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Aspects of String Theory: Higher-spins, Pure Spinors in AdS, and Ambitwistors

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Resumo

Nesta tese, nós estudamos ambos limites da correspondência AdS/CFT: o espaço de fase das teorias de spin alto, e a inserção de operadores de vértice nas funções de correlação da corda de espinores puros escrita no *background* $AdS_5 \times S^5$. Nós calculamos também as equações de movimento do setor de Ramond para a corda ambitwistor heterótica na descrição RNS. O estudo do espaço de fase das teorias de spin alto usa a formulação obtida por Penrose e dá indícios da presença de uma simetria conforme não-local. A amplitude de espalhamento em AdS é estudada com o formalismo BV, apropriado para este *background* onde o fantasma b não é holomórfico; nós escrevemos a amplitude com a inserção de um vértice de deformação beta. Para o sistema heterótico, nós obtemos as transformações de supersimetria da teoria.

Palavras-Chave: Teoria de Cordas, Teoria de Campos, Correspondencia AdS/CFT, Teoria de spins altos, Formalismo BV.

Abstract

In this thesis we study both limits of the AdS/CFT correspondence: the phase space of higher-spin theories, and the insertion of vertex operators in string correlation functions of the pure spinor formulation in $AdS_5 \times S^5$ background. We also compute the equations of motion for Ramond sector of the RNS heterotic ambitwistor string. The study the phase-space of free higher-spin theories uses a formulation obtained from twistors by Penrose and hints the presence of a non-local conformal symmetry. The string scattering in AdS is studied with the BV formalism, appropriate for this background given that the b-ghost is non-holomorphic; we write the amplitude with the insertion of a beta-deformation vertex. For the heterotic ambitwistor system, we obtain the supersymmetry algebra of this theory.

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Chapter 1

Introduction

String theory in AdS is one of the central topics of modern theoretical physics due to the AdS/CFT correspondence [1–3]. The correspondence states that maximally supersymmetric four-dimensional gauge theory (CFT) is equivalent to a theory of quantum gravity (type IIB superstring theory) on a space with negative cosmological constant in five dimensions, called Anti-de Sitter space (AdS). In this context, the energy scale in the conformal field theory provides the holographic direction, related to the radial direction of AdS [4]. The AdS/CFT is a strong/weak coupling duality and we can use this fact to explore new phenomena on both sides of the correspondence. In this thesis, we follow two approaches. The first one is higher spin theory of Fradkin and Vasiliev, which is supposed to work in the small radius limit.

The second is the pure spinor approach, and the related ambitwistor formalism. Pure spinors were developed by Berkovits [5] and aimed at the universal description of AdS background and its deformations. It is technically the most difficult one. The ambitwistor formalism can be thought of as a limit of the pure spinor string, which is promising and technically more accessible than the original pure spinor formalism. At this point, however, it is not clear to us if it can be extended beyond the tree level.

In the rest of this introductory section, we will briefly summarize what we have done following these three directions presenting the technical details.

1.1 On the phase space of free higher-spin theories and conformal transformations.

Higher-spin theories are a toy-model for string field theory. Historically, the first considerations can be traced back to Gross and Mende in [6, 7], where it is argued that higher-spin theories may govern the high energy limit of string scattering. Specific to the AdS/CFT context though, Witten [8] conjectured that when the AdS radius is small a subset of large string excitations decouples from the remaining degrees of freedom and is described by an interacting higher-spin theory.

Unfortunately, interactions are hard to construct. Powerful no-go theorems have been discovered which prevent simple extensions of free higher-spin theories [9–17]. Indeed, it seems that there are only three interacting theories which are generally agreed to be well-defined: Vasiliev's theory in space-times with a non-vanishing cosmological constant [18], Segal's conformal higher-spin theories [19], and string theory.

Vasiliev's theory, as well as Segal's theory, are based on symmetry. One has a higher-spin symmetry that is gauged, and interactions are built based on the gauge invariance principle. The formalism developed by Vasiliev is non-local, his equations are written for multiplets that contain all spin s fields as components; they lead to the higher-spin equations of motion provided they are partially solved with respect to certain auxiliary conditions and the gauge is fixed appropriately [20, 20, 21].

It might be troubling that the main higher-spin theory available is non-local. Consider that from the moment one allows non-local interactions, it is completely possible to also consider non-local field redefinitions. If no locality condition is imposed, then interactions which are invariant under a given gauge algebra can be formally solved up to any order and for any choice of couplings. That is, one can perform arbitrary non-local field redefinitions that map order by order in a weak field expansion any interaction to zero [22, 23][1]. Consistency then implies that there must be a way to truncate or better define what kind of non-localities are allowed.

This guide, if it exists, would be a breakthrough not just for higher-spin theories. Naturally, in the past couple of year, efforts have concentrated on understanding the possible consequences of these non-localities for vertices of flat space higher-spin theories, as well as other possible applications in the AdS/CFT correspondence [20, 22, 24, 25], which is a topic that we are not familiar. We highlight the paper by Sharapov and Skvortsov [26] that gives a geometric interpretation for the origin of this non-local behaviour based on the Hochschild cohomology of higher-spin algebras; it is interesting that all this complicated behaviour may find a succinct and clear explanation in one piece of mathematics.

Our work also investigated non-local behaviour of higher-spin theories. More precisely, it hinted the presence of *a non-local conformal symmetry* already at flat space: it satisfies the conformal algebra despite being non-locally realized.

Our focus, then, is on free higher-spin theories. They have two known descriptions, which we refer to as Fronsdal and Penrose formulation. In Fronsdal theory [27, 28], we have constrained spacetime tensors that form an irreducible representation of the Lorentz group on-shell, while in Penrose theory one uses twistor geometry to construct irreducible representations of the little group [29] of the Lorentz group. Both theories are well described by an action which is invariant under higher-spin gauge symmetries. It is interesting, however, that Penrose formulation *is* invariant under *local* conformal symmetries while Fronsdal formulation *is not* [30]. It is by mapping one formulation to another that we can stablish the presence of a non-local conformal symmetry in Fronsdal theory.

1.2 On worldsheet curvature coupling in the pure spinor sigma-model.

The pure spinor string is the most suitable formalism for computations of scattering amplitudes, specially in $AdS_5 \times S^5$ background due to the manifest PSU$(2,2|4)$ symmetry [5]. Moreover, the AdS background has some advantages when compared to the flat space. It is not necessary to introduce non-minimal variables, so in priciple amplitudes could be computed without regularizations [31].

But, there still exist complications. The evaluation of OPE's is difficult due to the interacting nature of the theory; supergravity vertex operators have not been completely described, we have covariant expressions for the dilaton [31], the beta-deformation [32], and the half-BPS states [33]. In addition to that, the b-ghost is a composite operator which is not holomorphic for general curved backgrounds [31, 34]; this poses a problem for the current prescription of the scattering amplitudes which could be argued to be even more fundamental than the complete knowledge of the vertex operators.

Consider that string theory amplitudes are defined as integrals of differential forms over the moduli space of Riemann surfaces with marked points [35]. In order to have a well-defined measure of integration, it is necessary that our form is closed and horizontal. Closedness implies

[1] See also references therein.

independence of the path of integration while horizontality means that for every diffeomorphism of the metric our string amplitude remains invariant. Mathematically, horizontality is translated into holomorphicity of the b-ghost.

Mikhailov and Schwarz [36, 37] suggested a geometric interpretation of the string amplitude using the BV formalism. The string amplitude is described as a pseudo-differential form defined over a family of Lagrangian submanifolds that live inside the BV space. It is interesting that this geometric interpretation generalizes the amplitude prescription for non-holomorphic b-ghosts.

In collaboration with Andrei Mikhailov, we study in this new prescription the insertion of vertex operators, integrated and unintegrated. We also explore the relationship between unintegrated vertex operators and the Fradkin-Tseytlin term in the pure spinor formalism. Explicitly construct the string measure for the $AdS_5 \times S^5$ background when the amplitude is deformed by the insertion of a beta deformation vertex.

1.3 On the spectrum and spacetime supersymmetry of heterotic ambitwistor string.

The ambitwistor paper is the only one that is not directly connected to $AdS_5 \times S^5$. It is connected to the techniques learned during the previous years. We wanted to understand the problem of describing Ramond backgrounds in RNS formalism, because for quite some time, we believed that the BV machinery was the appropriate language for this problem. From this point of view, ambitwistor theory just seemed a good place to test what one has learned about Ramond vertex operators.

Another motivation was to understand what these ambitwistor strings would look like in $AdS_5 \times S^5$. The space of null geodesics has been studied in [38] by Mikhailov, and perhaps the vertex operators of this sigma model would be easier to describe (in comparison with the standard pure spinor $AdS_5 \times S^5$ string) given the amount of symmetry that we have. To the best of our knowledge there is sigma model of written for this space, but it is interesting exercise.

In any case, these ambitwistor models have been on constant investigation recently due to their connection with the CHY formula for scattering amplitudes. In 2013, Cachazzo, Hei and Yuan published [39, 40] an expression for n-particle tree level amplitude of any massless theory as an integral over the moduli space of the Riemann sphere with n-punctures. These integrals are supported on a set of polynomial equations, today known as scattering equations, and the precise form of the integrand is determined by the theory. Again, to the best of our knowledge, no one has explained the origin of CHY using standard quantum field theory techniques, which leaves us only with the direct comparison method for knowing which integrand corresponds to which field theory [41, 42].

But, even though we don't have an explanation of the CHY formula via quantum field theory, the integral over the moduli space of Riemann surfaces with marked points suggests a string interpretation: a genus zero amplitude with n insertions of vertex operators. This is precisely what the ambitwistor string theory does. It reproduces the CHY prescription via a chiral worldsheet model with no free parameters. The target space is the space of null geodesics, known as ambitwistor space. The localization over the scattering equations happens naturally via path integration of the momentum eigenstates.

The model was developed by Mason and Skinner in [43], and is also described as the infinite tension limit ($\alpha' \to 0$) of the bosonic string. Infinite tension limits of supersymmetric strings – the pure spinor string and RNS string – also exist as well. It is interesting to mention that this description – the α' to zero limit of standard string theory – might be incomplete. The theory is known to have non-unitary states in all formulations with the exception of type IIB RNS sector. It would be really interesting to understand if the presence of these non-unitarities is unavoidable when considering these infinite tension limits, or if there is in fact some string theory that isolates precisely super Yang-Mills and supergravity, that is, there exists a well-defined $\alpha' \to 0$ limit.

This is not the main research direction of these ambitwistor models. Efforts have been on the generalization of a CHY-type formula for loops. Loop amplitudes were obtained by considering worldsheet correlators at higher genus surfaces in [44, 45] or what is called "nodal Riemann sphere". At this point, it is not clear to us the construction of this loop level prescription, because of the difficulties with gauge fixing. In particular, it is not clear how the worldsheet metric enters in the construction of the model.

Together with Matheus Lize, we computed the fermionic equations of motion that should determine the spectrum of the heterotic ambitwistor string. We also have written the supersymmetry algebra for the combined bosonic and fermionic system. We had hoped that the equations of motion would be easier to solve, and by supersymmetry we could determine the bosonic spectra of the theory. But unfortunately, we were not able to find a solution for these differential equations.

Chapter 2

On the phase space of free higher-spin theories and conformal transformations.

2.1 Outline.

We organize our presentation as follows. Section 2 is a brief review, where we explain the two approaches for free massless higher-spin theories.

In section 3 we write an action for Penrose higher-spin theory. To our knowledge, such action for general higher-spins has never appeared before in the literature. First-order formulations, however, were used by Fradkin and Vasiliev in [46] for AdS space, where they were also extended to interactions. More recently, Kirill Krasnov described full self-dual gravity in [47] using an action that resembles ours; but, in our case, this action is defined over complex field configurations, and it describes off-shell a doubled set of the higher-spin modes [1].

We look at some examples, so the spins 1, 3/2 and 2 cases are discussed in detail, each of which highlights a particular feature of our construction outlining our strategy for dealing with general spins. The spin s case is done in section 4; our construction is a particular instance of the prescription given in [48], where a set of equations of motion and a presymplectic structure are shown to lift to a well-defined Lagragian.

With the map defined, we can investigate conformal invariance. In section 5 we show that Penrose action does have conformal symmetry for every spin s. Therefore one is able to push forward these transformations to the Fronsdal case. For spins lower than 2, these new transformations agree with usual conformal change of coordinates. The first non-trivial case is linearized gravity. We write explicitly the resulting transformation, where one is able to see the difference from standard Lie derivatives.

On notation. Our conventions follow those of [49]; we are concerned with 4-dimensional Minskowski space; so, through out the paper, the various indices will always be running over fixed intervals. Small Latin letters, for example, are spacetime indices running from 0 to 3, so that A_m is a spacetime covector. Capital Latin letters, in turn, are spinor indices in Van der Warden notation, that is, dotted and undotted running from 0 to 1. In particular, a Dirac spinor is a two component Weyl and anti-Weyl spinor written like

$$\Psi = \begin{pmatrix} \psi_A \\ \overline{\chi}^{\dot{A}} \end{pmatrix} \tag{2.1.1}$$

for some chiral spinor ψ_A and anti-chiral $\overline{\chi}^{\dot{A}}$.

[1] In phase space, however, there is a well-defined notion of reality, and it is where we obtain a single copy of the spectrum.

Such notation is designed so that there is a correspondence between spacetime and spinor indices where, for instance, m will correspond to the pair $M\dot{M}$. The explicit realization is given by the Pauli matrices with index structure $\sigma^m_{M\dot{M}}$, where

$$\sigma^0 = -1 \quad \text{and} \quad \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3).$$

The epsilon symbol satisfies $\epsilon_{AB}\epsilon^{BC} = \delta_A{}^C$ for undotted and dotted indices. This enables one to raise the indices of $\sigma^m_{M\dot{M}}$ to obtain

$$\overline{\sigma}^{m\,M\dot{M}}, \quad \text{where} \quad \sigma^0 = -1 \quad \text{and} \quad \vec{\sigma} = (-\sigma^1, -\sigma^2, -\sigma^3).$$

Everything is combined to form the Weyl representation of the Dirac matrices:

$$\Gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \overline{\sigma}^m & 0 \end{pmatrix}, \tag{2.1.2}$$

which satisfy the Clifford algebra

$$\{\Gamma^m, \Gamma^n\} = -2\eta^{mn} \tag{2.1.3}$$

for the metric signature $(-,+,+,+)$.

2.2 Review of massless higher-spin formulations.

This section is an overview of some background material based on references [27] and [28]. It begins with Fronsdal theory and then proceeds to Penrose description [50].

2.2.1 Fronsdal theory of free massless higher-spin fields.

Let us begin with bosonic spins. Given a totally symmetric tensor of s indices, $h_{m_1\cdots m_s}$, which has higher-spin gauge freedom of the form

$$\delta h_{m_1\cdots m_s} = s\,\partial_{(m_1}\varepsilon_{m_2\cdots m_s)} \tag{2.2.1}$$

and is double-traceless:

$$\eta^{m_1m_2}\eta^{m_3m_4}h_{m_1m_2m_3m_4\cdots m_s} = 0; \tag{2.2.2}$$

one can form the so-called Fronsdal tensor:

$$F_{m_1\cdots m_s} = \Box h_{m_1\cdots m_s} - s\,\partial_{(m_1}\partial^p h_{|p|\,m_2\cdots m_s)} + \frac{s(s-1)}{2}\partial_{(m_1}\partial_{m_2}h^p{}_{|p|\,m_3\cdots m_s)}. \tag{2.2.3}$$

A higher-spin theory in flat spacetime is then described by the action

$$S = \frac{(-1)^{s+1}}{2}\int \mathrm{d}^4x \left(h^{m_1\cdots m_s}F_{m_1\cdots m_s} - \frac{s(s-1)}{4}h_n{}^{nm_3\cdots m_s}F^p{}_{pm_3\cdots m_s}\right), \tag{2.2.4}$$

which is symmetric in the higher-spin field $h_{m_1\cdots m_s}$ and gauge invariant under transformations (2.2.1).

The equations of motion read

$$F_{m_1\cdots m_s} - \frac{s(s-1)}{4}\eta_{(m_1m_2}F^p{}_{p\,m_3\cdots m_s)} = 0. \tag{2.2.5}$$

And these can be further simplified if (2.2.2) is taken into account. It implies

$$\eta^{m_1m_2}\eta^{m_3m_4}F_{m_1m_2m_3m_4\cdots m_s} = 0 \tag{2.2.6}$$

which, in turn, allows us to cast equation (2.2.5) as

$$F_{m_1\cdots m_s} = 0. \tag{2.2.7}$$

We see the Fronsdal tensor fixes $h_{m_1\cdots m_s}$ up to gauge transformations since both have the same number of degrees of freedom. The physical degrees of freedom, however, are obtained once we gauge fix the above description. It is possible to gauge away the trace part of the higher-spin field $h_{m_1\cdots m_s}$ as well as its divergence. Consider the gauge field ε which satisfies

$$h^p{}_{pm_3\cdots m_s} = \partial^n\varepsilon_{nm_3\cdots m_s} \tag{2.2.8}$$

and

$$\partial^p h_{pm_2\cdots m_s} = \Box\varepsilon_{m_2\cdots m_s}, \tag{2.2.9}$$

so that the remaining gauge symmetry obeys

$$\Box\varepsilon_{m_2\cdots m_s} = 0, \quad \partial^n\varepsilon_{nm_3\cdots m_s} = 0, \text{ and } \varepsilon^p{}_{pm_3\cdots m_s} = 0. \tag{2.2.10}$$

Once we choose (2.2.10), our higher-spin field satisfies

$$\Box h_{m_1\cdots m_s} = 0, \quad \partial^p h_{pm_2\cdots m_s} = 0, \text{ and } h^p{}_{pm_3\cdots m_s} = 0; \tag{2.2.11}$$

thus proving that $h_{m_1\cdots m_s}$ describes a spin s massless particle.

There are minor changes if one wants to describe fermions. For a spin $s = h + 1/2$, we have a Majorana spinor $\Psi_{m_1\cdots m_h}$ totally symmetric in its h indices which has gauge freedom

$$\delta\Psi_{m_1\cdots m_h} = h\,\partial_{(m_1}\chi_{m_2\cdots m_h)}, \tag{2.2.12}$$

and satisfies the triple Γ-trace condition:

$$\Gamma^{m_1}\Gamma^{m_2}\Gamma^{m_3}\Psi_{m_1m_2m_3\cdots m_h} = 0. \tag{2.2.13}$$

The fermionic Fronsdal tensor,

$$F_{m_1\cdots m_h} = \Gamma^a\partial_a\Psi_{m_1\cdots m_h} - h\,\partial_{(m_1}\Gamma^a\Psi_{m_2\cdots m_h)a}, \tag{2.2.14}$$

is the gauge invariant object used to construct the action

$$S = \frac{1}{2}\int \mathrm{d}^4x \left(\overline{\Psi}^{m_1\cdots m_h}F_{m_1\cdots m_h} - \frac{h}{2}\Gamma^p\overline{\Psi}_p{}^{m_2\cdots m_h}\Gamma^a F_{am_2\cdots m_h} - \frac{h(h-1)}{4}\overline{\Psi}_q{}^{qm_3\cdots m_h}F^p{}_{pm_3\cdots m_h}\right) \tag{2.2.15}$$

where $\overline{\Psi}^{m_1\cdots m_h}$ satisfies the Majorana condition:

$$\overline{\Psi}^{m_1\cdots m_h} = \Psi^T C, \quad \text{and} \quad C = \begin{pmatrix} \epsilon_{BA} & 0 \\ 0 & \epsilon^{\dot{B}\dot{A}} \end{pmatrix} \tag{2.2.16}$$

is the charge conjugation matrix. The equations of motion are

$$F_{m_1\cdots m_s} - \frac{h}{2}\Gamma_{(m_1}\Gamma^a F_{m_2\cdots m_s)a} - \frac{h(h-1)}{4}\eta_{(m_1m_2}F^p{}_{m_3\cdots m_s)p} = 0. \tag{2.2.17}$$

and they can be simplified once one notices (2.2.13) implies

$$\Gamma^{m_1}\Gamma^{m_2}\Gamma^{m_3}F_{m_1m_2m_3\cdots m_h} = 0, \tag{2.2.18}$$

which enables one to cast (2.2.17) in the form

$$F_{m_1\cdots m_h} = 0. \tag{2.2.19}$$

Notice that, again, the fermionic Fronsdal tensor fixes $\Psi_{m_1\cdots m_h}$ up to gauge transformations. The physical degrees of freedom are obtained from the gauge parameter $\chi_{m_2\cdots m_h}$ that satisfies

$$\Gamma^p\Psi_{pm_2\cdots m_h} = \Gamma^m\partial_m\chi_{m_2\cdots m_h}, \tag{2.2.20}$$

so that the remaining gauge symmetry obeys

$$\Gamma^m\partial_m\chi_{m_2\cdots m_h} = 0 \quad \text{and} \quad \Gamma^p\chi_{pm_2\cdots m_h} = 0. \tag{2.2.21}$$

The gauge fixing (2.2.20) ensures that $\Psi_{m_1\cdots m_h}$ is an irreducible representation of the little group. The on-shell degrees of freedom are then described by a field Ψ which satisfies

$$\Gamma^p\partial_p\Psi_{m_1\cdots m_h} = 0 \quad \text{and} \quad \Gamma^p\Psi_{pm_2\cdots m_h} = 0 \tag{2.2.22}$$

thus proving $\Psi_{m_1\cdots m_h}$ describes an spin $s = h + 1/2$ representation.

2.2.2 Penrose theory of free massless higher-spin fields.

Penrose's description of massless higher-spin fields is obtained from the Penrose transform. It relates homogeneous functions of definite degree in twistor space to massless higher-spin fields in Minkowski space. For an introduction to twistors, see reference [51] as well as references therein.

Here we describe the integral expressions obtained by Penrose in [50] only to give some context. These integral formulas are not necessary for the rest of this paper. We are only interested in the spacetime fields they define.

Let $Z = (\omega^A, \pi_{\dot{A}})$ be the coordinates of a twistor inside the complex projective line $\mathbf{P}_1$. These are constrained by the twistor equation:

$$\omega^A = x^{A\dot{A}}\pi_{\dot{A}}, \tag{2.2.23}$$

where $x^{A\dot{A}}$ parametrizes the Minkowski space. Consider also a point $\overline{Z} = (\lambda_A, \mu^{\dot{A}})$ in the dual twistor space and fix two closed cycles of integration: γ inside $\mathbf{P}_1$ and γ^* inside the dual line $\mathbf{P}_1^*$. Define the following spacetime spinors

$$\overline{\phi}_{\dot{A}\dot{B}\cdots\dot{D}}(x) = \frac{1}{2\pi i}\int_\gamma \underbrace{\pi_{\dot{A}}\pi_{\dot{B}}\ldots\pi_{\dot{D}}}_{2s} f(Z)\,\pi_{\dot{E}}\mathrm{d}\pi^{\dot{E}} \tag{2.2.24a}$$

and

$$\phi_{AB\cdots D}(x) = \frac{1}{2\pi i}\int_{\gamma^*} \underbrace{\lambda_A\lambda_B\ldots\lambda_D}_{2s} \overline{f}(\overline{Z})\,\lambda^A\mathrm{d}\lambda_A \tag{2.2.24b}$$

for some semi-integer number s.

Remark. These integrals are well defined over $\mathbf{P}_1$ if the integrands are homogeneous functions of degree 0. Hence, the complex functions $f(Z)$ and $\overline{f}(\overline{Z})$ must have homogeneity $-2s-2$ in $\pi_{\dot{A}}$ and λ_A respectively.

These spinors form an irreducible representation of the Lorentz group $SL(2,\mathbf{C})$ and satisfy, by consequence of their definitions, the differential equations

$$\partial^{A\dot{A}}\overline{\phi}_{\dot{A}\dot{B}\cdots\dot{D}}(x) = 0 \tag{2.2.25}$$

and

$$\partial^{\dot{A}A}\phi_{AB\cdots D}(x) = 0. \tag{2.2.26}$$

In view of the (anti-)self-duality conditions, we can see $\phi^{AB\cdots D}$ and $\overline{\phi}^{\dot{A}\dot{B}\cdots\dot{D}}$ describe right-handed massless free fields of spin s and left-handed massless free fields of spin $-s$ respectively.

Let $a_{\dot{A}B\cdots D}$ be the field given by

$$\overline{\phi}_{\dot{A}\dot{B}\cdots\dot{D}} = \partial^{B}{}_{(\dot{B}}\cdots\partial^{D}{}_{\dot{D}}\, a_{\dot{A})B\cdots D}. \tag{2.2.27}$$

It readily follows that equation (2.2.25) is automatically satisfied when

$$\partial_{(A}{}^{\dot{A}}a_{B\cdots D)\dot{A}} = 0. \tag{2.2.28}$$

Notice, however, that there is an ambiguity. There are gauge symmetries of the form

$$\delta a_{\dot{A}B\cdots D} = \partial_{\dot{A}(B}\xi_{\cdots D)} \tag{2.2.29}$$

for some symmetric spinor $\xi_{C\cdots D}$ of $2s-2$ indices. These are the higher-spin gauge symmetries which were also present in Fronsdal theory.

We will always refer to $\phi^{AB\cdots D}$ and $a_{\dot{A}B\cdots D}$ as the fundamental fields of Penrose description. And, for future reference, we call $\phi_{AB\cdots D}$ the curvature spinor and $a_{\dot{A}B\cdots D}$ the gauge field.

2.3 Higher-spin action in Penrose's description.

2.3.1 Higher-spin action.

We suggest the following higher-spin action for a massless spin s particle:

$$S = i\int \mathrm{d}^4x\,\left(\phi^{AB\cdots D}\partial_{A\dot{A}}a^{\dot{A}}{}_{B\cdots D}\right) \tag{2.3.1}$$

where $\phi^{AB\cdots D}$ and $a^{\dot{A}}{}_{B\cdots D}$ have $2s$ and $2s-1$ undotted indices respectively. Invariance under higher-spin gauge symmetries is respected, because if we consider the variation under (2.2.29) the action transforms into

$$\delta S = i\int \mathrm{d}^4x\,\left[\phi^{AB\cdots D}\partial_{A\dot{A}}\partial^{\dot{A}}{}_{(B}\xi_{\cdots D)}\right]. \tag{2.3.2}$$

From the identity

$$\partial_{A\dot{A}}\partial^{\dot{A}}{}_{B} = +\frac{1}{2}\epsilon_{AB}\Box, \tag{2.3.3}$$

we get $\delta S = 0$ since the curvature spinor $\phi^{AB\cdots D}$ is completely symmetric in its indices. The equations of motion obtained from (2.3.1) are precisely (2.2.25) and (2.2.28):

$$\partial_{\dot{A}A}\phi^{AB\cdots D} = 0 \quad \text{and} \quad \partial_{\dot{A}(A}a^{\dot{A}}{}_{B\cdots D)} = 0.$$

2.3.2 Reality conditions.

It is a good point to make some observations. First, although twistors were used as a motivation for this action, we are not integrating over twistor space. We are only using a spinor basis and it is possible to write this action with usual Lorentz indices too. The convenience of using spinors is the easier treatment of self-duality conditions. Second, and possibly a troublesome point, is that it appears that this action describes just one helicity, but this is not the case.

Let us discuss this point in detail. For the sake of argument, let us specialize our discussion to the spin 1 case. We want to show that the phase space spanned by these equations is equivalent to the phase space of Maxwell's electromagnetism. The natural route is to describe a canonical map. Therefore, given the data (ϕ, a), we are supposed to construct a map to the Maxwell gauge field A,

$$H : (\phi, a) \longmapsto A, \tag{2.3.4}$$

where solutions of the (ϕ, a) system are carried to solutions of the Maxwell's equations. In addition, we must verify two things: the kernel of this map must be zero, otherwise there are configurations of ϕ and a which would correspond to zero electromagnetic solution; and the cokernel should also be zero, that is the set of all Maxwell solutions, given by A, should be fully covered.

The canonical map H is constructed as follows. Given the equation of motion (2.2.25), locally by the Poincaré lemma, we can write ϕ as

$$\phi \;=\; \mathrm{d}\overline{a} \tag{2.3.5}$$

with some possible ambiguity given by the addition of a closed form. The second equation of motion, (2.2.28), is the statement that a does not contribute to the self-dual part, hence it must describe the anti-self-dual piece. It becomes natural to define

$$A \;=\; a + \overline{a} \tag{2.3.6}$$

since it satisfies Maxwell's equations as a consequence of self-duality:

$$\begin{aligned} \mathrm{d} \star \mathrm{d}A &= \mathrm{d} \star \mathrm{d}\,(a + \overline{a}) \\ &= \mathrm{d} \star (\mathrm{d}a + \mathrm{d}\overline{a}) \\ &= i\mathrm{d}\,(\mathrm{d}a - \mathrm{d}\overline{a}) \\ &= 0. \end{aligned} \tag{2.3.7}$$

Notice that the kernel of (2.3.6) indeed vanishes. One takes $-a + \mathrm{d}\alpha = \overline{a}$, for some α, and, by consequence of (2.2.28), $\phi = 0$, which forces a to be pure gauge. That the cokernel vanishes is a more subtle point. Because the Hodge star operator $\star$ satisfies $\star^2 = -1$ in four dimensions, it splits the bundle Λ^2, of two-forms in Minkowski space, into a direct sum,

$$\Lambda^2 \;=\; \Lambda^2_+ \oplus \Lambda^2_-, \tag{2.3.8}$$

where $\Lambda^2_\pm$ are the $\pm i$ eigenspaces of $\star$. Thus, any two form can be written as

$$F = \phi + \overline{\phi} \tag{2.3.9}$$

and, by the Poincaré lemma, we locally have the decomposition (2.3.6).

The analysis of this construction is special to the 4-dimensional Minkowski space and it carries through only for the equations of motion. It is not true that the action (2.3.1) is off-shell equivalent to the Maxwell action. One way to understand this is to notice that the action

(2.3.1) is not real. In general, equation (2.3.1) is defined over some complex infinite-dimensional manifold.

Such consideration raises the question if whether the map (2.3.6) defines a real A or not. It turns out that, in phase space, complex conjugation acts as an involution, where the complex conjugation map, denoted `c.c.`, is

$$\texttt{c.c.} \begin{pmatrix} a \\ \phi \end{pmatrix} = \begin{pmatrix} \mathrm{d}^{-1}\overline{\phi} \\ \mathrm{d}\overline{a} \end{pmatrix}. \tag{2.3.10}$$

It has fixed point given by

$$\overline{\phi_{AB}} = \overline{\phi}_{\dot{A}\dot{B}} = \partial_{C(\dot{A}} a^{C}{}_{\dot{B})}, \tag{2.3.11}$$

from where we see that the complex conjugate of a is $\overline{a}$ and vice-versa. To summarize our results: the action (2.3.1) is complex, but in phase space – that is, the space of classical solutions – there is a well-defined notion of reality, which is given by the fixed point of the involution (2.3.10), namely equation (2.3.11). Only in this submanifold, the two theories classically agree.

Outside the fixed point, the complex theory describes two photons. Self-duality of ϕ allows one to write

$$\phi = F + i \star F \tag{2.3.12}$$

for a real 2-form F. Hence, the equation of motion $\mathrm{d}\phi = 0$ implies Maxwell's equations:

$$\mathrm{d}F = 0 \quad \text{and} \quad \mathrm{d} \star F = 0. \tag{2.3.13}$$

On the other hand, the gauge field a on-shell gives an anti-self-dual 2-form:

$$\mathrm{d}a = G - i \star G \tag{2.3.14}$$

from where the second Maxwell equations come:

$$\mathrm{d}G = 0 \quad \text{and} \quad \mathrm{d} \star G = 0. \tag{2.3.15}$$

The reality conditions (2.3.11) impose $F = G$.

2.3.3 Making action real.

Consider the real part of the action[2] (2.3.1):

$$S = \int \left(\phi \wedge \mathrm{d}a + \overline{\phi} \wedge \mathrm{d}\overline{a}\right). \tag{2.3.16}$$

It turns out that the equations of motion are unchanged. To see this, consider the variation of this action under the real and imaginary parts of a, it gives

$$\mathrm{d}\left(\phi + \overline{\phi}\right) = 0 \quad \text{and} \quad \mathrm{d}\left(\phi - \overline{\phi}\right) = 0 \tag{2.3.17}$$

respectively. Self-duality of ϕ does not allow us to vary its real and imaginary parts independently, therefore we have a single equation of motion:

$$\mathrm{d}\left(a + \overline{a}\right) + i \star \mathrm{d}\left(a - \overline{a}\right) = 0. \tag{2.3.18}$$

[2]We would like to thank Arkady Tseytlin for suggesting this idea.

Inspection shows that the real and imaginary parts of a satisfy the Maxwell's equations while ϕ again satisfies $\mathrm{d}\phi = 0$. The two copies of the Maxwell theory can be identified with the reality condition (2.3.11). It is surprising that the addition of complex conjugatation does not change the field content of the theory.

2.3.4 Symplectic structure.

We wish to establish the above correspondence for every spin s field. The above consideration can be rephrased using the notion of symplectic structure. In this language, although the action is defined for complex field configurations, there is a real submanifold inside the phase space where the restriction of the symplectic form derived from (2.3.1) is non-degenerate. Then, we will construct a map H that becomes a canonical transformation to the phase space of Fronsdal.

The symplectic structure for action (2.3.1) is

$$\Omega = i\int_C \delta\phi^{AB\cdots D}\wedge\delta a^{\dot{A}}{}_{B\cdots D}\wedge \mathrm{d}^3x_{A\dot{A}}, \quad \text{where} \quad n_{A\dot{A}}\mathrm{d}^3x = \mathrm{d}^3x_{A\dot{A}}, \tag{2.3.19}$$

for a normal vector $n_{A\dot{A}}$ to the spacelike contour C. It is δ-closed and invariant under deformations of C, because

$$\partial_{A\dot{A}}\left(\delta\phi^{AB\cdots D}\wedge\delta a^{\dot{A}}{}_{B\cdots D}\right) = 0 \tag{2.3.20}$$

once we use the equations of motion. However, note that this symplectic structure is also degenerate. Degeneracies indicate the presence of gauge symmetries in the action. In our case, if we let

$$V = \partial_{\dot{A}(B}\xi_{\cdots D)}(x)\frac{\delta}{\delta a_{\dot{A}B\cdots D}(x)} \tag{2.3.21}$$

be a tangent vector field along gauge trajectories, we get

$$\begin{aligned}
\iota_V\Omega &= i\int_C \partial^{\dot{A}}{}_{(B}\xi_{\cdots D)}\delta\phi^{AB\cdots D}\mathrm{d}^3x_{A\dot{A}} \\
&= i\int_C \partial^{\dot{A}}{}_{(B}\left[\xi_{\cdots D)}\delta\phi^{AB\cdots D}\right]\mathrm{d}^3x_{A\dot{A}} - i\int_C \xi_{(C\cdots D}\partial_{B)}{}^{\dot{A}}\delta\phi^{ABC\cdots D}\mathrm{d}^3x_{A\dot{A}} \\
&= i\int_C \partial^{\dot{A}}{}_{(B}\left[\xi_{\cdots D)}\delta\phi^{AB\cdots D}\right]\mathrm{d}^3x_{A\dot{A}} = 0,
\end{aligned} \tag{2.3.22}$$

where the last line vanishes due to C being a closed contour. Degenerate symplectic structures descend to a reduced phase space. If we define $\ker\Omega$ to be the set of gauge generators, then the reduced phase space is given by the factor $M/\ker\Omega$. On-shell gauge-invariant functions are points in this space and they coincide with physical observables.

It still remains to be checked whether this symplectic structure is real over the fixed point defined by the involution[3]. The fixed point can be written as

$$\overline{\phi_{AB\cdots D}} = \overline{\phi}_{\dot{A}\dot{B}\cdots\dot{D}} = \partial_{(\dot{B}}{}^{B}\cdots\partial_{\dot{D}}{}^{D}\,a_{\dot{A})B\cdots D} \tag{2.3.23}$$

and it follows that

[3]See paragraph above equation (2.3.11)

$$\begin{aligned}
\overline{\Omega} &= -i \int_C \delta\overline{\phi}^{\dot{A}\dot{B}\cdots\dot{D}} \wedge \delta\overline{a}^{A}{}_{\dot{B}\cdots\dot{D}} \wedge \mathrm{d}^3x_{A\dot{A}} \\
&= -i \int_C \partial^{(\dot{B}}{}_B\, \partial^{\dot{C}}{}_C \ldots \partial^{\dot{D}}{}_D\, \delta a^{\dot{A})B\cdots D} \wedge \delta\overline{a}^{A}{}_{\dot{B}\cdots\dot{D}} \wedge \mathrm{d}^3x_{A\dot{A}} \\
&= (-)^{2s+1} i \int_C \delta a^{\dot{A}B\cdots D} \wedge \partial^{\dot{B}}{}_{(B}\, \partial^{\dot{C}}{}_C \ldots \partial^{\dot{D}}{}_D\, \delta\overline{a}_{A)\dot{B}\cdots\dot{D}} \wedge \mathrm{d}^3x_{\dot{A}}{}^{A} \\
&= -i \int_C \delta a^{\dot{A}}{}_{B\cdots D} \wedge \delta\phi^{AB\cdots D} \wedge \mathrm{d}^3x_{A\dot{A}} \\
&= +\Omega,
\end{aligned} \tag{2.3.24}$$

thus proving that indeed the symplectic structure is real.

Having the symplectic structure for Penrose theory, it remains to construct the canonical map which will relate the two descriptions. In doing so, we are ready to prove that the two phase spaces agree.

2.4 Canonical map between descriptions.

It is instructive to consider some examples before treating the general case. We specialize our discussion to Rarita-Schwinger and linearized gravity in the next two subsections. Each case will serve to emphasize the introduction of a new tool for the analysis.

In the Rarita-Schwinger case, for example, we will see how the spliting of the gauge field into self-dual and anti-self-dual connection – as it has already happened in electromagnetic case – comes about in the symplectic structure. The main objective is to demonstrate, on the real slice given by (2.3.23), that the canonical map indeed preserves the symplectic structure.

In linearized gravity, we show how the analysis can be made rather straightforward once we pass to momentum space. It will avoid dealing with integration by parts when we show that the symplectic structures agree.

2.4.1 Rarita-Schwinger case.

The Rarita-Schwinger theory is obtained when $h = 1$ in Section 2.2.1. We have the Majorana spinor

$$\Psi_m = \begin{pmatrix} \psi_{A\,m} \\ \overline{\psi}^{\dot{A}}_m \end{pmatrix} \tag{2.4.1}$$

with higher-spin gauge symmetries $\delta\Psi_m = \partial_m\varepsilon$ and gauge-invariant action

$$S = \int \mathrm{d}^4x \ \left(\overline{\Psi}^m F_m + \frac{1}{2}\, \overline{\Psi}_p \Gamma^p\, \Gamma^m F_m \right). \tag{2.4.2}$$

The equations of motion read

$$F_m = \Gamma^n \partial_n \Psi_m - \partial_m \Gamma^n \Psi_n = 0. \tag{2.4.3}$$

For our applications, it will be useful to consider the gauge-invariant combination

$$R_{mn} = \partial_m \Psi_n - \partial_n \Psi_m, \tag{2.4.4}$$

in order to make contact with the curvature spinors ϕ^{ABC} and $\overline{\phi}^{\dot{A}\dot{B}\dot{C}}$. To see how, let us introduce the following spinor counterpart of R_{mn}:

$$R_{M\dot{M}N\dot{N}} = \mathrm{d}\Psi_{(\dot{M}\dot{N})}\epsilon_{MN} - \mathrm{d}\Psi_{(MN)}\epsilon_{\dot{M}\dot{N}}, \tag{2.4.5}$$

where abbreviations have been used:

$$\partial^{A}{}_{(\dot{N}}\Psi_{\dot{M})A} = \mathrm{d}\Psi_{(\dot{M}\dot{N})} = \begin{pmatrix} \mathrm{d}\psi_{(\dot{M}\dot{N})B} \\ \mathrm{d}\overline{\psi}^{\dot{B}}{}_{(\dot{M}\dot{N})} \end{pmatrix} \tag{2.4.6a}$$

and

$$\partial_{\dot{A}(M}\Psi_{N)}{}^{\dot{A}} = \mathrm{d}\Psi_{(MN)} = \begin{pmatrix} \mathrm{d}\psi_{(MN)B} \\ \mathrm{d}\overline{\psi}^{\dot{B}}{}_{(MN)} \end{pmatrix}. \tag{2.4.6b}$$

It enables us to rewrite the equations of motion in the form

$$\Gamma^{m}R_{mn} = 0 \longmapsto \begin{pmatrix} 0 & \delta_{B}{}^{M}\delta^{\dot{M}}{}_{\dot{B}} \\ \epsilon^{MB}\epsilon^{\dot{M}\dot{B}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}\psi_{\dot{M}\dot{N}B}\epsilon_{MN} - \mathrm{d}\psi_{MNB}\epsilon_{\dot{M}\dot{N}} \\ \mathrm{d}\overline{\psi}^{\dot{B}}{}_{\dot{M}\dot{N}}\epsilon_{MN} - \mathrm{d}\overline{\psi}^{\dot{B}}{}_{MN}\epsilon_{\dot{M}\dot{N}} \end{pmatrix} = 0. \tag{2.4.7}$$

from where we obtain

$$\mathrm{d}\psi_{\dot{A}\dot{N}N} - \mathrm{d}\psi^{C}{}_{NC}\,\epsilon_{\dot{A}\dot{N}} = 0 \tag{2.4.8a}$$

and

$$\mathrm{d}\overline{\psi}^{\dot{C}}{}_{\dot{C}\dot{N}}\epsilon_{AN} + \mathrm{d}\overline{\psi}^{\dot{N}}{}_{AN} = 0. \tag{2.4.8b}$$

A quick inspection shows the only possible solutions for (2.4.8a) are

$$\mathrm{d}\psi_{\dot{A}\dot{N}N} = 0 \quad \text{and} \quad \mathrm{d}\psi^{C}{}_{NC} = 0 \tag{2.4.9}$$

since the first term is symmetric in $\dot{A}\dot{N}$ while the second one is anti-symmetric in $\dot{A}\dot{N}$. The same type of reasoning leads us to the solutions of (2.4.8b):

$$\mathrm{d}\overline{\psi}^{\dot{C}}{}_{\dot{C}\dot{N}} = 0 \quad \text{and} \quad \mathrm{d}\overline{\psi}^{\dot{N}}{}_{AN} = 0. \tag{2.4.10}$$

These solutions annihilate any components with dotted and undotted indices. Moreover they completely symmetrize the self-dual and anti-self-dual part. The remaining components split R_{mn} into

$$R_{mn} \longmapsto -\mathrm{d}\psi_{(MNA)}\epsilon_{\dot{M}\dot{N}} - \mathrm{d}\overline{\psi}_{(\dot{A}\dot{M}\dot{N})}\epsilon_{MN} \tag{2.4.11}$$

and we can identify

$$-\mathrm{d}\psi_{(MNA)} \quad \text{as} \quad \phi_{AMN}, \tag{2.4.12}$$

and

$$-\mathrm{d}\overline{\psi}_{\dot{A}\dot{M}\dot{N}} \quad \text{as} \quad \overline{\phi}_{\dot{A}\dot{M}\dot{N}}. \tag{2.4.13}$$

This procedure occurs for other spins as well. One defines a gauge-invariant combination, and once the equations of motion are imposed the spinors $\phi^{AB\cdots D}$ and $\overline{\phi}^{\dot{A}\dot{B}\cdots\dot{D}}$ are the only remaining components. Notice that

$$\partial_{[m}R_{np]} = 0 \tag{2.4.14}$$

is trivially satisfied in the presence of Ψ_m. As soon as we change pictures and use the curvature spinors, this equation turns into an equation of motion. The anti-symmetry is equivalent to a contraction of spinor indices, and so we recover (2.2.25) and (2.2.26):

$$\partial^{\dot{A}A}\phi_{AMN} = 0 \quad \text{and} \quad \partial^{A\dot{A}}\overline{\phi}_{\dot{A}\dot{M}\dot{N}} = 0.$$

The Penrose description splits the gauge field $h_{m_1\cdots m_s}$ into anti-self-dual and self-dual parts treating the self-dual part via the curvature while the anti-self-dual part is described with the anti-self-dual gauge field.

In the Rarita-Schwinger case, the gauge field $a_{\dot{A}BC}$ is mapped to the anti-chiral part $\overline{\psi}^{m\dot{A}}$ with the *ansatz*

$$\overline{\psi}^{m\dot{A}} = i\overline{\sigma}^{m\dot{E}E}\left(\partial^{\dot{A}C}a_{\dot{E}CE} + \frac{1}{2}\partial_{\dot{E}}{}^{C}a^{\dot{A}}{}_{CE}\right) \tag{2.4.15}$$

where the coefficients are fixed by requiring the higher-spin gauge symmetries to coincide. For consistency, it is also possible, with this choice, to check that $\overline{\psi}^{m\dot{A}}$ satisfies the equations of motion when $a_{\dot{A}BC}$ does. We should point out that this map is the non-trivial piece of our correspondence. For other higher-spins, it has to be constructed with the right coefficients case by case.

One can derive the symplectic structure from action (2.4.2) and it reads:

$$\begin{aligned}\Omega = \int \big(&2\delta\psi_m \wedge \sigma^m\overline{\sigma}^{np}\delta\overline{\psi}_p + 2\delta\overline{\psi}_m \wedge \overline{\sigma}^m\sigma^{np}\delta\psi_p \\ &+ \delta\psi_m \wedge \sigma^n\delta\overline{\psi}^m + \delta\overline{\psi}_m \wedge \overline{\sigma}^n\delta\psi^m - \delta\psi^n \wedge \sigma^m\delta\overline{\psi}_m - \delta\overline{\psi}^n \wedge \overline{\sigma}^m\delta\psi_m\big) \wedge d^3x_n.\end{aligned} \tag{2.4.16}$$

If we intend to describe the spin 3/2 piece, we are allowed to use the gauge

$$\Gamma^m\Psi_m = 0 \tag{2.4.17}$$

so the symplectic structure collapses to

$$\Omega = 2\int \delta\psi_m \wedge \sigma^n\delta\overline{\psi}^m \wedge \mathrm{d}^3x_n. \tag{2.4.18}$$

In Penrose case, the symplectic structure follows from (2.3.1), and it is

$$\Omega = i\int \delta\phi^{ABC} \wedge \delta a^{\dot{A}}{}_{BC} \wedge \mathrm{d}^3x_{A\dot{A}}. \tag{2.4.19}$$

Notice the gauge condition implies

$$\partial^{\dot{B}A}\psi_{\dot{E}EA} = 0, \tag{2.4.20}$$

and by consequence of (2.4.15):

$$\partial_{\dot{A}}{}^{B}a^{\dot{A}}{}_{BC} = 0. \tag{2.4.21}$$

When substitute our *ansatz* into the symplectic structure (2.4.18), we obtain

$$\Omega = +i\int \delta\psi^{\dot{E}EA} \wedge \partial_{\dot{E}}{}^{C}\delta a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}} \tag{2.4.22}$$

and there is a subtlety we must highlight. Despite the advantage of being able to use the equations of motion when dealing with a symplectic structure, we are not allowed to integrate by parts indiscriminately. If we assume, for the moment, that we can make such integration, then we would get the desired result:

$$\Omega = +i\int \delta\psi^{\dot{E}EA} \wedge \partial_{\dot{E}}{}^{C}\delta a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}} = -i\int \partial_{\dot{E}}{}^{C}\delta\psi^{\dot{E}EA} \wedge \delta a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}}, \tag{2.4.23}$$

because, by the equations of motion, the $\mathrm{d}\psi$ term is symmetric in the pair CE but also in EA – thus being symmetric in all of its indices – and we have

$$\Omega = -i\int \partial_{\dot{E}}{}^{C}\delta\psi^{\dot{E}EA} \wedge \delta a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}} = -i\int \delta\phi^{CEA} \wedge \delta\, a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}}. \tag{2.4.24}$$

The integration by parts is justified if we show that the two terms differ by an exact form. Consider

$$\begin{aligned}\int \partial_m\, \delta X^{[mn]} \wedge \mathrm{d}^3x_n &= \int \partial_{\dot{E}C}\, \delta X^{[\dot{E}C|\dot{A}A]} \wedge \mathrm{d}^3x_{\dot{A}A} \\ = -&\int \partial_{\dot{E}}{}^{C}\left(\delta\psi^{\dot{E}AE} \wedge \delta a_{\dot{A}CE} - \delta\psi_{\dot{A}}{}^{E}{}_{C} \wedge \delta a^{\dot{E}A}{}_{E}\right) \wedge \mathrm{d}^3x_A{}^{\dot{A}}\end{aligned} \tag{2.4.25}$$

and notice that (2.4.25) is exactly what we want:

$$-\int \left(\partial_{\dot{E}}{}^{C}\delta\psi^{\dot{E}EA} \wedge \delta a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}} + \delta\psi^{\dot{E}EA} \wedge \partial_{\dot{E}}{}^{C}\delta a_{\dot{A}EC} \wedge \mathrm{d}^3x_A{}^{\dot{A}}\right), \tag{2.4.26}$$

since all other terms cancel after we use (2.4.20) together with the equation of motion for the gauge field $a_{\dot{A}B\cdots D}$:

$$\partial^{\dot{A}}{}_{(A}a_{BC)\dot{A}} = 0. \tag{2.4.27}$$

In all other cases, the integration by parts will be the main issue. We circumvent the difficulty of finding appropriate exact forms by working in momentum space.

2.4.2 Linearized gravity case.

When $s = 2$ in section 2.2.1 we have linearized Einstein theory of gravity. The field h_{mn} has gauge invariance of the form

$$\delta_\xi h_{mn}(x) = \partial_m\xi_n(x) + \partial_n\xi_m(x) \tag{2.4.28}$$

and is described by the flat space action

$$S = -\frac{1}{2}\int \mathrm{d}^4x\left(h^{mn}R_{mn} - \frac{1}{2}h^p{}_p R^q{}_q\right). \tag{2.4.29}$$

The R_{mn} and $R^p{}_p$ represent the Ricci tensor and Ricci scalar respectively. Both can be obtained from the linearized curvature given by

$$R_{mnpq} = 4\,\partial_{[m}h_{n][p}\overleftarrow{\partial}_{q]}. \tag{2.4.30}$$

The equations of motion are the linearized Einstein field equations

$$R_{mn} = 0 \tag{2.4.31}$$

and the symplectic structure is

$$\begin{aligned}\Omega = -\frac{1}{2}\int \big(&2\delta h^m{}_n \wedge \partial^p \delta h_p{}^n - \delta h_{pn} \wedge \partial^m \delta h^{pn} + \delta h_p{}^p \wedge \partial^m \delta h_n{}^n \\ &- \partial_n \delta h^{mn} \wedge \delta h_p{}^p + \partial^p \delta h^n{}_n \wedge \delta h_p{}^m\big) \wedge \mathrm{d}^3 x_m. \end{aligned} \tag{2.4.32}$$

In order to change to Penrose description, we need to identify the (ϕ, a) fields. The self-dual part of R_{mnpq} gives ϕ_{MNPQ} via

$$\phi_{MNPQ} = \partial_{\dot{M}(M}\partial_{|\dot{N}|N}h^{\dot{M}\dot{N}}{}_{PQ)}, \tag{2.4.33}$$

while the anti-self-dual piece is described by the map

$$h_{M\dot{M}N\dot{N}} = -i\partial_{\dot{M}}{}^C a_{\dot{N}CMN} - i\partial_{\dot{N}}{}^C a_{\dot{M}CMN}. \tag{2.4.34}$$

Again, (2.4.34) is an *ansatz*. It is constructed by requiring gauge symmetries to coincide. An interesting feature we should stress is that h comes traceless since a is completely symmetric in its undotted indices. This is not a problem. In Fronsdal theory these degrees of freedom are pure gauge.

We will demonstrate that the phase spaces of these descriptions agree. In this on-shell counting, let us go into Fourier space and fix the only non-zero component of the momentum to be $p_2{}^{\dot{2}}$. From the spinor description, we have then

$$\partial^{\dot{A}A}\phi_{ABCD} = 0 \quad \Longrightarrow \quad p^{\dot{2}1}\phi_{1BCD} = 0, \tag{2.4.35}$$

which implies that every term with an 1 index vanishes. The only non-zero component of ϕ thus is ϕ_{2222}. For the gauge field a, we have

$$\partial_{(A}{}^{\dot{A}}a_{BCD)\dot{A}} = 0 \quad \Longrightarrow \quad p_{(2}{}^{\dot{2}}a_{BCD)\dot{2}} = 0, \tag{2.4.36}$$

which means that every a with a $\dot{2}$ and a 2 index vanishes. The only remaining degrees of freedom are $a_{\dot{1}BCD}$. However, we should account for the gauge invariance:

$$\delta a_{\dot{A}BCD} = \partial_{\dot{A}(B}\,\xi_{CD)} \quad \Longrightarrow \quad \delta a_{\dot{1}2CD} = p_{\dot{1}(2}\,\xi_{CD)}, \tag{2.4.37}$$

which makes the only non-zero component $a_{\dot{1}111}$. Finally the symplectic structure for spin 2 Penrose theory is

$$\Omega = i\int \delta\phi^{1111} \wedge \delta a^{\dot{2}}{}_{111} \wedge \mathrm{d}^3 x_{1\dot{2}}. \tag{2.4.38}$$

Let us turn to Fronsdal theory. Fix a gauge where h_{mn} is traceless, so the symplectic structure (2.4.32) reduces to

$$\Omega = -\frac{1}{2}\int \left(2\delta h^m{}_n \wedge \partial^p \delta h_p{}^n - \delta h_{pn} \wedge \partial^m \delta h^{pn}\right) \wedge \mathrm{d}^3 x_m. \tag{2.4.39}$$

The degrees of freedom of the self-dual part are fixed by Einstein's equation since ϕ is written in terms of h. For spin 2:

$$R_{(M\dot{M}|N\dot{N})} = p^2 h_{(M\dot{M}|N\dot{N})} + p_{(M\dot{M}}p^a h_{|a|\,N\dot{N})} = 0, \tag{2.4.40}$$

which gives, after we impose $p^2 = 0$,

$$p_{(M\dot{M}}h_{N\dot{N})1\dot{2}} = 0. \tag{2.4.41}$$

The general solution of this equation is

$$h_{(1\dot{2}|M\dot{M})} = 0. \tag{2.4.42}$$

So, for the self-dual part of the curvature, we have then

$$\phi_{22CD} = -p_{(2}{}^{\dot{2}}p_2{}^{\dot{2}}h_{CD)\dot{2}\dot{2}} \quad \Longrightarrow \quad \phi_{2222} = -p_2{}^{\dot{2}}p_2{}^{\dot{2}}h_{22\dot{2}\dot{2}}. \tag{2.4.43}$$

To connect the two descriptions, we split the gravitational field h into a self-dual and anti-self-dual part. The self-dual piece is already described by Einstein's equations while the anti-self-dual part is given by the *ansatz* (2.4.21). It implies:

$$h_{(1\dot{1}|1\dot{1})} = p_{\dot{1}}{}^{1}a_{\dot{1}111} + p_{\dot{1}}{}^{1}a_{\dot{1}111} = +2p_{\dot{1}}{}^{1}a_{\dot{1}111}. \tag{2.4.44}$$

These considerations collapse the symplectic structure to

$$\begin{aligned}
\Omega &= -\frac{i}{2}\int \left(2\delta h^m{}_n \wedge p^{1\dot{2}}\delta h_{1\dot{2}}{}^n\right) \wedge \mathrm{d}^3x_m - \left(\delta h_{pn} \wedge p^{1\dot{2}}\delta h^{pn}\right) \wedge \mathrm{d}^3x_{1\dot{2}} \\
&= +\frac{i}{2}\int \left(\delta h_{pn} \wedge p^{1\dot{2}}\delta h^{pn}\right) \wedge \mathrm{d}^3x_{1\dot{2}} \\
&= +i\int \left(p_{\dot{1}}{}^{1}\delta a_{\dot{1}111} \wedge p^{1\dot{2}}\delta h^{\dot{1}1\dot{1}1}\right) \wedge \mathrm{d}^3x_{1\dot{2}} \\
&= +i\int \left(\delta a_{\dot{1}111} \wedge p_{\dot{1}}{}^{1}p^{1\dot{2}}\delta h^{\dot{1}1\dot{1}1}\right) \wedge \mathrm{d}^3x_{1\dot{2}} \\
&= -i\int \left(\delta a_{\dot{1}111} \wedge p_2{}^{\dot{2}}p_2{}^{\dot{2}}\delta h_{22\dot{2}\dot{2}}\right) \wedge \mathrm{d}^3x_{1\dot{2}} \\
&= +i\int \left(\delta a_{\dot{1}111} \wedge \delta\phi_{2222}\right) \wedge \mathrm{d}^3x_{1\dot{2}}.
\end{aligned} \tag{2.4.45}$$

This computation highlights the usefulness of momentum space. We can work directly with physical degrees of freedom as it is suggested when dealing with symplectic structures.

2.4.3 Canonical map between formulations for general spin s.

In order to relate the two descriptions in general case, we split the Fronsdal field $h_{m_1\cdots m_s}$ into self-dual and anti-self-dual components. The anti-self-dual part is described by the gauge field $a_{\dot{M}A\cdots N}$ via

$$h_{M_1\dot{M}_1\cdots M_s\dot{M}_s} = (-i)^{2s-1}\partial_{(\dot{M}_s}{}^{N_s}\ldots\partial_{\dot{M}_2}{}^{N_2}a_{\dot{M}_1)N_2\cdots N_sM_1\cdots M_s}, \tag{2.4.46}$$

while the self-dual degrees of freedom are given by the curvature $\phi_{A\cdots D}$, which should come from the gauge-invariant tensor

$$R_{[m_1n_1]\cdots[m_sn_s]} = \partial_{[n_s|}\partial_{[n_{s-1}|}\ldots\partial_{|[n_1}h_{m_1]|\cdots|m_{s-1}]|m_s]}. \tag{2.4.47}$$

Once Fronsdal equations are imposed, we expect[4]

$$\phi_{M_1N_1\cdots M_sN_s} = R_{(M_1N_1\cdots M_sN_s)} = \partial_{(N_s}{}^{\dot{N}_s}\dots\partial_{N_1}{}^{\dot{N}_1}h_{M_1\dots M_s)\dot{N}_1\dots\dot{N}_s}. \tag{2.4.48}$$

We also expect that any component of $R_{m_1n_1\cdots m_sn_s}$ which contains mixed dotted and undotted indices should vanish. In what follows, we will prove that this is indeed the case.

For the moment, we should stress interesting features of this map. The anti-self-dual component gives a traceless $h_{m_1\cdots m_s}$. But this is not a problem since these degrees of freedom are pure gauge. Moreover, in order to show that the symplectic structures match, one does not need all coefficients in the anti-self-dual map. The Fronsdal equations will restrict these to a single component each.

2.4.4 Equivalent symplectic structures: Fourier counting.

We proceed to the symplectic structures. We circumvent the need to look for exact forms by going to momentum space, which also makes straightforward to work only with physical degrees of freedom.

Let us choose a non-zero $p_2{}^{\dot{2}}$ component. Hence, the equation of motion for $a_{\dot{A}B\cdots D}$ collapses into

$$p_{(2}{}^{\dot{2}}a_{\dot{2}B_2\cdots B_{2s})} = 0, \tag{2.4.49}$$

and we can see the only non zero component is $a_{\dot{1}B\cdots D}$. We can restrict further using the gauge transformations:

$$\delta a_{\dot{1}2\cdots D} = p_{\dot{1}(2}\xi_{\cdots D)}, \tag{2.4.50}$$

from where the only physical component which remains is $a_{\dot{1}1\dots 1}$. Thus, the map we described in (2.4.46) gives $h_{1\dot{1}\cdots 1\dot{1}}$ component of the Fronsdal gauge field.

The degrees of freedom which the curvature spinor describes are obtained from the Fronsdal equation. Together with the condition $p^2 = 0$, they imply

$$p_{(M_1\dot{M}_1}h_{1\dot{2}M_3\dot{M}_3\cdots M_s\dot{M}_s)} = 0, \tag{2.4.51}$$

since our map describes a traceless $h_{m_1\cdots m_s}$ field. This equation forces $h_{1\dot{2}\dots} = 0$, which also annihilates any component with mixed dotted and undotted indices, and so we have

$$\phi_{22\cdots 22} = i^s p_2{}^{\dot{2}}\dots p_2{}^{\dot{2}}h_{2\dots 2\dot{2}\dots\dot{2}}. \tag{2.4.52}$$

Such considerations are in line with the usual formulation of Fronsdal theory, where the degrees of freedom contained in the trace and divergence of $h_{m_1\cdots m_s}$ can be gauged away.

We combine all of such considerations to show the symplectic structures agree. Note that we are allowed to discard terms of the type

$$\int \delta h_{\dots}\wedge\partial^p h_{p\dots} \quad \text{and} \quad \int \delta h^p{}_{p\dots}\wedge\delta h_{\dots\dots}$$

because $h_{1\dot{2}\dots}$ vanishes and our canonical map gives a traceless $h_{m_1\cdots m_s}$. Thus the only allowed combination for the bosonic case is of the form

$$\Omega = \int(\delta h_{n_1\cdots n_s}\wedge\partial^m\delta h^{n_1\cdots n_s})\wedge \mathrm{d}^3x_m \tag{2.4.53}$$

and if we apply our results to (2.4.53) we obtain

[4]Remember, to a spacetime index m there corresponds a pair $M\dot{M}$.

$$\begin{aligned}\Omega &= \int \left(\delta h_{(1\dot{1}|\cdots|1\dot{1})} \wedge p_2{}^{\dot{2}} \delta h^{(1\dot{1}|\cdots|1\dot{1})}\right) \wedge \mathrm{d}^3 x_{\dot{2}}{}^2 \\ &= (-i)^{s-1} \int \left(p^1{}_{\dot{1}} \dots p^1{}_{\dot{1}} \delta a_{\dot{1}1\cdots 1} \wedge p_2{}^{\dot{2}} \delta h_{2\cdots 2\dot{2}\cdots\dot{2}}\right) \wedge \mathrm{d}^3 x_{\dot{2}}{}^2 \\ &= (-)^{s-1}(-i)^{s-1} \int \left(\delta a_{\dot{1}1\cdots 1} \wedge p_2{}^{\dot{2}} \dots p_2{}^{\dot{2}} \delta h_{2\cdots 2\dot{2}\cdots\dot{2}}\right) \mathrm{d}^3 x_{\dot{2}}{}^2 \\ &= (-)^{s-1}(-i)^{s-1}(-i)^{s} \int \left(\delta a_{\dot{1}1\cdots 1} \wedge \delta\phi_{22\cdots 22}\right) \wedge \mathrm{d}^3 x_{\dot{2}}{}^2 \\ &= -i \int \left(\delta a_{\dot{1}1\cdots 1} \wedge \delta\phi_{22\cdots 22}\right) \wedge \mathrm{d}^3 x_{\dot{2}}{}^2 \end{aligned} \tag{2.4.54}$$

thus proving the desired result.

2.5 Conformal Invariance.

The conformal generator v^c is

$$\mathrm{v}^c = a^c + \omega^{cb} x_b + \alpha x^c + 2\,(\rho \,.\, x)\, x^c - \rho^c (x \,.\, x), \tag{2.5.1}$$

where the first two terms are the usual Poincaré transformations; the third one describes dilatations and the last two generate special conformal transformations.

2.5.1 Lie derivation of spinors.

In treating Penrose action, we are going to need to vary spinor fields under conformal transformations. The Lie derivative of a spinor field is not widely used when compared with the usual tensor variations. This subsection explains briefly this terminology before applying it to our case.

In geometry, given a vector field v^c and a vector density u^b, the Lie derivative of u^b with respect to v^c is defined as

$$\mathcal{L}_{\mathrm{v}} u^b = \mathrm{v}^a \partial_a u^b - u^a \partial_a \mathrm{v}^b + \mathrm{w}_u\, (\partial_a \mathrm{v}^a)\, u^b, \tag{2.5.2}$$

where w_u is the density weight of u^b. When u^b is null, it can be written as product of two spinors, $u^b = \mu^B \overline{\mu}^{\dot{B}}$, and so we can use equation (2.5.2) to define the Lie derivative of μ^B.

Following this procedure, a general spinor density [51,52] μ^A flows along the flux of v^c such that its infinitesimal change is given by

$$\delta_{\mathrm{v}} \mu^A = \mathcal{L}_{\mathrm{v}} \mu^A = \mathrm{v}^m \partial_m \mu^A + \mu^B f^A{}_B + \mathrm{w}_\mu\, (\partial_m \mathrm{v}^m)\, \mu^A; \tag{2.5.3}$$

in here w_μ denotes the density weight of the μ field and $f^A{}_B$ is the self-dual part of v^c:

$$f_{AB} = -\frac{1}{2} \partial_{\dot{C}(A} \mathrm{v}_{B)}{}^{\dot{C}}. \tag{2.5.4}$$

In deriving (2.5.3) from (2.5.2), we must impose that v^c is a conformal generator. Indeed, the second term in (2.5.2) gives a contribution of the form:

$$
\begin{aligned}
-u^a\partial_a \mathrm{v}_b &= -\mu^A\overline{\mu}^{\dot{A}}\partial_{A\dot{A}}\mathrm{v}_{B\dot{B}} \\
&= -\mu^A\overline{\mu}^{\dot{A}}\partial_{[A\dot{A}}\mathrm{v}_{B\dot{B}]} - \mu^A\overline{\mu}^{\dot{A}}\partial_{(A\dot{A}}\mathrm{v}_{B\dot{B})} \\
&= -\mu^A\overline{\mu}^{\dot{A}}\left(f_{AB}\epsilon_{\dot{A}\dot{B}} + \overline{f}_{\dot{A}\dot{B}}\epsilon_{AB}\right) - \mu^A\overline{\mu}^{\dot{A}}\partial_{(A\dot{A}}\mathrm{v}_{B\dot{B})} \\
&= \overline{\mu}_{\dot{B}}\,\mu^A f_{AB} + \mu_B\,\overline{\mu}^{\dot{A}}\overline{f}_{\dot{A}\dot{B}} - \mu^A\overline{\mu}^{\dot{A}}\partial_{(A\dot{A}}\mathrm{v}_{B\dot{B})},
\end{aligned} \tag{2.5.5}
$$

in which the last term does not split into something dependent of B and $\dot{B}$ separately. It is precisely when v^c is a conformal generator, that is

$$
\partial_{(A\dot{A}}\,\mathrm{v}_{B\dot{B})} \;=\; \left(\frac{1}{2}\partial_m \mathrm{v}^m\right)\epsilon_{AB}\,\epsilon_{\dot{A}\dot{B}}. \tag{2.5.6}
$$

that we can identify the desired contributions to each spinor.

In our applications, of special interest is the self-dual part of the special conformal transformations. We write it explicitly for future use:

$$
f_{AB} = -2\,\rho_{\dot{C}(A}x_{B)}{}^{\dot{C}}. \tag{2.5.7}
$$

2.5.2 Weight conventions.

The weight of a density is a geometrical quantity, that is, it has fixed value independent of which transformation is made; and usually we would have

$$
\mathcal{L}_{\mathrm{v}}\epsilon_{AB} = \frac{\lambda}{2}\epsilon_{AB}. \tag{2.5.8}
$$

However, there is still freedom if we define ϵ_{AB} to be a density instead of a tensor. We choose the weight of ϵ_{AB} such that

$$
\mathcal{L}_{\mathrm{v}}\epsilon_{AB} = 0. \tag{2.5.9}
$$

From definition (2.5.3):

$$
\begin{aligned}
\mathcal{L}_{\mathrm{v}}\epsilon_{AB} = 0 &= \frac{\lambda}{2}\epsilon_{AB} + \mathrm{w}_\epsilon\partial_m \mathrm{v}^m\epsilon_{AB} \\
&= \left(\frac{1}{2} + 2\mathrm{w}_\epsilon\right)\epsilon_{AB}
\end{aligned} \tag{2.5.10}
$$

we see this amounts choosing $\mathrm{w}_\epsilon = -1/4$. Consistency, however, requires ϵ^{AB} to have weight $\mathrm{w}^\epsilon = +1/4$. Hence, given an arbitrary spinor μ^A, in our conventions it is true that

$$
\mathcal{L}_{\mathrm{v}}\mu_A = \epsilon_{AB}\mathcal{L}_{\mathrm{v}}\mu^B, \tag{2.5.11}
$$

which is equivalent to state that a spinor and its dual have the same conformal weight. All considerations apply equally for dotted indices.

2.5.3 Conformal invariance of Penrose action.

In this section we will state the conformal invariance of the action (2.3.1). This in turn ensures the existence of a set of conformal symmetries in Fronsdal description.

Let us begin with dilatations. The higher-spin fields vary under it according to

$$\delta_{\mathrm{v}}\phi^{AB\cdots D} = \alpha x^m \partial_m \phi^{AB\cdots D} + 4\alpha \mathrm{w}_\phi \phi^{AB\cdots D} \tag{2.5.12a}$$

and

$$\delta_{\mathrm{v}} a^{\dot{A}}{}_{B\cdots D} = \alpha x^m \partial_m a^{\dot{A}}{}_{B\cdots D} + 4\alpha \mathrm{w}_a a^{\dot{A}}{}_{B\cdots D}. \tag{2.5.12b}$$

These change the action by

$$\begin{aligned}\delta_{\mathrm{v}} S = \int \mathrm{d}^4x \, &\left(\alpha x^m \partial_m \phi^{AB\cdots D} + 4\alpha \mathrm{w}_\phi \phi^{AB\cdots D}\right) \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} \\ &+ \phi^{AB\cdots D} \partial_{A\dot{A}} \left(\alpha x^m \partial_m a^{\dot{A}}{}_{B\cdots D} + 4\alpha \mathrm{w}_a a^{\dot{A}}{}_{B\cdots D}\right).\end{aligned} \tag{2.5.13}$$

After a few simplifications, we get

$$\delta_{\mathrm{v}} S = \int \mathrm{d}^4x \, \left\{\alpha \left[-3 + 4\left(\mathrm{w}_\phi + \mathrm{w}_a\right)\right] \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D}\right\}, \tag{2.5.14}$$

which vanishes only when

$$\mathrm{w}_\phi + \mathrm{w}_a = \frac{3}{4}. \tag{2.5.15}$$

As we can see, dilatations are unable to fix completely the conformal weights. The remaining condition comes from the special conformal transformations.

Under special conformal transformations, generated by

$$\mathrm{v}^m = 2\left(\rho.x\right) x^m - \left(x.x\right) \rho^m, \tag{2.5.16}$$

the spin fields $\phi^{AB\cdots D}$ and $a^{\dot{A}}{}_{B\cdots D}$ vary according to

$$\delta_{\mathrm{v}}\phi^{AB\cdots D} = \mathrm{v}^m \partial_m \phi^{AB\cdots D} + 2s\phi^{C(AB\cdots} f^{D)}{}_C + 8\mathrm{w}_\phi \left(\rho.x\right) \phi^{AB\cdots D}, \tag{2.5.17a}$$

and

$$\delta_{\mathrm{v}} a^{\dot{A}}{}_{B\cdots D} = \mathrm{v}^m \partial_m a^{\dot{A}}{}_{B\cdots D} + \overline{f}^{\dot{A}}{}_{\dot{C}} a^{\dot{C}}{}_{B\cdots D} - \left(2s-1\right) f^C{}_{(B} a^{\dot{A}}{}_{\cdots D)C} + 8\mathrm{w}_a \left(\rho.x\right) a^{\dot{A}}{}_{B\cdots D}. \tag{2.5.17b}$$

The action becomes

$$\begin{aligned}\delta_{\mathrm{v}} S = \int \mathrm{d}^4x \, \Big(&\mathrm{v}^m \partial_m \phi^{AB\cdots D} \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} + 2s\phi^{C(AB\cdots} f^{D)}{}_C \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} + 8\mathrm{w}_\phi \left(\rho.x\right) \phi\partial a \\ &+ \phi^{AB\cdots D} \partial_{A\dot{A}} \mathrm{v}^m \partial_m a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D} \mathrm{v}^m \partial_m \partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D} \partial_{A\dot{A}} \overline{f}^{\dot{A}}{}_{\dot{C}} a^{\dot{C}}{}_{B\cdots D} \\ &+ \phi^{AB\cdots D} \overline{f}^{\dot{A}}{}_{\dot{C}} \partial_{A\dot{A}} a^{\dot{C}}{}_{B\cdots D} - \left(2s-1\right) \phi^{AB\cdots D} \partial_{A\dot{A}} f^C{}_{(B} a^{\dot{A}}{}_{\cdots D)C} \\ &- \left(2s-1\right) \phi^{AB\cdots D} f^C{}_{(B} \partial_{|A\dot{A}|} a^{\dot{A}}{}_{\cdots D)C} + 8\mathrm{w}_a \phi\rho a + 8\mathrm{w}_a \left(\rho.x\right) \phi\partial a\Big).\end{aligned} \tag{2.5.18}$$

In the second line, we open $\partial_a \mathrm{v}_m$ in its symmetric and anti-symmetric pieces and integrate by parts ∂_m in $\partial_{A\dot{A}} \partial_m a^{\dot{A}}{}_{B\cdots D}$. Then we obtain

$$\phi^{AB\cdots D}\partial_{A\dot{A}}\mathrm{v}^m\partial_m a^{\dot{A}}{}_{B\cdots D} = \phi^{AB\cdots D}\partial_{(A\dot{A}}\mathrm{v}_{m)}\partial^m a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D}\partial_{[A\dot{A}}\mathrm{v}_{m]}\partial^m a^{\dot{A}}{}_{B\cdots D}$$
$$= 2\,(\rho.x)\,\phi\partial a + \phi^{AB\cdots D} f_{AM}\partial^M{}_{\dot{A}} a^{\dot{A}}{}_{B\cdots D} + \phi^{AB\cdots D}\overline{f}_{\dot{A}\dot{M}}\partial_A{}^{\dot{M}} a^{\dot{A}}{}_{B\cdots D} \tag{2.5.19}$$

and

$$\phi^{AB\cdots D}\mathrm{v}^m\partial_m\partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} = -\mathrm{v}^m\partial_m\phi^{AB\cdots D}\partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} - \partial_m \mathrm{v}^m\phi^{AB\cdots D}\partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D}$$
$$= -\mathrm{v}^m\partial_m\phi^{AB\cdots D}\partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} - 8\,(\rho.x)\,\phi\partial a. \tag{2.5.20}$$

When we substitute everything back into the action, the only remaining terms are

$$\delta_{\mathrm{v}} S = \int \mathrm{d}^4x \left\{ [8\,(\mathrm{w}_\phi + \mathrm{w}_a) - 6]\,(\rho.x)\phi^{AB\cdots D}\partial_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D} \right\} + (8\mathrm{w}_a - 3)\,\phi^{AB\cdots D}\rho_{A\dot{A}} a^{\dot{A}}{}_{B\cdots D}$$
$$- (2s-1)\phi^{AB\cdots D}\partial_{A\dot{A}} f^C{}_{(B}\, a^{\dot{A}}{}_{\cdots D)C}. \tag{2.5.21}$$

We can use (2.5.7) so that

$$\partial_{A\dot{A}} f^C{}_B = -\rho_{\dot{A}}{}^C\epsilon_{AB} - \rho_{\dot{A}B}\delta_A{}^C. \tag{2.5.22}$$

At the end, we get two relations involving the weights. They are

$$8\,(\mathrm{w}_\phi + \mathrm{w}_a) - 6 = 0 \tag{2.5.23a}$$

and

$$8\mathrm{w}_a + 2s - 4 = 0. \tag{2.5.23b}$$

If we use (2.5.15), the first equation, (2.5.23a), is an identity. It gives no new information. However, the second equation fixes the weight of the gauge field. Finally, we have

$$\mathrm{w}_a = \frac{2-s}{4} \tag{2.5.24}$$

and

$$\mathrm{w}_\phi = \frac{s+1}{4}. \tag{2.5.25}$$

The following table lists a few values for weights given different spin s theories.

	w_ϕ	w_a
s = 0	1/4	1/2
s= 1/2	3/8	3/8
s = 1	1/2	1/4
s= 3/2	5/8	1/8
s = 2	3/4	0
s = 5/2	7/8	-1/8

2.5.4 The structure of conformal transformations.

Penrose theory is described by the set (ϕ, a) while Fronsdal theory is described by h. We have defined a map, which we name H, that takes one description into another:

$$H : h_{m_1\cdots m_s} \longmapsto \left(\phi^{AB\cdots D}, a_{\dot{A}B\cdots D}\right).$$

It was shown that this map preserves phase space, i.e., it is a canonical transformation.

A map between symplectic structures also carries through symmetries of one description to another. If a symplectic structure admits an action, then its symmetries must be also symmetries of the action. Therefore it is natural to define a conformal transformation of the form

$$\delta_{\mathrm{v}} h_{m_1\cdots m_s} = H^{-1}\mathcal{L}_{\mathrm{v}}\, H\;\, h_{m_1\cdots m_s}, \tag{2.5.26}$$

where v is the conformal generator (2.5.1). It can act non-trivially; its action, as equation (2.5.26) shows, is not obtained from standard Lie derivations. Moreover, additional complications may appear due to H^{-1}, which involves inverting derivatives, as (2.4.46) illustrates. For spins running from $s = 1/2$ to $s = 3/2$, it can be shown to agree with usual conformal transformations obtained by change of coordinates. At spin $s = 2$, however, since Fronsdal theory *is not* conformal invariant, our transformation exhibits the non-local behaviour.

We can work out this case explicitly. For special conformal transformations, if we plug the variation (2.5.17b) inside (2.4.34), we obtain

$$\delta_{\mathrm{v}} h_{(M\dot{M}|N\dot{N})} \;=\; \mathcal{L}_{\mathrm{v}} h_{(M\dot{M}|N\dot{N})} + 2\,(\rho.x)\, h_{(M\dot{M}|N\dot{N})} + 6\,\rho_{(\dot{M}}{}^{E}\, a_{\dot{N})MNE}(h), \tag{2.5.27}$$

where $\mathcal{L}_{\mathrm{v}}$, in this case, denotes the diffeomorphism Lie derivative and ρ is the special conformal parameter. The last term shows the non-local behaviour since it involves rewriting equation (2.5.17b) for $a_{\dot{M}MNE}$ in terms of $h_{M\dot{M}N\dot{N}}$, giving inverse powers of ∂_a. Notice that the conformal weight obtained from this expression, which reads $\mathrm{w} = +1/4$, does not agree with the usual Fronsdal theory, which is dilatation invariant for $\mathrm{w} = -1/4$ at every spin [30].

These differences may appear problematic. They raise suspicion whether this transformation satisfies the conformal algebra or not. The simplest way to answer this question is to notice that (2.5.26) is a conjugation; therefore, if H is well-defined, they must satisfy the same algebra of the vector field v in question.

The issue of conformal invariance is unrelated to the on-shell phase space, but is rather related to the off-shell description. The connection field of Fronsdal theory is not a representation of the conformal algebra, but the curvatures used in Penrose theory transform covariantly under the action of the conformal group. Then, it could be said that the non-local nature of the transformation follows from expressing the Fronsdal gauge field in terms of its curvature, as is usual in spin 2 case when one uses Riemann normal coordinates. It should be stressed though that, in this formalism, we still have a gauge field, $a_{\dot{M}MNE}$ in gravity case, and moreover, our map (2.4.34) is not obtained by a change of coordinates, it is the definition of a[5].

The conformal change of coordinates preserves the action (2.3.1) and its symplectic structure. We have shown that the two symplectic structures agree, so our transformations should be a symmetry of Fronsdal action. This analysis is straightforward for free theories. It is possible that this symmetry is not preserved by arbitrary interaction terms. It would be interesting to understand what kind of interactions, if there is any, would preserve these symmetries.

[5]We would like to thank the referee for pointing this out.

2.6 Conclusions.

We have defined an action for Penrose theory and constructed its symplectic structure. This action appears to be simpler than the usual one obtained by Fronsdal. Moreover, it depends only on the epsilon symbol, being possible to examine how it should extend to curved spaces. In particular, this action might be the description of the singular limit suggested in [53] for the AdS_4 higher-spin action.

We showed that this theory describes the same classical phase space as of Fronsdal. Moreover, the action of conformal change of coordinates can be push-forwarded from one Langragian to another. It, in turn, leads us to conjecture a set of non-trivial conformal symmetries for the Fronsdal higher-spin fields $h_{m_1 \cdots m_s}$. These are not generated by usual coordinate changes, although to lower spins – those which run from 1/2 to 3/2 – it is possible to show that both symmetries agree. The non-local behaviour appears only at spin 2.

The construction of conformal higher-spin theories in four-dimensional flat space was developed by Fradkin and Tseytlin in [54], and it was generalized for arbitrary curved backgrounds by Segal in [55]. These theories involve higher-derivatives and additional fields; so, it should be stressed that having identified a non-local realization of conformal symmetries is not enough to argue that this gives a non-trivial relation between conformal higher-spin theories and Fronsdal theory.

Chapter 3

On worldsheet curvature coupling in pure spinor sigma-model

3.1 Outline.

In a general curved background, the b-ghost of the pure spinor superstring [5, 56] is not holomorphic:

$$\bar{\partial} b = Q(\ldots) \tag{3.1.1}$$

On one hand, this is a problem, complicating the computation of scattering amplitudes. On the other hand, this is a tip of an interesting mathematical structure. It was suggested in [36, 37] that in such cases the definition of the string measure should be modified, so that the resulting measure should descend on the factorspace of metrics[1] over diffeomorphisms. The method of [36, 37] is to first construct a pseudodifferential form equivariant with respect to diffeomorphisms, and then obtain a base form using some connection.

This procedure can be also used to study the insertion of **un**integrated vertex operators. Once we inserted unintegrated vertex operators, we should then integrate over the moduli space of Riemann surfaces with marked points. Let us first integrate, for each *fixed* complex structure on Σ, over the positions of the marked points, postponing the integration over complex structure for later. We interpret the result as the insertion of the **in**tegrated vertex operator. It is usually assumed that to any **un**integrated vertex operator V corresponds some integrated vertex operator U. The naive formula is:

$$U = b_{-1}\bar{b}_{-1}V \tag{3.1.2}$$

However, this naive formula does not always work correctly. First of all, in the pure spinor formalism, b is a rational function of the pure spinor fields. This, generally speaking, leads to U being a rational function of the pure spinors, with non-constant denominators. It is not clear if such rational expressions should be allowed in the worldsheet action. We will leave this question open. Instead, we discuss another issue: Eq. (3.1.2) does not tell us the whole truth about the curvature coupling (the Fradkin-Tseytlin term in the worldsheet action). In this paper we will explain how to derive the Fradkin-Tseytlin term in the action starting from the insertion of the **un**integrated vertex operator V. We will construct, following the prescription of [36, 37], the integration measure for integrating over the point of insertion of V. We will show that the procedure of [36,37] simplifies. This is mostly due to the existence of a relatively straightforward construction of a connection on the space of Lagrangian submanifolds, as a principal bundle with the structure group diffeomorphisms. The curvature of this connection is essentially equal to the Riemann curvature of the worldsheet metric. The curvature term

[1] or, more generally, of Lagrangian submanifolds of BV phase space

in the base form generates, effectively, the dilaton coupling (the Fradkin-Tseytlin term) on the string worldsheet. Under certain conditions, this reasoning leads (Section 3.2) to the formula for the deformation of the dilaton superfield:

$$(b_0 - \bar{b}_0)V = Q\Phi \tag{3.1.3}$$

In general there are two contributions to Φ: one from Eq. (3.1.2) and another from Eq. (3.1.3).

Eqs. (3.1.2) and (3.1.3) in the case of bosonic string In the case of bosonic string (Section 3.4.2), the curvature coupling, generally speaking, comes from *both* Eq. (3.1.2) and Eq. (3.1.3). The contribution from Eq. (3.1.2) is due to the fact that already the **un**integrated vertex operator contains the curvature coupling: $c\bar{c}\sqrt{g}R\Phi$.

Eqs. (3.1.2) and (3.1.3) in the case of pure spinor superstring In Section 3.6 we discuss Eqs. (3.1.2) and (3.1.3) in the context of the pure spinor superstring on $AdS_5 \times S^5$. In this case, the only source of the curvature coupling is Eq. (3.1.3) — the second line of Eq. (3.2.33).

The b-ghost is a rational function of the pure spinor (not a polynomial). Therefore, the OPEs $b_{-1}\bar{b}_{-1}V$ and $(b_0 - \bar{b}_0)V$ are also non-polynomial. We explicitly evaluate $(b_0 - \bar{b}_0)V$ in the particular case when V is the beta-deformation vertex, using the b_0 and $\bar{b}_0$ from [34] — see Section 3.6. At this time, we do not know any specific application of the formulas of Section 3.6. However, these computations inspired us to make some conjectures about the unintegrated vertex operators — see Sections 3.5.6 and 3.5.7.

One interesting feature of the beta-deformation is the existence of non-physical vertex operators [32, 57]. They normally cannot be put on a curved worldsheet, because of the anomaly. However, once we allow denominators of the form $\frac{1}{\mathrm{STr}(\lambda_L\lambda_R)}$, it seems that there is no obstacle, and the nonphysical vertices can be included. This at least means, that the first few orders in the expansion in powers of ε in Eq. (3.3.1) actually make sense in string perturbation theory.

3.2 General theory of vertex insertions

In this Section we will apply the prescription of [36, 37] for the vertex operators insertion.

3.2.1 Use of BV formalism and notations

In BV formalism, instead of integrating over the worldsheet complex structures, we integrate over general families of Lagrangian submanifolds L in BV phase space. The space of all Lagrangian submanifolds is denoted LAG. In this paper, we will only consider a $6g-6$-dimensional subspace of LAG, which corresponds to variations of the complex structure.

We use the **notations** of [37]. The odd Poisson bracket will be denoted $\{\_, \_\}_{\mathrm{BV}}$, or just $\{\_, \_\}$. For a vector field ξ on the BV phase space, generated by a BV Hamiltonian, we denote that Hamiltonian $\underline{\xi}$:

$$\xi = \{\underline{\xi}, \_\}_{\mathrm{BV}} \tag{3.2.1}$$

3.2.2 Use of worldsheet metric

Classically, the string worldhseet action depends on the worldsheet metric only through its complex structure. Quantum mechanically, the computation of the path integral usually involves the choice of the worldsheet *metric* (and not just complex structure), and then showing that in critical dimension the result of the computation is actually Weyl-invariant (*i.e.* only depends on the complex structure).

In this paper, we will need a worldsheet metric also for another purpose: to define a connection [2] on the space of Lagrangian submanifolds as a principal bundle:

$$\mathrm{LAG} \longrightarrow \frac{\mathrm{LAG}}{\mathrm{Diff}} \tag{3.2.2}$$

which we need to convert an equivariant form into a base form. Suppose that we choose a metric for every complex structure. Then, we will explain in Section 3.2.5, this defines a choice of horizontal directions, *i.e.* a connection on (3.2.2) — see Eqs. (3.2.29) and (3.2.30).

Given a complex structure, we will use the constant curvature metric of unit volume, which always exists and is unique by the uniformization theorem [58]. (But other global choices of a metric would also be OK.)

3.2.3 String measure

Equivariant Master Equation String worldsheet theory, in the approach of [36, 37], comes with a PDF[3] Ω^{base} on LAG, which is base with respect to $H = \mathrm{Diff}$. It is obtained from the **equivariant half-density** ρ^{C}, which satisfies the equivariant Master Equation:

$$\Delta_{\mathrm{can}}\rho^{\mathsf{C}}(\xi) = \underline{\xi}\rho^{\mathsf{C}}(\xi) \tag{3.2.3}$$

where $\xi \in \mathbf{h} = \mathrm{Lie}(H)$ is the equivariant parameter, and $\underline{\xi}$ the corresponding BV Hamiltonian.

Expansion in powers of ξ Let us write $\rho^{\mathsf{C}}(\xi)$ as a product:

$$\rho^{\mathsf{C}}(\xi) = e^{a(\xi)}\rho_{1/2} \tag{3.2.4}$$

where $\rho_{1/2}$ is a half-density satisfying the usual (not equivariant) Master Equation:

$$\rho_{1/2} = \exp(S_{\mathrm{BV}}) \quad (\; S_{\mathrm{BV}} \texttt{ is string worldsheet Master Action }) \tag{3.2.5}$$

$$\Delta_{\mathrm{can}}\rho_{1/2} = 0 \tag{3.2.6}$$

and $a(\xi)$ is a function on the BV phase space, $a(0) = 0$. For any function f and half-density $\rho_{1/2}$, let us denote:

$$\Delta_{\rho_{1/2}} f = \rho_{1/2}^{-1}\Delta_{\mathrm{can}}(f\rho_{1/2}) - (-)^{\bar{f}} f\rho_{1/2}^{-1}\Delta_{\mathrm{can}}\rho_{1/2} \tag{3.2.7}$$

Eqs. (3.2.3) and (3.2.6) imply:

$$\Delta_{\rho_{1/2}} a(\xi) + \frac{1}{2}\{a(\xi), a(\xi)\}_{\mathrm{BV}} = \underline{\xi} \tag{3.2.8}$$

$a(\xi)$ for bosonic string and for pure spinor string

For bosonic string $a(\xi)$ is background-independent, linear in ξ, and given by a simple formula:

$$a(\xi) = a^{(1)}\langle\xi\rangle = \int_{\Sigma} \xi^{\alpha} c^{\star}_{\alpha} \tag{3.2.9}$$

[2]there might be other choices of a connection, not requiring a metric

[3]pseudo-differential form

For pure spinor string $a(\xi)$ is a complicated background-dependent expression. For background $AdS_5 \times S^5$, the $a^{(1)}\langle\xi\rangle$ **was constructed** in [59], where it was called Φ_ξ. Schematically:

$$a^{(1)}\langle\xi\rangle = \int_\Sigma (\xi \cdot \partial Z^M) A_{M\alpha} \lambda^{\star\alpha} + (\xi \cdot \partial Z^M) B_M^N Z_N^\star + \tag{3.2.10}$$

$$+ (\partial Z^M) C_M^\alpha \mathcal{L}_\xi w_\alpha + D^{\alpha\beta} w_\alpha \mathcal{L}_\xi w_\beta \tag{3.2.11}$$

where:

- Z are coordinates on super-$AdS_5 \times S^5$
- λ are pure spinors (both λ_L and λ_R)
- $A_{M\alpha}$, B_M^N, C_M^α and $D^{\alpha\beta}$ are some functions of Z, and rational functions of pure spinors

Some assumptions

BV formalism is **ill-defined in field-theoretic context** , because $\Delta^{(0)}$ is ill-defined. We will assume that on local functionals $\Delta^{(0)} = 0$. In other words, when f_{loc} is a local functional of the string worldsheet fields:

$$\Delta_{\rho_{1/2}} f_{\mathrm{loc}} = \{S_{\mathrm{BV}}, f_{\mathrm{loc}}\} \tag{3.2.12}$$

We believe that it is possible justify this assumption in worldsheet perturbation theory, but at this time our considerations are not rigorous.

3.2.4 Equivariant unintegrated vertex

Stabilizer of a point Insertion of unintegrated vertex operator V at a point on $p \in \Sigma$ leads to breaking of the diffeomorphisms down to the subgroup $\mathrm{St}(p) \subset \mathrm{Diff}$ which preserves p. Let $\mathrm{st}(p)$ denote the Lie algebra of $\mathrm{St}(p)$:

$$\mathrm{St}(p) = \{g \in \mathrm{Diff} \mid g(p) = p\} \tag{3.2.13}$$

$$\mathrm{st}(p) = \mathrm{Lie}(\mathrm{St}(p)) \tag{3.2.14}$$

We will now explain how to construct an $\mathrm{St}(p)$-equivariant form on LAG, and then in Sections 3.2.5 and 3.2.6 how to construct a base form.

Equivariantization of vertex Given an unintegrated vertex V, suppose that we can construct for any $\xi_0 \in \mathrm{st}(p)$ an equivariant vertex $V^{\mathtt{C}}(\xi_0)$, satisfying[4]:

$$V^{\mathtt{C}}(0) = V \tag{3.2.15}$$

$$\Delta_{\rho^{\mathtt{C}}(\xi_0)} V^{\mathtt{C}}(\xi_0) = 0 \tag{3.2.16}$$

and:

$$\{\underline{\xi_0}, V^{\mathtt{C}}(\eta_0)\}_{\mathrm{BV}} = \left.\frac{d}{dt}\right|_{t=0} V^{\mathtt{C}}(e^{t[\xi_0\,,\,\_]}\eta_0) \tag{3.2.17}$$

Under the conditions of Eqs. (3.2.16) and (3.2.17) the product $V^{\mathtt{C}}(\xi_0)\rho^{\mathtt{C}}(\xi_0)$ defines an $\mathrm{st}(p)$-equivariant half-density satisfying the $\mathrm{st}(p)$-equivariant Master Equation:

$$(\Delta_{\mathrm{can}} - \underline{\xi_0}) \left(V^{\mathtt{C}}(\xi_0)\rho^{\mathtt{C}}(\xi_0)\right) = 0 \tag{3.2.18}$$

[4] the subindex $\mathtt{C}$ stands for **C**artan model of equivariant cohomology

Any solution $V^{\mathsf{c}}(\xi_0)$ of Eq. (3.2.18) leads to st(p)-equivariant pseudo-differential form:

$$\Omega^{\mathsf{c}}(L, dL, \xi_0) = \int_{gL_0} \exp\left(\sigma\langle dL\rangle\right)\; V^{\mathsf{c}}(\xi_0)\rho^{\mathsf{c}}(\xi_0) \tag{3.2.19}$$

Here $\sigma\langle dL\rangle$ is any BV Hamiltonian generating the infinitesimal deformation dL of L.

We can think of $V^{\mathsf{c}}(\xi_0)\rho^{\mathsf{c}}(\xi_0)$ as correction of the first order in ϵ to $\rho^{\mathsf{c}}(\xi_0)$ under the deformation:

$$\rho\, \exp\left(a(\xi_0)\right) \to \rho\, \exp\left(a(\xi_0) + \varepsilon V^{\mathsf{c}}(\xi_0)\right) \tag{3.2.20}$$

Eqs. (3.2.16) and (3.2.17) imply:

$$\left(\Delta_{\rho_{1/2}} + \{a(\xi_0), \_\}_{\mathrm{BV}}\right) V^{\mathsf{c}}(\xi_0) \;=\; 0 \tag{3.2.21}$$

$$\{\underline{\xi_0}, V^{\mathsf{c}}(\eta_0)\}_{\mathrm{BV}} = \left.\frac{d}{dt}\right|_{t=0} V^{\mathsf{c}}(e^{t[\xi_0\,,\,\_]}\eta_0) \tag{3.2.22}$$

The exact deformations, of the form:

$$V^{\mathsf{c}}_{\mathtt{exact}}(\xi_0) \;=\; \left(\Delta_{\rho_{1/2}} + \{a(\xi_0), \_\}_{\mathrm{BV}}\right) v^{\mathsf{c}}(\xi_0) \tag{3.2.23}$$

with v^{c} satisfying the equivariance condition $\{\underline{\xi_0}, v^{\mathsf{c}}(\eta_0)\}_{\mathrm{BV}} = \frac{d}{dt}\big|_{t=0} v^{\mathsf{c}}(e^{t[\xi_0\,,\,\_]}\eta_0)$ are considered trivial.

Consider the expansion of $V^{\mathsf{c}}(\xi_0)$ in powers of ξ_0:

$$V^{\mathsf{c}}(\xi_0) = V^{(0)} + V^{(1)}\langle\xi_0\rangle + V^{(2)}\langle\xi_0 \otimes \xi_0\rangle + \ldots \tag{3.2.24}$$

(We use angular brackets $\langle\ldots\rangle$ to highlight linearity, *i.e.* $f\langle x\rangle$ instead of $f(x)$ when f is a linear functions of x.) In particular, Eq. (3.2.21) implies at the linear order in ξ:

$$\Delta_{\rho_{1/2}} V^{(1)}\langle\xi_0\rangle + \{a^{(1)}\langle\xi_0\rangle, V^{(0)}\}_{\mathrm{BV}} = 0 \tag{3.2.25}$$

Equivariant vertex operators form a representation of the $D\mathbf{g}$ algebra discussed in [60], the differential d of [60] being represented by $\Delta_{\rho_{1/2}}$.

For our purpose, we will use a slightly different form of Eq. (3.2.25). Let us return to Eq. (3.2.18). At the linear order in ξ_0 it becomes:

$$\Delta_{\rho_{1/2}}\left(a^{(1)}\langle\xi_0\rangle V^{(0)} + V^{(1)}\langle\xi_0\rangle\right) = \underline{\xi_0} V^{(0)} \tag{3.2.26}$$

An exact V corresponds to (see Eq. (3.2.23)):

$$V^{(0)}_{\mathrm{exact}} = \Delta_{\rho_{1/2}} v^{(0)} \tag{3.2.27}$$

$$V^{(1)}_{\mathrm{exact}}\langle\xi_0\rangle = \Delta_{\rho_{1/2}}(a^{(1)}\langle\xi_0\rangle v^{(0)} + v^{(1)}\langle\xi_0\rangle) - \underline{\xi}^{(0)} v^{(0)} \tag{3.2.28}$$

Eq. (3.2.26) is an equivalent form of Eq. (3.2.25). We will explain in Section 3.4, that in case of bosonic string it is more convenient to use Eq. (3.2.25). But in case of pure spinor string we use Eq. (3.2.26).

3.2.5 A connection on $\Lambda \to \Lambda/St(p)$

In order to integrate, we need to pass from equivariant Ω^{c} to base Ω^{base}. This requires a choice of a connection in the principal $St(p)$-bundle LAG $\to$ LAG$/St(p)$. We will now define the connection by specifying the distribution $\mathcal{H}_0 \subset TE|_S$ of horizontal vectors. We say that the vector belongs to $\mathcal{H}_0$, if it is a linear combination of vectors of the following two classes:

- The **first class** consists of the variations of the metric satisfying:

$$h^{\alpha\beta}\delta h_{\alpha\beta} = 0 \tag{3.2.29}$$
$$\nabla^{\alpha}\delta h_{\alpha\beta} = 0 \tag{3.2.30}$$

 Such $\delta h_{\alpha\beta}$ can be identified as holomorphic or antiholomorphic quadratic differentials.

- The **second class** by definition consists of infinitesimal isometric ("rigid") translations of the disk D_ϵ of the small radius ϵ. These are delta-function-like variations of the metric with the support on ∂D_ϵ. They are always trivial in LAG/Diff, but nontrivial in LAG$/St(p)$ when genus is greater than one.

 (This definition only works for the metric of constant negative curvature, because for generic metric D_ϵ does not have any infinitesimal isometries. In such cases, we can choose some lift to a vector field v which is approximately isometry, in the sense that $\mathcal{L}_v g_{\alpha\beta} = O(|z|^2)$. Formulas do not change.)

3.2.6 Base form and its integration

Given a connection, we can construct a base form out of the equivariant form of Eq. (3.2.19); it is given by the following expression [36, 37]:

$$\Omega^{\mathsf{base}}(L, dL) = \int_L \exp(\sigma\langle dL|_{\mathrm{hor}}\rangle)V^{\mathsf{C}}(F)\rho^{\mathsf{C}}(F) \tag{3.2.31}$$

where V^{C} must satisfy Eqs. (3.2.16) and (3.2.17), and F is the curvature of our connection. Here, as in Eq. (3.2.19), $\sigma\langle dL|_{\mathrm{hor}}\rangle$ is any BV Hamiltonian generating the infinitesimal deformation, but we have to "project" the variation dL to the horizontal subspace (using our connection).

Let us consider the fiber bundle:

$$\frac{\mathrm{MET}}{St(p)} \xrightarrow{\pi} \frac{\mathrm{MET}}{\mathrm{Diff}} \tag{3.2.32}$$

We want to integrate Ω over the cycle of the form $\pi^{-1}c_{6g-6}$ where c_{6g-6} is the fundamental cycle of the moduli space of Riemann surfaces. Let us first integrate over the fiber (which is Σ). Our connection, described in Section 3.2.5, lifts the tangent vectors to the fiber as horizontal vectors of the second class, *i.e.* as infinitesimal rigid translations of D_ϵ. The curvature of our connection, evaluated on a pair of vectors tangent to the fiber, takes values in infinitesimal rigid rotations of D_ϵ and equals to the curvature of Σ. Therefore Ω^{base} is:

$$\begin{aligned}\Omega^{\mathsf{base}} = \int e^S\Big[\, & V^{(0)}\sigma\,\langle dL_{\mathrm{hor}}\rangle \wedge \sigma\,\langle dL_{\mathrm{hor}}\rangle\ + \\ & +\ V^{(0)}a^{(1)}\langle R\rangle + V^{(1)}\langle R\rangle\,\Big]\end{aligned} \tag{3.2.33}$$

We will now explain this equation, first line first, and then the second.

First line of Eq. (3.2.33)

With our definition of the connection in Section 3.2.5, the horizontal projection $dL|_{\mathrm{hor}}$ is *an infinitesimal diffeomorphism*: an infinitesimal translation of the disk D_ϵ by $\begin{bmatrix} dz \\ d\bar{z}\end{bmatrix}$. Therefore, the corresponding BV Hamiltonian $\sigma\langle dL|_{\rm hor}\rangle$ is actually Δ-exact. Indeed, Eq. (3.2.8) implies that:

$$\sigma\langle dL|_{\rm hor}\rangle \;=\; \Delta_{\rho_{1/2}} a^{(1)}\langle u(dz,d\bar{z})\rangle \tag{3.2.34}$$

Here $u(dz,d\bar{z})$ is the vector field on Σ which is:

- at the center of D_ϵ equals to $\left[\begin{array}{c} dz \\ d\bar{z}\end{array}\right]$
- inside D_ϵ is an infinitesimal rigid translation
- outside of D_ϵ is zero

Since $a^{(1)}$ is a local functional on the string worldsheet, Eqs. (3.2.34) and (3.2.12) imply:

$$\sigma\langle dL|_{\rm hor}\rangle \;=\; \{S_{\rm BV}\,,\, a^{(1)}\langle u(dz,d\bar{z})\rangle\} \tag{3.2.35}$$

Lemma-definition 1: For any vector field v, the restriction of $\{S_{\rm BV}, a^{(1)}\langle v\rangle\}$ on L is $\int b^{\alpha\beta}\nabla_\alpha v_\beta$:

$$\{S_{\rm BV}, a^{(1)}\langle v\rangle\}\big|_L = \int b^{\alpha\beta}\nabla_\alpha v_\beta \tag{3.2.36}$$

We take Eq. (3.2.36) as the *definition* of $b^{\alpha\beta}$ (which is otherwise defined only up to a Q-closed expression).

Proof Let us consider the expansion of $S_{\rm BV}$ and the expansion of $\mathcal{V} = \{S_{\rm BV}, a^{(1)}\}$:

$$S_{\rm BV} = S_0 + Q^A\phi^\star_A + \ldots \tag{3.2.37}$$

$$\{S_{\rm BV}, a^{(1)}\} = \mathcal{V}_0 + \mathcal{V}^A_1\phi^\star_A + \ldots \tag{3.2.38}$$

From $\{S_{\rm BV},\{S_{\rm BV},a^{(1)}\}\} = 0$ we derive:

$$\mathcal{L}_Q\mathcal{V}_0 = \mathcal{L}_{\mathcal{V}_1}S_0 \tag{3.2.39}$$

Eq. (3.2.36) follows from the variation of S_0 under infinitesimal diffeomorphism being equal to $\int T^{\alpha\beta}\nabla_\alpha v_\beta$, and from the vanishing of the off-shell cohomology in ghost number -1 (we are working off-shell!).

Returning to Eq. (3.2.35), Since u is an isometry inside D_ϵ and zero outside D_ϵ, we have:

$$\{S_{\rm BV}, a^{(1)}\langle u\rangle\}|_L = \int_\Sigma \sqrt{g}\; b^{\alpha\beta}\nabla_\alpha u_\beta = \oint_{\partial D_\epsilon} dz^\alpha b_{\alpha\beta}u^\beta \tag{3.2.40}$$

Therefore the first line in Eq. (3.2.33) contributes:

$$b_{-1}\bar{b}_{-1}V^{(0)} \tag{3.2.41}$$

Second line of Eq. (3.2.33)

Expressions like $a^{(1)}\langle R\rangle$ and $V^{(1)}\langle R\rangle$ should be understood in the following way. We think of the curvature R as a two-form on the worldsheet with values in rotations of the tangent space:

$$R \in \Gamma\left(\Omega^2\Sigma\otimes so(T\Sigma)\right) \tag{3.2.42}$$

In particular, if $\xi\in T_p\Sigma$ and $\eta\in T_p\Sigma$ are two tangent vectors, then $R(\xi,\eta)$ at the point p is an infinitesimal rotations of $T_p\Sigma$. This infinitesimal rotation can be represented by a vector field

v with zero at the point p. Let us "truncate" v by putting it to zero outside D_ϵ, *i.e.* multiply v by the function χ_{D_ϵ} which is 1 inside D_ϵ and 0 outside. By definition:

$$\begin{aligned} a^{(1)}\langle R(\xi,\eta)\rangle &\stackrel{\text{def}}{=} a^{(1)}\langle \chi_{D_\epsilon} v\rangle \\ V^{(1)}\langle R(\xi,\eta)\rangle &\stackrel{\text{def}}{=} V^{(1)}\langle \chi_{D_\epsilon} v\rangle \end{aligned} \tag{3.2.43}$$

(This is an abbreviation, rather than a definition.) In this context, Eq. (3.2.26) becomes:

$$\Delta_{\rho_{1/2}}\left(V^{(0)}a^{(1)}\langle R\rangle + V^{(1)}\langle R\rangle\right) \;=\; \{S_{\mathrm{BV}}, a^{(1)}\langle R\rangle\}V^{(0)} \tag{3.2.44}$$

In the case of pure spinor string $\{a^{(1)}\langle R\rangle, V^{(0)}\} = 0$, because v in Eq. (3.2.43) is a vector field vanishing at the point of insertion of $V^{(0)}$, and $V^{(0)}$ *does not contain derivatives.* Therefore, the left hand side of Eq. (3.2.44) is $\left\{S_{\mathrm{BV}}\,,\; a^{(1)}\langle R\rangle V^{(0)} + V^{(1)}\langle R\rangle\right\}$. When restricted to the Lagrangian submanifold, up to equations of motion[5]:

$$Q\left(a^{(1)}\langle R\rangle|_L V^{(0)}|_L + V^{(1)}\langle R\rangle|_L\right) = \{S_{\mathrm{BV}}, a^{(1)}\langle R\rangle\}|_L V^{(0)}|_L \tag{3.2.45}$$

We must stress that this equation is only valid under assumption $\{a^{(1)}\langle R\rangle, V^{(0)}\} = 0$. Generally speaking, instead of Eq. (3.2.45):

$$\begin{aligned} & Q\left(a^{(1)}\langle R\rangle|_L V^{(0)}|_L + V^{(1)}\langle R\rangle|_L\right) \;= \\ &= \{S_{\mathrm{BV}}, a^{(1)}\langle R\rangle\}|_L V^{(0)}|_L - \left.\{a^{(1)}\langle R\rangle, V^{(0)}\}\right|_L \end{aligned} \tag{3.2.46}$$

The computation of $\{S_{\mathrm{BV}}, a^{(1)}\langle R\rangle\}|_L$ uses Eq. (3.2.40):

$$\{S_{\mathrm{BV}}, a^{(1)}\langle R\rangle\}|_L V^{(0)}|_L = (b_0 - \bar{b}_0)V^{(0)} \tag{3.2.47}$$

Therefore:

$$a^{(1)}\langle R\rangle|_L V^{(0)}|_L + V^{(1)}\langle R\rangle|_L = \sqrt{g}R\Phi \tag{3.2.48}$$

where Φ satisfies:

$$Q\Phi = (b_0 - \bar{b}_0)V^{(0)} \tag{3.2.49}$$

To summarize, the total **<u>in</u>**tegrated vertex insertion corresponding to the **<u>un</u>**integrated vertex $V^{(0)}$ is given by the expression:

$$\int_\Sigma d^2z\left(b_{-1}\bar{b}_{-1}V^{(0)} + \sqrt{g}R\Phi\right) \tag{3.2.50}$$
$$\texttt{where }\Phi\texttt{ satisfies:}\, Q\Phi = (b_0 - \bar{b}_0)V^{(0)}$$

3.3 Brief review of the conventional description of the curvature coupling

Here we will briefly review the "standard" derivation of the curvature coupling.

Consider the deformation of the worldsheet action by adding the *integrated* vertex operator:

$$S \mapsto S + \epsilon\int U \tag{3.3.1}$$

[5]In spite of the fact that $\chi_{D_\epsilon}v$ of Eq. (3.2.43) is zero at the point of insertion of $V^{(0)}$, we cannot claim that $a^{(1)}\langle R\rangle|_L V^{(0)}|_L$ is zero. This is because of the singularities in the OPE of the integrand of $a^{(1)}_L$ and $V^{(0)}$.

where ϵ is a small "deformation parameter". Suppose that the deformed action is classically BRST invariant. At the one loop level, we get:

$$\partial^\mu j_\mu^{\mathrm{BRST}} \;=\; \alpha'(X + \sqrt{g}RY) \tag{3.3.2}$$

where X is a BRST-closed operator of conformal dimension $(1,1)$ and ghost number one, and Y is a BRST-closed expression of conformal dimension zero and ghost number one[6] In generic curved target-spaces, there is no BRST cohomology at ghost number 1 and conformal dimension zero. Therefore, exists Φ such that:

$$Y = -Q_{\mathrm{BRST}}\Phi \tag{3.3.3}$$

Also, there is no cohomology in conformal dimension $(1,1)$ and ghost number 1, therefore exists U' such that $X = -QU'$. These U' and Φ can be absorbed into U:

$$U \mapsto U + \alpha' U' + \alpha'\sqrt{g}R\Phi \tag{3.3.4}$$

and the term Φ is the deformation of the dilaton.

3.4 Bosonic string *vs* pure spinor string

3.4.1 Main differences

In **pure spinor** string theory on $AdS_5 \times S^5$:

- simplification: $V^{(0)}$ does not contain derivatives
- complication: restriction of $a(\xi)$ on "standard" family of Lagrangian submanifolds is nonzero

In this case we need compute $\left(a^{(1)}V^{(0)} + V^{(1)}\right)\Big|_L$ (this is what deforms the equivariant density), and we get it from Eq. (3.2.45)

In **bosonic** string theory:

- complication: $V^{(0)}$ contains at least derivatives of matter fields, and sometimes derivatives of ghosts
- simplification: $a(\xi)$ is given by a simple formula: $a(\xi) = \xi^\alpha c^\star_\alpha$, and in particular its restriction to the standard Lagrangian submanifold is zero

In this situation we compute $V^{(1)}$ from Eq. (3.2.21):

$$\{S_{\mathrm{BV}}, V^{(1)}\} = -\{\xi^\alpha c^\star_\alpha\, ,\, V^{(0)}\} = -\xi^\alpha \frac{\partial}{\partial c^\alpha} V^{(0)} \tag{3.4.1}$$

The exact vertex has:

$$V^{(0)}_{\mathrm{exact}} \;= \{S_{\mathrm{BV}}, v^{(0)}\} \tag{3.4.2}$$

$$V^{(1)}_{\mathrm{exact}}\langle\xi\rangle \;= \{S_{\mathrm{BV}}, v^{(1)}\langle\xi\rangle\} + \xi^\alpha \frac{\partial}{\partial c^\alpha} v^{(0)} \tag{3.4.3}$$

$$\ldots \tag{3.4.4}$$

[6] Notice that there is no $\sqrt{g}RV$ term in Eq. (3.3.1), because there are no BRST-closed scalar operators V of ghost number zero, other than 1 (the 1 corresponding to the change in string coupling).

3.4.2 Bosonic string vertices as functions on BV phase space

Consider bosonic string on a general curved worldsheet. We work in BV formalism, our vertex operators are functions on the **BV phase space of bosonic string worldsheet**.

Let us start by considering the vertex corresponding to a "gravitational wave", *i.e.* an infinitesimal deformation of the target space metric $G_{\mu\nu}$. We assume that $G_{\mu\nu}$ satisfies transversality and linearized Einstein equations:

$$\partial^{\mu} G_{\mu\nu} = 0 \tag{3.4.5}$$

$$\Box G_{\mu\nu} = 0 \tag{3.4.6}$$

(almost all gravitational waves can be obtained like this, except for some zero modes). Let $h_{\alpha\beta}$ be the worldsheet metric, and I^{α}_{β} the corresponding complex structure. We claim that the following vertex operator:

$$V^{(0)} = (Ic \cdot \partial X^{\mu})(c \cdot \partial X^{\mu}) G_{\mu\nu}(x) \tag{3.4.7}$$

satisfies:

$$\{S_{\mathrm{BV}}, V^{(0)}\} = 0 \tag{3.4.8}$$

Let us prove this. The odd Poisson brackets with **BV Master Action** are:

$$\{S_{\mathrm{BV}}, X\} \; = \mathcal{L}_c X \tag{3.4.9}$$

$$\{S_{\mathrm{BV}}, c\} \; = \frac{1}{2}[c, c] \tag{3.4.10}$$

$$\{S_{\mathrm{BV}}, I\} \; = \mathcal{L}_c I \tag{3.4.11}$$

(Here $\mathcal{L}_c X$ is the same as $c \cdot \partial X$ — the Lie derivative of X.)

$$\begin{aligned}
&\{S_{\mathrm{BV}},\; (Ic \cdot \partial X)(c \cdot \partial X)\} \; = \\
&= (([\mathcal{L}_c, \mathcal{L}_{Ic}] - \mathcal{L}_{I[c,c]})X)\; \mathcal{L}_c X - (\mathcal{L}_{Ic}\mathcal{L}_c X)\; \mathcal{L}_c X + \frac{1}{2}(\mathcal{L}_{I[c,c]} X)\; \mathcal{L}_c X \; = \\
&= (\mathcal{L}_c \mathcal{L}_{Ic} X)\mathcal{L}_c X - \frac{1}{2}(\mathcal{L}_{I[c,c]} X)\mathcal{L}_c X
\end{aligned} \tag{3.4.12}$$

Eq. (3.4.12) follows from:

$$\mathcal{L}_c \mathcal{L}_{Ic} X - \frac{1}{2}\mathcal{L}_{I[c,c]} X \; = \; \frac{1}{2}\iota_c^2 \; d * dX \; = \; \frac{1}{2}\{S_{\mathrm{BV}}, \iota_c^2 X^{\star}\} \tag{3.4.13}$$

In Eq. (3.4.13) we identify $X^{\star}$ as a 2-form on the worldsheet, and contract it two times with c. This operation can be characterized by saying that for every local (*i.e.* given by a single integral over the worldsheet Σ) functional $F[X]$:

$$\{\iota_{\xi}\iota_{\eta} X^{\star}\, ,\; F[X]\} \; = \; \iota_{\xi}\iota_{\eta}\frac{\delta F}{\delta X} \tag{3.4.14}$$

To prove Eq. (3.4.13), let us choose the coordinates $(z, \bar{z})$ where the complex structure is: $I\frac{\partial}{\partial z} = i\frac{\partial}{\partial z}$. We denote $C = c^z$ and $\bar{C} = c^{\bar{z}}$, *i.e.* $c \cdot \partial = C\partial + \bar{C}\bar{\partial}$ (with a slight abuse of notations, we let ∂ denote also ∂_z). With these notations:

$$(C\partial + \bar{C}\bar{\partial})(iC\partial - i\bar{C}\bar{\partial})X - I(C\partial + \bar{C}\bar{\partial})^2 X \; = \; 2i\bar{C}C\partial\bar{\partial}X \tag{3.4.15}$$

In order to actually insert $V^{(0)}$ we have to regularize it. (Even when Eqs. (3.4.5) and (3.4.6) are satisfied, we have the product of two ∂X at the same point, which does not make sense without regularization.)

Regularization We *regularize* $V^{(0)}$ by replacing every X^μ (including those acted on by ∂) with the averaged value:

$$X(0,0) \;\mapsto\; \mathcal{N}_\epsilon \int d^2z \sqrt{g} \exp\left(-\frac{1}{\epsilon}\mathrm{dist}^2((z,\bar{z}),(0,0))\right) X(z,\bar{z}) \tag{3.4.16}$$

where dist is the distance measured by the worldsheet metric, $\epsilon \to 0$ the regularization parameter, and $\mathcal{N}_\epsilon$ is the normalization factor:

$$\mathcal{N}_\epsilon = \left[\int d^2z \sqrt{g} \exp\left(-\frac{1}{\epsilon}\mathrm{dist}^2((z,\bar{z}),(0,0))\right)\right]^{-1} \tag{3.4.17}$$

When c gets contracted with ∂x, we take the average of $c^\alpha \partial_\alpha x$.

Renormalization After specifying the regularization prescription, we have to subtract infinities.

Actually, with Eqs. (3.4.5) and (3.4.6) the subtraction is not even needed, because the regularized $V^{(0)}$ remains finite when $\epsilon \to 0$.

But suppose that (having in mind extensions to string field theory) we want to define our vertex in a way which requires smooth extension off-shell, *i.e.* relaxing of Eqs (3.4.5) and (3.4.6). Then, for our expression to remain finite off-shell, we have to do a regularization. We define the *subtraction* as follows:

$$\mathcal{O}_{ren} = \exp\left(-\int d^2z \int d^2w \, \frac{\alpha'}{2} \ln \mathrm{dist}^2(z,\bar{z};w,\bar{w}) \frac{\delta}{\delta X^\mu(z,\bar{z})} \frac{\delta}{\delta X_\mu(w,\bar{w})}\right) \mathcal{O} \tag{3.4.18}$$

— this removes the short distance singularity in $\langle X(z,\bar{z}) X(w,\bar{w})\rangle$. Although this subtraction is diffeomorphism invariant, it is not Weyl invariant, and therefore it does not commute with $\{S_{\mathrm{BV}}, \_\}$. The actual effect of the subtraction is:

$$\lim_{x\to y}\left(c^\alpha(x) c^\beta(y) \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} \log \mathrm{dist}^2(x,y)\right) = \frac{\alpha'}{3}(c, Ic) R(x) \tag{3.4.19}$$

This implies, that the unintegrated vertex annihilated by $\{S_{\mathrm{BV}}, \_\}$ is:

$$\left(\mathcal{L}_c x^\mu \mathcal{L}_{Ic} x^\nu G_{\mu\nu}(x)\right)_{ren} + \frac{\alpha'}{3}(c, Ic) \Phi_{ren} R \tag{3.4.20}$$

$$\texttt{where } \Phi = G^\mu_\mu \tag{3.4.21}$$

Therefore the curvature coupling arises from Eq. (3.1.2), as $b_{-1}\bar{b}_{-1}\left((c, Ic) R\, \Phi\right) = R\, \Phi$. (And this source of curvature coupling is not present in the pure spinor case.)

If we do not impose the condition (3.4.5), then Eq. (3.4.7) requires modification. Additional terms should be added, such as *e.g.* $\mathrm{div}\, c\, (\mathcal{L}_c x^\mu) A_\mu(x)$. With these extra terms, Eq. (3.1.3) also contributes to the curvature coupling.

3.4.3 Ghost number one

Cohomology at ghost number one is (*cp* Eq. (3.4.13)):

$$W^\mu = \mathcal{L}_{Ic} X^\mu - \frac{1}{2}\iota_c^2 X^{\mu\star} \tag{3.4.22}$$

$$W^{\mu\nu} = X^{[\mu} \mathcal{L}_{Ic} X^{\nu]} - \frac{1}{2} X^{[\mu} \iota_c^2 X^{\nu]\star} \tag{3.4.23}$$

They are both already equivariant, because $\{a(\xi), W\} = 0$, since W does not contain derivatives of c. Notice that:

$$dW^\mu = \{S_{\mathrm{BV}}\,,\, U^\mu\} \tag{3.4.24}$$
$$\texttt{where } U^\mu = *dX^\mu - \iota_c X^{\mu\star} \tag{3.4.25}$$

The proof of Eq. (3.4.24) uses:

$$d\mathcal{L}_{Ic}X^\mu - \mathcal{L}_c * dX^\mu = \{S_{\mathrm{BV}}, \iota_c X^{\mu\star}\} \tag{3.4.26}$$

As a consistency check, it should be true, at least in restriction to a reasonable Lagrangian submanifold, that:

$$\iota_\xi U^\mu = \left(\int_D \left\{S_{\mathrm{BV}}, a^{(1)}\langle\xi\rangle\right\}\right) W^\mu \tag{3.4.27}$$
$$\texttt{where } \left\{S_{\mathrm{BV}}, a^{(1)}\langle\xi\rangle\right\} = (\mathcal{L}_\xi X^\mu) X^\star_\mu + [\xi, c]c^\star + (\mathcal{L}_\xi g_{\alpha\beta}) b^{\alpha\beta} \tag{3.4.28}$$

This is true on the **standard Lagrangian submanifold** *i.e.* $c^\star = 0$, $X^\star = 0$. We did not explicitly check this for other Lagrangian submanifolds.

3.4.4 Dilaton zero mode

Ghost dilaton Let us lift the expression $\partial c - \bar\partial\bar c$ of [61] to the BV phase space as $v = \mathrm{div}(Ic)$. The Cartan differential of v is (see Eqs. (3.2.27) and (3.2.28)):

$$V^{(0)} = \{S_{\mathrm{BV}}, v\} = \mathcal{L}_c(\mathrm{div}(Ic)) - \frac{1}{2}\mathrm{div}(I[c,c]) \tag{3.4.29}$$
$$V^{(1)}\langle\xi_0\rangle = \{a^{(1)}\langle\xi_0\rangle, v\} = \mathrm{div}(I\xi_0) \tag{3.4.30}$$
$$V^{(\geq 2)}\langle\ldots\rangle = 0$$

The restriction of $V^{(0)}$ on the standard family is, on-shell, $c\partial^2 c - \bar c\bar\partial^2\bar c$.

The base form corresponding to $V^{(1)}$ by the procedure of Section 3.2.6 is $\sqrt{g}R$. Therefore, we should interpret $V^{(1)}$ as the unintegrated vertex operator corresponding to the dilaton zero mode. However, $V^{(1)}$ by itself is not $\{S_{\mathrm{BV}}, \_\}$-closed:

$$\{S_{\mathrm{BV}}, V^{(1)}\} = \mathrm{tr}\Big(I\,[\mathcal{L}_c I, \mathcal{L}_{\xi_0} I]\Big) \neq 0 \tag{3.4.31}$$
$$\texttt{(commutator as matrices in } T_p\Sigma\texttt{)} \tag{3.4.32}$$

What is going on? The construction of the base form consists of the substitution of the curvature 2-form in place of ξ_0. The way we construct connection in Section 3.2.5 it actually takes values in a smaller subalgebra $\mathrm{st}(p, I_p) \subset \mathrm{st}(p)$, which consists of those vector fields which preserve the complex structure *in the tangent space to the point p of insertion, i.e.* $I_p \in gl(T_p\Sigma)$. We observe that:

$$\xi_0 \in \mathrm{st}(p, I_p) \subset \mathrm{st}(p) \quad \Rightarrow \quad \{S_{\mathrm{BV}}, V^{(1)}\langle\xi_0\rangle\} = 0 \tag{3.4.33}$$

(We must stress that, since I is one of the BV fields, $\mathrm{st}(p, I_p)$ varies from point to point in the BV phase space.)

Equivalence of $V^{(0)}$ and $V^{(1)}$ Eqs. (3.4.29) and (3.4.30) imply that the integrated vertex obtained from $V^{(1)}$ should be same as the one obtained from $V^{(0)}$. We can check this explicitly:

$$\left(\oint dz^\alpha b_{\alpha\beta}\xi^\beta\right)\left(\oint dz^\alpha b_{\alpha\beta}\eta^\beta\right)V^{(0)} = \tag{3.4.34}$$

$$= \left(\mathcal{L}_\xi \mathrm{div}(I\eta) - \frac{1}{2}\mathrm{div}(I[\xi,\eta])\right) - (\xi \leftrightarrow \eta) = \tag{3.4.35}$$

$$= \mathrm{div}(I[\xi,\eta]) = R(\xi,\eta) \tag{3.4.36}$$

We used the fact that, by the prescription of Section 3.2.5, ξ and η are lifted as isometries of a small neighborhood of the insertion point; in particular, the Lie derivative $\mathcal{L}_\xi$ commutes with the operations I and div.

3.4.5 Semirelative cohomology

In our paper we identify the space of states as the cohomology of the *equivariant* complex, as defined in Section 3.2.4.

The usual definition is *via* the *semirelative* complex [61]. In the case of bosonic string, the cohomology is the same. Indeed, imposing the semirelative condition $(b_0 - \bar{b}_0)V = 0$ leads to two effects:

- **Effect 1.** There are ghost number 2 cocycles, which should be thrown away because they are not annihilated by $b_0 - \bar{b}_0$. Those are non-physical beta-deformations.[7]
- **Effect 2.** The ghost-dilaton is $Q(\partial C - \bar{\partial}\bar{C})$ – would be BRST exact in the naive BRST complex, but $Q(\partial C - \bar{\partial}\bar{C})$ is not annihilated by $b_0 - \bar{b}_0$. Therefore, the ghost-dilaton is actually nontrivial.

The equivariant complex gives the same result. For V a nonphysical beta-deformation (Effect 1), $\{a(\xi), V\}$ is not just nonzero, but actually not even $\{S_{\mathrm{BV}}, \_\}$-exact. Therefore, we cannot "equivariantize" such vertex in the sense of Section 3.2.4. Therefore, such states should be thrown away also in our approach.

In case of Effect 2, we do admit $\partial C - \bar{\partial}\bar{C}$ (we present it as $\mathrm{div}(Ic)$). It is a perfectly valid cochain for us. However, our differential is not just Q_{BRST}, or $\{S_{\mathrm{BV}}, \_\}$. We actually have the equivariant differential, which consists of two parts:

$$d_{\mathbb{C}} = \{S_{\mathrm{BV}}, \_\} + \{a\langle\xi\rangle, \_\} \tag{3.4.37}$$

$\left\{S_{\mathrm{BV}}, \partial C - \bar{\partial}\bar{C}\right\}$ is ghost dilaton, but the second term is also nonzero:

$$\{a\langle\xi\rangle, \partial C - \bar{\partial}\bar{C}\} = \mathrm{div}(I\xi) \tag{3.4.38}$$

Therefore, it is not the ghost-dilaton which is $d_{\mathbb{C}}$-exact, but a sum of the ghost-dilaton and the expression $\mathrm{div}(I\xi)$. In other words, in our approach the ghost-dilaton is not d-exact, but is d-equivalent to $\mathrm{div}(I\xi)$. Both expressions, when passing to the base form, result in $\sqrt{g}R$ — the dilaton zero-mode. This means that Effect 2 is also the same in our approach, as in the semirelative approach.

[7]For pure spinor string, they are described in [62] and references therein. The pure spinor case is similar.

3.5 Vertex operators of pure spinor superstring

3.5.1 Conventions and notations for $AdS_5 \times S^5$ string

We begin introducing some notation that will be useful through out the calculation. Our notation is largely based on references [32, 63].

Constant Grassmann parameters The target space is a supermanifold, a coset of the Lie supergroup $PSU(2,2|4)$. As usual [64], treating the supermanifold, we introduce a "pool" of constant Grassmann parameters $\epsilon, \epsilon', \epsilon'', \ldots$. We can construct the "$\epsilon, \epsilon', \epsilon'', \ldots$-points" of the supermanifold $PSU(2,2|4)$ as formal expressions of the form, for example $\exp\left(\epsilon\mu^\alpha t^3_\alpha + \epsilon'\mu^{\dot\alpha} t^1_{\dot\alpha}\right)$ where μ^α and $\mu^{\dot\alpha}$ are some spinors with real number components. In addition to these constant Grassmann parameters, there are string worldsheet fields θ^α_L and $\theta^{\dot\alpha}_R$; therefore we also have: $\exp\left(\theta^\alpha_L t^3_\alpha\right)$ — another element of the supergroup.

Superconformal generators and Casimir conventions An element in the superconformal algebra $\mathbf{g} = \text{psu}(2,2|4)$ will be represented according to its $\mathbf{Z}_4$ grading,

$$t = t^0_{[mn]} \oplus t^1_{\dot\alpha} \oplus t^2_m \oplus t^3_\alpha$$

$$\texttt{where}$$

$$t^0_{[mn]} \in \mathbf{g}_0, \quad t^1_{\dot\alpha} \in \mathbf{g}_1, \quad t^2_m \in \mathbf{g}_2 \quad \text{and} \quad t^3_\alpha \in \mathbf{g}_3 \tag{3.5.1}$$

Latin letters are vector indices and greek letters are spinor indices. The bosonic generators are boosts and rotations, given by $t^0_{[mn]}$, and translations denoted t^2_m. The fermionic generators are the right supersymmetries, $t^1_{\dot\alpha}$, and the left supersymmetries, t^3_α, with both spinors in the $d = 10$ Majorana-Weyl representation. The vector space $\mathbf{g}_2$ is the sum of the tangent vector spaces of AdS_5 and S^5; $m \in \{0, \ldots, 9\}$.

For a finite-dimensional representation, the invariant bilinear form is given by the supertrace:

$$\text{str}\left(t^2_m t^2_n\right) = \kappa_{mn}, \quad \text{str}\left(t^3_\alpha t^1_{\dot\alpha}\right) = \kappa_{\alpha\dot\alpha} \quad \texttt{and} \quad \text{str}\left(t^1_{\dot\alpha} t^3_\alpha\right) = \kappa_{\dot\alpha\alpha} \tag{3.5.2}$$

where $\kappa_{\alpha\dot\alpha}$ and κ_{mn} are Casimir tensors.

Parametrization of $\text{AdS}_5 \times S^5$ We will work with the conventions of [63]. The coordinates in $\text{AdS}_5 \times S^5$ are given by $\left(x, \theta, \widehat\theta\right)$ such that

$$x = x^m(z,\bar z) t^2_m, \quad \theta = \theta^\alpha(z,\bar z) t^3_\alpha, \quad \widehat\theta = \widehat\theta^{\dot\alpha}(z,\bar z) t^1_{\dot\alpha}. \tag{3.5.3}$$

Each of these coordinates lifts to an element in $PSU(2,2|4)$ given by

$$g(x,\theta,\widehat\theta) = \exp\left(\frac{1}{R}\theta + \frac{1}{R}\widehat\theta\right)\exp\left(\frac{1}{R}x\right) \tag{3.5.4}$$

where R is the AdS radius.

The pure spinor action The $AdS_5 \times S^5$ pure spinor string action is

$$S = \frac{R^2}{\pi}\int d^2z \ \text{str}\left(\frac{1}{2}J_{2z}J_{2\bar z} + \frac{3}{4}J_{1z}J_{3\bar z} + \frac{1}{4}J_{3z}J_{1\bar z} + \omega_{1z}D_{\bar z}\lambda_3 + \omega_{3z}D_{\bar z}\lambda_1 + N_{0z}N_{0\bar z}\right) \tag{3.5.5}$$

with the covariant derivatives defined as

$$D_{\bar z}\lambda_3 = \partial_{\bar z}\lambda_3 + [J_{0\bar z},\lambda_3], \quad D_z\lambda_1 = \partial_z\lambda_1 + [J_{0z},\lambda_1] \tag{3.5.6a}$$

and the Lorentz currents for the ghosts given by

$$N_{0z} = -\left\{\omega_{1z},\lambda_1\right\}, \quad N_{0\bar z} = -\left\{\omega_{3\bar z},\lambda_3\right\}. \tag{3.5.6b}$$

The pure spinor action is built out of the right-invariant currents:

$$J = -\mathrm{d}g\, g^{-1} = -\partial_z g\, g^{-1}\mathrm{d}z - \partial_{\bar z} g\, g^{-1}\mathrm{d}\bar z, \tag{3.5.7}$$

where g is given by Eq. (3.5.4). These currents decompose according to the conformal weight and the $\mathbf{Z}_4$ grading. We write $J = J_0 + J_1 + J_2 + J_3$ to highlight the grading structure, and we observe that under local Lorentz symmetry J_0 transforms as a connection while J_1, J_2 and J_3 transform in the adjoint representation.

3.5.2 Covariance of vertices

In this Section we will consider vertex operators of pure spinor superstring in $AdS_5 \times S^5$. We will restrict ourselves with only those vertex operators which transform in **finite-dimensional** representations of $\mathbf{g}$ [65, 66]. We mainly consider the simplest example, namely the beta-deformation, which transforms in $\frac{(\mathbf{g}\wedge\mathbf{g})_0}{\mathbf{g}}$. We also make some conjectures about deformations transforming in other representations ("higher" vertices, Section 3.5.7).

Let $\mathcal{H}$ denote some subspace in the space of deformations, closed as a representation of $\mathbf{g}$. We assume that the vertex is *covariant*. This means that exists a map from $\mathcal{H}$ to space of vertices, commuting with the action of $\mathbf{g}$. As was explained in [67], under these conditions the all the vertex operators in the given representation $\mathcal{H}$ are completely specified by a single λ-dependent vector v in the dual of $\mathcal{H}$:

$$v(\lambda_L,\lambda_R) \in \mathcal{H}' \tag{3.5.8}$$

It should satisfy:

$$\rho(\lambda_L + \lambda_R)v = 0 \tag{3.5.9}$$

where $\rho(\lambda_L+\lambda_R)$ is the action of the element $\lambda_L^\alpha t^3_\alpha + \lambda_R^{\dot\alpha} t^1_{\dot\alpha} \in \mathbf{g}$ in $\mathcal{H}'$. In this sense, the pure spinor BRST operator acts on $\mathcal{H}'$:

$$Q = \rho(\lambda_3 + \lambda_1) \tag{3.5.10}$$

In this Section we will study the case when $\mathcal{H}$ is finite-dimensional. Then $\mathcal{H}' = \mathcal{H}$. We will consider those $\mathcal{H}$ which can be constructed products of adjoint representations of $\mathbf{g}$, the simplest example being the beta-deformation $\frac{(\mathbf{g}\wedge\mathbf{g})_0}{\mathbf{g}}$. Such spaces are naturally related to the cochain complex of $\mathbf{g}$, which we will now discuss.

3.5.3 Lie algebra cohomology complex

Let us consider the Lie algebra cohomology complex of $\mathbf{g} = \mathbf{psu}(2,2|4)$ with coefficients in a trivial representation. As a linear space, it is the direct sum $\bigoplus\limits_{i=0}^{\infty}\Lambda^n\mathbf{g}'$, where $\mathbf{g}'$ is the dual space of $\mathbf{g}$. We use the fact that $\mathbf{g}$ has a supertrace, and identify $\mathbf{g}'$ with $\mathbf{g}$. The supertrace induces the pairing

$$\Lambda^n\mathbf{g} \otimes \Lambda^n\mathbf{g} \longrightarrow \mathbf{C} \tag{3.5.11}$$

For example:

$$\langle x\wedge y, z\wedge w\rangle \;= \tag{3.5.12}$$
$$= \mathrm{STr}(yz)\mathrm{STr}(xw) - (-1)^{\bar{x}\bar{y}}\mathrm{STr}(xz)\mathrm{STr}(yw) \tag{3.5.13}$$

The Lie superalgebra cohomology differential d_{Lie} acts as follows:

$$d_{\mathrm{Lie}} \;: \Lambda^n \mathbf{g} \to \Lambda^{n+1}\mathbf{g} \tag{3.5.14}$$
$$\langle d_{\mathrm{Lie}} x, y\wedge w\rangle \overset{\text{def of } d_{\mathrm{Lie}}}{=} \langle x, [y,w]\rangle = \mathrm{STr}(x[y,w]) \tag{3.5.15}$$

3.5.4 Vertex operators corresponding to global symmetries

The following element:

$$\lambda_3 - \lambda_1 \in C^1\mathbf{g} = \mathbf{g} \tag{3.5.16}$$

is a nontrivial cocycle of Q. It corresponds to the unintegrated vertex operator:

$$V_a^{(0)} = \mathrm{STr}(t_a g^{-1}(\lambda_3 - \lambda_1)g) \tag{3.5.17}$$

3.5.5 Interplay between Lie algebra cohomology and pure spinor cohomology

The Q-cocycle $\lambda_3 - \lambda_1$ is not a Q-coboundary. However the Lie algebra differential applied to it is a coboundary, if we allow denominator $\frac{1}{\mathrm{STr}(\lambda_3\lambda_1)}$:

$$d_{\mathrm{Lie}}(\lambda_3 - \lambda_1) = Q\left(k^{\alpha\dot\alpha} t^3_\alpha \wedge (\mathbf{1} - 2\mathbf{P}_{13}) t^1_{\dot\alpha}\right) \tag{3.5.18}$$

The internal commutator of $k^{\alpha\dot\alpha} t^3_\alpha \wedge (\mathbf{1} - 2\mathbf{P}_{13}) t^1_{\dot\alpha}$ is nonzero, but is Q-exact:

$$k^{\alpha\dot\alpha}\{t^3_\alpha, (\mathbf{1} - 2\mathbf{P}_{13}) t^1_{\dot\alpha}\} = \frac{3}{2}\{\lambda_3, \lambda_1\} = \frac{3}{4} Q(\lambda_3 + \lambda_1) \tag{3.5.19}$$

3.5.6 Beta-deformation and its generalizations

Definition

The definition of the unintegrated vertex for beta-deformation given in [32, 67] is:

$$V = B^{ab} W_a W_b \tag{3.5.20}$$
$$\texttt{where}\;\; W_a = \mathrm{STr}\left(t_a g^{-1}(\lambda_3 - \lambda_1) g\right) \tag{3.5.21}$$

where B^{ab} is a constant antisymmetric tensor, defined up to the equivalence relation:

$$B^{ab} \simeq B^{ab} + f^{ab}{}_c A^c \tag{3.5.22}$$

The beta-deformation transforms in the following the following representation of $\mathbf{psu}(2,2|4)$:

$$\frac{(\mathbf{g}\wedge\mathbf{g})_0}{\mathbf{g}} \tag{3.5.23}$$

where the factor over $\mathbf{g}$ accounts for the equivalence relation defined by the Eq. (3.5.22).

This vertex operator defined in Eq. (3.5.20) is not strictly speaking covariant, for the following reason. When we change B^{ab} to $B^{ab}+f^{ab}{}_c A^c$, it changes by a BRST exact expression:

$$V \longrightarrow V + QW \tag{3.5.24}$$

$$\texttt{where } W = \mathrm{STr}\left(Ag^{-1}(\lambda_3+\lambda_1)g\right) \tag{3.5.25}$$

It is possible to define the vertex which is strictly covariant:

$$V' = V - \left\langle\, B\,,\, g^{-1}\left([\Sigma,\lambda_3+\lambda_1]\wedge[\Sigma,\lambda_3+\lambda_1]\right)g\,\right\rangle \tag{3.5.26}$$

where

$$\Sigma = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1) \tag{3.5.27}$$

The difference between V and V' is a BRST-exact expression:

$$\left\langle\, B\,,\, g^{-1}\left([\Sigma,\lambda_3+\lambda_1]\wedge[\Sigma,\lambda_3+\lambda_1]\right)g\,\right\rangle \;=\; QX \tag{3.5.28}$$

$$\texttt{where } X = -\left\langle\, B\,,\, g^{-1}\left(\Sigma\wedge[\Sigma,\lambda_3+\lambda_1]\right)g\,\right\rangle \tag{3.5.29}$$

The definition of X requires some work, because Σ is not an element of $\mathbf{g} = \mathbf{psu}(2,2|4)$, because $\mathrm{STr}\Sigma \neq 0$. Therefore, in order to define X, we need to lift B from $\mathbf{g}\wedge\mathbf{g}$ to $\mathbf{su}(2,2|4)\wedge\mathbf{su}(2,2|4)$. There is no way to do it while preserving the $\mathbf{psu}(2,2|4)$-invariance. Therefore, X does not transform as Eq. (3.5.23). Still, Eq. (3.5.28) holds, thus V' *is* BRST-equivalent to V.

Alternative definition

When B satisfies the "physicality" condition $B^{ab}f_{ab}{}^c = 0$, we can use the alternative vertex:

$$\widetilde{V} = \mathrm{STr}(\lambda_3\lambda_1)B^{ab}\Big\langle t_a\wedge t_b\,,\; g^{-1}\left(k^{\alpha\dot\alpha}t^3_\alpha\wedge \mathbf{P}_{13}t^1_{\dot\alpha}\right)g\Big\rangle \tag{3.5.30}$$

This alternative beta-deformation vertex is "homogeneous", in the sense that it has a definite ghost number $(1,1)$. It is linear in λ_3 and in λ_1, because the pre-factor $\mathrm{STr}(\lambda_3\lambda_1)$ cancels the denominator in $\mathbf{P}_{13}$.

Conjecture The vertex operator $\widetilde{V}$ defined by Eq. (3.5.30) is not BRST-exact. If this is the case, then $\widetilde{V}$ is proportional to the beta-deformation vertex of Eq. (3.5.20). We leave the proof of this conjecture, and the computation of the proportionality coefficient, for future work.

3.5.7 Conjectures about higher finite-dimensional vertices

Recurrent construction of vertices

Eq. (3.5.30) calls for generalization for higher finite-dimensional vertices [65]. Let us consider the bicomplex:

$$d_{\mathrm{tot}} = Q + d_{\mathrm{Lie}} \tag{3.5.31}$$

Eq. (3.5.18) shows that:

$$Qv_2 = \;-\,d_{\mathrm{Lie}}v_1 \tag{3.5.32}$$

$$\texttt{where } v_1 = \lambda_3 - \lambda_1 \tag{3.5.33}$$

$$v_2 = t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot\alpha} \tag{3.5.34}$$

Notice that the ghost number of v_n is $2-n$.

Conjecture:

1. Exist $v_3, v_4, \ldots$ such that:
$$d_{\rm tot}\sum_{j=1}^{\infty} v_j = 0 \tag{3.5.35}$$

2. For $j \geq 2$: $(\mathrm{STr}(\lambda_3\lambda_1))^j\, v_{2j}$ is a polynomial in λ_3 and λ_1, and is a covariant ghost number 2 vertex for the deformation corresponding to $\int d^4x \mathrm{tr} Z^{2+j}$

3. For $j \geq 2$: $(\mathrm{STr}(\lambda_3\lambda_1))^{j+1}\, v_{2j+1}$ is a polynomial in λ_3 and λ_1, and is a covariant ghost number 3 vertex, also corresponding to $\int d^4x \mathrm{tr} Z^{2+j}$ as explained in [62].

We leave the verification of these conjectures for future work.

Infinitesimal deformations of worldsheet BV Master Action

We will now describe another recurrent construction. As explained in [59], the pure spinor superstring in $AdS_5 \times S^5$ is quasiisomorphic to the theory with the following Master Action:

$$S_{\rm BV} = \int \mathrm{STr}\left(J_1 \wedge (\mathbf{1} - 2\mathbf{P}_{31})J_3\right) \tag{3.5.36}$$

This is the integral over the worldsheet of the 2-form $\mathcal{B} = \mathrm{STr}\left(J_1 \wedge (\mathbf{1} - 2\mathbf{P}_{31})J_3\right)$ which satisfies the property:

$$\mathcal{L}_Q \mathcal{B} = d\mathcal{A} \tag{3.5.37}$$
$$\texttt{where}\ \ \mathcal{A} = \mathrm{STr}(\lambda_3 J_1 - \lambda_1 J_3) = \mathrm{Str}\left((\lambda_3 - \lambda_1)J\right) \tag{3.5.38}$$

It is natural to **conjecture** that a vertex operator will correspond to an infinitesimal deformation of the action defined by Eq. (3.5.36):

$$\Delta S_{\rm BV} = \int \langle \beta, J \wedge J\rangle \tag{3.5.39}$$

Here β is a rational function of λ with values in $\mathrm{Hom}\left(\mathcal{H}, \mathbf{g}\wedge\mathbf{g}\right)$, where $\mathcal{H}$ is the space of deformations. The BRST invariance of the deformed action implies:

$$Q\beta = d_{\rm Lie}\alpha \tag{3.5.40}$$

Suppose that $\mathrm{STr}(\lambda_3\lambda_1)\beta$ is a polynomial in λ. Then Eq. (3.5.40) implies that $\mathrm{STr}(\lambda_3\lambda_1)\beta$ defines a Q-closed equivariant vertex for $\mathcal{H}\otimes(\mathbf{g}\wedge\mathbf{g})_0$. We **conjecture** that this vertex is nontrivial (*i.e.* not BRST exact), although it may be BRST exact on a proper subspace $L \subset \mathcal{H}\otimes(\mathbf{g}\wedge\mathbf{g})_0$. That means that, given a covariant vertex transforming in the representation $\mathcal{H}$, we can build a new covariant vertex on the space of the larger spin representation $\widetilde{\mathcal{H}} = \frac{\mathcal{H}\otimes(\mathbf{g}\wedge\mathbf{g})_0}{L}$. This gives a recurrent procedure for producing covariant vertices. We leave verification of these conjectures for future work.

3.6 OPE of b-ghost with beta-deformation vertex

3.6.1 The b-ghost

The b-ghost satisfies:

$$Q_L b_{zz} = T_{zz}, \tag{3.6.1}$$
$$Q_R b_{zz} = 0 \tag{3.6.2}$$

where T_{zz} is the holomorphic stress-energy tensor. The $\bar{b}_{\bar{z}\bar{z}}$ is defined by the same formula with Q_L exchanged with Q_R and T_{zz} replaced with $T_{\bar{z}\bar{z}}$. The solutions of these equations are given by [31, 34]:

$$b_{zz} = -\frac{\mathrm{str}\left(\lambda_1 \left[J_{2z}\Sigma, J_{1z}\right]\right)}{\mathrm{str}\left(\lambda_3\lambda_1\right)} + \frac{1}{2}\mathrm{str}\left(\mathrm{P}_{13}\omega_{1z}J_{3z}\right) \tag{3.6.3}$$

and

$$\bar{b}_{\bar{z}\bar{z}} = +\frac{\mathrm{str}\left(\lambda_3 \left[J_{2\bar{z}}\Sigma, J_{3\bar{z}}\right]\right)}{\mathrm{str}\left(\lambda_3\lambda_1\right)} + \frac{1}{2}\mathrm{str}\left(\mathrm{P}_{31}\omega_{3\bar{z}}J_{1\,\bar{z}}\right) \tag{3.6.4}$$

where P_{13} and P_{31} are **some projectors** . These projectors are needed because the pure spinor momenta ω_{1z} and $\omega_{3\bar{z}}$ are defined up to gauge transformations of the form:

$$\delta_u\omega_{3z} = \left[u_z, \lambda_1\right], \quad \texttt{and} \quad \delta_u\omega_{1\bar{z}} = \left[u_{\bar{z}}, \lambda_3\right], \tag{3.6.5}$$

for both u_z and $u_{\bar{z}}$ in $\mathbf{g}_2$. Therefore, the projectors are constructed to satisfy

$$\mathrm{P}_{13}\delta_u\omega_{1\bar{z}} = 0 \qquad \texttt{and} \qquad \mathrm{P}_{31}\delta_u\omega_{3z} = 0. \tag{3.6.6}$$

Explicit formulas for P_{13} and P_{31} as rational functions of the pure spinor variables can be found in [59].

It is an open question to prove that the expressions $\mathrm{str}\left(\mathrm{P}_{13}\omega_{1z}J_{3z}\right)$ and $\mathrm{str}\left(\mathrm{P}_{31}\omega_{3\bar{z}}J_{1\,\bar{z}}\right)$ are well-defined in the quantum theory.

Lemma 3.2.6 implies that b given by Eqs. (3.6.3) and (3.6.4) coincides with $\Delta\Psi|_L$ up to a Q-closed expression. We have not verified this explicitly.

OPE between b-ghost and global vertex With these definitions, the OPE between the b-ghost and unintegrated global symmetry becomes to 1-loop order:

$$\begin{aligned}\left\langle \epsilon\left(b_0 - \bar{b}_0\right) V[\tilde{\epsilon}](0)e^{-S_i}\right\rangle &= \left\langle\left(\oint \frac{\mathrm{d}z}{2\pi i}\, z\epsilon b_{zz}(z) - \oint \frac{\mathrm{d}\bar{z}}{2\pi i}\, \bar{z}\epsilon\bar{b}_{\bar{z}\bar{z}}\right) V[\tilde{\epsilon}](0)\right\rangle \\ &\quad - \left\langle\left(\oint \frac{\mathrm{d}z}{2\pi i}\, z\epsilon b_{zz}(z) - \oint \frac{\mathrm{d}\bar{z}}{2\pi i}\, \bar{z}\epsilon\bar{b}_{\bar{z}\bar{z}}\right) V[\tilde{\epsilon}](0)\, S_i\right\rangle.\end{aligned} \tag{3.6.7}$$

We will calculate all Feynman diagrams considering the pure spinor action and the b-ghost as a power series in the AdS radius. For the parametrization (3.5.4), the expansion of the action can be found in reference [63]. In the above equation S_i represents all contributions of order $1/R$ or greater.

3.6.2 General considerations

At the leading order in α', we should have:

$$b_{zz}W_a = \frac{1}{z}(j_{az} + Ql_{az}) \tag{3.6.8}$$

$$b_{\bar{z}\bar{z}}W_a = -\frac{1}{\bar{z}}(j_{a\bar{z}} + Ql_{a\bar{z}}) \tag{3.6.9}$$

where l_{az}, $l_{a\bar{z}}$ are some operators, and $j_{az}dz + j_{a\bar{z}}d\bar{z}$ is the global charge density; our definition of the charge density is such that:

$$\left(\frac{1}{2\pi i}\oint j_{az}dz + j_{a\bar{z}}d\bar{z}\right) W_b = f_{ab}{}^{c}W_c \tag{3.6.10}$$

Notice:

$$j_{za}W_b = \frac{1}{2z}f_{ab}{}^cW_c + \dots \tag{3.6.11}$$

$$j_{\bar{z}a}W_b = -\frac{1}{2\bar{z}}f_{ab}{}^cW_c + \dots \tag{3.6.12}$$

(where $\dots$ can include $\log z$ but not z^{-1}) Therefore:

$$\begin{aligned}(b_0 - \bar{b}_0)V &= \oint (dz z b_{zz} - d\bar{z}\bar{z}b_{\bar{z}\bar{z}})\, V = \\ &= B^{ab}f_{ab}{}^cW_c + Q\left[B^{ab}\left(\oint l_a\right)W_b\right]\end{aligned} \tag{3.6.13}$$

One is tempted to say that Eq. (3.6.13) implies that V is annihilated by $b_0 - \bar{b}_0$, in cohomology, once B satisfies the physicality condition $B^{ab}f_{ab}{}^c = 0$. However, notice that the expression $f_{ab}{}^cW_c$ is anyway Q-exact (and even $\mathbf{g}$-covariantly Q-exact) since we allow denominator $\frac{1}{\mathrm{STr}(\lambda_3\lambda_1)}$, see Section 3.7.

3.6.3 Explicit computation

The operator $(b_0 - \bar{b}_0)V$ is a sum of two terms: the term with the ghost number $(1,0)$ and the term with the ghost number $(0,1)$. The term with the ghost number $(0,1)$ is:

$$\frac{2}{\mathrm{STr}(\lambda_3\lambda_1)}\left(\{[\lambda_1, t^2_{-m}], \lambda_3\} \wedge [t^2_m, \lambda_1] - \{[\lambda_3, t^2_{-m}], \lambda_1\} \wedge [t^2_m, \lambda_1]\right)$$

$$- \kappa^{\beta\dot{\beta}}\{t^3_\beta, \lambda_1\} \wedge t^1_{\dot{\beta}} \tag{3.6.14}$$

and the term with the ghost number $(1,0)$ is equal, with the minus sign, to the same expression with $\lambda_3 \leftrightarrow \lambda_1$ and exchanged dotted and undotted indices. Transform:

$$\begin{aligned} & -\frac{2}{\mathrm{STr}(\lambda_3\lambda_1)}\{[\lambda_3, t^2_{-m}], \lambda_1\} \wedge [t^2_m, \lambda_1] \\ &= -\frac{2\kappa^{\dot{\beta}\beta}}{\mathrm{STr}(\lambda_3\lambda_1)}\{[\lambda_3, \overline{\{\lambda_1, t^1_{\dot{\beta}}\}}_{\mathrm{STL}}], \lambda_1\} \wedge t^3_\beta \\ &= \frac{2\kappa^{\dot{\beta}\beta}}{\mathrm{STr}(\lambda_1\lambda_3)}\{[\lambda_3, \overline{\{\lambda_1, t^1_{\dot{\beta}}\}}_{\mathrm{STL}}], \lambda_1\} \wedge t^3_\beta \\ &= -\frac{2\kappa^{\dot{\beta}\beta}}{\mathrm{STr}(\lambda_1\lambda_3)}\{[\overline{\{\lambda_1, t^1_{\dot{\beta}}\}}_{\mathrm{STL}}, \lambda_3], \lambda_1\} \wedge t^3_\beta \\ &= \kappa^{\dot{\beta}\beta}\{t^1_{\dot{\beta}}, \lambda_1\} \wedge t^3_\beta \end{aligned} \tag{3.6.15}$$

where we used the explicit form of the pure spinor projector P_{31} that can be found in [59]. Thus we arrive at:

$$\frac{2}{\mathrm{STr}(\lambda_3\lambda_1)}\{[\lambda_1, t^2_{-m}], \lambda_3\} \wedge [t^2_m, \lambda_1] - Q_R\left(\kappa^{\beta\dot{\beta}}t^3_\beta \wedge t^1_{\dot{\beta}}\right) \tag{3.6.16}$$

Adding the "mirror" term with the ghost number $(1,0)$, we arrive at:

$$(b_0 - \bar{b}_0)\left\langle B^{ab} g(t_a \wedge t_b) g^{-1}\,,\, (\lambda_3 - \lambda_1)\wedge(\lambda_3 - \lambda_1)\right\rangle \;=\; Q\Phi \tag{3.6.17}$$

where:

$$\Phi \;=\; \Big\langle B^{ab} g(t_a \wedge t_b) g^{-1}\,, \tag{3.6.18}$$

$$\frac{2[t^2_{-m},\lambda_1]\wedge[t^2_m,\lambda_1] + 2[t^2_{-m},\lambda_3]\wedge[t^2_m,\lambda_3]}{\mathrm{Str}(\lambda_3\lambda_1)} - \kappa^{\beta\dot{\beta}} t^3_\beta \wedge t^1_{\dot{\beta}}\Big\rangle$$

Up to Q-exact terms, we can also take:

$$\Phi \;=\; \Big\langle B^{ab} g(t_a \wedge t_b) g^{-1}\,, \tag{3.6.19}$$

$$\frac{-4[t^2_{-m},\lambda_3]\wedge[t^2_m,\lambda_1] + 2[\{\lambda_3,\lambda_1\},t^2_{-m}]\wedge t^2_m}{\mathrm{Str}(\lambda_3\lambda_1)} - \kappa^{\beta\dot{\beta}} t^3_\beta \wedge t^1_{\dot{\beta}}\Big\rangle$$

3.6.4 Discussion

In this Section we will compare our proposed Eq. (3.2.50):

$$\int U \;=\; \int_\Sigma d^2 z\left(b_{-1}\bar{b}_{-1}V^{(0)} + \sqrt{g}R\Phi\right) \tag{3.6.20}$$

$$\text{where } \Phi \text{ is given by Eq. (3.6.18)} \tag{3.6.21}$$

with the standard approach to the beta-deformation [32]. The most obvious observation is that the "dilaton superfield" Φ of Eq. (3.6.18) contains pure spinors (while the "standard" dilaton superfield, obviously, does not). Therefore, they are certainly not the same. We will now explain that there are two reasons for the difference.

<u>First reason:</u> $b_{-1}\bar{b}_{-1}V^{(0)}$ is different from the standard integrated vertex on flat worldsheet. The standard integrated vertex on flat worldsheet is [32, 67]:

$$B^{ab} j_a \wedge j_b \tag{3.6.22}$$

In our approach here, it is the $b_{-1}\bar{b}_{-1}V^{(0)}$ of Eq. (3.6.20). This is *not* equal to $B^{ab}j_a\wedge j_b$, but differs from it by a Q-exact expression, which we have not explicitly computed[8]:

$$B^{ab} j_a \wedge j_b = dz\wedge d\bar{z}\, b_{-1}\bar{b}_{-1}V^{(0)} + QX \tag{3.6.23}$$

Notice that the BRST operator is only nilpotent on-shell:

$$Q^2 = \frac{\partial S}{\partial w_1}\frac{\partial}{\partial w_3} + (1 \leftrightarrow 3) \tag{3.6.24}$$

Therefore, the QX on the RHS of Eq. (3.6.23) deforms the BRST operator:

$$Q \mapsto Q + \left(\frac{\partial X}{\partial w_1}\frac{\partial}{\partial w_3} + (1\leftrightarrow 3)\right) \tag{3.6.25}$$

This leads to the change in the BRST anomaly, and, by the mechanism of Eqs. (3.3.3), (3.3.4), to the change of the Fradkin-Tseytlin term.

[8] since we have not explicitly computed $b_{-1}\bar{b}_{-1}V^{(0)}$

When we modify the unintegrated vertex:

$$V^{(0)} \mapsto \widetilde{V}^{(0)} = V^{(0)} + QW^{(0)} \tag{3.6.26}$$

The change in Φ, *i.e.* $\widetilde{\Phi} - \Phi$, should satisfy:

$$Q(\widetilde{\Phi} - \Phi) = (b_0 - \bar{b}_0)QW^{(0)} \tag{3.6.27}$$

Under the assumption that $(L_0 - \bar{L}_0)W^{(0)} = 0$ this can be solved by taking:

$$\widetilde{\Phi} = \Phi - (b_0 - \bar{b}_0)W^{(0)} \tag{3.6.28}$$

Suppose that we were able to find such $W^{(0)}$ that $\widetilde{V}^{(0)}$ is polynomial in pure spinors. Then, the curvature coupling also changes, according to Eq. (3.6.28),

Second reason: we have not required the vanishing of $B^{ab} f_{ab}{}^c$. In fact, Φ of Eq. (3.6.19) can be presented as:

$$\Phi = B^{ab} \left(X_{[ab]} + \left\langle g(t_a \wedge t_b) g^{-1} \,,\, \frac{2[\{\lambda_3, \lambda_1\}, t^2_{-m}] \wedge t^2_m}{\mathrm{Str}(\lambda_3 \lambda_1)} \right\rangle \right) \tag{3.6.29}$$

where $X_{[ab]}$ is defined in Eq. (3.7.3). Since $QX_{[ab]}$ is proportional to $f_{ab}{}^c$, the term $B^{ab}X_{[ab]}$ can be dropped when B has zero internal commutator, *i.e.* $B^{ab} f_{ab}{}^c = 0$. In that case, we have just:

$$\Phi = B^{ab} \left\langle g(t_a \wedge t_b) g^{-1} \,,\, \frac{2[\{\lambda_3, \lambda_1\}, t^2_{-m}] \wedge t^2_m}{\mathrm{Str}(\lambda_3 \lambda_1)} \right\rangle \tag{3.6.30}$$

We see that imposing the condition $B^{ab} f_{ab}{}^c = 0$ "considerably simplifies" the expression for the dilaton superfield. But still the resulting expression is a rational function of λ's.

3.6.5 Computation.

The free field propagators can be read from [63]:

$$\langle x^m(z, \bar{z}) x^n(0) \rangle = -\kappa^{mn} \log |z|^2 \tag{3.6.31}$$

$$\langle \theta^{\alpha}_L(z, \bar{z}) \theta^{\dot{\beta}}_R(0) \rangle = -\kappa^{\alpha\dot{\beta}} \log |z|^2 \tag{3.6.32}$$

$$\langle \theta^{\dot{\alpha}}_R(z, \bar{z}) \theta^{\beta}_L(0) \rangle = -\kappa^{\dot{\alpha}\beta} \log |z|^2 \,. \tag{3.6.33}$$

The propagator λw can be characterized by saying that for any $A^{\alpha}(\lambda)$ such that $A^{\alpha}\Gamma^m_{\alpha\beta}\lambda^{\beta} = 0$ (*i.e.* tangent to the pure spinor cone):

$$\langle A_{\dot{\alpha}} \left(\lambda(z, \bar{z}) \right) w^{\dot{\alpha}}_+(z, \bar{z}) \; \lambda^{\beta} \rangle = -\kappa^{\dot{\alpha}\beta} z^{-1} \tag{3.6.34}$$

Current Vertex

Let us focus, for the moment, on contractions that take only one V in $V \wedge V$; that is, we are going to compute the OPE of $(b_0 - \bar{b}_0)$ with $\epsilon(\lambda_3 - \lambda_1)$. The contributions we are interested are represented in the diagrams below:

$$\mathrm{str}(\epsilon\lambda_1[J_2\Sigma, J_1]) \;\to\; \mathrm{str}(\epsilon\lambda_1[\partial x\Sigma, \partial\hat{\theta}])$$

$$g^{-1}\tilde{\epsilon}(\lambda_3-\lambda_1)g \;\to\; [x,[\theta,\tilde{\epsilon}(\lambda_3-\lambda_1)]]$$

Figure 3.1: Disconnected contractions for the OPE between b_{zz} and $V[\tilde{\epsilon}]$.

$$\mathrm{str}(\epsilon \mathrm{P}_{13}\omega_1 J_3) \;\to\; -\mathrm{str}(\epsilon \mathrm{P}_{13}\omega_1\partial\theta)$$

$$g^{-1}\tilde{\epsilon}(\lambda_3-\lambda_1)g \;\to\; -[\widehat{\theta}, \tilde{\epsilon}\lambda_3]$$

Figure 3.2: Disconnected contractions for the OPE between b_{zz} and $V[\tilde{\epsilon}]$.

Contribution from the diagram of Fig. 3.1

$$-\frac{1}{\mathrm{str}(\lambda_3\lambda_1)}\frac{\kappa^{mn}\kappa^{\dot{\alpha}\alpha}}{R^4(z-w)^2}\,\mathrm{str}\Big(\epsilon\lambda_1\left[t^2_m\Sigma, t^1_{\dot{\alpha}}\right]\Big)\Big[t^2_n, \left\{t^3_\alpha, \tilde{\epsilon}\left(\lambda_3-\lambda_1\right)\right\}\Big]. \tag{3.6.35}$$

Let us use the identity:

$$-\kappa^{mn}\kappa^{\dot{\alpha}\alpha}\,\mathrm{str}\Big(\epsilon\lambda_1\left[t^2_m\Sigma, t^1_{\dot{\alpha}}\right]\Big)\Big[t^2_n, \left\{t^3_\alpha, \tilde{\epsilon}\left(\lambda_3-\lambda_1\right)\right\}\Big] = \tag{3.6.36}$$

$$= \kappa^{mn}\Big[t^2_n, \left[\left[t^2_m\Sigma, \epsilon\lambda_1\right], \tilde{\epsilon}\left(\lambda_3-\lambda_1\right)\right]\Big]$$

$$= \kappa^{mn}\Big[t^2_n, \left[\left[t^2_m\Sigma, \epsilon\lambda_1\right], \tilde{\epsilon}\lambda_3\right]\Big]. \tag{3.6.37}$$

Contribution from the diagram of Fig. 3.2

$$\frac{\kappa^{\alpha\dot{A}}\kappa^{\dot{B}\beta}}{2R^4(z-\mathrm{w})^2}\mathrm{str}\left(\epsilon \mathrm{P}_{13}t^1_{\dot{\beta}}t^3_\alpha\right)\left\{t^3_\beta, \tilde{\epsilon}t^1_{\dot{\alpha}}\right\} = \frac{\kappa^{\dot{B}\beta}}{2R^4(z-\mathrm{w})^2}\left[\epsilon t^3_\beta, \tilde{\epsilon}\mathrm{P}_{13}t^1_{\dot{\beta}}\right]. \tag{3.6.38}$$

Sum of first and second diagram

$$\epsilon b_0 V[\tilde{\epsilon}] = +\frac{1/R^4}{\mathrm{str}(\lambda_3\lambda_1)}\kappa^{mn}\Big[t^2_n, \left[\left[t^2_m\Sigma, \epsilon\lambda_1\right], \tilde{\epsilon}\lambda_3\right]\Big] - \frac{\kappa^{\dot{B}\beta}}{2R^4}\left[\tilde{\epsilon}\mathrm{P}_{13}t^1_{\dot{\beta}}, \epsilon t^3_\beta\right]. \tag{3.6.39}$$

Anti-holomorphic b-ghost A similar computation gives for the anti-holomorphic term:

$$\epsilon\bar{b}_0 V[\tilde{\epsilon}] = +\frac{1/R^4}{\mathrm{str}(\lambda_3\lambda_1)}\kappa^{mn}\Big[t^2_n, \left[\left[t^2_m\Sigma, \epsilon\lambda_3\right], \tilde{\epsilon}\lambda_1\right]\Big] - \frac{\kappa^{\dot{B}\beta}}{2R^4}\left[\tilde{\epsilon}\mathrm{P}_{13}t^1_{\dot{\beta}}, \epsilon t^3_\beta\right]. \tag{3.6.40}$$

Contribution of $b_0 - \bar{b}_0$ We can simplify the total contrubution of the diagrams of Figures 3.1 and 3.2 to

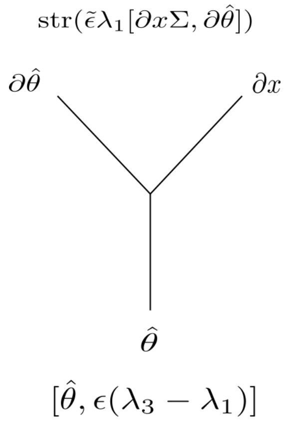

Figure 3.3: Vertex contribution b_{zz} and $V[\tilde{\epsilon}]$.

$$\epsilon(b_0 - \bar{b}_0)V[\tilde{\epsilon}] = \left(\kappa^{mn}\left[t_n^2, \left[[t_m^2\Sigma, \epsilon\lambda_1], \tilde{\epsilon}\lambda_3\right]\right] - \kappa^{mn}\left[t_n^2, \left[[t_m^2\Sigma, \epsilon\lambda_3], \tilde{\epsilon}\lambda_1\right]\right]\right)$$

$$= \left(\frac{5}{2}\left[\epsilon\lambda_1, \tilde{\epsilon}\lambda_3\right] - \frac{5}{2}\left[\epsilon\lambda_3, \tilde{\epsilon}\lambda_1\right]\right)$$

$$+ \left(\kappa^{mn}\left[\left[[t_m^2\Sigma, \epsilon\lambda_1\right], \left[t_n^2, \tilde{\epsilon}\lambda_3\right]\right] - \kappa^{mn}\left[\left[[t_m^2\Sigma, \epsilon\lambda_3\right], \left[t_n^2, \tilde{\epsilon}\lambda_1\right]\right]\right)$$

$$= \left(\kappa^{mn}\left[\left[[t_m^2\Sigma, \epsilon\lambda_1\right], \left[t_n^2, \tilde{\epsilon}\lambda_3\right]\right] - \kappa^{mn}\left[\left[[t_m^2\Sigma, \epsilon\lambda_3\right], \left[t_n^2, \tilde{\epsilon}\lambda_1\right]\right]\right)$$

$$= -\frac{3}{2}\left[\epsilon\lambda_1, \tilde{\epsilon}\lambda_3\right] + \frac{3}{2}\left[\epsilon\lambda_3, \tilde{\epsilon}\lambda_1\right] = 0 \tag{3.6.41}$$

In this derivation, we used the identities

$$\begin{aligned}\kappa^{mn}\left[t_n^2, \left[t_m^2\Sigma, \epsilon\lambda_1\right]\right] &= \frac{\Sigma}{2}\kappa^{mn}\left[\left\{t_m^2, t_n^2\right\}, \epsilon\lambda_1\right] \\ &= \kappa^{mn}\kappa_{mn}\frac{\Sigma}{8}\left[\Sigma, \epsilon\lambda_1\right] \\ &= \frac{1}{4}\kappa^{mn}\kappa_{mn}\,\epsilon\lambda_1 \\ &= \frac{5}{2}\epsilon\lambda_1\end{aligned} \tag{3.6.42a}$$

together with

$$\kappa^{mn}\left[\left[t_m^2\Sigma, \epsilon\lambda_1\right], \left[t_n^2, \tilde{\epsilon}\lambda_3\right]\right] = -\frac{3}{2}\left[\epsilon\lambda_1, \tilde{\epsilon}\lambda_3\right] \tag{3.6.42b}$$

and

$$-\kappa^{mn}\left[\left[t^2_m\Sigma,\epsilon\lambda_3\right],\left[t^2_n,\tilde\epsilon\lambda_1\right]\right]=\frac{3}{2}\left[\epsilon\lambda_3,\tilde\epsilon\lambda_1\right] \tag{3.6.42c}$$

Contribution of the diagram of Figure 3.3 There only remains the contractions that get contributions from the interaction vertices:

$$\frac{1}{2\pi R^4}\frac{1}{\mathrm{str}(\lambda_3\lambda_1)}\mathrm{str}\left(\epsilon\lambda_1\left[\partial x\Sigma,\partial\widehat{\theta}\right]\right)(z)\left[\widehat{\theta},\tilde\epsilon\left(\lambda_3-\lambda_1\right)\right](\mathrm{w})\int \mathrm{d}^2u\ \mathrm{str}\left(\partial x\left[\theta,\overline{\partial}\theta\right]\right) \tag{3.6.43}$$

We use:

$$\int \mathrm{d}^2u\,\frac{1}{(z-u)^3}\frac{1}{(\bar{\mathrm{w}}-\bar u)}=\frac{\pi}{2}\frac{1}{(z-\mathrm{w})^2} \tag{3.6.44}$$

and obtain:

$$-\kappa^{mn}\kappa^{\dot B\beta}\kappa^{\alpha\dot A}\mathrm{str}\left(\epsilon\lambda_1\left[t^2_m\Sigma,t^1_{\dot\beta}\right]\right)\left\{t^1_{\dot\alpha},\tilde\epsilon(\lambda_3-\lambda_1)\right\}\mathrm{str}\left(t^2_n\left\{t^3_\beta,t^3_\alpha\right\}\right). \tag{3.6.45}$$

We temporarily do not write the factor of $1/4R^2\mathrm{str}(\lambda_3\lambda_1)$ since it only observes the calculation. This answer can be rewritten as

$$-\kappa^{mn}\kappa^{\dot B\beta}\kappa^{\alpha\dot A}\mathrm{str}\left(\epsilon\lambda_1\left[t^2_m\Sigma,t^1_{\dot\beta}\right]\right)\left\{t^1_{\dot\alpha},\tilde\epsilon(\lambda_3-\lambda_1)\right\}\mathrm{str}\left(t^2_n\left\{t^3_\beta,t^3_\alpha\right\}\right)=$$

$$-\kappa^{mn}\kappa^{\dot B\beta}\kappa^{\alpha\dot A}\mathrm{str}\left([\epsilon\lambda_1,t^2_m\Sigma]t^1_{\dot\beta}\right)\left\{t^1_{\dot\alpha},\tilde\epsilon(\lambda_3-\lambda_1)\right\}\mathrm{str}\left([t^2_n,t^3_\beta]t^3_\alpha\right)=$$

$$-\kappa^{mn}\kappa^{\dot B\beta}\mathrm{str}\left([\epsilon\lambda_1,t^2_m\Sigma]t^1_{\dot\beta}\right)\left[[t^2_n,t^3_\beta],\tilde\epsilon(\lambda_3-\lambda_1)\right]=$$

$$-\kappa^{mn}\left[\left[t^2_n,[\epsilon\lambda_1,t^2_m\Sigma]\right],\tilde\epsilon(\lambda_3-\lambda_1)\right]=$$

$$+\kappa^{mn}\left[\left[t^2_n,[t^2_m\Sigma,\epsilon\lambda_1]\right],\tilde\epsilon(\lambda_3-\lambda_1)\right]=\frac{5}{2}\left[\epsilon\lambda_1,\tilde\epsilon(\lambda_3-\lambda_1)\right]=$$

$$\frac{5}{2}\left[\epsilon\lambda_1,\tilde\epsilon\lambda_3\right]=\frac{5}{2}\left[\epsilon\lambda_3,\tilde\epsilon\lambda_1\right] \tag{3.6.46}$$

to give the contribution – with all factors restored –

$$\frac{5}{8R^4}\frac{g^{-1}\left[\epsilon\lambda_3,\tilde\epsilon\lambda_1\right]g}{\mathrm{str}(\lambda_3\lambda_1)}. \tag{3.6.47}$$

Notice that in deriving equation (3.6.46) we used identity (3.6.42a). To summarize, the contribution of Figure 3.3 is given by equation (3.6.47).

Anti-holormophic b-ghost One can compute the contribution of $\bar{b}_{\bar{z}\bar{z}}(\bar{z})$ in the same way and it gives

$$\epsilon \bar{b}_0 V[\tilde{\epsilon}](\mathrm{w}) = \frac{5}{8R^4} \frac{g^{-1}\Big[\epsilon\lambda_3, \tilde{\epsilon}\lambda_1\Big] g}{\mathrm{str}(\lambda_3\lambda_1)} \tag{3.6.48}$$

Final answer Combining the three diagrams we arrive at

$$\epsilon \left(b_0 - \bar{b}_0\right) V[\tilde{\epsilon}] = 0 \tag{3.6.49}$$

for the current vertex.

Beta-deformation Vertex.

In order to finish the calculation, we only have to compute contractions where the b-ghost hits both V in $V \wedge V$. These mixed contractions are given by the diagrams below:

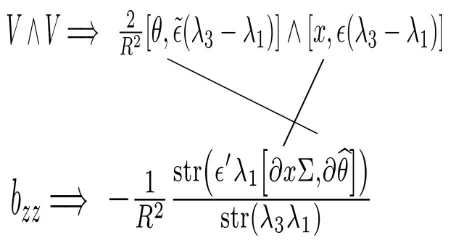

Figure 3.4: Disconnected contractions for the OPE between b_{zz} and $V[\tilde{\epsilon}] \wedge V[\epsilon]$.

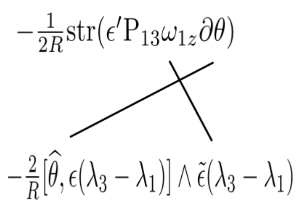

Figure 3.5: Disconnected contractions for the OPE between b_{zz} and $V[\tilde{\epsilon}] \wedge V[\epsilon]$.

We stress that there are no contributions from the action up to 1-loop.

Contribution of diagram in figure 3.4 The diagram in figure 3.4 contributes as

$$-\kappa^{\dot{A}\alpha}\kappa^{mn} \frac{2}{R^4 \mathrm{str}\,(\lambda_3\lambda_1)} \mathrm{str}\,\left(\epsilon'\lambda_1\,\left[t^2_m\Sigma, t^1_{\dot{\alpha}}\right]\right)\,\{t^3_\alpha, \tilde{\epsilon}(\lambda_3 - \lambda_1)\} \wedge [t^2_n, \epsilon(\lambda_3 - \lambda_1)] \tag{3.6.50}$$

And this result can be simplified to:

$$-\frac{2\kappa^{\dot{A}\alpha}\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\mathrm{str}\left(\epsilon'\lambda_1\left[t^2_m\Sigma,t^1_{\dot{\alpha}}\right]\right)\,\{t^3_\alpha,\tilde{\epsilon}(\lambda_3-\lambda_1)\}\wedge[t^2_n,\epsilon(\lambda_3-\lambda_1)]=$$

$$-\frac{2\kappa^{\dot{A}\alpha}\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\mathrm{str}\left([\epsilon'\lambda_1,t^2_m\Sigma]t^1_{\dot{\alpha}}\right)\,\{t^3_\alpha,\tilde{\epsilon}(\lambda_3-\lambda_1)\}\wedge[t^2_n,\epsilon(\lambda_3-\lambda_1)]=$$

$$-\frac{2\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\left[[\epsilon'\lambda_1,t^2_m\Sigma],\tilde{\epsilon}(\lambda_3-\lambda_1)\right]\wedge\left[t^2_n,\epsilon(\lambda_3-\lambda_1)\right]=$$

$$-\frac{2\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\left[[\epsilon'\lambda_1,t^2_m\Sigma],\tilde{\epsilon}\lambda_3\right]\wedge\left[t^2_n,\epsilon(\lambda_3-\lambda_1)\right]\tag{3.6.51}$$

Contribution of diagram in figure 3.5 Likewise, we obtain:

$$\frac{1}{R^2}\mathrm{str}\,(\epsilon'\mathrm{P}_{13}\omega_{1z}\partial\theta)\,[\widehat{\theta},\tilde{\epsilon}(\lambda_3-\lambda_1)]\wedge\epsilon(\lambda_3-\lambda_1)=$$

$$-\frac{1}{R^2}\kappa^{\alpha\dot{A}}\mathrm{str}\left(\epsilon'\mathrm{P}_{13}\omega_{1z}t^3_\alpha\right)\{t^1_{\dot{\alpha}},\tilde{\epsilon}(\lambda_3-\lambda_1)\}\wedge\epsilon(\lambda_3-\lambda_1)=$$

$$\frac{1}{R^4}\kappa^{\alpha\dot{A}}\kappa^{\dot{B}\beta}\mathrm{str}\left(\epsilon'\mathrm{P}_{13}t^1_{\dot{\beta}}t^3_\alpha\right)\{t^1_{\dot{\alpha}},\tilde{\epsilon}(\lambda_3-\lambda_1)\}\wedge\epsilon t^3_\beta=$$

$$\frac{1}{R^4}\kappa^{\dot{B}\beta}\left[\epsilon'\mathrm{P}_{13}t^1_{\dot{\beta}},\tilde{\epsilon}(\lambda_3-\lambda_1)\right]\wedge\epsilon t^3_\beta=$$

$$\frac{1}{R^4}\kappa^{\dot{B}\beta}\left[\epsilon'\mathrm{P}_{13}t^1_{\dot{\beta}},\tilde{\epsilon}\lambda_3\right]\wedge\epsilon t^3_\beta=$$

$$\frac{1}{R^4}\kappa^{\dot{B}\beta}\left[\epsilon't^1_{\dot{\beta}},\tilde{\epsilon}\lambda_3\right]\wedge\epsilon t^3_\beta\tag{3.6.52}$$

Holormorphic b-ghost The sum of these contribution gives us:

$$\epsilon'b_0V[\tilde{\epsilon}]\wedge V[\epsilon]=-\frac{2\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\left[[\epsilon'\lambda_1,t^2_m\Sigma],\tilde{\epsilon}\lambda_3\right]\wedge\left[t^2_n,\epsilon(\lambda_3-\lambda_1)\right]$$

$$+\frac{1}{R}\kappa^{\dot{B}\beta}\left[\epsilon't^1_{\dot{\beta}},\tilde{\epsilon}\lambda_3\right]\wedge\epsilon t^3_\beta\tag{3.6.53}$$

Anti-holomorphic b-ghost The same can be done for the anti-holomorphic b-ghost, and we obtain

$$\epsilon'\bar{b}_0 V[\tilde{\epsilon}]\wedge V[\epsilon] = -\frac{2\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\left[\left[\epsilon'\lambda_3, t^2_m\Sigma\right], \tilde{\epsilon}\lambda_1\right]\wedge\left[t^2_n, \epsilon(\lambda_3-\lambda_1)\right]$$

$$+\frac{1}{R}\kappa^{\beta\dot{B}}\left[\epsilon' t^3_\beta, \tilde{\epsilon}\lambda_1\right]\wedge\epsilon t^1_{\dot{\beta}} \tag{3.6.54}$$

Final answer

The sum of all contributions from the current and the mixed contractions gives us the final answer:

$$\epsilon'\left(b_0-\bar{b}_0\right)V[\tilde{\epsilon}]\wedge V[\epsilon] =$$

$$\frac{-2\kappa^{mn}}{R^4\mathrm{str}\,(\lambda_3\lambda_1)}\left(\left[\left[\epsilon'\lambda_1, t^2_m\Sigma\right], \tilde{\epsilon}\lambda_3\right]\wedge\left[t^2_n, \epsilon(\lambda_3-\lambda_1)\right]-\left[\left[\epsilon'\lambda_3, t^2_m\Sigma\right], \tilde{\epsilon}\lambda_1\right]\wedge\left[t^2_n, \epsilon(\lambda_3-\lambda_1)\right]\right)$$

$$+\frac{1}{R^4}\left(\kappa^{\dot{B}\beta}\left[\epsilon' t^1_{\dot{\beta}}, \tilde{\epsilon}\lambda_3\right]\wedge\epsilon t^3_\beta-\kappa^{\beta\dot{B}}\left[\epsilon' t^3_\beta, \tilde{\epsilon}\lambda_1\right]\wedge\epsilon t^1_{\dot{\beta}}\right) \tag{3.6.55}$$

3.7 BRST triviality of $f_{ab}{}^c W_c$

The projectors **P** were used in [32] to prove that BRST triviality of the ghost number 1 vertices corresponding to the global symmetries. Once we allow denominators, the BRST cohomology is zero anyway. But in highly supersymmetric backgrounds, it is meaningful to ask to which extent resolving $Q\phi=\psi$ preserves the global supersymmetries. The ghost number 1 vertex for a global symmetry $t_a\in\mathbf{psu}(\mathbf{2},\mathbf{2}|\mathbf{4})$ is:

$$W_a(\epsilon)=\left(g^{-1}(\epsilon\lambda_3-\epsilon\lambda_1)g\right)_a \tag{3.7.1}$$

for a Grassmann odd constant parameter[9] ϵ. It was proven in [32] that

$$\begin{aligned} f_{ab}{}^c W_c &= -\epsilon Q X_{ab} = -\epsilon Q X_{[ab]} \quad \texttt{where} \\ X_{ab} &= \mathrm{Str}\left(g t_a g^{-1}\ \left((g t_b g^{-1})_{\bar{3}}+2(g t_b g^{-1})_{\bar{2}}+3(g t_b g^{-1})_{\bar{1}}-4\mathbf{P}_{13}(g t_b g^{-1})_{\bar{1}}\right)\right)\end{aligned} \tag{3.7.2}$$

where f^c_{ab} are the structure constants of $\mathbf{psu}(2,2|4)$. This implies that $f_{ab}{}^c W_c$ is Q-exact in a way preserving symmetries. However, W_c cannot be obtained from $f_{ab}{}^c W_c$ preserving symmetries. (Notice that $f_{ab}{}^c f^{ab}_d=0$.) In this sense, $f_{ab}{}^c W_c$ is BRST-exact but W_c is not.

Notice that:

$$X_{[ab]} = \left\langle t_a\wedge t_b\ ,\ g^{-1}Ag\right\rangle \tag{3.7.3}$$

$$\texttt{where}\ A = -2k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}} = \tag{3.7.4}$$

$$= 2k^{\alpha\dot{\alpha}}t^3_\alpha\wedge t^1_{\dot{\alpha}}+8\frac{k^{\alpha\dot{\alpha}}t^3_\alpha\wedge[\overline{\{\lambda_1,t^1_{\dot{\alpha}}\}_{\mathrm{STL}}}\,,\,\lambda_3]}{\mathrm{STr}\lambda_1\lambda_3} =$$

$$= 2k^{\alpha\dot{\alpha}}t^3_\alpha\wedge t^1_{\dot{\alpha}}+8\frac{[\lambda_1,t^2_m]\wedge[t^2_{-m},\lambda_3]}{\mathrm{STr}\lambda_1\lambda_3} \tag{3.7.5}$$

[9] As usual in supergeometry, we use a sufficiently large pool of constant fermionic parameters

In other words, in the covariant complex (see Section 3.5.2, Eq. (3.5.10)):

$$Q\left(k^{\alpha\dot{\alpha}}t^3_\alpha \wedge (\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right) = d_{\mathrm{Lie}}(\lambda_3-\lambda_1) \tag{3.7.6}$$

where d_{Lie} is defined in Section 3.5.3.

Relation to the "minimalistic action" We will now explain that Eq. (3.7.6) is equivalent to the BV Master Equation for the minimalistic action of [59]. Let us consider the scalar product, as defined in Section 3.5.3, with $J_3\wedge J_1$:

$$\left\langle J_3\wedge J_1\;,\;k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right\rangle \;= \tag{3.7.7}$$

$$= \mathrm{STr}\left(J_1t^3_\alpha\right)k^{\alpha\dot{\alpha}}\wedge\;\mathrm{STr}\left(t^1_{\dot{\alpha}}(\mathbf{1}-2\mathbf{P}_{31})J_3\right)\;= \tag{3.7.8}$$

$$= \mathrm{STr}\left(J_1\wedge(\mathbf{1}-2\mathbf{P}_{31})J_3\right) \tag{3.7.9}$$

$$\epsilon Q\left\langle J_3\wedge J_1\;,\;k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right\rangle \;= \tag{3.7.10}$$

$$= \;\left\langle[\epsilon\lambda_1,J_2]\wedge J_1 + J_3\wedge[\epsilon\lambda_3,J_2]\;,\;k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right\rangle +$$

$$+\;\left\langle -\epsilon D_0\lambda_3\wedge J_1 - J_3\wedge\epsilon D_0\lambda_1\;,\;k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right\rangle$$

The first line of the RHS of Eq. (3.7.10) equals to (in the sense of Section 3.5.2, Eq. (3.5.10)):

$$-\left\langle J_2\wedge J_1 + J_3\wedge J_2\;,\;\epsilon Q\left(k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right)\right\rangle \;=$$

$$= \left\langle J_2\wedge J_1 + J_3\wedge J_2\;,\;d_{\mathrm{Lie}}(\epsilon\lambda_3-\epsilon\lambda_1)\right\rangle \;= \tag{3.7.11}$$

$$= \left\langle [J_2,J_1]+[J_3,J_2]\;,\;\epsilon\lambda_3-\epsilon\lambda_1\right\rangle \;=\; \mathrm{STr}\left([J_3,J_2]\epsilon\lambda_3 - [J_2,J_1]\epsilon\lambda_1\right)$$

The second line of the RHS of Eq. (3.7.10) is:

$$\left\langle -\epsilon D_0\lambda_3\wedge J_1 - J_3\wedge\epsilon D_0\lambda_1\;,\;k^{\alpha\dot{\alpha}}t^3_\alpha\wedge(\mathbf{1}-2\mathbf{P}_{13})t^1_{\dot{\alpha}}\right\rangle \;= \tag{3.7.12}$$

$$= \left\langle -\epsilon D_0\lambda_3\wedge J_1 - J_3\wedge\epsilon D_0\lambda_1\;,\;k^{\alpha\dot{\alpha}}t^3_\alpha\wedge t^1_{\dot{\alpha}}\right\rangle \;=$$

$$= \mathrm{STr}\left((D_0\lambda_1)J_3 - (D_0\lambda_3)J_1\right) \tag{3.7.13}$$

The sum is a total derivative:

$$Q\mathrm{STr}\left(J_1\wedge(\mathbf{1}-2\mathbf{P}_{31})J_3\right) \;=\; d\mathrm{STr}\left((\lambda_3-\lambda_1)J\right) \tag{3.7.14}$$

This shows that Eq. (3.5.36) is Q-invariant.

3.8 MATHEMATICA code

MATHEMATICA code for computations in $AdS_5\times S^5$ sigma-model is **available on GitHub**.

Chapter 4

On the Spectrum and Spacetime Supersymmetry of Heterotic Ambitwistor String

4.1 Outline.

Sections 4.2 and 4.3 contain a mini review on the Ramond sector and the ambitwistor model. Their main purpose is to set our notation and make our presentation self-contained. It doesn't contain anything new, and it can be skipped for those who already know the subject.

We start in section 4.4, where we use the standard BRST method to compute the equations of motion of the Ramond sector for the heterotic system. These represent the fermionic degrees of freedom of the theory, and our analysis shows that they also follow non-unitary equations of motion. We write a gauge-invariant version of theory in terms of Fronsdal fields [68]. The kinetic term of the fermionic ambitwistor string field theory action is also computed in section 4.5. It is expressed in terms of gauge-invariant objects and resembles Fronsdal's free action despite having more derivatives.

Finally, in section 4.6 we write the supersymmetry transformations of the system. In RNS language, the supersymmetry operator is defined on-shell and thus gives the supersymmetry transformations up to equations of motion. Then we prove the invariance of the action under supersymmetry transformations.

4.2 Ramond sector, cocycles and Gamma matrices.

Spinor indices in 10 dimensions can be distinguished between chiral and anti-chiral. We denote chiral indices by undotted greek letters, α, while anti-chiral indices are represented by dotted greek letters, $\dot{\alpha}$. Both run from 1 to 16. Spinor indices are 5-dimensional vector representations of $u(5)$:

$$\dot{\alpha} = \frac{1}{2}\begin{pmatrix} - & - & - & - & - \\ - & - & - & + & + \\ - & + & + & + & + \end{pmatrix} \quad \text{and} \quad \beta = \frac{1}{2}\begin{pmatrix} + & + & + & + & + \\ + & + & + & - & - \\ + & - & - & - & - \end{pmatrix}. \tag{4.2.1}$$

where an anti-chiral index, $\dot{\alpha}$, must have an even number of plus signs, and a chiral index, β, must have an odd number of plus signs. Each of these combinations has 16 independent components represented as $\mathbf{16} = \mathbf{1} + \mathbf{10} + \mathbf{5}$.

4.2.1 The Ramond Sector.

The Ramond sector of the Ambitwistor string is defined by the antiperiodic boundary conditions of ψ^m:

$$\psi^m(e^{2\pi i}z) = -\psi^m(z). \tag{4.2.2}$$

We follow [69] and implement these boundary conditions via spin fields. That is, we have a conformal primary $S(z)$ that twists a periodic ψ:

$$\psi^m\left(z + (w-z)e^{2\pi i}\right) S(z) = -\psi^m(w)\, S(z). \tag{4.2.3}$$

This implies that a state $|\alpha\rangle$ created from the vacuum $|0\rangle$ via

$$|\alpha\rangle = S^\alpha(0)|0\rangle \tag{4.2.4}$$

should transform as a spacetime spinor. Notice that, due to the presence of S forcing ψ to be in the Ramond sector, this state must belong to an irreducible representation of the zero-mode Clifford algebra of ψ^m: $\{\psi_0^m, \psi_0^n\} = \eta^{mn}$, which implies

$$\psi_0^m|\alpha\rangle = \frac{1}{\sqrt{2}}\Gamma^{m\,\alpha}_{\dot\beta}|\dot\beta\rangle. \tag{4.2.5}$$

4.2.2 Bosonization and cocycles.

Because S^α twists the boundary conditions of ψ^m, the system is not free and OPE's are difficult to compute. Bosonization is a technique that allows us to deal with free fields only. Bosonization assigns for a pair of complex fermions one chiral boson, which means that we have to break manifest $so(10)$ invariance down to $u(5)$.

Spin Fields. The bosonization of spin fields is given by

$$S^\alpha(z) = \exp\Big(\alpha\cdot\phi(z)\Big)c_\alpha \tag{4.2.6}$$

where α is a chiral spinor index. The same expression is valid for anti-chiral spin fields by just replacing α for $\dot\alpha$. The factor c_α is a cocycle phase that guarantees the correct anticommutation relations.

Cocycles. The anticommuting fermionic algebra is reproduced in the bosonic system via the Baker-Campbell-Hausdorff formula:

$$e^{\phi(z)}e^{\pm\phi(z')} = e^{\pm\phi(z')}e^{\mp\phi(z')}e^{\phi(z)}e^{\pm\phi(z')} = -e^{\pm\phi(z')}e^{\phi(z)} \tag{4.2.7}$$

provided for $|z'| = |z|$ we have

$$\Big[\phi(z'),\phi(z)\Big] = \pm i\pi \quad \texttt{which implies} \quad \phi(z)\phi(0)\sim\ln z \tag{4.2.8}$$

Now, if we are given more than one pair of fermions, they won't naturally anticommute because $[\phi_i,\phi_j] = 0$. This is corrected by the introduction of cocycles [70]:

- Order all bosons of the theory: ϕ_i where $i = 1,\ldots,N$;

- Then multiply each exponential by a factor $(-)^{N_1+\ldots+N_{i-1}}$, where N_i is the fermion number operator:

$$N_i = -\oint \frac{\mathrm{d}z}{2\pi i}\,\overline{\psi}_i\psi_i = \oint \frac{\mathrm{d}z}{2\pi i}\,\partial\phi_i. \tag{4.2.9}$$

For example, if we consider two pairs of fermions, the bosonization becomes

$$\psi_1 = e^{\phi_1}, \quad \overline{\psi}_1 = e^{-\phi_1} \tag{4.2.10}$$

with

$$\psi_2 = e^{\phi_2}(-)^{N_1}, \quad \overline{\psi}_2 = e^{-\phi_2}(-)^{N_1} \tag{4.2.11}$$

where now ψ_1 and ψ_2 anticommute

$$e^{\phi_1}e^{\phi_2}(-)^{N_1} = e^{\phi_2}e^{\phi_1}(-)^{N_1} = e^{\phi_2}(-)^{N_1}(-)^{-N_1}e^{\phi_1}(-)^{N_1} = -e^{\phi_2}(-)^{N_1}e^{\phi_1} \tag{4.2.12}$$

provided

$$\left[N_i, e^{n\phi_j}\right] = n\delta_{ij}e^{n\phi_j}. \tag{4.2.13}$$

Thus, for more than one pair of fermions, we need to introduce the cocycle phase factors:

$$c_i = (-)^{N_1+\ldots+N_{i-1}}. \tag{4.2.14}$$

Consider the vector

$$\partial\phi = (N_1, N_2, \ldots, N_5) \tag{4.2.15}$$

then the cocycle factor can be written as

$$c_{\pm e_i} = \exp\left[\pm i\pi\langle e_i M \partial\phi\rangle\right] \tag{4.2.16}$$

where e_i is 1 in the ith component and zero elsewhere, $\langle\ \rangle$ is a matrix inner product and M is a lower triangular matrix with entries ±1:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

The signs of M are arbitrary at this point, but they can be specified studying the charge conjugation matrix [70].

The cocycle factors of spin fields, c_α and $c_{\dot\alpha}$, are given the following expressions:

$$c_\alpha = \exp\left[i\pi\langle \alpha M\partial\phi\rangle\right] \quad \texttt{and} \quad c_{\dot\alpha} = \exp\left[i\pi\langle \dot\alpha M\partial\phi\rangle\right] \tag{4.2.17}$$

Gamma Matrices. To motivate the construction of gamma matrices and show how cocycles work, let us consider the OPE between ψ^i and S^α. Using expressions (4.3.4) and (4.2.6) we have to compute the OPE of $e^{\phi_i(z)}c_i$ with $e^{\alpha\phi(\mathrm{w})}c_\alpha$. Notice that c_i will pass through $e^{\alpha\phi}$ and due to Baker-Campbell-Hausdorff we obtain an extra phase:

$$c_i e^{\alpha\phi} = e^{i\pi\langle e_i M \partial\phi\rangle} e^{\alpha\phi} = e^{i\pi\langle e_i M\alpha\rangle} e^{\alpha\phi} c_i \tag{4.2.18}$$

so that our OPE becomes

$$e^{\phi_i(z)}c_i \; e^{\alpha\phi(\mathrm{w})}c_\alpha \sim (z-\mathrm{w})^{\alpha\cdot e_i}\, e^{i\pi\langle e_i M\alpha\rangle} e^{(e_i+\alpha)\phi} c_{i+\alpha}. \tag{4.2.19}$$

Notice that we obtain a branch-cut if $\alpha\cdot e_i = \alpha_i = -1/2$ which in turn implies that the sum $e_i+\alpha$ must be an anti-chiral index $\dot\beta$. Therefore given

$$e^{\phi_i(z)}c_i \; e^{\alpha\phi(\mathrm{w})}c_\alpha \sim (z-\mathrm{w})^{-1/2}\, e^{i\pi\langle e_i M\alpha\rangle} e^{\dot\beta\phi} c_{\dot\beta}, \tag{4.2.20}$$

we see that it becomes natural to define the gamma matrices as

$$(\Gamma^j)_{\dot\alpha}^{\;\beta} = \sqrt{2}\delta\left(e_j+\beta-\dot\alpha\right) e^{i\pi\langle e_j M\dot\alpha\rangle} \tag{4.2.21a}$$

and

$$(\Gamma^j)_{\alpha}^{\;\dot\beta} = \sqrt{2}\delta\left(e_j+\dot\beta-\alpha\right) e^{i\pi\langle e_j M\alpha\rangle} \tag{4.2.21b}$$

giving us the final result:

$$\psi^i(z)S^\alpha(\mathrm{w}) \sim \frac{1}{\sqrt{2}}\frac{\Gamma^{i\,\alpha}_{\;\dot\beta} S^{\dot\beta}(\mathrm{w})}{(z-\mathrm{w})^{1/2}}. \tag{4.2.22}$$

The explicit representation is written in terms of the Pauli-matrices via

$$\Gamma^{\pm e_j} = (\pm i)^{j-1}\sqrt{2}\left(\sigma^3\otimes\right)^{j-1}\sigma^{\mp}\left(\otimes 1\right)^{5-j} \tag{4.2.23}$$

and one can convert between $u(5)$ and covariant $so(10)$ using

$$\Gamma^{2j-1} = \frac{1}{\sqrt{2}}\left(\Gamma^{e_j}+\Gamma^{-e_j}\right) \tag{4.2.24a}$$

and

$$\Gamma^{2j} = \frac{i}{\sqrt{2}}\left(\Gamma^{e_j}-\Gamma^{-e_j}\right) \tag{4.2.24b}$$

Notice that in our construction, the notation γ^μ is reserved for the symmetric gamma matrices:

$$\gamma^\mu_{\alpha\beta} = \Gamma^{\mu\,\dot\beta}_{\alpha} C_{\dot\beta\beta} \tag{4.2.25a}$$

$$\gamma^{\mu\,\alpha\beta} = \Gamma^{\mu\,\alpha}_{\dot\beta} C^{\dot\beta\beta} \tag{4.2.25b}$$

as it is common in the literature. In above equations, C denotes the charge conjugation matrix which is the next topic in our discussion.

Charge Conjugation Matrix. We define C as

$$C^{\beta\dot{\beta}} = \delta\left(\beta + \dot{\beta}\right) e^{i\pi\beta M\dot{\beta}} \tag{4.2.26a}$$

and

$$C^{\dot{\beta}\beta} = -\delta\left(\dot{\beta} + \beta\right) e^{i\pi\dot{\beta}M\beta} \tag{4.2.26b}$$

and with these convetions we have $C^{\beta\dot{\beta}} = C^{\dot{\beta}\beta}$. These expressions can be motivated by studying the OPE of S^{α} and $S^{\dot{\beta}}$.

It is also common to use only undotted indices when describing spinors in 10d. Charge matrices act as metrics on the spinor space and can remove all dotted indices. For us all spinors are defined with upper indices and then anti-chiral ones are written as

$$S_{\beta} = C_{\beta\dot{\beta}} S^{\dot{\beta}}. \tag{4.2.27}$$

This notation is used together with the symmetric gamma representation.

4.3 Ambitwistor Action and Ramond Sector.

We first review the ambitwistor model. Its main purpose is to set the basic definitions and notation.

The heterotic ambitwistor model is defined by the free action

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z\Big(\mathrm{p}_m\bar{\partial}x^m + \psi_m\bar{\partial}\psi^m + b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c} + \beta\bar{\partial}\gamma + S_{\mathrm{j}}\Big) \tag{4.3.1}$$

where p_m is a worldsheet holomorphic one-form and x^m is an holomorphic coordinate function. The b and c fields together with β and γ are the Faddeev-Poppov ghosts of superconformal worldsheet symmetry. Particular to the heterotic model, we have the current action S_{j}; its specific form is irrelevant for us, we only require the existence of a current j^a with conformal weight 1 that satisfies the OPE

$$\mathrm{j}^a(z)\mathrm{j}^b(w) \sim \frac{\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_c\mathrm{j}^c(w)}{(z-w)}, \tag{4.3.2}$$

being f^{ab}_c the structure constants of the Lie algebra in question. The Ambitwistor model differs from the superstring due to the presence of the $\tilde{b}$ and $\tilde{c}$ ghosts related to the gauge symmetries of the light-cone constrain: $\mathrm{p}^2 = 0$. These ghosts have conformal weights 2 and -1 respectively and both are worldsheet fermions.

Our Majorana spinors ψ^m will be rewritten in the complex linear combinations:

$$\psi^{\pm i} = \frac{1}{\sqrt{2}}\left(\psi^{2i-1} \mp \psi^{2i}\right) \tag{4.3.3}$$

for $i = 1, \ldots, 5$ that are subsequently bosonized to

$$\psi^{\pm i}(z) = \exp\Big(\pm\phi_i(z)\Big)c_{\pm e_i} \tag{4.3.4}$$

with ϕ's satisfying

$$\phi_i(z)\phi_j(\mathrm{w}) \sim +\delta_{ij}\ln(z-\mathrm{w}) \tag{4.3.5}$$

The (β,γ) system is bosonized with extra fermions (ξ,η) [69], both primaries of conformal weight 0 and 1 respectively:

$$\beta = \partial\xi e^{-\phi_6} c_{e_6} \quad \text{and} \quad \gamma = \eta e^{\phi_6} c_{e_6}. \tag{4.3.6}$$

This choice follows the conventions of [70] and [71] where we have introduced the cocycles c_{e_i} and c_{e_6}. During the computation of cohomology, cocycle factors are important and must be taken into account. The definition of cocycles depends on the way we order the different ϕ_i. For us the chiral bosons corresponding to ψ^m are ordered from 1 to 5 while the boson coming from the $\beta\gamma$ system is labeled as 6. The sixth boson has OPE:

$$\phi_6(z)\phi_6(\mathrm{w}) \sim -\ln(z-\mathrm{w}) \tag{4.3.7}$$

while (ξ,η) form a free system:

$$\xi(z)\eta(\mathrm{w}) \sim \frac{1}{(z-\mathrm{w})} \tag{4.3.8}$$

The symmetries of this action are encoded in the following BRST charge:

$$Q = \oint \frac{dz}{2\pi i}\left[c\Big(\mathrm{T}_{\text{matter}} + T_{\tilde{b}\tilde{c}} + T_{\beta\gamma} + \mathrm{T}_{\mathrm{j}}\Big) + bc\partial c + \frac{1}{2}\tilde{c}\mathrm{p}^2 + \gamma \mathrm{p}^m\psi_m - \gamma^2\tilde{b}\right] \tag{4.3.9}$$

provided

$$\mathrm{T}_{\text{matter}} = -\mathrm{p}_m\partial x^m + \frac{1}{2}\sum_{i=1}^{5}\partial\phi_i\partial\phi_i, \qquad \mathrm{T}_{\tilde{b}\tilde{c}} = \tilde{c}\partial\tilde{b} - 2\tilde{b}\partial\tilde{c}, \tag{4.3.10a}$$

$$T_{\beta\gamma} = -\frac{1}{2}\partial\phi_6\partial\phi_6 - \partial^2\phi_6 - \eta\partial\xi, \quad \texttt{and} \quad \gamma^2 = \eta\partial\eta e^{+2\phi_6}. \tag{4.3.10b}$$

These are all the stress-energy tensors for $(x^m, \mathrm{p}_m, \psi^m)$, (β,γ) and $(\tilde{b},\tilde{c})$. We only require for the stress tensor of the current sector, T_{j}, that the following OPE is satisfied:

$$\mathrm{T}_{\mathrm{j}}(z)\mathrm{T}_{\mathrm{j}}(w) \sim \frac{c_j}{2(z-w)^4} + \frac{2\mathrm{T}_{\mathrm{j}}(w)}{(z-w)^2} + \frac{\partial\mathrm{T}_{\mathrm{j}}(w)}{(z-w)}. \tag{4.3.11}$$

Then, provided the central charge of the current system is 16, it is possible to show that $Q^2=0$ when the spacetime is 10-dimensional.

4.4 Cohomology.

In this section, we compute the ghost number 2 BRST cohomology of the Ambitwistor string for states in the Ramond sector. The cohomology of the Neveu-Schwartz sector has already been computed in [72].

We start by writing the most general vertex operator and the most general gauge parameter. Once all equations of motion and gauge transformations are obtained, we solve the algebraic gauge conditions to obtain a set of independent field equations.

4.4.1 Vertex operators.

States are defined by picture number $-1/2$ and ghost number 2 BRST cohomology. We define ghost and picture numbers by the expressions:

$$N_{\text{ghost}} = -\oint \frac{\mathrm{d}z}{2\pi i}\Big(bc + \tilde{b}\tilde{c} + \xi\eta\Big) \quad \text{and} \quad N_{\text{picture}} = \oint \frac{\mathrm{d}z}{2\pi i}\Big(\xi\eta - \partial\phi_6\Big). \tag{4.4.1}$$

Vertex Operator. The most general ghost number 2 and picture number $-1/2$ vertex operator that is annihilated by b_0 is given by the sum,

$$V_R = V_+ + V_-, \tag{4.4.2}$$

where V_+ and V_- are the $GSO(+)$ and $GSO(-)$ combinations. The $GSO(+)$ vertex operator is given by:

$$\begin{aligned} V_+ &= c\eta S^\alpha e^{\phi/2}\mathbf{A}_\alpha + \tilde{c}\eta S^\alpha e^{\phi/2}\mathbf{B}_\alpha + c\tilde{c}S^{\dot\alpha}e^{-\phi/2}\partial x^m \mathbf{C}_{m\dot\alpha} + c\tilde{c}S^{\dot\alpha}e^{-\phi/2}\mathrm{p}_m\mathbf{D}^m_{\dot\alpha} + c\tilde{c}S^{\dot\alpha}e^{-\phi/2}\mathrm{j}^a\mathbf{E}^a_{\dot\alpha} \\ &+ c\partial\tilde{c}S^{\dot\alpha}e^{-\phi/2}\mathbf{F}_{\dot\alpha} + c\tilde{c}S^{\dot\alpha}\partial e^{-\phi/2}\mathbf{G}_{\dot\alpha} + c\tilde{c}\psi_m(\not\psi S)^\alpha e^{-\phi/2}\mathbf{H}^m_\alpha + c\tilde{c}\partial\tilde{c}\partial\xi S^\alpha e^{-3\phi/2}\mathbf{I}_\alpha + \tilde{c}\partial\tilde{c}S^{\dot\alpha}e^{-\phi/2}\mathbf{J}_{\dot\alpha} \end{aligned} \tag{4.4.3}$$

while V_- is obtained from V_+ by changing the chirality of our spinors. Notice that the vertices $\psi^m\psi^n S^{\dot\alpha}$ and $\partial S^{\dot\alpha}$ have not been written. In bosonized form, these combinations are related to $\psi\not\psi S$ via field redefinitions [70]; there is no need to worry about them.

Gauge vertex. As for the gauge transformations, we parametrize them by ghost number 1 and picture number $-1/2$ vertex operators:

$$\Lambda = cS^{\dot\alpha}e^{-\phi_6/2}\lambda_{\dot\alpha} + \tilde{c}S^{\dot\alpha}e^{-\phi_6/2}\omega_{\dot\alpha} + c\tilde{c}\partial\xi S^\alpha e^{-3\phi_6/2}\mu_\alpha. \tag{4.4.4}$$

Both expressions (4.4.2) and (4.4.4) constitute the basic field content of BRST cohomology.

4.4.2 Equations of motion and gauge symmetries.

For clarity we consider only the $GSO(+)$ sector. The $GSO(-)$ is obtained by replacing chiral indices for anti-chiral and vice-versa. We present the equations of motion organized by ghost number as they were obtained from the OPE of Q and V_+. We also write the worldsheet operator that multiplies the resulting equation of motion.

- For $(2c, 1\tilde{c})$ multiplying $(S^{\dot\alpha}e^{-\phi_6/2}c\tilde{c}\partial^2 c)$:

$$+\frac{1}{2}\partial_m\mathbf{D}^m_{\dot\alpha} + \mathbf{F}_{\dot\alpha} - \frac{3}{8}\mathbf{G}_{\dot\alpha} - \frac{9}{4}(\Gamma^m)^\beta_{\dot\alpha}\mathbf{H}_{m\beta} = 0 \tag{4.4.5}$$

- For $(0c, 1\tilde{c})$ multiplying $(S^{\dot\alpha}e^{3\phi_6/2}\tilde{c}\eta\partial\eta)$:

$$+\mathbf{J}_{\dot\alpha} - \frac{i}{\sqrt{2}}(\Gamma^m)^\beta_{\dot\alpha}\partial_m\mathbf{B}_\beta = 0 \tag{4.4.6}$$

- For $(1c, 0\tilde{c})$ multiplying $(S^{\dot\alpha}e^{3\phi_6/2}c\eta\partial\eta)$:

$$-\frac{i}{\sqrt{2}}(\Gamma^m)_{\dot{\alpha}}^{\beta}\partial_m\mathbf{A}_\beta - \mathbf{G}_{\dot{\alpha}} + \mathbf{F}_{\dot{\alpha}} = 0 \quad (4.4.7)$$

- For $(1c, 2\tilde{c})$
  - multiplying $(S^\alpha e^{-\phi_6/2}c\tilde{c}\partial\tilde{c}\mathrm{p}_m)$:

$$-\frac{1}{2}\Box\mathbf{D}_{\dot{\alpha}}^m + \mathbf{C}_{\dot{\alpha}}^m - \partial^m\mathbf{F}_{\dot{\alpha}} - \frac{i}{\sqrt{2}}(\Gamma^m)_{\dot{\alpha}}^{\beta}\mathbf{I}_\beta = 0 \quad (4.4.8)$$

  - multiplying $(S^{\dot{\alpha}} e^{-\phi_6/2}c\tilde{c}\partial^2\tilde{c})$:

$$-\frac{1}{2}\partial^m\mathbf{C}_{m\dot{\alpha}} - \mathbf{J}_{\dot{\alpha}} = 0 \quad (4.4.9)$$

  - multiplying $(S^{\dot{\alpha}} e^{-\phi_6/2}c\tilde{c}\partial\tilde{c}\partial x^m)$:

$$-\frac{1}{2}\Box\mathbf{C}_{m\dot{\alpha}} - \partial_m\mathbf{J}_{\dot{\alpha}} = 0 \quad (4.4.10)$$

  - multiplying $(S^{\dot{\alpha}} e^{-\phi_6/2}c\tilde{c}\partial\tilde{c}\partial\phi_6)$:

$$+\frac{1}{4}\Box\mathbf{G}_{\dot{\alpha}} + \frac{1}{2}\mathbf{J}_{\dot{\alpha}} + \frac{i}{\sqrt{2}}(\Gamma^m)_{\dot{\alpha}}^{\beta}\partial_m\mathbf{I}_\beta = 0 \quad (4.4.11)$$

  - multiplying $(c\tilde{c}\partial\tilde{c}\psi^m(\psi S)^\alpha)$:

$$+\frac{1}{2}\Box\mathbf{H}_\alpha^m - \frac{i}{4\sqrt{2}}\partial_m\mathbf{I}_\alpha + \frac{i}{8\times 9\sqrt{2}}(\Gamma_m)_\alpha^{\dot{\beta}}(\partial\!\!\!/\mathbf{I})_{\dot{\beta}} + \frac{1}{9\times 4}(\Gamma_m)_\alpha^{\dot{\beta}}\mathbf{J}_{\dot{\beta}} = 0 \quad (4.4.12)$$

- For $1c, 1\tilde{c}$
  - multiplying $(S^\alpha e^{\phi_6/2}c\eta\partial\tilde{c})$:

$$-\frac{1}{2}\Box\mathbf{A}_\alpha + \mathbf{B}_\alpha + 2\mathbf{I}_\alpha - \frac{i}{\sqrt{2}}(\Gamma^m)_\alpha^{\dot{\beta}}\partial_m\mathbf{F}_{\dot{\beta}} = 0 \quad (4.4.13)$$

  - multiplying $(S^\alpha e^{\phi_6/2}c\tilde{c}\eta\partial x^m)$:

$$-\partial_m\mathbf{B}_\alpha + \frac{i}{\sqrt{2}}(\Gamma^n)_\alpha^{\dot{\beta}}\partial_n\mathbf{C}_{m\dot{\beta}} = 0 \quad (4.4.14)$$

  - multiplying $(S^\alpha e^{\phi_6/2}c\tilde{c}\eta\mathrm{p}_m)$:

$$-\partial^m\mathbf{A}_\alpha + \frac{i}{\sqrt{2}}(\Gamma^n)_\alpha^{\dot{\beta}}\partial_n\mathbf{D}_{\dot{\beta}}^m + \frac{i}{2\sqrt{2}}(\Gamma^m)_\alpha^{\dot{\beta}}\mathbf{G}_{\dot{\beta}} - \frac{i8}{\sqrt{2}}\mathbf{H}_\alpha^m - \frac{i}{\sqrt{2}}\mathbf{H}_{\beta n}(\Gamma^n)_{\dot{\alpha}}^{\beta}(\Gamma^m)_\alpha^{\dot{\alpha}} = 0 \quad (4.4.15)$$

– multiplying $(S^\alpha e^{\phi_6/2} c\tilde{c}\partial\eta)$:

$$-\mathbf{B}_\alpha + 3\mathbf{I}_\alpha + \frac{i}{\sqrt{2}}(\Gamma^m)_\alpha^{\dot\beta}\mathbf{C}_{m\dot\beta} - \frac{i}{2\sqrt{2}}(\Gamma^m)_\alpha^{\dot\beta}\partial_m\mathbf{G}_{\dot\beta} + \frac{8i}{\sqrt{2}}\partial^m\mathbf{H}_{\alpha m} + \frac{i}{\sqrt{2}}(\Gamma^n)_\alpha^{\dot\beta}(\Gamma^m)_{\dot\beta}^\tau\partial_n\mathbf{H}_{\tau m} = 0 \tag{4.4.16}$$

– multiplying $(S^\alpha e^{\phi_6/2} c\tilde{c}\eta\partial\phi_6)$:

$$\frac{1}{2}\mathbf{B}_\alpha + 4\mathbf{I}_\alpha + \frac{i}{\sqrt{2}}(\Gamma^m)_\alpha^{\dot\beta}\mathbf{C}_{m\dot\beta} - \frac{i}{\sqrt{2}}(\Gamma^m)_\alpha^{\dot\beta}\partial_m\mathbf{G}_{\dot\beta} + \frac{8i}{\sqrt{2}}\partial^m\mathbf{H}_{\alpha m} + \frac{i}{\sqrt{2}}(\Gamma^n)_\alpha^{\dot\beta}(\Gamma^m)_{\dot\beta}^\tau\partial_n\mathbf{H}_{\tau m} = 0 \tag{4.4.17}$$

– multiplying $\big(\eta c\tilde{c}\psi^m(\psi\!\!\!/ S)^{\dot\alpha} e^{\phi_6/2}\big)$:

$$+\frac{1}{36}(\Gamma_m)_{\dot\alpha}^\alpha\mathbf{B}_\alpha + \frac{i}{2\sqrt{2}}\left[\frac{1}{4}\partial_m\mathbf{G}_{\dot\beta} - \frac{1}{9\times 8}(\Gamma_m)_{\dot\alpha}^\tau(\partial\!\!\!/\mathbf{G})_\tau\right] - \frac{i}{\sqrt{2}}\left[\frac{1}{4}\mathbf{C}_{\dot\alpha m} - \frac{1}{9\times 8}(\Gamma_m)_{\dot\alpha}^\tau(\mathbf{C}\!\!\!/)_\tau\right]$$

$$-\frac{i}{\sqrt{2}}\left[-\frac{1}{4}(\Gamma^n)_{\dot\alpha}^\beta\partial_m\mathbf{H}_{\beta n} - (\Gamma^n)_{\dot\alpha}^\beta\partial_n\mathbf{H}_{\beta m} + \frac{1}{9}(\Gamma_m)_{\dot\alpha}^\beta\partial^n\mathbf{H}_{\beta n} + \frac{1}{9\times 8}(\Gamma_m)_{\dot\alpha}^\beta(\Gamma^l)_\beta^{\dot\beta}(\Gamma^p)_{\dot\beta}^\tau\partial_l\mathbf{H}_{\tau p}\right] = 0 \tag{4.4.18}$$

These 14 equations of motion are all invariant under the following 10 gauge transformations:

$$\delta\mathbf{A}_\alpha = +\frac{i}{\sqrt{2}}(\Gamma^m)_\alpha^{\dot\beta}\partial_m\lambda_{\dot\beta} + 2\mu_\alpha \tag{4.4.19a}$$

$$\delta\mathbf{B}_\alpha = +\frac{i}{\sqrt{2}}(\Gamma^m)_\alpha^{\dot\beta}\partial_m\omega_{\dot\beta} \tag{4.4.19b}$$

$$\delta\mathbf{I}_\alpha = \frac{1}{2}\Box\mu_\alpha \tag{4.4.19c}$$

$$\delta\mathbf{H}_\alpha^m = \frac{1}{9\times 4}(\Gamma^m)_\alpha^{\dot\beta}\omega_{\dot\beta} + \frac{i}{4\sqrt{2}}\partial_m\mu_\alpha - \frac{i}{8\times 9\sqrt{2}}(\Gamma_m)_\alpha^{\dot\beta}(\partial\!\!\!/\mu)_{\dot\beta} \tag{4.4.19d}$$

$$\delta\mathbf{C}_{m\dot\alpha} = \partial_m\omega_{\dot\alpha} \tag{4.4.19e}$$

$$\delta\mathbf{D}_{\dot\alpha}^m = \partial_m\lambda_{\dot\alpha} - \frac{i}{\sqrt{2}}(\Gamma^m)_{\dot\alpha}^\beta\mu_\beta \tag{4.4.19f}$$

$$\delta\mathbf{E}_{\dot\alpha}^A = 0 \tag{4.4.19g}$$

$$\delta\mathbf{F}_{\dot\alpha} = -\frac{1}{2}\Box\lambda_{\dot\alpha} + \omega_{\dot\alpha} \tag{4.4.19h}$$

$$\delta\mathbf{G}_{\dot\alpha} = \omega_{\dot\alpha} - \frac{2i}{\sqrt{2}}(\Gamma^m)_{\dot\alpha}^\beta\partial_m\mu_\beta \tag{4.4.19i}$$

$$\delta\mathbf{J}_{\dot\alpha} = -\frac{1}{2}\Box\omega_{\dot\alpha} \tag{4.4.19j}$$

We determined the basic content of ghost number 2 BRST cohomology; all equations of motion have been written between (4.4.5) and (4.4.18). This set is highly redundant, and the next step is to use (4.4.19) to stablish the independent field equations.

4.4.3 Gauge-fixing and independent equations of motion.

In order to find the independent set of equations of motion, we begin by fixing algebraic gauge conditions and solving auxiliary field equations. Let us gauge-fix $\mathbf{A}$ and $\mathbf{F}$ to zero using the parameters μ and ω, that is, we choose $\mu = -\mathbf{A}$ and $\omega = -\mathbf{F}$ so that the residual gauge parameters μ' and ω' must satisfy:

$$\mu'_{\alpha} + \frac{i}{2\sqrt{2}}(\Gamma^m)^{\dot\beta}_{\alpha}\partial_m\lambda_{\dot\beta} = 0, \tag{4.4.20}$$

and

$$\omega'_{\dot\alpha} - \frac{1}{2}\Box\lambda_{\dot\alpha} = 0. \tag{4.4.21}$$

After this gauge fixing, the following auxiliary field conditions can be imposed:

$$\mathbf{G}^m_{\dot\alpha} = 0, \tag{4.4.22a}$$

$$\mathbf{B}_{\alpha} = -2\mathbf{I}_{\alpha}, \tag{4.4.22b}$$

$$\mathbf{C}^m_{\dot\alpha} = +\frac{1}{2}\Box\mathbf{D}^m_{\dot\alpha} + \frac{i}{\sqrt{2}}\left(\Gamma^m\mathbf{I}\right)_{\dot\alpha}, \tag{4.4.22c}$$

$$\mathbf{J}_{\dot\alpha} = -\frac{1}{4}\Box\,\partial_m\mathbf{D}^m_{\dot\alpha} - \frac{i}{2\sqrt{2}}\left(\partial\!\!\!/\mathbf{I}\right)_{\dot\alpha}, \tag{4.4.22d}$$

$$\mathbf{H}^m_{\alpha} = \frac{1}{8}\partial\!\!\!/^{\dot\beta}_{\alpha}\mathbf{D}^m_{\dot\beta} - \frac{1}{18\times 8}(\Gamma^m\Gamma_n\,\partial\!\!\!/)^{\dot\beta}_{\alpha}\mathbf{D}^n_{\dot\beta}. \tag{4.4.22e}$$

At this point it is already clear that there only remains two independent fields given by $\mathbf{D}^m_{\dot\alpha}$ and $\mathbf{I}_{\alpha}$. Moreover, the only remaining gauge parameter is λ. We leave the gluino field $\mathbf{E}^a_{\dot\beta}$ out of the discussion since its equation of motion is already the Dirac equation and it has no gauge transformations.

Finally, the following set of 3 equations,

$$\frac{i}{\sqrt{2}}\partial^m\mathbf{I}_{\alpha} = \Box\left(\frac{1}{4}\partial\!\!\!/^{\dot\beta}_{\alpha}\mathbf{D}^m_{\dot\beta} - \frac{1}{12}(\Gamma^m)^{\dot\beta}_{\alpha}\partial_n\mathbf{D}^n_{\dot\beta}\right) \tag{4.4.23a}$$

$$2\partial_m\mathbf{D}^m_{\dot\alpha} + (\Gamma_n\Gamma_p)^{\dot\beta}_{\dot\alpha}\partial^n\mathbf{D}^p_{\dot\beta} = 0 \tag{4.4.23b}$$

$$\partial\!\!\!/^{\dot\alpha}_{\alpha}\mathbf{E}^a_{\dot\alpha} = 0 \tag{4.4.23c}$$

with the corresponding gauge transformations:

$$\delta\mathbf{D}^m_{\dot\alpha} = \frac{3}{4}\partial^m\lambda_{\dot\alpha} - \frac{1}{4}(\Gamma^{mn})_{\dot\alpha}{}^{\dot\beta}\partial_n\lambda_{\dot\beta} \tag{4.4.24a}$$

$$\delta\mathbf{I}_{\alpha} = -\frac{i}{4\sqrt{2}}\partial\!\!\!/^{\dot\beta}_{\alpha}\Box\lambda_{\dot\beta} \tag{4.4.24b}$$

defines the spectrum of the theory.

Gauge-invariant description. Consider the following field redefinitions:

$$\mathbf{d}^m_{\dot\alpha} = \mathbf{D}^m_{\dot\alpha} - \frac{1}{6}(\Gamma^m)^{\alpha}_{\dot\alpha}\not{\mathbf{D}}_\alpha \tag{4.4.25a}$$

$$\mathbf{i}_\alpha = +\frac{i4}{\sqrt{2}}\mathbf{I}_\alpha + \frac{1}{6}\Box\not{\mathbf{D}}_\alpha \tag{4.4.25b}$$

such that our gauge transformations are mapped to

$$\delta\mathbf{d}^m_{\dot\alpha} = \partial^m\lambda_{\dot\alpha} \quad \texttt{and} \quad \delta\mathbf{i}_\alpha = 0. \tag{4.4.26}$$

The gauge-invariant object is then naturally defined as:

$$\mathbf{F}_{mn\dot\alpha} = \partial_m\mathbf{d}_{n\dot\alpha} - \partial_n\mathbf{d}_{m\dot\alpha} \tag{4.4.27}$$

which allows us to write the equations of motion in the following form:

$$\partial_m\mathbf{i}_\alpha = \Box\mathbf{F}_{m\alpha} \tag{4.4.28a}$$

and

$$(\not{\mathbf{F}})_{\dot\alpha} = 0 \tag{4.4.28b}$$

where

$$\mathbf{F}_{m\alpha} \equiv (\Gamma^n)^{\dot\alpha}_{\alpha}\mathbf{F}_{mn\dot\alpha} = (\not\partial\mathbf{d}_m - \partial_m\not{\mathbf{d}})_\alpha. \tag{4.4.29}$$

In the formulation of free higher-spin theories $\mathbf{F}_m$ is called Fronsdal tensor [68], it is the analog of the Ricci curvature in spin 2 formulation.

This section started with the most general ghost number 2 picture $-1/2$ vertex operator. Then we obtained all equations of motion from the BRST method together with all gauge transformations parametrized by ghost number 1 picture $-1/2$ vertex operators. By fixing some of this gauge freedom, we have found a independent set of equations of motion that can be parametrized by Fronsdal fields. The next natural step is to write the spacetime action that gives the dynamics of this system.

4.5 Action

The kinetic term of the ambitwistor string field theory was defined in [72]:

$$S[V] = \langle I \circ V^{(-3/2)}(0)\,\partial c\, QV^{(-1/2)}(0)\rangle, \tag{4.5.1}$$

where $V^{-1/2}$ is the vertex operator (4.4.2) introduced in the previous section, an element of the small Hilbert space that is also constrained to satisfy $L_0V = b_0V = 0$. The RNS string has one additional feature: the picture number. It is necessary to saturate the background charge of supermoduli space to -2, and that is why we need a string field with picture $-1/2$, $V^{-1/2}$, together with a string field with picture $-3/2$, $V^{-3/2}$. We define picture raising, Z, and picture lowering, Y, by the following expressions:

$$Z = c\partial\xi + e^{\phi_6}\mathrm{p}_m\psi^m - \partial(e^{2\phi_6}\eta\tilde b) - e^{2\phi_6}\partial\eta\tilde b, \tag{4.5.2}$$

$$Y(z) = \tilde c\partial\xi e^{-2\phi_6}, \tag{4.5.3}$$

so that we can obtain $V^{-3/2}$ from $V^{-1/2}$ via

$$V^{-3/2}(z) = \frac{1}{2\pi i}\oint \frac{dw}{(w-z)} Y(w) V^{-1/2}(z). \tag{4.5.4}$$

Using the auxiliary gauge-fixing conditions imposed on the previous section, we obtain

$$\begin{aligned} V^{-3/2} =& + \tilde{c}\partial\tilde{c}S^{\alpha}e^{-3\phi_6/2}\mathbf{B}_{\alpha} - c\tilde{c}\partial\tilde{c}\partial\xi S^{\dot{\alpha}}e^{-5\phi_6/2}\partial x^m \mathbf{C}_{m\dot{\alpha}} - c\tilde{c}\partial\tilde{c}\partial\xi S^{\dot{\alpha}}e^{-5\phi_6/2}\mathrm{p}_m \mathbf{D}^m_{\dot{\alpha}} \\ & - c\tilde{c}\partial\tilde{c}\partial\xi S^{\dot{\alpha}}e^{-5\phi_6/2} j^a \mathbf{E}^a_{\dot{\alpha}} - c\tilde{c}\partial\tilde{c}\partial\xi\psi_m(\not{\psi}S)^{\alpha}e^{-5\phi_6/2}\mathbf{H}^m_{\alpha} - \frac{1}{2}c\tilde{c}\partial\tilde{c}\partial\xi\partial^2\tilde{c}\partial^2\xi S^{\alpha}e^{-7\phi_6/2}\mathbf{I}_{\alpha} \end{aligned} \tag{4.5.5}$$

The composition $I \circ V^{-3/2}$ is the BPZ conjugate of the picture $-3/2$ field with $I = -1/z$. We should be careful when computing the conformal transformation $I \circ V^{-3/2}$ because $V^{-1/2}$ is not primary. From the OPE with the stress-energy tensor

$$T(z)V^{-1/2}(0) \sim z^{-3}S^{\dot{\alpha}}e^{-\phi_6/2}c\tilde{c}\left(\frac{1}{2}\partial_m \mathbf{D}^m_{\dot{\alpha}} + \mathbf{F}_{\dot{\alpha}} - \frac{3}{8}\mathbf{G}_{\dot{\alpha}} - \frac{9}{4}\not{\mathbf{H}}_{\dot{\alpha}}\right) + \cdots \tag{4.5.6}$$

we obtain a cubic pole contribution that changes the finite conformal transformation to

$$I \circ V = \left[V\big(I(z)\big) + \frac{1}{2}\frac{I''(z)}{[I'(z)]^2}\,\#\big(I(z)\big)\right]. \tag{4.5.7}$$

where # is cubic pole coefficient. Even after the auxiliary conditions are imposed we still have non-primary contributions that must be taken into account.

To calculate the free action, we fix the normalization $\langle c\partial c\partial^2 c\tilde{c}\partial\tilde{c}\partial^2\tilde{c}e^{-2\phi_6}\rangle = 4$, then the correlation function (4.5.1) gives the following gauge-invariant action:

$$S_R = -\int d^{10}x\left[\frac{1}{2}\mathbf{d}^{m\alpha}\Box\left(\mathbf{F}_{m\alpha} - \frac{1}{2}(\gamma_m)_{\alpha\beta}\not{\mathbf{F}}^{\beta}\right) + \frac{1}{2}(\not{\mathbf{F}})^{\alpha}\mathbf{i}_{\alpha} - \frac{i}{2}\mathrm{Tr}\Big(\mathbf{E}\not{\partial}\mathbf{E}\Big)\right]. \tag{4.5.8}$$

In this expression we used the symmetric gamma matrices $(\gamma^m_{\alpha\beta}, \gamma^{\alpha\beta}_m)$ defined in section 4.2. When using these symmetric matrices, the charge conjugation is used to eliminate all dotted indices; different chiralities are just represented by upper and lower indices, *i.e.* $(C^{\alpha\dot{\alpha}}\mathbf{d}^m_{\dot{\alpha}} = \mathbf{d}^{m\alpha})$.

We have written a non-unitary action that gives the equations of motion obtained in (4.4.28). It closely resembles the gauge-invariant formulation of spin 3/2, the difference being the presence of more derivatives. Let us proceed and study the supersymmetry of this non-unitary system.

4.6 Supersymmetry.

Let us define the supersymmetry generator as

$$\mathbf{Q}^{-1/2}_{\alpha} = \frac{1}{2\pi i}\oint \mathrm{d}z\, S_{\alpha}e^{-\phi_6/2} \tag{4.6.1}$$

Notice that it carries picture, which means that supersymmetry algebra only closes on-shell. We need the picture 1/2 supersymmetric charge:

$$\mathbf{Q}^{1/2}_{\alpha} = \frac{1}{2\pi i}\oint \mathrm{d}z\left[i\mathrm{p}_m(\gamma^m)_{\alpha\beta}S^{\beta}e^{\phi_6/2} + \tilde{b}\eta S_{\alpha}e^{3\phi_6/2}\right]. \tag{4.6.2}$$

to obtain $\{Q_\alpha^{-1/2}, Q_\beta^{1/2}\} = 2\gamma^m_{\alpha\beta}\mathrm{p}_m$. In practice, supersymmetry transformations are written up to equations of motion. One also needs to choose a GSO sector to have well-defined supersymmetry transformations, otherwise there will be branch cuts. Given the generator (4.6.1), we need use the GSO(+) vertex operator.

4.6.1 Supersymmetry transformations of NS and R sectors.

The Neveu-Schwarz vertex operator in picture -1 was written in [72]:

$$V_{NS}^{-1} = e^{-\phi_6}c\tilde{c}\Big[\Big(G^{(1)}_{(mn)} + B^{(1)}_{[mn]}\Big)\,\mathrm{p}^m\psi^n + \Big(G^{(2)}_{(mn)} + B^{(2)}_{[mn]}\Big)\,\partial x^m\psi^n + C_{mnp}\psi^m\psi^n\psi^p + \mathrm{j}^a\psi^m A^a_m\Big] + \\ + e^{-\phi_6}c\tilde{c}\partial\psi^m A^{(4)}_m + \partial\phi_6 e^{-\phi_6}c\tilde{c}A^{(3)}_m\psi^m + \partial\xi e^{-2\phi_6}\partial^2\tilde{c}\tilde{c}cS^{(4)} + \eta c S^{(1)} + \partial\xi e^{-2\phi_6}\partial^2 cc\tilde{c}S^{(2)} \\ + \dots \tag{4.6.3}$$

where $\dots$ depends only on the previous fields. In [72], the fields $(B^{(1)}_{mn}, A^{(3)}_m, A^{(4)}, S^{(1)}, S^{(2)})$ of (4.6.3) were gauged to zero. If we choose to keep this gauge, we must observe that in general supersymmetry does not preserve a given gauge condition. Therefore when calculating supersymmetry transformations, we have to choose the gauge parameter Λ:

$$\delta_\zeta V_{NS}^{-1} = \Big[\zeta\mathbf{Q}^{-1/2}, V_R^{-1/2}\Big] + \Big[Q_{BRST}, \Lambda^{-1}\Big], \tag{4.6.4}$$

which is a vertex operator of ghost number 1 and picture -1, to ensure that $\delta_\zeta(B^{(1)}_{mn}, A^{(3)}_m, A^{(4)}, S^{(1)}, S^{(2)})$ all give zero. In the transformations below, the contributions of $\mathbf{H}$ are due to the gauge-fixing of these auxiliary fields:

$$\delta_\zeta G^{(1)}_{mn} = 2(\zeta\gamma_{(m}\mathbf{D}_{n)}) \tag{4.6.5}$$

$$\delta_\zeta G^{(2)}_{mn} = \frac{2}{5}(\zeta\gamma_{(n}\mathbf{C}_{m)}) - \frac{48}{5}\partial_{(n}\zeta\mathbf{H}_{m)} \tag{4.6.6}$$

$$\delta_\zeta B^{(2)}_{mn} = -4(\zeta\gamma_{[n}\mathbf{C}_{m]}) - \frac{48}{5}(\zeta\partial_{[m}\mathbf{H}_{n]}) \tag{4.6.7}$$

$$\delta_\zeta C_{mnp} = \frac{3}{2}\partial_{[p}(\zeta\gamma_m\mathbf{D}_{n]}) - 24(\zeta\gamma_{[np}\mathbf{H}_{m]}) + 6(\zeta\gamma_{mnp}\not{\mathbf{H}}) \tag{4.6.8}$$

and using the field redefinitions of [72]:

$$h_{mn} = G^{(1)}_{mn} + \frac{1}{4}\eta_{mn}h^r_r, \quad t = \frac{1}{4}\Box h^m_m + G^{m(2)}_m \quad \texttt{and} \quad B^{(2)}_{mn} = B_{mn} \tag{4.6.9}$$

we arrive at

$$\delta_\zeta h_{mn} = 2\zeta\gamma_{(m}\mathbf{d}_{n)} \tag{4.6.10}$$

$$\delta_\zeta t = \zeta\mathbf{i} \tag{4.6.11}$$

$$\delta_\zeta C_{mnp} = -3(\zeta\gamma_{t[mn}\mathbf{F}^t{}_{p]}) - 3(\zeta\gamma_{[m}\mathbf{F}_{np]}) \tag{4.6.12}$$

$$\delta_\zeta B_{mn} = -2\Box(\zeta\gamma_{[m}\mathbf{d}_{n]}) - (\zeta\gamma_{mn}\mathbf{i}) + \frac{1}{6}(\zeta\gamma_{mn}\partial_p\mathbf{F}^p) \tag{4.6.13}$$

$$\delta_\zeta A^a_m = \frac{i}{2}(\zeta\gamma_m\mathbf{E}^a). \tag{4.6.14}$$

The term $(\zeta\gamma_{mn}\partial_p\mathbf{F}^p)$ is zero if we use the equation of motion $\not{\mathbf{F}} = 0$, and so could not have been obtained from the supersymmetry generator (4.6.1). This term was added by hand in order to make the action invariant under supersymmetry.

For the Ramond sector the same can be done if we use instead the picture +1/2 supersymmetry generator (4.6.2):

$$\delta_\zeta \mathbf{d}^\alpha_m = +(\gamma^{rs}\zeta)^\alpha\partial_s h_{mr} - 2(\gamma^{np}\zeta)^\alpha C_{mnp} + \frac{1}{3}(\gamma_{mnps}\zeta)C^{nps} \tag{4.6.15}$$

$$\delta_\zeta \mathbf{i}_\alpha = 2(\zeta\not{\partial})_\alpha t - (\gamma^{mnp}\zeta)_\alpha H_{mnp} + \frac{1}{3}(\gamma^{mnp}\zeta)_\alpha \Box C_{mnp} \tag{4.6.16}$$

$$\delta_\zeta \mathbf{E}^{a\beta} = -\frac{1}{4}F_{mn}(\gamma^{mn}\zeta)^\beta \tag{4.6.17}$$

At this point, we have obtain the supersymmetry transformations of both NS and R system for the independent fields of the theory in equations (4.6.10) to (4.6.9). Let us proceed and check that indeed the total $GSO(+)$ action is supersymmetric invariant.

4.6.2 Supersymmetry invariance of the action.

The action that describes the Neveu-Schwarz sector is

$$S_{NS} = -\int d^{10}x \left[\frac{1}{2}h^{mn}\Box\left(R_{mn} - \frac{1}{2}\eta_{mn}R\right) - tR + \frac{1}{4}\mathrm{Tr}(F^{mn}F_{mn}) + \right.$$
$$\left. -C^{mnp}H_{mnp} + \frac{1}{2}C^{mnp}\left(\Box C_{mnp} - \frac{1}{2}\partial_{[p}\partial^r C_{mn]r}\right)\right] \tag{4.6.18}$$

where H_{mnp} is the field strength for B_{mn} and R_{mn} is the Ricci tensor. This expression is equivalent to the action written in equation (4.13) of [72] if we shift t by $t \mapsto t + R^2$. The equations of motion derived from (4.6.18) are

$$\Box R_{mn} - \partial_m\partial_n t = 0\ , \quad R = 0\ , \quad \Box C_{mnp} - H_{mnp} = 0\ ,$$

$$\partial^m C_{mnp} = 0\ , \quad \texttt{and} \quad \partial_m F^{mn} = 0. \tag{4.6.19}$$

Now, the Ramond sector is described by equation (4.5.8):

$$S_R = -\int d^{10}x \left[\frac{1}{2}\mathbf{d}^{m\alpha}\Box\left(\mathbf{F}_{m\alpha} - \frac{1}{2}(\gamma_m)_{\alpha\beta}\not{\mathbf{F}}^\beta\right) + \frac{1}{2}(\not{\mathbf{F}})^\alpha\mathbf{i}_\alpha - \frac{i}{2}\mathrm{Tr}\left(\mathbf{E}\not{\partial}\mathbf{E}\right)\right]. \tag{4.6.20}$$

from which we obtain the following set of equations of motion – (4.4.28):

$$\partial_m\mathbf{i}_\alpha = \Box\mathbf{F}_{m\alpha}\ , \quad \not{\mathbf{F}}^\alpha = 0 \quad \texttt{and} \quad i\not{\partial}_{\alpha\beta}\mathbf{E}^{a\,\beta} = 0. \tag{4.6.21}$$

From now on, we leave the Yang-Mills system out of the discussion because its supersymmetry transformations and action are already standard. For later use, let us write the supersymmetry transformation for all field strengths:

$$\delta_\zeta R_{mn} = (\zeta\partial_{(m}\mathbf{F}_{n)}) + (\zeta\gamma_{(m}\partial^p\mathbf{F}_{n)p}) \tag{4.6.22a}$$

$$\delta_\zeta H_{mnp} = 3\Box(\zeta\gamma_{[m}\mathbf{F}_{np]}) - 3(\zeta\gamma_{[mn}\partial_{p]}\mathbf{i}) + \frac{1}{2}(\zeta\gamma_{[mn}\partial_{p]}\partial_\ell\mathbf{F}^\ell) \tag{4.6.22b}$$

$$\delta_\zeta \mathbf{F}^\alpha_{mn} = -2(\gamma^{rs}\zeta)^\alpha R_{mrsn} + 4(\gamma^{rp}\zeta)^\alpha\partial_{[n}C_{m]rp} - \frac{2}{3}(\partial_{[n}\gamma_{m]rps}\zeta)^\alpha C^{rps} \tag{4.6.22c}$$

$$\begin{aligned}\delta_\zeta \mathbf{F}_{m\alpha} = &+2(\gamma^n\zeta)_\alpha R_{mn} - 2(\gamma^{lnp}\zeta)_\alpha\partial_l C_{mnp} + \frac{1}{3}(\gamma_{lmnps}\zeta)_\alpha\partial^l C^{nps}\\ &+ 4(\gamma^n\zeta)_\alpha\partial^p C_{mnp} - (\gamma_{mps}\zeta)_\alpha\partial_n C^{nps}\end{aligned} \tag{4.6.22d}$$

$$\delta\not{\mathbf{F}}^\beta = 2\zeta^\beta R - 6(\gamma^{np}\zeta)^\beta\partial^m C_{npm} \tag{4.6.22e}$$

4.6.3 Supersymmetry for $(h_{mn}, t, \mathbf{i}, \mathbf{d})$

Let us consider the system:

$$\begin{aligned}\mathbf{S} = -\int d^{10}x\Bigg(&\frac{1}{2}h^{mn}\Box\left(R_{mn} - \frac{1}{2}\eta_{mn}R\right) - tR\\ &+ \frac{1}{2}\mathbf{d}^{m\alpha}\Box\left(\mathbf{F}_{m\alpha} - \frac{1}{2}(\gamma_m)_{\alpha\beta}\not{\mathbf{F}}^\beta\right) + \frac{1}{2}(\not{\mathbf{F}})^\alpha\mathbf{i}_\alpha\Bigg)\end{aligned} \tag{4.6.23}$$

such that the

S_{NS} **variation** is given by

$$\delta_\zeta\,(-tR) = -\zeta^\alpha\mathbf{i}_\alpha R - 2t\zeta^\alpha\partial_p\mathbf{F}^p_\alpha$$

$$\delta_\zeta\left[\frac{1}{2}h^{mn}\Box\left(R_{mn} - \frac{1}{2}\eta_{mn}R\right)\right] = 2(\zeta\gamma^m\mathbf{d}^n)\Box\left(R_{mn} - \frac{1}{2}\eta_{mn}R\right)$$

and the

S_R **variation** is given by

$$\begin{aligned}\delta_\zeta\left(\frac{1}{2}\mathbf{d}^{m\alpha}\Box\left(\mathbf{F}_{m\alpha} - \frac{1}{2}(\gamma_m)_{\alpha\beta}\not{\mathbf{F}}^\beta\right)\right) &= 2\mathbf{d}^{m\alpha}\Box\left((\zeta\gamma^n)_\alpha R_{mn} - \frac{1}{2}(\zeta\gamma_m)_\alpha R\right)\\ &= -2(\zeta\gamma^n\mathbf{d}^m)\Box\left(R_{mn} - \frac{1}{2}\eta_{mn}R\right)\end{aligned}$$

$$\begin{aligned}\delta_\zeta\left(\frac{1}{2}(\not{\mathbf{F}})^\alpha\mathbf{i}_\alpha\right) &= \zeta^\alpha R\mathbf{i}_\alpha + (\not{\mathbf{F}})^\alpha\not{\partial}_{\alpha\beta}\zeta^\beta t\\ &= \zeta^\alpha\mathbf{i}_\alpha R + 2\zeta^\alpha(\partial_p\mathbf{F}^p_\alpha)t + \partial(...)\end{aligned}$$

where we have used (4.6.22) and $(\not{\partial}\not{\mathbf{F}}\zeta) = 2(\zeta\partial_p\mathbf{F}^p)$. It is clear that the sum of all terms cancels and invariance of this system is stablished.

4.6.4 Supersymmetry for $(H_{mnp}, C_{mnp}, \mathbf{d}^{\alpha}_{m}, \mathbf{i}_{\alpha})$

It remains for consideration the following system:

$$\mathbf{S} = -\int d^{10}x\Bigg(- C^{mnp}H_{mnp} + \frac{1}{2}C^{mnp}\left(\Box C_{mnp} - \frac{1}{2}\partial_{[p}\partial^{r}C_{mn]r}\right) + \frac{1}{2}\mathbf{d}^{m\alpha}\Box\left(\mathbf{F}_{m\alpha} - \frac{1}{2}(\gamma_m)_{\alpha\beta}\not{\mathbf{F}}^{\beta}\right) + \frac{1}{2}(\not{\mathbf{F}})^{\alpha}\mathbf{i}_{\alpha}\Bigg) \tag{4.6.24}$$

In order to check supersymmetric invariance we have to gather all independent combination of gamma matrices $(\gamma^m, \gamma^{mn}, \gamma^{mnp}, \gamma^{mnpp}, \gamma^{mnpqr})$. So consider the

S_{NS} **variation:**

$$\delta_\zeta(-C^{mnp}H_{mnp}) = +3\left[(\zeta\gamma_{tmn}\mathbf{F}^{t}{}_{p}) + (\zeta\gamma_m\mathbf{F}_{np})\right]H^{mnp} - 3(\zeta\gamma_m\mathbf{F}_{np})\Box C^{mnp} - 3(\zeta\gamma_{mn}\mathbf{i})\partial_p C^{mnp} - \frac{1}{2}(\zeta\gamma_{mn}\partial_p\mathbf{F}^{p})\partial_p C^{mnp} + \partial(\dots)$$

$$\delta_\zeta\left[\frac{1}{2}C^{mnp}\left(\Box C_{mnp} - \frac{1}{2}\partial_{[p}\partial^{r}C_{mn]r}\right)\right] = -3\left[(\zeta\gamma_{tmn}\mathbf{F}^{t}{}_{p}) + (\zeta\gamma_m\mathbf{F}_{np})\right]\Box C^{mnp} + \frac{1}{2}\left[(\zeta\gamma_{mn}\partial^{p}\mathbf{F}_{p}) + (\zeta\gamma_m\partial_n\not{\mathbf{F}})\right]\partial_r C^{mnr} + \partial(\dots)$$

and the

S_R **variation:**

$$\begin{aligned}\delta_\zeta\left(\frac{1}{2}\not{\mathbf{F}}\mathbf{i}\right) &= -3(\gamma^{np}\zeta)^{\beta}\partial^{m}C_{npm}\mathbf{i}_\beta - \frac{1}{2}\not{\mathbf{F}}^{\alpha}(\gamma^{mnp}\zeta)_\alpha H_{mnp} + \frac{1}{6}\not{\mathbf{F}}^{\alpha}(\gamma^{mnp}\zeta)_\alpha\Box C_{mnp}\\ &= +3(\zeta\gamma^{nm}\mathbf{i})\partial^{p}C_{nmp}+\\ &\quad -\left[\frac{1}{6}(\zeta\gamma^{mnpts}\mathbf{F}_{ts})\Box C_{mnp} - (\zeta\gamma^{tmn}\mathbf{F}_{t}{}^{p})\Box C_{mnp} - (\zeta\gamma^{m}\mathbf{F}^{np})\Box C_{mnp}\right]\\ &\quad +\left[\frac{1}{2}(\zeta\gamma^{mnpts}\mathbf{F}_{ts})H_{mnp} - 3(\zeta\gamma^{tmn}\mathbf{F}_{t}{}^{p})H_{mnp} - 3(\zeta\gamma^{m}\mathbf{F}^{np})H_{mnp}\right]\end{aligned}$$

$$\begin{aligned}\delta_\zeta\left(\frac{1}{2}\mathbf{d}^{\alpha}_{m}\Box\left(\mathbf{F}^{m}_{\alpha} - \frac{1}{2}(\gamma^m)_{\alpha\beta}\not{\mathbf{F}}^{\beta}\right)\right) &= \left(-2(\gamma^{np}\zeta)^{\alpha}C_{mnp} + \frac{1}{3}(\gamma_{mnps}\zeta)^{\alpha}C^{nps}\right)\Box\left(\mathbf{F}^{m}_{\alpha} - \frac{1}{2}(\gamma^m)_{\alpha\beta}\not{\mathbf{F}}^{\beta}\right)\\ &= +2(\mathbf{F}_{l}{}^{m}\gamma^{lnp}\zeta)\Box C_{mnp} - 4(\mathbf{F}^{mn}\gamma^{p}\zeta)\Box C_{mnp}+\\ &\quad -\frac{1}{3}(\mathbf{F}_{lm}\gamma^{lmnps}\zeta)\Box C_{nps} - (\mathbf{F}_{m}{}^{n}\gamma^{psm}\zeta)\Box C_{nps}+\\ &\quad -\left[\frac{1}{6}(\zeta\gamma^{mnpts}\mathbf{F}_{ts})\Box C_{mnp} - (\zeta\gamma^{tmn}\mathbf{F}_{t}{}^{p})\Box C_{mnp} - (\zeta\gamma^{m}\mathbf{F}^{np})\Box C_{mnp}\right]\end{aligned}$$

Recall that the γ^{mnpqr} is symmetric and γ^{mnp} is antisymmetric under the spinor indices. Gathering all independent terms we confirm the system is supersymmetric.

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