above 2.83, then μ_0^2 should be less than about 2.4. In this manner, at any rate, the spin-0 mass is constrained to be $< \mathcal{O}(1)$ if $\mu_2 \ll 1$.

We conclude this section with some remarks on the screening mechanism in the FP theory considering the case of the Sun. In the studies of the case with $n = 0, 1$, we found that the stellar radius R has a value of the order of the characteristic length scale ℓ as long as the graviton mass m is smaller than ℓ^{-1}, that is, $\mu \equiv m\,\ell \lesssim 1$. Main-sequence stars like the Sun, on the other hand, are usually modeled as a polytrope star with

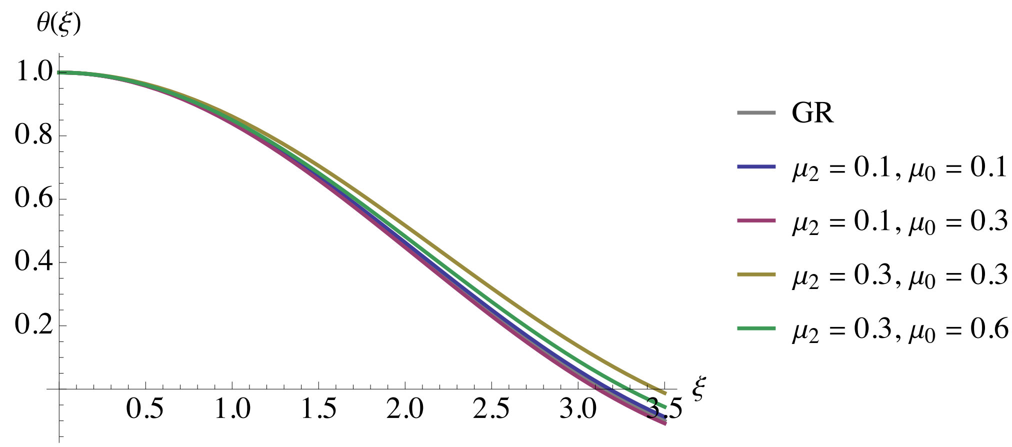

Figure 3.7: The profile function θ in non-FP generic linear MG and GR for the polytropic index $n = 1$.

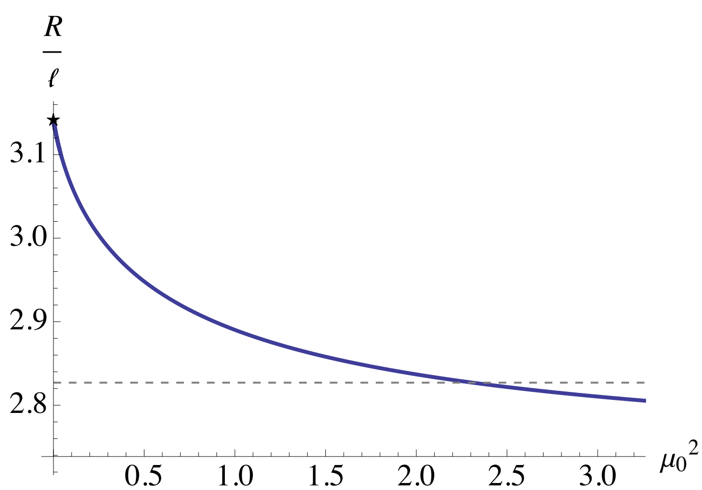

Figure 3.8: The stellar radius R/ℓ against the spin-0 graviton mass μ_0^2. The gray dashed line is the 90 % value of GR.

$n = 3$, which we did not consider in this thesis. Nevertheless, we can expect a similar estimate of R holds for $n = 3$. Now, if we take the dRGT theory as a specific example of the nonlinear completion of the FP theory, the screening radius $r_{\rm V} \equiv (r_{\rm S}/m^2)^{1/3}$ for the Sun with its Schwarzschild radius $r_{{\rm S},\odot} \approx 3\,{\rm km}$ and stellar radius $R_\odot \approx 7 \times 10^5\,{\rm km}$ is evaluated as

$$\frac{r_{{\rm V},\odot}}{R_\odot} = \frac{(r_{{\rm S},\odot}/R_\odot)^{1/3}}{(m\,R_\odot)^{2/3}} \sim 10^{-2}\,\mu^{-2/3}\,, \tag{3.75}$$

where we have assumed $R_\odot = \mathcal{O}(1) \times \ell$ in the last approximate equality. This implies that the whole Sun is enclosed within the screening radius if $\mu \lesssim 10^{-3}$ and the stellar calculations then has to be modified by some nonlinear screening effect. In Fig. 3.9, this situation corresponds to the shaded region, in which the stellar radius R/ℓ (blue) is below the screening radius $r_{\rm V}/\ell$ (red). Incidentally, an observational upper bound on μ may be set if one wishes the linear FP theory is valid over the solar system, where the inverse-square law has been observationally confirmed to hold. The inverse-square force by the Sun would not reach, e.g., Pluto's aphelion $r_{\rm P}$ at a distance of about 7×10^{12} meters unless the Yukawa length $r_{\rm Y} \equiv m^{-1}$ exceeds it, so the graviton mass m is constrained to be below $r_{\rm P}^{-1}$. This leads to an upper bound on the graviton mass as $m\,\ell \sim m\,R_\odot \lesssim R_\odot/r_{\rm P} \sim 10^{-4}$ represented by the vertical dashed line in Fig. 3.9.

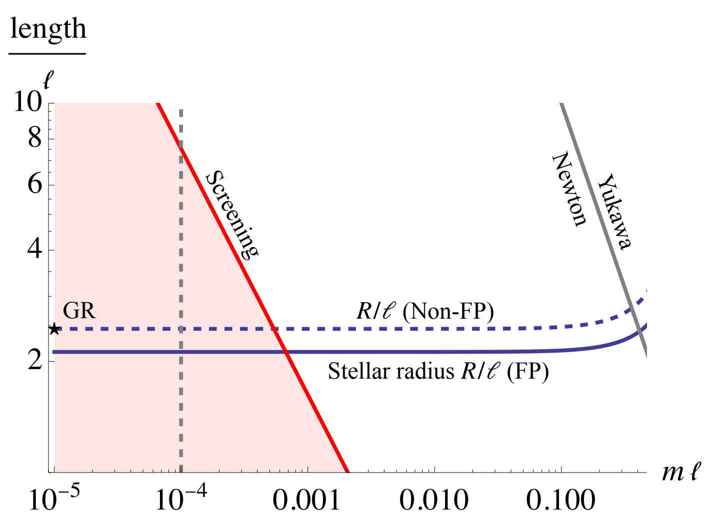

Figure 3.9: A schematic comparison of the stellar radius R/ℓ for the FP theory (solid blue) and a non-FP theory (dashed blue) with the screening radius $r_{\rm V}/\ell$ (red) for astronomical parameters compatible with the solar system. Some screening effect is expected to work in the shaded region in the case of the FP theory, the corresponding mass range being $m\,\ell \lesssim 10^{-3}$. The rightmost line (grey) represents the Yukawa length $1/(m\ell)$.

3.2 Non-relativistic stellar structure in higher-curvature gravity

Next, we study the structure of static spherical stars made up of a non-relativistic polytropic fluid in generic HCG. The method for MG can be partially applied because the massive gravitons also exist in HCG. We first formulate the modified LE equation for the stellar profile function in a gauge-invariant manner, finding it boils down to a sixth-order differential equation in the generic case of HCG, while it reduces to a fourth-order equation in two special cases, reflecting the number of additional massive gravitons arising in each theory. Moreover, as shown in the previous section, the existence of massive gravitons renders the nature of the boundary-value problem unlike the standard LE. As physical outcomes, we clarify how the stellar radius, mass, and Yukawa charges depend on the theory parameters and how these observables are mutually related. We obtain reasonable upper bounds on the Weyl-squared correction.

3.2.1 Massive graviton corrections to the Newtonian potential

In order to derive linear EOMs for the scalar modes in generic HCG, we begin with calculating the second-order perturbation of the gravitational action (1.9) plus a minimally coupled conservative matter. Scalar-type metric perturbations are characterised by four variables (3.1), and the gauge-invariant linear scalar perturbations Ψ and Φ are constructed as (3.2), see Appendix A. As the matter source, we adopt a perfect fluid with mass density ρ and pressure P coupled with scalar-type gravitational perturbations, as introduced in (3.5) with the conservation law (3.6). Thus, the scalar sector of the second-order perturbative action is then given by

$$\begin{aligned} S^{(2)}_{\mathrm{HCG}} = \frac{1}{16\pi G} \int \mathrm{d}^4x \left[-6\Phi\,\Box\Phi - 4\Phi\,\triangle(\Psi-\Phi) - \frac{4}{3}\,\alpha\,[\triangle(\Psi-\Phi)]^2 + \beta\,\left({}^{(1)}R\right)^2 \right] \\ + \int \mathrm{d}^4x\,\left[-\Psi\,\rho + 3\Phi\,P\right]\,, \end{aligned} \tag{3.76}$$

where

$${}^{(1)}R = -6\Box\Phi - 2\triangle(\Psi-\Phi) \tag{3.77}$$

is the Ricci scalar at the linear order. Varying the action with respect to Ψ and Φ, we obtain the field equations

$$\begin{aligned} \triangle\left[-2\Phi - \frac{4}{3}\alpha\,\triangle(\Psi-\Phi) - 2\beta\,{}^{(1)}R\right] = 8\pi\,G\,\rho\,, \\ {}^{(1)}R + 2\triangle\Phi + \frac{4}{3}\,\alpha\,\triangle^2(\Psi-\Phi) + 2\beta\,(-3\Box+\triangle)\,{}^{(1)}R = -24\pi\,G\,P\,. \end{aligned} \tag{3.78}$$

In the static case, the coupled equations (3.78) can be reduced and reorganised in a neat way. To see this, we introduce an alternative set of gauge-invariant variables

$$\begin{aligned}\Psi_2 &\equiv \frac{1}{2}\,\Psi - \frac{1}{2}\,\Phi\,,\\ \Psi_0 &\equiv -\Psi - 2\Phi\,,\end{aligned} \tag{3.79}$$

where, as will be clarified shortly, the subscripts refer to the spin s of the massive gravitons. For later convenience, we denote the inversion of the above as

$$\begin{aligned}\Psi &= -\alpha_2\,\Psi_2 - \alpha_0\,\Psi_0\,,\\ \Phi &= \beta_2\,\Psi_2 + \beta_0\,\Psi_0\end{aligned} \tag{3.80}$$

with

$$\alpha_2 = -\frac{4}{3}\,,\quad \alpha_0 = \frac{1}{3}\,,\quad \beta_2 = -\frac{2}{3}\,,\quad \beta_0 = -\frac{1}{3}\,. \tag{3.81}$$

Note that these coefficients are normalised so that $\alpha_2+\alpha_0 = -1$ and $\beta_2+\beta_0 = -1$. By taking linear combinations of (3.78) after reducing $\Box$ to $\triangle$, we obtain the decoupled equations for the gauge-invariant variables Ψ_s as

$$\begin{aligned}\triangle\Psi_2 - 2\alpha\,\triangle^2\Psi_2 &= 4\pi\,G\,\left(\rho + \frac{3}{2}\,P\right)\,,\\ \triangle\Psi_0 - 6\beta\,\triangle^2\Psi_0 &= 4\pi\,G\,\left(\rho - 3P\right)\,.\end{aligned} \tag{3.82}$$

One may already notice the similarity of the above two equations. Furthermore, with the non-relativistic approximation $|P| \ll \rho$ being the rest mass density of the fluid, these reduce to exactly the same form

$$\left(\triangle - m_s^2\right)\,\triangle\Psi_s = -4\pi\,G\,\rho\,m_s^2 \tag{3.83}$$

with $s = 2, 0$, where we have introduced the “mass” parameters $m_2^2 \equiv 1/(2\alpha)$ and $m_0^2 \equiv 1/(6\beta)$.

The key property inherent in these fourth-order equations is that the gauge-invariant potentials Ψ_s can be expressed as the difference of two variables

$$\Psi_s = \psi_s - \tilde{\psi}_s\,, \tag{3.84}$$

where ψ_s and $\tilde{\psi}_s$ are required to satisfy the following Poisson and Helmholtz equations,

$$\triangle\psi_s = 4\pi\,G\,\rho\,, \tag{3.85}$$

$$\left(\triangle - m_s^2\right)\,\tilde{\psi}_s = 4\pi\,G\,\rho\,, \tag{3.86}$$

respectively. An immediate implication of this decomposition being possible is that ψ_s is the massless graviton in GR while $\tilde{\psi}_s$ originates from the extra massive DOF with spin s arising in HCG. Indeed, these Helmholtz equations are reminiscent of the Klein–Gordon equations satisfied by the helicity-0 DOFs with the same masses identified in

2.2. Note, however, that there is a certain difference in the current definition of the variables from the vacuum case. See also [CS01, CSC01] for an analogous discussion for decomposing a higher-order equation into second-order equations in the presence of matter. Since the Poisson equations (3.85) for both s are identical, and so are their solutions, we may omit the subscript s for ψ_s: $\psi \equiv \psi_2 = \psi_0$. Then, the original gauge-invariant variables can be represented as

$$\Psi = \psi + \alpha_2\,\tilde{\psi}_2 + \alpha_0\,\tilde{\psi}_0\,, \tag{3.87}$$

$$\Phi = -\psi - \beta_2\,\tilde{\psi}_2 - \beta_0\,\tilde{\psi}_0 \tag{3.88}$$

with the coefficients given by (3.81).

It may be worth mentioning here the "massive" limit with $m_s^2 \to \infty$. Then, only terms in proportion to m_s^2 in the fourth-order equation (3.83) remain, reducing to a conventional Poisson equation for Ψ_s. Also, Eq. (3.86) in this limit imposes $\tilde{\psi}_s = 0$, and we have $\Psi_s = \psi$. Another interesting limit is the massless limit, $m_s^2 \to 0$, where one expects $\tilde{\psi}_s \to \psi$. However, a caution to be given here is that a vanishingly small value of the mass parameter m_s^2 corresponds to a huge absolute value of the expansion coefficient in the action (1.9) (α or β), for which the validity of the small-curvature approximation may be questioned.

Our next task is to express the gravitational potentials in a compatible way with adequate boundary conditions; On the one hand, asymptotic flatness requires $\lim_{|\vec{x}|\to\infty} \Psi = \lim_{|\vec{x}|\to\infty} \Phi = 0$, so the boundary conditions to be imposed on Ψ_s are $\lim_{|\vec{x}|\to\infty} \Psi_2 = \lim_{|\vec{x}|\to\infty} \Psi_0 = 0$. Also, all these perturbative variables must be bounded in the whole spatial domain. The same should also hold for ψ and $\tilde{\psi}_s$.

Hereafter we specialise in a spherically symmetric configuration, where every quantity becomes a function of the radial coordinate r and the Laplace operator reduces to $\triangle = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)$. We shall impose regularity at the stellar centre $r = 0$ and flatness at $r \to \infty$. The solution to the Poisson equation (3.85) satisfying these demands is the conventional Newtonian potential:

$$\psi_s = \psi = -G\int_r^\infty \mathrm{d}r'\,\frac{m(r')}{r'^2} \tag{3.89}$$

with the enclosed mass $m(r)$ being

$$m(r) \equiv 4\pi\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')\,. \tag{3.90}$$

If the matter is confined within a finite radius R like a star, $\rho(r \geq R) = 0$, then the enclosed mass is constant outside the stellar radius, $m(r \geq R) = m(R) \equiv M$, so the exterior potential is

$$\psi(r \geq R) = -\frac{G\,M}{r}\,. \tag{3.91}$$

The total mass M can be measured at remote distances using Kepler's law, for instance.

As shown in the previous section, the formal spherically symmetric solution to the Helmholtz equation (3.86) fulfilling the asymptotic boundary condition is given by

$$\tilde{\psi}_s = -\frac{G}{r}\,\left(\sigma_s(r)\,\cosh m_s\,r + (I_s - \chi_s(r))\,\sinh m_s\,r\right) \tag{3.92}$$

with

$$\sigma_s(r) \equiv \frac{4\pi}{m_s}\int_0^r \mathrm{d}r'\,r'\,\sinh m_s r'\,\rho(r')\,, \quad \chi_s(r) \equiv \frac{4\pi}{m_s}\int_0^r \mathrm{d}r'\,r'\,\cosh m_s r'\,\rho(r')\,, \tag{3.93}$$

where the integration constant I_s must be so tuned to kill the exponentially growing mode. When the matter distribution is confined within a finite stellar radius R, these functions have a constant value outside the star: $\sigma_s(r \geq R) = \sigma_s(r = R) \equiv \Sigma_s$, $\chi_s(r \geq R) = \chi_s(r = R) \equiv X_s$. Then, the integration constant is determined as

$$I_s \equiv X_s - \Sigma_s = \frac{4\pi}{m_s}\int_0^R \mathrm{d}r\,r\,\rho(r)\,\mathrm{e}^{-m_s r}\,. \tag{3.94}$$

Outside the star, it reduces to a Yukawa-type potential characterised by the graviton mass m_s and the total "charge" Σ_s as

$$\tilde{\psi}_s(r \geq R) = -\frac{G\,\Sigma_s\,\mathrm{e}^{-m_s r}}{r}\,. \tag{3.95}$$

As a consequence, the total gravitational potential Ψ within the distance m_s^{-1} from the star surface acquires Yukawa-type modifications.

Note that the enclosed "charges" σ_s and χ_s are both positive-semidefinite for positive mass density ρ as is the case for the enclosed mass m. They are divergent in the massive limit, $m_s \to \infty$, but the extra potential $\tilde{\psi}_s$ converges to 0 in this limit. On the other hand, in the massless limit, $m_s \to 0$, the enclosed charge σ_s tends to the same value as the enclosed mass as $\sigma_s = m + \mathcal{O}(m_s^2)$, while χ_s is divergent as $\chi_s = \mathcal{O}(m_s^{-1})$, and $\tilde{\psi}_s$ tends to the conventional potential ψ as expected.

Since these gravitational potentials have been so constructed to satisfy Eq. (3.85) or (3.86), their derivatives are necessarily continuous up to second order everywhere. On the other hand, the third derivatives involve $\mathrm{d}\rho/\mathrm{d}r$, and therefore, are discontinuous at the stellar surface.

3.2.2 Modified Lane–Emden equation

For the time being, we assume the graviton mass parameters m_s^2's are both bounded, leaving discussions of the limiting cases to the next subsection, in which either or both of m_s^2's are taken to infinity.

Given the gauge-invariant potential Ψ, the hydrostatic equilibrium condition in the Newtonian gauge reads

$$\frac{1}{\rho}\,\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,, \tag{3.96}$$

where the (outward) radial acceleration $-\frac{\mathrm{d}\Psi}{\mathrm{d}r}$ is obtained by differentiating (3.87) as

$$-\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\frac{G\,m}{r^2} - \alpha_2\,\frac{\mathrm{d}\tilde{\psi}_2}{\mathrm{d}r} - \alpha_0\,\frac{\mathrm{d}\tilde{\psi}_0}{\mathrm{d}r}\,. \tag{3.97}$$

One can deduce from the above expression that the sign of the coefficient α_s is crucial in determining the direction of the force exerted by the massive graviton of spin s, i.e., attractive or repulsive. Due to the reverse signs of $\alpha_2 = -4/3$ and $\alpha_0 = 1/3$, the two massive gravitons work in opposite ways.

By supplying an equation of state $P = P(\rho)$, the hydrostatic equilibrium condition (3.96) reduces to an equation for the single function ρ. At this stage, however, it is an integro-differential equation for ρ since the functions m and $\tilde{\psi}_s$ on the right-hand side involve radial integrations. In other words, in order to obtain an equivalent differential equation, one has to extract ρ from m and $\tilde{\psi}_s$. Let us first follow the standard procedure in GR, i.e., operate $\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}r^2$ on the both sides:

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\,\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi\,G\,\rho - \alpha_2\,\triangle\tilde{\psi}_2 - \alpha_0\,\triangle\tilde{\psi}_0\,. \tag{3.98}$$

Since we have assumed both m_s^2's are bounded, we can safely use the Helmholtz equations (3.86) for $\tilde{\psi}_2$ and $\tilde{\psi}_0$ to obtain

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\,\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -\alpha_2\,m_2^2\,\tilde{\psi}_2 - \alpha_0\,m_0^2\,\tilde{\psi}_0\,. \tag{3.99}$$

Here, a crucial difference from the case of GR is that this is still an integro-differential equation since $\tilde{\psi}_s$ involves integrals of ρ in a non-trivial manner. Another striking difference from GR is that the standard source term $-4\pi G\,\rho$ has been canceled out. These would appear to imply a thorough change in the structure of the governing equation compared to GR. We will see, however, one can take an appropriate limit of the above equation to find the correct expression in GR.

Hereafter we adopt the polytropic relation $P = K\,\rho^{1+\frac{1}{n}}$ as the EOS for the stellar matter, where K is the normalization constant and n is the constant called the polytropic index. As shown in the previous section, introduce a length scale

$$\ell \equiv \sqrt{\frac{(n+1)\,P_\mathrm{c}}{4\pi\,G\,\rho_c^2}}\,, \tag{3.100}$$

where the quantities with the subscript “c” denote the values at the stellar center, such as $\rho_\mathrm{c} \equiv \rho(r=0)$ and $P_\mathrm{c} \equiv P(r=0) = K\,\rho_\mathrm{c}^{1+\frac{1}{n}}$. The non-dimensional radial coordinate ξ and graviton mass parameters μ_s are defined by

$$\xi \equiv \frac{r}{\ell}\,, \quad \mu_0 \equiv m_0\,\ell\,, \quad \mu_2 \equiv m_2\,\ell\,. \tag{3.101}$$

The stellar mass density and pressure are conveniently replaced by the non-dimensional profile function $\theta(\xi)$ as

$$\rho = \rho_\mathrm{c}\, \theta^n\,, \quad P = P_\mathrm{c}\, \theta^{n+1}\,. \tag{3.102}$$

By definition, θ is normalized to unity at the center, $\theta_\mathrm{c} \equiv \theta(\xi = 0) = 1$. The value of ξ at the first positive zero of θ is denoted as ξ_R and is related to the stellar radius as $R = \xi_R\, \ell$. The laplacian operator is non-dimensionalized as $\triangle_\xi \equiv \frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi} \right) = \ell^2 \triangle$, and the hydrostatic equilibrium condition (3.99) for the polytropic EOS reads

$$\triangle_\xi \theta = -\frac{\alpha_0\, \mu_0^2\, \tilde{\psi}_0 + \alpha_2\, \mu_2^2\, \tilde{\psi}_2}{4\pi\, G\, \rho_\mathrm{c}\, \ell^2}\,. \tag{3.103}$$

Now, let us complete our task to obtain a differential equation for θ from the above expression. Since $\tilde{\psi}_s$ as given by (3.92) is a formal integration of the Helmholtz equation (3.86), it reduces in turn to the source term $4\pi\, G\, \rho$ by an operation of the Helmholtz operator $\triangle - m_s^2$. Note that the Helmholtz operators are commutable. Thus, operating $(\triangle_\xi - \mu_2^2)\,(\triangle_\xi - \mu_0^2)$ and using the Helmholtz equations for $\tilde{\psi}_2$ and $\tilde{\psi}_0$, we obtain a sixth-order differential equation for θ:

$$(\triangle_\xi - \mu_0^2)\,(\triangle_\xi - \mu_2^2)\, \triangle_\xi \theta + (\alpha_0\, \mu_0^2 + \alpha_2\, \mu_2^2)\, \triangle_\xi \theta^n + \mu_0^2\, \mu_2^2\, \theta^n = 0\,. \tag{3.104}$$

This is our master differential equation for the non-relativistic stellar structure in HCG. The fact that this is a sixth-order differential equation, as opposed to the second-order Lane–Emden equation in GR, is a direct consequence of the existence of three physical degrees of freedom in generic HCG. For a derivation of (3.104) via a different route, see [CS01].

While we assumed finiteness of the graviton masses m_s^2's in the derivation of the full equation (3.104), one would expect it contains various limiting cases corresponding to subclasses of HCG: when either of the mass parameters μ_2^2 or μ_0^2 is taken to infinity, i.e., either of α or β appearing in the HCG action (1.9) vanishes, the gravity theory should reduce to "$R + R^2$" ($\mu_2^2 \to \infty$) or "$R + C^2$" ($\mu_0^2 \to \infty$), and when both of μ_s^2 blow up, i.e., both of α and β go to zero, the theory should recover GR. In particular, in the last case, the standard LE equation (see below) should be reproduced somehow. As we confirm shortly, this is indeed the case: taking either or both of μ_s^2's to infinity leads to correct variants of the LE equation in the corresponding subclasses of gravitational theories, although one should be fairly cautious when taking such limits since they lead to changes in the order of differentiation, reflecting the changes in the number of dynamical DOFs.

As announced, when $\mu_0^2 \to \infty$ and $\mu_2^2 \to \infty$, the theory reduces to GR. In this limit, only terms in proportion to the product $\mu_0^2\, \mu_2^2$ in (3.104) should remain, reducing it to a second-order differential equation

$$\triangle_\xi \theta + \theta^n = 0\,. \tag{3.105}$$

This is nothing but the standard LE equation in the Newtonian limit of GR.

Next, when only $\mu_2^2 \to \infty$, the theory reduces to "$R + R^2$" gravity. Then, retaining the terms in proportion to μ_2^2 in (3.104) leads to a fourth-order equation

$$(\triangle_\xi - \mu_0^2)\, \triangle_\xi \theta - (\alpha_2\, \triangle_\xi + \mu_0^2)\, \theta^n = 0\,. \tag{3.106}$$

By recalling $\alpha_2 = -4/3$, one can straightforwardly show that the same equation derives by starting from the "$R + R^2$" action, i.e., (1.9) with $\alpha = 0$. This equation was first derived by Chen *et al.* [CSC01] in 2001, but their paper seems to have drawn little attention for long. For example, the authors of Ref. [CDLOS11, FDLCO14] have made attempts to solve an equivalent but relatively more intricate integro-differential equation.

The other limit $\mu_0^2 \to \infty$ corresponding to the "$R + C^2$" gravity reduces (3.104) to a similar fourth-order equation

$$(\triangle_\xi - \mu_2^2)\, \triangle_\xi \theta - (\alpha_0\, \triangle_\xi + \mu_2^2)\, \theta^n = 0\,. \tag{3.107}$$

To our best knowledge, this equation has not been obtained in the literature. A crucial qualitative difference from (3.106) is the positive sign of the coefficient $\alpha_0 = 1/3$, which, as remarked before, determines whether the extra gravitational force is attractive or repulsive.

For later convenience, we shall write the above fourth-order equations in the common form

$$(\triangle_\xi - \mu_s^2)\, \triangle_\xi \theta + ((1 + \alpha_s)\, \triangle_\xi - \mu_s^2)\, \theta^n = 0\,, \tag{3.108}$$

where we have used $\alpha_2 + \alpha_0 = -1$.

It might be useful to give a formula for the total stellar mass M in terms of derivatives of the profile function θ evaluated at the stellar surface. By virtue of (3.104), M can be written as

$$\begin{aligned} M &= 4\pi\, \ell^3\, \rho_{\rm c} \int_0^{\xi_R} \mathrm{d}\xi\, \xi^2\, \theta(\xi)^n \\ &= -\frac{4\pi\, \ell^3\, \rho_{\rm c}}{\mu_2^2\, \mu_0^2}\, \xi_R^2\, \frac{\mathrm{d}}{\mathrm{d}\xi} \left[(\alpha_2\, \mu_2^2 + \alpha_0\, \mu_0^2)\, \theta^n + (\triangle_\xi - \mu_2^2)\, (\triangle_\xi - \mu_0^2)\, \theta \right]_{\xi_R}\,. \end{aligned} \tag{3.109}$$

In the fourth order limit, i.e., either μ_0 or μ_2 goes to ∞, the expression reduces to

$$M = \frac{4\pi\, \ell^3\, \rho_{\rm c}}{\mu_s^2}\, \xi_R^2\, \frac{\mathrm{d}}{\mathrm{d}\xi} \left[(\triangle_\xi - \mu_s^2)\, \theta + (1 + \alpha_s) \theta^n \right]_{\xi_R}\,. \tag{3.110}$$

In the GR limit, it recovers the standard formula

$$M = -4\pi\, \ell^3\, \rho_{\rm c}\, \xi_R^2\, \left. \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right|_{\xi_R}\,. \tag{3.111}$$

Now, we move on to discuss boundary conditions for the profile function θ. Since the master equation (3.104) is sixth order in differentiation, there is a need for six independent conditions, a priori. Here, we take the same strategy as in MG in the sense that we aim to impose all the conditions on the values of θ and its derivatives at the stellar center.

Let us express the equilibrium condition (3.27) in terms of θ as

$$\theta' = -\frac{\ell^{-1}}{4\pi\, G\, \rho_c}\, \frac{\mathrm{d}\Psi}{\mathrm{d}r}\,, \tag{3.112}$$

where and hereafter, the prime denotes differentiation with respect to ξ. In order for this condition to hold at a given point, one has to ensure all the derivatives of both sides match at that position. Therefore, apart from the normalization $\theta_\mathrm{c} = \theta(0) = 1$, the derivatives at the stellar center $\theta_\mathrm{c}^{(n)} \equiv \frac{\mathrm{d}^n\theta}{\mathrm{d}\xi^n}(0)$ are restricted to be consistent with the behaviour of the potential $\Psi = \psi + \alpha_2\, \tilde{\psi}_2 + \alpha_0\, \tilde{\psi}_0$ there. Indeed, one finds the expansion of the potentials to a sufficient order as

$$\begin{aligned} G^{-1}\,\psi &= G^{-1}\,\phi(0) + \frac{2\pi}{3}\,\rho(0)\,r^2 + \frac{\pi}{3}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}(0)\,r^3 + \frac{\pi}{10}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}(0)\,r^4 + \mathcal{O}(r^5)\,, \\ G^{-1}\,\tilde{\psi}_s &= -m_s\,I_s + \left(\frac{2\pi}{3}\,\rho(0) - \frac{m_s^3\,I_s}{6}\right) r^2 + \frac{\pi}{3}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}(0)\,r^3 \\ &\quad + \left(\frac{\pi}{10}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}(0) + \frac{\pi\, m_s^2\,\rho(0)}{30} - \frac{m_s^5\,I_s}{120}\right) r^4 + \mathcal{O}(r^5)\,. \end{aligned} \tag{3.113}$$

From above, it is immediately seen that the radial acceleration at the stellar center vanishes, $\lim_{r\to 0} -\frac{\mathrm{d}\Psi}{\mathrm{d}r} = 0$, which implies $\theta'_\mathrm{c} = 0$ via (3.36). Hence, the following two boundary conditions have so far been obtained:

$$\theta_\mathrm{c} = 1\,, \quad \theta'_\mathrm{c} = 0\,. \tag{3.114}$$

These two conditions just suffice in the case of the second-order LE equation (3.47). In this case, all the higher derivatives at the stellar center are readily read off as

$$\theta''_\mathrm{c} = -\frac{1}{3}\,, \quad \theta'''_\mathrm{c} = 0\,, \quad \theta_\mathrm{c}^{(4)} = \frac{n}{5}\,, \quad \theta_\mathrm{c}^{(5)} = 0\,, \quad \cdots\,. \tag{3.115}$$

By contrast, four more boundary conditions than (3.38) are required in order to solve the full sixth-order differential equation (3.104). They are found from derivatives of (3.36) as

$$\begin{aligned} \theta''_\mathrm{c} &= -\frac{1}{4\pi\, G\, \rho_\mathrm{c}}\, \lim_{r\to 0} \frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} = \frac{1}{3} \sum_{s=0,2} \alpha_s\, \mu_s^2\, \iota_s\,, \\ \theta'''_\mathrm{c} &= -\frac{\ell}{4\pi\, G\, \rho_\mathrm{c}}\, \lim_{r\to 0} \frac{\mathrm{d}^3\Psi}{\mathrm{d}r^3} = 0\,, \\ \theta_\mathrm{c}^{(4)} &= -\frac{\ell^2}{4\pi\, G\, \rho_\mathrm{c}}\, \lim_{r\to 0} \frac{\mathrm{d}^4\Psi}{\mathrm{d}r^4} = -\frac{1}{5} \sum_{s=0,2} \alpha_s\, \mu_s^2 + \frac{1}{5} \sum_{s=0,2} \alpha_s\, \mu_s^4\, \iota_s\,, \\ \theta_\mathrm{c}^{(5)} &= -\frac{\ell^3}{4\pi\, G\, \rho_\mathrm{c}}\, \lim_{r\to 0} \frac{\mathrm{d}^5\Psi}{\mathrm{d}r^5} = 0\,, \end{aligned} \tag{3.116}$$

where we have normalized the constant I_s appearing in the massive potential ψ_s to

$$\iota_s \equiv \frac{\mu_s\, I_s}{4\pi\, \ell^3\, \rho_{\mathrm{c}}} = \int_0^{\xi_R} \mathrm{d}\xi\, \xi\, \theta(\xi)^n\, \mathrm{e}^{-\mu_s \xi} \tag{3.117}$$

and iteratively applied the conditions arising from lower derivatives to the higher ones. It is observed that some quantities at the stellar centre, θ''_{c} and $\theta^{(4)}_{\mathrm{c}}$, are now related to the stellar global quantity I_s, which was moreover introduced to guarantee flatness at spatial infinity. Of course, the values of I_s are as yet undetermined since the profile function θ has not been solved at this stage. Therefore, these expressions of the boundary values should be considered merely formal; The problem is not formulated as a simple initial value problem as in GR, but we will have to perform some matching procedures between the boundary values at the stellar center and the integrals of the solution over the whole domain. Note that the stellar radius ξ_R is simultaneously determined by this procedure.

We also derive the boundary conditions for the fourth-order case (3.108), where four boundary conditions are required in total, two of which are (3.38). The remaining two are

$$\begin{aligned} \theta''_{\mathrm{c}} &= -\frac{1}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} = -\frac{1}{3} - \frac{\alpha_s}{3}\, (1 - \mu_s^2\, \iota_s)\,, \\ \theta'''_{\mathrm{c}} &= -\frac{\ell}{4\pi\, G\, \rho_{\mathrm{c}}} \lim_{r\to 0} \frac{\mathrm{d}^3\Psi}{\mathrm{d}r^3} = 0\,. \end{aligned} \tag{3.118}$$

Note that θ''_{c} differs from that for the sixth-order equation.

For comparison, we would like to clarify the differences between past studies and ours with respect to the boundary conditions. In the full sixth-order case, Chen and Shao [CS01] state that they imposed continuity of (an analogue of) the gravitational potential and its derivatives at the stellar surface $r = R$. In our construction, on the contrary, their (dis)continuity is already inherent in Eqs. (3.89) and (3.92) as remarked at the end of the section. So, we have failed to find a direct analogue of our Eq. (3.116) in their paper. In the reduced fourth-order case, Chen *et al.* [CSC01] again mention continuity at the stellar surface, and we faced the same difficulty. Capozziello *et al.* [CDLOS11] (and Farinelli *et al.* [FDLCO14]) are less talkative about the boundary conditions except for (3.38).

3.2.3 Case I: Fourth-order limits for "$R + R^2$" or "$R + C^2$" theories

Next, we discuss the cases where either α or β is zero, for which the gravitational theory reduces to "$R + R^2$" ($\alpha = 0$) or "$R + C^2$" ($\beta = 0$), respectively. Then, the sixth-order master equation (3.104) reduces to the fourth-order equation (3.108).

For $n = 0$, Eq. (3.108) reduces to an inhomogeneous linear equation

$$(\triangle_\xi - \mu_s^2)\, \triangle_\xi \theta - \mu_s^2 = 0, \tag{3.119}$$

which depends on the mass parameter μ_s but not on the coefficient α_s. The solution, however, depends on the spin of the massive graviton, i.e., gravitational theory, as α_s appears in the boundary conditions (3.118). A procedure to find a general solution to equations of the type of (3.119) is summarized in Appendix F. Following it, we find the general solution

$$\theta = 1 - \frac{\xi^2}{6} + A\,\frac{\sinh \mu_s \xi}{\xi} + B\,\frac{\cosh \mu_s \xi}{\xi} + C + \frac{D}{\xi}\,, \tag{3.120}$$

where A, B, C, D are arbitrary constants of integration. It may be interesting to note that Eq. (3.119) admits the LE solution in GR as a particular solution, but, as we will see shortly, it cannot satisfy the boundary conditions at the stellar center.

The "LE" boundary conditions (3.38), $\theta_\mathrm{c} = 1$ and $\theta'_\mathrm{c} = 0$, are so restrictive than they appear that three constants can be determined as $B = D = 0$ and $C = -\mu_s\, A$, and we are left with the form with only one constant:

$$\theta = 1 - \frac{\xi^2}{6} + A\,\frac{\sinh \mu_s \xi - \mu_s\, \xi}{\xi}\,. \tag{3.121}$$

One of the two extra conditions (3.118), $\theta'''_\mathrm{c} = 0$, has been already satisfied at this stage, and the last condition on θ''_c plays the role in fixing A. Indeed, the second derivative evaluated at the centre $\xi = 0$ is

$$\theta''_\mathrm{c} = -\frac{1}{3} + \frac{A\,\mu_s^3}{3}\,. \tag{3.122}$$

Comparing this with (3.118), we find the relation between A and ι_s as

$$A = -\alpha_s\,(\mu_s^{-3} - \mu_s^{-1}\,\iota_s)\,. \tag{3.123}$$

As already discussed, the value of ι_s (3.117) in general involves an integral of the profile function over the stellar radius, which can never be determined before the solution is known. In this sense, the determination of A is subject to an appropriate matching procedure for the overall consistency. In the case of the polytropic index $n = 0$, however, constancy of the stellar density, $\rho = \rho_\mathrm{c}$, makes the matching procedure considerably simpler, as the integral ι_s does not explicitly depend on θ. Nonetheless, even in this case, evaluation of ι_s is not trivial since the undetermined stellar radius ξ_R appears in its expression as

$$\iota_s = \frac{1 - (\mu_s \xi_R + 1)\,\mathrm{e}^{-\mu_s \xi_R}}{\mu_s^2}\,. \tag{3.124}$$

At any rate, after eliminating A, the solution satisfying all the boundary conditions is obtained as

$$\theta = 1 - \frac{\xi^2}{6} - \alpha_s\,\frac{(\mu_s\,\xi_R + 1)\,\mathrm{e}^{-\mu_s \xi_R}}{\mu_s^2}\,\frac{\sinh \mu_s \xi - \mu_s\,\xi}{\mu_s\,\xi}\,. \tag{3.125}$$

This expression indicates there is always non-zero deviation from the $n = 0$ LE solution.

The above solution is still considered "formal" since the stellar radius ξ_R must be fixed by solving the consistency condition $\theta(\xi_R) = 0$, which cannot be done analytically even in this simplest case. In each case of the limiting theories, "$R + C^2$" or "$R + R^2$", given the corresponding value of α_s, the radius ξ_R becomes a function of the mass parameter μ_s. The solution in "$R + R^2$" gravity, with $s = 0$ and $\alpha_0 = 1/3$, is identical with Eq. (30) of Ref. [CDLOS11], while the solution in "$R+C^2$" gravity, with $s = 2$ and $\alpha_2 = -4/3$, seems to have been undiscovered in the literature. Once ξ_R is determined, the stellar mass M and charge Σ_s can be evaluated via their expressions for $n = 0$:

$$M = \frac{4\pi\, \ell^3\, \rho_c\, \xi_R^3}{3}\,, \quad \Sigma_s = 4\pi\, \ell^3\, \rho_c\, \frac{\mu_s\, \xi_R\, \cosh \mu_s \xi_R - \sinh \mu_s \xi_R}{\mu_s^3}\,. \tag{3.126}$$

For the sake of completeness, we also present the gravitational potential inside the star

$$\Psi(r \leq R) = -\frac{GM\,(3R^2 - r^2)}{2R^3} - \alpha_s\, \frac{G\Sigma_s}{r}\, \frac{m_s\, r - (1 + m_s\, R)\, \mathrm{e}^{-m_s R}\, \sinh m_s r}{m_s\, R\, \cosh m_s R - \sinh m_s R}\,. \tag{3.127}$$

Before showing numerical results, let us overview some analytical properties of the solution (3.125). Considering $\xi < \xi_R$, it reduces as $\theta \to \theta_0^{\mathrm{LE}} = 1 - \xi^2/6$ when the GR limit $\mu_s^2 \to \infty$ is taken. When the massless limit $\mu_s^2 \to 0$, at the opposite extreme, is taken, the profile function reduces as $\theta \to 1 - (1 + \alpha_s)\, \xi^2/6$, where the coefficient α_s plays a crucial role. In "$R + R^2$" gravity, $\alpha_0 = 1/3$, the modification to the profile merely amounts to a moderate shrinkage of the stellar radius ξ_R from the Lane–Emden value of $\sqrt{6}$ to $3/\sqrt{2}$, a decrease by a factor of $1/\sqrt{1 + \alpha_0} = \sqrt{3}/2$. This reflects the attractive nature of the spin-0 graviton in HCG. On the other hand, in "$R + C^2$" gravity, $\alpha_2 = -4/3$, the profile function no longer acquires a positive zero, failing to express a star with a finite radius. This is explained by the repulsive nature of the spin-2 graviton, whose strength exceeds that of the attractive force mediated by the ordinary massless graviton. In reality, however, as long as these gravitons have a finite mass, their effects are restricted within a finite range $r \lesssim m_s^{-1}$, and the stellar radius remains finite in any case.

Figure 3.10 shows typical examples of the solution (3.125) in each gravity theory with different mass parameters μ_s^2. We see that the stellar radius in "$R+R^2$" gravity is smaller than the value in GR, whereas larger in "$R+C^2$" gravity. This reveals that, as anticipated, the massive spin-0 graviton arising from the addition of R^2 term provides an attractive force, and the massive spin-2 from the C^2 term provides a repulsive force.

The modifications in the stellar global quantities, i.e., radius, mass, and charge, are depicted in Fig. 3.11. The top panel shows the normalized stellar radius $\xi_R = R/\ell$ against the mass parameter μ_s^2. When μ_s^2 is increased, R/ℓ quickly converges to the GR value $R/\ell = \sqrt{6}$ in both gravity cases, as expected. On the other hand, when μ_s^2 is decreased, the difference between the natures of the two theories signifies: in the massless limit of "$R + R^2$" gravity (red), the limiting value of the radius is finite, $R/\ell \to 3/\sqrt{2}$, while in the same limit of "$R + C^2$" gravity (blue), in contrast, it

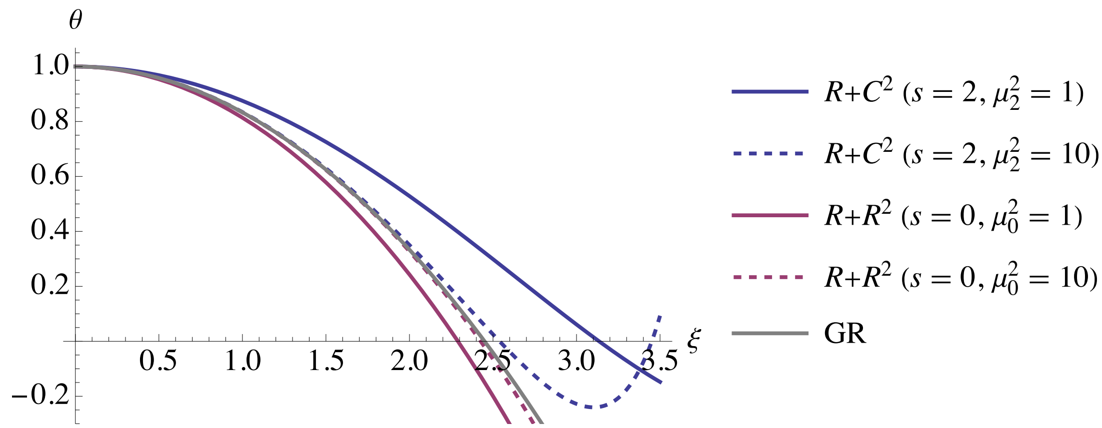

Figure 3.10: The profile function θ in "$R+C^2$" gravity (blue) and "$R+R^2$" gravity (red) for the polytropic index $n = 0$. The massive spin-2 (spin-0) graviton provides a repulsive (attractive) force. As the graviton mass μ_s^2 increases, the profile converges to the $n = 0$ LE solution in GR, $\theta_0^{\rm LE} = 1 - \xi^2/6$ (grey).

blows up. The bottom panel shows the dependences of the total mass M and the total charge Σ_s, both appropriately normalised, on the graviton mass μ_s^2. When μ_s^2 goes to infinity, M quickly converges to the values of GR, $M/(4\pi\,\ell^3\,\rho_{\rm c}) = (R/\ell)^3/3 = 2\sqrt{6}$ as expected, whereas Σ_s increases unboundedly. This is phenomenologically not problematic because the observable gravitational potential ψ_s securely converges to 0 in this limit. A similar behavior of the charge is observed in the study of neutron stars in [BS21]. In the massless limit of "$R + R^2$" gravity (red), the limiting values of the stellar mass and charge are $M \simeq \Sigma_s \to (\sqrt{3}/2)^3\, M_0^{\rm LE}$, while they both grow unboundedly in "$R+C^2$" gravity (blue) in accordance with the increase in the radius.

Finally, Fig. 3.12 shows the relations between M and R (solid) and Σ_s and R (dashed), where the variables are appropriately normalized. The M–R curve reproduces the trivial relation for $n = 0$: $M = \frac{4\pi}{3}\,R^3\,\rho_{\rm c}$. Σ_s diverges when the radius approaches to the GR value $R/\ell = \sqrt{6}$ as expected from the bottom panel of Fig. 3.11. The fact that the value of Σ_s is comparable to M as long as $\mu_s^2 \lesssim \mathcal{O}(1)$ implies that there is a significant modification to the Newton law within distances shorter than m_s^{-1} from the stellar surface.

The radius R of a polytrope star is in proportion to the length scale ℓ related to the physical conditions at the stellar centre, such as the central density $\rho_{\rm c}$ and pressure $P_{\rm c}$, see (3.31). In this sense, there is a degeneracy between these physical conditions and gravity, including any possible modifications to GR, in measurements of stellar radius, for which it is incapable of quantifying the effects of the massive gravitons independently of the properties and individual conditions of stellar matter. Nevertheless, we here argue that, without going into direct comparisons with observational data, huge deviations in radius from the GR value with the same physical condition, as represented

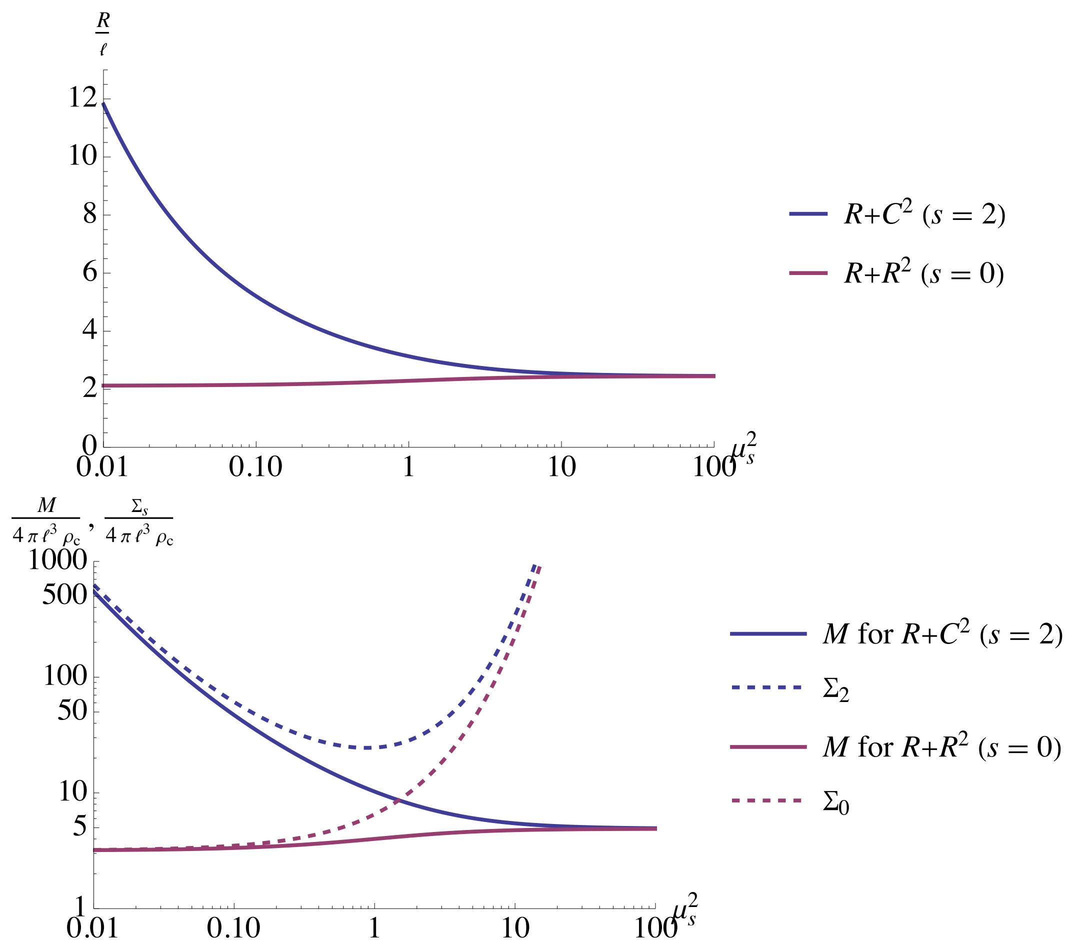

Figure 3.11: The dependences of the stellar radius R (top), mass M (bottom, solid), and charge Σ_s (bottom, dashed), each appropriately normalized, on the graviton mass μ_s^2 for the polytropic index $n = 0$. The stellar radius and mass are larger in "$R + C^2$" gravity (blue) and smaller in "$R + R^2$" gravity (red), but they both approach the GR values of $\sqrt{6}$ and $2\sqrt{6}$, respectively, as the graviton mass μ_s^2 increases. The charge Σ_s blows up in the GR limit, but the potential ψ_s then tends to 0. In the massless limit of "$R + R^2$" gravity, the limiting values are: $R/\ell \to 3/\sqrt{2}$ and $M \simeq \Sigma_s \to (\sqrt{3}/2)^3\, M_0^{\mathrm{LE}}$. In "$R + C^2$" gravity, these quantities grow unboundedly as the spin-2 graviton mass μ_2^2 approaches to 0.

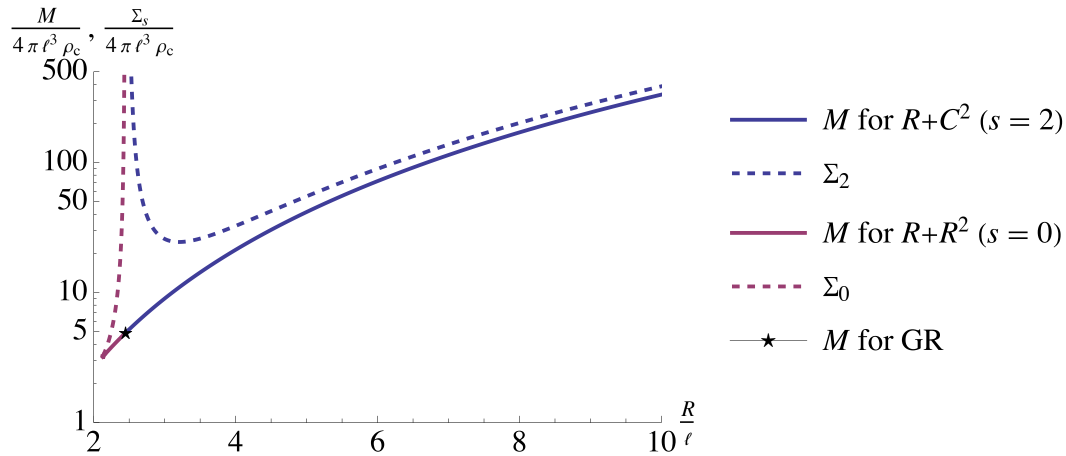

Figure 3.12: The normalized mass $M/(4\pi\,\ell^3\,\rho_c)$ (solid) and the normalized charge $\Sigma_s/(4\pi\,\ell^3\,\rho_c)$ (dashed) versus the normalized radius R/ℓ for the polytropic index $n = 0$. The mass M and radius R in both gravity cases approach the values in GR (black star) as the graviton mass μ_s^2 increases.

by ℓ, should be disfavored. For instance, in "$R+C^2$" gravity, we may consider a radius R which is twice as large as the GR value, $R_{\mathrm{LE}} = \sqrt{6}\,\ell$, to be unlikely enough. In order to have the ratio $R/R_{\mathrm{LE}} \leq 2$, the spin-2 graviton mass must exceed 0.12, which can be interpreted as an upper bound on the parameter $\sqrt{\alpha}/\ell = 1/\sqrt{2\mu_2^2} < 2.0$.

For the polytropic index $n = 1$, the fourth-order equation (3.108) reduces to a linear homogeneous equation

$$\triangle_\xi^2\theta + (1 + \alpha_s - \mu_s^2)\,\triangle_\xi\theta - \mu_s^2\,\theta = 0\,. \tag{3.128}$$

In order to find the solution, we "factorise" the differential operator to rewrite the above equation as

$$(\triangle_\xi + \lambda_+^2)\,(\triangle_\xi - \lambda_-^2)\,\theta = 0 \tag{3.129}$$

with the "roots"

$$\lambda_\pm = \sqrt{\frac{\sqrt{(1 + \alpha_s - \mu_s^2)^2 + 4\mu_s^2} \pm (1 + \alpha_s - \mu_s^2)}{2}}\,. \tag{3.130}$$

It is obvious from the expression that $\lambda_\pm$ are positive real irrespective of α_s and μ_s^2 (as long as $\mu_s^2 > 0$). The general solution is a superposition of the fundamental solutions for the (homogeneous) Helmholtz equations with eigenvalues $-\lambda_+^2$ and λ_-^2, hence

$$\theta = A_+\,\frac{\sin\lambda_+\,\xi}{\xi} + B_+\,\frac{\cos\lambda_+\,\xi}{\xi} + A_-\,\frac{\sinh\lambda_-\,\xi}{\xi} + B_-\,\frac{\cosh\lambda_-\,\xi}{\xi}\,. \tag{3.131}$$

Let us determine the integration constants one by one. By imposing the LE boundary conditions (3.38), $\theta_c = 1$ and $\theta_c' = 0$, three constants are fixed as $B_+ = B_- = 0$

and $A_- = (1 - A_+\lambda_+)/\lambda_-$. Thus we find

$$\theta = \frac{\sinh\lambda_-\,\xi + A_+\,(\lambda_-\,\sin\lambda_+\,\xi - \lambda_+\,\sinh\lambda_-\,\xi)}{\lambda_-\,\xi}\,. \tag{3.132}$$

As in the case of $n = 0$, one more boundary condition $\theta'''_{\rm c} = 0$, being one of the remaining two in Eq. (3.118), is already satisfied at this stage. On the other hand, the yet unused second derivative $\theta''_{\rm c}$ is given in terms of the constant A_+ as

$$\theta''_{\rm c} = \frac{\lambda_-^2 - A_+\,\lambda_+\,(\lambda_+^2 + \lambda_-^2)}{3}\,. \tag{3.133}$$

From (3.118), we find the relation between A_+ and the stellar global quantity ι_s as

$$A_+ = \frac{\lambda_-^2 + 1 + \alpha_s\,(1 - \mu_s^2\,\iota_s)}{\lambda_+\,(\lambda_+^2 + \lambda_-^2)}\,. \tag{3.134}$$

Here, unlike the $n = 0$ case, ι_s involves integration of θ, Eq. (3.132), so it necessarily contains the undetermined integration constant A_+ . Such an intermediate expression for ι_s looks somewhat tedious, but has a simple linear (since $n = 1$) dependence on A_+ :

$$\begin{aligned}\iota_s = {} & \frac{\lambda_- - {\rm e}^{-\mu\xi_R}\,(\lambda_-\,\cosh\lambda_-\xi_R + \mu\,\sinh\lambda_-\xi_R)}{\lambda_-\,(\mu_s^2 - \lambda_-^2)} \\ & + \frac{A_+}{\lambda_-\,(\mu_s^2 - \lambda_-^2)\,(\mu_s^2 + \lambda_+^2)}\Big\{-\lambda_+\,\lambda_-\,(\lambda_+^2 + \lambda_-^2) \\ & \quad + {\rm e}^{-\mu\xi_R}\left[\lambda_+\,(\mu_s^2 + \lambda_+^2)\,(\lambda_-\,\cosh\lambda_-\xi_R + \mu_s\,\sinh\lambda_-\xi_R)\right. \\ & \quad \left. + \lambda_-\,(\lambda_-^2 - \mu_s^2)\,(\lambda_+\,\cos\lambda_+\xi_R + \mu_s\,\sin\lambda_+\xi_R)\right]\Big\}\,.\end{aligned} \tag{3.135}$$

Substituting this into (3.134) gives back a linear equation for A_+ , which can be explicitly solved as

$$A_+ = \left[\lambda_+ + \frac{\lambda_-\,(\lambda_-^2 - \mu_s^2)\,(\lambda_+\,\cos\lambda_+\xi_R + \mu_s\,\sin\lambda_+\xi_R)}{(\mu_s^2 + \lambda_+^2)\,(\lambda_-\,\cosh\lambda_-\xi_R + \mu_s\,\sinh\lambda_-\xi_R)}\right]^{-1}\,, \tag{3.136}$$

where we have used the characteristic equations for $\lambda_\pm$ to reduce the expression. In this way, we have found the profile function θ for $n = 1$ satisfying all the boundary conditions. One should recall here that this expression is "formal" because it involves the stellar radius ξ_R , which can only be found numerically by solving the consistency condition $\theta(\xi_R) = 0$. Nonetheless, the expression ceases to contain ξ_R in the massive and massless limits. In both gravity theories, the massive limit, $\mu_s^2 \to \infty$, is the $n = 1$ LE solution in GR, $\theta \to \theta_1^{\rm LE} = \xi^{-1}\,\sin\xi$. On the other hand, in the massless limit, it reduces as $\theta \to \sin(\sqrt{1 + \alpha_s}\,\xi)/(\sqrt{1 + \alpha_s}\,\xi)$. This represents a rescaled LE solution for "$R + R^2$" gravity with $\alpha_0 = 1/3$, whereas it no longer has a finite radius for "$R + C^2$" gravity with $\alpha_2 = -4/3$.

Figure 3.13 compares the $n = 1$ solutions for "$R + C^2$" (blue) and "$R + R^2$" (red) theories with different values of μ_s^2 with the $n = 1$ LE solution (grey). As in the $n = 0$ case, in "$R + R^2$" ("$R + C^2$") gravity, the radius becomes smaller (larger) than GR. In the GR limit, $\mu_s^2 \to \infty$, they all reduce to the LE solution.

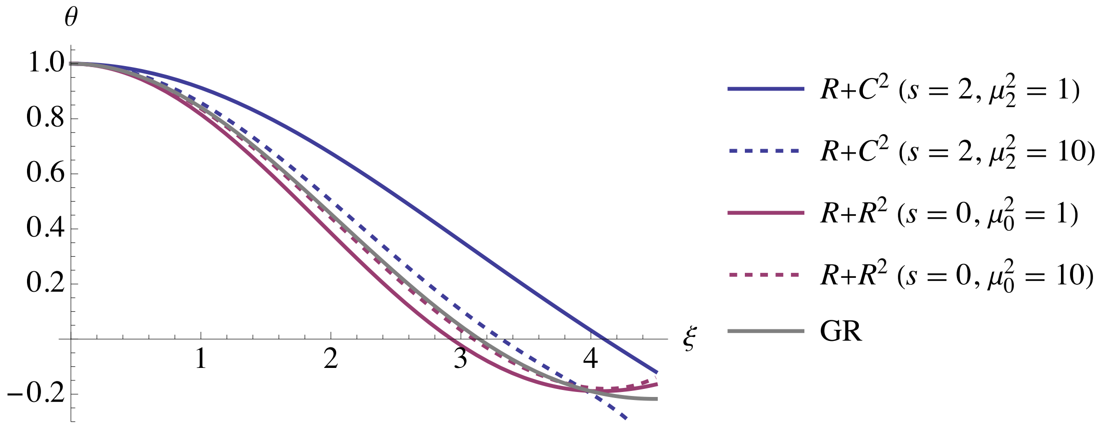

Figure 3.13: The solutions in "$R + C^2$" gravity (blue) and "$R + R^2$" gravity (red) together with the LE solution in GR (grey) for the polytropic index $n = 1$.

The modification to the stellar radius, mass, and charge is shown in Fig. 3.14, where we plot the values of the normalized radius $\xi_R = R/\ell$, the normalized mass $M/(4\pi\, \ell^3\, \rho_c)$ (solid), and the normalized charge $\Sigma_s/(4\pi\, \ell^3\, \rho_c)$ (dashed) against the normalized mass parameter μ_s for "$R + C^2$" (blue) and "$R + R^2$" (red). Both curves converge to the GR value of π as μ_s^2 blows up. The massless limit for "$R + R^2$" gravity is $R/\ell \to \sqrt{3}\,\pi/2$, whereas R/ℓ increases unboundedly in "$R + C^2$" gravity as μ_2^2 approaches to 0. The asymptotic value of M in the massive limit, $\mu_s^2 \to \infty$, is the GR value of π. In the massless limit for "$R + R^2$" gravity (red), M and Σ_0 converge to the rescaled LE mass $M \simeq \Sigma_0 \to (\sqrt{3}/2)^3\, M_1^{\text{LE}}$. In contrast, both M and Σ_2 grow unboundedly for "$R + C^2$" gravity (blue) as $\mu_2^2 \to 0$. These behaviours can be understood in a much similar fashion to the $n = 0$ case.

Unlike the $n = 0$ case, here again, the stellar mass M and charge Σ_s explicitly depend on θ, and hence one has to express them by substituting (3.132) together with (3.136) into (3.90) and (3.19), respectively, and evaluating them at the surface $r = R$ using the numerical value of ξ_R for each choice of the mass parameter μ_s^2 . Fortunately, in the current case, the integrations can be carried out analytically, giving explicit expressions for the mass and charge:

$$M = \frac{4\pi\, \ell^3\, \rho_c}{\lambda_+^2\, \lambda_-^3} \left[-A_+ \lambda_-^3 (\lambda_+ \xi_R \cos \lambda_+ \xi_R - \sin \lambda_+ \xi_R) - \lambda_+^2 (A_+ \lambda_+ - 1)(\lambda_- \xi_R \cosh \lambda_- \xi_R - \sinh \lambda_- \xi_R) \right] ,$$

$$\Sigma_s = \frac{4\pi\, \ell^3\, \rho_c\, e^{\mu_s \xi_R} \, (\lambda_- \cosh \lambda_- \xi_R \sin \lambda_+ \xi_R - \lambda_+ \sinh \lambda_- \xi_R \cos \lambda_+ \xi_R)}{\lambda_+ \, (\lambda_+^2 + \mu_s^2)\, (\lambda_- \cosh \lambda_- \xi_R + \mu_s \sinh \lambda_- \xi_R) + \lambda_- \, (\lambda_-^2 - \mu_s^2)\, (\lambda_+ \cos \lambda_+ \xi_R + \mu_s \sin \lambda_+ \xi_R)} , \tag{3.137}$$

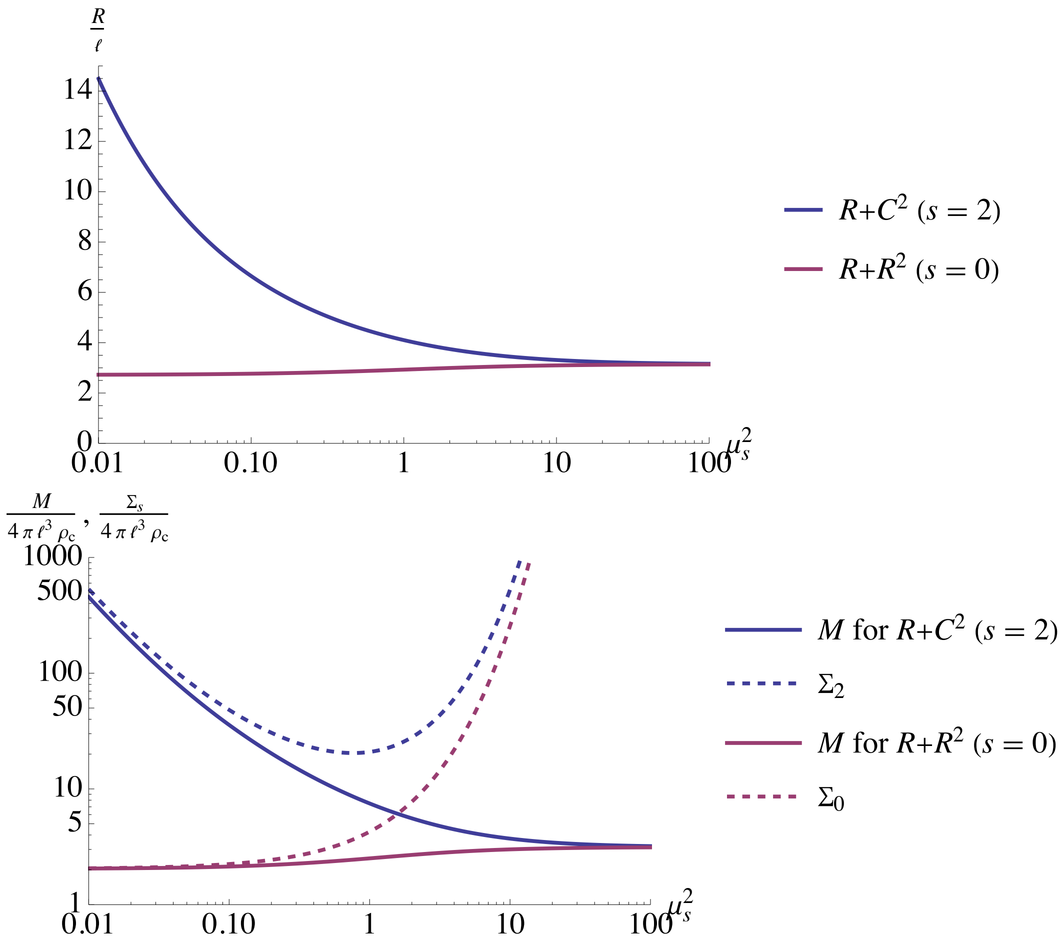

Figure 3.14: The μ_s dependences of the stellar radius R/ℓ, the stellar mass M (solid) and charge Σ_s (dashed) for the polytropic index $n = 1$.

where we have taken advantage of maintaining A_+ in the expression of M. Similarly, one can also evaluate the gravitational potential $\Psi = \phi + \alpha_s \psi_s$ inside a star by manipulating (3.89) and (3.92), which also have analytical expressions in this case:

$$
\begin{aligned}
&\psi(r \le R) \\
&= -\frac{GM}{R}\left[1 + \frac{R-r}{r}\,\frac{(A_+\lambda_+ - 1)\,\lambda_+^2\,(\lambda_-\,\xi\,\cosh\lambda_-\xi - \sinh\lambda_-\xi) + A_+\,\lambda_-^3\,(\lambda_+\,\xi\,\cos\lambda_+\xi - \sin\lambda_+\xi)}{(A_+\lambda_+ - 1)\,\lambda_+^2\,(\lambda_-\,\xi_R\,\cosh\lambda_-\xi_R - \sinh\lambda_-\xi_R) + A_+\,\lambda_-^3\,(\lambda_+\,\xi\,\cos\lambda_+\xi_R - \sin\lambda_+\xi_R)}\right], \\
&\tilde{\psi}_s(r \le R) \\
&= -\frac{\Sigma_s\,\mathrm{e}^{-m_s R}}{r}\,\frac{(\lambda_+\,\cos\lambda_+\xi_R + \mu_s\,\sin\lambda_+\xi_R)\,\sinh\lambda_-\xi - (\lambda_-\,\cosh\lambda_-\xi_R + \mu_s\,\sinh\lambda_-\xi_R)\,\sin\lambda_+\xi}{\lambda_+\,\cos\lambda_+\xi_R\,\sinh\lambda_-\xi_R - \lambda_-\,\sin\lambda_+\xi_R\,\cosh\lambda_-\xi_R}\,.
\end{aligned}
\tag{3.138}
$$

Finally, Fig. 3.15 shows the M–R (solid) and Σ_s–R (dashed) relations, where the quantities are appropriately normalized. Similar trends to the $n = 0$ case show up in each diagram.

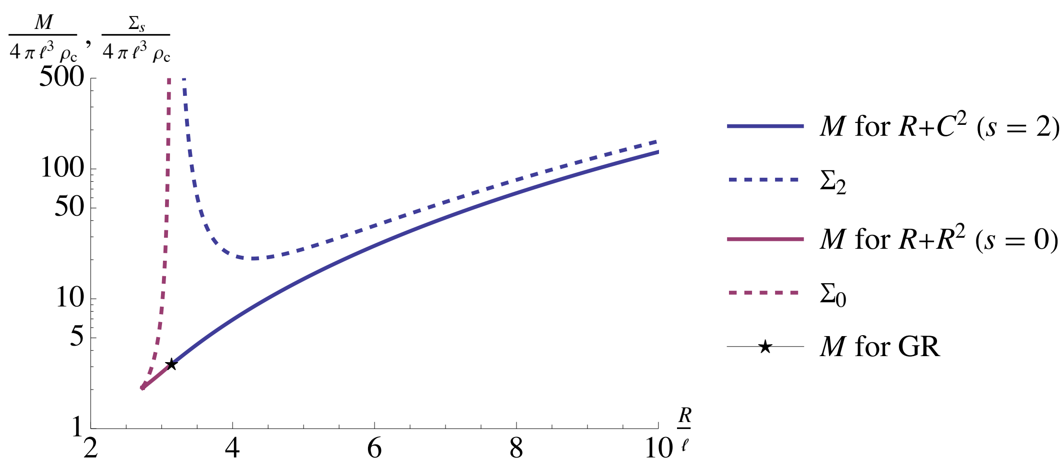

Figure 3.15: The M–R (solid) and Σ_s–R (dashed) relations for the polytropic index $n = 1$.

In the case of "$R + C^2$" gravity, from an argument that $n = 1$ polytrope stars should not acquire a radius and mass much larger than those in GR, we can place a reasonable lower bound on the spin-2 graviton mass. For instance, in order to have $R/R_{\rm LE} < 2$, we obtain $\mu_2^2 > 0.12$. This can be converted into an upper bound on the theory parameter α in the "$R + C^2$" action: $\sqrt{\alpha} < 2.0\,\ell$.

3.2.4 Case II: Generic higher-curvature theories

Now, we would like to tackle the full master equation (3.104) in generic HCG. The co-existence of the two extra DOFs of spin-2 and -0 renders the analysis considerably messy, but most features of the solutions will be reasonably understood as collective contributions from the spin-2 and -0 DOFs. Indeed, in most occasions treated in this paper, either of the DOFs dominates, and the obtained solution, therefore, mimics some of those that appeared in the previous fourth-order cases.

In the case of $n = 0$, the master equation (3.104) reduces to an inhomogeneous linear equation

$$(\triangle_\xi - \mu_0^2)\,(\triangle_\xi - \mu_2^2)\,\triangle_\xi\theta + \mu_0^2\,\mu_2^2 = 0\,. \tag{3.139}$$

The general solutions with six arbitrary constants are found following the procedure in Appendix F. For non-degenerate eigenvalues $\mu_0 \neq \mu_2$, it is

$$\theta = 1 - \frac{\xi^2}{6} + A_0\,\frac{\sinh\mu_0\,\xi}{\xi} + B_0\,\frac{\cosh\mu_0\,\xi}{\xi} + A_2\,\frac{\sinh\mu_2\,\xi}{\xi} + B_2\,\frac{\cosh\mu_2\,\xi}{\xi} + C + \frac{D}{\xi}\,. \tag{3.140}$$

Although we are not so much concerned with the degenerate case $\mu_0 = \mu_2 \equiv \mu$, the general solution in such a special case is

$$\theta = 1 - \frac{\xi^2}{6} + A\,\frac{\sinh\mu\,\xi}{\xi} + B\,\frac{\cosh\mu\,\xi}{\xi} + \tilde{A}\,\sinh\mu\,\xi + \tilde{B}\,\cosh\mu\,\xi + C + \frac{D}{\xi}\,. \tag{3.141}$$

As in the fourth-order case, the $n = 0$ LE solution is again a particular solution but it will turn out not to satisfy the boundary conditions.

We shall concentrate on the non-degenerate case (3.140). This time we are to impose six boundary conditions in total as given by (3.38) and (3.116) By imposing first the LE boundary condition (3.38), we can fix three constants as $C = -\mu_0\,A_0 - \mu_2\,A_2$ and $D = B_0 = B_2 = 0$, and get a reduced form of the solution

$$\theta = 1 - \frac{\xi^2}{6} + A_0\,\frac{\sinh\mu_0\,\xi - \mu_0\,\xi}{\xi} + A_2\,\frac{\sinh\mu_2\,\xi - \mu_2\,\xi}{\xi}\,. \tag{3.142}$$

At this point, the above solution already satisfies two of the four extra conditions in (3.116), $\theta_c''' = \theta_c^{(5)} = 0$, and we are left with the requirements for θ_c'' and $\theta_c^{(4)}$. These derivatives are written in terms of the remaining constants A_0 and A_2 as

$$\theta_c'' = -\frac{1 - A_0\,\mu_0^3 - A_2\,\mu_2^3}{3}\,, \quad \theta_c^{(4)} = \frac{A_0\,\mu_0^5 + A_2\,\mu_2^5}{5}\,. \tag{3.143}$$

Then from (3.116), A_0 and A_2 are related to the stellar integrals ι_0 and ι_2 as

$$A_0 = -\alpha_0\,(\mu_0^{-3} - \mu_0^{-1}\,\iota_0)\,, \quad A_2 = -\alpha_2\,(\mu_2^{-3} - \mu_2^{-1}\,\iota_2)\,, \tag{3.144}$$

respectively. Thanks to the constancy of ρ for the polytropic index $n = 0$, ι_s are found the same, being independent of A_0 or A_2, as in the fourth-order case,

$$\iota_s = \frac{1 - (\mu_s\,\xi_R + 1)\,\mathrm{e}^{-\mu_s\xi_R}}{\mu_s^2}\,, \tag{3.145}$$

thereby fixing the remaining constants as

$$A_0 = -\alpha_0\,\frac{(\mu_0\,\xi_R + 1)\,\mathrm{e}^{-\mu_0\xi_R}}{\mu_0^3}\,, \quad A_2 = -\alpha_2\,\frac{(\mu_2\,\xi_R + 1)\,\mathrm{e}^{-\mu_2\xi_R}}{\mu_2^3}\,. \tag{3.146}$$

Therefore, we get the solution satisfying all the boundary conditions:

$$\theta = 1 - \frac{\xi^2}{6} + \alpha_0 \, \frac{(\mu_0\,\xi_R + 1)\,\mathrm{e}^{-\mu_0 \xi_R}}{\mu_0^3} \, \frac{\sinh \mu_0\,\xi - \mu_0\,\xi}{\xi} + \alpha_2 \, \frac{(\mu_2\,\xi_R + 1)\,\mathrm{e}^{-\mu_2 \xi_R}}{\mu_2^3} \, \frac{\sinh \mu_2\,\xi - \mu_2\,\xi}{\xi} \,. \tag{3.147}$$

The remaining parameter ξ_R is numerically determined by solving the consistency condition $\theta(\xi_R) = 0$ for given mass parameters μ_2 and μ_0. After all, it is clear from the above expression that the gravitational effects from individual DOFs are separated and purely additive in this case. Moreover, due to the specialness of the $n = 0$ EOS, where the mass density ρ is constant, the analytical expressions of the total stellar mass M and two charges Σ_2 and Σ_0 are identical with the ones in the fourth-order case:

$$M = \frac{4\pi\,\ell^3\,\rho_\mathrm{c}\,\xi_R^3}{3}\,, \quad \Sigma_s = 4\pi\,\ell^3\,\rho_\mathrm{c}\,\frac{\mu_s\,\xi_R\,\cosh \mu_s \xi_R - \sinh \mu_s \xi_R}{\mu_s^3}\,. \tag{3.148}$$

The gravitational potential inside a star is then found as

$$\Psi(r \leq R) = -\frac{GM\,(3R^2 - r^2)}{2R^3} - \sum_{s=0,2} \alpha_s \, \frac{G\Sigma_s}{r} \, \frac{m_s\,r - (1 + m_s\,R)\,\mathrm{e}^{-m_s R}\,\sinh m_s r}{m_s\,R\,\cosh m_s R - \sinh m_s R}\,. \tag{3.149}$$

Various massive and massless limits of the solution (3.147) can be understood from the properties of the fourth-order $n = 0$ solutions. Among others, the spurious double massless limit $\theta \to 1$ seems to reflect some profound aspect of the full purely quadratic gravity.

Some examples of the solution are shown in Fig. 3.16 together with the $n = 0$ LE solution $\theta_0^{\mathrm{LE}} = 1 - \xi^2/6$ (grey). The general tendency is that the lower the spin-2 (spin-0) graviton mass is, the more effectively the repulsive (attractive) force works. Quantitatively, though, there is a significant difference between these two graviton effects, which shows up representatively in the case of $\mu_2^2 = \mu_0^2 = 1$ (green): repulsion by spin-2 graviton is much more noticeable than attraction by spin-0, which was observed as well in the study of neutron stars [BS21]. This could be understood as a direct consequence of the ratio of the coefficients being $\alpha_2/\alpha_0 = -4$: in the case of comparable graviton masses, $\mu_2^2 \approx \mu_0^2$, the influence coming from the massive spin-2 graviton is four-fold stronger in magnitude compared to spin-0. Moreover, as the spin-2 mass μ_2^2 decreases below $\mathcal{O}(1)$, the repulsive force can even overcome the attractive force of the massless graviton so that a star can puff up unboundedly, whereas the spin-0 attractive force can merely strengthen gravity by at most a factor of a few, resulting in a bounded shrinkage of a star.

By solving the consistency condition $\theta(\xi_R) = 0$ numerically, we have the dependence of the normalised radius R/ℓ on the mass parameters, some examples being plotted in Fig. 3.17. In the figure, either of the two masses are varied while the rest is fixed. The massive limit in this case corresponds to either of the reduced theories, "$R + R^2$" or "$R + C^2$", so the radius does not converge to the GR value of $\sqrt{6}$; It is only realised when the both masses are taken to infinity. The radius remains finite in the massless limit

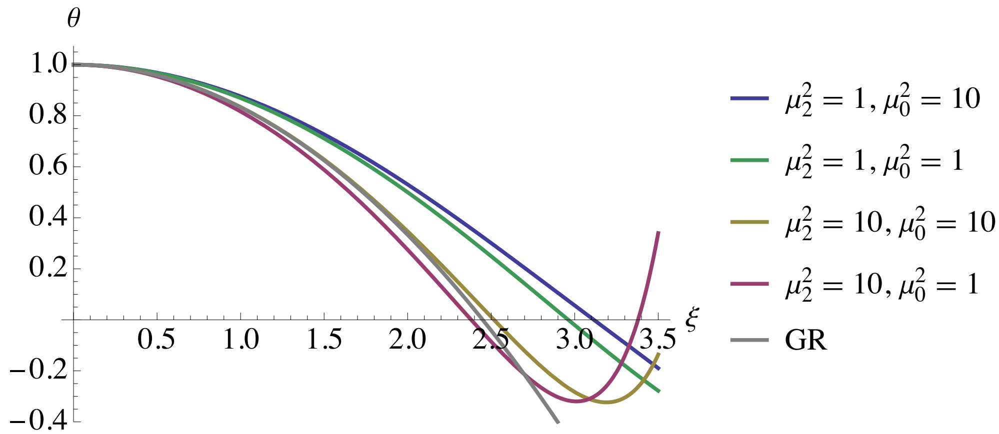

Figure 3.16: Examples of the profile functions for the polytropic index $n = 0$ compared with the LE solution (grey).

of spin-0 (red and yellow), whereas it blows up as the spin-2 graviton mass approaches to 0 (blue and green).

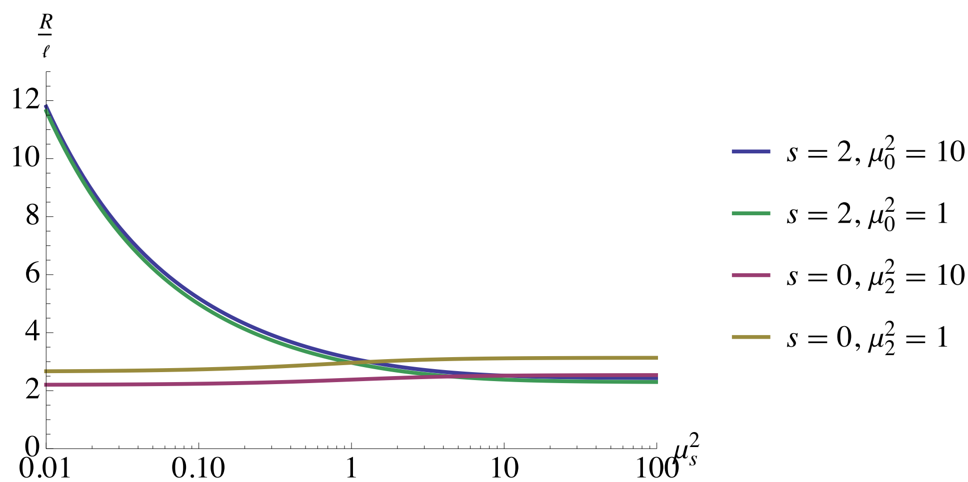

Figure 3.17: The μ_s^2 dependences of the normalized radius R/ℓ for the polytropic index $n = 0$.

Figure 3.18 shows typical dependences on the graviton mass μ_s of the stellar mass M and charges Σ_s, where they are appropriately normalized. In the top (bottom) panel, μ_2^2 (μ_0^2) is varied while μ_0^2 (μ_2^2) is fixed to 1. The behavior of M can be understood in a similar way to the radius. As for the charges, when the spin-s graviton mass μ_s^2 goes to infinity, the corresponding spin-s charge Σ_s diverges, while the other charge $\Sigma_{s'}$ ($s' \neq s$) remains finite. These divergences do not matter because the potential ψ_s vanishes in the massive limits. On the other hand, in the massless limit of the spin-s

graviton, the spin-s charge Σ_s tends to the mass M. The other charge $\Sigma_{s'}$ has similar tendency as it is correlated with the stellar mass M.

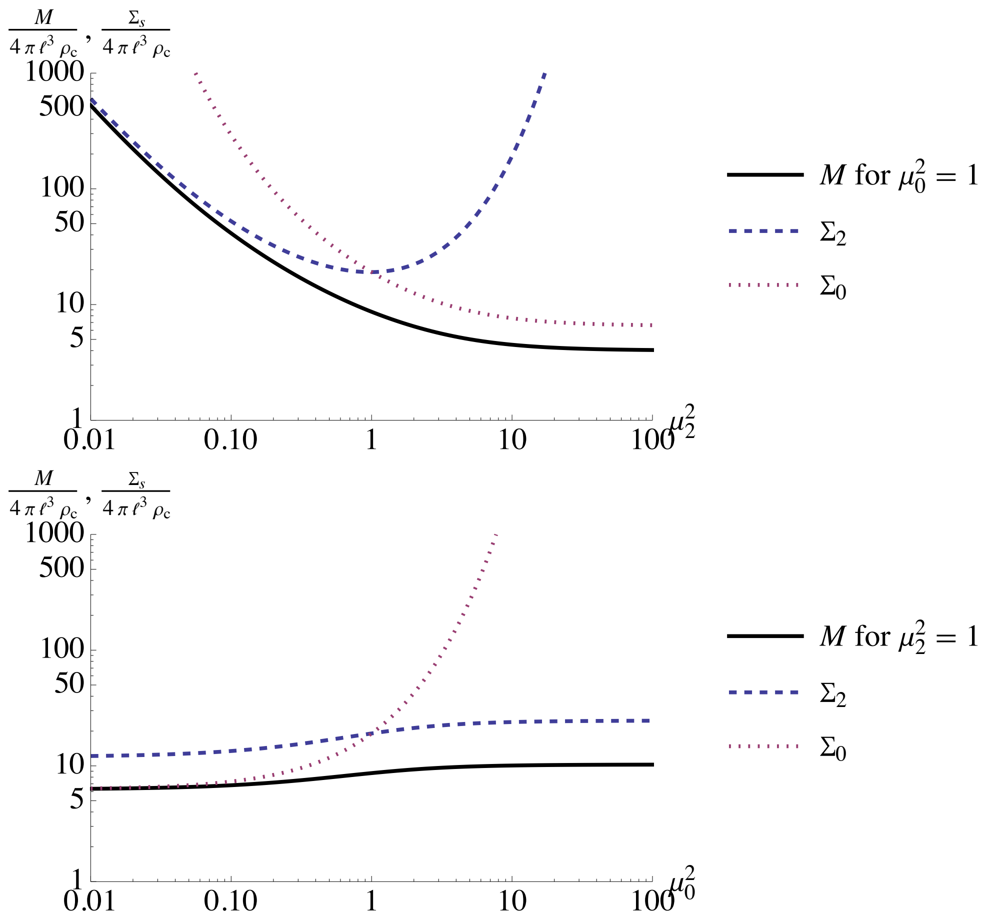

Figure 3.18: Typical μ_s dependences of M (solid black), Σ_2 (dashed blue), and Σ_0 (dotted red) for $n = 0$. In the top (bottom) panel, μ_2 (μ_0) is varied while the other is fixed.

After all, each panel in Fig. 3.19 shows typical relations between M and R (solid black) and Σ_s and R (dashed blue and dotted red), where the values are appropriately normalized. In the top (bottom) panel, μ_2^2 (μ_0^2) is varied while μ_0^2 (μ_2^2) is fixed. One can confirm from the top panel that the possible ranges of the stellar radius R and mass M for varying μ_2^2 are enormously large. On the other hand, for a given value of μ_2^2, the stellar radius and mass can only vary within a rather tiny range as seen in the bottom panel.

Figure 3.20 shows contours of the stellar radius R in the parameter plane (μ_0^2, μ_2^2), where on each contour, R has a multiple of the GR value $R_{\rm LE} = \sqrt{6}\,\ell$. By demanding any $n = 0$ polytrope stars in the universe to have a radius no larger than some multiple, say 2, of the GR value, one finds a constraint on the combination of the theory param-

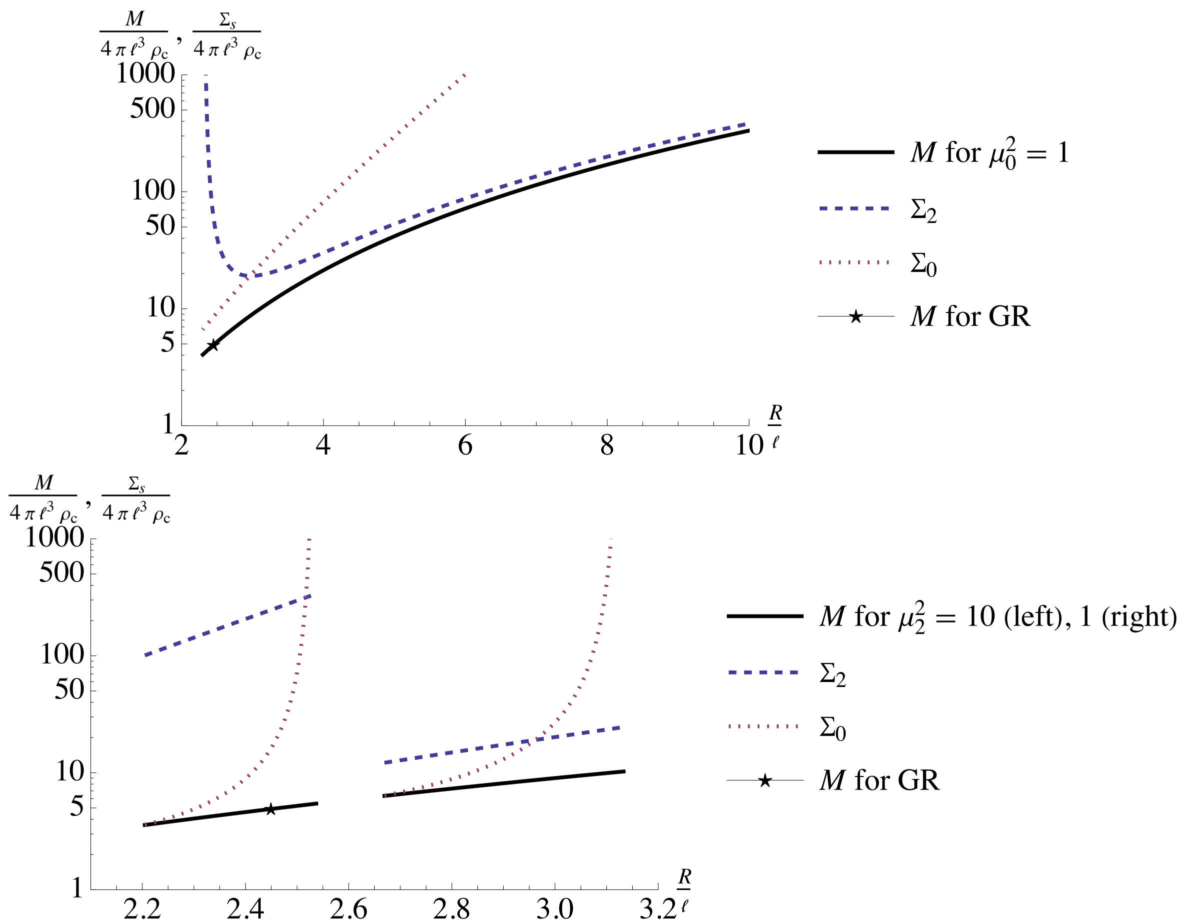

Figure 3.19: Typical relations between M and R (solid black) and Σ_s and R (dashed blue for Σ_2 and dotted red for Σ_0) for $n = 0$. In the top panel, μ_2^2 is varied while μ_0^2 is fixed to 1. In the bottom panel, μ_0^2 is varied while μ_2^2 is fixed to 10 (left) or 1 (right).

eters (μ_0^2, μ_2^2), or equivalently (α, β). Since the radius is sensitive to μ_2^2 for $R \gtrsim R_{\rm LE}$, $\sqrt{\alpha}$ is generally constrained to below a few ℓ, whereas $\sqrt{\beta}$ is virtually not restricted.

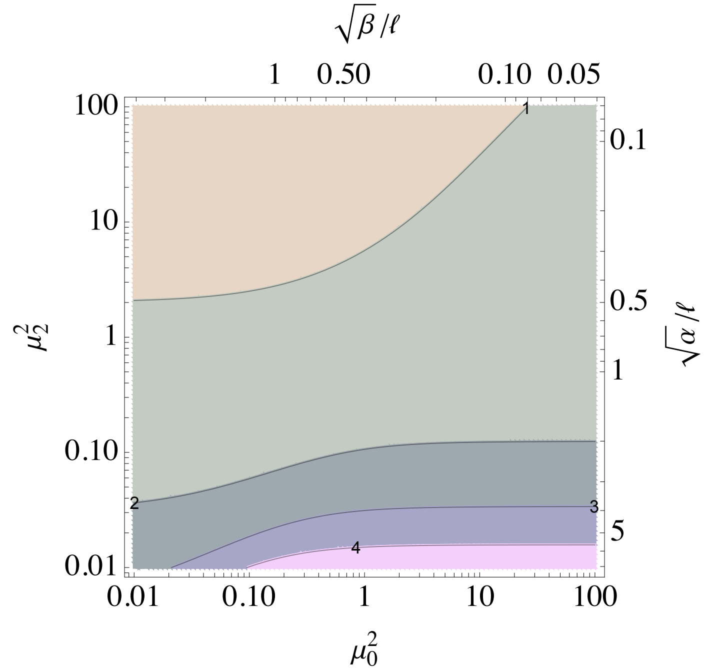

Figure 3.20: Contours of the stellar radius R in the (μ_0^2, μ_2^2) plane for the polytropic index $n = 0$. On the contours from top to bottom, the ratio of the calculated stellar radius to the GR value, $R/R_{\rm LE}$, is $1, 2, 3, 4$.

In the case of $n = 1$, the master equation (3.104) becomes a linear homogeneous equation, which reads

$$f(\triangle_\xi)\,\theta = 0 \tag{3.150}$$

with the characteristic polynomial f being

$$f(x) = x^3 - (\mu_2^2 + \mu_0^2)\,x^2 + (\mu_2^2\,\mu_0^2 + \alpha_2\,\mu_2^2 + \alpha_0\,\mu_0^2)\,x + \mu_2^2\,\mu_0^2\,. \tag{3.151}$$

This problem can be treated in parallel to the fourth-order cases, where we factorised the differential operator into a product of two Helmholtz operators. Here, the sixth-order differential operator $f(\triangle_\xi)$ can be cast into a product of three Helmholtz operators with eigenvalues given by the three roots of the characteristic equation $f(x) = 0$, and these roots characterise the solution of (3.150). Having that the inflection point of $f(x)$ lies at $x = (\mu_2^2 + \mu_0^2)/3 > 0$ and $f(0) = \mu_2^2\,\mu_0^2 > 0$, the cubic equation $f(x) = 0$ turns out to have one and only one negative real root, which we denote as $x = -\lambda_1^2$ with λ_1 being real. Whether the other two roots are positive real or complex depends on the sign of the discriminant

$$\mathcal{D} \equiv (\mu_2^2+\mu_0^2)^2\,(\mu_2^2\,\mu_0^2+\alpha_2\,\mu_2^2+\alpha_0\,\mu_0^2)^2 - 4\,(\mu_2^2\,\mu_0^2+\alpha_2\,\mu_2^2+\alpha_0\,\mu_0^2)^3 - 4\,(\mu_2^2+\mu_0^2)^3\,\mu_2^2\,\mu_0^2 \\ - 27\mu_2^4\,\mu_0^4 - 18\,(\mu_2^2+\mu_0^2)\,(\mu_2^2\,\mu_0^2+\alpha_2\,\mu_2^2+\alpha_0\,\mu_0^2)\,\mu_2^2\,\mu_0^2\,. \tag{3.152}$$

The sign of $\mathcal{D}$ in the parameter plane is shown in Fig. 3.21. In terms of the area, having $\mathcal{D} \geq 0$ (blue) is more likely as it generally realises in the presence of a large hierarchy between the graviton masses, that is, when $\mu_0 \gg \mu_2$ or $\mu_0 \ll \mu_2$. In this case, less massive graviton is expected to dominate. The region of $\mathcal{D} < 0$ (red) is only seen around (to the slight right of) the equality line $\mu_0^2 = \mu_2^2$, in which the massive gravitons are expected to compete with each other. In either case, we denote the two remaining roots as $x = \lambda_2^2\,,\lambda_3^2$, which are positive real if $\mathcal{D} \geq 0$, or complex and conjugate of each other if $\mathcal{D} < 0$. Using Viète's formula, we may write the roots as

$$\begin{aligned} -\lambda_1^2 &= \frac{\mu_0^2+\mu_2^2}{3} + \frac{2}{3}\sqrt{P}\,\cos\left[\frac{1}{3}\,\cos^{-1}\left(\frac{Q}{2P\sqrt{P}}\right) - \frac{4\pi}{3}\right], \\ \lambda_2^2 &= \frac{\mu_0^2+\mu_2^2}{3} + \frac{2}{3}\sqrt{P}\,\cos\left[\frac{1}{3}\,\cos^{-1}\left(\frac{Q}{2P\sqrt{P}}\right) - \frac{2\pi}{3}\right], \\ \lambda_3^2 &= \frac{\mu_0^2+\mu_2^2}{3} + \frac{2}{3}\sqrt{P}\,\cos\left[\frac{1}{3}\,\cos^{-1}\left(\frac{Q}{2P\sqrt{P}}\right)\right] \end{aligned} \tag{3.153}$$

with

$$\begin{aligned} P &\equiv \mu_0^4+\mu_2^4-\mu_0^2\,\mu_2^2 - 3\sum_s \alpha_s\,\mu_s^2\,, \\ Q &\equiv 2\,(\mu_0^6+\mu_2^6) - 3\mu_0^2\,\mu_2^2\,(\mu_0^2+\mu_2^2) - 9\sum_s \alpha_s\,\mu_s^4 - 18\mu_0^2\,\mu_2^2\,. \end{aligned} \tag{3.154}$$

With the use of these roots, the master equation settles down to the form

$$(\triangle_\xi+\lambda_1^2)\,(\triangle_\xi-\lambda_2^2)\,(\triangle_\xi-\lambda_3^2)\,\theta = 0\,, \tag{3.155}$$

which is ready to solve.

For $\mathcal{D} > 0$, the general solution is written in terms of real-valued functions as

$$\theta = A_1\,\frac{\sin\lambda_1\,\xi}{\xi} + B_1\,\frac{\cos\lambda_1\,\xi}{\xi} + A_2\,\frac{\sinh\lambda_2\,\xi}{\xi} + B_2\,\frac{\cosh\lambda_2\,\xi}{\xi} + A_3\,\frac{\sinh\lambda_3\,\xi}{\xi} + B_3\,\frac{\cosh\lambda_3\,\xi}{\xi}\,. \tag{3.156}$$

In the current case, unlike when $n = 0$, contributions from the two massive gravitons do not simply separate, as the eigenvalues λ_i depend on both of μ_0 and μ_2. By imposing three boundary conditions $\theta_c = 1\,,\theta'_c = \theta'''_c = 0$, we determine four constants as $B_1 = B_2 = B_3 = 0$ and $A_1 = (1 - A_2\,\lambda_2 - A_3\,\lambda_3)/\lambda_1$, obtaining the reduced expression

$$\theta = \frac{\sin\lambda_1\,\xi + A_2\,(\lambda_1\,\sinh\lambda_2\,\xi - \lambda_2\,\sin\lambda_1\,\xi) + A_3\,(\lambda_1\,\sinh\lambda_3\,\xi - \lambda_3\,\sin\lambda_1\,\xi)}{\lambda_1\,\xi}\,. \tag{3.157}$$

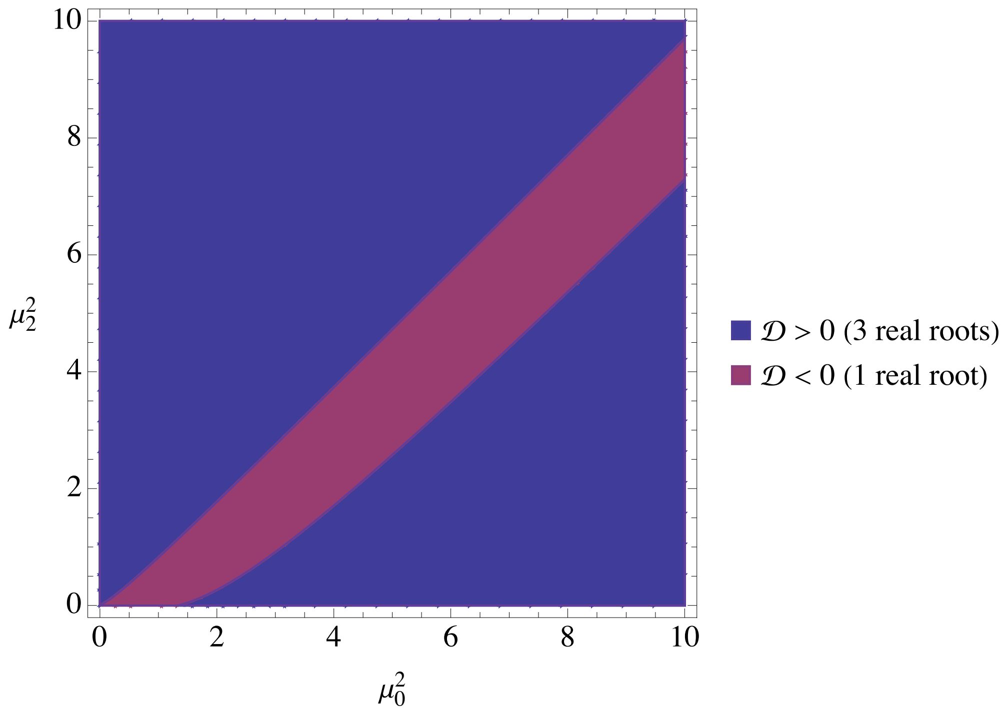

Figure 3.21: The sign of the discriminant $\mathcal{D}$ (3.152).

To determine the remaining constants A_2 and A_3, we follow the same scheme as we employed in the $n = 1$ fourth-order case as follows. On the one hand, these constants appear in the yet unused second and fourth derivatives as

$$\begin{aligned} \theta_{\mathrm{c}}'' &= \frac{-\lambda_1^2 + A_2\,\lambda_2\,(\lambda_1^2+\lambda_2^2) + A_3\,\lambda_3\,(\lambda_1^2+\lambda_3^2)}{3}\,, \\ \theta_{\mathrm{c}}^{(4)} &= \frac{\lambda_1^4 + A_2\,\lambda_2\,(\lambda_2^4-\lambda_1^4) + A_3\,\lambda_3\,(\lambda_3^4-\lambda_1^4)}{5}\,. \end{aligned} \tag{3.158}$$

Then the boundary conditions on these derivatives in (3.116) provide us with the linear relations between the constants and the stellar integrals ι_s:

$$\begin{aligned} A_2 &= \frac{\lambda_1^2\,\lambda_3^2 + \sum_{s=0,2}\alpha_s\,\mu_s^2\,[1+(\lambda_3^2-\lambda_1^2-\mu_s^2)\,\iota_s]}{\lambda_2\,(\lambda_1^2+\lambda_2^2)\,(\lambda_3^2-\lambda_2^2)}\,, \\ A_3 &= \frac{\lambda_1^2\,\lambda_2^2 + \sum_{s=0,2}\alpha_s\,\mu_s^2\,[1+(\lambda_2^2-\lambda_1^2-\mu_s^2)\,\iota_s]}{\lambda_3\,(\lambda_1^2+\lambda_3^2)\,(\lambda_2^2-\lambda_3^2)}\,. \end{aligned} \tag{3.159}$$

On the other hand, ι_s may be calculated by substituting the profile function (3.157) into Eq. (3.40). Thanks to the simpleness of $n = 1$ polytrope, these integrations can be analytically done, and we are allowed to express ι_s analytically in a form linear in A_2 and A_3, which however, we do not present here as their expressions are too messy and not illuminating. Then, substituting them into ι_s's in (3.159) and solving for the integration constants A_2 and A_3 just algebraically, we obtain their expressions that include μ_s and ξ_R only. As a result, we arrive at the final analytical expression of the profile function θ parametrically depending on μ_s and ξ_R.

The case with $\mathcal{D} < 0$ can be analysed in parallel, or by means of analytical continuation, so we do not redo the procedure here but only show the general solution. In this case, denoting the two complex conjugate roots as $x = (p + q\,\mathrm{i})^2\,, (p - q\,\mathrm{i})^2$, the general solution in terms of real-valued functions is written down as

$$\begin{aligned}\theta = A\,\frac{\sin\lambda_1\,\xi}{\xi} + B\,\frac{\cos\lambda_1\,\xi}{\xi} + C_1\,\frac{\sinh p\,\xi\,\sin q\,\xi}{\xi}\\ + C_2\,\frac{\cosh p\,\xi\,\sin q\,\xi}{\xi} + C_3\,\frac{\sinh p\,\xi\,\cos q\,\xi}{\xi} + C_4\,\frac{\cosh p\,\xi\,\cos q\,\xi}{\xi}\,.\end{aligned} \tag{3.160}$$

Lastly, in the special case with $\mathcal{D} = 0$, where the remaining roots degenerate, $\lambda_2 = \lambda_3$, the general solution is

$$\theta = A\,\frac{\sin\lambda_1\,\xi}{\xi} + B\,\frac{\cos\lambda_1\,\xi}{\xi} + C\,\frac{\sinh\lambda_2\,\xi}{\xi} + D\,\frac{\cosh\lambda_2\,\xi}{\xi} + \tilde{C}\,\sinh\lambda_2\,\xi + \tilde{D}\,\cosh\lambda_2\,\xi\,. \tag{3.161}$$

Figure 3.22 shows typical solutions for the polytropic index $n = 1$ together with the LE solution in GR (grey). The tendencies can be well understood as consequences of the competition between the two massive gravitons, viz., attraction by the spin-0 graviton dominates for $\mu_0 \ll \mu_2$ (red) while repulsion by the spin-2 graviton dominates for $\mu_2 \lesssim \mu_0$ (rest).

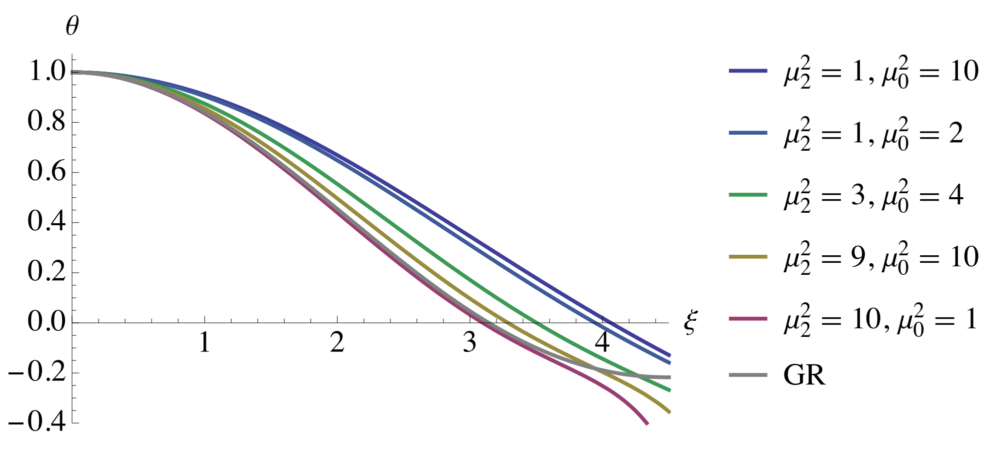

Figure 3.22: Typical solutions and the LE solution in GR (grey) for $n = 1$.

Shown in Fig. 3.23 are the μ_s dependences of R (top), M, and Σ_s (middle and bottom). Figure 3.24 shows the relationships between M and R (solid black) and Σ_s and R (dashed blue and dotted red). All these appearances can be well understood analogously to the previous cases.

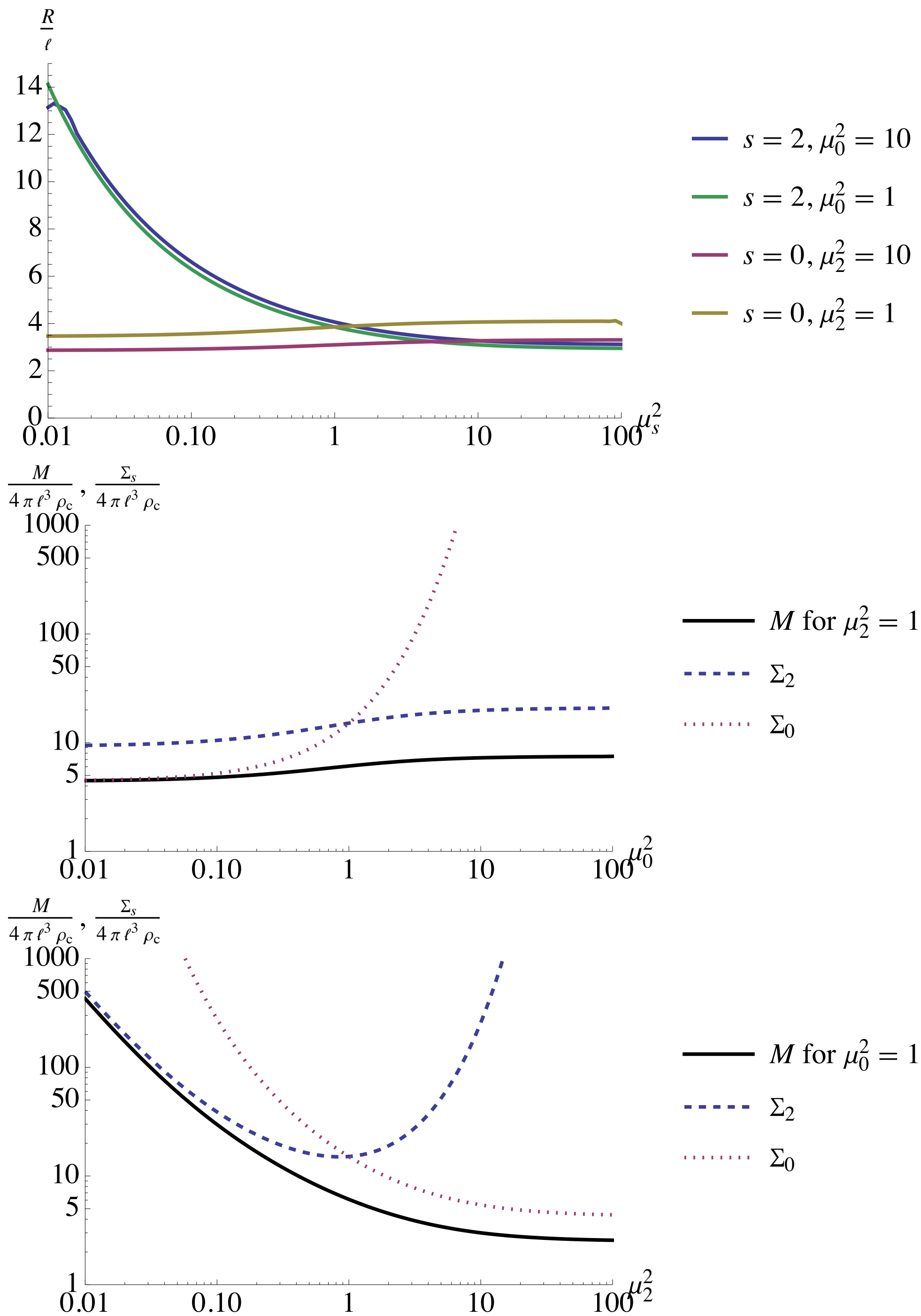

Figure 3.23: Typical μ_s dependences of R (top), M, and Σ_s (middle and bottom) for $n = 1$. We believe the appearance of a kink near $\mu_2^2 = 0$ on the blue curve in the top panel is not physical but due to a lack of numerical precision in our calculation.

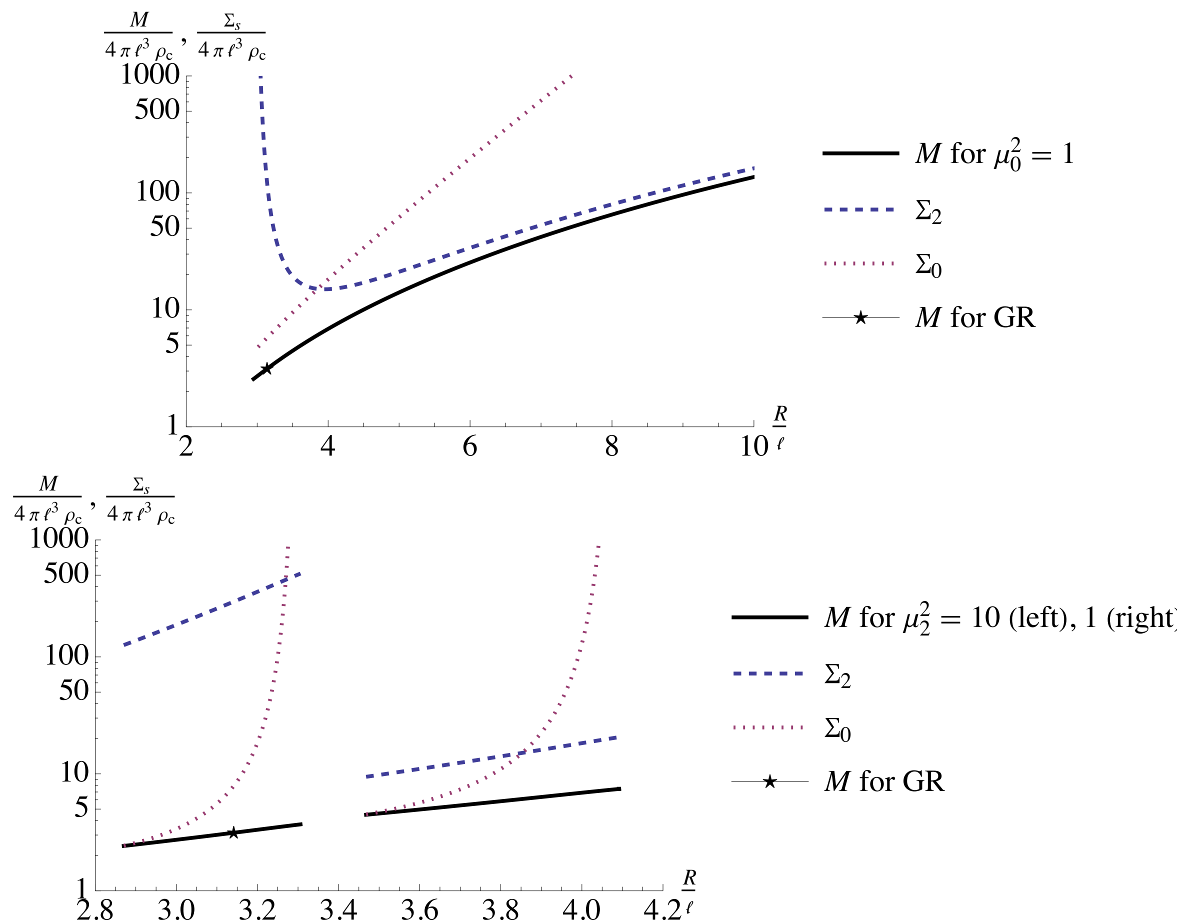

Figure 3.24: Relationships between M and R (solid black) and Σ_s and R (dashed blue and dotted red) for $n = 1$.

Two radius contours are plotted in Fig. 3.25, where on each curve the ratio $R/R_{\rm LE}$ is 1 (top) and 2 (bottom). From this diagram, one can conclude that the parameter $\sqrt{\alpha}$ cannot exceed a few times ℓ if one requires $n = 1$ polytrope stars have radius no larger than 2 or 3 times the GR value.

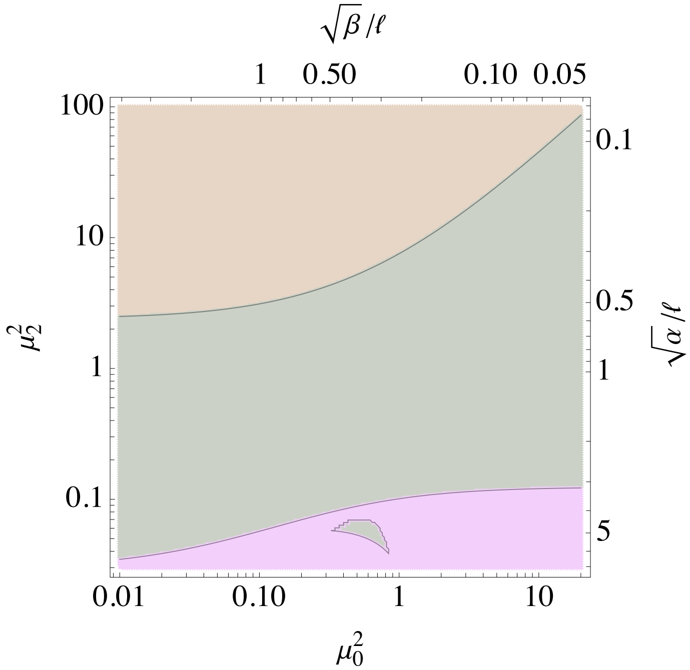

Figure 3.25: Contours of the stellar radius R in the parameter plane for $n = 1$, on which the ratio $R/R_{\rm LE}$ takes 1 (top) or 2 (bottom). We believe the appearance of a spot beneath the bottom contour is not physical but due to a lack of numerical precision in our calculation.

Chapter 4

Conclusion

In this thesis, we studied GW polarizations and non-relativistic stellar structure in effective quantum gravity.

In Chapter 2, we studied gravitational-wave polarizations in generic linear MG and generic HCG in the Minkowski background. We defined and analysed the GW polarizations in terms of the components of the Riemann tensor governing the geodesic deviation.

In Section 2.1, we formulated the polarizations in linear MG with generic, non-Fierz–Pauli-type masses. We identified all the independent variables that obey Klein–Gordon-type equations. The dynamical DOFs in the generic MG consist of spin-2 and spin-0 modes; the former breaks down into two tensor (helicity-2), two vector (helicity-1) and one scalar (helicity-0) polarizations, while the latter just corresponds to a scalar polarization. We found convenient ways of decomposing the two scalar modes of each spin into distinct linear combinations of the transverse and longitudinal polarizations as in (2.18). This expression contains the graviton masses as the coefficients, so we expect it will serve as a useful tool in measuring the masses of GWs.

In Section 2.2, we analysed the linear perturbations of generic HCG whose Lagrangian is an arbitrary polynomial of the Riemann tensor. When expanded around a flat background, the linear dynamical DOFs in this theory were identified as massless spin-2, massive spin-2 and massive spin-0 modes. The massive spin-2 arises from the Weyl-squared term in the Lagrangian and the massive spin-0 from the Ricci scalar squared. The massless spin-2 is characterised by the tensor-type (helicity-2) polarization modes. As its massive part encompasses the identical structure to the generic MG, GWs in the generic HCG provide six massive polarizations on top of the ordinary two massless modes. In parallel to MG, we found convenient representations for the scalar polarizations directly connected to the coupling constants of HCG as in (2.26).

In the analysis of HCG, we used two methods and showed that the two results agree. One took full advantage of the partial equivalence between the generic HCG and MG at the linear level, whereas the other relied upon a gauge-invariant formalism originally developed for cosmological perturbation theories. The present result about the scalar part can be compared with the case of inflationary cosmological perturbations in Einstein–Weyl gravity studied in [DSSY11], where the conformal analogue of the gauge-invariant variable $W = \Psi - \Phi$ becomes dynamical.

In Section 2.3, we gave a brief discussion about possible methods to determine the theory parameters by means of GW-polarization measurements with emphasis on the merit that they do not require measuring the propagation speeds, whether absolute or relative to other signals, or the details of the waveforms of the GWs. In any case, the full development in this direction is left to future work.

In Chapter 3, we studied the structure of static spherical stars made up of a non-relativistic polytropic fluid in generic linear MG and generic HCG.

In Section 3.1, we studied the structure of non-relativistic polytropic stars in the Fierz–Pauli (FP) theory and generic linear massive gravity (MG) theories. Our aim was to study the effects of the graviton mass m and the "non-Fierz–Pauli" parameter ϵ incorporated in the MG action (1.13) on the stellar structures. The spin-2 graviton is the only content in the FP theory with $\epsilon = 0$, while the spin-0 ghost graviton is also included in generic non-FP MG with $\epsilon \neq 0$.

First, we formulated useful variables and their governing equations. The scalar-type metric perturbations on a flat background could be neatly reorganised into the two helicity variables ϕ_2 and ϕ_0 as defined in (3.8). These variables constitute the gauge-invariant potential Ψ as in Eq. (3.15) while obeying the Helmholtz-type equations (3.14) in static configurations. The negative coefficient in front of ϕ_0 in (3.15) illuminates its ghost nature. We pointed out the absence of the van Dam–Veltman–Zakharov (vDVZ) discontinuity for the external gravitational potential in the limit of vanishing graviton masses in non-FP MG.

Then, together with the polytropic EOS, the hydrostatic equilibrium condition (3.27) leads to the modified Lane–Emden (LE) equation (3.34) and boundary conditions (3.38)–(3.39). The reason for the master equation being a sixth-order differential equation is the presence of the two extra gravitational DOFs. The boundary conditions were given in terms of the derivatives at the stellar centre but involved integrals of the profile function through ι_s (3.40). The reduced set of equations for the case of the FP theory ($\epsilon = 0$) was derived as in Eqs. (3.42)–(3.43), where the master equation was fourth order because the spin-0 potential ϕ_0 had been turned off. As expected, the massless limit of the master equation for the FP theory did not recover the original LE equation in GR, signaling the emergence of discontinuity analogous to vDVZ. On the other hand, as imagined, the doubly massless limit of the non-FP equation smoothly connected to GR.

We found exact solutions to the modified LE equations (3.34) in the cases of the polytropic indices $n = 0$ and 1. Typical solutions in the FP theory ($\epsilon = 0$) were depicted in Figs. 3.1 ($n = 0$) and 3.3 ($n = 1$). As these figures illustrated, the radius of a star monotonically decreases with the decreasing spin-2 graviton mass μ_2, with the minimum value reached in the massless limit being $\sqrt{3}/2$ times the GR value. Figure 3.2 showed the dependence of the stellar radius R/ℓ (blue), mass $M/(4\pi\ell^3\rho_c)$ (red) and Yukawa charge $\Sigma_2/(4\pi\ell^3\rho_c)$ (yellow) on μ_2 for $n=0$. In the limit of vanishing graviton mass, $\mu_2 \to 0$, both R and M did not converge to the values of GR, proving the presence of discontinuities analogous to vDVZ.

Typical solutions in non-FP theories ($\epsilon \neq 0$) were presented in Figs. 3.4 ($n=0$) and 3.7 ($n=1$). Although the attractive spin-2 and repulsive spin-0 competed in these cases, we could confirm that the solutions had a finite radius within an observationally reasonable range of parameters. Figure 3.5 showed the dependence of the stellar radius R/ℓ (blue) and the mass $M/(4\pi\ell^3\rho_c)$ (red) on the spin-2 graviton mass parameter μ_2^2 for several values of ϵ. The convergence to the GR values for $\epsilon \neq 0$ was observed, visualising the absence of the vDVZ-like discontinuity. The contour plot of the ratio R/R_{LE} in Fig. 3.6 told us that there were many possible combinations of the mass parameters that had the same radius as GR. We argued that, if we demand that the radius must not deviate significantly from GR, the masses should be constrained to fall into a region around the $R/R_{\mathrm{LE}}=1$ contour, say between 0.9 and 1.1. Considering that the spin-2 mass must be tiny, $\mu_2 \ll 1$, as implied by the GW experiments, the spin-0 mass was constrained as $\mu_0 \lesssim \mathcal{O}(1)$ as was read off from Figs. 3.6 and 3.8.

The presence of the vDVZ-like discontinuity in the massless limit of the FP theory would again imply the need for some screening mechanism so as to make the FP theory compatible with any observations which are consistent with the predictions of GR. On the other hand, the fact that the spin-0 ghost in generic non-FP theories can take a role in smoothly recovering GR in the doubly massless limit might suggest that the ghost offers a different mechanism to give a tiny but non-zero mass to the spin-2 graviton. Of course, one might think the absence of the discontinuity might be at a cost of security against possible instabilities brought by the ghost. We leave the stability issue to future work, where our analytical solutions in this study will serve as background on which stability can be examined.

In Section 3.2, we studied non-relativistic polytropic stars in HCG. We analyzed the hydrostatic equilibrium condition, starting with the gauge-invariant equations of motion (3.78) derived from the second-order perturbative action (3.76). In the static configuration, a particular set of gauge-invariant variables Ψ_2 and Ψ_0 as defined in (3.79) turned to be useful to reduce the EOMs into the decoupled form (3.83). These fourth-order eoms have the general solution in the form of difference of massless part ψ and massive part $\tilde{\psi}_s$, which respectively satisfies the Poisson equation (3.85) and the Helmholtz equation (3.86). As a result, the gauge-invariant gravitational potential Ψ, which appears in the hydrostatic equilibrium condition (3.27), was found as in (3.87). The equilibrium condition is an integro-differential equation at this stage. Applying an adequate higher-order differential operator on the both sides and adopting the polytropic equation of state, we obtained a sixth-order differential equation (3.104) for the Lane–Emden-like variable θ. When either of the graviton masses is taken to infinity, it reduces to a fourth-order equation (3.108) corresponding to "$R+R^2$" or

"$R + C^2$" gravity. When both go to infinity, it recovers the second-order Lane–Emden equation (3.47) in GR.

Then, we solved the fourth-order equation (3.108) imposing (3.118). As shown in Figs. 3.10 ($n = 0$) and 3.13 ($n = 1$), the dimensionless radius of the star increases (decreases) compared to GR for "$R + C^2$" ("$R + R^2$") gravity, reflecting the repulsive (attractive) nature of the massive graviton. In all cases, as $\mu_s^2 \to \infty$, these solutions recover the Lane–Emden profile in GR.The massless limit can be understood as a GR-like theory with a "renormalised" Newton constant $(1 + \alpha_s)\,G$. In "$R + R^2$" gravity, $\alpha_0 = 1/3$, it mimics GR with a larger Newton constant $4G/3$, leading to shrinkage of the radius by a factor of $\sqrt{3}/2$, while the same limit of "$R + C^2$" gravity, $\alpha_2 = -4/3$, is antigravity with negative Newton constant $-G/3$, leading to an infinite radius. We have clarified how the stellar radius R, mass M, and charge Σ_s depend on the graviton mass μ_s in Figs. 3.11 ($n = 0$) and 3.14 ($n = 1$). Diagrams relating the mass M and the charge Σ_s to the radius R were obtained in Figs. 3.12 ($n = 0$) and 3.15 ($n = 1$). We argued that, in "$R + C^2$" gravity, upper limits on the parameter α in the Lagrangian (1.8) can be obtained by requiring the stellar radius R should not exceed several multiples of the GR values $R_{\rm LE}$, which generally leads to $\sqrt{\alpha} \lesssim$ a few $\times\, \ell$.

We solved the sixth-order equation (3.104) in generic HCG with the boundary conditions (3.116). Most of the modification trends as compared to GR arise as a result of the competition of the opposite contributions from the co-existing massive gravitons. In particular, when the masses have a large hierarchy, $\mu_2 \ll \mu_0$ or $\mu_0 \ll \mu_2$, the graviton with smaller mass dominates. Because the coefficient of the massive gravitational potential for spin-2, α_2, is four times as large in magnitude as that of spin-0, α_0, the contribution from the former is generally more prominent than the latter when the two graviton masses are at the same order. Typical solutions were presented in Figs. 3.16 ($n = 0$) and 3.22 ($n = 1$). The dependences of R, M, and Σ_s on μ_s were shown in Figs. 3.17–3.18 ($n = 0$) and 3.23 ($n = 1$). M–R and Σ_s–R relations were shown in Figs. 3.19 ($n = 0$) and 3.24 ($n = 1$). The dependence of the stellar radius R, in the units of the GR value $R_{\rm LE}$, on the mass parameters (μ_0^2, μ_2^2) were illustrated in Figs. 3.20 ($n = 0$) and 3.25 ($n = 1$). These will be useful to find allowed regions for the QCG parameters (α, β) once an upper bound on the stellar radius of polytrope stars is established.

Let us give some prospects for future studies. On the theoretical side, the development of an additional numerical procedure for imposing the boundary conditions will become necessary if one wishes to construct solutions for an arbitrary polytropic index n. For $n \neq 0, 1$, since no analytical solution is known, one has to somehow numerically make a derivative at the stellar centre and integrals of a solution over the stellar radius match. On the observational side, the observable characteristics such as M–R and Σ_s–R diagrams, as well as the radius contours in the parameter plane, should offer a way to test HCG through comparisons with the distribution of known stellar populations.

Finally, to conclude this thesis, we summarize the gravitational waves and non-relativistic stellar structure in generic HCG, a candidate for quantum gravity at low energies. GWs are powerful tools that can determine the theory parameters α and β in generic HCG, but it requires more detectors than are currently available. If we assume that no massive GWs have been observed since 2015, we can place upper bounds on α and β, but this remains future work. On the other hand, since there are many non-relativistic stars in the present universe, it is expected that a lot of observational data will be avaiable soon. However, they will only provide a partial constraint because only some combinations of α and β can be determined. Moreover, in order to severely constrain α and β using real observational data, it is necessary to take into account the effect of stellar rotation, which is left for future work. In conclusion, if GW polarizations become observable in the future, we will be able to determine the theory parameters α and β. Any information about the combination of the theory parameters α and β from observations of the stellar radii will gain or lose confidence in the constraints based on GW observations.

Appendices

Appendix A

Perturbations on a flat background

This appendix presents perturbed geometric quantities on a flat background and constructs gauge-invariant variables.

A.1 Perturbations of geometric quantities

Decompose perturbed spacetime into a flat background spacetime $\eta_{\mu\nu}$ and perturbed components $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{A.1}$$

The inverse matrix is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}{}^{\nu} + \mathcal{O}(h^3)\,, \tag{A.2}$$

and $g_{\mu\alpha}g^{\alpha\nu} = \delta_{\mu}{}^{\nu}$ is satisfied up to the second order of $h_{\mu\nu}$. The up to the second order of $h_{\mu\nu}$, $\sqrt{-g}$ is expanded as

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}(h^2 - 2h_{\mu\nu}h^{\mu\nu}) + \mathcal{O}(h^3)\,. \tag{A.3}$$

The Christoffel symbols in the first and second order are

$$\begin{aligned} {}^{(1)}\Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu})\,, \\ {}^{(2)}\Gamma^{\alpha}{}_{\mu\nu} &= -\frac{1}{2}h^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu})\,. \end{aligned} \tag{A.4}$$

The Riemann tensor is

$$\begin{aligned} {}^{(1)}R^{\alpha}{}_{\mu\beta\nu} &= \partial_\beta {}^{(1)}\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu {}^{(1)}\Gamma^{\alpha}{}_{\mu\beta}\,, \\ {}^{(2)}R^{\alpha}{}_{\mu\beta\nu} &= \partial_\beta {}^{(2)}\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu {}^{(2)}\Gamma^{\alpha}{}_{\mu\beta} + {}^{(1)}\Gamma^{\alpha}{}_{\lambda\beta}{}^{(1)}\Gamma^{\lambda}{}_{\mu\nu} - {}^{(1)}\Gamma^{\alpha}{}_{\lambda\nu}{}^{(1)}\Gamma^{\lambda}{}_{\mu\beta}\,. \end{aligned} \tag{A.5}$$

The Ricci tensor is

$$\begin{aligned}
{}^{(1)}R_{\mu\nu} &= {}^{(1)}R^{\alpha}{}_{\mu\alpha\nu} \\
&= \partial^{\alpha}\nabla_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\partial_{\mu}\partial_{\nu}h - \frac{1}{2}\Box h_{\mu\nu}\,, \\
{}^{(2)}R_{\mu\nu} &= {}^{(2)}R^{\alpha}{}_{\mu\alpha\nu} \\
&= \partial_{\alpha}{}^{(2)}\Gamma^{\alpha}{}_{\mu\nu} - \partial_{\nu}{}^{(2)}\Gamma^{\alpha}{}_{\mu\alpha} + {}^{(1)}\Gamma^{\alpha}{}_{\lambda\alpha}{}^{(1)}\Gamma^{\lambda}{}_{\mu\nu} - {}^{(1)}\Gamma^{\alpha}{}_{\lambda\nu}{}^{(1)}\Gamma^{\lambda}{}_{\mu\alpha}\,.
\end{aligned} \tag{A.6}$$

The Ricci scalar is

$$\begin{aligned}
{}^{(1)}R &= \eta^{\mu\nu\,(1)}R_{\mu\nu} \\
&= \partial_{\mu}\partial_{\nu}h^{\mu\nu} - \Box h\,, \\
{}^{(2)}R &= -h^{\mu\nu\,(1)}R_{\mu\nu} + \eta^{\mu\nu\,(2)}R_{\mu\nu} \\
&= -\frac{1}{4}\partial_{\nu}h_{\alpha\beta}\partial^{\nu}h^{\alpha\beta} - \frac{1}{4}\partial_{\alpha}h\partial^{\alpha}h + \frac{1}{2}\partial_{\nu}h_{\alpha\beta}\partial^{\alpha}h^{\beta\nu} \\
&\quad + \partial_{\alpha}(h^{\beta\nu}\partial^{\alpha}h_{\beta\nu} + h^{\alpha\beta}\partial_{\beta}h - h^{\alpha\beta}\partial^{\nu}h_{\nu\beta} - h_{\nu\beta}\partial^{\nu}h^{\beta\alpha})\,.
\end{aligned} \tag{A.7}$$

A.2 Gauge transformations and gauge-invariant variables

In this section, we shall introduce gauge-invariant perturbations. This methodology effectively isolates the physical degrees of freedom and has been developed within a cosmological framework.

Let us consider Minkowski spacetime as a background. Decompose the perturbation spacetime into Minkowski spacetime, and the perturbation components

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{A.8}$$

The perturbation components are defined as

$$\begin{aligned}
h_{00} &= -2A\,, \\
h_{0i} &= -\partial_i B - B_i\,, \\
h_{ij} &= 2(\delta_{ij}C + \partial_i\partial_j E + \partial_{(i}E_{j)} + H_{ij})\,,
\end{aligned} \tag{A.9}$$

where the vector and tensor variables are satisfy

$$\partial^i B_i = 0\,, \partial^i E_i = 0\,, H_i{}^i = 0\,, \partial^j H_{ij} = 0\,. \tag{A.10}$$

Then, the metric of the perturbed spacetime is

$$\begin{aligned}
ds^2 &= g_{\mu\nu}dx^{\mu}dx^{\nu} \\
&= -(1+2A)dt^2 - 2(\partial_i B + B_i)dt dx^i \\
&\quad + [(1+2C)\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + 2H_{ij}]dx^i dx^j\,.
\end{aligned} \tag{A.11}$$

To establish gauge invariant variables, it is necessary to contemplate an active transformation of the coordinate system. The coordinates of any point change according to

$$x^\mu \to x^\mu + \xi^\mu(x^\nu)\,, \tag{A.12}$$

where ξ is a vector field as small as the perturbation, and the right–arrow means the active transformation of the coordinate system. Accordingly, the spacetime metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}\,, \tag{A.13}$$

where $\mathcal{L}_\xi$ is the Lie derivative of $g_{\mu\nu}$ along $-\xi^\mu$. It follows that

$$h_{\mu\nu} \to h_{\mu\nu} - \mathcal{L}_\xi \eta_{\mu\nu}\,, \tag{A.14}$$

of first order in the perturbations. The vector field ξ^μ can be decomposed into a scalar and vector part as

$$(\xi^0, \xi^i) = (T, \partial^i L + L^i)\,, \tag{A.15}$$

with $\partial_i L^i = 0$. The transformations of the metric perturbations are given as

$$\begin{aligned} A &\to A - \dot{T}\,, \\ B &\to B - T + \dot{L}\,, \\ C &\to C\,, \\ E &\to E - L\,, \end{aligned} \tag{A.16}$$

for the scalar variables, and as

$$\begin{aligned} B_i &\to B_i + \dot{L}_i\,, \\ E_i &\to E_i - L_i\,, \end{aligned} \tag{A.17}$$

for the vector variables. The tensor variable is gauge invariant,

$$H_{ij} \to H_{ij}\,. \tag{A.18}$$

Then we construct gauge-invariant scalar and vector variables as

$$\begin{aligned} \Psi &:= A - (\dot{B} + \ddot{E})\,, \\ \Phi &:= C\,, \\ \Sigma_i &:= B_i + \dot{E}_i\,. \end{aligned} \tag{A.19}$$

At this point, we have found six gauge invariant variables out of the original ten in $h_{\mu\nu}$.

A.3 Expressions for curvature tensors and higher-curvature Lagrangian

This appendix delineates the essential expressions about the linear-order curvature within the Minkowski framework, along with the quadratic-curvature action integrals expanded to the second order. Due to the topological nature of the Gauss–Bonnet combination in four dimensions, the Weyl-squared action can be reformulated as follows:

$$\begin{aligned}
S_C &\equiv \frac{-\alpha}{2\kappa} \int \mathrm{d}^4x \sqrt{-g}\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} \\
&= \frac{-\alpha}{2\kappa} \int \mathrm{d}^4x \sqrt{-g} \left(R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}\, R^{\mu\nu} + \frac{1}{3}\, R^2 \right) \\
&= \frac{-\alpha}{2\kappa} \int \mathrm{d}^4x \sqrt{-g} \left(2R_{\mu\nu}\, R^{\mu\nu} - \frac{2}{3}\, R^2 \right) ,
\end{aligned} \tag{A.20}$$

up to irrelevant surface integrals. Therefore, in order to compute the second-order expansion, it is sufficient to consider only the first-order Ricci tensor

$$\begin{aligned}
{}^{(1)}R_{\mu\nu} &= -\frac{1}{2}\, \Box h_{\mu\nu} + \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2}\, \partial_\mu \partial_\nu h \,, \\
{}^{(1)}R_{ij} &= -\Box H_{ij} + \partial_{(i} \dot{\Sigma}_{j)} - \partial_i \partial_j \Psi - \partial_i \partial_j \Phi - \delta_{ij}\, \Box \Phi \,, \\
{}^{(1)}R_{i0} &= \frac{1}{2}\, \triangle \Sigma_i - 2\partial_i \dot{\Phi} \,, \\
{}^{(1)}R_{00} &= \triangle \Psi - 3\ddot{\Phi}
\end{aligned} \tag{A.21}$$

and the Ricci scalar

$${}^{(1)}R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h = -2\triangle(\Psi - \Phi) - 6\Box\Phi \,. \tag{A.22}$$

The Weyl-squared action expanded up to the second order is given in terms of the perturbative variables as

$$\begin{aligned}
{}^{(2)}S_C &= \frac{-\alpha}{2\kappa} \int \mathrm{d}^4x \left(2\,{}^{(1)}R_{\mu\nu}\, {}^{(1)}R^{\mu\nu} - \frac{2}{3}\, {}^{(1)}R^2 \right) \\
&= \frac{-\alpha}{2\kappa} \int \mathrm{d}^4x \left[\frac{1}{2}\, \Box h_{\mu\nu}\, \Box h^{\mu\nu} - \Box h_{\mu\nu}\, \partial_\alpha \partial^\mu h^{\alpha\nu} \right. \\
&\qquad \left. + \frac{1}{3}\, \Box h\, \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{3}\, (\partial_\mu \partial_\nu h^{\mu\nu})^2 - \frac{1}{6}\, (\Box h)^2 \right] \\
&= \frac{-\alpha}{2\kappa} \int \mathrm{d}^4x \left[2(\Box H_{ij})^2 + (\partial_i \dot{\Sigma}_j)^2 - (\triangle \Sigma_i)^2 + \frac{4}{3}\, [\triangle(\Psi - \Phi)]^2 \right] ,
\end{aligned} \tag{A.23}$$

where surface terms have been discarded. The computation of the second-order expansion of the Ricci-squared action

$$S_R \equiv \frac{\beta}{2\kappa} \int \mathrm{d}^4x \sqrt{-g}\, R^2 \tag{A.24}$$

is straightforward:

$$\begin{aligned} {}^{(2)}S_R &= \frac{\beta}{2\kappa} \int \mathrm{d}^4x \, {}^{(1)}R^2 \\ &= \frac{\beta}{2\kappa} \int \mathrm{d}^4x \, \left[(\partial_\mu \partial_\nu h^{\mu\nu})^2 - 2\Box h \, \partial_\mu \partial_\nu h^{\mu\nu} + (\Box h)^2 \right] \\ &= \frac{\beta}{2\kappa} \int \mathrm{d}^4x \, 4 \left[\triangle(\Psi - \Phi) + 3\Box\Phi \right]^2 . \end{aligned} \tag{A.25}$$

Appendix B

Decoupling DOFs in quadratic curvature gravity on arbitrary Einstein manifolds

In this Appendix, we describe the equivalence of the quadratic curvature gravity (QCG) and GR "minus" MG at the linear level on arbitrary Einstein manifolds [NNZ+]. Let us begin with the generic quadratic curvature action with a cosmological constant

$$S_{\rm QCG}[g_{\mu\nu}] = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,\left(R - 2\Lambda - \alpha\, C_{\mu\nu\rho\sigma}\,C^{\mu\nu\rho\sigma} + \beta\, R^2\right)\,, \tag{B.1}$$

where we have dropped the topological Gauss–Bonnet term. This theory admits any metric $\bar{g}_{\mu\nu}$ satisfying

$$R_{\mu\nu}[\bar{g}_{\mu\nu}] = \Lambda\,\bar{g}_{\mu\nu} \tag{B.2}$$

as a solution to the EOM. It is useful to introduce a Lovelock tensor

$$\mathcal{G}_{\mu\nu} \equiv G_{\mu\nu} + \Lambda\, g_{\mu\nu}\,, \tag{B.3}$$

which vanishes when evaluated with $\bar{g}_{\mu\nu}$. The action can be rewritten as

$$S_{\rm QCG}[g_{\mu\nu}] = \frac{\zeta}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,\left(2\Lambda - \mathcal{G} - \frac{\tilde{\alpha}}{2}\,\mathcal{G}_{\mu\nu}\,\mathcal{G}^{\mu\nu} + \frac{\tilde{\beta}}{2}\,\mathcal{G}^2\right)\,, \tag{B.4}$$

where $\mathcal{G} = g^{\mu\nu}\,\mathcal{G}_{\mu\nu}$, $\zeta \equiv 1 + (8\beta + 4\alpha/3)\,\Lambda$, $\tilde{\alpha} \equiv 4\alpha/\zeta$ and $\tilde{\beta} \equiv (2\beta + 4\alpha/3)/\zeta$ and where we have again discarded the Gauss–Bonnet term. Taking $\bar{g}_{\mu\nu}$ as the background and expanding the action up to quadratic order in $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$, we obtain the second-order action for $h_{\mu\nu}$

$$\begin{aligned}&{}^{(2)}S_{\rm QCG}[h_{\mu\nu}]\\ &= \frac{\zeta}{4\kappa}\int \mathrm{d}^4x\,\sqrt{-\bar{g}}\,\left(-h^{\mu\nu}\,{}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}] - \tilde{\alpha}\,{}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}]\,{}^{(1)}\mathcal{G}^{\mu\nu}[h_{\mu\nu}] + \tilde{\beta}\,{}^{(1)}\mathcal{G}[h_{\mu\nu}]^2\right)\,,\end{aligned} \tag{B.5}$$

where

$$ {}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}] \equiv {}^{(1)}G_{\mu\nu}[h_{\mu\nu}] + \Lambda\, h_{\mu\nu} \tag{B.6} $$

and ${}^{(1)}\mathcal{G}[h_{\mu\nu}] \equiv \bar{g}^{\mu\nu}\, {}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}]$. As usual, the tensor indices are raised and lowered with the background metric $\bar{g}_{\mu\nu}$. Replacing ${}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}]$ in (B.5) with an auxiliary variable $A_{\mu\nu}$ and adding a constraint leads to

$$ \begin{aligned} &{}^{(2)}S_{\rm QCG}[h_{\mu\nu}, A_{\mu\nu}, \lambda_{\mu\nu}] \\ &= \frac{\zeta}{4\kappa} \int \mathrm{d}^4x \sqrt{-\bar{g}}\, \Big(-h^{\mu\nu} A_{\mu\nu} - \tilde{\alpha}\, A_{\mu\nu} A^{\mu\nu} + \tilde{\beta}\, A^2 + \lambda^{\mu\nu} \left(A_{\mu\nu} - {}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}] \right) \Big) \,, \end{aligned} \tag{B.7} $$

where $A \equiv \bar{g}^{\mu\nu} A_{\mu\nu}$ and $\lambda^{\mu\nu}$ is a Lagrange multiplier. The variation of the above action with respect to $A_{\mu\nu}$ gives an algebraic constraint

$$ \lambda_{\mu\nu} = h_{\mu\nu} + 2\tilde{\alpha}\, A_{\mu\nu} - 2\tilde{\beta}\, A\, \bar{g}_{\mu\nu} \,, \tag{B.8} $$

which can be substituted back to the action to eliminate $A_{\mu\nu}$ to give

$$ \begin{aligned} &{}^{(2)}S_{\rm QCG}[h_{\mu\nu}, \lambda_{\mu\nu}] \\ &= \frac{\zeta}{4\kappa} \int \mathrm{d}^4x \sqrt{-\bar{g}} \left[-\lambda^{\mu\nu}\, {}^{(1)}\mathcal{G}_{\mu\nu}[h_{\mu\nu}] + \frac{m^2}{8} \left((h_{\mu\nu} - \lambda_{\mu\nu})(h^{\mu\nu} - \lambda^{\mu\nu}) - (1-\epsilon)(h-\lambda)^2 \right) \right] \,, \end{aligned} \tag{B.9} $$

where $\epsilon \equiv 1 + \tilde{\beta}/(\tilde{\alpha} - 4\tilde{\beta}) = 9\beta/(2\alpha + 12\beta)$, $m^2 \equiv 2/\tilde{\alpha} = \zeta/(2\alpha)$ and $\lambda \equiv \bar{g}^{\mu\nu}\lambda_{\mu\nu}$. Finally, by transforming

$$ h_{\mu\nu} \to \phi_{\mu\nu} + \tilde{\phi}_{\mu\nu} \,, \quad \lambda_{\mu\nu} \to \phi_{\mu\nu} - \tilde{\phi}_{\mu\nu} \,, \tag{B.10} $$

we arrive at

$$ \begin{aligned} &{}^{(2)}S_{\rm QCG}[\phi_{\mu\nu}, \tilde{\phi}_{\mu\nu}] \\ &= \frac{\zeta}{4\kappa} \int \mathrm{d}^4x \sqrt{-\bar{g}} \left[-\phi^{\mu\nu}\, {}^{(1)}\mathcal{G}_{\mu\nu}[\phi_{\mu\nu}] + \tilde{\phi}^{\mu\nu}\, {}^{(1)}\mathcal{G}_{\mu\nu}[\tilde{\phi}_{\mu\nu}] + \frac{m^2}{2} \left(\tilde{\phi}_{\mu\nu}\tilde{\phi}^{\mu\nu} - (1-\epsilon)\,\tilde{\phi}^2 \right) \right] \,, \end{aligned} \tag{B.11} $$

where $\tilde{\phi} \equiv \bar{g}^{\mu\nu}\tilde{\phi}_{\mu\nu}$. Equation (1.9) is obtained as the Minkowski version of this.

Appendix C

Pure quadratic curvature gravity

We consider the special case $\chi = 0$ where the Lagrangian is given by pure quadratic curvature terms. In this case, the theory cannot be seen as GR with corrections and, moreover, there arise instabilities in the tensor and scalar parts. This class contains conformal gravity and R^2 gravity. At first glance one might expect this is the massless limit, but it is not. The action is

$$S^{(\mathrm{T})}_{\mathrm{HCG}}[H_{ij}] = \frac{1}{2\kappa} \int \mathrm{d}^4x \left[-2\alpha \, \Box H_{ij} \, \Box H^{ij}\right] , \tag{C.1}$$

$$S^{(\mathrm{V})}_{\mathrm{HCG}}[\Sigma_i] = \frac{1}{2\kappa} \int \mathrm{d}^4x \left[-\alpha \left(\partial_j \dot{\Sigma}_i \, \partial^j \dot{\Sigma}^i - \triangle \Sigma_i \, \triangle \Sigma^i\right)\right] , \tag{C.2}$$

$$S^{(\mathrm{S})}_{\mathrm{HCG}}[\Phi, \Theta, \Xi] = \frac{1}{2\kappa} \int \mathrm{d}^4x \left[-3\alpha \, \Theta^2 - \beta \, \Xi^2 - 12\beta \, \Xi \, \Box \Phi + 6\beta \, \Xi \, \Theta\right] , \tag{C.3}$$

where we have already introduced Ξ to replace ${}^{(1)}R$.

The EOMs for the tensor and vector variables can be easily found:

$$\alpha \, \Box^2 H_{ij} = 0 \, , \quad \alpha \, \Box \triangle \Sigma_i = 0 \, . \tag{C.4}$$

When $\alpha \neq 0$, these EOMs admit plane-wave solutions

$$H_{ij} = A_{ij} \, \mathrm{e}^{\mathrm{i}\omega_A (z-t)} + t \, B_{ij} \, \mathrm{e}^{\mathrm{i}\omega_B (z-t)} \, , \quad \Sigma_i = C_i \, \mathrm{e}^{\mathrm{i}\omega_C (z-t)} \, , \tag{C.5}$$

with A_{ij} , B_{ij} and C_i arbitrary constants. The tensor wave indicates the emergence of an instability.

The scalar part is more involved. The variations of the action with respect to each scalar variable give a set of equations

$$\beta \, \Box \Xi = 0 \, , \quad -6\beta \, \Box \Phi + 3\beta \, \Theta - \beta \, \Xi = 0 \, , \quad \alpha \, \Theta - \beta \, \Xi = 0 \, . \tag{C.6}$$

If $\beta = 0$, then we have $\Theta = 0$ hence $\Phi = \Psi$, but these cannot be determined. If $\beta \neq 0$ but $\alpha = 0$, then $\Xi = 0$ and one cannot determine Φ and Θ. If both α and β are nonzero but $\alpha = 3\beta$, we obtain equations for Φ and Ψ as

$$\Box \Phi = \Box \triangle \Psi = 0 \, . \tag{C.7}$$

Finally, in the most generic case when both α and β are nonzero and $\alpha \neq 3\beta$, we can eliminate Ξ and Θ to have the EOM for Φ as

$$\Box^2 \Phi = 0 \,, \tag{C.8}$$

which suggests that Φ is unstable.

Appendix D

Detector responses

This Appendix summarizes the angular pattern functions for the six polarizations defined by

$$F_\alpha(\boldsymbol{\Omega}) \equiv \boldsymbol{D} : \boldsymbol{e}_\alpha(\boldsymbol{\Omega}) \,, \tag{D.1}$$

where $\boldsymbol{D}$ is the so-called detector tensor, $\boldsymbol{e}_\alpha$ the polarization tensor, $\boldsymbol{\Omega}$ the unit vector pointing the impinging direction of a GW, and the symbol : denotes contraction between tensors.

In Chapter 2, we used an inertial coordinate system such that a GW propagates in the z direction. We call it the gravitational-wave frame and denote its orthonormal basis as $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{\Omega})$, where $\boldsymbol{\Omega}$ is the unit vector along the z direction. Note that there is a rotation degree of freedom along the $\boldsymbol{\Omega}$ axis, denoted as the polarisation angle ψ. The polarization tensors for $\alpha \in \{+, \times, x, y, \mathrm{B}, \mathrm{L}, \mathrm{T}, \mathbb{T}\}$ can be written using the unit vectors as

$$\begin{aligned}
\boldsymbol{e}_+ &= \boldsymbol{m} \otimes \boldsymbol{m} - \boldsymbol{n} \otimes \boldsymbol{n} \,, \\
\boldsymbol{e}_\times &= \boldsymbol{m} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{m} \,, \\
\boldsymbol{e}_x &= \boldsymbol{m} \otimes \boldsymbol{\Omega} + \boldsymbol{\Omega} \otimes \boldsymbol{m} \,, \\
\boldsymbol{e}_y &= \boldsymbol{n} \otimes \boldsymbol{\Omega} + \boldsymbol{\Omega} \otimes \boldsymbol{n} \,, \\
\boldsymbol{e}_\mathrm{B} &= \boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{n} \otimes \boldsymbol{n} \,, \\
\boldsymbol{e}_\mathrm{L} &= \sqrt{2}\, \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \,, \\
\boldsymbol{e}_\mathrm{T} &= \sqrt{\frac{2}{3}}\, \left(\boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \right) \,, \\
\boldsymbol{e}_\mathbb{T} &= \frac{1}{\sqrt{3}}\, \left(\boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{n} \otimes \boldsymbol{n} - 2\boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \right) \,.
\end{aligned} \tag{D.2}$$

To characterise ground-based interferometers or pulsar timing arrays, we introduce an inertial coordinate system specified by an orthonormal basis $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ such that $\boldsymbol{w}$ is the upward normal to the Earth's surface. We call it the detector frame and introduce the usual polar angles (θ, ϕ) in this frame to point the GW propagation direction. We then rotate the detector frame $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ to $(\boldsymbol{u}', \boldsymbol{v}', \boldsymbol{w}')$ so that $\boldsymbol{w}'$ points toward the

GW propagation as shown in Fig. 2 of Ref. [NTH+09]. Their relationship is

$$\begin{cases} \boldsymbol{u}' = \cos\theta\,\cos\phi\,\boldsymbol{u} + \cos\theta\,\sin\phi\,\boldsymbol{v} - \sin\theta\,\boldsymbol{w}\,, \\ \boldsymbol{v}' = -\sin\phi\,\boldsymbol{u} + \cos\phi\,\boldsymbol{v}\,, \\ \boldsymbol{w}' = \sin\theta\,\cos\phi\,\boldsymbol{u} + \sin\theta\,\sin\phi\,\boldsymbol{v} + \cos\theta\,\boldsymbol{w}\,. \end{cases} \tag{D.3}$$

Finally, introducing ψ as the angle from $\boldsymbol{u}'$ to $\boldsymbol{m}$ in the plane perpendicular to $\boldsymbol{w}'$, we arrive at the relationship between the mediation coordinates and the GW coordinates

$$\begin{cases} \boldsymbol{m} = \cos\psi\,\boldsymbol{u}' + \sin\psi\,\boldsymbol{v}'\,, \\ \boldsymbol{n} = -\sin\psi\,\boldsymbol{u}' + \cos\psi\,\boldsymbol{v}'\,, \\ \boldsymbol{\Omega} = \boldsymbol{w}'\,. \end{cases} \tag{D.4}$$

Let us consider L-shaped interferometers such as LIGO, Virgo and KAGRA. The detector tensor, in this case, is [NTH+09]

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{v}\otimes\boldsymbol{v}\right)\,. \tag{D.5}$$

The antenna pattern functions F_α are calculated as

$$\begin{aligned} F_+(\theta,\phi,\psi) &= \frac{1}{2}\left(1+\cos^2\theta\right)\,\cos 2\phi\,\cos 2\psi - \cos\theta\,\sin 2\phi\,\sin 2\psi\,, \\ F_\times(\theta,\phi,\psi) &= -\frac{1}{2}\left(1+\cos^2\theta\right)\,\cos 2\phi\,\sin 2\psi - \cos\theta\,\sin 2\phi\,\cos 2\psi\,, \\ F_x(\theta,\phi,\psi) &= \sin\theta\,\left(\cos\theta\,\cos 2\phi\,\cos\psi - \sin 2\phi\,\sin\psi\right)\,, \\ F_y(\theta,\phi,\psi) &= -\sin\theta\,\left(\cos\theta\,\cos 2\phi\,\sin\psi + \sin 2\phi\,\cos\psi\right)\,, \\ F_{\rm B}(\theta,\phi) &= -\frac{1}{2}\,\sin^2\theta\,\cos 2\phi\,, \\ F_{\rm L}(\theta,\phi) &= \frac{1}{\sqrt{2}}\,\sin^2\theta\,\cos 2\phi\,, \\ F_{\rm T} &= 0\,, \\ F_{\mathbb{T}}(\theta,\phi) &= -\frac{\sqrt{3}}{2}\,\sin^2\theta\,\cos 2\phi\,. \end{aligned} \tag{D.6}$$

The scalar functions are degenerate since they have the same dependence on the angles.

If we consider a pulsar frequency shift as a signal, the detector tensor is [YS13, LJP+10, QBK21]

$$\boldsymbol{D} = \frac{1}{2}\,\frac{1}{1+v\,\boldsymbol{\Omega}\cdot\boldsymbol{w}}\,\boldsymbol{w}\otimes\boldsymbol{w}\,, \tag{D.7}$$

where $\boldsymbol{w}$ points in the direction of the pulsar. It was pointed out in Ref. [LJP+10, QBK21] that there should be a modification factor v representing the subluminal velocity of GWs in the denominator. In the luminal ($v = 1$) case, the antenna pattern functions are

$$
\begin{aligned}
F_{+}(\theta,\psi) &= \frac{1}{2}\,\frac{\sin^2\theta}{1+\cos\theta}\;\cos 2\psi\,,\\
F_{\times}(\theta,\psi) &= -\frac{1}{2}\,\frac{\sin^2\theta}{1+\cos\theta}\;\sin 2\psi\,,\\
F_{x}(\theta,\psi) &= -\frac{1}{2}\,\frac{\sin 2\theta}{1+\cos\theta}\;\cos\psi\,,\\
F_{y}(\theta,\psi) &= \frac{1}{2}\,\frac{\sin 2\theta}{1+\cos\theta}\;\sin\psi\,,\\
F_{\mathrm{B}}(\theta) &= \frac{1}{2}\,\frac{\sin^2\theta}{1+\cos\theta}\,,\\
F_{\mathrm{L}}(\theta) &= \frac{1}{\sqrt{2}}\,\frac{\cos^2\theta}{1+\cos\theta}\,,\\
F_{\mathrm{T}}(\theta) &= \frac{1}{\sqrt{6}}\,\frac{1}{1+\cos\theta}\,,\\
F_{\mathbb{T}}(\theta) &= \frac{1}{2\,\sqrt{3}}\,\frac{1-3\cos^2\theta}{1+\cos\theta}\,.
\end{aligned}
\tag{D.8}
$$

The scalar functions are not degenerate.

Appendix E

Properties of stellar matter

This Appendix provides an overview of the properties of stellar matter. The treatment of this Appendix follows [KWW12].

E.1 Eddington standard model

We consider the case where the polytropic coefficient K is a free parameter. If there is a radiation pressure, the total pressure is given as

$$\begin{aligned} P &= P_{\rm gas} + P_{\rm rad} \\ &= \frac{\mathscr{R}}{\mu}\rho T + \frac{a}{3}T^4 \,. \end{aligned} \tag{E.1}$$

Supposing the ratio is constant all over the star

$$\beta \equiv \frac{P_{\rm gas}}{P} = \text{const.}\,, \tag{E.2}$$

we have

$$1 - \beta = \frac{P_{\rm rad}}{P} = \frac{a\,T^4}{3P}\,, \tag{E.3}$$

where the range of β is $0 \leq \beta \leq 1$. Thus, the total pressure is approximately given by $P \sim T^4$. When the ratio takes $\beta = 1$, the radiation pressure is zero. On the other hand, when the ratio takes $\beta = 0$, the gas pressure is zero, and the total pressure (E.1) can be written as

$$P = \frac{\mathscr{R}}{\mu\,\beta}\rho T\,. \tag{E.4}$$

Using the formula

$$\frac{1-\beta}{\beta} = \frac{P_{\rm rad}}{P_{\rm gas}} = \frac{a\,\mu\,T^3}{3\,\mathscr{R}\,\rho}\,, \tag{E.5}$$

we obtain the polytropic relation with $n = 3$,

$$P = \left(\frac{3\mathscr{R}^4}{a\,\mu^4}\right)^{\frac{1}{3}} \left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{3}} \rho^{\frac{3}{4}} \,. \tag{E.6}$$

Since β is a free parameter, the polytropic coefficient K is also a free parameter. This is known as the Eddington standard model for stellar model.

E.2 Degenerate electron gas

The phase space density of an electron is given by

$$\begin{aligned} f(p) &= \frac{8\pi\,p^2}{h^3} \quad \text{for} \quad p \le p_{\rm F} \,, \\ f(p) &= 0 \quad \text{for} \quad p > p_{\rm F} \,, \end{aligned} \tag{E.7}$$

where $p_{\rm F}$ is the Fermi momentum. The total number of electrons in the volume dV is given by

$$n_{\rm e}\, dV = dV \int_0^{p_{\rm F}} {\rm d}p\, \frac{8\pi\,p^3}{h^3} = \frac{8\pi}{3h^3} p_{\rm F}^3\, dV \,, \tag{E.8}$$

and the pressure of the electron is

$$P_{\rm e} = \frac{8\pi}{3h^3} \int_0^{p_{\rm F}} {\rm d}p\, p^3 v(p) \,. \tag{E.9}$$

The special relativity tells that the momentum of the electron is given by

$$p = \frac{m_{\rm e} v}{\sqrt{1 - v^2/c^2}} \,, \tag{E.10}$$

and thus, the pressure of the electron is

$$P_{\rm e} = \frac{8\pi}{3h^3} \int_0^{p_{\rm F}} {\rm d}p\, p^3 \frac{p/(m_{\rm e}\, c)}{\sqrt{1 + p^2/(m_{\rm e}^2\, c^2)}} \,. \tag{E.11}$$

We introduce parameters defined by

$$x \equiv \frac{p_{\rm F}}{m_{\rm e}\, c} = \frac{v_{\rm F}/c}{\sqrt{1 - v_{\rm F}^2/c^2}} \,, \quad y \equiv \frac{p}{m_{\rm e}\, c} \,. \tag{E.12}$$

Then, the pressure of the electron is written in terms of these parameters as

$$\begin{aligned} P_{\rm e} &= \frac{8\pi\, c^5\, m_{\rm e}^4}{3h^3} \int_0^x {\rm d}y\, \frac{y^4}{\sqrt{1+y^2}} \\ &= \frac{\pi\, c^5\, m_{\rm e}^4}{3h^3} \left[x(2x^2 - 3)(1 + x^2)^{\frac{1}{2}} + 3 \operatorname{arcsinh} x \right] \\ &= \frac{\pi\, c^5\, m_{\rm e}^4}{3h^3} \left[x(2x^2 - 3)(1 + x^2)^{\frac{1}{2}} + 3 \ln \left[x + (1 + x^2)^{\frac{1}{2}} \right] \right] \,. \end{aligned} \tag{E.13}$$

Since the parameter x is rewritten as

$$v_{\mathrm{F}}^2/c^2 = \frac{x^2}{1+x^2}\,, \tag{E.14}$$

we can evaluate the limiting cases of the pressure of the electron. If x is sufficiently small $x \ll 1$, i.e., $v_{\mathrm{F}}/c \ll 1$, all electrons are moving very slowly compared to the speed of light (non-relativistic motion). On the other hand, if x is much larger than 1, i.e., $x \gg 1$ and then $v_{\mathrm{F}}/c \gg 1$, all electrons are moving at velocities close to the speed of light. (relativistic motion). Therefore, the equation of state for a completely degenerate non-relativistic electron gas is given by

$$\lim_{x\to 0} P_{\mathrm{e}} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} \frac{h^2}{m_{\mathrm{e}}\, m_{\mu}^{5/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{\frac{5}{3}} \tag{E.15}$$

and extreme relativistic complete degenerate electron gas is given by

$$\lim_{x\to\infty} P_{\mathrm{e}} = \left(\frac{3}{\pi}\right)^{\frac{1}{3}} \frac{h\,c}{8 m_{\mu}^{4/3}} \left(\frac{\rho}{\mu_{\mathrm{e}}}\right)^{\frac{4}{3}} \tag{E.16}$$

Hence, we can see that the non-relativistic electron gas is the polytropic relation with $n = 1.5$, and the relativistic electron gas is $n = 3$. The significant difference from the Eddington standard model is that the polytropic coefficient K is not a free parameter. Since the mass of a polytropic star with $n = 3$ does not depend on the stellar centre density and the polytropic coefficient K is fixed, an object supported by the degeneracy pressure of relativistic electrons takes only one mass (Chandrasekhar mass).

E.3 Polytropic relation

The case where pressure P is expressed as a function only of density ρ is called a barotropic relation $P = P(\rho)$. In particular, when pressure is expressed as a power of density, as in

$$P = K\,\rho^{\Gamma}\,, \tag{E.17}$$

it is called the polytropic relation. Although this is a simple expression of an equation of state, it can be used as an equation of state for many astronomical objects and is a very useful relation. In this section, we show that Poisson's equation (polytropic relationship for ideal gases) can be derived by assuming an adiabatic state for ideal gases. This allows us to interpret the parameter Γ as the effective specific heat ratio γ.

Let us consider a one-component ideal gas with molecular weight μ. The equation of state for an ideal gas of n mol is

$$PV = n\mathscr{R}T\,, \tag{E.18}$$

where T is temperature, and $\mathscr{R}$ is the gas constant. V is the volume per n mol, but since it depends on the number of mol and is not convenient, so we express it in terms of density. Since the molecular weight of the fluid is μ, the mass (molar mass) of 1 mol of fluid is $\mu\,[g]$. Therefore, the density of n mol fluid is given by $\rho = (n\mu)/V$, and the equation of state for an ideal gas can be written as the form

$$P = \frac{\mathscr{R}}{\mu}\rho T\,. \tag{E.19}$$

We consider the variation in internal energy dU of an ideal gas. The variation in internal energy of an ideal gas is determined only by the change in temperature, not by the change in volume, and is

$$dU = C_V\,dT\,, \tag{E.20}$$

where C_V is the heat capacity at constant volume defined as

$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V\,. \tag{E.21}$$

In the case of an adiabatic process, the change in heat quantity is zero ($dQ = 0$), so the change in internal energy is

$$dU = -P\,dV \tag{E.22}$$

from the first law of thermodynamics. Thus, we obtain

$$C_V\,dT + P\,dV = 0\,. \tag{E.23}$$

Using the equation of state $PV = n\mathscr{R}T$ for the ideal gas, it becomes

$$C_V\,dT + \frac{n\mathscr{R}T}{V}dV = 0\,. \tag{E.24}$$

Using Mayer's relation, which holds for ideal gases

$$C_V(\gamma - 1) = n\mathscr{R}\,, \tag{E.25}$$

we get

$$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0\,, \tag{E.26}$$

where γ is the heat capacity ratio $\gamma \equiv C_P/C_V$ and C_P is the heat capacity at constant pressure. Integrating the equation, we have

$$\log T + (\gamma - 1)\log V = const. \tag{E.27}$$

Therefore, we obtain

$$T\,V^{\gamma-1} = const.\,. \tag{E.28}$$

Using the equation of state $PV = n\mathscr{R}T$ for the ideal gas, it can be written as

$$P = K\,\rho^{\gamma}\,, \tag{E.29}$$

and

$$T = K\,\rho^{\gamma-1}\,. \tag{E.30}$$

Not only in adiabatic processes, pressure can often be written as a power of density. Therefore, the specific heat ratio γ can be considered as a parameter Γ, including cases where there is some heat exchange. In this case, the parameter Γ is interpreted as the effective specific heat ratio. Also, it can be seen that the closer γ is to 1, the closer it is to isothermal.

Appendix F

Solution for higher-order Helmholtz equations

We consider a linear inhomogeneous equation of the form

$$f(\triangle)\,\varphi = S\,, \tag{F.1}$$

where f is an n-th order polynomial, which we call the characteristic function, $\triangle$ the flat-space Laplace operator, and S a given source function. Without loss of generality, using the n roots for the characteristic equation $f(x) = 0$, $x = \lambda_i$ $(i = 1, \cdots, n)$, the problem reduces to solving

$$\prod_{i=1}^{n} (\triangle - \lambda_i)\ \varphi = S\,. \tag{F.2}$$

We assume $\lambda_i \neq \lambda_j$ for $i \neq j$ for simplicity, but extending the formula to degenerate cases is straightforward. The above equation admits a solution of the form $\varphi = \sum_{i=1}^{n} \varphi_i$, where each φ_i solves a single Helmholtz equation

$$(\triangle - \lambda_i)\ \varphi_i = c_i\, S \tag{F.3}$$

with the coefficients c_i $(i = 1, \cdots, n)$ satisfying the following system of n linear equations

$$
\begin{aligned}
&\sum_{i=1}^{n} c_i = 0\,, \\
&\sum_{i=2}^{n} c_i\,(\lambda_i - \lambda_1) = 0\,, \\
&\sum_{i=3}^{n} c_i\,(\lambda_i - \lambda_1)\,(\lambda_i - \lambda_2) = 0\,, \\
&\vdots \\
&\sum_{i=k}^{n} c_i \prod_{j=1}^{k-1}(\lambda_i - \lambda_j) = 0\,, \\
&\vdots \\
&\sum_{i=n-1}^{n} c_i \prod_{j=1}^{n-2}(\lambda_i - \lambda_j) = 0\,, \\
&c_n \prod_{j=1}^{n-1}(\lambda_n - \lambda_j) = 1\,.
\end{aligned}
\tag{F.4}
$$

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