Constraint propagation revisited
— Adjusted ADM formulation for numerical relativity —

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Abstract

Formulation of the Einstein equations is one of the necessary implements for realizing long-term stable and accurate numerical simulations. We re-examine our formulation scheme, that intends to construct a dynamical system which evolves toward the constraint surface as the attractor, by adjusting evolution equations with constraints. We propose an additional guideline which may delay the final blow-up. The new idea is to avoid multipliers to the evolution equations which produce non-linear growth of the constraints in the later stage. Slight but actual improvement can be seen in our test simulations.

1 Introduction

With the purpose of the predictions of precise gravitational waveforms from the coalescence of the binary neutron-stars and/or black-holes, the research field of "numerical relativity" has been developed for the past three decades. The difficulty of numerical integrations of the Einstein equations arises both from its mathematical complexity of the equations and from high-level requirements for computational skills and technology.

In 2005-2006, several groups independently announced that the success of the inspiral black-hole binary merger [1, 2, 3, 4, 5]. There are many implements for their successes, such as gauge conditions, coordinate selections, boundary treatments, singularity treatments, numerical discretization, and mesh refinements, together with the re-formulation of the Einstein equations which we will discuss here.

There are many approaches to re-formulate the Einstein equations for obtaining a long-term stable and accurate numerical evolution (e.g. see references in [6]). In a series of our works, we have proposed to construct a system that has its constraint surface as an attractor. By applying eigenvalue analysis of constraint propagation equations, we showed that there *is* a constraint-violating mode in the standard Arnowitt-Deser-Misner (ADM) evolution system [7, 8], which has been used for simulations over 20 years, when it is applied to a single non-rotating black-hole space-time [10]. We also found that such a constraint-violating mode can be compensated if we adjust the evolution equations with a particular modification using constraint terms like the one proposed by Detweiler [9].

Our predictions are borne out in simple numerical experiments using the Maxwell, Ashtekar, and ADM systems [10, 11, 12, 13]. There are also several numerical experiments to confirm our predictions are effective [14, 15].

The recent binary black-hole simulations also applies such ideas. Pretorius [1] uses harmonic decomposition of the Einstein equations with constraint damping terms. NASA/Goddard, UTB, and LSU groups applied modified BSSN formulation [16], while PSU group applied another modified BSSN formulation [15]. Here, BSSN is the widely used modification of the ADM formulation which was originally proposed by Kyoto group[17, 18].

In this report, we re-examine our formulation scheme and propose the additional guideline which may delay the final blow-up. The new idea is to avoid multipliers to the evolution equations which may produce non-linear growth of the constraints in the later stage. We applied the idea to the adjusted ADM formulation, and also show our test simulations. We think that killing or compensating the constraint violation mode is the essential to this formulation problem, and for that purpose, the ADM formulation is the best benchmark to work with.

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2 Adjusted system and Constraint Propagation

2.1 Our idea of adjusted system

Suppose we have a dynamical system of variables $u^a(x^i, t)$, which has evolution equations,

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots), \tag{1}$$

and the (first class) constraints,

$$C^\alpha(u^a, \partial_i u^a, \cdots) \approx 0. \tag{2}$$

Note that we do not require (1) to form a first-order hyperbolic form. We propose to investigate the evolution equation of C^α (constraint propagation, CP),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots), \tag{3}$$

for evaluating violation features of constraints.

The character of constraint propagation, (3), will vary when we modify the original evolution equations. Suppose we modify (adjust) (1) using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots) + F(C^\alpha, \partial_i C^\alpha, \cdots), \tag{4}$$

then (3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots) + G(C^\alpha, \partial_i C^\alpha, \cdots). \tag{5}$$

Therefore, finding a proper adjustment $F(C^\alpha, \cdots)$ is a quite important problem.

Hyperbolicity analysis may be a way to evaluate constraint propagation, (3) and (5) [19]. However, this requires (3) to be a first-order system which is easy to be broken. (See e.g. Detweiler-type adjustment [9] in the ADM formulation [10]). Furthermore hyperbolicity analysis only concerns the principal part of the equation, that may fail to analyze the detail evaluation of evolution.

Alternatively, we have proceeded an eigenvalue analysis of the whole RHS in (3) and (5) after a suitable homogenization,

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \text{ where } \quad C(x,t)^\alpha = \int \hat{C}(k,t)^\alpha \exp(ik \cdot x) d^3k, \tag{6}$$

and conjectured that the system is more stable, if the eigenvalues of $M^\alpha{}_\beta$ [we call them constraint amplification factors (CAFs)] has a *negative real-part* or *non-zero imaginary-part* [10, 11, 12, 13].

2.2 Additional idea

Suppose that RHS of the constraint propagation equation (5) accidentally includes C^2 terms,

$$\partial_t C = -aC + bC^2, \tag{7}$$

then the solution will blow-up as

$$C = \frac{-aC_0 \exp(-at)}{-a + bC_0 - bC_0 \exp(-at)}. \tag{8}$$

The blow-up will appear when C^2-term is comparable to C-term, that is, the last stage of simulation supposing the constraint surface is the attractor. We therefore have to prohibit the adjustments which simply produce self-growing terms (C^2) in constraint propagation, $\partial_t C$.

3 Adjusted ADM system

In the ADM system, we can write possible adjustments generally as [10, 11, 12]:

$$\begin{aligned}\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \\ &\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \qquad (9)\\ \partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k)K_{kj} + (\nabla_j \beta^k)K_{ki} + \beta^k \nabla_k K_{ij} \\ &\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \qquad (10)\end{aligned}$$

with constraint equations

$$\begin{aligned}\mathcal{H} &:= R^{(3)} + K^2 - K_{ij}K^{ij}, \qquad (11)\\ \mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. \qquad (12)\end{aligned}$$

Along to the discussion in §2.2, by carefully observing the constraint propagation equations $\partial_t \mathcal{H} = \cdots$ and $\partial_t \mathcal{M}_i = \cdots$ with above adjustments, we conclude that the adjustments using p, q, P, Q-terms in the above (9) and (10) may produce non-linear terms in constraint propagation equations. Therefore, we have not to put too much confidence for adjustments using these terms. Conversely, several adjustments are *safe* for this points, and are expected to compensate such non-linear term effects.

In Fig.1, we demonstrate numerical evolutions of such an adjusted ADM system. We plot violation of Hamiltonian constraints versus time for Teukolsky wave evolution with harmonic slicing, and with periodic boundary condition, which is one of the benchmark test of the formulation problem proposed by the Mexico NR workshop in 2002 [20]. We apply two sets of adjusted ADM equations: The Case (I)

$$\begin{aligned}\partial_t \gamma_{ij} &= (\text{first line of (9)}) - \kappa_1 \alpha \gamma_{ij} \mathcal{H} \qquad (13)\\ \partial_t K_{ij} &= (\text{first line of (10)}) + \kappa_2 \alpha \gamma_{ij} \gamma^{kl}\, \partial_k \mathcal{M}_l \qquad (14)\end{aligned}$$

and the Case (II):

$$\begin{aligned}\partial_t \gamma_{ij} &= (\text{first line of (9)}) - \kappa_1 \alpha^3 \gamma_{ij} \mathcal{H} \qquad (15)\\ \partial_t K_{ij} &= (\text{first line of (10)}) + \kappa_1 \alpha^3 (K_{ij} - (1/3) K \gamma_{ij})\mathcal{H} + \kappa_2 \alpha \gamma_{ij} \gamma^{kl}\, \partial_k \mathcal{M}_l \\ &\quad + \kappa_1 \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}]\mathcal{M}_k + \kappa_1 \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}](\nabla_k \mathcal{M}_l), \qquad (16)\end{aligned}$$

where the terms with coefficient κ_1 in Case (II) were those in Detweiler [9], while the terms with coefficient κ_2 in Case (I/II) are newly introduced adjustment along to the above discussion.

We see in Fig.1 that the evolution with the standard ADM system is the shortest lifetime in simulation, while Case (II) makes twice as much longer evolution available. Moreover, the newly added term helps to extend the lifetime of simulation. The new term works effectively, makes 10% longer evolution available, while this is not yet perfect nor drastic.

We are now investigating CAFs of these new adjustments, blow-up time estimations, together with other numerical demonstrations. These will be reported elsewhere near future.

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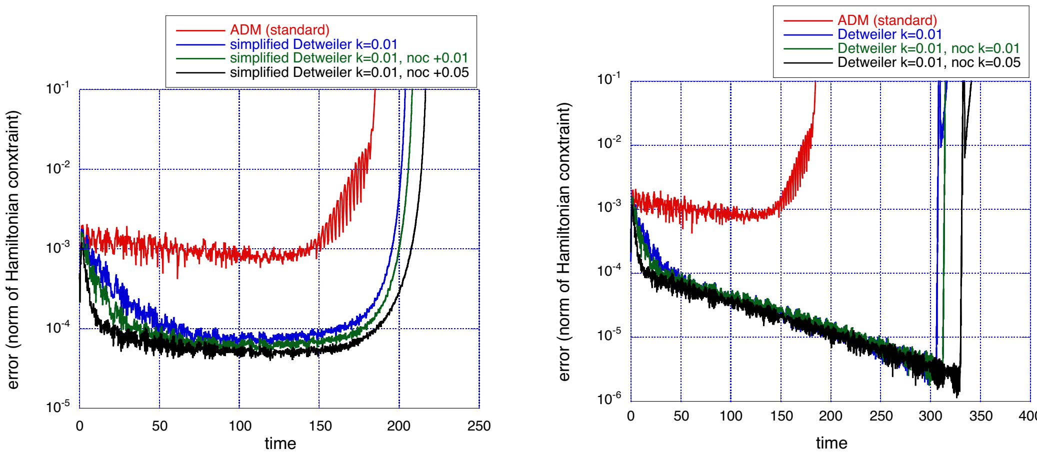

Figure 1: Comparisons of numerical evolutions of adjusted ADM systems, using Teukolsky wave propagation. L2 norm of the Hamiltonian constraint $\mathcal{H}$ is plotted. (Left panel) The Case (I). (Right panel) The Case (II). Cactus-based original (3+1)-dimensional code was applied.

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