A remarkable dynamical symmetry of the Landau problem

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Abstract. We show that the dynamical group of an electron in a constant magnetic field is the group of symplectomorphisms $Sp(4,\mathbb{R})$. It is generated by the spinorial realization of the conformal algebra $\mathfrak{so}(2,3)$ considered in Dirac's seminal paper "A Remarkable Representation of the 3 + 2 de Sitter Group". The symplectic group $Sp(4,\mathbb{R})$ is the double covering of the conformal group $SO(2,3)$ of 2+1 dimensional Minkowski spacetime which is in turn the dynamical group of a hydrogen atom in 2 space dimensions. The Newton-Hooke duality between the 2D hydrogen atom and the Landau problem is explained via the Tits-Kantor-Koecher construction of the conformal symmetries of the Jordan algebra of real symmetric 2×2 matrices. The connection between the Landau problem and the 3D hydrogen atom is elucidated by the reduction of a Dirac spinor to a Majorana one in the Kustaanheimo-Stiefel spinorial regularization.

1. Introduction

In the heart of the quantum theory is the passionate history of the hydrogen atom spectrum. By good fortune the Coulomb potential describing the "the action at a distance" is the twin brother of the potential of the Newton universal interaction. Therefore the quantum motion of electron in the hydrogen atom amounts to the quantization of the Kepler orbits. In some poetical sense, the heavenly spheres mirror the depth of the micro-cosmos.

In his famous paper, Vladimir Fock [13] explained the accidental degeneracies in the spectrum of the hydrogen atom by the existence of an additional integral of motion given by the Laplace-Runge-Lenz. Fock's result allows for a generalization. Namely, a n-dimensional hydrogen atom whose Schrödinger equation for the bounded states in momentum space is transformed by stereographic projection to the Laplace's equation on a n-dimensional sphere S^n[9].

In this work we study the duality between the 2-dimensional (2D) hydrogen atom on one hand and the Landau problem for the quantization of the electronic orbits in the uniform magnetic field on the other:

$$\text{2D } e^- \text{ in electric field} \quad \overset{Newton-Hooke}{\leftrightarrow} \quad \text{2D } e^- \text{ in magnetic field} \;. \tag{1}$$

The classical geodesic motion on the sphere, say on the equator, is seen as the periodic circular Larmour motion in the magnetic field directed between the poles. The Fock method applied for

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2D hydrogen system with potential $-k/r$ sends orbital eigenfunctions to spherical harmonics in momentum space [6, 28, 4, 26, 31] which are the harmonic functions of the sphere S^2, *i.e.,* eigenfunctions of an isotropic harmonic oscillator with two degrees of freedom. The mapping from 2D harmonic oscillator to 2D Coulomb-Kepler problem is known to be the Levi-Civita transform [4]. In this paper we show that the dynamical group of the Landau problem is the double covering $Sp(4,\mathbb{R})$ of the conformal group $SO(2,3)$ of the 2D hydrogen atom and arises from the exponentiation of the remarkable spinorial representation of $SO(2,3)$ found by Dirac in his famous paper "A Remarkable Representation of the 3 + 2 de Sitter Group"[11].

Different dualities between Coulomb-Kepler motion and harmonic oscillators has been already studied by many autors[1][4, 9, 1] in various dimensions, however to the best of our knowledge the implications about the Landau problem were not highlighted. A tremendous amount of work on the spectrum generating algebra of the hydrogen atom has been done by Asım Barut and his collaborators [2, 3, 5](See also [22]). Gradually it became clear that the spectrum of 3D hydrogen atom carries a minimal (massless) representation with helicity $\lambda = 0$ of the conformal group $SO(2,4)$. It allows for a dual description in terms of 4D harmonic oscillator via a ladder $U(2,2)$-representation that has been put forward by Mack and Todorov[21]. The correspondence with the massless $U(2,2)$-representation having arbitrary helicity λ (isomorphic to a $SO(2,4)$-representation) turns out to be the quantization of the Kustaanheimo-Stiefel regularization [20] in celestial mechanics. This correspondence holds true not only for the quantum Kepler motion of electron in the field of the positively charged nucleus but also for more general binary systems of coupled dyons [3, 32].

A powerful approach to the conformal dynamical symmetries of generalized quantum Kepler problems based on Jordan algebras was proposed by Guowu Meng [23]. To an Euclidean Jordan algebra one associates a symmetric null ray cone whose automorphisms form a conformal group. From the minimal data of the Jordan algebra $\mathfrak{J}_2^C$ of 2×2 complex Hermitian matrices one is able to build the whole conformal algebra $\mathfrak{so}(2,4)$ of 3D hydrogen atom[23, 27]. We show that the reduction from 3D to 2D hydrogen atom dynamical group is done by simply imposing the reality condition on the Jordan algebra of observables $\mathfrak{J}_2^C$:

$$\begin{array}{ccccc} \mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) & = & \mathfrak{so}(2,4) & \xleftarrow{Kustaanheimo-Stiefel} & \mathfrak{su}(2,2) \\ \big\uparrow{\scriptstyle x=x^t} & & & & \big\uparrow{\scriptstyle \psi=\psi^c} \\ \mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) & = & \mathfrak{so}(2,3) & \xleftarrow{Levi-Civita} & \mathfrak{sp}(4,\mathbb{R}) \end{array} \quad . \tag{2}$$

The projection $\pi : \mathfrak{J}_2^{\mathbb{C}} \to \mathfrak{J}_2^{\mathbb{R}}$ simply projects out the complex Pauli matrix σ_2. On the other hand the 3D hydrogen atom is in duality with the 4D harmonic oscillator (the quantization of the harmonic motion on the sphere S^3) the duality mapping being the Kustaanheimo-Stiefel transformation [20], the so called spinorial regularization removing collision Kepler orbits from the phase space. Similarly the Levi-Civita transformation connects the 2D harmonic oscillator with the 2D hydrogen atom (1). We found out that the spinorial reduction from 4D harmonic oscillator (regularized motion on S^3) to 2D harmonic oscillator (Landau problem, *i.e.,* motion on S^2) is the reduction from Dirac to the Majorana spinor. In other words on the spinorial side of the diagram we also apply the reality condition $\psi = \psi^c$.

2. Landau Problem and Harmonic oscillator

An electron in an uniform magnetic field propagates on circular orbits with Larmour frequency $\omega = \frac{eB}{mc}$. The Landau quantization of the circular orbits leads to an isotropic quantum harmonic

[1] For comprehensive lecture notes and further references we send the reader to [30].

oscillator with two modes. The minimal coupling of the electron's charge density to the external magnetic field described by the electromagnetic vector potential $\boldsymbol{\mathcal{A}}$ leads to the Hamiltonian

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2 =: \frac{1}{2m}\boldsymbol{P}^2 \ . \tag{3}$$

The constant uniform magnetic field $\boldsymbol{B} = B\hat{z}$ along the z-direction can be obtained from different potentials $\boldsymbol{\mathcal{A}} = (\mathcal{A}_x, \mathcal{A}_y)$ in the plane. We choose the symmetric gauge

$$\boldsymbol{\mathcal{A}} = (\mathcal{A}_x, \mathcal{A}_y) = \frac{B}{2}(-y, x) \ , \qquad \mathcal{A}_i = -\frac{B}{2}\epsilon_{ij}x^j \tag{4}$$

but most of our conclusions are gauge independent. The gauge independent kinetical momenta $\boldsymbol{P}$ and the coordinates $\boldsymbol{X}$ of the center of the cyclotron motion are related to the phase space canonical coordinates (x, y, p_x, p_y) by

$$P_i = m\dot{x}_i = m\frac{\partial H}{\partial p_i} = p_i - \frac{e}{c}\mathcal{A}_i, \qquad X_i = x_i + \frac{1}{m\omega}\epsilon_{ij}P^j \ . \tag{5}$$

The center of mass coordinates $\boldsymbol{X} = (X, Y)$ are integrals of motion $\dot{X} = 0 = \dot{Y}$ and decouple from the system. One has two independent Heisenberg algebras $[\boldsymbol{P}, \boldsymbol{X}] = 0$

$$[P_x, P_y] = i\frac{\hbar e}{c}B = i\hbar m\omega = \frac{i\hbar^2}{\ell^2} \ , \qquad\qquad [X, Y] = -i\frac{\hbar}{m\omega} = -i\ell^2 \ .$$

Here ℓ stands for the magnetic length $\ell^2 = \frac{\hbar}{m\omega} = \frac{\hbar c}{eB}$. We introduce also dimensionless canonical coordinates of the phase space

$$x = \sqrt{2}\xi\ell \qquad y = \sqrt{2}\eta\ell \qquad p_x = p_\xi\sqrt{m\omega\hbar}/\sqrt{2} \qquad p_y = p_\eta\sqrt{m\omega\hbar}/\sqrt{2}$$

satisfying the canonical commutation relations

$$[\xi, p_\xi] = i = [\eta, p_\eta] \qquad [\xi, p_\eta] = 0 = [\eta, p_\xi] \ .$$

The Hamiltonian H in term of the new variables boils down to an isotropic harmonic oscillator Hamiltonian plus a "magnetic" term proportional to the angular momentum:

$$H = \frac{P_x^2 + P_y^2}{2m} = \frac{\hbar\omega}{4}\left\{p_\xi^2 + p_\eta^2 + (\xi^2 + \eta^2)\right\} - \frac{\hbar\omega}{2}(\xi p_\eta - \eta p_\xi) \ .$$

The kinetic momenta $\boldsymbol{P} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}$ are quantized by the energy creation and annihilation operators $a^\pm$. The guiding center coordinates X and Y are integrals of motion, they are quantized by the *magnetic translation operators*[2] $b^\pm$:

$$\begin{aligned} a^\pm &= \tfrac{a_x^\pm \mp i a_y^\pm}{\sqrt{2}} = \tfrac{-P_y \mp iP_x}{\sqrt{2m\omega\hbar}} \ , \qquad & P_x &= \tfrac{p_\xi + \eta}{\sqrt{2}}\sqrt{m\omega\hbar} \ , \qquad & P_y &= \tfrac{p_\eta - \xi}{\sqrt{2}}\sqrt{m\omega\hbar} \ , \\ b^\pm &= \tfrac{a_x^\pm \pm i a_y^\pm}{\sqrt{2}} = \tfrac{X \pm iY}{\ell\sqrt{2}} \ , & X &= \left(\tfrac{\xi + p_\eta}{\sqrt{2}}\right)\ell \ , & Y &= \left(\tfrac{\eta - p_\xi}{\sqrt{2}}\right)\ell \end{aligned} \tag{6}$$

where we have used two commuting Heisenberg algebras with generators

$$a_x^\pm = (\xi \mp ip_\xi)/\sqrt{2} \ , \qquad a_y^\pm = (\eta \mp ip_\eta)/\sqrt{2} \qquad \text{such that} \qquad [a_i^-, a_i^+] = 1 \ . \tag{7}$$

[2] The geometrical meaning of the Zak's magnetic translations in the Landau problem has been clarified in the work [10].

The operators $a^{\pm}$ shift between different energy levels of the Hamiltonian

$$H = \frac{\hbar\omega}{2}\{a^+, a^-\} , \qquad [H, a^{\pm}] = \pm a^{\pm} , \qquad [H, b^{\pm}] = 0 . \tag{8}$$

The "zero mode" generators $b^{\pm}$ commute with $a^{\pm}$ since the momenta $\boldsymbol{P} = (P_x, P_y)$ commute with the guiding center coordinates $\boldsymbol{X} = (X, Y)$. Hence the magnetic translations $b^{\pm}$ are responsible for the degeneracy of the Landau levels. The angular momentum operator $L_z = xp_y - yp_x$ is the generator of the rotational symmetry, the operators $b^{\pm}$ increase (decrease) the angular momentum eigenvalue

$$\frac{L_z}{\hbar} = \frac{1}{2}\{b^-, b^+\} - \frac{1}{2}\{a^-, a^+\} , \qquad [\frac{L_z}{\hbar}, b^{\pm}] = \pm b^{\pm} , \qquad [L_z, H] = 0 .$$

The angular momentum commutes with the energy operator H, however, there is a larger accidental group $SO(3)$ of transformations preserving the energy. Its origin is rooted in the analog of the Laplace-Runge-Lenz vector in the Landau problem, the so called magnetic LRL vector [33] which is an integral of motion beside the angular momentum. We proceed by exploring the dynamical group of the Landau problem in a more systematic way.

3. Dirac dynamical algebra $\mathfrak{so}(2,3)$

In his seminal paper "A Remarkable Representation of the 3 + 2 de Sitter Group"[11] Dirac came out with a realization of $\mathfrak{so}(2,3)$ which is quadratic in the phase space coordinates on a plane: p_ξ, p_η, ξ, η. The symplectic group $Sp(4,\mathbb{R})$ is a natural symmetry for the Landau problem since the magnetic field is encoded into the symplectic form of the phase space.

We now show that the Dirac's algebra $\mathfrak{so}(2,3)$ of the De Sitter group $SO(2,3)$ can be thought of as the group of symplectomorphisms, in view of the isomorphism

$$SO(2,3) \cong Sp(4,\mathbb{R})/\mathbb{Z}_2$$

thus yielding a dynamical group of the Landau's problem. The group $SO(2,3)$ has a homogeneous space dS_4 defined by the Dirac quadric in 5-dimensional flat ambient space with coordinates y^A

$$y_{-1}^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 = R^2 = \eta_{ab} y^a y^b , \qquad \eta_{ab} = diag(+1, +1, -1, -1, -1) .$$

The group of motions of the 4-dimensional hyperboloid is the orthogonal de Sitter group $SO(2,3)$, the conformal group of the Minkowski space $\mathbb{R}^{1,2}$. The generators m_{ab} of the Lie algebra $\mathfrak{so}(2,3)$ satisfy the commutations relations

$$[m_{ab}, m_{cd}] = 0 , \qquad [m_{ab}, m_{bc}] = -i\eta_{bb} m_{ac} \tag{9}$$

where the indices a, b, c, d are assumed to be all distinct from the set $\{-1, 0, 1, 2, 3\}$.

The Dirac's remarkable representation [11] is given by the following quadratic generators[3]

[3] We adopt a different convention for the indices of the matrix m_{ab}, the mapping between our convention and the Dirac's one reads $\{1, 2, 3, -1, 0\} \to \{1, 2, 3, 4, 5\}$.

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alternatively written with the Heisenberg algebras generators $a_i^\pm$ (7):

$$
\begin{array}{lclcl}
m_{12} & = & \frac{1}{2}(\xi p_\eta - \eta p_\xi) & = & \frac{1}{2i}\left(a_x^+ a_y^- - a_y^+ a_x^-\right) \;, \\
m_{23} & = & \frac{1}{4}\left(p_\xi^2 - p_\eta^2 + \xi^2 - \eta^2\right) & = & -\frac{1}{2}\left(a_x^+ a_x^- - a_y^+ a_y^-\right) \;, \\
m_{31} & = & -\frac{1}{2}\left(\xi\eta + p_\xi p_\eta\right) & = & -\frac{1}{2}\left(a_x^+ a_y^- + a_y^+ a_x^-\right) \;, \\
m_{1-1} & = & \frac{1}{2}\left(\xi\eta - p_\xi p_\eta\right) & = & \frac{1}{2}\left(a_x^- a_y^- + a_x^+ a_y^+\right) \;, \\
m_{2-1} & = & \frac{1}{4}\left(\xi^2 - \eta^2 + p_\eta^2 - p_\xi^2\right) & = & \frac{1}{4}\left(a_x^- a_x^- + a_x^+ a_x^+ - a_y^- a_y^- - a_y^+ a_y^+\right) \\
m_{3-1} & = & \frac{1}{2}\left(\xi p_\xi + \eta p_\eta\right) - \frac{i}{2} & = & \frac{i}{4}\left(a_x^+ a_x^+ - a_x^- a_x^- + a_y^+ a_y^+ - a_y^- a_y^-\right) \;, \\
m_{01} & = & \frac{i}{2}\left(\xi p_\eta + \eta p_\xi\right) & = & -\frac{1}{2}\left(a_x^+ a_y^+ - a_x^- a_y^-\right) \;, \\
m_{02} & = & \frac{1}{2}\left(\xi p_\xi - \eta p_\eta\right) & = & \frac{i}{4}\left(a_x^+ a_x^+ - a_x^- a_x^- - a_y^+ a_y^+ + a_y^- a_y^-\right) \;, \\
m_{03} & = & \frac{1}{4}\left(p_\xi^2 + p_\eta^2 - \xi^2 - \eta^2\right) & = & \frac{1}{4}\left(a_x^+ a_x^+ + a_x^- a_x^- + a_y^+ a_y^+ + a_y^- a_y^-\right) \;, \\
m_{-10} & = & \frac{1}{4}\left(p_\xi^2 + p_\eta^2 + \xi^2 + \eta^2\right) & = & \frac{1}{2}\left(a_x^+ a_x^- + a_y^- a_y^+\right) \;.
\end{array}
\tag{10}
$$

Therefore the Dirac $\mathfrak{so}(2,3)$-representation is rooted in the oscillator algebra with two modes $a_x^\pm$ and $a_y^\pm$ [7]. We will further related these two modes $a_x^\pm$ and $a_y^\pm$ through eqs (6) to the energy and magnetic translation creation and annihilation operators $a^\pm$ and $b^\pm$. We will get the Dirac conformal algebra $\mathfrak{so}(2,3)$ playing the role of infinitesimal symplectomorphisms of the Landau problem.

It is convenient to introduce (anti)holomorphic coordinates z ($\bar{z}$) on the phase space such that

$$
\begin{array}{lcll}
z & = & (x+iy)/2\ell = \frac{1}{\sqrt{2}}(\xi + i\eta) & \partial = \frac{\partial}{\partial z} = \ell\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \frac{1}{\sqrt{2}}\left(\partial_\xi - i\partial_\eta\right) \;, \\
\bar{z} & = & (x-iy)/2\ell = \frac{1}{\sqrt{2}}(\xi - i\eta) & \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \ell\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{\sqrt{2}}\left(\partial_\xi + i\partial_\eta\right) \;,
\end{array}
$$

where the momenta are related to the (anti)holomorphic derivatives ∂ ($\bar{\partial}$) through

$$
\partial = \frac{i}{\sqrt{2}}(p_\xi - ip_\eta) \;, \quad \bar{\partial} = \frac{i}{\sqrt{2}}(p_\xi + ip_\eta) \;, \quad ip_\xi = \frac{1}{\sqrt{2}}(\partial + \bar{\partial}) \;, \quad p_\eta = \frac{1}{\sqrt{2}}(\partial - \bar{\partial}) \;.
$$

The energy and magnetic translation creation and annihilation operators $a^\pm$ and $b^\pm$ in the Landau problem from equation (6) are then expressed in the holomorphic phase space coordinates as follows [4]

$$
\begin{array}{lcllcl}
a^- & = & \frac{1}{\sqrt{2}}(z + \bar{\partial}) & b^- & = & \frac{1}{\sqrt{2}}(\bar{z} + \partial) \;, \\
a^+ & = & \frac{1}{\sqrt{2}}(\bar{z} - \partial) & b^+ & = & \frac{1}{\sqrt{2}}(z - \bar{\partial}) \;.
\end{array}
\tag{11}
$$

We are now able to recast the Dirac's generators m_{ab} from eq (10) in the form of quadratic

[4] The operators $a^\pm$ and $b^\pm$ yields one more parametrization of the phase space $\mathbb{R}^4$. The inverse transformation reads

$$
\begin{array}{lcllcl}
z & = & \frac{1}{\sqrt{2}}(a^- + b^+) & \bar{\partial} & = & \frac{1}{\sqrt{2}}(a^- - b^+) \\
\bar{z} & = & \frac{1}{\sqrt{2}}(a^+ + b^-) & \partial & = & \frac{1}{\sqrt{2}}(b^- - a^+)
\end{array}
$$

polynomials of the creation and annihilation operators $a^\pm$ and $b^\pm$ as follows

$$\begin{aligned}
m_{12} &= \tfrac{1}{2}(z\partial - \bar{z}\bar{\partial}) &&= \tfrac{1}{4}\left(\{b^-, b^+\} - \{a^-, a^+\}\right) ,\\
m_{23} &= \tfrac{1}{4}(z^2 + \bar{z}^2 - \partial^2 - \bar{\partial}^2) &&= \tfrac{1}{4}\left(\{a^-, b^+\} + \{a^+, b^-\}\right) ,\\
m_{31} &= \tfrac{i}{4}(z^2 - \bar{z}^2 + \partial^2 - \bar{\partial}^2) &&= \tfrac{i}{4}\left(\{a^-, b^+\} - \{a^+, b^-\}\right) ,\\
m_{1-1} &= \tfrac{1}{4i}(z^2 - \bar{z}^2 - \partial^2 + \bar{\partial}^2) &&= \tfrac{i}{4}\left(a^+a^+ - a^-a^- + b^-b^- - b^+b^+\right) ,\\
m_{2-1} &= \tfrac{1}{4}(z^2 + \bar{z}^2 + \partial^2 + \bar{\partial}^2) &&= \tfrac{1}{4}\left(a^-a^- + a^+a^+ + b^-b^- + b^+b^+\right) ,\\
m_{3-1} &= -\tfrac{i}{2}\left(z\partial + \bar{z}\bar{\partial} - 1\right) &&= -\tfrac{i}{4}\left(\{a^-, b^-\} - \{a^+, b^+\}\right) ,\\
m_{01} &= -\tfrac{1}{2}\left(z\bar{\partial} - \bar{z}\partial\right) &&= -\tfrac{1}{4}\left(a^-a^- + a^+a^+ - b^-b^- - b^+b^+\right) ,\\
m_{02} &= -\tfrac{i}{2}\left(\bar{z}\partial + z\bar{\partial}\right) &&= -\tfrac{i}{4}\left(a^-a^- - a^+a^+ + b^-b^- - b^+b^+\right) ,\\
m_{03} &= \tfrac{1}{2}\left(z\bar{z} + \partial\bar{\partial}\right) &&= -\tfrac{1}{4}\left(\{a^+, b^+\} + \{a^-, b^-\}\right) ,\\
m_{-10} &= \tfrac{1}{2}\left(z\bar{z} - \partial\bar{\partial}\right) &&= \tfrac{1}{4}\left(\{a^-, a^+\} + \{b^-, b^+\}\right) .
\end{aligned} \tag{12}$$

Weyl Spinors and $Sp(4,\mathbb{R})$. The whole dynamic Dirac algebra $\mathfrak{so}(2,3) \cong \mathfrak{sp}(4,\mathbb{R})$ can be compactly represented if we pack the Landau's creation and annihilation operators $a^\pm$ and $b^\pm$ into a two-component Weyl spinor

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} + \partial \\ z + \bar{\partial} \end{pmatrix} , \qquad \chi^*_\alpha = (b^+ \;\; a^+) = \frac{1}{\sqrt{2}}(z - \bar{\partial} \;,\; \bar{z} - \partial) . \tag{13}$$

Barut and Duru [4] have found that the conformal algebra $\mathfrak{so}(2,3)$ generated by two oscillators (12) can be written by the spinorial operators

$$\begin{aligned}
m_{ij} &= \tfrac{1}{2}\epsilon_{ijk}\chi^*\sigma_k^T\chi , \qquad & m_{-1i} &= \tfrac{i}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T - \chi^T\epsilon\sigma_i^T\chi) ,\\
m_{-10} &= \tfrac{1}{2}(\chi^*\chi + 1) , \qquad & m_{0i} &= \tfrac{1}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T + \chi^T\epsilon\sigma_i\chi)
\end{aligned} \tag{14}$$

with the help of the Pauli matrices $(\sigma_i)_{\alpha\beta}$ and the charge operator $\epsilon = i\sigma_2$. They have obtained the isomorphism of the spinorial $\mathfrak{so}(2,3)$ representation (14) with the spectrum generating algebra of the 2D hydrogen atom. The latter spectrum has been derived as a solution of infinite-component Majorana equation (see also the work of Stoyanov and Todorov [29]).

The subalgebra generated by m_{-10}, m_{-13} and m_{03} is the radial subalgebra $\mathfrak{so}(1,2)$. The generator m_{-10} is the conformal Hamiltonian attached to the harmonic oscillator while the true hamiltonian is $\frac{H}{\hbar\omega} = m_{-10} - m_{12}$. As the motion of a harmonic oscillator is periodic the evolution parameter τ attached to the operator m_{-10} will be compactified to S^1 and will be referred to as *conformal time* τ .

The $\mathfrak{so}(3)$-algebra spanned by the operators m_{12}, m_{23}, m_{31} is commuting with the conformal Hamiltonian m_{-10}

$$[m_{-10}, m_{ij}] = 0 \qquad i,j = 1,2,3 \ .$$

The $\mathfrak{so}(3)$-algebra is the dynamical "accidental" symmetry extending the rotational symmetry generated by the angular momentum operator $2m_{12} = L_z/\hbar$. The rotational group element $R(\phi) = \exp im_{12}\phi$ would actually live in a spinor representation of $\mathfrak{so}(3)$.

4. Jordan algebra toolbox

The tight connection between the two quantum problems sharing the same dynamical conformal symmetry $\mathfrak{so}(2,3)$: Landau problem and 2D hydrogen atom find their natural formulations in the setting of Jordan algebras. We shall follow the ideas of Murat Günaydin [14] to employ a Jordan algebra in the construction of a regular linear representation of the conformal group. For the sake of completeness we first introduce the main notions related to Jordan algebra in general and only then we specialize to the important examples of the Jordan algebras of the 2×2 real Hermitian matrices $\mathfrak{J}_2^{\mathbb{R}}$ and complex Hermitian matrices $\mathfrak{J}_2^{\mathbb{C}}$ yielding the linear representation of $SO(2,3)$ and $SO(2,4)$, respectively.

Jordan algebras. A commutative multiplication law $\circ : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$, satisfying the Jordan identity

$$a \circ b = b \circ a \qquad a \circ \left(a^2 \circ b\right) = a^2 \circ (a \circ b) \tag{15}$$

defines a Jordan algebra $(\mathfrak{J}, \circ)$. Any associative matrix algebra (over $\mathbb{R}, \mathbb{C}, \mathbb{H}$) can be converted into a *special* Jordan algebra via the product $a \circ b = \frac{1}{2}(ab + ba)$, where ab is the standard associative matrix multiplication.

Jordan triple product and conformal group $Co(\mathfrak{J})$ representations. Any Jordan algebra $\mathfrak{J}$ has a Jordan triple product $(\bullet, \bullet, \bullet) : \mathfrak{J} \times \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$

$$(abc) = a \circ (b \circ c) - b \circ (a \circ c) + (a \circ b) \circ c \ . \tag{16}$$

The Jordan triple product defines a Jordan triple system with identities

$$(abc) \quad = \quad (cba) \tag{17}$$

$$(ab(cdx)) - (cd(abx)) \quad = \quad (a(dcb)x) - ((cda)bx) \ . \tag{18}$$

For any pair of elements $(x, y) \in \mathfrak{J} \times \mathfrak{J}$ one has a linear map $S_x^y : \mathfrak{J} \to \mathfrak{J}$ with a matrix fixed by the structure constants $(\Sigma_\mu^\nu)_\rho^\sigma$ of the Jordan triple product

$$S_x^y(z) = (xyz) \qquad\qquad (e_\mu, e_\nu, e_\rho) = \Sigma_{\mu\rho}^{\nu\sigma} e_\sigma \ . \tag{19}$$

Tits, Kantor and Koecher (TKK) construction of $\mathfrak{co}(\mathfrak{J})$. The conformal group $Co(\mathfrak{J})$ is generated by vector fields closing a *conformal* Lie algebra $\mathfrak{co}(\mathfrak{J})$. The general construction of a conformal Lie algebra $\mathfrak{co}(\mathfrak{J})$ from a given Jordan algebra $\mathfrak{J}$ is due to Tits, Kantor and Koecher.

Any Jordan triple system generated in $x \in \mathfrak{J}$ gives rise to a 3-graded Lie algebra $\mathfrak{co}(\mathfrak{J})$, endowed with an involution $\dagger$ via

$$(x, y, z) := [[x, y^\dagger], z] \ .$$

Conversely, any 3-graded Lie algebra $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (with $\mathfrak{g}_i = 0$ when $i \neq 0, \pm 1$) endowed with a graded involution $\dagger$, $\mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}$ determines a Jordan triple system. The 3-graded Lie algebra $\mathfrak{co}(\mathfrak{J})$ has the graded decomposition

$$\mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} := \mathfrak{J}^* \oplus \mathfrak{str}(\mathfrak{J}) \oplus \mathfrak{J} \tag{20}$$

where the abelian subalgebra $\mathfrak{g}_{-1}(\mathfrak{g}_{+1})$ is generated in the space $\mathfrak{J}(\mathfrak{J}^*)$. The grading operator $D \in \mathfrak{g}_0$ is the dilatation $[D, g] = kg$, for any $g \in \mathfrak{g}_k$. The grading alone implies that $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$ are abelian Lie subalgebras of $\mathfrak{co}(\mathfrak{J})$ and their mutual commutators belong to $\mathfrak{g}_0$, $[\mathfrak{g}_{-1}, \mathfrak{g}_{+1}] \subset \mathfrak{g}_0$. It turns out that all elements in $\mathfrak{g}_0$ can be represented as commutators, the structure algebra of $\mathfrak{J}$, $\mathfrak{str}(\mathfrak{J}) := \mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_{+1}]$ and $\mathfrak{g}_{\pm 1}$ are its fundamental and antifundamental representations in view of

$$[\mathfrak{g}_0, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 1} \ .$$

Coordinate-free definition of $\mathfrak{co}(\mathfrak{J})$. The relations[5] of the conformal algebra $\mathfrak{co}(\mathfrak{J})$ are compactly written with the help of the Jordan triple product conveying the essence of the

[5] The normalization $[U_a, U^b] = -2S_a^b$ is adopted for future convenience following [23].

TKK construction [14]

$$
\begin{array}{rclrclr}
[U_a, U^b] & = & -2S_a^b\ , & [U_a, U_b] & = & 0\ , & U_a \in \mathfrak{g}_{-1} \\
[S_a^b, U_c] & = & U_{(abc)}\ , & [S_a^b, S_c^d] & = & S^d_{(abc)} - S_c^{(bad)}\ , & S_a^b \in \mathfrak{g}_0 \\
[S_a^b, U^c] & = & -U^{(bac)}\ , & [U^a, U^b] & = & 0\ , & U^b \in \mathfrak{g}_{+1}\ .
\end{array} \tag{21}
$$

The involution † acts as $U_a^\dagger = U^a$ and consequently $(S_a^b)^\dagger = S_b^a$.

TKK for Jordan algebra $\mathfrak{J}_2^{\mathbb{C}} = H_2(\mathbb{C})$. The associative algebra of 2×2 Hermitian matrices with complex elements $H_2(\mathbb{C})$ has a basis of Pauli matrices σ_μ. The symmetric product $\circ$ gives rise to the (special) Jordan algebra $\mathfrak{J}_2^C$

$$
\sigma_i \circ \sigma_j := \frac{1}{2}\{\sigma_i, \sigma_j\} = \delta_{ij}\ , \qquad \sigma_0 \circ \sigma_i = \sigma_i\ , \qquad \sigma_0 \circ \sigma_0 = \sigma_0\ .
$$

An element $x \in H_2(\mathbb{C})$ is parametrized by its Minkowski coordinates

$$
x = x^\mu \sigma_\mu = x^0 \mathbb{1} + x^i \sigma_i
$$

where σ_0 is the unit matrix $\mathbb{1}$ and σ_i are the Pauli matrices. Hence

$$
x = x^\mu \sigma_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}\ , \qquad x^\mu := \frac{1}{2} tr(x\sigma_\mu) \tag{22}
$$

and the Minkowski metric g with a signature $(+,-,-,-)$ is given by the determinant

$$
\det x = x_\mu x^\mu = (x^0)^2 - \delta_{ij} x^i x^j = g_{\mu\nu} x^\mu x^\nu .
$$

The evaluation of the Jordan triple structure constants $\Sigma_{\alpha\gamma}^{\beta\rho}$ in the $\mathfrak{J}_2^{\mathbb{C}}$ basis

$$
(\sigma_\alpha, \sigma_\beta, \sigma_\gamma) = \Sigma_{\alpha\gamma}^{\beta\rho} \sigma_\rho = \sigma_\alpha \circ (\sigma_\beta \circ \sigma_\gamma) - \sigma_\beta \circ (\sigma_\gamma \circ \sigma_\alpha) + (\sigma_\alpha \circ \sigma_\beta) \circ \sigma_\gamma
$$

gives back the concise formula,

$$
\Sigma_{\alpha\gamma}^{\beta\rho} = \delta_\gamma^\rho \delta_\alpha^\beta + \delta_\alpha^\rho \delta_\gamma^\beta - g^{\beta\rho} g_{\alpha\gamma}\ . \tag{23}
$$

The latter formula establishes via TKK construction a dichotomy between Minkowski's spacetime Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$ and the 3-graded Lie algebra $\mathfrak{so}(2,4)$.

Indeed the Jordan triple product induces via the TKK construction (Eq. (21)) a $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}})$-representation [14, 25] by the following vector fields:

$\mathfrak{co}(\mathfrak{J})$	operator	$\in \mathfrak{co}(\mathfrak{J})$	mapping	$x-$rep basis $\mathfrak{so}(2,4)$	deg(x)
$\mathfrak{J}$	$U_a = -ia^\mu P_\mu$	$\in \mathfrak{g}_{-1}$	$x \mapsto a$	$P_\nu = i\partial_\nu$	0
$\mathfrak{str}(\mathfrak{J})$	$S_a^b = ia^\nu b_\mu S_\nu^\mu$	$\in \mathfrak{g}_0$	$x \mapsto (a,b,x)$	$S_\nu^\mu = -i\Sigma_{\nu\alpha}^{\mu\beta} x^\alpha \partial_\beta$	1
$\mathfrak{J}^*$	$U^b = ib_\mu K^\mu$	$\in \mathfrak{g}_{+1}$	$x \mapsto -(x,b,x)$	$K^\mu = i\Sigma_{\nu\alpha}^{\mu\beta} x^\nu x^\alpha \partial_\beta$	2

The 15 generators of the conformal group $Co(\mathfrak{J}_2^{\mathbb{C}}) = SO(2,4)$ are obtained after the evaluation of the Jordan structure constants $\Sigma_{\nu\alpha}^{\mu\beta}$ (23)

$$
\begin{array}{llcll}
\mathfrak{g}_{-1} & -iP_\nu & = & \partial_\nu & \text{translations} \\
\mathfrak{g}_0 & iM^\mu{}_\nu & = & -x^\mu \partial_\nu + x_\nu \partial^\mu & \text{Lorentz transformations} \\
\mathfrak{g}_0 & iD & = & x^\mu \partial_\mu & \text{dilatation} \\
\mathfrak{g}_{+1} & iK^\mu & = & -2x^\mu x^\nu \partial_\nu + x^\nu x_\nu \partial^\mu & \text{special conformal transformations}
\end{array} \tag{24}
$$

yielding a minimal representation the Lie algebra $\mathfrak{so}(2,4)$. The Lorentz generators $M^{\mu}{}_{\nu}$ are the basis of the reduced structure algebra $\mathfrak{str}_0(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(1,3)$ (generating the Lorentz norm $\det x$ preserving group). The structure algebra $\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})$ is generated by S_b^a. When we choose coordinates as in the table above $\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})$ is the span of the generators $S^{\mu}_{\nu} = \frac{i}{2}[K^{\mu}, P_{\nu}] = M^{\mu}{}_{\nu} - \delta^{\mu}_{\nu} D$. Then the TKK construction (21) for $\mathfrak{J}_2^{\mathbb{C}}$ yields the commutation relations of the conformal algebra $\mathfrak{so}(2,4)$ of the Minkowski space $\mathbb{R}^{1,3}$ $(\mu, \nu, \lambda = 0,1,2,3)$

$$\begin{array}{rclccc} [K_\mu, P_\nu] & = & 2i(g_{\mu\nu}D - M_{\mu\nu}) \ , & [D, P_\mu] = iP_\mu \ , & [D, K_\mu] = -iK_\mu \ , & \\ [K_\lambda, M_{\mu\nu}] & = & i(g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu) \ , & [D, M_{\mu\nu}] = 0 \ , & [P_\mu, P_\nu] = 0 = [K_\mu, K_\nu] \ , & (25) \\ [P_\lambda, M_{\mu\nu}] & = & i(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu) \ , & [M_{\mu\alpha}, M_{\beta\nu}] = 0 \ , & [M_{\mu\alpha}, M_{\alpha\nu}] = ig_{\alpha\alpha}M_{\mu\nu} \ . & \end{array}$$

In the last two relations the indices are assumed distinct.

The conformal inversion $I(x^0, \boldsymbol{x}) = \left(\frac{x^0}{x^2}, -\frac{\boldsymbol{x}}{x^2}\right)$ is an involution, $I^2 = \mathbb{1}$. It anticommutes with the grading operator, $ID = -DI$ and induces the involution † through

$$K^{\mu} = IP_{\mu}I = P_{\mu}^{\dagger} \ .$$

The conformal inversion changes dimensions; namely, the length to inverse length.

TKK for Jordan algebra $\mathfrak{J}_2^{\mathbb{R}} = H_2(\mathbb{R})$. The real symmetric matrices are spanned by the subset $\{\sigma_0, \sigma_1, \sigma_3\}$ of real Pauli matrices with coordinates $\{y_0, y_1, y_2\}$

$$y = \sum_{\mu=0,1,3} x^{\mu}\sigma_{\mu} = \begin{pmatrix} y_0 + y_2 & y_1 \\ y_1 & y_0 - y_2 \end{pmatrix} \ , \quad y^T = y \, . \tag{26}$$

We simply skip the Pauli matrix σ_2 (which is not symmetric) from the $3+1$ Minkowski spinor $x = x^{\mu}\sigma_{\mu}$. Renaming the components $x^0 = y^0$, $x^1 = y^1$and $x^3 = y^2$ we end up with a real spinorial representation of the Minkowski spacetime $\mathbb{R}^{1,2}$ such that the deteminant yields the metric

$$\det y = y_{\tilde{\mu}} y^{\tilde{\mu}} = (y^0)^2 - (y^1)^2 - (y^2)^2 = g_{\tilde{\mu}\tilde{\nu}} y^{\tilde{\mu}} y^{\tilde{\nu}} \qquad \tilde{\mu}, \tilde{\nu} = 0,1,2 \ .$$

Tits-Kantor-Koecher construction applied to the Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$ and $\mathfrak{J}_2^{\mathbb{R}}$ yields, respectively, the conformal algebra of the Minkowski spacetime $\mathbb{R}^{1,3}$, eq. (25):

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4) = \underbrace{(\mathfrak{J}_2^{\mathbb{C}})^*}_{K^{\mu}} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,3)}_{M^{\mu}_{\nu}} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{C}}}_{P_{\nu}} \tag{27}$$

and the conformal algebra of the Minkowski space $\mathbb{R}^{1,2}$, eq. (25):

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) = \mathfrak{so}(2,3) = \underbrace{(\mathfrak{J}_2^{\mathbb{R}})^*}_{K^{\tilde{\mu}}} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,2)}_{M^{\tilde{\mu}}_{\tilde{\nu}}} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{R}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{R}}}_{P_{\tilde{\nu}}} \qquad \mu, \nu = 0,1,2. \tag{28}$$

Geometry of null cones. In his famous Erlangen program Félix Klein associated any geometric space with its group of motion, *i.e.,* its underlying group of symmetries.

Proposition 4.1 *Let the compactified Minkowski space $\mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^*$ be the space of the isotropic rays in $\mathbb{R}^{2,4}$ (proportional isotropic vectors are identified)*

$$\mathcal{N} = \{\vec{x} \in \mathbb{R}^{2,4} | x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_5^2 = 0; x \neq 0\}.$$

The compactified null cone $\mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^* \cong (S^1 \times S^3)/\mathbb{Z}_2$ *is a homogeneous space for the conformal group* $SO(2,4)$ *generated in the Lie algebra* $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4)$*, eqs (31).*

The intersection of $\mathcal{M}_{1,3}$ *by hyperplane* $x_2 = 0$ *yields the space* $\mathcal{M}_{1,2} = \tilde{\mathcal{N}}/\mathbb{R}^*$ *of the isotropic rays in* $\mathbb{R}^{2,3}$

$$\tilde{\mathcal{N}} = \{\vec{y} \in \mathbb{R}^{2,3} | y_{-1}^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0; y \neq 0\}$$

where the remaining coordinates after the reduction are renamed according to

$$\{x_{-1}, x_0, x_1, x_3, x_5\} \leftrightarrow \{y_{-1}, y_0, y_1, y_2, y_3\} .$$

The compactified null cone $\mathcal{M}_{1,2} = \tilde{\mathcal{N}}/\mathbb{R}^* \cong (S^1 \times S^2)/\mathbb{Z}_2$ *of the isotropic rays in* $\mathbb{R}^{2,3}$ *is a homogeneous space for the conformal group* $SO(2,3)$ *generated in the Lie algebra* $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) = \mathfrak{so}(2,3)$ *(see eqs (31)) .*

In this sense, the geometry of the compactified Minkowski spaces $\mathcal{M}_{1,2} = (S^1 \times S^2)/\mathbb{Z}_2$ and $\mathcal{M}_{1,3} = (S^1 \times S^3)/\mathbb{Z}_2$ are associated with the conformal symmetry algebras $\mathfrak{so}(2,3)$ and $\mathfrak{so}(2,4)$ of the Jordan algebras of real and complex hermitian 2×2 matrices. The intersection with the plane $x_2 = 0$ corresponds to the reduction of the complex Jordan algebra to real $\mathfrak{J}_2^{\mathbb{R}}$ Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$

$$\mathfrak{J}_2^{\mathbb{R}} = \{y \in \mathfrak{J}_2^{\mathbb{C}} | y^T = y\} ,$$

thus reducing the conformal symmetry from $\mathfrak{so}(2,4)$ to $\mathfrak{so}(2,3)$.

The advantage of the TKK construction of the Jordan algebra symmetries is that it yields explicitly a conformal group representationof the conformal spacetime symmetries as linear transformations of a null-ray cone.

5. Hydrogen atom from 3D to 2D

The non-relativistic hydrogen atom in three space dimensions is central in the development of the quantum mechanics. Barut and collaborators (see e.g. [3, 5]) have shown that the states in the 3D hydrogen atom spectrum transform in a helicity zero massless irreducible representation [21] of the dynamical group $SO(2,4)$ with generators

$$\begin{array}{rclrclrcl} \boldsymbol{L} &=& \boldsymbol{r} \times \boldsymbol{p} & B_0 - A_0 &=& r & B_0 + A_0 &=& r\boldsymbol{p}^2 \\ \boldsymbol{\Gamma} &=& r\boldsymbol{p} & \boldsymbol{B} - \boldsymbol{A} &=& \boldsymbol{r} & \boldsymbol{B} + \boldsymbol{A} &=& \boldsymbol{r}\boldsymbol{p}^2 - 2\boldsymbol{p}(\boldsymbol{r}\cdot\boldsymbol{p}) \\ D &=& \boldsymbol{r}\cdot\boldsymbol{p} - i & & & & & & \end{array} . \qquad (29)$$

On the other hand the conformal symmetry $SO(2,4)$ of the hydrogen spectrum is also the group of causal space-time automorphisms of the Minkowski space $\mathbb{R}^{1,3}$. The dictionary between the two is given by the table

$\mathfrak{co}(\mathfrak{J})$	space-time cone	hydrogen atom
$\mathfrak{g}_0$	$M_{\mu\nu}$; D	L_{ij}, Γ_i; D
$\mathfrak{g}_{-1}$	K_μ	$B_\mu + A_\mu$
$\mathfrak{g}_{+1}$	P_μ	$B_\mu - A_\mu$

.

The dynamical group $SO(1,2)$ generated by $\{A_0, D, B_0\}$ is the "radial" group of the conformal transformations of the time coordinate $\mathbb{R}^{0,1}$ [17]

$$\begin{array}{lllll} t \rightarrow t' &=& t + t_0 & P_0 = B_0 - A_0 & \text{time translations} \\ t \rightarrow t' &=& \lambda t & D & \text{time dilation} \\ t \rightarrow \frac{1}{t'} &=& \frac{1}{t} + \frac{1}{t_0} & K_0 = A_0 + B_0 & \text{special conformal} \end{array} .$$

The conformal transformation group $SO(1,2)$ arises from the TKK construction of the Jordan algebra $\mathfrak{J}_1^{\mathbb{R}} = \mathbb{R}$ yielding

$$\mathfrak{co}(\mathbb{R}) = \mathfrak{so}(1,2) = (\mathfrak{J}_1^{\mathbb{R}})^* \oplus \mathfrak{str}(\mathfrak{J}_1^{\mathbb{R}}) \oplus \mathfrak{J}_1^{\mathbb{R}} = \mathbb{R}K_0 \oplus \mathbb{R}D \oplus \mathbb{R}P_0 \ .$$

The sign of energy chooses different $SO(1,2)$-generator to be a conformal hamiltonian:

$$\begin{array}{rclll} B_0 & = & r(p^2+1)/2 & E<0 & \text{bound states ,} \\ A_0 & = & r(p^2-1)/2 & E>0 & \text{scattering states ,} \\ A_0+B_0 & = & rp^2 & E=0 & \text{free motion .} \end{array}$$

The conformal $SO(1,2)$ group can be seen also as the even generators of the Schrödinger group in the context of non-relativistic conformal symmetries (for the correspondence between Schrödinger and conformal group see [16]).

The 15 generators of the 3D hydrogen dynamical group $SO(2,4)$ act by linear transformations of the null-ray cone. They have been identified in [27] (see the table above) with the causal automorphisms of the light cone $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}})$, see eqns (27, 31)

$$\begin{pmatrix} 0 & B_0 & B_1 & B_2 & B_3 & D \\ & 0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & A_0 \\ & & 0 & L_3 & -L_2 & A_1 \\ & & & 0 & L_1 & A_2 \\ & & & & 0 & A_3 \\ & & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & L_{-10} & L_{-11} & L_{-12} & L_{-13} & L_{-15} \\ & 0 & L_{01} & L_{02} & L_{03} & L_{05} \\ & & 0 & L_{12} & L_{13} & L_{15} \\ & & & 0 & L_{23} & L_{25} \\ & & & & 0 & L_{35} \\ & & & & & 0 \end{pmatrix} . \tag{30}$$

Stated differently the compactified Minkowski space $\mathcal{M}_{1,3} = (S^1 \times S^3)/\mathbb{Z}_2$ carries a minimal representation of the conformal group $SO(2,4)$ stemming from the TKK construction of the Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$.

The conformal algebra generators L_{AB} satisfy the commutation relations

$$[L_{AB}, L_{CD}] = -i(\eta_{AC}L_{BD} + \eta_{BD}L_{AC} - \eta_{AD}L_{BC} - \eta_{BC}L_{AD}) \tag{31}$$

where the set of indices contains the auxiliary indices -1 and 5 in addition to the spacetime indices $\mu = 0,1,2,3$

$$L_{AB} \in \mathfrak{so}(2,4) \ , \qquad \eta_{AB} = diag(1,1,-1,-1,-1,-1) \ , \qquad A,B \in \{-1,0,1,2,3,5\} \ .$$

While reducing to the spacetime $\mathbb{R}^{1,2}$ with indices $\mu = 0,1,2$ we get an algebra representation equivalent to the Dirac's remarkable $\mathfrak{so}(2,3)$-representation (9)

$$L_{ab} \in \mathfrak{so}(2,3) \ , \qquad \eta_{ab} = diag(1,1,-1,-1,-1) \ , \qquad a,b \in \{-1,0,1,2,3=5\}^6 \ .$$

When the motion of the electron is constrained to a plane we obtain a system which we will refer to as 2D hydrogen atom, the reduction of the usual 3D atom to two space dimensions. The dynamical algebras of the 2D hydrogen atom and the Landau problem are isomorphic, the isomorphism is simply the transposition

$$L_{ab} \quad \leftrightarrow \qquad m_{ba} = -m_{ab} \ . \tag{32}$$

[6] The auxiliary index 5 comes with the Dirac matrix $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. In $\mathbb{R}^{1,2}$ we have $\gamma_5 = \gamma_0\gamma_1\gamma_2$ and similarly $i\sigma_3 = \sigma_0\sigma_1\sigma_2$ so we adopt a double notation $3 = 5$.

The isotropic rays in the five dimensional space $\mathbb{R}^{2,3}$ carry a linear representation of the conformal group $SO(2,3)$ with generators :

$$L_{ab} = \begin{pmatrix} 0 & B_0 & B_1 & B_2 & D \\ & 0 & \Gamma_1 & \Gamma_2 & A_0 \\ & & 0 & L_3 & A_1 \\ & & & 0 & A_2 \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & L_{-10} & L_{-11} & L_{-12} & L_{-15} \\ & 0 & L_{01} & L_{02} & L_{05} \\ & & 0 & L_{12} & L_{15} \\ & & & 0 & L_{25} \\ & & & & 0 \end{pmatrix}.$$

The dynamical symmetry group of the $2D$ hydrogen atom is the conformal group $SO(2,3)$ stemming from the TKK algebra $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}})$ (28). The compactified Minkowski space $\mathcal{M}_{1,2} \cong (S^1 \times S^2)/\mathbb{Z}_2$ is a homogeneous space for the dynamical $SO(2,3)$ group of the 2D hydrogen atom. We conclude that in the Jordan algebra language the reduction from $SO(2,4)$ to $SO(2,3)$ is projecting the complex Jordan algebras $\mathfrak{J}_2^{\mathbb{C}}$ to the real $\mathfrak{J}_2^{\mathbb{R}}$.

The sphere S^3 in $\mathcal{M}_{1,3}$ arises as compactification of the flat momentum space via the Cayley transform whereas S^1 stays for the compatified time coordinate. The maximally compact subgroup $SO(4) \subset SO(2,4)$ stabilizes the sphere S^3 interpolating between states with equal energy. A dual point of view is advocated in a recent paper [19] where S^3 in $\mathcal{M}_{1,3}$ is thought as a configuration space of the quark-antiquark system. In that way the compactified Minkowski space $\mathcal{M}_{1,3}$ becomes a toy model for the simplest QCD system: a meson. In other words one obtains the "QCD hydrogen atom" where the interaction potential is the curved analogue of the Coulomb potential, inherently introduced by the Green function of the Laplace-Beltrami operator on S^3. On closed surfaces charges appear in pairs thus the charge neutrality is naturally leading to a confinement. The phenomenologically observed degeneracies of the spectrum of meson masses can be then attributed to the conformal symmetries [18] of the compactified Minkowski space $(S^1 \times S^3)/\mathbb{Z}_2$ seen as a configuration space for one color charge degree of freedom. From that perspective the mesons are the QCD cousins of the system "electron-constant magnetic field" living in the space $\mathcal{M}_{1,2} \cong (S^1 \times S^2)/\mathbb{Z}_2$.

6. Reduction of Kustaanheimo-Steifel transform

The Kustaanheimo-Stiefel transform [20] in celestial mechanics removes the singular trajectories due to binary collisions from the phase space of the 3D Kepler motion and estabilishes a correspondence between the 4D isotropic harmonic oscillator 3D Coulomb-Kepler problem. The spectrum of bounded states for the 3D hydrogen atom (*i.e.* the quantum Kepler problem) is then symplectically equivalent to the spectrum of the harmonic oscillator with 4 bosonic modes. One has an inclusion $SU(2,2) \subset Sp(8,\mathbb{R})$ [21].

The elliptic Kepler orbits for negative energies $E < 0$ correspond to geodesic motion on 3-dimensional sphere S^3. The sphere S^3 arises through the stereographic projection of the momenta. A great circle in S^3 is the hodograph[7] of an elliptic orbit [24]. The regularized Kepler orbits live on the cotangent bundle T^*S^3 to the sphere S^3 with the zero section removed [9]

$$T^+S^3 = T^*S^3 - \{0_{sec}\} .$$

Definition 6.1 *The Kustaanheimo-Stiefel (KS) transform is the mapping between the cotangent bundles (with north poles deleted)* [8]

$$KS: \quad T^+S^3 \to T^+S^2 \qquad \subset (\mathbb{R}^*)^4 \times \mathbb{R}^4 \to (\mathbb{R}^*)^3 \times \mathbb{R}^3$$

[7] The hodograph is a curve drawn by the velocity vector, that is, the trajectory in the momentum space. It turns out the the hodographs of Kepler orbits for hyperbolic ($E > 0$) and parabolic $E = 0$ motions are segments of circles.

[8] We adopt the notation $\mathbb{R}^* = \mathbb{R} - \{0\}$.

where the 4D harmonic oscillator phase space coordinates $(u, w) \in (\mathbb{R}^*)^4 \times \mathbb{R}^4$ *are related with the 3D Kepler phase space* $(x, p) \in (\mathbb{R}^*)^3 \times \mathbb{R}^3$ *coordinates through the Hopf fibration map (see eq. (35))*

$$\begin{array}{rcl} x_1 &=& u_1u_3 + u_2u_4 \ , \\ x_2 &=& u_2u_3 - u_1u_4 \ , \\ x_3 &=& -u_1^2 - u_2^2 + u_3^2 + u_4^2 \ , \end{array} \tag{33}$$

extended with the derived momentum relations ($|\boldsymbol{z}|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2$)

$$\begin{array}{rcl} p_1 &=& -(u_1w_3 + w_1u_3 + u_2w_4 + w_2u_4)/|\boldsymbol{z}|^2 \ , \\ p_2 &=& -(u_2w_3 + w_2u_3 - w_1u_4 - u_1w_4)/|\boldsymbol{z}|^2 \ , \ . \\ p_3 &=& (u_1w_1 + u_2w_2 - u_3w_3 - u_4w_4)/|\boldsymbol{z}|^2 \ . \end{array} \tag{34}$$

The coordinates on T^+S^3 *are subject to the constraint*

$$K = u_1w_2 - u_2w_1 + u_3w_4 - u_4w_3 = 0 \ .$$

This constraint is in fact a constant of motion as a sum of two angular momentum components. The non-vanishing of the constraint K is playing a crucial role in the generalizations of the magnetised problem of a electric charge in the field of a Dirac monopole and systems of two dyonic particles [32].

The Hopf fibration is essentially representing a 3D vector $\boldsymbol{x} \in \mathbb{R}^3$ as a "square root" of a spinor $\boldsymbol{z} \in \mathbb{C}^2$ in view of

$$|\boldsymbol{x}| = |\boldsymbol{z}|^2 \ , \qquad |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \ , \qquad \boldsymbol{z} = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} \ , \qquad |\boldsymbol{z}|^2 = \boldsymbol{z}^\dagger \boldsymbol{z}. \tag{35}$$

The KS transformation can be seen as a phase space extension of the Hopf fibration

$$0 \to S^1 \hookrightarrow S^3 \to S^2 \to 0 \ ,$$

the kernel consists of the spinors $e^{i\theta}\boldsymbol{z} \in S^1$.

Reduction of KS transform to the Levi-Civita transform is done by the choice

$$u_2 = u_4 = 0 \ , \qquad w_2 = w_4 = 0$$

which amounts to taking a square root of a vector in the 2D plane $x_1 = \eta$ and $x_3 = \xi$.

The Levi-Civita transform stems from the change to parabolic coorindates u_1 and u_3 in the complex plane

$$\xi + i\eta = Z^2 \ , \qquad \begin{array}{rcl} \xi &=& u_1^2 - u_3^2 \ , \\ \eta &=& 2u_1u_3 \ , \end{array} \qquad \begin{array}{l} p_\xi = (u_1w_1 - w_3u_3)/|Z|^2 \ , \\ p_\eta = -(u_1w_3 + w_1u_3)/|Z|^2 \end{array} \tag{36}$$

where the real spinor $\psi = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$ is written with one complex number $Z = u_1 + iu_3$. Levi-Civita mapping is then a sympletic extension of the the trivial Hopf fibration

$$0 \to S^0 \hookrightarrow S^1 \to S^1 \to 0$$

where the dimension zero sphere $S^0 = \mathbb{Z}_2$ is in the kernel thus reflecting the fact that any pair of parabolic (spinor) coordinates (u_1, u_3) and $(-u_1, -u_3)$ parametrize one and the same point (ξ, η) in the complex plane.

The Levi-Civita transform (36) is then a 2-to-1 mapping between 2D harmonic oscillator phase-space (u_1, u_3, w_1, w_3) and the phase space of the 2D Kepler problem $(\xi, \eta, p_\xi, p_\eta)$. Upon quantization it yields the Newton-Hooke duality between the Landau problem and the 2D hydrogen atom (1). We are now going to show that the phase space (u_1, u_3, w_1, w_3) is naturally parametrized by a Majorana spinor in dimension 4.

7. Ladder $U(2,2)$ representation

Let $\psi = (\psi^\alpha)$ be operator-valued Dirac spinor with 4 components satisfying the canonical commutation relations

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta^\alpha_\beta \ , \qquad [\psi^\alpha, \psi^\beta] = 0 \ .$$

A canonical representation of a pair of 2D harmonic oscillators with two complex variables z^α and the holomorphic derivatives $iw_\alpha = \frac{\partial}{\partial z^\alpha}$ is given by

$$\boldsymbol{z} = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \qquad \boldsymbol{\partial} = \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \frac{\partial}{\partial z^2} \end{pmatrix} \qquad \psi = \begin{pmatrix} \bar{z} \\ \partial \end{pmatrix} \qquad \bar{\psi} = (-\bar{\partial}, z) \ . \tag{37}$$

The ladder representation of $U(2,2)$ [21] is a spinorial representation realized by the operators[9]

$$J^{AB} = \bar{\psi}\sigma^{AB}\psi \ , \qquad C_1 = \bar{\psi}\psi \tag{38}$$

where the 4×4 matrices σ^{AB} close a defining representation of $\mathfrak{su}(2,2)$: these are the matrices in $\mathfrak{sl}(4, \mathbb{C})$ preserving a pseudo-Hermitian form β with signature $(+ + --)$

$$(\phi, \psi) = \phi_1^*\psi_1 + \phi_2^*\psi_2 - \phi_3^*\psi_3 - \phi_4^*\psi_4 = \phi^\dagger \beta \psi = \bar{\phi}\psi \ .$$

The Hermitian matrix $\beta = \beta^\dagger$ depends on the basis, it fixes the choice of the Dirac matrix $\gamma_0 := \beta$ and the invariance of the form implies

$$\beta \sigma^{AB} \beta^{-1} = (\sigma^{AB})^\dagger \ .$$

The linear Casimir operator C_1 is the center of $U(2,2)$. It is represented by an integer multiple of the unit $\mathbb{1}$:

$$C_1 + 2 = -2\lambda = z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} \qquad \lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$$

where the half-integer helicities λ are labelling the zero-mass representations of $U(2,2)$. The hydrogen atom is described by helicity $\lambda = 0$ representation [21].

A finite-dimensional representation of $\mathfrak{su}(2,2)$ is generated by the 4×4 Dirac gamma matrices γ^μ,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3 \ .$$

The 15 $\mathfrak{su}(2,2)$-generators σ^{AB} can be concisely written[10] (see e.g. [21]) :

$$\begin{aligned} \sigma^{\mu\nu} &= \tfrac{i}{4}[\gamma^\mu, \gamma^\nu] \ , & \sigma^{-15} &= -\tfrac{1}{2}\gamma^5 \ , \\ \sigma^{\mu 5} &= \tfrac{i}{4}[\gamma^\mu, \gamma^5] \ , & \sigma^{-1\mu} &= -\tfrac{1}{2}\gamma^\mu \ . \end{aligned} \tag{39}$$

Here we denoted $\gamma^5 := \gamma^0\gamma^1\gamma^2\gamma^3$. In view of the isomorphism $\mathfrak{su}(2,2) \cong \mathfrak{so}(2,4)$ the operators σ^{AB} satisfy the $\mathfrak{so}(2,4)$ commutation relations (31). They give rise to a spinorial representation of the conformal algebra $\mathfrak{so}(2,4)$, eq. (30) through the Kustaanheimo-Stiefel correspondence between the quantum Coulomb-Kepler problem and the 4D harmonic oscillator

$$L_{AB} \qquad \leftrightarrow \qquad J^{AB} = \bar{\psi}\sigma^{AB}\psi \ , \qquad A, B \in \{-1, 0, 1, 2, 3, 5\} \ .$$

[9] Here the summation on spinorial indices is implicit.
[10] We get another concise expression $\sigma^{AB} = \frac{i}{2}\gamma^A\gamma^B$ for $A < B$ in $\{-1, 0, 1, 2, 3, 5\}$ if we set the unit matrix multiplier $\gamma^{-1} := i\mathbb{1}$.

Proposition 7.1 *The Majorana condition $\psi = \psi^c$ is reducing the Dirac $SU(2,2)$-spinor ψ_α to a real $Sp(4,\mathbb{R})$-spinor. The reality condition is essentially reducing the 4D harmonic oscillator (Dirac spinor) to the 2D oscillator (Weyl spinor).*

Proof. The Majorana condition is the invariance under the charge-conjugation involution $C = i\gamma^0\gamma^2$. We choose the Dirac representation for the Clifford algebra generators γ^μ

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} , \qquad C = i\gamma^0\gamma^2 = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} .$$

A Majorana spinor is a Weyl spinor written in 4-dimensional form. The Majorana condition $\psi = \psi^c$ yields

$$\psi = \begin{pmatrix} \chi \\ \epsilon^T (\chi^*)^T \end{pmatrix} , \qquad \bar{\psi} = \psi^* \gamma^0 = \begin{pmatrix} \chi^* & -\chi^T \epsilon \end{pmatrix} . \tag{40}$$

The Dirac's remarkable spinorial $\mathfrak{so}(2,3)$-algebra represention (14) is nothing else than the ladder representation for the Majorana spinor $\psi = \psi^c$

$$m_{ba} \quad \leftrightarrow \quad J^{ab} = \bar{\psi}^c \sigma^{ab} \psi^c \qquad\qquad a, b \in \{-1, 0, 1, 2, 3\} .$$

The Weyl spinor $\chi_\alpha^{(*)}$ has components identified in eq. (13) with the Heisenberg algebra generators $a^\pm$ and $b^\pm$ in Landau problem. Hence the dynamical algebra $\mathfrak{so}(2,3)$ of the Landau problem (12) is the span of the infinitesimal symplectomorphisms $\mathfrak{sp}(4,\mathbb{R})$ of the phase space $(\xi, \eta, p_\xi, p_\eta) \in \mathbb{R}^4$ obtained from the Majorana reduction of the ladder representation (38) of $\mathfrak{su}(2,2) \cong \mathfrak{sp}(8,\mathbb{R}) \cap \mathfrak{so}(4,4)$.

8. Conclusion and outlook

In celestial mechanics the regularization is the transformation of the singular equations of the Kepler motion to the regular equations of the harmonic oscillator. The duality between the Newton's universal law and Hooke's law of harmonic motion has been know since the time of Newton and Hooke [8]. Its reincarnation in quantum mechanics is the duality between the hydrogen atom and the harmonic oscillator via the Fock's method of quantization. At the quantum level, the Levi-Civita regularization induces the duality transform between the 2D hydrogen atom and 2D harmonic oscillator, whereas the Kustaanheimo-Stiefel regularization bridges between the 3D hydrogen atom and the 4D harmonic oscillator. We have given an unified picture (2) of the dynamical groups in duality derived from the conformal symmetries on Jordan algebras and the TKK construction. We point out that the dynamical group $Sp(4,\mathbb{R})$ of the quantum motion of an electron in a constant magnetic field (Landau problem) is mapped via the Levi-Civita regularization to the conformal $SO(2,3)$ spectrum generating algebra of the 2D hydrogen atom. The symplectic group $Sp(4,\mathbb{R})$ is a natural symmetry for the Landau problem since the magnetic field is encoded into the symplectic form on the phase space of the planar motion. On the other hand the carrier of the electromagnetic interaction is the zero-mass photon propagating on the light cone in Minkowski spacetime $\mathbb{R}^{1,2}$ whose causal automorphisms form the group $SO(2,3)$. In a similar fashion, the Kustaanheimo-Stiefel regularization connects the zero-mass spinorial representation of $SU(2,2)$ with the minimal $SO(2,4)$-representation of light cone automorphisms in Minkowski spacetime $\mathbb{R}^{1,3}$.

The merit of the Jordan algebra formalism is that the reduction from 3D to 2D hydrogen atom is done by the reduction of the Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$ to $\mathfrak{J}_2^{\mathbb{R}}$. We have shown that on the spinorial side of the Newton-Hooke correspondence (1), a 4D Dirac spinor is reduced to the Majorana spinor with 4 real independent components (or alternatively Weyl spinor with 2 complex), that is, we impose the natural reality condition for spinors.

We have also noted that the Levi-Civita transform connecting the Landau problem and the 2D hydrogen atom is an extension of the Hopf fibration $S^1 \to S^1$. We have drawn the parallel with the Kustaanheimo-Stiefel transform thought as an extension of the Hopf fibration $S^3 \to S^2$. We believe that phase space extentions of the remaining Hopf fibrations

$$\begin{array}{llllllllll} \mathbb{H} & 0 & \to & S^3 & \hookrightarrow & S^7 & \to & S^4 & \to & 0\,, \\ \mathbb{O} & 0 & \to & S^7 & \hookrightarrow & S^{15} & \to & S^8 & \to & 0\,, \end{array}$$

are of potential interest for high energy physics in view of the exceptional role that the octonions could play in the quark-lepton symmetry of the Standard Model [12, 15].

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References

[1] Bars, Itzhak, and Jonathan L. Rosner. "Duality between hydrogen atom and oscillator systems via hidden $SO(d, 2)$ symmetry and 2T-physics." *Journal of Physics A: Mathematical and Theoretical* **53** (2020): 234001.

[2] Barut, A. O. and Hagen Kleinert. "Transition probabilities of the hydrogen atom from noncompact dynamical groups." *Physical Review* **156** (1967): 1541.

[3] Barut, A.O. and G.L. Bornzin. "$SO(4, 2)$-Formulation of the Symmetry Breaking in Relativistic Kepler Problems with or without Magnetic Charges", *Journal of Mathematical Physics* **12** (1971) 841-846.

[4] Barut, A. O., and I. H. Duru. "Introduction of internal coordinates into the infinite-component Majorana equation." *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **333** (1973): 217-224.

[5] Barut, A. O., C. K. E. Schneider, and Raj Wilson. "Quantum theory of infinite component fields." *Journal of Mathematical Physics* **20** (1979): 2244-2256.

[6] Cisneros, Arturo, and Harold V. McIntosh. "Symmetry of the two?dimensional hydrogen atom." *Journal of Mathematical Physics* **10**, no. 2 (1969): 277-286.

[7] Başkal, Sibel, Young S. Kim, and Marilyn E. Noz. "Poincaré Symmetry from Heisenbergs Uncertainty Relations." Symmetry 11 (2019): 409.

[8] Chandrasekhar, Subrahmanyan. Newton's Principia for the common reader. Oxford University Press, 2003.

[9] Cordani, Bruno. The Kepler Problem: Group Theotretical Aspects, Regularization and Quantization, with Application to the Study of Perturbations. Vol. 29. Springer Science Business Media, 2003.

[10] Dereli, Tekin and Popov, Todor. Bloch waves and non-commutative tori of magnetic translations. *Journal of Mathematical Physics* **62** (2021), 101701.

[11] Dirac, P. A. M., A remarkable representation of the 3+ 2 de Sitter group. *Journal of Mathematical Physics* (1963), 901-909.

[12] Dubois-Violette, Michel. "Exceptional quantum geometry and particle physics." *Nuclear Physics B***912** (2016), 426-449.

[13] Fock, Vladimir. *Z. Phys.* **98**(1935), 145.

[14] Günaydin, Murat. "Generalized conformal and superconformal group actions and Jordan algebras." *Modern Physics Letters A* **8**(1993), 1407-1416.

[15] Günaydin, Murat and Feza Gürsey. "Quark structure and octonions" *J. Math. Phys.* **14**, 1651-1667.

[16] Henkel, Malte, and Jeremie Unterberger. "Schrödinger invariance and spacetime symmetries." *Nuclear Physics* **B 660** (2003): 407-435.

[17] Jackiw, R. "Dynamical symmetry of the magnetic vortex." *Annals of Physics* 201.1 (1990): 83-116.

[18] Kirchbach, M., and C. B. Compean. "Modelling duality between bound and resonant meson spectra by means of free quantum motions on the de Sitter space-time dS_4." *The European Physical Journal A***52** (2016): 1-21.

[19] Kirchbach, M., Popov, T. Vallejo, J.A. "Color confinement at the boundary of the conformally compactified AdS_5" *J. High Energ. Phys.* (2021), 1-35.

[20] Kustaanheimo, Paul and E. Stiefel. "Perturbation theory of Kepler motion based on spinor regularization." *J. Reine Angew. Math.* **218** (1965), 204-219.

Journal of Physics: Conference Series **2191** (2022) 012009 doi:10.1088/1742-6596/2191/1/012009

[21] Mack, G. and Ivan Todorov, "Irreducibility of the ladder representations of $U(2,2)$ when restricted to the Poincaré subgroup", *Journal of Mathematical Physics* **10**(1969), 2078-2085.
[22] Malkin, I.A. and V.I. Manko, Symmetry of the Hydrogen Atom, *JETP Letters* (1966), 146148.
[23] Meng, Guowu. "Euclidean Jordan algebras, hidden actions, and J-Kepler problems." *Journal of mathematical physics* **52** (2011): 112104.
[24] Milnor, John. "On the geometry of the Kepler problem." *The American Mathematical Monthly* **90**(1983): 353-365.
[25] Palmkvist, Jakob. "A generalization of the Kantor-Koecher-Tits construction." arXiv preprint arXiv:1308.3761 (2013).
[26] Parfitt, D. G. W., and M. E. Portnoi. "The two-dimensional hydrogen atom revisited." *Journal of Mathematical Physics* **43**, (2002): 4681-4691.
[27] Popov, Todor. "Jordan Algebra and Hydrogen Atom." In Quantum Theory And Symmetries, pp. 231-244. Springer, Singapore, 2017.
[28] Shibuya, Tai-ichi, and Carl E. Wulfman. "The Kepler problem in two-dimensional momentum space." *American Journal of Physics* **33**, (1965): 570-574.
[29] Stoyanov, D. Tz, and I. T. Todorov. "Majorana Representations of the Lorentz Group and Infinite-Component Fields." *Journal of Mathematical Physics* **9**,(1968): 2146-2167.
[30] Ter-Antonyan, V. "Dyon-oscillator duality." arXiv preprint quant-ph/0003106 (2000).
[31] Torres Del Castillo, G. F. "On the connection between the Kepler problem and the isotropic harmonic oscillator in classical mechanics." *Revista Mexicana de Fisica* **44**, (1998): 333-338.
[32] Torres del Castillo, Gerardo Francisco, and Ortega A. Lopez. "The three-dimensional Kepler problem amd the four-dimensional isotropic harmonic oscillator" *Revista Mexicana De Fisica* **45**, (1999): 1-8.
[33] Velasco-Martinez, D., V. G. Ibarra-Sierra, J. C. Sandoval-Santana, A. Kunold, and J. L. Cardoso. "Constants of motion for the planar orbit of a charged particle in a static and uniform magnetic field: the magnetic LaplaceRungeLenz vector." *European Journal of Physics* **35**, (2014): 055018.