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Dynamical Symmetry and Hadron Spectrum

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Abstract

Symmetry plays an important role in hadron physics. The predictions of baryon Ω^- and dibaryon d^* in the 1960s, which were confirmed by experiments, are based on the dynamical symmetry of quark systems. In this pedagogical article, the dynamical symmetry of hadrons is explored. The multiplets of color, spin, and flavor symmetries of two- to six-quark systems are discussed and the phase-consistent wave functions of these systems are presented. Based on the dynamical symmetry of the system, the mass formulas for these systems are constructed, which give a good description of the hadron spectra. Compared with quark–antiquark mesons and three-quark baryons, multi-quark systems have richer structures. It is expected that the symmetry can provide a good guide for exploring multi-quark systems (exotic hadrons) systematically.

Keywords: dynamic symmetry; irreducible representation; phase consistent; wave function; mass formula

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1. Introduction

In 1962, Gell-Mann proposed the eightfold way to classify the hadrons [1]. In 1964, Gell-Mann and Zweig independently proposed the quark model [2–4], which incorporates eight baryons with spin parity $\frac{1}{2}^+$ in an eight-dimensional irreducible representation (irrep) of the SU_3 group based on the symmetry of flavor, and, by incorporating the nine baryons with spin parity $\frac{3}{2}^+$, discovered, in the experiment into a ten-dimensional representation, the tenth baryon of $J^P = \frac{3}{2}^+$ ($I = 0, S = -3, M \approx 1680$ MeV). The tenth baryon of $J^P = \frac{3}{2}^+$, which is Ω^-, had not yet been discovered but was predicted. Later, the successful observation of Ω^- ($M = 1672.5$ MeV) in the experiment in [5] marked a huge success for a quark model based on symmetry.

With the improvement of experimental equipment and the increase in accuracy, people have detected more and more particles. Since 2003, many new hadron states have been reported, some of which cannot be explained with traditional meson $q\bar{q}$ and baryon qqq pictures, and they are called "exotic" states. Various explanations are proposed, such as multi-quark states, quark–gluon hybrids, hadronic molecules, etc. There are many excellent review papers on exotic hadron states available [6–20]. This paper focused on multi-quark states. Because of the requirement of color singlets, three-quark baryons and quark–antiquark mesons have unique color structures. This uniqueness streamlines model building but limits direct access to the full-color dynamics of quantum chromodynamics (QCD), the fundamental theory of strong interaction. Multi-quark states admit diverse color structures beyond the singlet, such as octet, sextet, or higher-dimensional representations of SU_3 symmetry in color space, so they can act as a "laboratory" for QCD-complementing studies of conventional hadrons. Their rich color structures unlock a deeper understanding of QCD and make it indispensable for probing the strong interaction's fundamental nature. In this work, we try to emphasize the symmetry of systems from two body systems to six body systems and discuss the phase-consistent wave functions of these systems in orbit, color, spin, and flavor spaces.

In the past twenty years, major international experimental cooperation groups in high-energy physics and nuclear physics have reported many discoveries about exotic hadron states. What is an exotic hadron state? From the fundamental theory of strong interaction, quantum chromodynamics (QCD), we can know that the basic building blocks of strong-interacting matter are quarks and gluons. Starting from this point of view, we now believe that all hadrons are composed of quarks (q), antiquarks $(\bar{q})$, and gluons (g), that is, $(q^n\bar{q}^m g^l)$. The usual hadrons—mesons and baryons—are composed of quark–antiquark $(q\bar{q})$ pairs and three quarks (q^3), respectively. When the sum of quark and antiquark numbers is greater than three $(n+m>3)$, or the number of gluons is not 0 ($l \neq 0$), the states can be called exotic states. There have been abundant experimental discoveries of exotic states, such as the tetraquark state named the "XYZ" particle with $n=m=2$ and $l=0$ [21–25], where the total number of the quarks and antiquarks is 4, the number of gluons is 0, and the hidden-charm pentaquarks $(n+m=5, l=0)$ are like the P_c state discovered by LHCb [26,27], as well as the dibaryon state, namely the hexaquark state $(n+m=6, l=0)$, the most typical example of which is the deuteron, which is the only confirmed multi-quark state in the world so far, although the hexaquark state d^* was discovered in the WASA-at-COSY experimental cooperation [28–33]. The A2 collaboration at MAMI also reported the signatures of the $d^*(2380)$ hexaquark in d(γ, $p\vec{n}$) [34]; further experimental verification is still needed. There are also signals of $N\Omega$ reported in experiments [35]. When the number of gluons is not zero, the states are called hybrid $(n \text{ or } m \neq 0, l \neq 0)$ [36]. Three possible single-gluon light hybrid mesons, $\pi_1(1400)$ [37], $\pi_1(1600)$ [38], $\pi_1(2015)$ [39], composed of $q\bar{q}$ and one gluon with $I^G J^{PC} = 1^-1^{-+}$ were observed, as well as glueballs composed entirely of gluons $(n=m=0, l \geq 2)$ [40–42], which have at least two gluons due to the constraint of color singlets.

As the only widely recognized multi-quark state, the mass of deuteron has been listed as a physical constant in PDG [43], which is obviously separated into two sub-clusters due to its large charge radius of 2.1 fm [44] and hence is a very loose structure that can be described by the degrees of freedom of neutrons and protons. People are more interested in the existence of spatially compact multi-quark state, where several quarks stay closely together, and the compact multi-quark state may be a new form of material existence. Recently, some people have discussed whether the dibaryon can be considered as a candidate for dark matter [45–49] because once the dibaryon is combined stably, it may have very weak interactions with other objects, making it a candidate for dark matter.

In quantum electrodynamics (QED), the combination of electrons and nucleus forms an atom. With atoms, various molecules can form, which is QED chemistry. In QCD, quarks can form hadrons; for example, quarks and anti-quarks pair $(q\bar{q})$ to form mesons with baryon number $B=0$, and three quarks, qqq, form baryons with baryon number $B=1$. Therefore, whether there are four quark states $(q\bar{q}q\bar{q}, B=0)$, five quark states $(q^4\bar{q}, B=1)$, and six quark states $(q^6, B=2)$ has attracted widespread interest. If they really exist, it may lead to a new disciplinary branch, namely QCD chemistry, which is listed in Table 1.

In the 1960s, based on the SU_3 symmetry, Gell-Mann and Zweig independently proposed a new classification method for hadrons [2–4] and predicted the existence of the Ω^- baryon with mass around 1680 MeV. This state was observed in experiments two years later [5]. Another example occurred in 1964, shortly after the birth of the quark model, when Dyson et al. proposed the possible existence of dibaryons (six-quark states) based on the flavor–spin SU_6 symmetry within strong interactions. He predicted several possible dibaryon states, especially D_{03}, whose isospin and spin quantum numbers are 0 and 3, respectively, and the mass is assumed to be 2350 MeV [50]. In 2014, the WASA-at-COSY experimental cooperation confirmed the existence of d^* with $M = 2380$ MeV [28–33], which is very close to the value under Dyson's theoretical prediction. From these two examples, it can be seen that the symmetry based on the quark model is very successful in describing experimental data and predicting hadron states.

Table 1. Difference between QED and QCD.

QED	QCD		
nucleus + electron $\rightarrow$ atoms	quarks $\rightarrow$	hadrons	
molecules: di-atom	meson:	$q\bar{q}$	B = 0
tri-atom	baryon:	qqq	B = 1
⋮	tetraquark:	$q\bar{q}q\bar{q}$	B = 0
⋮	pentaquark:	$q^4\bar{q}$	B = 1
⋮	dibaryon:	q^6	B = 2
QED chemistry	QCD chemistry?		

At the beginning of establishing the quark model, people only assumed that there were three kinds of quarks in nature, named u (up), d (down), and s (strange) quarks. In 1964,Bjorken and Glashow predicted the possible existence of the fourth type of quark and named it as the charm quark [51]. In 1970, Glashow, Iliopoulos and Maiani predicted charm quark based on GIM mechanism [52]. Few years later, high-energy physics experimental teams led by Ting and Richter independently discovered the charm quark [53,54]. In 1977, Lederman discovered the fifth type of quark at Fermilab, known as the 'bottom' or 'beauty' quark [55,56]. In 1995, the sixth type of quark—the 'top' quark—was also discovered at Fermilab, which is the heaviest particle in the Standard Model [57,58].

2. The Symmetry of Hadrons

When it comes to symmetry, what we are familiar with is geometric symmetry in fact, which means that the shape or structure of an object remains unchanged under geometric operations (rotation, inversion, etc.). For example, benzene molecules have a ring structure and form a regular hexagon in space, which has point group symmetry D_{6h}. There is a popular molecule called C_{60} made of sixty carbon atoms, which has point group symmetry I_h. When studying crystals in solid physics, space groups are used frequently. The symmetry group of crystals is composed of some rotations, inversions, and spatial translations.

Geometric symmetry has a characteristic from the perspective of quantum mechanics; the Hamiltonian of a system with geometric symmetry is commutative with every element of its symmetry group. Geometric symmetry brings us a lot of convenience in studying physical systems, allowing us to obtain many qualitative results without specific quantitative calculations and also simplifying the calculation process.

For physical systems, there exists another type of symmetry—dynamical symmetry—wherein the system's Hamiltonian can be expressed as a function of specific operators. These operators are termed Casimir operators in the context of continuous groups and class operators in discrete groups, collectively characterized by the key property of commuting with every element of their respective groups. A system is considered to possess dynamical symmetry if its Hamiltonian can be formulated as a function of the class operators or Casimir operators of a group G. A key distinction between dynamical symmetry and geometric symmetry lies in the commutation relations of the Hamiltonian with group elements. For a system with dynamical symmetry, the Hamiltonian generally does not commute with arbitrary elements of group G; instead, it commutes specifically with the class operators or Casimir operators of G. Notably, the energy spectrum of the system can be readily determined from the eigenvalues of these class operators or Casimir operators.

$$\begin{aligned} &H = F(C, C_1, C_2, \cdots) = F(C, C(s)) \\ &[H, G] \neq 0, \quad [H, C] = 0, \quad [H, C(s)] = 0 \end{aligned} \tag{1}$$

where $C, C_1, C_2 \cdots$ are the Casimir operators (class operators) of group chains $G \supset G_1 \supset G_2 \supset \cdots$.

The dynamical symmetry has been widely used in physics [59], for example, in molecular physics [60,61] and nuclear physics [62,63]. The present work focuses on the quark model of particle physics; more applications of symmetry in particle physics can be found in [64–69]. There are four types of degrees of freedom for quarks, coordinate (orbital), color, flavor, and spin. The symmetries for color and spin are SU_3^c and SU_2^σ, respectively, which are exact. The three color states are denoted by r, g, b ($\bar{r}, \bar{g}, \bar{b}$ for antiquarks, and the same method is also applied to flavor) and two spin states are denoted by α, β. Spin, quarks, and antiquarks share the same notation due to the SU_2 symmetry. The quarks involved in hadron physics are u, d, s, c, b. The t-quark has a lifetime that is too short to form a bound state which is observable. The masses of c and b quarks are too large compared to the masses of u, d, s quarks. So the symmetry for flavor used in the present work is SU_3^f, which is an approximate one because the s quark is a little heavier than u, d quarks. SU_3^f symmetry breaking to $SU_2^\tau \times U_1^Y$ is considered. τ stands for isospin and Y for hypercharge; $\mathcal{B}, S, C, B, T$ are baryon number, strangeness, charm, bottomness, and topness, respectively. The definition of hypercharge is [43]

$$Y = \mathcal{B} + S - \frac{C - B + T}{3}, \tag{2}$$

bythis definition, Y is $\frac{1}{3}$ for u and d quarks, and 0 for other quarks, the diagram of light quarks and antiquarks is shown in Figure 1.

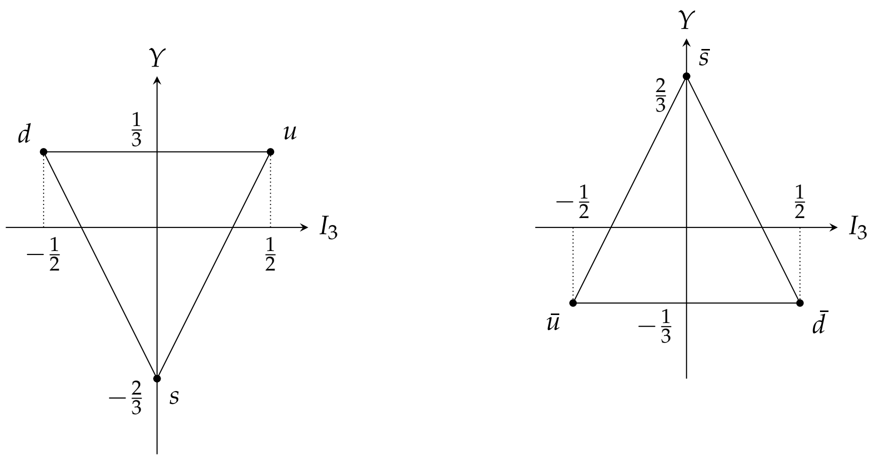
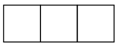
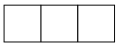
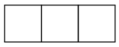
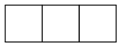
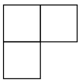
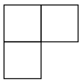
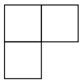
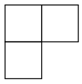
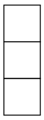
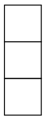
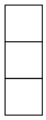


Figure 1. The fundamental representation **3** and conjugate representation $\bar{\mathbf{3}}$ of SU_3 flavor group; **3** = (u, d, s), $\bar{\mathbf{3}}$ = $(\bar{u}, \bar{d}, \bar{s})$.

The symmetry for coordinate motion is generally taken as U_1^x to simplify the discussion. To illustrate the extension, sometimes SU_2^x is also employed.

Three types of notations for the irreducible representation of groups S_f and SU_n are used here: the partition $[\nu_1\nu_2 \ldots \nu_f]$ with $\nu_1 + \nu_2 + \ldots + \nu_f = f$ (ν_i is a integer), the Young diagram, and the dimension of the irreducible representation (only for SU_n). These three types of notations are equivalent, and the relations among them in the three particle cases are shown in Table 2. The solution of irreducible representation dimensions for permutation groups S_f and unitary groups SU_n can be seen in [67,70].

Table 2. The notations for irreps of S_3 and $SU_n, n = 2, 3, 6$. P, Y, and D stand for partition, Young diagram, and dimension of irrep, respectively. Number in [ ] represents partition, linked boxes is the Young diagram and number in bold format stand for the dimension of the irrep.

S_3		SU_2			SU_3			SU_6		
P	**Y**	**P**	**Y**	**D**	**P**	**Y**	**D**	**P**	**Y**	**D**
[3]		[3]		**4**	[3]		**10**	[3]		**56**
[21]		[21]		**2**	[21]		**8**	[21]		**70**
[111]		-	-	-	[111]		**1**	[111]		**20**

In Young diagrams, one box represents a quark. For antiquarks, we use the concept of Dirac; under SU_n symmetry, the antiquark state is $n-1$ quark state $[1^{n-1}]$ with the requirement that n quarks form the vaccuum state $[1^n]$. So, the antiquark with SU_2 symmetry is denoted by $[1]$ with dimension **2**, the same as for the quark; with SU_3, it is $[11]$ with dimension **3**, and with SU_6, it is $[1^5]$ with dimension **6** [67].

For f identical particle states, the symmetry group is S_f, and the total anti-symmetrization of wave functions is required. For example, for the baryon states with the three-light-quark state, the combined wave function of coordinate, color, flavor, and spin must be totally anti-symmetrized under the exchange of any two quarks, and the total symmetry is $[1^3]$ in partition of group S_3.

For each degree of freedom, the symmetry can be deduced based on the Clebsch–Gordan (CG) series of permutation group S_f. For the coordinate, the symmetry is $[3]$ due to the constraint of U_1^x. For the color, the color singlet states require that the symmetry is $[111]$. Hence, the symmetry for flavor–spin is $[3]$, and then the two combinations of flavor and spin symmetries are possible: flavor $[21]$ with spin $[21]$ and flavor $[3]$ with spin $[3]$.

For $q^m\bar{q}^n$ states with m identical quarks and n identical antiquarks, the total anti-symmetrization requires that the total symmetry for quark is $[1^m]$ and for antiquarks is $[1^n]$. There is an additional requirement, a color singlet. The color symmetry for quarks $[c_1]$ coupled with the symmetry for antiquarks $[c_2]$ must give the color singlet $[111]$ for the $q\bar{q}$ and qqq states and $[222]$ for the $qq\bar{q}\bar{q}$, $q^4\bar{q}$ and q^6 states.

Combining four degrees of freedom, one arrives at a symmetry group chain for quark systems. For a baryon composed of three light quarks in the ground state (in this article, the u, d, s quarks are called light quarks), where all three quarks stay in the same state in coordinate space (ground state), U_1^x is employed to describe the spatial symmetry. Dibaryons in molecular structure are approximately divided into two sub-clusters, so there are two ground states. We adopt SU_2^x to describe the spatial symmetry of the dibaryon system in this case. For the intrinsic degrees of freedom of quarks, the color symmetry is SU_3, and the flavor of the light baryon system is approximately the SU_3 symmetry, and the spin symmetry is SU_2, and the two group chains of the baryons (3) and dibaryons (4) can be obtained. In fact, the following group chains stand for any systems of quarks, tetraquarks, pentaquarks, novem quarks, etc.:

$$SU_{18} \supset U_1^x \otimes \left\{ SU_{18}^{cf\sigma} \supset SU_3^c \otimes \left[SU_6^{f\sigma} \supset \left(SU_3^f \supset SU_2^\tau \otimes U_1^Y \right) \otimes SU_2^\sigma \right] \right\} \tag{3}$$

$$SU_{36} \supset SU_2^x \otimes \left\{ SU_{18}^{cf\sigma} \supset SU_3^c \otimes \left[SU_6^{f\sigma} \supset \left(SU_3^f \supset SU_2^\tau \otimes U_1^Y \right) \otimes SU_2^\sigma \right] \right\} \tag{4}$$

where U_1^x, SU_2^x stand for the spatial (orbital) symmetry of the baryon and dibaryon respectively, and SU_3^c, SU_3^f, and SU_2^σ stand for the symmetry of color, flavor, and spin. The total antisymmetrization requires that the symmetry of the total wave functions is $[1^f]$. For group chain Equation (3), the symmetry of $SU_{18}^{cf\sigma}$ is $[1^f]$ because the symmetry of U_1^x is symmetric $[f]$. The color singlet demands that the symmetry of SU_3^c is [111] for triquarks; [222] for tetraquarks, pentaquarks, and hexaquarks; and [333] for novem-quarks. The CG series of S_f tells us that the symmetry $[\mu]$ for $SU_6^{f\sigma}$ is the conjugate of the symmetry of color SU_3^c, [3] or [33] or [333]. Similarly, the CG series of S_f will give us the contents of the symmetry of SU_3^f and SU_2^σ in the symmetry $[\mu]$. For group chain Equation (4), the same procedure will find the possible combinations of coordinate, color, flavor, and spin symmetries for the dibaryon system.

Applying the dynamical symmetry to f-quark systems, i.e., the Hamiltonian of the system is the function of the Casimir operators of involved groups.

$C(SU_2^\sigma) = \sum_{i,j}^{f} \frac{\sigma_i}{2} \cdot \frac{\sigma_j}{2}$, $\boldsymbol{\sigma}$ are Pauli matrices, the generators of SU_2 group, the familiar operator in Hamiltonian is $\sum_{j>i=1}^{f} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$, which is related to the Casimir operator by

$$\sum_{j>i=1}^{f} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = 2\left(C(SU_2^\sigma) - \frac{3f}{4} \right). \tag{5}$$

The Casimir operator of SU_2^σ is

$$C(SU_2^\sigma) = \boldsymbol{S}^2 \, . \tag{6}$$

whose eigenvalue is $S(S+1)$.

$C(SU_3^{c,f}) = \sum_{i,j}^{f} \frac{\lambda_i}{2} \cdot \frac{\lambda_j}{2}$, $\boldsymbol{\lambda}$ are Gell-Mann matrices, the generators of SU_3 group, the relation between operator $\sum_{j>i=1}^{f} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$, and the Casimir operator of SU_3 is

$$\sum_{j>i=1}^{f} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j = 2\left(C(SU_3^{c,f}) - \frac{4f}{3} \right). \tag{7}$$

$C(SU_6^{\sigma f}) = \sum_{i,j}^{f} \frac{\mathbf{A}_i}{2} \cdot \frac{\mathbf{A}_j}{2}$, $\mathbf{A}$ are the generators of SU_6 group with $\mathrm{Tr}(A^a)^2 = 1$, the relation between $\sum_{j>i=1}^{f} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$, and the Casimir operator of SU_6 is

$$\sum_{j>i=1}^{f} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \frac{1}{2}\left(C(SU_6^{\sigma f}) - \frac{35f}{6} \right). \tag{8}$$

Then one has

$$H = F\Big(C(SU_3^c), C(SU_6^{\sigma f}), C(SU_3^f), C(SU_2^\tau), C(U_1^Y), C(SU_2^\sigma) \Big) \tag{9}$$

Because of the color singlet requirement, the contribution of operator $C(SU_3^c)$ is a constant, as is the operator $C(SU_6^{\sigma f})$, and the spin–flavor symmetry is conjugate to the color singlet for the ground states of the coordinate space due to the total anti-symmetrization. The simplest realization of Equation (9) is the linear one, i.e.,

$$H = H_0 + X_1 C(SU_3^f) + X_2 C(SU_2^\sigma) + X_3 C(U_1^Y) + X_4 C(SU_2^\tau). \quad (10)$$

Considering the flavor symmetry SU_3^f breaking to $SU_2^\tau \times U_1^Y$, at last one has the following mass formula (Gursey–Radicati mass formula [71]):

$$M = M_0 + X_1 C(SU_3^f) + X_2 J(J+1) + X_3 Y + X_4 [I(I+1) - \frac{Y^2}{4}]. \quad (11)$$

It can be extended to the following expression in the heavy quark sector [72,73]:

$$M = M_0 + X_1 C(SU_3^f) + X_2 J(J+1) + X_3 Y + X_4 [I(I+1) - \frac{Y^2}{4}] + \sum_{i=c,b} X_i N_i. \quad (12)$$

where $C(SU_3^f)$, J, Y, I, N_c (N_b) are the eigenvalues of the SU_3^f Casimir operator, spin, hypercharge, isospin, number of charm (beauty) quarks, and antiquarks of a hadron. The value of $C(SU_3^f)$ can be calculated as:

$$C(SU_3^f) = \frac{1}{3}\Big(p^2 + pq + q^2 + 3p + 3q\Big), \quad (13)$$

the symmetry (irreducible representations) of SU_3 is specified to $[\nu_1, \nu_2, \nu_3] = (p, q)$, with $p = \nu_2 - \nu_3$, $q = \nu_1 - \nu_2$.

Due to the Hamitonians of the systems being functions of Casimir operators of involved groups, the wave functions of systems are irreducible bases, which are the eigen functions of the Casimir operators of the groups.

In the configuration space, according to Theorem 7.1 on page 311 of [70], in the tensor space V_n^f, if $\psi^{(\nu)}$ belongs to the irrep (ν) of one of four groups (unitary group in coordinate space U_n, unitary group in state space $\mathcal{U}_n$, coordinate permutation group S_f, and state permutation group $\mathcal{S}_f$), then $\psi^{(\nu)}$ also belongs to the irrep(ν) of any one of the other three groups.

So, the irreducible basis of the permutation group S_f in the configuration space is equivalent to the irreducible basis of SU_n, the irreducible bases of S_f can be easily obtained with the group theory method. Generally, to obtain the irreducible bases of finite groups is easier than those of continuous groups. So, in most cases, one prefers to find the irreducible bases of S_f. The computational details can be found in many books. For example, in Ref. [70], it can be easily extended from a two-body system to a multi-body system after we write the calculation process into a program. In the following, we just quote the results.

Next, we will discuss the symmetry of multi-quark systems wit two to six bodies in detail. This paper is organized as follows: Section 1 gives a brief introduction about the motivation to study the symmetry of multi-quark systems. Section 2 discusses symmetry of hadrons generally. In Sections 3–7, the symmetries of each degree of freedom and the symmetries after coupling these degrees of freedom are discussed in detail; by solving the irreducible basis in different configuration spaces, we have provided the necessary color, spin, and flavor wave functions.

3. Mesons

Meson spectroscopy encompasses phenomena across an enormous mass range, from the relatively light pseudoscalar mesons with masses of just a few hundred MeV to the substantially heavier bottomonia $b\bar{b}$ systems approaching the 10 GeV scale. This remarkably broad energy span provides unique opportunities to investigate both perturbative and non-perturbative aspects of quantum chromodynamics (QCD). The abundance of experimental data from modern collider facilities and the deceptively simple structure of mesons have established them as a nearly ideal place for exploring and elucidating the fundamental properties and dynamics of QCD across different energy regimes. The systematic study of meson spectra consequently serves as a powerful tool for testing theoretical predictions, validating computational approaches like lattice QCD, and potentially revealing new phenomena beyond our current understanding of strong interaction physics.

Systematic calculation of meson spectra can be seen in [74], where many theoretical methods like unquenched quark models, lattice QCD, etc., are used to distinguish traditional meson pictures $q\bar{q}$ and glueballs and hybrid or multi-quark states [75–78].

In the constraint of the ground state in coordinate space, there are $5 \times 5 \times 4 = 100$ states, in which there are 36 light mesons, 48 light–heavy mesons, and 16 heavy mesons.

3.1. Color

Each quark has three possible colors, r, g, b, which construct the three-dimensional fundamental representation of SU_3^c:

$$\begin{aligned} \chi_1^{c1}(q) &= \boxed{r} = r, \\ \chi_2^{c1}(q) &= \boxed{g} = g, \\ \chi_3^{c1}(q) &= \boxed{b} = b. \end{aligned} \tag{14}$$

Additionally, the antiquark with colors, $\bar{r}, \bar{g}, \bar{b}$, constructs the anti-three-dimensional representation.

$$\begin{aligned} \chi_1^{c2}(\bar{q}) &= \begin{array}{|c|}\hline g \\ \hline b \\ \hline \end{array} = \frac{1}{\sqrt{2}}(gb - bg) = \bar{r}, \\ \chi_2^{c2}(\bar{q}) &= - \begin{array}{|c|}\hline r \\ \hline b \\ \hline \end{array} = \frac{1}{\sqrt{2}}(br - rb) = \bar{g}, \\ \chi_3^{c2}(\bar{q}) &= \begin{array}{|c|}\hline r \\ \hline g \\ \hline \end{array} = \frac{1}{\sqrt{2}}(rg - gr) = \bar{b}. \end{aligned} \tag{15}$$

Coupling representations of quark and antiquark, one arrives at

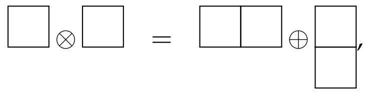
$$\begin{array}{ccc} \begin{array}{|c|}\hline \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} & = & \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \\ \mathbf{3} \otimes \bar{\mathbf{3}} & = & \mathbf{8} \oplus \mathbf{1} \end{array} \tag{16}$$

The color singlet requirement picks up the representation [111]. By using the Clebsch–Gordan (CG) coefficient of SU_3, the wave function for mesons of the color singlet can be written as

$$\begin{aligned} \chi^c_{[111]}(q\bar{q}) &= \frac{1}{\sqrt{3}}[r\frac{1}{\sqrt{2}}(gb - bg) + g\frac{1}{\sqrt{2}}(br - rb) + b\frac{1}{\sqrt{2}}(rg - gr)] \\ &= \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}). \end{aligned} \tag{17}$$

3.2. Spin

Two particles with spin $\frac{1}{2}$ can form a triplet state with spin 1 and a singlet state with spin 0. In terms of the Young diagram and dimension notation, they are



$$\mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{3} \oplus \mathbf{1}. \tag{18}$$

By referring to the CG coefficients of SU_2, the spin wave function χ^{σ}_{S,M_S} of a two-particle system can be written as

$[\sigma_2] = [2], S = 1$:

$$\begin{aligned} \chi^{\sigma}_{1,1} &= \alpha\alpha \\ \chi^{\sigma}_{1,0} &= \frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha) \\ \chi^{\sigma}_{1,-1} &= \beta\beta \end{aligned} \tag{19}$$

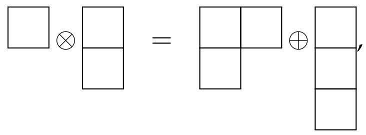
$[\sigma_2] = [11], S = 0$:

$$\chi^{\sigma}_{0,0} = \frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha) \tag{20}$$

3.3. Flavor

3.3.1. Light Mesons

For the light mesons, $q\bar{q}, q = u, d, s$, in the SU^f_3 picture, they are classified as singlet and octet.



$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{21}$$

By considering the group chain $SU^f_3 \supset SU^{\tau}_2 \otimes U^Y_1$, the flavor wave functions of $q\bar{q}$ can be written as

$[f] = [21],\ I = 1$:

$$\begin{aligned} \chi^f_{11} &= u\bar{d} & (\pi^+, \rho^+) \\ \chi^f_{10} &= \sqrt{\frac{1}{2}}(d\bar{d} - u\bar{u}) & (\pi^0, \rho^0) \\ \chi^f_{1-1} &= -d\bar{u} & (\pi^-, \rho^-) \end{aligned} \tag{22}$$

$[f] = [21],\ I = \frac{1}{2}$:

$$\begin{aligned} \chi^f_{\frac{1}{2}\frac{1}{2}} &= \begin{cases} u\bar{s} & (K^+, K^{*+}) \\ s\bar{d} & (\bar{K}^0, \bar{K}^{*0}) \end{cases} \\ \chi^f_{\frac{1}{2}-\frac{1}{2}} &= \begin{cases} d\bar{s} & (K^0, K^{*0}) \\ -s\bar{u} & (K^-, K^{*-}) \end{cases} \end{aligned} \tag{23}$$

$[f] = [21],\ I = 0$:

$$\chi^f_{00} = \sqrt{\frac{1}{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \quad (\eta_8, \omega_8). \tag{24}$$

$[f] = [111]$, $I = 0$:

$$\chi^f_{00} = \sqrt{\frac{1}{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \ \ (\eta_1, \omega_1). \tag{25}$$

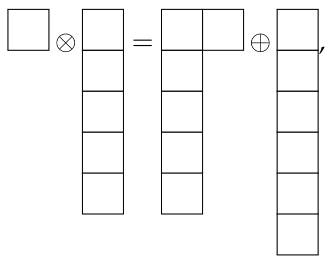
Because the flavor SU^f_3 is broken, two χ^f_{00} states will be mixed up, resulting in the physical η and η', ω and ϕ mesons

$$\begin{aligned}
|\eta\rangle &= \cos\theta_P|\eta_8\rangle + \sin\theta_P|\eta_1\rangle, \\
|\eta'\rangle &= -\sin\theta_P|\eta_8\rangle + \cos\theta_P|\eta_1\rangle, \\
|\omega\rangle &= \cos\theta_V|\omega_8\rangle + \sin\theta_V|\omega_1\rangle, \\
|\phi\rangle &= -\sin\theta_V|\omega_8\rangle + \cos\theta_V|\omega_1\rangle.
\end{aligned} \tag{26}$$

In obtaining the above wave functions, the isospin doublets for quark and antiquark are used.

$$\begin{pmatrix} |IM_I\rangle = |\frac{1}{2}\frac{1}{2}\rangle \\ |IM_I\rangle = |\frac{1}{2}\text{-}\frac{1}{2}\rangle \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} \text{ or } \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}. \tag{27}$$

In the light meson sector, sometimes the spin–flavor symmetry $SU^{f\sigma}_6 \supset SU^f_3 \otimes SU^\sigma_2$ is used, and then the mesons in the $SU^{f\sigma}_6$ scheme are



(28)

$$\mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{35} \oplus \mathbf{1}.$$

To extract the SU^f_3 flavor and SU^σ_2 spin symmetry from the spin–flavor symmetry $SU^{f\sigma}_6$, the CG series of the permutation group can be employed, and one has

$$\begin{aligned}
&\mathbf{1} = (\mathbf{1},\mathbf{1}) \rightarrow \text{flavor singlet meson with } J^P = 0^-, \eta_1, \\
&\mathbf{35} = (\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{8},\mathbf{3}). \\
&\qquad\quad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\
&\quad (K,\pi,\eta_8) \quad \omega_1 \quad (\rho, K^*, \omega_8)
\end{aligned} \tag{29}$$

The notation $(\boldsymbol{D}, 2S+1)$ represents the symmetries of flavor SU^f_3 and spin SU^σ_2, the singlet $\mathbf{1}$ of $SU^{f\sigma}_6$ corresponds to an $S = 0$ flavor singlet meson, and multiplets $\mathbf{35}$ consist of an $S = 0$ flavor octet, an $S = 1$ flavor octet, and an $S = 1$ flavor singlet [79].

The weight diagrams of octet and singlet mesons under SU^f_3 are shown in Figures 2 and 3 with coupling process $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$. A point represents a state, a circle is added to the point if there is another state located at the same place, and so on.

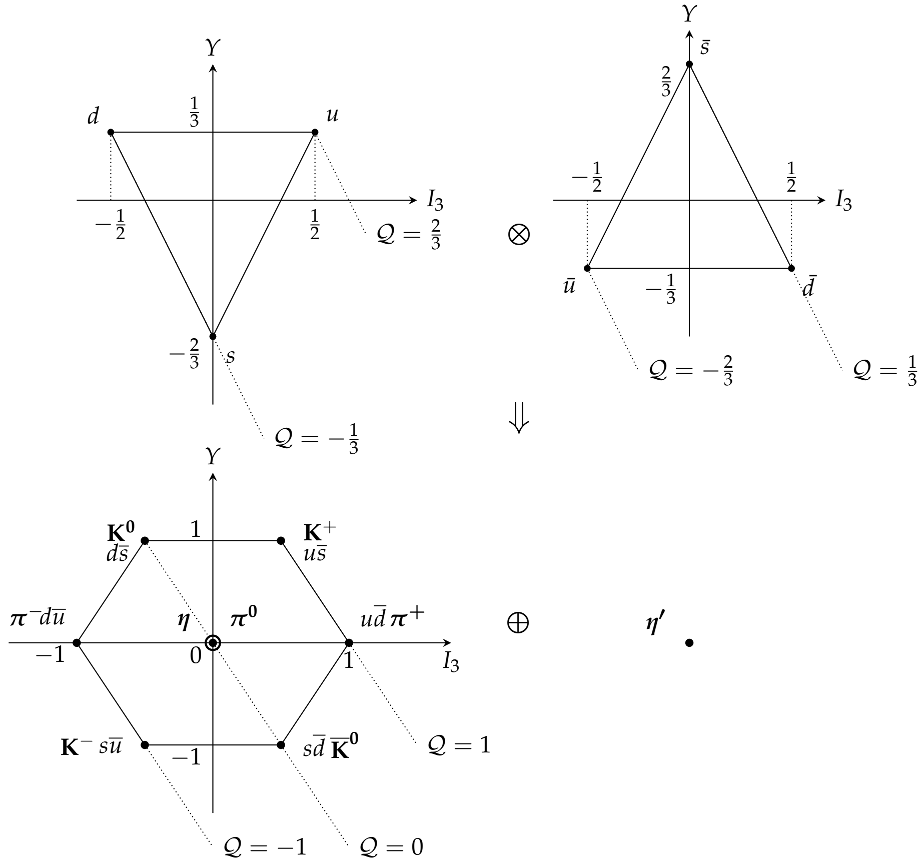


Figure 2. Pseudoscalar mesons of octet and singlet under SU_3^f symmetry with $J^P = 0^-$.

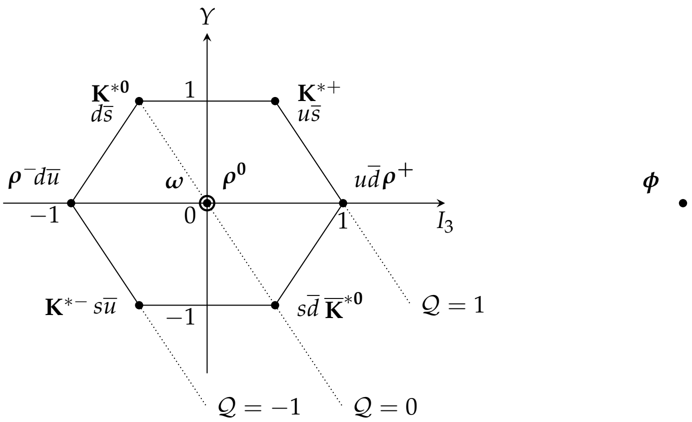


Figure 3. Vector mesons of octet and singlet under SU_3^f symmetry with $J^P = 1^-$.

3.3.2. Light–Heavy Mesons

For the light–heavy mesons, the flavor symmetry is carried by one particle and no coupling in the flavor part is needed. The wave functions are listed as follows.

$[f] = [1], I = \frac{1}{2}$:

$$\begin{aligned} \chi^f_{\frac{1}{2}\frac{1}{2}} &= u\bar{Q} \;\; (\bar{D}^0, \bar{D}^{*0}, B^+, B^{*+}), \\ \chi^f_{\frac{1}{2}-\frac{1}{2}} &= d\bar{Q} \;\; (D^-, D^{*-}, B^0, B^{*0}). \end{aligned} \tag{30}$$

$[f] = [11], I = \frac{1}{2}$:

$$\chi^f_{\frac{1}{2}\frac{1}{2}} = Q\bar{d} \ \ (D^+, D^{*+}, \bar{B}^0, \bar{B}^{*0}),$$
$$\chi^f_{\frac{1}{2}-\frac{1}{2}} = -Q\bar{u} \ \ (D^0, D^{*0}, B^-, B^{*-}). \tag{31}$$

$[f] = [1], I = 0$:

$$\chi^f_{00} = s\bar{Q} \ \ (D_s^-, D_s^{*-}, B_s^0, B_s^{*0}). \tag{32}$$

$[f] = [11], I = 0$:

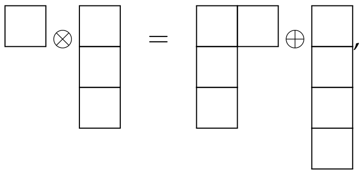
$$\chi^f_{00} = Q\bar{s} \ \ (D_s^+, D_s^{*+}, \bar{B}_s^0, \bar{B}_s^{*0}). \tag{33}$$

3.3.3. Heavy Mesons

For the heavy mesons, no flavor symmetry is applied. The wave functions are just the product of the quark and antiquark states.

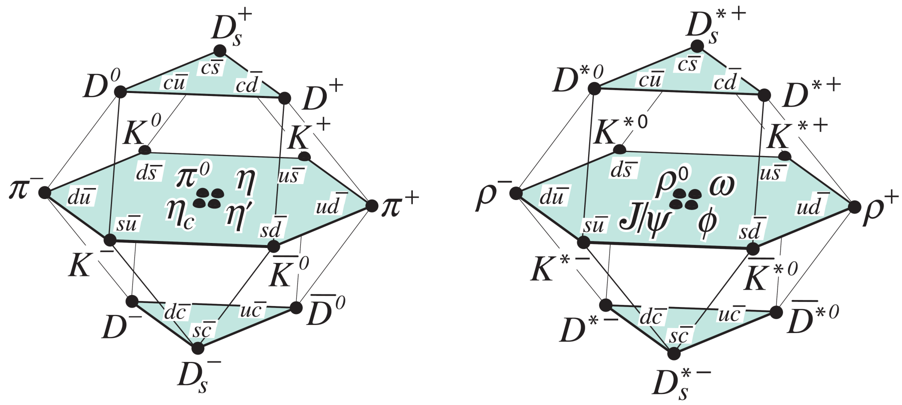
$$\chi^f_{00} = Q\bar{Q}' \ \ (\eta_c, J/\psi, B_c^+, B_c^{*+}, B_c^-, B_c^{*-}, \eta_b, \Upsilon). \tag{34}$$

Expanding the flavor symmetry to SU_4^f by including c quark, the meson states are classified into a 15-plet and a singlet. Light and light-charm mesons are demonstrated in Figure 4.



(35)

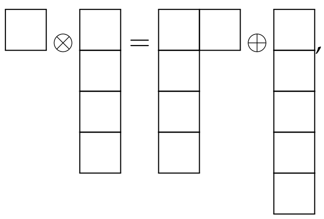
$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{15} \oplus \mathbf{1}.$$



(**a**) Pseudoscalar mesons with $J^P = 0^-$. (**b**) Vector mesons with $J^P = 1^-$.

Figure 4. Mesons under SU_4^f symmetry [43].

Adding the b quark to the flavor box, SU_5^f symmetry employed, and one has 25 flavor states, which are classified into a 24-plet and a singlet.



(36)

$$\mathbf{5} \otimes \bar{\mathbf{5}} = \mathbf{24} \oplus \mathbf{1}.$$

The total wave function of the mesons can be obtained by multiplying the ones from each degree of freedom. No further coupling is needed for the coordinate ground state.

A mass-squared formula for an SU_8 supermultiplet of mesons and a linear mass formula of baryons is proposed in [80]. After the charm quark is introduced, the symmetry of spin–flavor space is extended to $SU_8^{f\sigma} \supset SU_4^f \otimes SU_2^\sigma$; similar to Equation (29), we have the following composition:

$$\begin{aligned} \mathbf{8} \otimes \bar{\mathbf{8}} &= \mathbf{63} \oplus \mathbf{1}, \\ \mathbf{63} &= (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{15}, \mathbf{3}). \end{aligned} \tag{37}$$

Mesons satisfy the Klein–Gordan equation, and most of the literature gave a mass squared formula for it [71,80]:

$$M^2 = M_0^2 + X_1 J(J+1) + X_2[I(I+1) - \frac{Y^2}{4}]. \tag{38}$$

According to the Equation (38), one can get [80]:

$$\begin{aligned} K^{*2} - \rho^2 &= K^2 - \pi^2 \\ D^{*2} - J/\psi^2 &= D^2 - \eta_c^2 \end{aligned} \tag{39}$$

The application of Gürsey–Radicati (GR) mass formula in mesons can be seen in [81,82]. They construct a mass formula for the string-like properties of $q\bar{q}$ mesons, and the result of the mass formula offered a reference for them to distinguish mesons which are not $q\bar{q}$ type like $a_0(980)$.

Here, due to the existence of heavy quarks and antiquarks in mesons, we perform a little modification with formula Equation (12).

$$M = M_0 + X_1 C(SU_3^f) + X_2 J(J+1) + X_3 I(I+1) + \sum_{i=s,c,b} X_i N_i \,, \tag{40}$$

where N_s, N_c, N_b stand for the sum of the number of $s/\bar{s}$, $c/\bar{c}$, $b/\bar{b}$, respectively. Table 3 shows the result of some light mesons, light–heavy mesons, and heavy mesons in ground orbit with formula Equation (40). A gross feature of the meson spectrum is obtained.

Table 3. The parameters and masses of ground mesons.

Parameter	M_0	X_1		X_2		X_3	
	516.47 ± 0.52	-10.282 ± 0.076		221.17 ± 0.071		-133.68 ± 0.12	
	X_s	X_c		X_b			
	128.01 ± 0.23	1330.7 ± 0.28		4437.3 ± 0.37			
Meson	Y (hypercharge)	I (isospin)	J (spin)	$[f]$ (flavor)	Theo. (MeV)	Exp. [43] (MeV)	Ref. (MeV)
π	0	1	0	[21]	187.42 ± 1.18	139.57	146.4 [83]
K	1	$\frac{1}{2}$	0	[21]	482.53 ± 1.28	493.67	524.7 [83]
η	0	0	0	[21]	454.78 ± 0.97	547.86	543.5 ± 5.2 [84]
η'	0	0	0	[111]	772.50 ± 0.97	957.78	986 ± 35 [84]
ρ	0	1	1	[21]	629.76 ± 1.32	775.26	796 [85]
K^*	1	$\frac{1}{2}$	1	[21]	924.87 ± 1.42	892.00	893 [85]
ω	0	0	1	[21]	897.12 ± 1.11	782.66	691 [74]
ϕ	0	0	1	[111]	1214.84 ± 1.11	1019.46	1020 [74]

Table 3. *Cont.*

D	$-\frac{1}{3}$	$\frac{1}{2}$	0	[11]	1719.53 ± 1.08	1869.66	1883 [74]
D_s	$\frac{2}{3}$	0	0	[11]	1947.80 ± 1.23	1968.35	1998.2 [86]
D^*	$-\frac{1}{3}$	$\frac{1}{2}$	1	[11]	2161.87 ± 1.22	2006.85	2010 [74]
D_s^*	$\frac{2}{3}$	0	1	[11]	2390.14 ± 1.37	2112.20	2086.5 [86]
η_c	0	0	0	[0]	3177.95 ± 1.08	2984.10	3022.8 [83]
J/ψ	0	0	1	[0]	3620.29 ± 1.22	3096.90	3121.1 [83]
B	$-\frac{1}{3}$	$\frac{1}{2}$	0	[11]	4826.10 ± 1.16	5279.41	5281 [74]
B_s	$\frac{2}{3}$	0	0	[11]	5054.37 ± 1.31	5366.93	5355 [74]
B^*	$-\frac{1}{3}$	$\frac{1}{2}$	1	[11]	5268.44 ± 1.30	5324.75	5321 [74]
B_s^*	$\frac{2}{3}$	0	1	[11]	5496.71 ± 1.45	5415.40	5400 [74]
η_b	0	0	0	[0]	9391.08 ± 1.24	9398.70	9454 [74]
Υ	0	0	1	[0]	9833.43 ± 1.38	9460.40	9505 [74]
B_c	$-\frac{2}{3}$	0	0	[0]	6284.52 ± 1.16	6274.47	6277 [74]

4. Baryon

Baryons are the cornerstone of the matter that constitutes the visible universe. As composite particles tightly bound by the strong interaction among three quarks, they are the main form of ordinary matter in the universe. From protons and neutrons that make up atomic nuclei to particles such as Λ and Σ, the baryon family has a rich variety of members, and their properties and structures are directly related to the basic laws of the material world. According to the differences in quark composition, baryons can be divided into ordinary baryons containing light quarks (up, down, strange quarks) and heavy-flavored baryons containing heavy quarks (charm, bottom quarks). Their ground states and excited states together form a complex baryon spectrum.

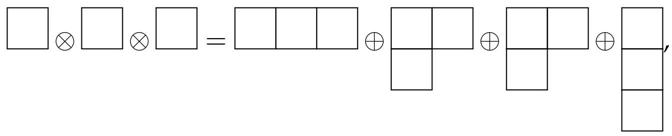
The study of baryons is not only an important testing place for the non-perturbative effects of QCD but also a key to uncovering the nature of the strong interaction and exploring the evolution of matter in the early universe. However, due to the complexity of non-perturbative QCD and the difficulty in accurately describing multi-quark systems, many mysteries about the internal structure and dynamic behavior of baryons remain unsolved, driving physicists to continuously develop new theoretical models and experimental techniques to explore deeper into the baryon world. Many ground baryons have been identified. However, a large number of excited baryons predicted by theory have not been observed in experiments, which is the well-known 'missing state' problem [43,87–89]. Searching for baryon missing states is an important direction in particle physics. The baryon spectrum is a direct manifestation of the non-perturbative effects of QCD. The discovery or exclusion of missing states can verify the accuracy of the theory in describing strong interactions. By comparing the theoretical and experimental baryon spectra, we can gain a deeper understanding of the interactions between quarks (such as spin–orbit coupling and color charge interaction). Similar to mesons, baryons may also mix with multi-quark states [90].

$$\begin{aligned} |Baryon\rangle &= |qqq\rangle + |qqqq\bar{q}\rangle + |qqqg\rangle + \dots \\ |Meson\rangle &= |q\bar{q}\rangle + |q\bar{q}q\bar{q}\rangle + |q\bar{q}g\rangle + \dots . \end{aligned} \tag{41}$$

In quark models, a baryon is composed of three valence quarks. The dimension of the configuration space for color is $3^3 = 27$, and the color singlet picks up only one state. The dimension of the flavor space is $5 \times 5 \times 5 = 125$, in which there are 27 light states, 54 heavy–light–light states, 36 heavy–heavy–light states, and 8 full heavy states, and the dimension for spin space is 8. However the total anti-symmetrization will reduce the number of baryon states. For example, there are only 56 light ground state baryons, 8 baryons with spin $\frac{1}{2}$, and 10 baryons with spin $\frac{3}{2}$. The details of the baryon states are listed below.

4.1. Color

The color symmetry is SU_3^c, and the possible representations for a three-particle system can be derived by the CG series of SU_3^c or the outer product of permutation group S_3. One has the following:



$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{MS} \oplus \mathbf{8}_{MA} \oplus \mathbf{1}_A. \tag{42}$$

The subscripts denote the permutation symmetry. S means the state is symmetric under particle permutation, $MS(MA)$ stands for the state being symmetric (antisymmetric) only under the particle 1 and 2 exchange, A stands for total antisymmetry under particle permutation. The physical state must be a color singlet, and the color symmetry of the three-quark system can only be a color singlet **1**.

By using CG coefficients of SU_3, or according to Equations (15) and (17), the color wave function of the baryon can be written as

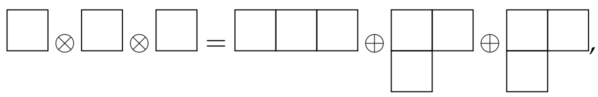
$$\begin{aligned} \chi^c_{[111]}(qqq) &= \frac{1}{\sqrt{3}}[r\frac{1}{\sqrt{2}}(gb-bg)+g\frac{1}{\sqrt{2}}(br-rb)+b\frac{1}{\sqrt{2}}(rg-gr)] \\ &= \frac{1}{\sqrt{6}}(rgb-rbg+gbr-grb+brg-bgr). \end{aligned} \tag{43}$$

4.2. Spin

According to (19), the spin of the first two particles can be 1 or 0, and then we can couple the third particle.

$$\frac{1}{2}\otimes\frac{1}{2}\rightarrow\begin{cases}1\otimes\frac{1}{2}\rightarrow\frac{1}{2},\frac{3}{2}\\ 0\otimes\frac{1}{2}\rightarrow\frac{1}{2}\end{cases}. \tag{44}$$

So for a baryon composed of three quarks, its total spin is $\frac{3}{2}$ or $\frac{1}{2}$; the CG series of SU_2 gives



$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_S \oplus \mathbf{2}_{MS} \oplus \mathbf{2}_{MA}. \tag{45}$$

The spin wave functions $\chi^{\sigma_i}_{SM_S}(qqq)$ of three-quark system can be obtained with the help of the CG coefficients of SU_2. From the wave functions of two particles and one particle, one gets $2S+1$ symmetric or anti-symmetric wave functions for different total spin under the exchange of the first two particles.

$[\sigma_3] = [3], S = \frac{3}{2}$:

$$\begin{aligned}
\chi^{\sigma}_{\frac{3}{2},\frac{3}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline \alpha & \alpha & \alpha \\ \hline \end{array} \right\rangle = \alpha\alpha\alpha, \\
\chi^{\sigma}_{\frac{3}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline \alpha & \alpha & \beta \\ \hline \end{array} \right\rangle = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha), \\
\chi^{\sigma}_{\frac{3}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline \alpha & \beta & \beta \\ \hline \end{array} \right\rangle = \frac{1}{\sqrt{3}}(\alpha\beta\beta + \beta\alpha\beta + \beta\beta\alpha), \\
\chi^{\sigma}_{\frac{3}{2},-\frac{3}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline \beta & \beta & \beta \\ \hline \end{array} \right\rangle = \beta\beta\beta.
\end{aligned} \tag{46}$$

$[\sigma_3] = [21], S = \frac{1}{2}$:

$$\begin{aligned}
\chi^{\sigma_1}_{\frac{1}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\ \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{6}}(2\alpha\alpha\beta - \alpha\beta\alpha - \beta\alpha\alpha), \\
\chi^{\sigma_2}_{\frac{1}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\ \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\alpha - \beta\alpha\alpha), \\
\chi^{\sigma_1}_{\frac{1}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\ \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \beta \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{6}}(\alpha\beta\beta + \beta\alpha\beta - 2\beta\beta\alpha), \\
\chi^{\sigma_2}_{\frac{1}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\ \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \beta \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\beta - \beta\alpha\beta).
\end{aligned} \tag{47}$$

For example, using the familiar CG coefficients, one gets

$$\begin{aligned}
\left| \frac{3}{2}\frac{1}{2} \right\rangle &= \sqrt{\frac{1}{3}}|11\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}}|10\rangle \left| \frac{1}{2}\frac{1}{2} \right\rangle \\
&= \sqrt{\frac{1}{3}}\alpha\alpha\beta + \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}(\alpha\beta + \beta\alpha)\alpha \\
&= \sqrt{\frac{1}{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha).
\end{aligned} \tag{48}$$

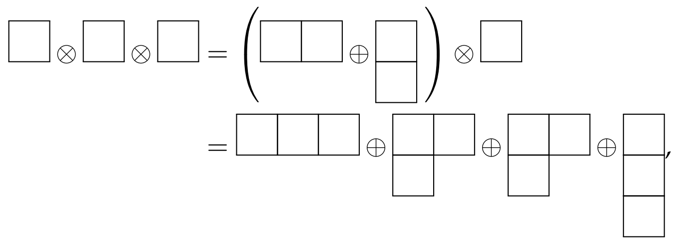
Other spin wave functions can be obtained with the same procedures.

4.3. *Flavor*

4.3.1. *qqq*

In the exploration process of particle physics, the study of nucleons, Roper resonances, and Δ has always been at the forefront and critical position, which greatly promotes our in-depth understanding of the microstructure and fundamental interactions of matter. The nucleus, as the fundamental unit of the atomic nucleus, is composed of protons and neutrons, and its properties and interactions are the cornerstone of understanding the structure and behavior of atomic nuclei. Roper resonance, as an excited state of nucleons, has been a hot topic in particle physics research since its announcement in 1964 [91]. Its mass is about 1440 MeV and its spin parity is $\frac{1}{2}^+$. Its unique properties are of great significance for testing and improving the theory of strong interactions. However, due to the extremely short lifetime of the Roper resonance state, precise detection and study of it in experiments face many challenges. For many years, scientists have attempted to accurately measure the various properties of Roper resonance through various experimental methods such as electron nucleon scattering and π-meson nucleon scattering, in order to verify theoretical predictions and reveal its internal structure.

For a system composed of three light quarks, the flavor symmetry SU_3^f states can be expressed as follows:



(49)

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{MS} \oplus \mathbf{8}_{MA} \oplus \mathbf{1}_A.$$

The weight diagrams of octet and decuplet baryons under SU_3^f are presented in Figures 5 and 6. The coupling processes are also shown in the figures.

There are 27-flavor wave functions for baryons composed of three light quarks, which can be obtained by using the CG coefficients of the SU_3 group, or by finding the irreducible bases of permutation group S_3 in the configuration spaces, they are denoted by $\chi^{f_i}_{[f]IM_I}(qqq)$. More theory about constructing octet baryons under the SU_3 group can be seen in [92].

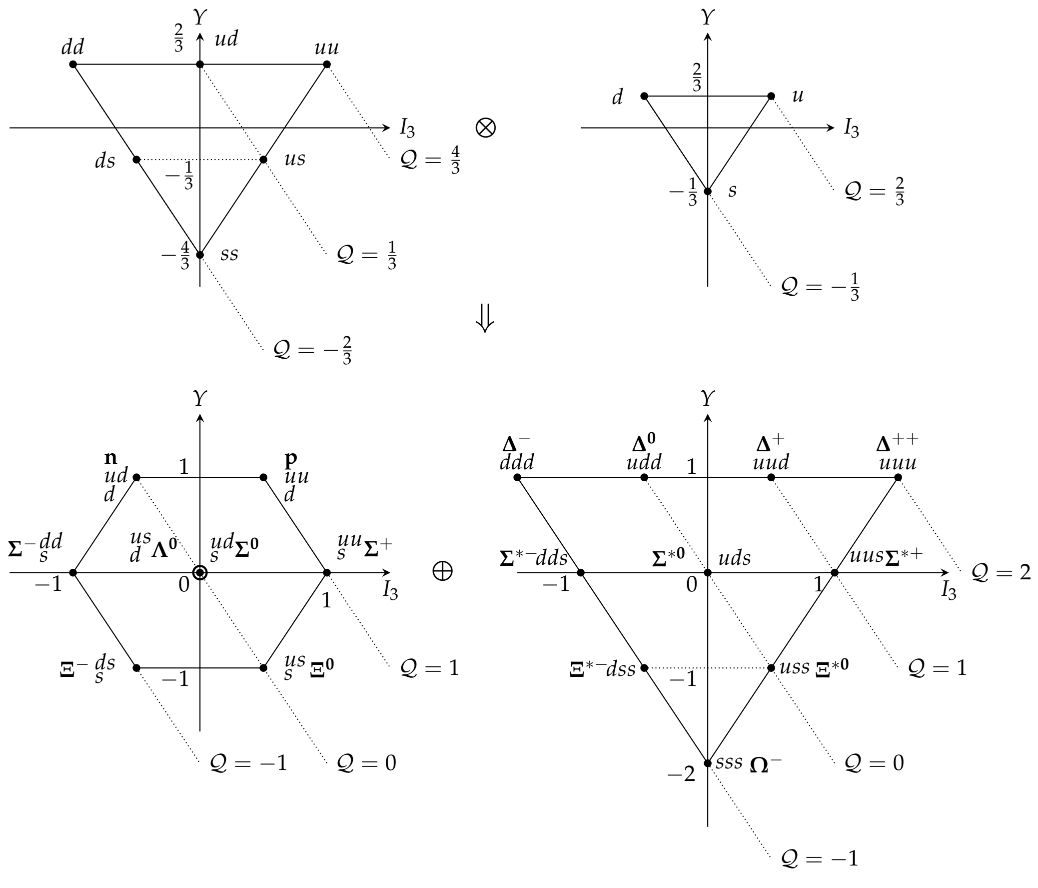


Figure 5. $J^P = \frac{1}{2}^+$ octet baryons for the partition [21] of SU_3^f and $J^P = \frac{3}{2}^+$ decuplet baryons for the partition [3] of SU_3^f; they come from $[2]_{\mathbf{6}_S} \otimes [1]_{\mathbf{3}} = [21]_{\mathbf{8}_{MS}} \oplus [3]_{\mathbf{10}_S}$.

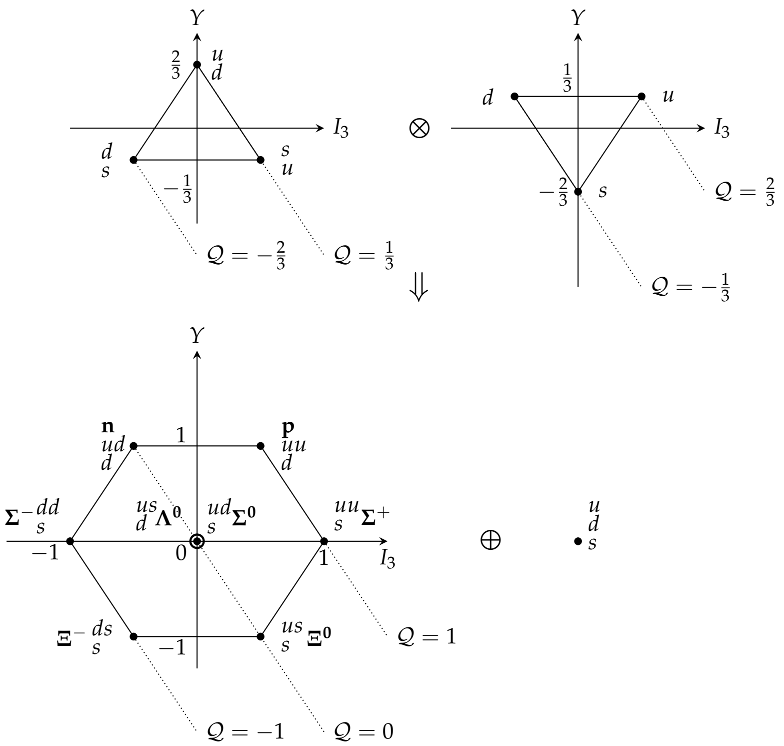


Figure 6. $J^P = \frac{1}{2}^+$ octet baryons for the partition [21] of SU_3^f and flavor singlet [111] of SU_3^f; they come from $[11]_{\mathbf{3}_A} \otimes [1]_{\mathbf{3}} = [21]_{\mathbf{8}_{MA}} \oplus [3]_{\mathbf{1}_A}$.

$[f] = [3], I = \frac{3}{2}, Y = 1$:

$$\begin{aligned}
\chi^{f_1}_{[3]\frac{3}{2},\frac{3}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline u&u&u\\\hline\end{array} \right\rangle = uuu, \\
\chi^{f_2}_{[3]\frac{3}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline u&u&d\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{3}}(uud + udu + duu), \\
\chi^{f_3}_{[3]\frac{3}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline u&d&d\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{3}}(udd + dud + ddu), \\
\chi^{f_4}_{[3]\frac{3}{2},-\frac{3}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline d&d&d\\\hline\end{array} \right\rangle = ddd.
\end{aligned} \tag{50}$$

$[f] = [3], I = 1, Y = 0$:

$$\begin{aligned}
\chi^{f_1}_{[3]11}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline u&u&s\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{3}}(uus + usu + suu), \\
\chi^{f_2}_{[3]10}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline u&d&s\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{6}}(uds + usd + dus + dsu + sud + sdu), \\
\chi^{f_3}_{[3]1-1}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline d&d&s\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{3}}(dds + dsd + sdd).
\end{aligned} \tag{51}$$

$[f] = [3], I = \frac{1}{2}, Y = -1$:

$$\begin{aligned}
\chi^{f_1}_{[3]\frac{1}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline u&s&s\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{3}}(uss + sus + ssu), \\
\chi^{f_2}_{[3]\frac{1}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline d&s&s\\\hline\end{array} \right\rangle = \frac{1}{\sqrt{6}}(dss + sds + ssd).
\end{aligned} \tag{52}$$

$[f] = [3], I = 0, Y = -2$:

$$\chi^{f}_{[3]00}(qqq) = \left| \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \ \begin{array}{|c|c|c|}\hline s & s & s \\ \hline \end{array} \right\rangle = sss. \tag{53}$$

$[f] = [21], I = \frac{1}{2}, Y = 1$:

$$\begin{aligned}
\chi^{f_1}_{[21]\frac{1}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & u \\ \hline d \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{6}}(2uud - udu - duu), \\
\chi^{f_1}_{[21]\frac{1}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & d \\ \hline d \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{6}}(udd + dud - 2ddu), \\
\chi^{f_2}_{[21]\frac{1}{2},\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & u \\ \hline d \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{2}}(udu - duu) \\
\chi^{f_2}_{[21]\frac{1}{2},-\frac{1}{2}}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & d \\ \hline d \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{2}}(udd - dud).
\end{aligned} \tag{54}$$

$[f] = [21], I = 1, Y = 0$:

$$\begin{aligned}
\chi^{f_1}_{[21]11}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & u \\ \hline s \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{6}}(2uus - usu - suu), \\
\chi^{f_1}_{[21]10}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & d \\ \hline s \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{12}}(2uds + 2dus - usd - dsu - sud - sdu), \\
\chi^{f_1}_{[21]1-1}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline d & d \\ \hline s \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{6}}(2dds - dsd - sdd).
\end{aligned} \tag{55}$$

$$\begin{aligned}
\chi^{f_2}_{[21]11}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & u \\ \hline s \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{2}}(usu - suu), \\
\chi^{f_2}_{[21]10}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline u & d \\ \hline s \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{4}}(usd + dsu - sud - sdu), \\
\chi^{f_2}_{[21]1-1}(qqq) &= \left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \ \begin{array}{|c|c|}\hline d & d \\ \hline s \\ \cline{1-1} \end{array} \right\rangle = \frac{1}{\sqrt{2}}(dsd - sdd).
\end{aligned} \tag{56}$$

$[f]=[21], I=\frac{1}{2}, Y=-1$:

$$
\begin{aligned}
\chi^{f_1}_{[21]\frac{1}{2},\frac{1}{2}}(qqq) &= \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline u & s \\ \hline s \\ \cline{1-1}\end{array}\right\rangle = \frac{1}{\sqrt{6}}(uss+sus-2ssu), \\
\chi^{f_1}_{[21]\frac{1}{2},-\frac{1}{2}}(qqq) &= \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline d & s \\ \hline s \\ \cline{1-1}\end{array}\right\rangle = \frac{1}{\sqrt{6}}(dss+sds-2ssd), \\
\chi^{f_2}_{[21]\frac{1}{2},\frac{1}{2}}(qqq) &= \left|\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline u & s \\ \hline s \\ \cline{1-1}\end{array}\right\rangle = \frac{1}{\sqrt{2}}(uss-sus), \\
\chi^{f_2}_{[21]\frac{1}{2},-\frac{1}{2}}(qqq) &= \left|\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline d & s \\ \hline s \\ \cline{1-1}\end{array}\right\rangle = \frac{1}{\sqrt{2}}(dss-sds).
\end{aligned}
\tag{57}
$$

$[f]=[21], I=0, Y=0$:

$$
\begin{aligned}
\chi^{f_1}_{[21]00}(qqq) &= \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline u & s \\ \hline d \\ \cline{1-1}\end{array}\right\rangle = \frac{1}{\sqrt{4}}(usd-dsu+sud-sdu), \\
\chi^{f_2}_{[21]00}(qqq) &= \left|\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline u & s \\ \hline d \\ \cline{1-1}\end{array}\right\rangle = \frac{1}{\sqrt{12}}(2uds-2dus+usd-dsu-sud+sdu).
\end{aligned}
\tag{58}
$$

There is a spin–flavor locking for ground state light baryons; flavor octets have 1/2 spin, and decuplets have 3/2 spin. By using the CG coefficients of S_3, the wave functions of the ground baryons (spin–flavor part) are as follows:

$$
\begin{aligned}
&\text{For the decuplet:} \\
&\quad \chi^{\sigma_f}_{IM_ISM_S}(qqq) = \chi^{\sigma}_{\frac{3}{2}M_S}(qqq)\chi^{f}_{[3]IM_I}(qqq), \\
&\text{For the octet:} \\
&\quad \chi^{\sigma_f}_{IM_ISM_S}(qqq) = \frac{1}{\sqrt{2}}\left[\chi^{\sigma_1}_{\frac{1}{2}M_S}(qqq)\chi^{f_1}_{[21]IM_I}(qqq) + \chi^{\sigma_2}_{\frac{1}{2}M_S}(qqq)\chi^{f_2}_{[21]IM_I}(qqq)\right],
\end{aligned}
\tag{59}
$$

Coupling four degrees of freedom together, the total antisymmetrized wave functions are obtained. For the ground state baryons, the coupling is illustrated in the following.

$$
\begin{aligned}
\ydiagram{1,1,1}^{\,xc\sigma f} &= \ydiagram{3}^{\,x} \otimes \ydiagram{1,1,1}^{\,c} \otimes \ydiagram{3}^{\,\sigma} \otimes \ydiagram{3}^{\,f}, \\
\mathbf{1}^{xc\sigma f} &= \mathbf{1}^{x}\otimes\mathbf{1}^{c}\otimes\mathbf{4}^{\sigma}\otimes\mathbf{10}^{f}. \\
\ydiagram{1,1,1}^{\,xc\sigma f} &= \ydiagram{3}^{\,x} \otimes \ydiagram{1,1,1}^{\,c} \otimes \ydiagram{2,1}^{\,\sigma} \otimes \ydiagram{2,1}^{\,f}, \\
\mathbf{1}^{xc\sigma f} &= \mathbf{1}^{x}\otimes\mathbf{1}^{c}\otimes\mathbf{2}^{\sigma}\otimes\mathbf{8}^{f}.
\end{aligned}
\tag{60}
$$

The octet and decuplet baryons are shown in Figures 5 and 6. In the weight diagrams of the $\frac{1}{2}^+$ baryon and the $\frac{3}{2}^+$ baryon states, the horizontal axis I_3 represents the third component of the isospin, and the vertical axis represents the hypercharge of the baryon states. For the ground states, all quarks occupy the same state in the coordinate space; only

56 baryon states are allowed because of the total anti-symmetrization. Based on the $SU_4^{\sigma\tau}$ symmetry and baryon, Dyson and Xuong predicted several possible non-strange dibaryon states [50]; we will discuss the dibaryon in Section 7.

4.3.2. *qqQ*

For light–light–heavy baryons, the flavor symmetry analysis only applies to the light quarks, the flavor structures of singly charmed baryons are shown in Figure 7.

$$\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\ \square\end{array}, \qquad \mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}. \tag{61}$$

The wave functions $\chi^f_{[f]IM_I}(qqQ)$ are $(Q = c, b)$

$[f] = [2], I = 1, Y = \frac{2}{3}$:

$$\begin{aligned} \chi^f_{[2]11}(qqQ) &= uuQ, \\ \chi^f_{[2]10}(qqQ) &= \frac{1}{\sqrt{2}}(ud + du)Q, \\ \chi^f_{[2]1-1}(qqQ) &= ddQ. \end{aligned} \tag{62}$$

$[f] = [2], I = \frac{1}{2}, Y = -\frac{1}{3}$:

$$\begin{aligned} \chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qqQ) &= \frac{1}{\sqrt{2}}(us + su)Q, \\ \chi^f_{[2]\frac{1}{2},-\frac{1}{2}}(qqQ) &= \frac{1}{\sqrt{2}}(ds + sd)Q. \end{aligned} \tag{63}$$

$[f] = [2], I = 0, Y = -\frac{4}{3}$:

$$\chi^f_{[2]00}(qqQ) = ssQ. \tag{64}$$

$[f] = [11], I = \frac{1}{2}, Y = -\frac{1}{3}$:

$$\begin{aligned} \chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qqQ) &= \frac{1}{\sqrt{2}}(us - su)Q, \\ \chi^f_{[11]\frac{1}{2},-\frac{1}{2}}(qqQ) &= \frac{1}{\sqrt{2}}(ds - sd)Q. \end{aligned} \tag{65}$$

$[f] = [11], I = 0, Y = \frac{2}{3}$:

$$\chi^f_{[11]00}(qqQ) = \frac{1}{\sqrt{2}}(ud - du)Q. \tag{66}$$

For the ground state, the anti-symmetrization locks spin $\frac{3}{2}$ to flavor [2], but spin $\frac{1}{2}$ can combine with flavors [2] and [11]. By using the CG coefficients of S_3, the wave functions of ground light–light–heavy baryons (spin–flavor part) are as follows:

$$\begin{aligned} \chi^{\sigma f}_{IM_ISM_S}(qqQ) &= \chi^{\sigma}_{\frac{3}{2}M_S}(qqq)\chi^{f}_{[2]IM_I}(qqQ) \quad \text{(for sextet)}, \\ \chi^{\sigma f}_{IM_ISM_S}(qqQ) &= \chi^{\sigma_1}_{\frac{1}{2}M_S}(qqq)\chi^{f}_{[2]IM_I}(qqQ) \quad \text{(for sextet)}, \\ \chi^{\sigma f}_{IM_ISM_S}(qqQ) &= \chi^{\sigma_2}_{\frac{1}{2}M_S}(qqq)\chi^{f}_{[11]IM_I}(qqQ) \quad \text{(for anti-triplet)}. \end{aligned} \tag{67}$$

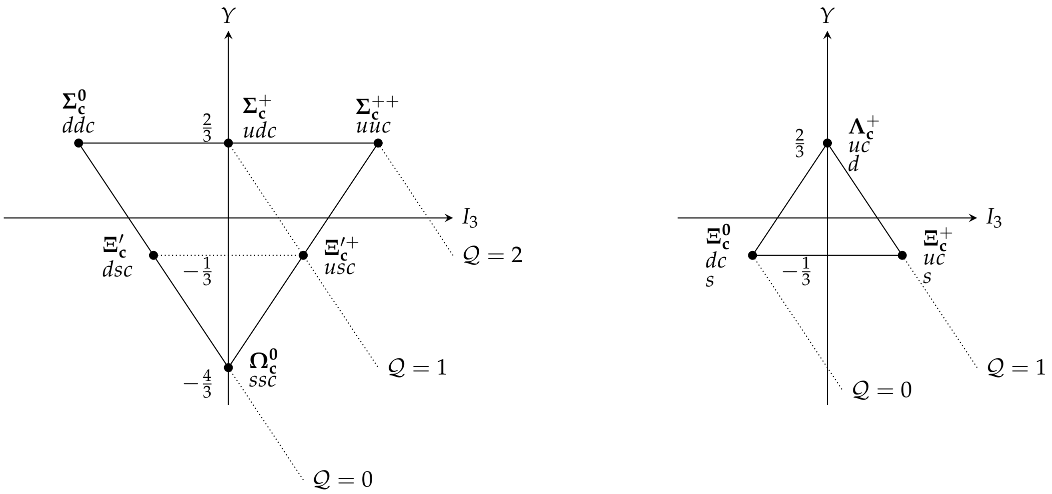


Figure 7. The weight diagrams of sextet and anti-triplet for single charm baryons. (Here we assume the heavy quark to be a charm quark; if the heavy quark is a bottom quark, the graph will be exactly the same, only the charge will be different.)

4.3.3. *QQq*

The spectroscopy of doubly heavy baryons provides a unique window into strong interaction dynamics, establishing them as a focal point in contemporary hadron physics. Over the past two decades, this field has witnessed significant advancements both theoretically and experimentally. The SELEX collaboration's 2002 report of the Ξ_{cc}^{++} baryon marked the initial experimental endeavor, though its confirmation remained inconclusive [93]. A major milestone was achieved with LHCb's observation of Ξ_{cc}^{++} [94–96], now officially recognized in the Particle Data Group (PDG) review [43]. The mass of $\Xi_{cc}^{++} = ccu$ reported by LHCb is 3621.40 ± 0.72 MeV, which is close to the theoretical estimated value—3627 ± 12 MeV, the same methods were used to predict Ω_{cc}, Ω_{cc}^*, Ω_{bb}, and Ω_{bc} [97,98]. Subsequent efforts have targeted other doubly heavy states, including Ξ_{bc}^0 and Ω_{bc}^0 [99]. It is believed that double-bottom baryons Ξ_{bb} or Ω_{bb} can be detected in the near future [100,101].

For heavy–heavy–light baryons, the flavor symmetry is trivial, $[f] = [1]$. So the charm and bottomness are used to characterize the states. The anti-symmetrization is applied to the states with two identical heavy quarks.

The flavor wave functions of $\chi^{f}_{IM_I,CB}(QQq)$ are as follows:

$I = \frac{1}{2}, Y = \frac{1}{3}, C = 2, B = 0$:

$$\begin{aligned} \chi^{f}_{\frac{1}{2},\frac{1}{2},20}(QQq) &= \quad ccu, \\ \chi^{f}_{\frac{1}{2},-\frac{1}{2},20}(QQq) &= \quad ccd. \end{aligned} \tag{68}$$

$I = 0, Y = -\frac{2}{3}, C = 2, B = 0$:

$$\chi^f_{00,20}(QQq) = ccs. \tag{69}$$

$I = \frac{1}{2}, Y = \frac{1}{3}, C = 1, B = -1$:

$$\begin{aligned} \chi^f_{\frac{1}{2},\frac{1}{2},1-1}(QQq) &= cbu, \\ \chi^f_{\frac{1}{2},-\frac{1}{2},1-1}(QQq) &= cbd. \end{aligned} \tag{70}$$

$I = 0, Y = -\frac{2}{3}, C = 1, B = -1$:

$$\chi^f_{00,1-1}(QQq) = cbs. \tag{71}$$

$I = \frac{1}{2}, Y = \frac{1}{3}, C = 0, B = -2$:

$$\begin{aligned} \chi^f_{\frac{1}{2},\frac{1}{2},0-2}(QQq) &= bbu, \\ \chi^f_{\frac{1}{2},-\frac{1}{2},0-2}(QQq) &= bbd. \end{aligned} \tag{72}$$

$I = 0, Y = -\frac{2}{3}, C = 0, B = -2$:

$$\chi^f_{00,0-2}(QQq) = bbs. \tag{73}$$

For the ground state, the anti-symmetrization requires that the $C = 2$ and $B = -2$ states can only combine with $\chi^{\sigma}_{\frac{3}{2}M_S}(qqq)$ or $\chi^{\sigma_1}_{\frac{1}{2}M_S}(qqq)$, not with $\chi^{\sigma_2}_{\frac{1}{2}M_S}(qqq)$. The wave functions of ground heavy–heavy–light baryons (spin–flavor part) are as follows:

$$\begin{aligned} \chi^{\sigma f}_{IM_I,CB,SM_S}(QQq) &= \chi^{\sigma}_{\frac{3}{2}M_S}(qqq)\chi^f_{IM_I,CB}(QQq), \\ \chi^{\sigma f}_{IM_I,CB,SM_S}(QQq) &= \chi^{\sigma_1}_{\frac{1}{2}M_S}(qqq)\chi^f_{IM_I,CB}(QQq). \end{aligned} \tag{74}$$

4.3.4. *QQQ*

The existence of numerous singly heavy baryons has been firmly established through experimental studies conducted by the Belle, BABAR, CLEO, and LHCb collaborations [43]. These observations have significantly enriched our understanding of their mass spectra and decay properties. A major breakthrough was achieved in 2017 with the discovery of the Ξ_{cc}^{++} baryon (ccu) by LHCb [96], marking the first confirmed observation of a doubly heavy baryon. To date, no triply heavy baryon (e.g., Ω_{ccc}, Ω_{bbb}, Ω_{bcc}, Ω_{bbc}) has been experimentally observed, leaving this sector of the baryon family unexplored. The ground states, excited-state mass spectra, and decay properties of fully heavy baryons have been extensively investigated within various theoretical frameworks like the constituent quark model, QCD sum rules, Regge trajectories, etc. [102–113]. It is anticipated that experimental detection of these fully heavy baryons will be achieved in the near future, thereby further enriching the baryon family.

For fully heavy baryons, isospin is zero, and only the anti-symmetrization is applicable when there are identical heavy quarks.

The wave functions $\chi^f_{CB}(QQQ)$ are as follows:

$I = 0, Y = 0, C = 3, B = 0$:

$$\chi^f_{30}(QQQ) = ccc. \tag{75}$$

$I = 0, Y = 0, C = 2, B = -1$:

$$\chi^{f}_{2-1}(QQQ) \;=\; ccb. \tag{76}$$

$I = 0, Y = 0, C = 1, B = -2$:

$$\chi^{f}_{1-2}(QQQ) \;=\; bbc. \tag{77}$$

$I = 0, Y = 0, C = 0, B = -3$:

$$\chi^{f}_{0-3}(QQQ) \;=\; bbb. \tag{78}$$

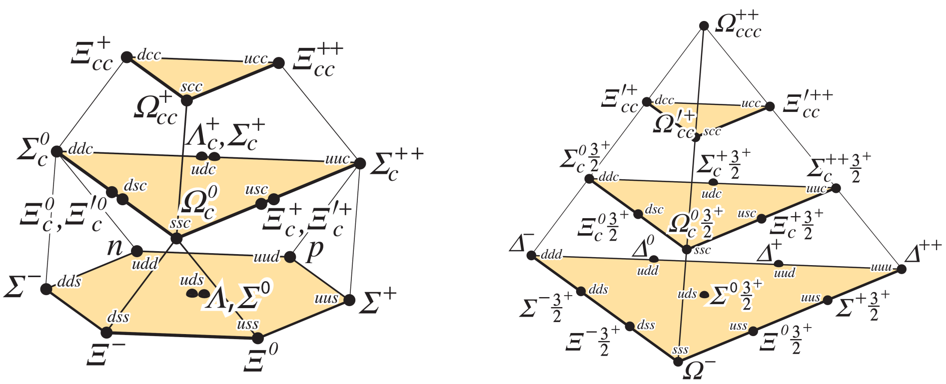
For the ground state, the anti-symmetrization requires that all the fully heavy baryon states can only combine with $\chi^{\sigma}_{\frac{3}{2}M_S}(qqq)$ or $\chi^{\sigma_1}_{\frac{1}{2}M_S}(qqq)$, not with $\chi^{\sigma_2}_{\frac{1}{2}M_S}(qqq)$. The wave functions of ground fully heavy baryons (spin–flavor part) are as follows:

$$\begin{aligned} \chi^{\sigma f}_{CB,SM_S}(QQQ) &= \chi^{\sigma}_{\frac{3}{2}M_S}(qqq)\chi^{f}_{CB}(QQQ), \\ \chi^{\sigma f}_{CB,SM_S}(QQQ) &= \chi^{\sigma_1}_{\frac{1}{2}M_S}(qqq)\chi^{f}_{CB}(QQQ). \end{aligned} \tag{79}$$

If the flavor symmetry is extended to SU^f_4 or SU^f_5, the flavor symmetry of the baryon is as follows:

$$\begin{aligned} \mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4} &= \mathbf{20}_S \oplus \mathbf{20}_{MS} \oplus \mathbf{20}_{MA} \oplus \mathbf{4}_A, \\ \mathbf{5}\otimes\mathbf{5}\otimes\mathbf{5} &= \mathbf{35}_S \oplus \mathbf{40}_{MS} \oplus \mathbf{40}_{MA} \oplus \mathbf{10}_A. \end{aligned} \tag{80}$$

Baryons in SU^f_4 flavor symmetry is demonstrated in Figure 8. Compared with light baryons, there are additional states, Ξ'_c. If SU^f_3 symmetry is a good one, then Ξ_c and Ξ'_c will not be mixed. It is well known that the SU^f_3 symmetry is broken, so the mixing between these two states is inevitable. So far, the mixing angle is not fixed experimentally, and several methods have been proposed to determine the $\Xi_c - \Xi'_c$ mixing [114–117]. A recent work indicated that the QED contribution to $\Xi_c - \Xi'_c$ mixing is smaller than the mass difference between the strange and up/down quarks [118].



The $\mathbf{20}_M$-plet baryons. The $\mathbf{20}_S$-plet baryons.

Figure 8. Ground baryons under SU^f_4 flavor symmetry [43].

For the light quarks, the spin and flavor symmetry can be combined to form a group $SU^{f\sigma}_6$; then, the light baryons with $SU^{f\sigma}_6$ flavor–spin symmetry are as follows [119,120]:

$$\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_{MA} \oplus \mathbf{70}_{MS} \oplus \mathbf{20}_A. \tag{81}$$

The flavor–spin contents in $SU_6^{f\sigma}$ can be obtained by the S_3 CG series (the first digit in parentheses represents flavor symmetry, and the last digit represents spin symmetry).

$$\begin{aligned} \mathbf{56}_S &= (\mathbf{10},\mathbf{4}) \oplus (\mathbf{8},\mathbf{2}), \\ \mathbf{70}_M &= (\mathbf{10},\mathbf{2}) \oplus (\mathbf{8},\mathbf{4}) \oplus (\mathbf{8},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2}), \\ \mathbf{20}_A &= (\mathbf{8},\mathbf{2}) \oplus (\mathbf{1},\mathbf{4}). \end{aligned} \tag{82}$$

The ground state baryons belong to $\mathbf{56}_S$. The excitation in coordinate space must be involved for other representations. For example, if we put one quark in p-wave orbit ($L = 1$), then one has states with negative parity. Now the symmetry in coordinate space [21] ([3] is for the states with center of mass motion in p-wave), and its combining with [21] symmetry in flavor–spin space gives symmetry [3] in coordinate–flavor–spin space. At last the symmetry [3] combines with [111] in the color space and gives the symmetry [111] of the system. To describe physical states, it is better to combine orbital angular momentum with spin first, and then the following possible negative parity states can be obtained [67]:

$$\mathbf{70}_M \begin{cases} & S \otimes L \to J^P \\ (\mathbf{10},\mathbf{2}) & \frac{1}{2} \otimes 1 \to \frac{1}{2}^-, \frac{3}{2}^- \\ (\mathbf{8},\mathbf{4}) & \frac{3}{2} \otimes 1 \to \frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^- \\ (\mathbf{8},\mathbf{2}) & \frac{1}{2} \otimes 1 \to \frac{1}{2}^-, \frac{3}{2}^- \\ (\mathbf{1},\mathbf{2}) & \frac{1}{2} \otimes 1 \to \frac{1}{2}^-, \frac{3}{2}^- \end{cases} . \tag{83}$$

Similarly, when c-quarks are introduced, spin and flavor symmetry can form $SU_8^{f\sigma}$:

$$\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8} = \mathbf{120}_S \oplus \mathbf{168}_{MA} \oplus \mathbf{168}_{MS} \oplus \mathbf{56}_A. \tag{84}$$

The flavor–spin contents can be extracted by using the CG series of the permutation group S_3. The results are shown below [121].

$$\begin{aligned} \mathbf{120}_S &= (\mathbf{20},\mathbf{4}) \oplus (\mathbf{20},\mathbf{2}), \\ \mathbf{168}_M &= (\mathbf{20},\mathbf{2}) \oplus (\mathbf{20},\mathbf{4}) \oplus (\mathbf{20},\mathbf{2}) \oplus (\mathbf{4},\mathbf{2}), \\ \mathbf{56}_A &= (\mathbf{4},\mathbf{2}) \oplus (\mathbf{20},\mathbf{4}). \end{aligned} \tag{85}$$

The results of the ground baryons from the light quark sector to the heavy quark sector with mass formula Equation (4) are shown in Table 4. The mass formula gives us a rough baryon spectrum. More precise calculations of the baryon spectrum with other theoretical methods such as quark models [122–124], the Faddeev equation [125–127], the QCD sum rule [128], Regge theory [129–131], and Lattice QCD [132] can be seen in the references.

The masses of both the radial excitations and the ground state masses of the baryons are required to conform to the spacing rule. The Gell-Mann–Okubo mass formula is satisfied as follows [1,133–135].

$$\begin{aligned} &\text{For light baryons:} \\ &\qquad \frac{1}{2}[m_N(uud) + m_\Xi(ssu)] = \frac{1}{4}[3m_\Lambda(uds) + m_\Sigma(uus)]. \\ &\text{For single heavy baryons:} \\ &\qquad m_{\Sigma_Q} + m_{\Omega_Q} = 2m_{\Xi_Q}. \end{aligned} \tag{86}$$

Here, with the help of CERN routine MINUIT, we try to fit the masses from N to Ξ_b^* according to their well-established values from PDG [43]. With the determined parameters, we calculate several other baryons without experimental values and compare them with some theoretical predictions.

Diquarks are often used to simplify problems for systems composed of more than two particles [136]. In the diquark model, two quarks are approximated to be a single particle, which may have a lager size and a form factor [137]. The light diquarks, heavy–light diquarks, and doubly heavy diquarks are investigated with Regge trajectories, and it is expected that the Regge trajectory can provide an easier way to study baryons, tetraquarks, and pentaquarks in ρ-mode excitations with diquark pictures [138–140].

Regarding color structure, a diquark in the color-antitriplet representation $(qq)_{\bar{3}}$ is regarded as "good" on account of its attractive confinement potential [141]. In contrast, the color-sextet $(qq)_6$ (the so-called "bad" diquark) shows repulsive interactions. It is only the color-singlet configurations that are physically acceptable. The color-sextet $(qq)_6$ and color-antitriplet $(qq)_{\bar{3}}$ is marked in Figure 9.

Table 4. The parameters and masses of ground state baryons.

Parameter	M_0	X_1		X_2		X_3	
	983.14 ± 0.72	28.401 ± 0.11		1.5894 ± 0.14		-215.51 ± 0.26	
	X_s	X_c		X_b			
	52.530 ± 0.22	1277.7 ± 0.37		-4529.1 ± 0.37			
Baryon	Y (hypercharge)	I (isospin)	J (spin)	$[f]$ (flavor)	Theo. (MeV)	Exp. [43] (MeV)	Refs. [142,143] (MeV)
N	1	$\frac{1}{2}$	$\frac{1}{2}$	[21]	965.49 ± 1.86	939.00	936.25 ± 0.22
Λ	0	0	$\frac{1}{2}$	[21]	1154.74 ± 1.49	1116.00	1114.15 ± 0.5
Σ	0	1	$\frac{1}{2}$	[21]	1259.80 ± 1.94	1193.00	1169.18 ± 0.15
Ξ	-1	$\frac{1}{2}$	$\frac{1}{2}$	[21]	1396.51 ± 1.86	1315.00	1317.16 ± 0.13
Δ	1	$\frac{3}{2}$	$\frac{3}{2}$	[3]	1298.26 ± 3.62	1232.00	1234.82 ± 0.81
Σ^*	0	1	$\frac{3}{2}$	[3]	1434.97 ± 3.03	1385.00	1427.46 ± 0.35
Ξ^*	-1	$\frac{1}{2}$	$\frac{3}{2}$	[3]	1571.68 ± 2.96	1530.00	1565.26 ± 0.15
Ω	-2	0	$\frac{3}{2}$	[3]	1708.39 ± 3.33	1672.00	1676.20 ± 0.15
Λ_c	$\frac{2}{3}$	0	$\frac{1}{2}$	[11]	2188.30 ± 1.69	2286.00	2254.48 ± 0.31
Σ_c	$\frac{2}{3}$	1	$\frac{1}{2}$	[2]	2406.96 ± 2.53	2455.00	2474.41 ± 0.25
Ξ_c	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	[11]	2447.58 ± 1.74	2471.00	2433.35 ± 0.30
Ξ_c'	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	[2]	2561.19 ± 2.19	2576.00	2574.37 ± 0.23
Ω_c	$-\frac{4}{3}$	0	$\frac{1}{2}$	[2]	2715.41 ± 2.38	2695.00	2679.37 ± 0.20
Ω_c^*	$-\frac{4}{3}$	0	$\frac{3}{2}$	[2]	2720.17 ± 2.81	2770.00	2755.37 ± 0.24
Ω_{cc}	$-\frac{2}{3}$	0	$\frac{1}{2}$	[1]	3753.39 ± 2.06	-	3738.20 ± 0.20
Ω_{cc}^*	$-\frac{2}{3}$	0	$\frac{3}{2}$	[1]	3758.15 ± 2.49	-	3822.20 ± 0.22
Σ_c^*	$\frac{2}{3}$	1	$\frac{3}{2}$	[2]	2411.73 ± 2.96	2520.00	2551.43 ± 0.25
Ξ_c^*	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{2}$	[2]	2565.95 ± 2.61	2645.00	2648.38 ± 0.25
Λ_b	$\frac{2}{3}$	0	$\frac{1}{2}$	[11]	5439.67 ± 1.69	5620.00	5626.52 ± 0.29
Σ_b	$\frac{2}{3}$	1	$\frac{1}{2}$	[2]	5658.33 ± 2.53	5813.00	5856.56 ± 0.27
Ξ_b	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	[11]	5698.95 ± 1.74	5797.00	5771.41 ± 0.24
Ξ_b'	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	[2]	5812.56 ± 2.19	5935.00	5933.47 ± 0.24
Ω_b	$-\frac{4}{3}$	0	$\frac{1}{2}$	[2]	5966.78 ± 2.38	6046.00	6056.47 ± 0.20
Ω_b^*	$-\frac{4}{3}$	0	$\frac{3}{2}$	[2]	5971.55 ± 2.81	-	6085.47 ± 20
Ω_{bb}	$-\frac{2}{3}$	0	$\frac{1}{2}$	[1]	$10{,}256.13 \pm 2.06$	-	$10{,}273.27 \pm 0.20$
Ω_{bb}^*	$-\frac{2}{3}$	0	$\frac{3}{2}$	[1]	$10{,}260.90 \pm 2.49$	-	$10{,}308.27 \pm 0.21$

Table 4. *Cont.*

Σ_b^*	$\frac{2}{3}$	1	$\frac{3}{2}$	[2]	5663.10 ± 2.96	5832.00	5877.55 ± 0.27
Ξ_b^*	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{2}$	[2]	5817.32 ± 2.61	5945.00	5960.47 ± 0.25
Ω_{ccc}	0	0	$\frac{3}{2}$	[0]	4822.33 ± 2.36	-	4796.8 ± 0.18
Ω_{ccb}	0	0	$\frac{1}{2}$	[0]	8068.93 ± 1.94	-	8007.9 ± 0.20
Ω_{ccb}^*	0	0	$\frac{3}{2}$	[0]	8073.70 ± 2.36	-	8037.9 ± 0.20
Ω_{bbc}	0	0	$\frac{1}{2}$	[0]	$11{,}320.30 \pm 1.94$	-	$11{,}195.9 \pm 0.20$
Ω_{bbc}^*	0	0	$\frac{3}{2}$	[0]	$11{,}325.07 \pm 2.36$	-	$11{,}229.8 \pm 0.20$
Ω_{bbb}	0	0	$\frac{3}{2}$	[0]	$14{,}576.44 \pm 2.36$	-	$14{,}366.9 \pm 0.20$

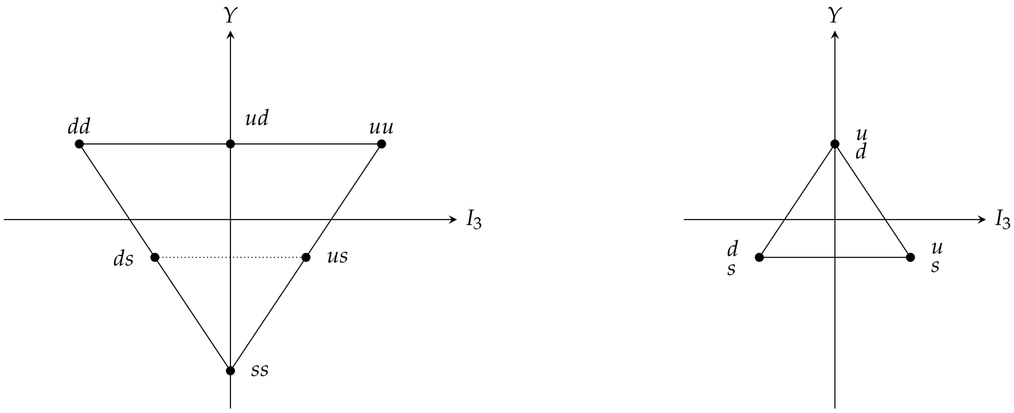


Figure 9. The weight diagrams of scalar diquarks and vector diquarks.

5. Tetraquark

Tetraquark states are exotic states. Although they are predicted in Gell-Mann and Zweig's work [2], they are not confirmed experimentally so far. Since the Belle collaboration reported their discovery about charmed $X(3872)$ close to the $\bar{D}^0 D^{*0}$ threshold in 2003 [25], the tetraquark has always been one of the focuses in hadron physics. About ten years after the discovery of $X(3872)$, another charmonium-like state, $Z_c(3900)$, was reported by two independent collaborations (BESIII and Belle) and corresponds to the partner of $Z_b(10610)$ in the charm quark sector [144–147], which lies near $\bar{D}^0 D^{*+}$ mass thresholds and may imply the existence of a virtual $D\bar{D}^*$ molecule-like structure [148–151]. Its true nature and internal configuration continue to be debated [6,152,153]. BESIII also reported the $Z_c(4020)$ in the $\pi^{\pm} h_c$ mass spectrum corresponds to $Z_b(10650)$ in the charmed sector [144,154]. Besides the molecular state structure, another possible explanation is the diquark–antidiquark structure [155–157].

The LHCb collaboration's recent detection of the doubly charmed $T_{cc}^+(3875)$ state close to the $D^{*+}D^0$ threshold in the $D^0 D^0 \pi^+$ invariant mass distribution represents an important breakthrough [158]. Subsequent analysis of the $T_{cc}^+(3875)$ state uncovers its $cc\bar{u}\bar{d}$ quark content, establishing it as a tetraquark with a double-charm quark [159–162] instead of a hidden-charm state (like $X(3872)$) with a $c\bar{c}$ pair. The nature of the $T_{cc}^+(3875)$ state is theoretically ambiguous. Several theoretical interpretations exist, such as a compact tetraquark [163–167], a hadronic molecule [168–173], and a virtual state [174–176].

The observed $X(3872)(\bar{D}^0 D^{*0})$, $Z_c(3900)(\bar{D}^0 D^{*+})$, $Z_c(4020)(\bar{D}^{*0} D^{*+})$, $Z_{cs}(\bar{D}^0 D_s^{*+})$ and $T_{cc}^+(3875)(D^0 D^{*+})$ with diverse quark compositions all reside near two-meson thresholds, in contrast to $Z_c(4430)$, $Z_{cs}(4220)$, and $X(6900)$, which lie significantly above the threshold energies. This dichotomy implies observations of both threshold-proximate hadronic molecules and compact tetraquark structures [177]. These phenomena of tetraquarks offer a unique chance to investigate the non-perturbative aspects of strong interaction dynamics by thoroughly examining their characteristics and configurations.

In quark models, it is easy to construct tetraquark states. Because of the color-singlet requirement, a tetraquark state must be two quarks and two antiquarks. The dimension of the configuration space for color is 81, and the color singlet picks up only the state with color symmetry [222]. The dimension of the flavor space is $5 \times 5 \times 5 \times 5 = 625$, in which there are 81 light states, 216 heavy–light–light–light states, 216 heavy–heavy–light–light states, 96 heavy–heavy–heavy–light states, and 16 full heavy states, and the dimension for spin space is 16. However, the total anti-symmetrization will reduce the number of tetraquark states.

For multi-quark systems with $n + m > 3$, there are at least two ways to construct the wave functions; one is called the physical basis, and the other is called the symmetry basis. The physical basis is constructed as follows (taking the tetraquark as an example): first, one quark and one antiquark are coupled to form a meson; the remaining quark and antiquark form another meson; secondly, two mesons are coupled to form the bases for the tetraquark system; at last the antisymmetrization operator is applied to the bases if there are identical particles. For the symmetry basis, first all quarks are coupled together, as are antiquarks; secondly quark clusters and antiquark clusters are coupled to form the bases of the tetraquark system. In this case, the anti-symmetrization operator is not needed because the antisymmetrization requirement has been imposed (the irreducible representation of four degrees of freedom is $[1^f]$) in constructing the bases of the quark (antiquark) clusters. Clearly, two bases are related by a unitary transformation. The transformations have been given for pentaquark systems [178] and hexaquark systems [179], and the transformation between physical bases $\Psi_{\alpha k}(q\bar{q}q\bar{q})$ and symmetry bases $\Phi_{\alpha K}(q^2\bar{q}^2)$ for tetraquark systems are given below.

$$\Psi_{\alpha k}(q\bar{q}q\bar{q}) \quad = \quad \sum_{\tilde{\nu}_2 \mu_2 f_2} C_{9j} \{CCC\}_{qq} \{CCC\}_{\bar{q}\bar{q}} \Phi_{\alpha K}(q^2\bar{q}^2). \tag{87}$$

where $9j$-coefficient re-coupling coefficients are among four representations.

$$C_{9j} = \left\{ \begin{array}{ccc} c_1 & c_2 & c_{q\bar{q}} \\ c_3 & c_4 & c_{q\bar{q}} \\ c_{qq} & c_{\bar{q}\bar{q}} & c \end{array} \right\} \left\{ \begin{array}{ccc} I_1 & I_2 & I_{q\bar{q}} \\ I_3 & I_4 & I_{q\bar{q}} \\ I_{qq} & I_{\bar{q}\bar{q}} & I \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & J_{q\bar{q}} \\ j_3 & j_4 & J_{q\bar{q}} \\ J_{qq} & J_{\bar{q}\bar{q}} & J \end{array} \right\}, \tag{88}$$

$$\{CCC\}_{qq} = C^{[\nu_{qq}][c_{qq}][\mu_{qq}]}_{[\nu_1][c_1][\mu_1],[\nu_3][c_3][\mu_3]} C^{[\mu_{qq}][f_{qq}][J_{qq}]}_{[\mu_1][f_1][J_1],[\mu_3][f_3][J_3]} C^{[f_{qq}]Y_{qq}I_{qq}}_{[f_1]Y_1I_1,[f_3]Y_3I_3}, \tag{89}$$

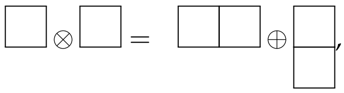
$$\{CCC\}_{\bar{q}\bar{q}} = C^{[\nu_{\bar{q}\bar{q}}][c_{\bar{q}\bar{q}}][\mu_{\bar{q}\bar{q}}]}_{[\nu_2][c_2][\mu_2],[\nu_4][c_4][\mu_4]} C^{[\mu_{\bar{q}\bar{q}}][f_{\bar{q}\bar{q}}][J_{\bar{q}\bar{q}}]}_{[\mu_2][f_2][J_2],[\mu_4][f_4][J_4]} C^{[f_{\bar{q}\bar{q}}]Y_{\bar{q}\bar{q}}I_{\bar{q}\bar{q}}}_{[f_2]Y_2I_2,[f_4]Y_4I_4}. \tag{90}$$

In the above equations, $\nu_i, c_i, \mu_i, f_i, Y_i, I_i, j_i$ are representations of the i-th particle in coordinate, color, spin–flavor, flavor, hypercharge, isospin, and spin spaces. The coefficients in Equations (89) and (90) are called isoscalar factors of group chain $SU_{mn} \supset SU_m \times SU_n$ [70]. For the two-particle case, isoscalar factors are very simple, and they take values 0 or 1.

In the present article, the symmetry bases are used.

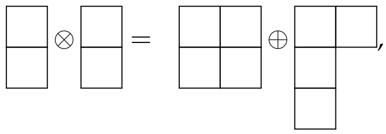
5.1. Color

For a tetraquark system composed of two quarks and two antiquarks, the color symmetries for two quarks are as follows:



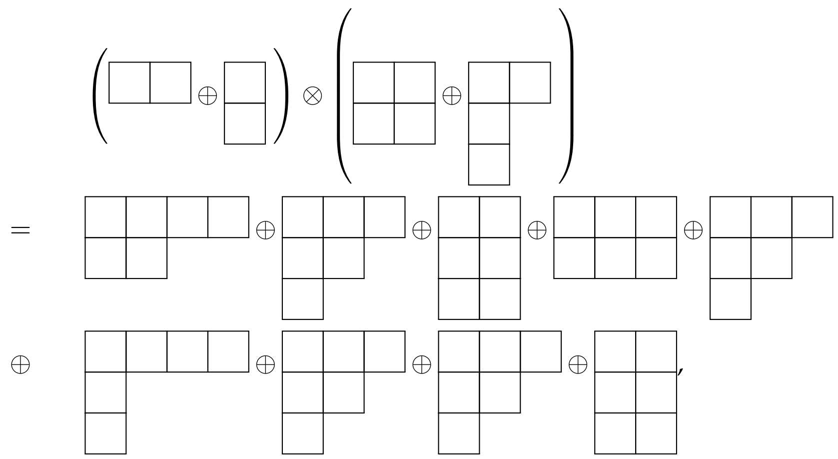
$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6}_S \oplus \bar{\mathbf{3}}_A. \tag{91}$$

Similarly, the color symmetries for two antiquatks are as follows:



$$\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{6}}_S \oplus \mathbf{3}_A. \tag{92}$$

Coupling two quarks and two antiquark together, one has



(93)

$$\begin{aligned} &[1] \otimes [1] \otimes [11] \otimes [11] \\ &= ([2] \oplus [11]) \otimes ([22] \oplus [211]) \\ &= [42] \oplus [321] \oplus [222] \oplus [33] \oplus [321] \oplus [411] \oplus [321] \oplus [321] \oplus [222] \\ &(\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes (\bar{\mathbf{6}} \oplus \mathbf{3}) = \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \end{aligned} \tag{94}$$

The color singlet picks up [222], which comes from [2] $\otimes$ [22] ($\mathbf{6}_S \otimes \bar{\mathbf{6}}_S$) and [11] $\otimes$ [211] ($\bar{\mathbf{3}}_A \otimes \mathbf{3}_A$). The wave functions for two-quarks are as follows:

$$\begin{array}{lll} \chi^{c_1}_{[2]}(qq) = rr & \chi^{c_2}_{[2]}(qq) = \frac{1}{\sqrt{2}}(rg + gr) & \chi^{c_3}_{[2]}(qq) = \frac{1}{\sqrt{2}}(rb + br) \\ \chi^{c_4}_{[2]}(qq) = gg & \chi^{c_5}_{[2]}(qq) = \frac{1}{\sqrt{2}}(gb + bg) & \chi^{c_6}_{[2]}(qq) = bb \\ \chi^{c_1}_{[11]} = \frac{1}{\sqrt{2}}(rg - gr) & \chi^{c_2}_{[11]} = \frac{1}{\sqrt{2}}(rb - br) & \chi^{c_3}_{[11]} = \frac{1}{\sqrt{2}}(gb - bg) \end{array} \tag{95}$$

The similar expressions of wave functions for two antiquarks are as follows:

$$
\begin{aligned}
&\chi^{c_1}_{[22]}(\bar{q}\bar{q}) = \bar{r}\bar{r} && \chi^{c_2}_{[22]}(\bar{q}\bar{q}) = -\frac{1}{\sqrt{2}}(\bar{r}\bar{g}+\bar{g}\bar{r}) && \chi^{c_3}_{[22]}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(\bar{r}\bar{b}+\bar{b}\bar{r}) \\
&\chi^{c_4}_{[22]}(\bar{q}\bar{q}) = \bar{g}\bar{g} && \chi^{c_5}_{[22]}(\bar{q}\bar{q}) = -\frac{1}{\sqrt{2}}(\bar{g}\bar{b}+\bar{b}\bar{g}) && \chi^{c_6}_{[22]}(\bar{q}\bar{q}) = \bar{b}\bar{b} \\
&\chi^{c_1}_{[211]}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(\bar{r}\bar{g}-\bar{g}\bar{r}) && \chi^{c_2}_{[211]}(\bar{q}\bar{q}) = -\frac{1}{\sqrt{2}}(\bar{r}\bar{b}-\bar{b}\bar{r}) && \chi^{c_3}_{[211]}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(\bar{g}\bar{b}-\bar{b}\bar{g})
\end{aligned}
\tag{96}
$$

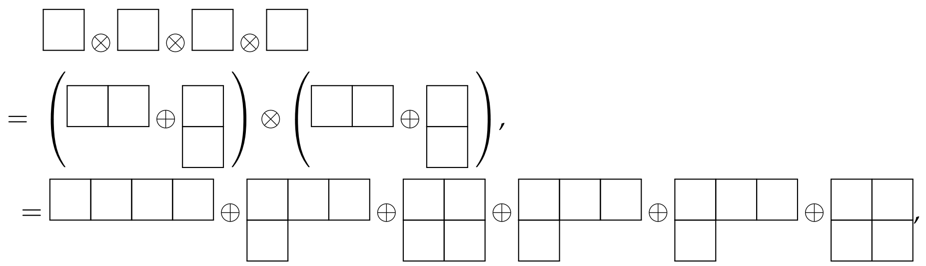
The color wave functions for a tetraquark system can be obtained by consulting the CG coefficients of SU_3 and the above wave functions for two quarks and two antiquarks: it reads

$$
\begin{aligned}
\chi^{c_1}_{[222]}(\overline{qq}\overline{\bar{q}\bar{q}}) = {}& \frac{1}{\sqrt{6}}\Big(\chi^{c_1}_{[2]}(qq)\chi^{c_1}_{[22]}(\bar{q}\bar{q}) - \chi^{c_2}_{[2]}(qq)\chi^{c_2}_{[22]}(\bar{q}\bar{q}) + \chi^{c_3}_{[2]}(qq)\chi^{c_3}_{[22]}(\bar{q}\bar{q}) \\
& + \chi^{c_4}_{[2]}(qq)\chi^{c_4}_{[22]}(\bar{q}\bar{q}) - \chi^{c_5}_{[2]}(qq)\chi^{c_5}_{[22]}(\bar{q}\bar{q}) + \chi^{c_6}_{[2]}(qq)\chi^{c_6}_{[22]}(\bar{q}\bar{q})\Big), \\
\chi^{c_2}_{[222]}(\widetilde{qq}\widetilde{\bar{q}\bar{q}}) = {}& \frac{1}{\sqrt{3}}\Big(\chi^{c_1}_{[11]}(qq)\chi^{c_1}_{[211]}(\bar{q}\bar{q}) - \chi^{c_2}_{[11]}(qq)\chi^{c_2}_{[211]}(\bar{q}\bar{q}) + \chi^{c_3}_{[11]}(qq)\chi^{c_3}_{[211]}(\bar{q}\bar{q})\Big).
\end{aligned}
\tag{97}
$$

The line or tilde over qq ($\bar{q}\bar{q}$) means that the wave function is symmetric or antisymmetric under the permutation of quarks (antiquarks).

5.2. Spin

For a tetraquark system composed of four quarks, its total spin is 2, 1, 0.



(98)

$$
\begin{aligned}
[1]\otimes[1]\otimes[1]\otimes[1] &= [4]\oplus[31]\oplus[22]\oplus[31]\oplus[31]\oplus[22] \\
\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} &= \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{1}.
\end{aligned}
\tag{99}
$$

The spin wave functions of $qq\bar{q}\bar{q}$ system can be setup from the wave functions of two clusters according to Equation (19) and with the help of the CG coefficients of SU_2. To save space, only the wave functions $\chi^{\sigma}_{SM_S}$ with component $M_S = S$ are given below.

$[\sigma_4] = [4], S = 2$:

$$
\chi^{\sigma_1}_{22}(\overline{qq}\overline{\bar{q}\bar{q}}) = \chi^{\sigma}_{1,1}\chi^{\sigma}_{1,1} = \alpha\alpha\alpha\alpha. \tag{100}
$$

$[\sigma_4] = [31], S = 1$:

$$
\begin{aligned}
\chi^{\sigma_1}_{11}(\overline{qq}\overline{\bar{q}\bar{q}}) &= \frac{1}{\sqrt{2}}\left[\chi^{\sigma}_{1,1}\chi^{\sigma}_{1,0} - \chi^{\sigma}_{1,0}\chi^{\sigma}_{1,1}\right] \\
&= \frac{1}{\sqrt{2}}\left[\alpha\alpha\frac{1}{\sqrt{2}}(\alpha\beta+\beta\alpha) - \frac{1}{\sqrt{2}}(\alpha\beta+\beta\alpha)\alpha\alpha\right] \\
&= \frac{1}{2}(\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha - \alpha\beta\alpha\alpha - \beta\alpha\alpha\alpha).
\end{aligned}
$$

$$
\begin{aligned}
\chi_{11}^{\sigma_2}(\overline{qq}\widetilde{\bar{q}\bar{q}}) &= \chi_{1,1}^{\sigma}\chi_{0,0}^{\sigma} \\
&= \alpha\alpha\frac{1}{\sqrt{2}}(\alpha\beta-\beta\alpha) = \frac{1}{\sqrt{2}}(\alpha\alpha\alpha\beta-\alpha\alpha\beta\alpha). \\
\chi_{11}^{\sigma_3}(\widetilde{qq}\overline{\bar{q}\bar{q}}) &= \chi_{0,0}^{\sigma}\chi_{1,1}^{\sigma} \\
&= \frac{1}{\sqrt{2}}(\alpha\beta-\beta\alpha)\alpha\alpha = \frac{1}{\sqrt{2}}(\alpha\beta\alpha\alpha-\beta\alpha\alpha\alpha).
\end{aligned} \tag{101}
$$

$[\sigma_4] = [22], S = 0$:

$$
\begin{aligned}
\chi_{00}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}}) =& \frac{1}{\sqrt{3}}[\chi_{1,1}^{\sigma}\chi_{1,-1}^{\sigma} - \chi_{1,0}^{\sigma}\chi_{1,0}^{\sigma} + \chi_{1,-1}^{\sigma}\chi_{1,1}^{\sigma}] \\
=& \frac{1}{\sqrt{3}}\left[\alpha\alpha\beta\beta - \frac{1}{\sqrt{2}}(\alpha\beta+\beta\alpha)\frac{1}{\sqrt{2}}(\alpha\beta+\beta\alpha) + \beta\beta\alpha\alpha\right] \\
=& \frac{1}{\sqrt{12}}[2\alpha\alpha\beta\beta - \alpha\beta\alpha\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta - \beta\alpha\beta\alpha + 2\beta\beta\alpha\alpha]. \\
\chi_{00}^{\sigma_2}(\widetilde{qq}\widetilde{\bar{q}\bar{q}}) =& \chi_{0,0}^{\sigma}\chi_{0,0}^{\sigma} \\
=& \frac{1}{\sqrt{2}}(\alpha\beta-\beta\alpha)\frac{1}{\sqrt{2}}(\alpha\beta-\beta\alpha) \\
=& \frac{1}{2}(\alpha\beta\alpha\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta + \beta\alpha\beta\alpha).
\end{aligned} \tag{102}
$$

The permutation symmetry with respect to quark exchange is symmetric for the wave functions $\chi_{22}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}})$, $\chi_{11}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}})$, $\chi_{11}^{\sigma_2}(\overline{qq}\widetilde{\bar{q}\bar{q}})$, $\chi_{00}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}})$, antisymmetric for wave functions $\chi_{11}^{\sigma_3}(\widetilde{qq}\overline{\bar{q}\bar{q}})$ and $\chi_{00}^{\sigma_2}(\widetilde{qq}\widetilde{\bar{q}\bar{q}})$, while the wave functions $\chi_{22}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}})$, $\chi_{11}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}})$, $\chi_{11}^{\sigma_3}(\widetilde{qq}\overline{\bar{q}\bar{q}})$, $\chi_{00}^{\sigma_1}(\overline{qq}\overline{\bar{q}\bar{q}})$ are symmetric under the antiquark exchange and antisymmetric for wave functions $\chi_{11}^{\sigma_2}(\overline{qq}\widetilde{\bar{q}\bar{q}})$ and $\chi_{00}^{\sigma_2}(\widetilde{qq}\widetilde{\bar{q}\bar{q}})$.

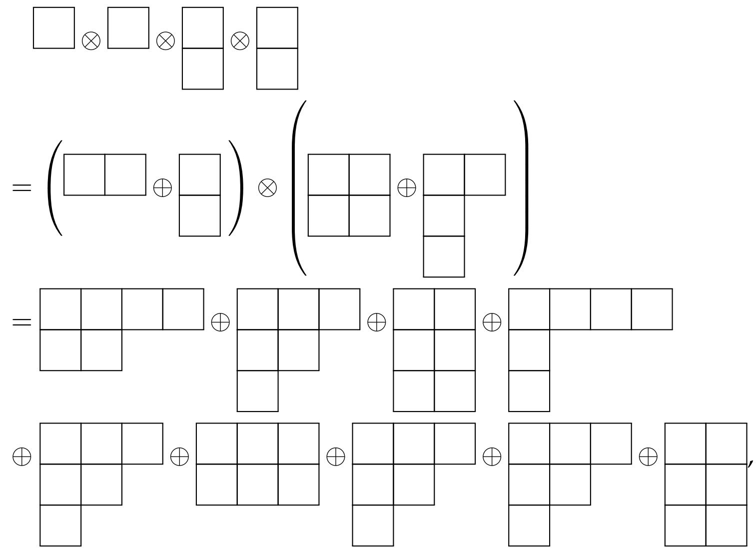
5.3. Flavor

5.3.1. $qq\bar{q}\bar{q}$

In the light quark sector, discriminating experimental signatures of authentic tetraquark ($qq\bar{q}\bar{q}$) states from ordinary $q\bar{q}$ mesons remains inherently difficult, and the nature of light scalar mesons $\sigma/f_0(500)$, $f_0(980)$, $\kappa/K_0^*(700)$ and $a_0(980)$ remain undetermined in experiment and theory. The mass of $a_0(980)$ and $f_0(980)$ is close to the threshold of $K\bar{K}$; whether they are conventional two-quark $q\bar{q}$ or exotic tetraquarks is still controversial [180–187]. Investigating possible $a_0(980)(I=1)$ and $f_0(980)(I=0)$ mixing would offer crucial insights for constraining hadronic models and optimizing their parameter sets [188,189]. $D \to a_0(980)\pi$ modes observed by BESIII in the decays of $D^0 \to \pi^+\pi^-\eta$ and $D^+ \to \pi^+\pi^0\eta$ provided strong support that $a_0(980)$ can be well explained with the tetraquark model $a_0(980)$, which serves as an intermediate state in the three-body D decays of $D \to a_0(980)P \to P_1P_2P$ (where P represents a pseudoscalar meson) [190,191].

$Y(2175)$ was assumed to be a possible candidate with an $s\bar{s}s\bar{s}$ component in the light quark sector with $J^{PC} = 1^{--}$. It was first reported by BaBar collaboration in $e^+e^- \to \phi(1020)f_0(980)$ [192–195] and then confirmed by BESII [22], BESIII [196,197], and the Belle group [198]. $Y(2175)$ can be well explained as $s\bar{s}s\bar{s}$ by constructing tetraquark currents to perform QCD sum rule analysis [199–201].

For the light tetraquark system, the flavor symmetry SU_3^f classifies 81 states into the following multiplets:



(103)

$$[1] \otimes [1] \otimes [11] \otimes [11] = ([2]_S \oplus [11]_A) \otimes ([22]_S \oplus [211]_A) \\ = [42]_{SS} \oplus [321]_{SS} \oplus [222]_{SS} \oplus [411]_{SA} \oplus [321]_{SA} \oplus \\ [321]_{AS} \oplus [33]_{AS} \oplus [222]_{AA} \oplus [321]_{AA}. \quad (104)$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{27}_{SS} \oplus \mathbf{8}_{SS} \oplus \mathbf{1}_{SS} \oplus \mathbf{10}_{SA} \oplus \mathbf{8}_{SA} \oplus \mathbf{8}_{AS} \oplus \mathbf{10}_{AS} \oplus \mathbf{1}_{AA} \oplus \mathbf{8}_{AA}. \quad (105)$$

The subscripts in Equation (104) denote the permutation symmetry under the quark (antiquark) exchanges. Generalized to the flavor symmetry SU_n, the coupling is depicted in Figure 10.

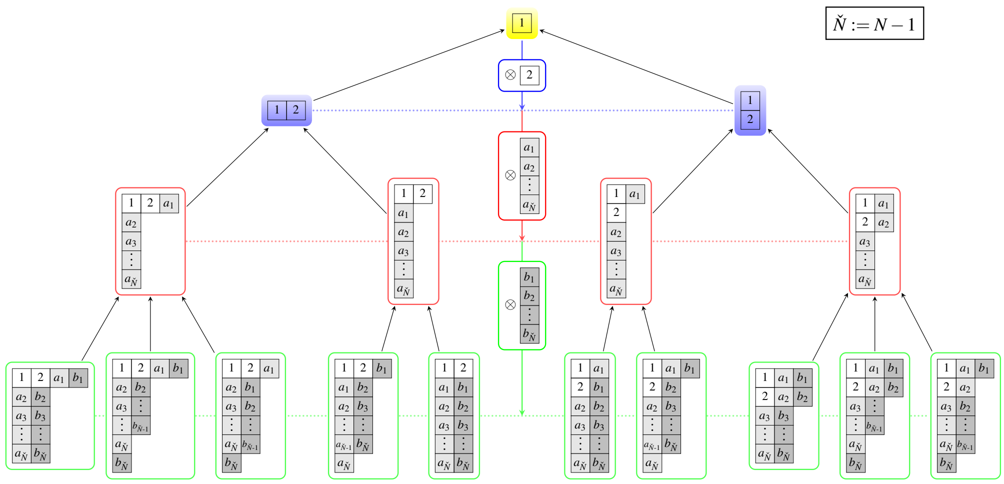


Figure 10. The coupling of $2q + 2\bar{q}$ for SU_n symmetry [202].

The flavor multiplets for tetraquark system are shown in Figures 11–14. Several multiplets appear more than once; we do not list all the possibilities. It is worth noting that the tetraquark states labeled next to each point in the weight diagram Figures 11–14 only represent a possible quark compositions corresponding to that point. The authentic flavor wave function is a linear combination of possible contents (e.g., $uu\bar{d}\bar{s}$, $uu\bar{s}\bar{d}$), and for simplicity, we did not label the combination on the diagram because some combinations are rather long. The specific flavor wave function is determined based on isospin and its third component, as shown in Equations (108)–(119). For example, in Figure 11, 27-plet states form the 27-dimensional representation of [42]; for states with $I=1, M_I=1, Y=2$, its quark content and flavor wave function are both $uu\bar{s}\bar{s}$, corresponding to Equation (111). For states with $I=\frac{3}{2}, M_I=\frac{3}{2}, Y=1$, its flavor wave function is Equation (109), but we only marked $uu\bar{s}\bar{d}$ in Figure 11.

The wave functions of tetraquark systems can be obtained from the wave functions of quark–quark clusters and antiquark–antiquark clusters by using the CG coefficients of SU_3. The nine wave functions $\chi^f_{[f_1]I_1M_{I_1}}(qq)$ for qq cluster are as follows:

$$\begin{aligned}
&\chi^f_{[2]11}(qq)=uu && \chi^f_{[2]10}(qq)=\frac{1}{\sqrt{2}}(ud+du) && \chi^f_{[2]1-1}(qq)=dd\\
&\chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)=\frac{1}{\sqrt{2}}(us+su) && \chi^f_{[2]\frac{1}{2},-\frac{1}{2}}(qq)=\frac{1}{\sqrt{2}}(ds+sd) && \chi^f_{[2]00}(qq)=ss\\
&\chi^f_{[11]00}(qq)=\frac{1}{\sqrt{2}}(ud-du) && \chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)=\frac{1}{\sqrt{2}}(us-su) && \chi^f_{[11]\frac{1}{2},-\frac{1}{2}}(qq)=\frac{1}{\sqrt{2}}(ds-sd).
\end{aligned} \tag{106}$$

Similarly, one has the following nine wave functions $\chi^f_{[f_2]I_2M_{I_2}}(\bar{q}\bar{q})$ for $\bar{q}\bar{q}$ cluster:

$$\begin{aligned}
&\chi^f_{[22]11}(\bar{q}\bar{q})=\bar{d}\bar{d} && \chi^f_{[22]10}(\bar{q}\bar{q})=-\frac{1}{\sqrt{2}}(\bar{u}\bar{d}+\bar{d}\bar{u}) && \chi^f_{[22]1-1}(\bar{q}\bar{q})=\bar{u}\bar{u}\\
&\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})=-\frac{1}{\sqrt{2}}(\bar{d}\bar{s}+\bar{s}\bar{d}) && \chi^f_{[22]\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{q})=\frac{1}{\sqrt{2}}(\bar{u}\bar{s}+\bar{s}\bar{u}) && \chi^f_{[22]00}(\bar{q}\bar{q})=\bar{s}\bar{s}\\
&\chi^f_{[211]00}(\bar{q}\bar{q})=\frac{1}{\sqrt{2}}(\bar{u}\bar{d}-\bar{d}\bar{u}) && \chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})=\frac{1}{\sqrt{2}}(\bar{d}\bar{s}-\bar{s}\bar{d}) && \chi^f_{[211]\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{q})=-\frac{1}{\sqrt{2}}(\bar{u}\bar{s}-\bar{s}\bar{u}).
\end{aligned} \tag{107}$$

According to the flavor symmetry SU^f_3, the quark pairs $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ will be mixed up. But mixing in the existing hadrons is very small, so we do not give the wave functions with SU^f_3 symmetry; instead, the wave functions $\chi^f_{IM_I,Y}(qq\bar{q}\bar{q})$ $(M_I=I)$ with $SU^\tau_2\times U^Y_1$ symmetry are listed below.
$I=2, M_I=2, Y=0$:

$$\chi^{f_1}_{22,0}(\overline{qq}\overline{\bar{q}\bar{q}})=\chi^f_{[2]11}(qq)\chi^f_{[22]11}(\bar{q}\bar{q})=uu\bar{d}\bar{d}. \tag{108}$$

$I=\frac{3}{2}, M_I=\frac{3}{2}, Y=1$:

$$\begin{aligned}
\chi^{f_1}_{\frac{3}{2},\frac{3}{2},1}(\overline{qq}\overline{\bar{q}\bar{q}})&=\chi^f_{[2]11}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})=-\frac{1}{\sqrt{2}}(uu\bar{d}\bar{s}+uu\bar{s}\bar{d}),\\
\chi^{f_2}_{\frac{3}{2},\frac{3}{2},1}(\overline{qq}\widetilde{\bar{q}\bar{q}})&=\chi^f_{[2]11}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})=\frac{1}{\sqrt{2}}(uu\bar{d}\bar{s}-uu\bar{s}\bar{d}).
\end{aligned} \tag{109}$$

$I=\frac{3}{2}, M_I=\frac{3}{2}, Y=-1$:

$$\begin{aligned}
\chi^{f_1}_{\frac{3}{2},\frac{3}{2},-1}(\overline{qq}\overline{\bar{q}\bar{q}})&=\chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]11}(\bar{q}\bar{q})=\frac{1}{\sqrt{2}}(us\bar{d}\bar{d}+su\bar{d}\bar{d}),\\
\chi^{f_2}_{\frac{3}{2},\frac{3}{2},-1}(\widetilde{qq}\overline{\bar{q}\bar{q}})&=\chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]11}(\bar{q}\bar{q})=\frac{1}{\sqrt{2}}(us\bar{d}\bar{d}-su\bar{d}\bar{d}).
\end{aligned} \tag{110}$$

$I=1, M_I=1, Y=2$:

$$\chi^{f_1}_{11,2}(\overline{qq}\overline{\bar{q}\bar{q}})=\chi^f_{[2]11}(qq)\chi^f_{[22]00}(\bar{q}\bar{q})=uu\bar{s}\bar{s}. \tag{111}$$

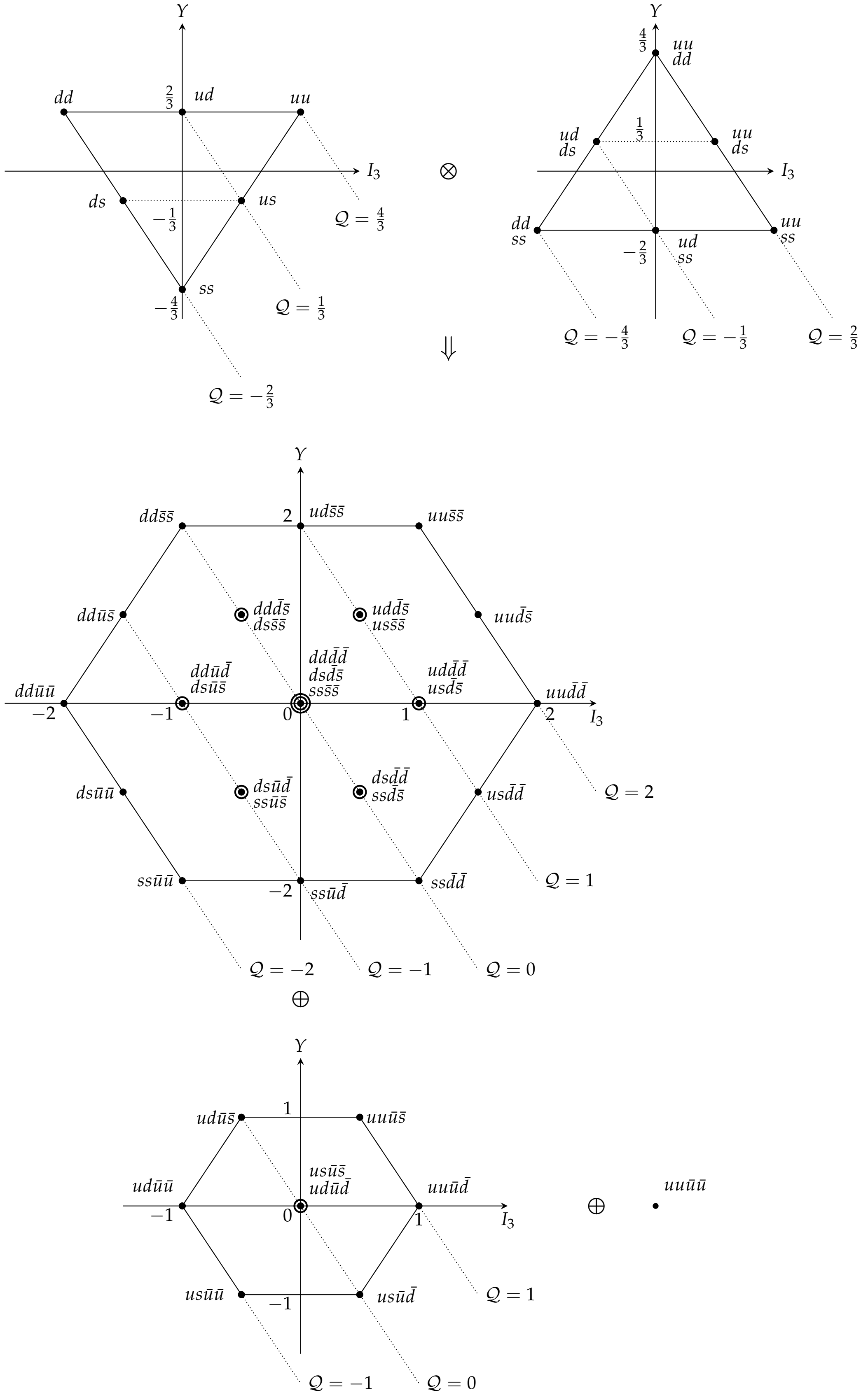


Figure 11. The 27-plet states of tetraquarks for partition [42] and 8-plet states of tetraquarks for partition [321], and singlet for partition [222]. They come from $[2]_S \otimes [22]_S = [42]_{SS} \oplus [321]_{SS} \oplus [222]_{SS}$, and the corresponding dimension is $\mathbf{6}_S \otimes \mathbf{6}_S = \mathbf{27}_{SS} \oplus \mathbf{8}_{SS} \oplus \mathbf{1}_{SS}$.

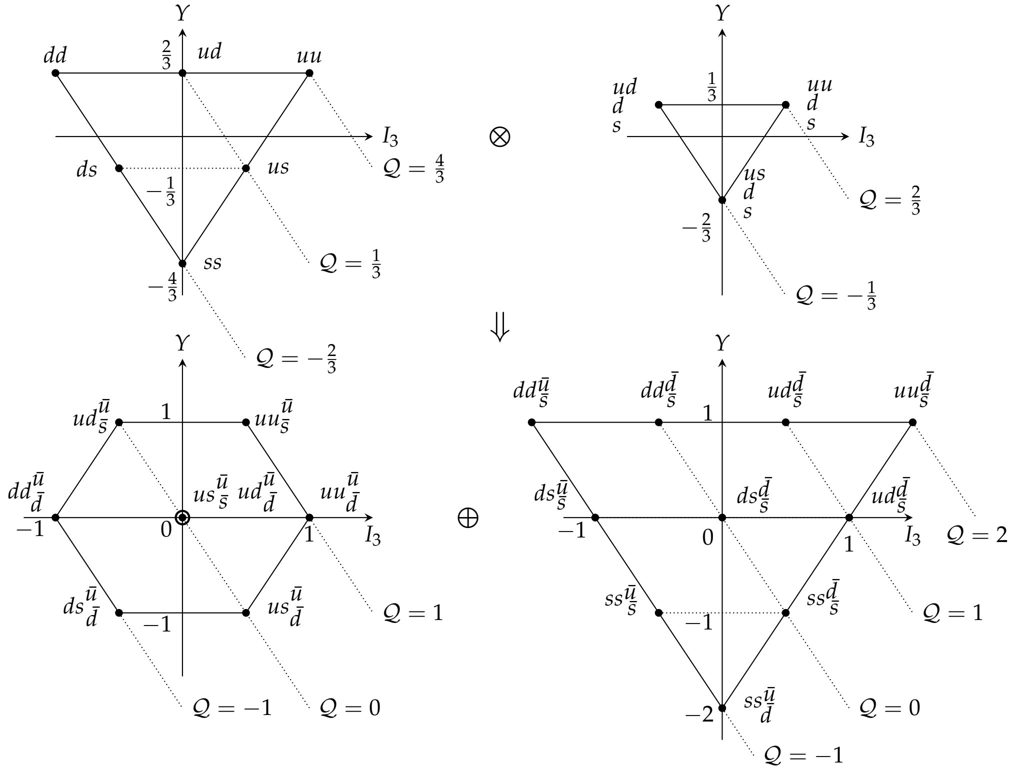


Figure 12. The 8-plet states of tetraquarks for partition [321] and 10-plet states of tetraquarks for partition [411]. They come from $[2]_S \otimes [211]_A = [321]_{SA} \oplus [411]_{SA}$, and the corresponding dimension is $\mathbf{6}_S \otimes \mathbf{3}_A = \mathbf{8}_{SA} \oplus \mathbf{10}_{SA}$.

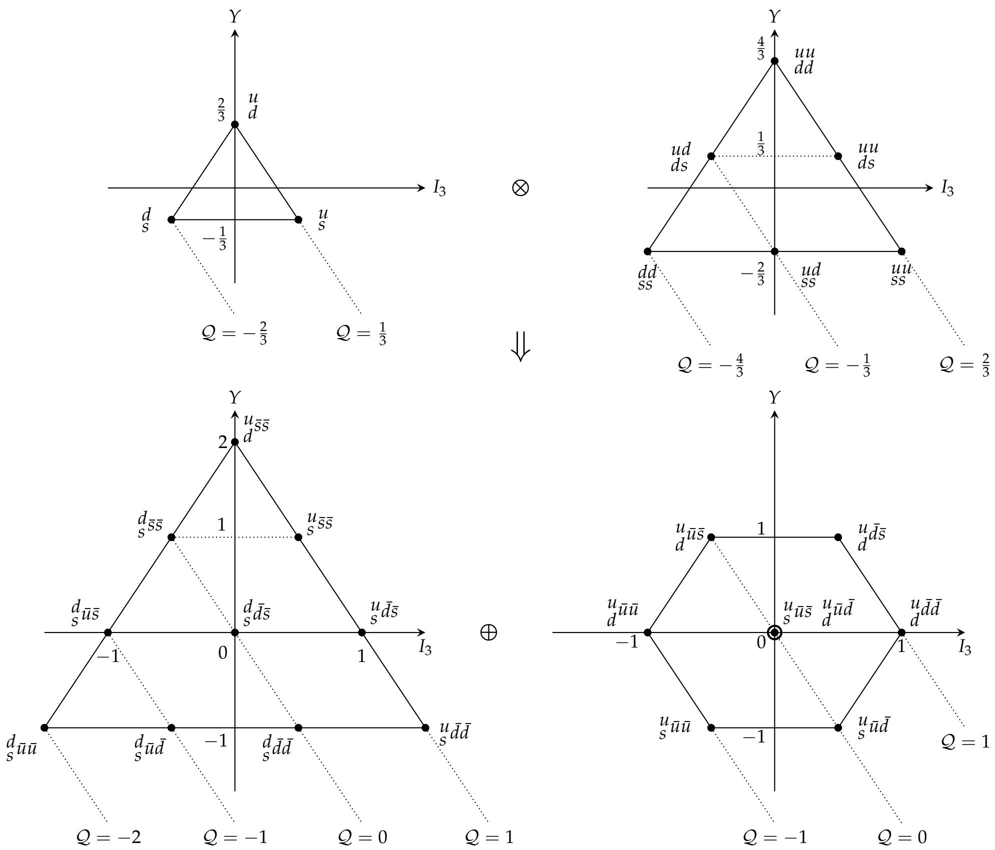


Figure 13. The 10-plet states of tetraquarks for partition [33] and 8-plet states of tetraquarks for partition [321]. They come from $[11]_A \otimes [22]_S = [33]_{AS} \oplus [321]_{AS}$, and the corresponding dimension is $\mathbf{3}_A \otimes \mathbf{6}_S = \mathbf{10}_{AS} \oplus \mathbf{8}_{AS}$.

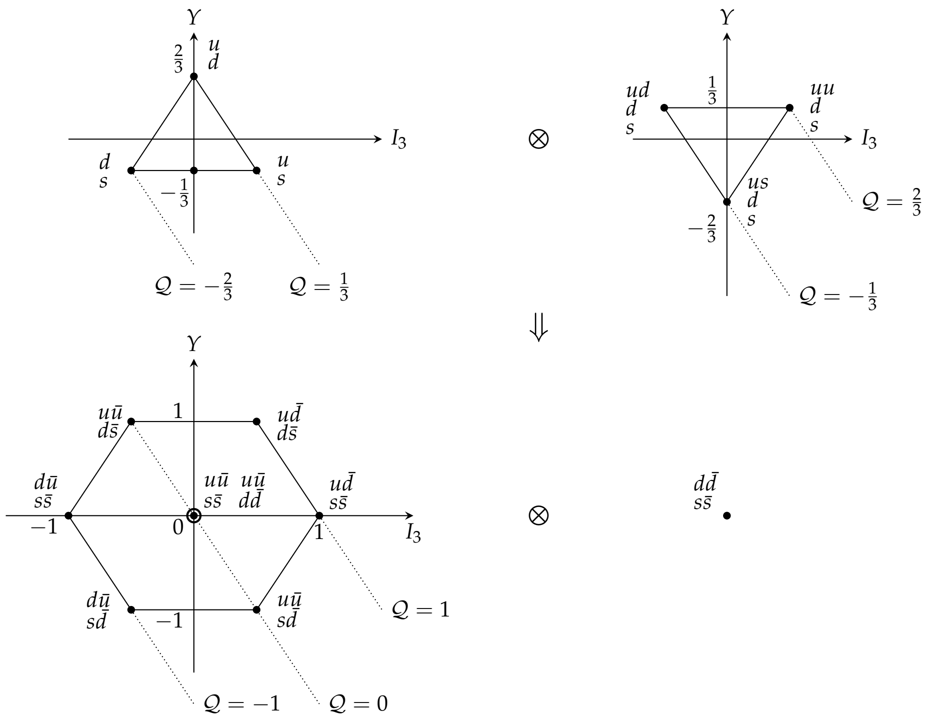


Figure 14. The 8-plet states of tetraquarks for partition [321] and singlet state of tetraquarks for partition [222]. They come from $[11]_A \otimes [211]_A = [321]_{AA} \oplus [222]_{AA}$, and the corresponding dimension is $\mathbf{3}_A \otimes \mathbf{3}_A = \mathbf{8}_{AA} \oplus \mathbf{1}_{AA}$.

$I = 1, M_I = 1, Y = 0$:

$$\begin{aligned}
\chi^{f_1}_{11,0}(\overline{qq}\,\overline{\bar{q}\bar{q}}) &= \frac{1}{\sqrt{2}}\Big[\chi^f_{[2]11}(qq)\chi^f_{[22]10}(\bar{q}\bar{q}) - \chi^f_{[2]10}(qq)\chi^f_{[22]11}(\bar{q}\bar{q})\Big] \\
&= -\frac{1}{2}(uu\bar{u}\bar{d} + uu\bar{d}\bar{u} + ud\bar{d}\bar{d} + du\bar{d}\bar{d}), \\
\chi^{f_2}_{11,0}(\overline{qq}\,\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[2]11}(qq)\chi^f_{[211]00}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(uu\bar{u}\bar{d} - uu\bar{d}\bar{u}), \\
\chi^{f_3}_{11,0}(\overline{qq}\,\overline{\bar{q}\bar{q}}) &= \chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = -\frac{1}{2}(us\bar{d}\bar{s} + us\bar{s}\bar{d} + su\bar{d}\bar{s} + su\bar{s}\bar{d}), \\
\chi^{f_4}_{11,0}(\overline{qq}\,\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = \frac{1}{2}(us\bar{d}\bar{s} - us\bar{s}\bar{d} + su\bar{d}\bar{s} - su\bar{s}\bar{d}), \\
\chi^{f_5}_{11,0}(\widetilde{qq}\,\overline{\bar{q}\bar{q}}) &= \chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = -\frac{1}{2}(us\bar{d}\bar{s} + us\bar{s}\bar{d} - su\bar{d}\bar{s} - su\bar{s}\bar{d}), \\
\chi^{f_6}_{11,0}(\widetilde{qq}\,\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = \frac{1}{2}(us\bar{d}\bar{s} - us\bar{s}\bar{d} - su\bar{d}\bar{s} + su\bar{s}\bar{d}), \\
\chi^{f_7}_{11,0}(\widetilde{qq}\,\overline{\bar{q}\bar{q}}) &= \chi^f_{[11]0,0}(qq)\chi^f_{[22]11}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(ud\bar{d}\bar{d} - du\bar{d}\bar{d}).
\end{aligned} \tag{112}$$

$I = 1, M_I = 1, Y = -2$:

$$\chi^{f_1}_{11,-2}(\overline{qq}\,\overline{\bar{q}\bar{q}}) = \chi^f_{[2]00}(qq)\chi^f_{[22]11}(\bar{q}\bar{q}) = ss\bar{d}\bar{d}. \tag{113}$$

$I=\frac{1}{2}, M_I=\frac{1}{2}, Y=1$:

$$\begin{aligned}
\chi^{f_1}_{\frac{1}{2},\frac{1}{2},1}(\overline{qq}\overline{\bar{q}\bar{q}}) &= \sqrt{\frac{2}{3}}\chi^f_{[2]11}(qq)\chi^f_{[22]\frac{1}{2}-\frac{1}{2}}(\bar{q}\bar{q}) - \sqrt{\frac{1}{3}}\chi^f_{[2]10}(qq)\chi^f_{[22]\frac{1}{2}\frac{1}{2}}(\bar{q}\bar{q}) \\
&= \frac{1}{\sqrt{12}}(2uu\bar{u}\bar{s}+2uu\bar{s}\bar{u}+ud\bar{d}\bar{s}+ud\bar{s}\bar{d}+du\bar{d}\bar{s}+du\bar{s}\bar{d}), \\
\chi^{f_2}_{\frac{1}{2},\frac{1}{2},1}(\overline{qq}\widetilde{\bar{q}\bar{q}}) &= \sqrt{\frac{2}{3}}\chi^f_{[2]11}(qq)\chi^f_{[211]\frac{1}{2}-\frac{1}{2}}(\bar{q}\bar{q}) - \sqrt{\frac{1}{3}}\chi^f_{[2]10}(qq)\chi^f_{[211]\frac{1}{2}\frac{1}{2}}(\bar{q}\bar{q}) \\
&= -\frac{1}{\sqrt{12}}(2uu\bar{u}\bar{s}-2uu\bar{s}\bar{u}+ud\bar{d}\bar{s}-ud\bar{s}\bar{d}+du\bar{d}\bar{s}-du\bar{s}\bar{d}), \\
\chi^{f_3}_{\frac{1}{2},\frac{1}{2},1}(\overline{qq}\overline{\bar{q}\bar{q}}) &= \chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[2]00}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(us\bar{s}\bar{s}+su\bar{s}\bar{s}), \\
\chi^{f_4}_{\frac{1}{2},\frac{1}{2},1}(\widetilde{qq}\overline{\bar{q}\bar{q}}) &= \chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[2]00}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(us\bar{s}\bar{s}-su\bar{s}\bar{s}), \\
\chi^{f_5}_{\frac{1}{2},\frac{1}{2},1}(\widetilde{qq}\overline{\bar{q}\bar{q}}) &= \chi^f_{[11]0,0}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = -\frac{1}{2}(ud\bar{d}\bar{s}+ud\bar{s}\bar{d}-du\bar{d}\bar{s}-du\bar{s}\bar{d}), \\
\chi^{f_6}_{\frac{1}{2},\frac{1}{2},1}(\widetilde{qq}\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[11]0,0}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = \frac{1}{2}(ud\bar{d}\bar{s}-ud\bar{s}\bar{d}-du\bar{d}\bar{s}+du\bar{s}\bar{d}).
\end{aligned} \tag{114}$$

$I=\frac{1}{2}, M_I=\frac{1}{2}, Y=-1$:

$$\begin{aligned}
\chi^{f_1}_{\frac{1}{2},\frac{1}{2},-1}(\overline{qq}\overline{\bar{q}\bar{q}}) &= \sqrt{\frac{1}{3}}\chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]1,0}(\bar{q}\bar{q}) - \sqrt{\frac{2}{3}}\chi^f_{[2]\frac{1}{2},-\frac{1}{2}}(qq)\chi^f_{[22]1,1}(\bar{q}\bar{q}) \\
&= -\frac{1}{\sqrt{12}}(us\bar{u}\bar{d}+us\bar{d}\bar{u}+2ds\bar{d}\bar{d}+su\bar{u}\bar{d}+su\bar{d}\bar{u}+2sd\bar{d}\bar{d}), \\
\chi^{f_2}_{\frac{1}{2},\frac{1}{2},-1}(\overline{qq}\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[211]0,0}(\bar{q}\bar{q}) \\
&= \frac{1}{2}(us\bar{u}\bar{d}-us\bar{d}\bar{u}+su\bar{u}\bar{d}-su\bar{d}\bar{u}), \\
\chi^{f_3}_{\frac{1}{2},\frac{1}{2},-1}(\overline{qq}\overline{\bar{q}\bar{q}}) &= \chi^f_{[2]0,0}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) \\
&= -\frac{1}{\sqrt{2}}(ss\bar{d}\bar{s}+ss\bar{s}\bar{d}), \\
\chi^{f_4}_{\frac{1}{2},\frac{1}{2},-1}(\overline{qq}\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[2]0,0}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) \\
&= \frac{1}{\sqrt{2}}(ss\bar{d}\bar{s}-ss\bar{s}\bar{d}), \\
\chi^{f_5}_{\frac{1}{2},\frac{1}{2},-1}(\widetilde{qq}\overline{\bar{q}\bar{q}}) &= \sqrt{\frac{1}{3}}\chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]1,0}(\bar{q}\bar{q}) - \sqrt{\frac{2}{3}}\chi^f_{[11]\frac{1}{2},-\frac{1}{2}}(qq)\chi^f_{[22]1,1}(\bar{q}\bar{q}) \\
&= -\frac{1}{\sqrt{12}}(us\bar{u}\bar{d}+us\bar{d}\bar{u}+2ds\bar{d}\bar{d}-su\bar{u}\bar{d}-su\bar{d}\bar{u}-2sd\bar{d}\bar{d}), \\
\chi^{f_6}_{\frac{1}{2},\frac{1}{2},-1}(\widetilde{qq}\widetilde{\bar{q}\bar{q}}) &= \chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[211]0,0}(\bar{q}\bar{q}) \\
&= \frac{1}{2}(us\bar{u}\bar{d}-us\bar{d}\bar{u}-su\bar{u}\bar{d}+su\bar{d}\bar{u}).
\end{aligned} \tag{115}$$

$I=0, M_I=0, Y=2$:

$$\chi^{f_1}_{00,-2}(\widetilde{qq}\overline{\bar{q}\bar{q}}) = \chi^f_{[11]00}(qq)\chi^f_{[22]00}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(ud\bar{s}\bar{s}-du\bar{s}\bar{s}). \tag{116}$$

$I=0, M_I=0, Y=0$:

$$\chi^{f_1}_{00,0}(\overline{qq}\,\overline{\bar{q}\bar{q}}) = \sqrt{\frac{1}{3}}\left[\chi^f_{[2]11}(qq)\chi^f_{[22]1-1}(\bar{q}\bar{q}) - \chi^f_{[2]10}(qq)\chi^f_{[22]10}(\bar{q}\bar{q}) + \chi^f_{[2]1-1}(qq)\chi^f_{[22]11}(\bar{q}\bar{q})\right] \tag{117}$$
$$= \frac{1}{\sqrt{12}}(2uu\bar{u}\bar{u} + ud\bar{u}\bar{d} + ud\bar{d}\bar{u} + du\bar{u}\bar{d} + du\bar{d}\bar{u} + 2dd\bar{d}\bar{d}),$$
$$\chi^{f_2}_{00,0}(\overline{qq}\,\overline{\bar{q}\bar{q}}) = \sqrt{\frac{1}{2}}\left[\chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{q}) - \chi^f_{[2]\frac{1}{2},-\frac{1}{2}}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})\right]$$
$$= \frac{1}{\sqrt{8}}(us\bar{u}\bar{s} + us\bar{s}\bar{u} + ds\bar{d}\bar{s} + ds\bar{s}\bar{d} + su\bar{u}\bar{s} + su\bar{s}\bar{u} + sd\bar{d}\bar{s} + sd\bar{s}\bar{d}),$$
$$\chi^{f_3}_{00,0}(\overline{qq}\,\widetilde{\bar{q}\bar{q}}) = \sqrt{\frac{1}{2}}\left[\chi^f_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[211]\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{q}) - \chi^f_{[2]\frac{1}{2},-\frac{1}{2}}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})\right]$$
$$= -\frac{1}{\sqrt{8}}(us\bar{u}\bar{s} - us\bar{s}\bar{u} + su\bar{u}\bar{s} - su\bar{s}\bar{u} + ds\bar{d}\bar{s} - ds\bar{s}\bar{d} + sd\bar{d}\bar{s} - sd\bar{s}\bar{d}),$$
$$\chi^{f_4}_{00,0}(\overline{qq}\,\overline{\bar{q}\bar{q}}) = \chi^f_{[2]00}(qq)\chi^f_{[22]00}(\bar{q}\bar{q}) = ss\bar{s}\bar{s}, \tag{118}$$
$$\chi^{f_5}_{00,0}(\widetilde{qq}\,\overline{\bar{q}\bar{q}}) = \sqrt{\frac{1}{2}}\left[\chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[22]\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{q}) - \chi^f_{[11]\frac{1}{2},-\frac{1}{2}}(qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})\right]$$
$$= \frac{1}{\sqrt{8}}(us\bar{u}\bar{s} + us\bar{s}\bar{u} + ds\bar{d}\bar{s} + ds\bar{s}\bar{d} - su\bar{u}\bar{s} - su\bar{s}\bar{u} - sd\bar{d}\bar{s} - sd\bar{s}\bar{d}),$$
$$\chi^{f_6}_{00,0}(\widetilde{qq}\,\widetilde{\bar{q}\bar{q}}) = \sqrt{\frac{1}{2}}\left[\chi^f_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^f_{[211]\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{q}) - \chi^f_{[11]\frac{1}{2},-\frac{1}{2}}(qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q})\right]$$
$$= -\frac{1}{\sqrt{8}}(us\bar{u}\bar{s} - us\bar{s}\bar{u} + ds\bar{d}\bar{s} - ds\bar{s}\bar{d} - su\bar{u}\bar{s} + su\bar{s}\bar{u} - sd\bar{d}\bar{s} + sd\bar{s}\bar{d}),$$
$$\chi^{f_7}_{00,0}(\widetilde{qq}\,\widetilde{\bar{q}\bar{q}}) = \chi^f_{[11]00}(qq)\chi^f_{[211]00}(\bar{q}\bar{q}) = \frac{1}{2}(ud\bar{u}\bar{d} - ud\bar{d}\bar{u} - du\bar{u}\bar{d} + du\bar{d}\bar{u}).$$

$I = 0, M_I = 0, Y = -2$:

$$\chi^{f_1}_{00,-2}(\overline{qq}\,\widetilde{\bar{q}\bar{q}}) = \chi^f_{[2]00}(qq)\chi^f_{[211]00}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(ss\bar{u}\bar{d} - ss\bar{d}\bar{u}). \tag{119}$$

If the ground state in the coordinate space is imposed, the allowed states are listed in Table 5. We can also get the following decomposition in the flavor–spin space with $SU_6^{f\sigma}$ symmetry [203–206]:

$$\begin{aligned}&[1]\otimes[1]\otimes[11111]\otimes[11111]\\ =&[42222]\oplus[422211]\oplus[33222]\oplus[332211]\oplus 4*[322221]\oplus 2*[222222],\end{aligned} \tag{120}$$

$$\mathbf{6}\otimes\mathbf{6}\otimes\bar{\mathbf{6}}\otimes\bar{\mathbf{6}} = \mathbf{405}\oplus\mathbf{280}\oplus\overline{\mathbf{280}}\oplus\mathbf{189}\oplus 4*\mathbf{35}\oplus 2*\mathbf{1}. \tag{121}$$

The flavor–spin contents in the given symmetry of $SU_6^{f\sigma}$ are as follows:

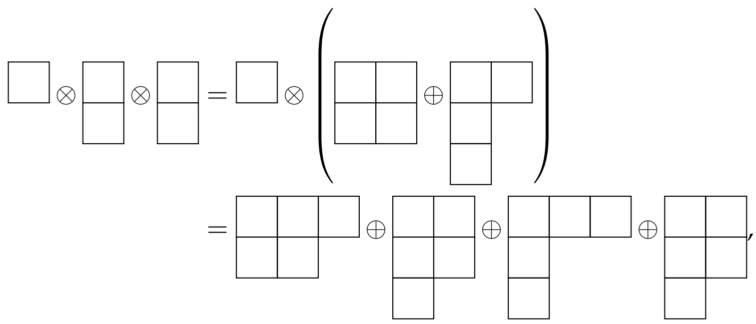
$$\begin{aligned}
\mathbf{405} &= (\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{5})\oplus(\mathbf{8},\mathbf{5})\oplus 2(\mathbf{8},\mathbf{3})\oplus(\mathbf{27},\mathbf{1})\oplus(\mathbf{8},\mathbf{1})\oplus(\mathbf{27},\mathbf{3})\oplus(\mathbf{10},\mathbf{3})\oplus(\overline{\mathbf{10}},\mathbf{3})\oplus(\mathbf{27},\mathbf{5})\\
\mathbf{280} &= (\mathbf{10},\mathbf{5})\oplus(\mathbf{8},\mathbf{5})\oplus(\mathbf{27},\mathbf{3})\oplus(\mathbf{10},\mathbf{3})\oplus 2(\mathbf{8},\mathbf{3})\oplus(\mathbf{10},\mathbf{1})\oplus(\overline{\mathbf{10}},\mathbf{1})\oplus(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\\
\overline{\mathbf{280}} &= (\overline{\mathbf{10}},\mathbf{5})\oplus(\mathbf{8},\mathbf{5})\oplus(\mathbf{27},\mathbf{3})\oplus(\overline{\mathbf{10}},\mathbf{3})\oplus 2(\mathbf{8},\mathbf{3})\oplus(\overline{\mathbf{10}},\mathbf{1})\oplus(\mathbf{10},\mathbf{1})\oplus(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\\
\mathbf{189} &= (\mathbf{8},\mathbf{5})\oplus(\mathbf{10},\mathbf{3})\oplus(\mathbf{27},\mathbf{1})\oplus(\overline{\mathbf{10}},\mathbf{3})\oplus 2(\mathbf{8},\mathbf{3})\oplus(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{5})\\
\mathbf{35} &= (\mathbf{1},\mathbf{3})\oplus(\mathbf{8},\mathbf{1})\oplus(\mathbf{8},\mathbf{3})\\
\mathbf{1} &= (\mathbf{1},\mathbf{1})
\end{aligned} \tag{122}$$

Table 5. The allowed ground states for given isospin and spin. The numbers ijk in parentheses are the indices of color, spin, and flavor wave functions, $\chi^{ci}, \chi^{\sigma j}_{SM_S}$ and $\chi^{fk}_{IM_I,Y}$.

I	2	$\frac{3}{2}$	$\frac{3}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
Y	0	1	-1	2	0	-2	1	-1	2	0	-2
$S=2$	(211)	(211)	(211)	(211)	(116)	(211)	(116)	(116)	-	(116)	-
					(211)		(211)	(211)		(117)	
					(213)		(213)	(213)		(211)	
										(212)	
										(214)	
$S=1$	(211)	(132)	(122)	(211)	(116)	(211)	(116)	(116)	(121)	(116)	(131)
		(211)	(211)		(125)		(124)	(125)	(231)	(117)	(221)
		(222)	(232)		(127)		(125)	(132)		(125)	
					(132)		(132)	(134)		(133)	
					(134)		(211)	(211)		(211)	
					(211)		(213)	(213)		(212)	
					(213)		(222)	(222)		(214)	
					(222)		(234)	(224)		(223)	
					(224)		(235)	(235)		(235)	
					(235)						
					(237)						
$S=0$	(121)	(121)	(121)	(121)	(116)	(121)	(116)	(116)	-	(116)	-
	(211)	(211)	(211)	(211)	(121)	(211)	(121)	(121)		(117)	
					(123)		(123)	(123)		(121)	
					(211)		(211)	(211)		(122)	
					(213)		(213)	(213)		(124)	
					(226)		(226)	(226)		(211)	
										(212)	
										(214)	
										(226)	
										(227)	

5.3.2. $Qq\bar{q}\bar{q}$

For the tetraquark system with one heavy quark, the flavor symmetry SU^f_3 classifies 27 states (as an example, the heavy quark is fixed to the first position here) into the following multiplets:



(123)

$$[1] \otimes [11] \otimes [11] = [1] \otimes ([22]_S \oplus [211]_A) = [32]_{-S} \oplus [221]_{-S} \oplus [311]_{-A} \oplus [221]_{-A}, \quad (124)$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{15}_{-S} \oplus \mathbf{3}_{-S} \oplus \mathbf{6}_{-A} \oplus \mathbf{3}_{-A}. \quad (125)$$

The symbol "−" in the subscripts means that there is no permutation symmetry under the quark exchanges. The weight diagram of Equation (125) is shown in Figures 15 and 16, it should be noted that the flavor symmetry we used in this article is SU_3^f. In the multi-quark diagram later, we did not mark heavy quarks. If we want to add heavy quarks, corresponding changes need to be made to the charge and hypercharge.

For Qq cluster, the possible three wave functions $\chi^f_{I_1 M_{I_1}}(Qq)$ are as follows:

$$\chi^f_{\frac{1}{2},\frac{1}{2}}(Qq) = Qu, \quad \chi^f_{\frac{1}{2},-\frac{1}{2}}(Qq) = Qd, \quad \chi^f_{00}(Qq) = Qs. \quad (126)$$

Using CG coefficients of SU_3, and combining Equations (126) and (107), the wave functions of the $Qq\bar{q}\bar{q}$ systems can be constructed and listed as follows.

$I = \frac{3}{2}, M_I = \frac{3}{2}, Y = -\frac{1}{3}$:

$$\chi^{f1}_{\frac{3}{2},\frac{3}{2},-\frac{1}{3}}(Qq\overline{\bar{q}\bar{q}}) = \chi^f_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^f_{[22]11}(\bar{q}\bar{q}) = Qu\bar{d}\bar{d}. \quad (127)$$

$I = 1, M_I = 1, Y = \frac{2}{3}$:

$$\chi^{f1}_{11,\frac{2}{3}}(Qq\overline{\bar{q}\bar{q}}) = \chi^f_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^f_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = -\frac{1}{\sqrt{2}}(Qu\bar{d}\bar{s} + Qu\bar{s}\bar{d}),$$
$$\chi^{f2}_{11,\frac{2}{3}}(Qq\widetilde{\bar{q}\bar{q}}) = \chi^f_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^f_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) = \frac{1}{\sqrt{2}}(Qu\bar{d}\bar{s} - Qu\bar{s}\bar{d}). \quad (128)$$

$I = 1, M_I = 1, Y = -\frac{4}{3}$:

$$\chi^{f1}_{11,-\frac{4}{3}}(Qq\overline{\bar{q}\bar{q}}) = \chi^f_{0,0}(Qq)\chi^f_{[22]1,1}(\bar{q}\bar{q}) = Qs\bar{d}\bar{d}. \quad (129)$$

$I = \frac{1}{2}, M_I = \frac{1}{2}, Y = \frac{5}{3}$:

$$\chi^{f1}_{\frac{1}{2},\frac{1}{2},\frac{5}{3}}(Qq\overline{\bar{q}\bar{q}}) = \chi^f_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^f_{[22]0,0}(\bar{q}\bar{q}) = Qu\bar{s}\bar{s}. \quad (130)$$

$I = \frac{1}{2}, M_I = \frac{1}{2}, Y = -\frac{1}{3}$:

$$\chi^{f1}_{\frac{1}{2},\frac{1}{2},-\frac{1}{3}}(Qq\overline{\bar{q}\bar{q}}) = \sqrt{\frac{1}{3}}\chi^f_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^f_{[22]1,0}(\bar{q}\bar{q}) - \sqrt{\frac{2}{3}}\chi^f_{\frac{1}{2},-\frac{1}{2}}(Qq)\chi^f_{[22]1,1}(\bar{q}\bar{q})$$
$$= -\frac{1}{\sqrt{6}}(Qu\bar{u}\bar{d} + Qu\bar{d}\bar{u} + 2Qd\bar{d}\bar{d}),$$
$$\chi^{f2}_{\frac{1}{2},\frac{1}{2},-\frac{1}{3}}(Qq\widetilde{\bar{q}\bar{q}}) = \chi^f_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^f_{[211]0,0}(\bar{q}\bar{q})$$
$$= \sqrt{\frac{1}{2}}(Qu\bar{u}\bar{d} - Qu\bar{d}\bar{u}),$$

$$\begin{aligned}
\chi^{f1}_{\frac{1}{2},\frac{1}{2},-\frac{1}{3}}(Qq\overline{\bar{q}\bar{q}}) &= \sqrt{\frac{1}{3}}\chi^{f}_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^{f}_{[22]1,0}(\bar{q}\bar{q}) - \sqrt{\frac{2}{3}}\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(Qq)\chi^{f}_{[22]1,1}(\bar{q}\bar{q}) \\
&= -\frac{1}{\sqrt{6}}(Qu\bar{u}\bar{d} + Qu\bar{d}\bar{u} + 2Qd\bar{d}\bar{d}), \\
\chi^{f2}_{\frac{1}{2},\frac{1}{2},-\frac{1}{3}}(Qq\widetilde{\bar{q}\bar{q}}) &= \chi^{f}_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^{f}_{[211]0,0}(\bar{q}\bar{q}) \\
&= \sqrt{\frac{1}{2}}(Qu\bar{u}\bar{d} - Qu\bar{d}\bar{u}), \\
\chi^{f3}_{\frac{1}{2},\frac{1}{2},-\frac{1}{3}}(Qq\overline{\bar{q}\bar{q}}) &= \chi^{f}_{0,0}(Qq)\chi^{f}_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) \\
&= -\sqrt{\frac{1}{2}}(Qs\bar{d}\bar{s} + Qs\bar{s}\bar{d}), \\
\chi^{f4}_{\frac{1}{2},\frac{1}{2},-\frac{1}{3}}(Qq\widetilde{\bar{q}\bar{q}}) &= \chi^{f}_{0,0}(Qq)\chi^{f}_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) \\
&= \sqrt{\frac{1}{2}}(Qs\bar{d}\bar{s} - Qs\bar{s}\bar{d}).
\end{aligned} \tag{131}$$

$I = 0, M_I = 0, Y = \frac{2}{3}$:

$$\begin{aligned}
\chi^{f1}_{00,\frac{2}{3}}(Qq\overline{\bar{q}\bar{q}}) &= \sqrt{\frac{1}{2}}\chi^{f}_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^{f}_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) - \sqrt{\frac{1}{2}}\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(Qq)\chi^{f}_{[22]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) \\
&= \frac{1}{2}(Qu\bar{u}\bar{s} + Qu\bar{s}\bar{u} + Qd\bar{d}\bar{s} + Qd\bar{s}\bar{d}), \\
\chi^{f3}_{00,\frac{2}{3}}(Qq\widetilde{\bar{q}\bar{q}}) &= \sqrt{\frac{1}{2}}\chi^{f}_{\frac{1}{2},\frac{1}{2}}(Qq)\chi^{f}_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) - \sqrt{\frac{1}{2}}\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(Qq)\chi^{f}_{[211]\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{q}) \\
&= -\frac{1}{2}(Qu\bar{u}\bar{s} - Qu\bar{s}\bar{u} + Qd\bar{d}\bar{s} - Qd\bar{s}\bar{d}), \\
\chi^{f4}_{00,\frac{2}{3}}(Qq\overline{\bar{q}\bar{q}}) &= \chi^{f}_{0,0}(Qq)\chi^{f}_{[22]0,0}(\bar{q}\bar{q}) \\
&= Qs\bar{s}\bar{s}.
\end{aligned} \tag{132}$$

$I = 0, M_I = 0, Y = -\frac{4}{3}$:

$$\begin{aligned}
\chi^{f1}_{00,-\frac{4}{3}}(Qq\widetilde{\bar{q}\bar{q}}) &= \chi^{f}_{0,0}(Qq)\chi^{f}_{[211]0,0}(\bar{q}\bar{q}) \\
&= \frac{1}{\sqrt{2}}(Qs\bar{u}\bar{d} - Qs\bar{d}\bar{u}).
\end{aligned} \tag{133}$$

The constraint of ground state (the wave function of the antiquark pair is symmetric in the coordinate space) only comes from the antiquark part. For example, in the spin 2 and isospin 3/2 state with color symmetry [11], the anti-triplet is allowed. We do not list them one by one here to save space.

5.3.3. $qq\bar{q}\bar{Q}$

For the tetraquark system with one heavy antiquark, the flavor symmetry SU_3^f classifies 27 states (as an example, the heavy antiquark is fixed to the fourth position here) into the following multiplets:

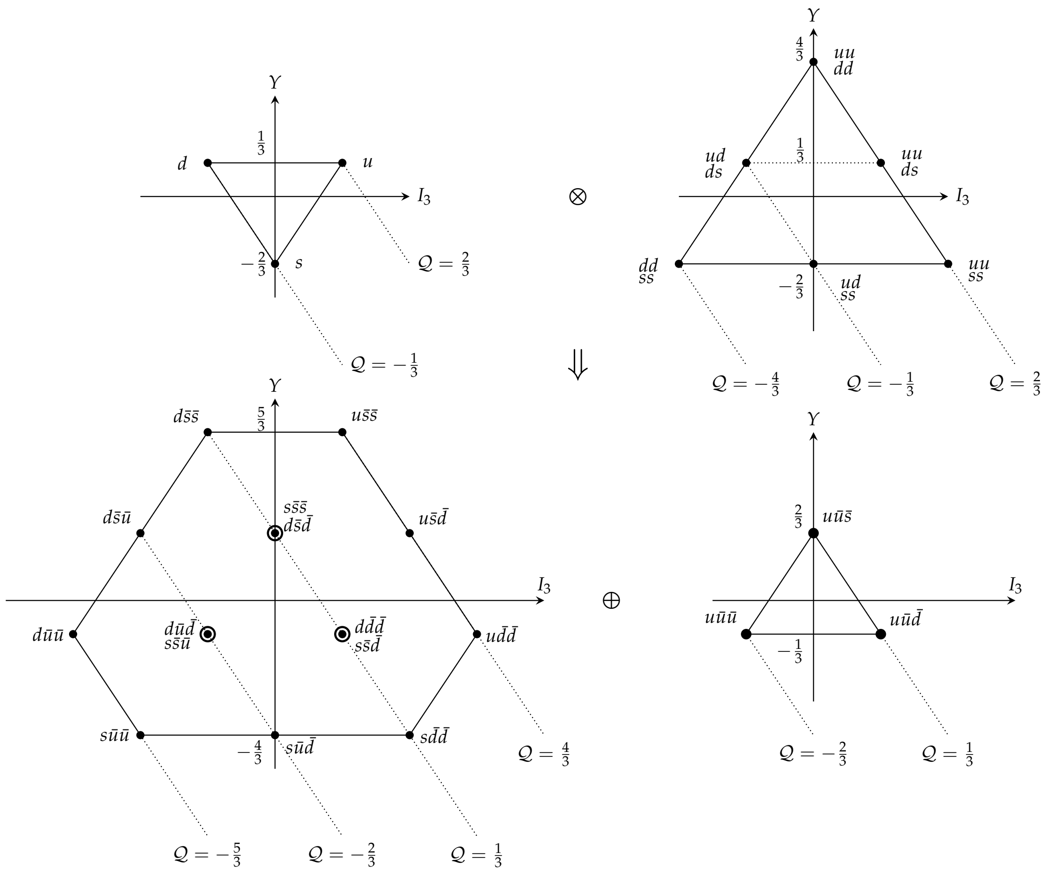


Figure 15. The 15-plet states of tetraquarks for partition [32] and 3-plet states of tetraquarks for partition [221]. They come from $[1]_- \otimes [22]_S = [32]_{-S} \oplus [221]_{-S}$, and the corresponding dimension is $\mathbf{3}_- \otimes \mathbf{6}_S = \mathbf{15}_{-S} \oplus \mathbf{3}_{-S}$.

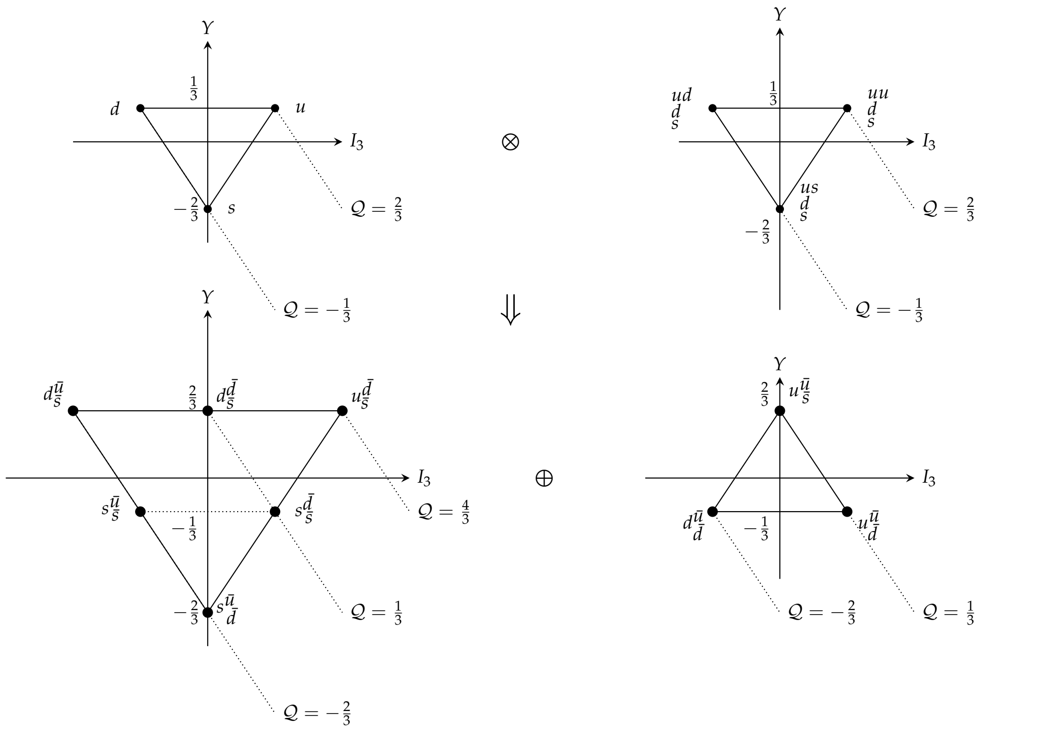
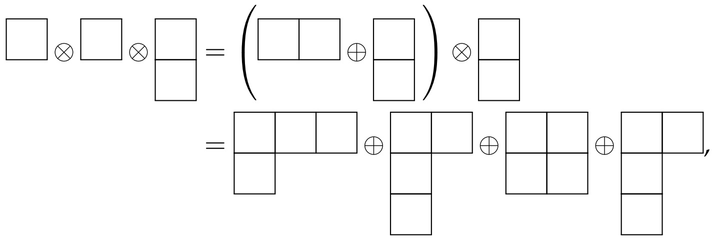


Figure 16. The 6-plet states of tetraquarks for partition [311] and 3-plet states of tetraquarks for partition [221]. They come from $[1]_- \otimes [211]_A = [311]_{-A} \oplus [221]_{-A}$, and the corresponding dimension is $\mathbf{3}_- \otimes \mathbf{3}_A = \mathbf{6}_{-A} \oplus \mathbf{3}_{-A}$.


(134)

$$\mathbf{3}\otimes\mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{15}_{S-}\oplus\mathbf{3}_{S-}\oplus\bar{\mathbf{6}}_{A-}\oplus\mathbf{3}_{A-}.$$

The symbol "−" in the subscripts also means that there is no permutation symmetry under the antiquark exchanges,the diagram of Equation (134) is displayed in Figures 17 and 18.

For $\bar{q}\bar{Q}$ cluster,the possible 3 wave functions $\chi^{f}_{I_1M_{I_1}}(\bar{q}\bar{Q})$ are as follows:

$$\begin{aligned}
&\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q})=\bar{d}\bar{Q},\\
&\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{Q})=-\bar{u}\bar{Q},\\
&\chi^{f}_{00}(\bar{q}\bar{Q})=\bar{s}\bar{Q}.
\end{aligned}\tag{135}$$

Using CG coefficients of SU_3, combine Equations (135) and (106), and the wave functions of the $qq\bar{q}\bar{Q}$ systems can be constructed and listed as follows.

$I=\frac{3}{2},M_I=\frac{3}{2},Y=\frac{1}{3}$:

$$\chi^{f1}_{\frac{3}{2},\frac{3}{2},\frac{1}{3}}(\overline{qq}\bar{q}\bar{Q})=\chi^{f}_{[2]11}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q})=uu\bar{d}\bar{Q}.\tag{136}$$

$I=1,M_I=1,Y=-\frac{2}{3}$:

$$\begin{aligned}
\chi^{f1}_{11,-\frac{2}{3}}(\overline{qq}\bar{q}\bar{Q})&=\chi^{f}_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q})=\frac{1}{\sqrt{2}}(us\bar{d}\bar{Q}+su\bar{d}\bar{Q}),\\
\chi^{f2}_{11,-\frac{2}{3}}(\widetilde{q}q\bar{q}\bar{Q})&=\chi^{f}_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q})=\frac{1}{\sqrt{2}}(us\bar{d}\bar{Q}-su\bar{d}\bar{Q}).
\end{aligned}\tag{137}$$

$I=1,M_I=1,Y=\frac{4}{3}$:

$$\chi^{f1}_{11,-\frac{4}{3}}(\overline{qq}\bar{q}\bar{Q})=\chi^{f}_{[2]11}(qq)\chi^{f}_{00}(\bar{q}\bar{Q})=uu\bar{s}\bar{c}.\tag{138}$$

$I=\frac{1}{2},M_I=\frac{1}{2},Y=-\frac{5}{3}$:

$$\chi^{f1}_{\frac{1}{2},\frac{1}{2},-\frac{5}{3}}(\overline{qq}\bar{q}\bar{Q})=\chi^{f}_{[2]00}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q})=ss\bar{d}\bar{c}.\tag{139}$$

$I=\frac{1}{2},M_I=\frac{1}{2},Y=\frac{1}{3}$:

$$\begin{aligned}
\chi^{f1}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(\overline{qq}\bar{q}\bar{Q})&=\sqrt{\frac{2}{3}}\chi^{f}_{[2]1,1}(qq)\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{Q})-\sqrt{\frac{1}{3}}\chi^{f}_{[2]1,0}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q})\\
&=-\frac{1}{\sqrt{6}}(2uu\bar{u}\bar{c}+ud\bar{d}\bar{c}+du\bar{d}\bar{c}),\\
\chi^{f2}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(\overline{qq}\bar{q}\bar{Q})&=\chi^{f}_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^{f}_{0,0}(\bar{q}\bar{Q})=\sqrt{\frac{1}{2}}(us\bar{s}\bar{c}+su\bar{s}\bar{c}),
\end{aligned}\tag{140}$$

$$\chi^{f3}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(\tilde{q}q\bar{q}\bar{Q}) = \chi^{f}_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^{f}_{0,0}(\bar{q}\bar{Q}) = \sqrt{\frac{1}{2}}(us\bar{s}\bar{c} - su\bar{s}\bar{c}),$$
$$\chi^{f4}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(\tilde{q}q\bar{q}\bar{Q}) = \chi^{f}_{[11]0,0}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q}) = \sqrt{\frac{1}{2}}(ud\bar{d}\bar{c} - du\bar{d}\bar{c}).$$

$I = 0, M_I = 0, Y = -\frac{2}{3}$:

$$\begin{aligned}
\chi^{f1}_{00,-\frac{2}{3}}(\overline{qq}\bar{q}\bar{Q}) &= \sqrt{\frac{1}{2}}\chi^{f}_{[2]\frac{1}{2},\frac{1}{2}}(qq)\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{Q}) - \sqrt{\frac{1}{2}}\chi^{f}_{[2]\frac{1}{2},-\frac{1}{2}}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q}) \\
&= -\frac{1}{2}(us\bar{u}\bar{c} - ds\bar{d}\bar{c} + su\bar{u}\bar{c} - sd\bar{d}\bar{c}), \\
\chi^{f3}_{00,-\frac{2}{3}}(\tilde{q}q\bar{q}\bar{Q}) &= \sqrt{\frac{1}{2}}\chi^{f}_{[11]\frac{1}{2},\frac{1}{2}}(qq)\chi^{f}_{\frac{1}{2},-\frac{1}{2}}(\bar{q}\bar{Q}) - \sqrt{\frac{1}{2}}\chi^{f}_{[11]\frac{1}{2},-\frac{1}{2}}(qq)\chi^{f}_{\frac{1}{2},\frac{1}{2}}(\bar{q}\bar{Q}) \\
&= -\frac{1}{2}(us\bar{u}\bar{c} + ds\bar{d}\bar{c} - su\bar{u}\bar{c} - sd\bar{d}\bar{c}), \\
\chi^{f4}_{00,-\frac{2}{3}}(\overline{qq}\bar{q}\bar{Q}) &= \chi^{f}_{[2]00}(qq)\chi^{f}_{00}(\bar{q}\bar{Q}) = ss\bar{s}\bar{Q}.
\end{aligned} \tag{141}$$

$I = 0, M_I = 0, Y = \frac{4}{3}$:

$$\chi^{f1}_{00,\frac{4}{3}}(\tilde{q}q\bar{q}\bar{Q}) = \chi^{f}_{[11]00}(qq)\chi^{f}_{00}(\bar{q}\bar{Q}) = \frac{1}{\sqrt{2}}(ud\bar{s}\bar{c} - du\bar{s}\bar{c}). \tag{142}$$

The constraint of the ground state only comes from the quark part because $\bar{q}$ and $\bar{Q}$ are not identical particles. We do not list them here to save space.

There have also been some discoveries about singly heavy tetraquarks in experiments. The LHCb detector collects data in proton–proton collisions to perform a combined amplitude analysis of the $B^0 \to \bar{D}^0 D_s^+ \pi^-$ and $B^+ \to D^- D_s^+ \pi^+$ decays. The neutral $D_s^+\pi^-$ and the doubly charged $D_s^+\pi^+$ resonance are reported, which marks the first observation of a doubly charged tetraquark $T^a_{c\bar{s}0}(2900)^{++}([c\bar{s}u\bar{d}])$ and its neutral isospin partner $T^a_{c\bar{s}0}(2900)^0([c\bar{s}\bar{u}d])$ with spin-parity of 0^+, the mass of $T^a_{c\bar{s}0}(2900)$ aligns with the open-charm tetraquark $X_0(2900)^0([cs\bar{u}\bar{d}])$ with 0^+ observed in D^+K^- decays, though exhibiting distinct widths and flavor configurations [207–210].

Under the framework of chiral quark models, S-wave $q\bar{q}\bar{s}Q$ ($q = u, d; Q = c, b$) tetraquarks with $J^P = 0^+, 1^+, 2^+$ in iso-scalar ($I = 0$) and iso-vector ($I = 1$) sectors were systematically investigated; several allowed narrow resonances for $q\bar{q}\bar{s}c$ with energy ranging from 2.4 to 3.4 GeV, and $q\bar{q}\bar{s}b$ with energy ranging from 5.7 to 6.7 GeV was obtained theoretically [211]. In the low-lying system $Qq\bar{q}\bar{q}$ ($q = u, d; Q = c, b$), the real-scaling method is utilized to identify authentic resonance states. For the single-channel analysis, no bound states are found below the minimum threshold in either $cq\bar{q}\bar{q}$ or $bq\bar{q}\bar{q}$ systems. However, upon coupling all channels, a bound $cq\bar{q}\bar{q}$ ($IJ^P = \frac{1}{2}0^+$) state with binding energy of 4 MeV emerged. In the $bq\bar{q}\bar{q}$ systems, two bound states are identified with binding energies of 5 MeV ($IJ^P = \frac{1}{2}0^+$) and 2 MeV ($IJ^P = \frac{1}{2}1^+$), respectively [212].

5.3.4. $QQ\bar{q}\bar{q}$, $qq\bar{Q}\bar{Q}$, $qQ\bar{q}\bar{Q}$

The systems with two heavy quarks, $QQ\bar{q}\bar{q}$, or two heavy antiquarks, $qq\bar{Q}\bar{Q}$, can be obtained from the products of the wave functions of two light quarks (Equation (106)) or antiquarks (Equation (107)) and the wave functions of heavy quarks, cc, cb, bb (antiquarks, $\bar{c}\bar{c}, \bar{c}\bar{b}, \bar{b}\bar{b}$). The weight diagram of tetraquark $qq\bar{Q}\bar{Q}$ is the same as that of the qq diquark [213].

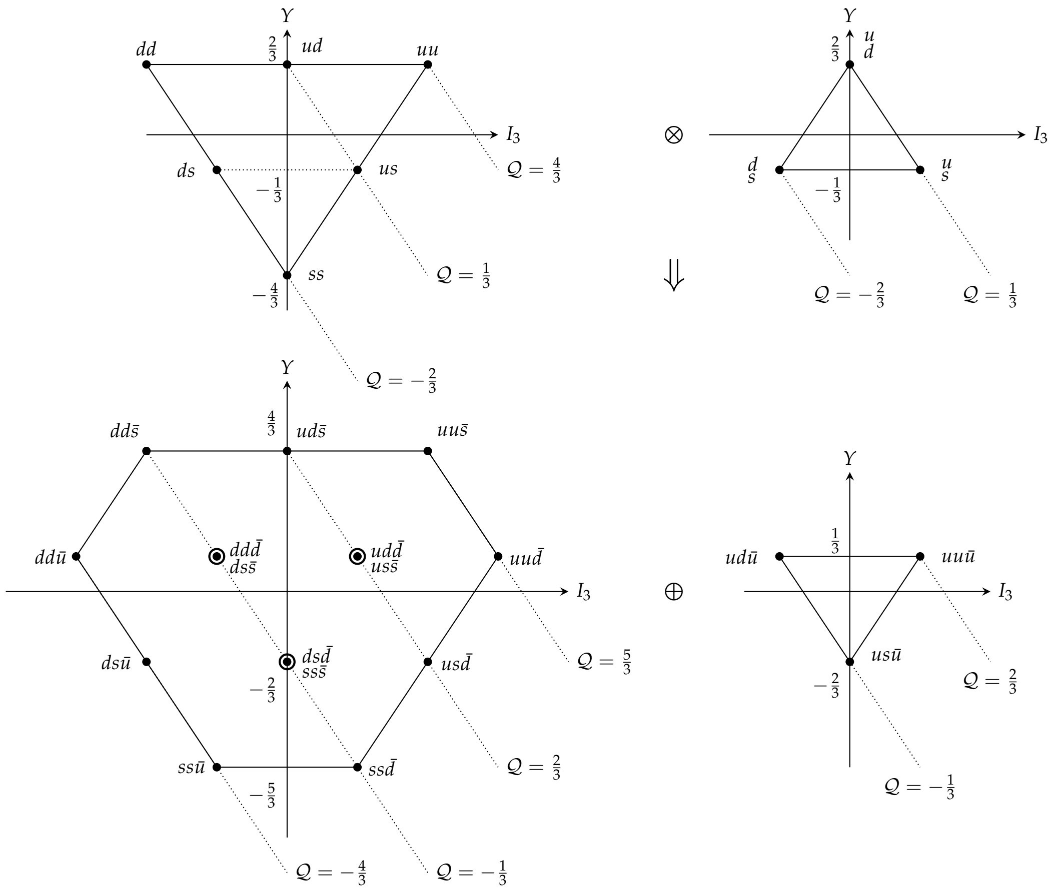


Figure 17. The 15-plet states of tetraquarks for partition [31] and 3-plet states of tetraquarks for partition [211]. They come from $[2]_{\mathbf{6}_S} \otimes [11]_{\mathbf{3}_-} = [31]_{\mathbf{15}_{S-}} \oplus [211]_{\mathbf{3}_{S-}}$.

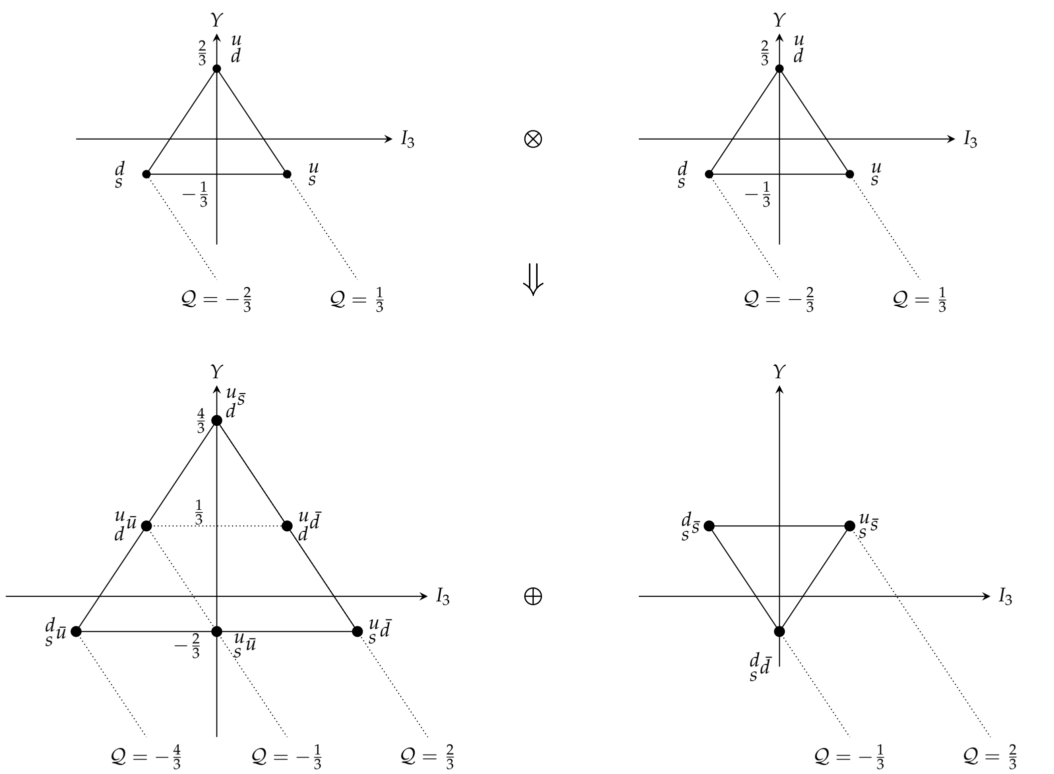


Figure 18. The 6-plet states of tetraquarks for partition [22] and 3-plet states of tetraquarks for partition [211]. They come from $[11]_{\mathbf{3}_A} \otimes [11]_{\mathbf{6}_-} = [22]_{\mathbf{6}_{A-}} \oplus [211]_{\mathbf{3}_{A-}}$.

For the systems with one heavy quark and one heavy antiquark $qQ\bar{q}\bar{Q}$, the wave functions are the products of the light $q\bar{q}$ mesons (Equations (22)–(26)) and $Q\bar{Q}'$, $Q, Q' = c, b$. In the flavor space, we consider $SU_f(3)$ symmetry, so the weight diagram of the tetraquark $qQ\bar{q}\bar{Q}$ is the same as that of the $q\bar{q}$ meson, which can be seen in [206,214].

Since the LHCb collaboration's discovery of $T_{cc} = cc\bar{u}\bar{d}$ in 2021 [158,215], the doubly heavy tetraquarks $QQ\bar{u}\bar{d}$ (where $Q = c$, b) have been extensively studied [216–220]. This exotic hadron is a resonance lying just below the $D^{*+}D^0$ threshold and exhibits a narrow decay width into $D^0D^0\pi^+$. In later research, $X(4140)$ has been established as a $J/\psi\phi$ resonance by several experiments, with quantum numbers $J^{PC} = 1^{++}$ [221–224]. The exotic state $X(4274)$ was subsequently observed in the $J/\psi\phi$ channel by both the CDF and LHCb collaborations [225,226], sharing identical quantum numbers with $X(4140)$. These states are theoretically consistent as partner states within the compact $cs\bar{c}\bar{s}$ tetraquark picture [227,228].

5.3.5. $qQ\bar{Q}\bar{Q}$, $QQ\bar{q}\bar{Q}$

For the systems with three heavy quarks/antiquarks, the wave functions can be constructed easily; they are just the products of the wave functions of four quarks/antiquarks. For example, $uc\bar{b}\bar{b}$, $sc\bar{c}\bar{b}$, etc.

Singly, doubly, and fully heavy tetraquarks have corresponding candidates such as $T_{c\bar{s}(2900)}$,$T_{cc(3875)}$, and $X(6900)$ reported by the LHCb collaboration, but triply heavy tetraquarks remain experimentally unobserved. A systematic theoretical investigation on triply heavy tetraquarks $\bar{Q}Q\bar{q}Q(q = u, d, s; Q = c, b)$ with spin-parity $J^P = 0^+, 1^+, 2^+$ and isospin 0 and $\frac{1}{2}$ was performed in a constituent quark model with the Gaussian expansion method (GEM) combined with the complex scaling method (CSM). Triple-charm tetraquark $\bar{c}c\bar{q}c$ and bottom tetraquark $\bar{b}b\bar{q}b$ resonances are obtained in 5.6–5.9 GeV and 15.3–15.7 GeV, respectively [229]. With similar methods, a series of resonant states with mass ranging from 5.6–15.8 Gev were obtained in $cc\bar{c}\bar{q}/\bar{s}$, $bb\bar{b}\bar{q}/\bar{s}$, $cc\bar{b}\bar{q}/\bar{s}$, $bb\bar{c}\bar{q}/\bar{s}$, $cb\bar{b}\bar{q}/\bar{s}$, $bc\bar{c}\bar{q}/\bar{s}$ systems, but there is no bound state [230]. There are no stable triply heavy tetraquarks in a modified chromomagnetic interaction model either [231]. More study on triply heavy tetraquark spectroscopy can be seen in [232].

5.3.6. $QQ\bar{Q}\bar{Q}$

For the fully heavy tetraquark system, the flavor wave functions χ^f_{CB} are simple, the products of four singlet particle states. The allowed states can be picked up according to the symmetry requirement. For example, for $cc\bar{c}\bar{c}$ ground state with spin $\chi^{\sigma_1}_{22}(\overline{qq}\overline{\bar{q}\bar{q}})(J = 2)$ and $\chi^{\sigma_1}_{11}(\overline{qq}\overline{\bar{q}\bar{q}})(J = 1)$, the color wave function must be $\chi^{c_2}_{[222]}(\widetilde{qq}\widetilde{\bar{q}\bar{q}})$. For $cc\bar{c}\bar{c}$ ground state with $J = 0$, two allowed states are $\chi^{c_1}_{[222]}(\overline{qq}\overline{\bar{q}\bar{q}})\chi^{\sigma_2}_{[222]}(\widetilde{qq}\widetilde{\bar{q}\bar{q}})$ and $\chi^{c_2}_{[222]}(\widetilde{qq}\widetilde{\bar{q}\bar{q}})\chi^{\sigma_1}_{[222]}(\overline{qq}\overline{\bar{q}\bar{q}})$.

Fully heavy tetraquarks consist of four heavy quarks/antiquarks and offer an excellent platform for investigating the dynamics of multi-quark states within nonrelativistic limits. With more and more experimental discoveries, we also have richer data in the region of fully heavy tetraquarks. The stability of the tetraquark $cc\bar{c}\bar{c}$ (T_{4c}) and $bb\bar{b}\bar{b}$ (T_{4b}) states has always been a concern for people. If the $cc\bar{c}\bar{c}(T_{4c})$ or $bb\bar{b}\bar{b}(T_{4b})$ states have relatively low masses, falling below the thresholds of heavy charmonium or bottomonium pairs, they could potentially be considered "stable". This is due to the fact that direct decays into heavy quarkonium pairs via quark rearrangements would not be energetically permitted [233–237]. The mass differences in the 1^{+-} and 0^{++} states for T^+_{cc} are found to be small enough, only a few MeV [238]. In addition, $cc\bar{c}\bar{c}$ has been discovered in experiments, and other fully heavy tetrquarks were also predicted in many works [239–242]. The Gürsey–Radicati mass formula has also been used to investigate tetarquark systems [206,243]. Here we choose some states to see if the mass formula can give a rough description of tetraquarks. The result is listed in Table 6.

Table 6. The parameters and masses of group-state tetraquarks.

Parameter	M_0	X_1		X_2		X_3	
	750.46 ± 0.29	5.46 ± 0.06		110.19 ± 0.07		141.14 ± 0.26	
	X_4	X_5		X_6			
	-16.93 ± 0.45	1388.9 ± 0.19		3200.0 ± 0.14			
states	Y (hypercharge)	I (isospin)	J (spin)	$[f]$ (flavor)	Theo. (MeV)	Exp. (MeV)	Ref. (MeV)
f_0	0	0	0	[222]	750.46 ± 0.28	600.00	
f_2	0	0	2	[222]	1411.57 ± 0.71	1430.00	
h_1	0	0	1	[222]	970.83 ± 0.43	1170.00	
a_0	0	1	0	[321]	1065.49 ± 1.16	980.00	
b_1	0	1	1	[321]	1285.86 ± 1.30	1235.00	
K_0^*	1	$\frac{1}{2}$	0	[321]	872.15 ± 1.28	800.00	
K_1^*	1	$\frac{1}{2}$	1	[321]	1092.52 ± 1.42	1270.00	
K_2^*	1	$\frac{1}{2}$	2	[321]	1533.26 ± 1.71	1430.00	
$qc\bar{q}\bar{c}$	0	1	1	[21]	4063.65 ± 1.68	-	$Zc(4200)$ [244]
$qc\bar{s}\bar{c}$	1	0	2	[21]	4311.04 ± 2.10	-	4135 [206]
$cc\bar{c}\bar{c}$	0	0	2	[0]	6967.14 ± 1.47	-	$X(6900)$ [24]

In 2020, the LHCb collaboration first reported fully charmed resonances in the di-J/ψ channel. Among them, there was a resonance with a narrow structure, designated as $X(6900)$, and another with a broad structure situated in the mass range of 6.2–6.8 GeV [24]. Later, experiments conducted by the CMS and ATLAS collaborations not only verified the presence of $X(6900)$ but also discovered and reported two additional resonances, namely $X(6600)$ and $X(7300)$ [245,246]. One possible theoretical explanation of $X(6600)$, $X(6900)$, and $X(7300)$ indicated that they may be different excitations of $[cc][\bar{c}\bar{c}]$ [247].

For tetraquarks composed of charm and bottom quarks, the lifetime of $T_{\bar{c}\bar{c}}^{bb}$ is found to be rather short, 0.1–0.3 $\times 10^{-12}$ s, with the operator product expansion rooted in heavy quark expansion. The golden channels, such as $T_{\bar{c}\bar{c}}^{bb} \to B^- K^0 B_c^-$ for the charm quark decay and $T_{\bar{c}\bar{c}}^{bb} \to B^- D^-$ for the bottom quark decay, have been collected, and these results may offer some help in the search of fully heavy tetraquarks in future experiments [248]. It is still in debate whether the $\bar{b}c\bar{b}c$ tetraquark state lies above or below the $B_c^+ B_c^-$ dissociation threshold [249–254].

6. Pentaquarks

Pentaqaurk states are also exotic states. The color-singlet requirement dictates that the system is composed by four quarks (antiquarks) and one antiquark (quark). In SU_3 symmetry, the anti-particle is represented by the anti-symmetric combination of two particles. So, the color symmetry of the pentaquark must be the [222] color singlet. The dimension of the spin space is $2^5 = 32$ and the dimension of the flavor space is $5^5 = 3125$. The anti-symmetrization for the identical particles will reduce the number of pentaquark states. A more comprehensive review of pentaquarks can be seen in [12,15].

6.1. Color

Due to the requirement for color singlets, the color symmetry of one antiquark is [11], and the CG series of SU_3 states require that the color symmetry of the four-quark part is [211]. To save space, we use partitions to denote the symmetry of the states instead of Young diagrams. Using the Littlewood rule for the four-quark states, one has

$$\begin{aligned} &[1] \otimes [1] \otimes [1] \otimes [1] = ([2] \oplus [11]) \otimes ([2] \oplus [11]) = [4] \oplus 3\,[31] \oplus 2\,[22] \oplus 3\,[211], \\ &\quad \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{15} \oplus 3 * \mathbf{15} \oplus 2 * \bar{\mathbf{6}} \oplus 3 * \mathbf{3}. \end{aligned} \tag{143}$$

Only the symmetry [211] combined with [11] gives color singlet [222]. The color wave functions for the four-quark part are listed in Table 7, which are obtained by finding the irreducible representation of permutation group S_4 in the configuration space $rrgb, rggb, rgbb$ (The detail of finding irreps of S_4 in configuration space $abbc$ is presented in Appendix A).

Table 7. The color wave function with symmetry [211] of four quarks. $\chi^{ci}_{[211]j} = \sum_k a^k_{ij} \varphi^k_j$.

φ^k_1	$rrgb$	$rrbg$	$rgrb$	$rgbr$	$rbrg$	$rbgr$	$grrb$	$grbr$	$gbrr$	$brrg$	$brgr$	$bgrr$
$\chi^{c1}_{[211]1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	0
$\chi^{c2}_{[211]1}$	0	0	$\frac{3}{\sqrt{48}}$	$-\frac{1}{\sqrt{48}}$	$-\frac{3}{\sqrt{48}}$	$\frac{1}{\sqrt{48}}$	$-\frac{3}{\sqrt{48}}$	$\frac{1}{\sqrt{48}}$	$\frac{2}{\sqrt{48}}$	$\frac{3}{\sqrt{48}}$	$-\frac{1}{\sqrt{48}}$	$-\frac{2}{\sqrt{48}}$
$\chi^{c3}_{[211]1}$	0	0	0	$\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$
φ^k_2	$rggb$	$rgbg$	$rbgg$	$grgb$	$grbg$	$ggrb$	$ggbr$	$gbrg$	$gbgr$	$brgg$	$bgrg$	$bggr$
$\chi^{c1}_{[211]2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$
$\chi^{c2}_{[211]2}$	$\frac{3}{\sqrt{48}}$	$-\frac{1}{\sqrt{48}}$	$-\frac{2}{\sqrt{48}}$	$-\frac{3}{\sqrt{48}}$	$\frac{1}{\sqrt{48}}$	0	0	$-\frac{1}{\sqrt{48}}$	$\frac{3}{\sqrt{48}}$	$\frac{2}{\sqrt{48}}$	$\frac{1}{\sqrt{48}}$	$-\frac{3}{\sqrt{48}}$
$\chi^{c3}_{[211]2}$	0	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$	0	0	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$	0
φ^k_3	$rgbb$	$rbgb$	$rbbg$	$grbb$	$gbrb$	$gbbr$	$brgb$	$brbg$	$bgrb$	$bgbr$	$bbrg$	$bbgr$
$\chi^{c1}_{[211]3}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\chi^{c2}_{[211]3}$	$\frac{2}{\sqrt{48}}$	$\frac{1}{\sqrt{48}}$	$-\frac{3}{\sqrt{48}}$	$-\frac{2}{\sqrt{48}}$	$-\frac{1}{\sqrt{48}}$	$\frac{3}{\sqrt{48}}$	$-\frac{1}{\sqrt{48}}$	$\frac{3}{\sqrt{48}}$	$\frac{1}{\sqrt{48}}$	$-\frac{3}{\sqrt{48}}$	0	0
$\chi^{c3}_{[211]3}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$	0	0	0

The color wave functions of pentaquarks are obtained by using the CG coefficients of the SU_3 group.

$$\chi^{ci}_{[222]}(q^4\bar{q}) = \frac{1}{\sqrt{3}}\left(\chi^{ci}_{[211]1}\bar{r} + \chi^{ci}_{[211]2}\bar{g} + \chi^{ci}_{[211]3}\bar{b}\right), \quad i = 1, 2, 3. \tag{144}$$

6.2. Spin

For a pentaquark system composed of five quarks, its total spin is $\frac{5}{2}$, $\frac{3}{2}$, $\frac{1}{2}$.

$$\begin{aligned} [1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] &= ([4] \oplus 3\,[31] \oplus 2\,[22]) \otimes [1] \\ &= [5] + 4[41] + 5[32] \\ \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} &= \mathbf{6} \oplus 4 * \mathbf{4} \oplus 5 * \mathbf{2}. \end{aligned} \tag{145}$$

For the wave functions, first the four-quark part is constructed by finding the irreps of S_4 in the configuration space $\alpha\alpha\alpha\alpha, \alpha\alpha\alpha\beta, \alpha\alpha\beta\beta$ and then it couples to the wave function of the antiquark to obtain the spin wave functions of the pentaquark system (the permutation symmetry for the four-part $[\sigma_4]$ is attached to each wave function. To save space, the coupling between the four-quark part and antiquark part is given only for $[\sigma_4] = [4]$).

$[\sigma_4] = [4], S_{4q} = 2$:

$$\chi^{\sigma_1}_{2,2}(q^4) = \alpha\alpha\alpha\alpha,$$
$$\chi^{\sigma_1}_{2,1}(q^4) = \frac{1}{\sqrt{4}}(\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha). \tag{146}$$

$2 \otimes \frac{1}{2} \to \frac{5}{2}$:

$$\chi^{\sigma_1}_{\frac{5}{2},\frac{5}{2}}(q^4\bar{q}) = \chi^{\sigma_1}_{2,2}(q^4)\chi_{\frac{1}{2}\frac{1}{2}} = \alpha\alpha\alpha\alpha\alpha. \tag{147}$$

$2 \otimes \frac{1}{2} \to \frac{3}{2}$:

$$\begin{aligned}\chi^{\sigma_1}_{\frac{3}{2},\frac{3}{2}}(q^4\bar{q}) =& \sqrt{\frac{1}{5}}[2\chi^{\sigma_1}_{2,2}(q^4)\chi_{\frac{1}{2}-\frac{1}{2}} - \chi^{\sigma_1}_{2,1}(q^4)\chi_{\frac{1}{2}\frac{1}{2}}] \\ =& \sqrt{\frac{1}{5}}\left[2\alpha\alpha\alpha\alpha\beta - \frac{1}{\sqrt{4}}(\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)\alpha\right] \\ =& \frac{1}{\sqrt{20}}(4\alpha\alpha\alpha\alpha\beta - \alpha\alpha\alpha\beta\alpha - \alpha\alpha\beta\alpha\alpha - \alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha).\end{aligned} \tag{148}$$

$[\sigma_4] = [31]$:

$$\begin{aligned}\chi^{\sigma_2}_{\frac{3}{2},\frac{3}{2}} =& \frac{1}{\sqrt{12}}(3\alpha\alpha\alpha\beta\alpha - \alpha\alpha\beta\alpha\alpha - \alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha), \\ \chi^{\sigma_3}_{\frac{3}{2},\frac{3}{2}} =& \frac{1}{\sqrt{6}}(2\alpha\alpha\beta\alpha\alpha - \alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha), \\ \chi^{\sigma_4}_{\frac{3}{2},\frac{3}{2}} =& \frac{1}{\sqrt{2}}(\alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha), \\ \chi^{\sigma_1}_{\frac{1}{2},\frac{1}{2}} =& \frac{1}{\sqrt{18}}(3\alpha\alpha\alpha\beta\beta - \alpha\alpha\beta\alpha\beta - \alpha\beta\alpha\alpha\beta - \beta\alpha\alpha\alpha\beta - \alpha\alpha\beta\beta\alpha - \alpha\beta\alpha\beta\alpha \\ &+ \alpha\beta\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha + \beta\beta\alpha\alpha\alpha), \\ \chi^{\sigma_2}_{\frac{1}{2},\frac{1}{2}} =& \frac{1}{6}(4\alpha\alpha\beta\alpha\beta - 2\alpha\beta\alpha\alpha\beta - 2\beta\alpha\alpha\alpha\beta - 2\alpha\alpha\beta\beta\alpha + \alpha\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha \\ &+ \beta\alpha\alpha\beta\alpha - \beta\alpha\beta\alpha\alpha + 2\beta\beta\alpha\alpha\alpha), \\ \chi^{\sigma_3}_{\frac{1}{2},\frac{1}{2}} =& \frac{1}{\sqrt{12}}(2\alpha\beta\alpha\alpha\beta - 2\beta\alpha\alpha\alpha\beta - \alpha\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha + \beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha).\end{aligned} \tag{149}$$

$[\sigma_4] = [22]$:

$$\begin{aligned}\chi^{\sigma_4}_{\frac{1}{2},\frac{1}{2}} =& \frac{1}{\sqrt{12}}(2\alpha\alpha\beta\beta\alpha - \alpha\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha - \beta\alpha\beta\alpha\alpha + 2\beta\beta\alpha\alpha\alpha), \\ \chi^{\sigma_5}_{\frac{1}{2},\frac{1}{2}} =& \frac{1}{\sqrt{4}}(\alpha\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha).\end{aligned} \tag{150}$$

6.3. Flavor

In total, there are $5^5 = 3125$ flavor states. According to the number of heavy quarks, these states can be classified into 10 categories, with 243 full light pentaquarks; 162 pentaquarks with four light quarks and one heavy antiquark; 648 states with three light quarks, one heavy quark, and one light antiquark; 648 states with two light quarks, two heavy quarks, and one light antiquark; 432 states with three light quarks, one heavy quark, and one heavy antiquark; 288 states with one light quark, three heavy quarks, and one light antiquark; 432 states with two light quarks, two heavy quarks, and one heavy antiquark;

48 states with four heavy quarks and one light antiquark; 192 states with one light quark, three heavy quarks, and one heavy antiquark; and 32 full heavy pentaquarks.

6.3.1. Pentaquarks with Four Light Quarks and One Heavy Antiquark

In this case, one does not need to consider permutation symmetry with respect to the exchange between the four-quark part and the antiquark. For the four-quark part, one has

$$\begin{gathered}[1]\otimes[1]\otimes[1]\otimes[1]=[4]\oplus 3[31]\oplus 2[22]\oplus 3[211],\\ \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{15}\oplus 3*\mathbf{15}\oplus 2*\bar{\mathbf{6}}\oplus 3*\mathbf{3}.\end{gathered} \tag{151}$$

To obtain the flavor wave functions of the four quarks, one can find the irreps of S_4 in the configuration spaces $uuuu$, $uuud$, $uuus$, …, $ssss$. The results are shown below. To construct the totally antisymmetrized wave functions, the flavor symmetry of the four-quark part is added to the subscript of the wave functions, $\chi^{fi}_{[f_4]IM_I,Y}$.

$I=2, M_I=2, Y=\frac{4}{3}$:

$$\chi^{f_1}_{[4]22,\frac{4}{3}}(q^4)=uuuu. \tag{152}$$

$I=\frac{3}{2}, M_I=\frac{3}{2}, Y=\frac{1}{3}$:

$$\begin{aligned}\chi^{f_1}_{[4]\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4)&=\frac{1}{\sqrt{4}}(uuus+uusu+usuu+suuu),\\ \chi^{f_2}_{[31]\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4)&=\frac{1}{\sqrt{12}}(3uuus-uusu-usuu-suuu),\\ \chi^{f_3}_{[31]\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4)&=\frac{1}{\sqrt{6}}(2uusu-usuu-suuu),\\ \chi^{f_4}_{[31]\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4)&=\frac{1}{\sqrt{2}}(usuu-suuu).\end{aligned} \tag{153}$$

$I=1, M_I=1, Y=\frac{4}{3}$:

$$\begin{aligned}\chi^{f_1}_{[31]11,\frac{4}{3}}(q^4)&=\frac{1}{\sqrt{12}}(3uuud-uudu-uduu-duuu),\\ \chi^{f_2}_{[31]11,\frac{4}{3}}(q^4)&=\frac{1}{\sqrt{6}}(2uudu-uduu-duuu),\\ \chi^{f_3}_{[31]11,\frac{4}{3}}(q^4)&=\frac{1}{\sqrt{2}}(uduu-duuu).\end{aligned} \tag{154}$$

$I=1, M_I=1, Y=-\frac{2}{3}$:

$$\begin{aligned}\chi^{f_1}_{[4]11,-\frac{2}{3}}(q^4)&=\frac{1}{\sqrt{6}}(uuss+usus+ussu+suus+susu+ssuu),\\ \chi^{f_2}_{[31]11,-\frac{2}{3}}(q^4)&=\frac{1}{\sqrt{6}}(uuss+usus-ussu+suus-susu-ssuu),\\ \chi^{f_3}_{[31]11,-\frac{2}{3}}(q^4)&=\frac{1}{\sqrt{12}}(2uuss-usus+ussu-suus+susu-2ssuu),\\ \chi^{f_4}_{[31]11,-\frac{2}{3}}(q^4)&=\frac{1}{2}(usus+ussu-suus-susu),\\ \chi^{f_5}_{[22]11,-\frac{2}{3}}(q^4)&=\frac{1}{\sqrt{12}}(2uuss-usus-ussu-suus-susu+2ssuu),\\ \chi^{f_6}_{[22]11,-\frac{2}{3}}(q^4)&=\frac{1}{2}(usus-ussu-suus+susu).\end{aligned} \tag{155}$$

$I = \frac{1}{2}, M_I = \frac{1}{2}, Y = \frac{1}{3}$:

$$\begin{aligned}
\chi^{f_1}_{[31]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{\sqrt{18}}(2uusd - udsu + 2usud - usdu - dusu - dsuu \\
&\quad + 2suud - sudu - sduu), \\
\chi^{f_2}_{[31]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{12}(6uuds + 2uusd - 3udus - udsu - usud + 5usdu \\
&\quad - 3duus - dusu - 4dsuu - suud + 5sudu - 4sduu), \\
\chi^{f_3}_{[31]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{\sqrt{48}}(3udus + 3udsu + usud + usdu - 3duus - 3dusu \\
&\quad - 2dsuu - suud - sudu + 2sduu), \\
\chi^{f_4}_{[22]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{\sqrt{24}}(2uuds + 2uusd - udus - udsu - usud - usdu \\
&\quad - duus - dusu + 2dsuu - suud - sudu + 2sduu), \\
\chi^{f_5}_{[22]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{\sqrt{8}}(udus - udsu + usud - usdu - duus + dusu \\
&\quad - suud + sudu), \\
\chi^{f_6}_{[211]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{4}(2uuds - 2uusd - udus + udsu + usud - usdu \\
&\quad - duus + dusu + suud - sudu), \\
\chi^{f_7}_{[211]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{\sqrt{48}}(3udus - udsu - 3usud + usdu - 3duus + dusu \\
&\quad + 2dsuu + 3suud - sudu - 2sduu), \\
\chi^{f_8}_{[211]\frac{1}{2}\frac{1}{2},\frac{1}{3}}(q^4) &= \frac{1}{\sqrt{6}}(udsu - usdu - dusu + dsuu + sudu - sduu).
\end{aligned} \tag{156, 157}$$

$I = \frac{1}{2}, M_I = \frac{1}{2}, Y = -\frac{5}{3}$:

$$\begin{aligned}
\chi^{f1}_{[31]\frac{1}{2}\frac{1}{2},-\frac{5}{3}}(q^4) &= \frac{1}{\sqrt{4}}(usss + suss + ssus + sssu), \\
\chi^{f2}_{[31]\frac{1}{2}\frac{1}{2},-\frac{5}{3}}(q^4) &= \frac{1}{\sqrt{12}}(usss + suss + ssus - 3sssu), \\
\chi^{f3}_{[31]\frac{1}{2}\frac{1}{2},-\frac{5}{3}}(q^4) &= \frac{1}{\sqrt{6}}(usss + suss - 2ssus), \\
\chi^{f4}_{[31]\frac{1}{2}\frac{1}{2},-\frac{5}{3}}(q^4) &= \frac{1}{\sqrt{2}}(usss - suss).
\end{aligned} \tag{158}$$

$I = 0, M_I = 0, Y = \frac{4}{3}$:

$$\begin{aligned}
\chi^{f1}_{[22]00,\frac{4}{3}}(q^4) &= \frac{1}{\sqrt{12}}(2uudd - udud - uddu - duud - dudu + 2dduu), \\
\chi^{f2}_{[22]00,\frac{4}{3}}(q^4) &= \frac{1}{2}(udud - uddu - duud + dudu).
\end{aligned} \tag{159}$$

$I = 0, M_I = 0, Y = -\frac{2}{3}$:

$$\chi^{f1}_{[31]00,-\frac{2}{3}}(q^4) = \frac{1}{\sqrt{6}}(ussd - dssu + susd - sdsu + ssud - ssdu), \quad (160)$$

$$\chi^{f2}_{[31]00,-\frac{2}{3}}(q^4) = \frac{1}{\sqrt{48}}(3usds + ussd - 3dsus - dssu + 3suds + suds - 3sdus - sdsu - 2ssud + 2ssdu), \quad (161)$$

$$\chi^{f3}_{[31]00,-\frac{2}{3}}(q^4) = \frac{1}{4}(2udss + usds + ussd - 2duss - dsus - dssu - suds - suds + sdus + sdsu),$$

$$\chi^{f4}_{[211]00,-\frac{2}{3}}(q^4) = \frac{1}{4}(usds - ussd - dsus + dssu + suds - susd - sdus + sdus + 2ssud - 2ssdu),$$

$$\chi^{f5}_{[211]00,-\frac{2}{3}}(q^4) = \frac{1}{\sqrt{48}}(2udss - usds + 3ussd + 2duss + dsus - 3dssu + suds - 3susd - sdus + 3sdsu),$$

$$\chi^{f6}_{[211]00,-\frac{2}{3}}(q^4) = \frac{1}{\sqrt{6}}(udss - usds - duss + dsus + suds - sdus).$$

$I = 0, M_I = 0, Y = -\frac{8}{3}$:

$$\chi^{f1}_{[4]00,-\frac{8}{3}}(q^4) = ssss. \quad (162)$$

Multiplying the antiquark state by the above wave functions, one can obtain the flavor wave functions for pentaquarks as follows: $\chi^{fi}_{IM_I,Y}(q^4Q) = \chi^{fi}_{[f]IM_I,Y}\bar{Q}$. Due to the color symmetry of four-quark part [211], the allowed states of the pentaquarks are those satisfying the following rule:

$$[x_4] \otimes [f_4] = [31] \otimes [\sigma_4], \quad (163)$$

where $[x_4]$ denotes the permutation symmetry of orbital wave functions of the four quarks. For the ground state, $[x_4] = [4]$, the rule simplifies to the following:

$$[f_4] \otimes [\sigma_4] = [31]. \quad (164)$$

The possible spin–flavor combinations $[f_4] \otimes [\sigma_4]$ are as follows:

$$[f_4] \otimes [\sigma_4] = [4] \otimes [31], [31] \otimes [4], [31] \otimes [31], [31] \otimes [22], [22] \otimes [31], [211] \otimes [31], [211] \otimes [22] \quad (165)$$

The total wavefucntions of pentaquark states can be arrived by using the CG coefficients of S_4. As an example, the wave functions $\psi_{JM_J,IM_I,Y}(q^4\bar{Q})$ for the $I = \frac{3}{2}$, $J = \frac{3}{2}$, $Y = \frac{2}{3}$ pentaquark are given below,here,we need to use the wavefunction of antiquark with Equation (15).

$$
\begin{aligned}
&\psi^{1}_{\frac{3}{2}\frac{3}{2},\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4\bar{Q}) \\
&=\frac{1}{\sqrt{3}}\left[\chi^{c_1}_{[211]}\chi^{c_2}_{1}(\bar{q})\chi^{\sigma_1}_{\frac{3}{2}\frac{3}{2}}\chi^{f_4}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\chi^{c_2}_{[211]}\chi^{c_2}_{2}(\bar{q})\chi^{\sigma_1}_{\frac{3}{2}\frac{3}{2}}\chi^{f_3}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}+\chi^{c_3}_{[211]}\chi^{c_2}_{3}(\bar{q})\chi^{\sigma_1}_{\frac{3}{2}\frac{3}{2}}\chi^{f_2}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}\right]\bar{Q}_{00,\frac{1}{3}}, \\
&\psi^{2}_{\frac{3}{2}\frac{3}{2},\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4\bar{Q}) \\
&=\frac{1}{\sqrt{3}}\left\{\chi^{c1}_{[211]}\chi^{c_2}_{1}(\bar{q})\left[-\frac{1}{\sqrt{6}}\chi^{\sigma_2}_{\frac{3}{2}\frac{3}{2}}\chi^{f_4}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{3}}\chi^{\sigma_3}_{\frac{3}{2}\frac{3}{2}}\chi^{f_2}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{6}}\chi^{\sigma_4}_{\frac{3}{2}\frac{3}{2}}\chi^{f_2}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{3}}\chi^{\sigma_4}_{\frac{3}{2}\frac{3}{2}}\chi^{f_3}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}\right]\bar{Q}_{00,\frac{1}{3}}\right. \\
&\quad-\chi^{c2}_{[211]}\chi^{c_2}_{2}(\bar{q})\left[-\frac{1}{\sqrt{6}}\chi^{\sigma_2}_{\frac{3}{2}\frac{3}{2}}\chi^{f_3}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{6}}\chi^{\sigma_3}_{\frac{3}{2}\frac{3}{2}}\chi^{f_2}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}+\frac{1}{\sqrt{3}}\chi^{\sigma_3}_{\frac{3}{2}\frac{3}{2}}\chi^{f_3}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{3}}\chi^{\sigma_4}_{\frac{3}{2}\frac{3}{2}}\chi^{f_4}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}\right]\bar{Q}_{00,\frac{1}{3}} \\
&\quad\left.+\chi^{c_3}_{[211]}\chi^{c_2}_{3}(\bar{q})\left[\frac{2}{\sqrt{3}}\chi^{\sigma_2}_{\frac{3}{2}\frac{3}{2}}\chi^{f_2}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{6}}\chi^{\sigma_3}_{\frac{3}{2}\frac{3}{2}}\chi^{f_3}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\frac{1}{\sqrt{6}}\chi^{\sigma_4}_{\frac{3}{2}\frac{3}{2}}\chi^{f_4}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}\right]\bar{Q}_{00,\frac{1}{3}}\bar{Q}_{00,\frac{1}{3}}\right\}, \\
&\psi^{3}_{\frac{3}{2}\frac{3}{2},\frac{3}{2}\frac{3}{2},\frac{1}{3}}(q^4\bar{Q}) \\
&=\frac{1}{\sqrt{3}}\left[\chi^{c_1}_{[211]}\chi^{c_2}_{1}(\bar{q})\chi^{\sigma_4}_{\frac{3}{2}\frac{3}{2}}\chi^{f_1}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}-\chi^{c_2}_{[211]}\chi^{c_2}_{2}(\bar{q})\chi^{\sigma_3}_{\frac{3}{2}\frac{3}{2}}\chi^{f_1}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}+\chi^{c_3}_{[211]}\chi^{c_2}_{3}(\bar{q})\chi^{\sigma_2}_{\frac{3}{2}\frac{3}{2}}\chi^{f_1}_{\frac{3}{2}\frac{3}{2},\frac{1}{3}}\right]\bar{Q}_{00,\frac{1}{3}}.
\end{aligned}
\tag{166}
$$

The flavor SU_3 multiplets of the four-quark part are shown in Figures 19–21. Multiplying by heavy antiquark state $\bar{Q}$, which is not attached in the figures, gives the pentaquark multiplets. If the heavy antiquark $\bar{Q}$ is added to the figures,the charge $\mathcal{Q}$ and hypercharge Y need to change.

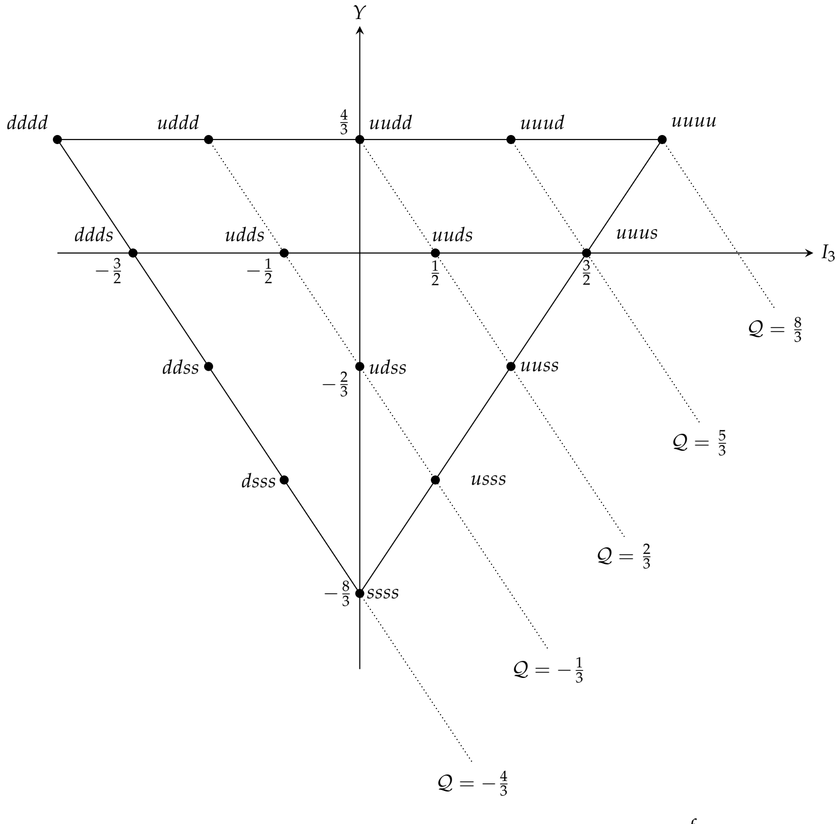


Figure 19. The 15-plet states of pentaquark for partition [4] of SU_3^f.

6.3.2. Full Light Pentaquark

Experimental investigations of light pentaquark candidates have been pursued for multiple decades, but no unambiguous detection with statistical significance has been achieved to date. The LEPS collaboration announced a state $\Theta(M_\Theta = 1.54 \pm 0.01$ GeV, $\Gamma_\Theta < 25$ MeV) in 2003. This state may be interpreted either as a meson–baryon molecular resonance or as an exotic pentaquark configuration $(uudd\bar{s})$ decaying into K^+n [255]. Its properties align with the predicted lightest member of the antidecuplet in the chiral soliton model [256]. More details about the search for and discussion of Θ can be seen in [257–259].

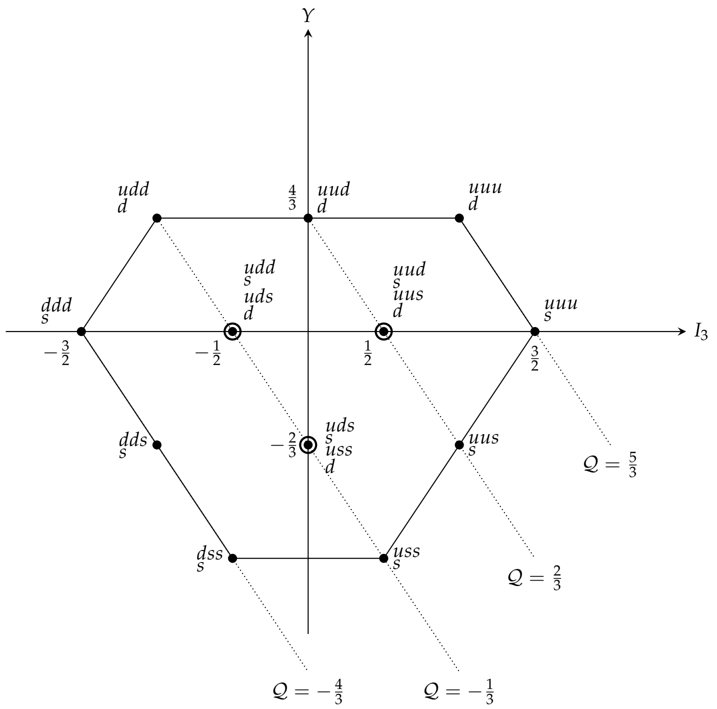


Figure 20. The 15-plet states of pentaquark for partition [31] of SU_3^f.

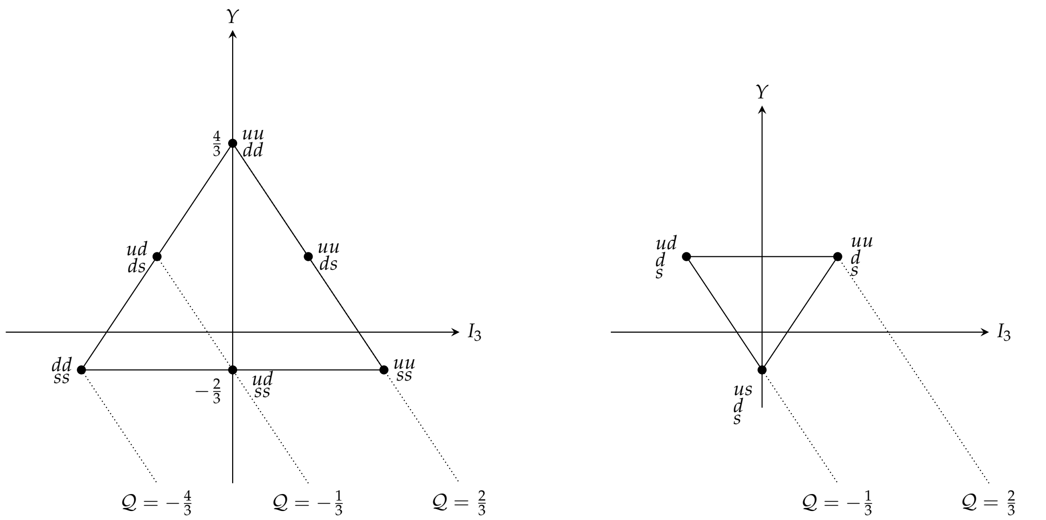


Figure 21. The 6-plet states of pentaquark for partition [22] and 3-plet states for partition [211] of SU_3^f.

Multiplying the flavor symmetries, Equation (151), of the four-quark part by an antiquark, one can obtain the flavor symmetries of the full light pentaquarks.

$$\begin{aligned}[1]\otimes[1]\otimes[1]\otimes[1]\otimes[11] &= [51]\oplus 3[42]\oplus 4[411]\oplus 2[33]\oplus 8[321]\oplus 3[222], \\ \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\bar{\mathbf{3}} &= \mathbf{35}\oplus 3*\mathbf{27}\oplus 4*\mathbf{10}\oplus 2*\overline{\mathbf{10}}\oplus 8*\mathbf{8}\oplus 3*\mathbf{1}.\end{aligned} \tag{167}$$

The flavor multiplets of full light pentaquark are shown in Figures 22–28.

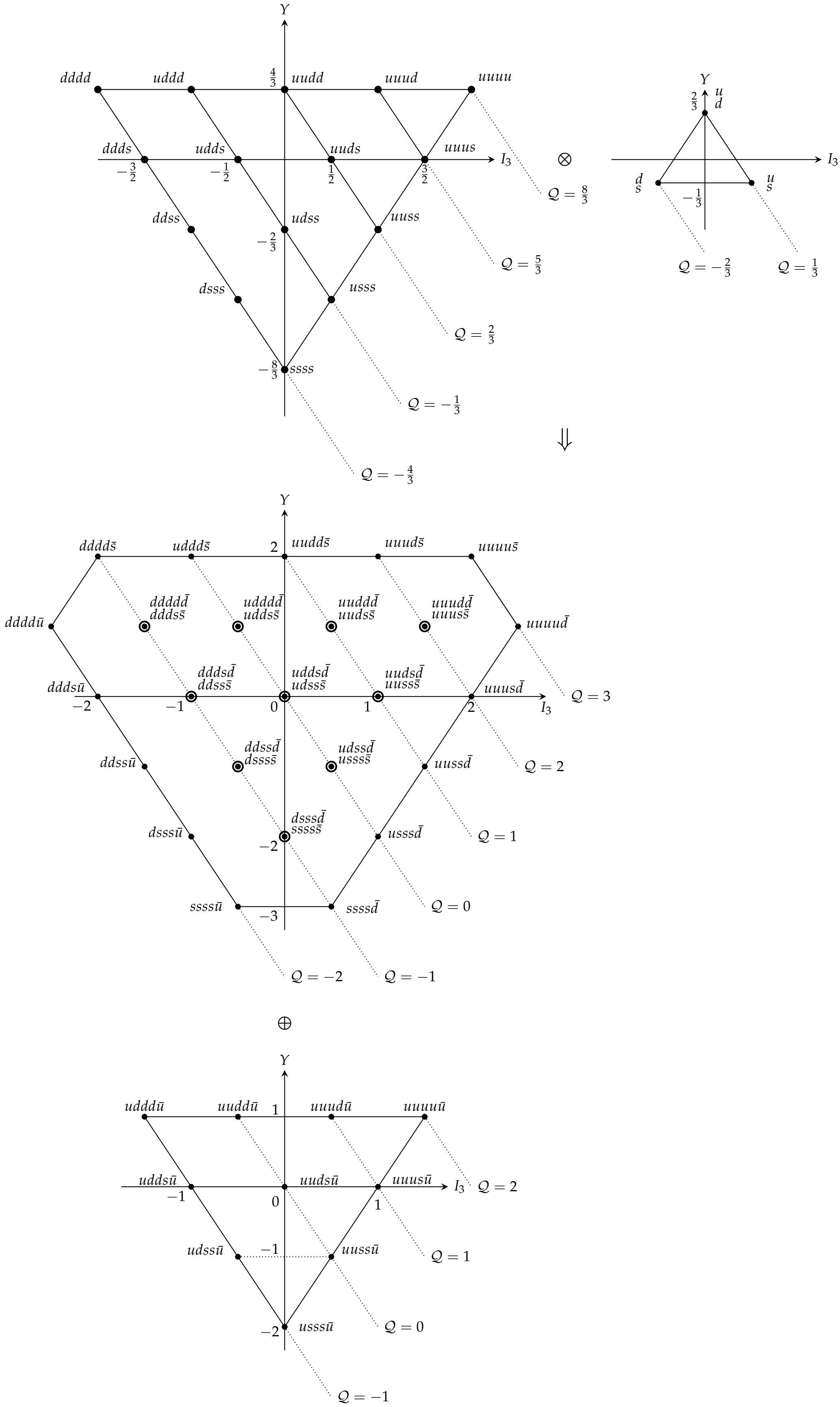


Figure 22. The 10-plet states of pentaquarks for partition [411] and 35-plet states of pentaquarks for partition [51]. They come from $[4]_{\mathbf{15}} \otimes [11]_{\mathbf{3}} = [51]_{\mathbf{35}} \oplus [411]_{\mathbf{10}}$.

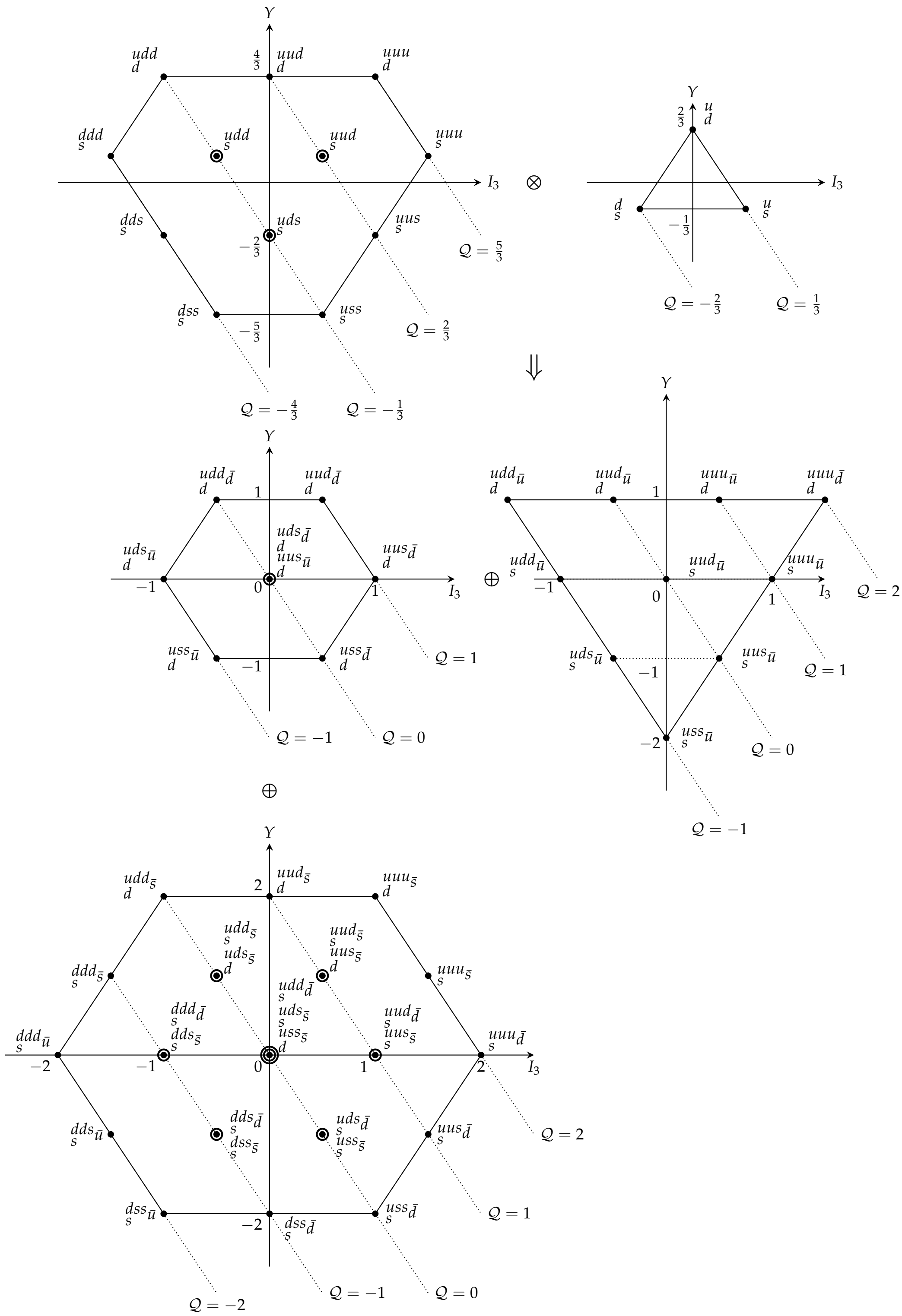


Figure 23. The 8-plet states of pentaquarks for partition [321], 10-plet states of pentaquarks for partition [411], and 27-plet states of pentaquarks for partition [42]. They come from $[31]_{\mathbf{15}} \otimes [11]_{\mathbf{3}} = [321]_{\mathbf{8}} \oplus [411]_{\mathbf{10}} \oplus [42]_{\mathbf{27}}$.

The flavor wave functions of pentaquarks are obtained by coupling the four-quark states to antiquark state using CG coefficients of SU_3.

$$\chi^{fi}_{IM_I,Y}(q^4\bar{q}) \quad = \quad \sum_{M_{I_{\bar{1}}}} C^{IM_I}_{I_4M_I-M_{I_{\bar{1}}},I_{\bar{1}}M_{I_{\bar{1}}}} \chi^{fi}_{I_4M_I-M_{I_{\bar{1}}},Y_4}(q^4)\chi_{I_{\bar{1}}M_{I_{\bar{1}}},Y_{\bar{1}}}(\bar{q}), \tag{168}$$

where $\chi_{\frac{1}{2}\frac{1}{2},-\frac{1}{3}} = \bar{d}$, $\chi_{\frac{1}{2}-\frac{1}{2},-\frac{1}{3}} = -\bar{u}$, $\chi_{00,\frac{2}{3}} = \bar{s}$. For example, the flavor wave functions for $I = \frac{3}{2}, M_I = \frac{3}{2}, Y = 1$ are

$$\begin{aligned}
\chi^{f_1}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \sqrt{\frac{4}{5}}uuuu(-\bar{u}) + \sqrt{\frac{1}{5}}\sqrt{\frac{1}{4}}(uuud + uudu + uduu + duuu)\bar{d}, \\
\chi^{f_2}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{4}}(uuus + uusu + usuu + suuu)\bar{s}, \\
\chi^{f_3}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{12}}(3uuus - uusu - usuu - suuu)\bar{s}, \\
\chi^{f_4}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{6}}(2uusu - usuu - suuu)\bar{s}, \\
\chi^{f_5}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{2}}(usuu - suuu)\bar{s}, \\
\chi^{f_6}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{12}}(3uuud - uudu - uduu - duuu)\bar{d}, \\
\chi^{f_7}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{6}}(2uudu - uduu - duuu)\bar{d}, \\
\chi^{f_8}_{\frac{3}{2}\frac{3}{2},1}(q^4\bar{q}) &= \frac{1}{\sqrt{2}}(uduu - duuu)\bar{d}.
\end{aligned} \tag{169}$$

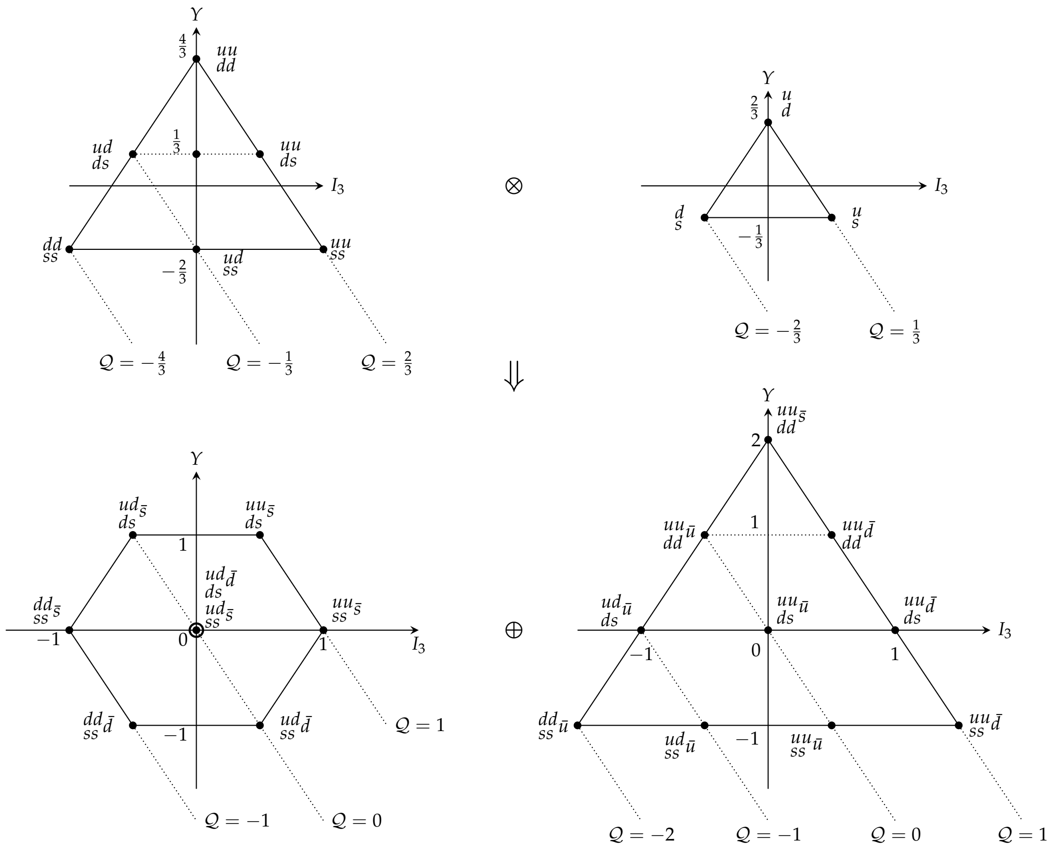


Figure 24. The 8-plet states of pentaquarks for partition [321] and 10-plet states of pentaquarks for partition [321]. They come from $[22]_{\mathbf{6}} \otimes [11]_{\mathbf{3}} = [321]_{\mathbf{8}} \oplus [33]_{\mathbf{10}}$.

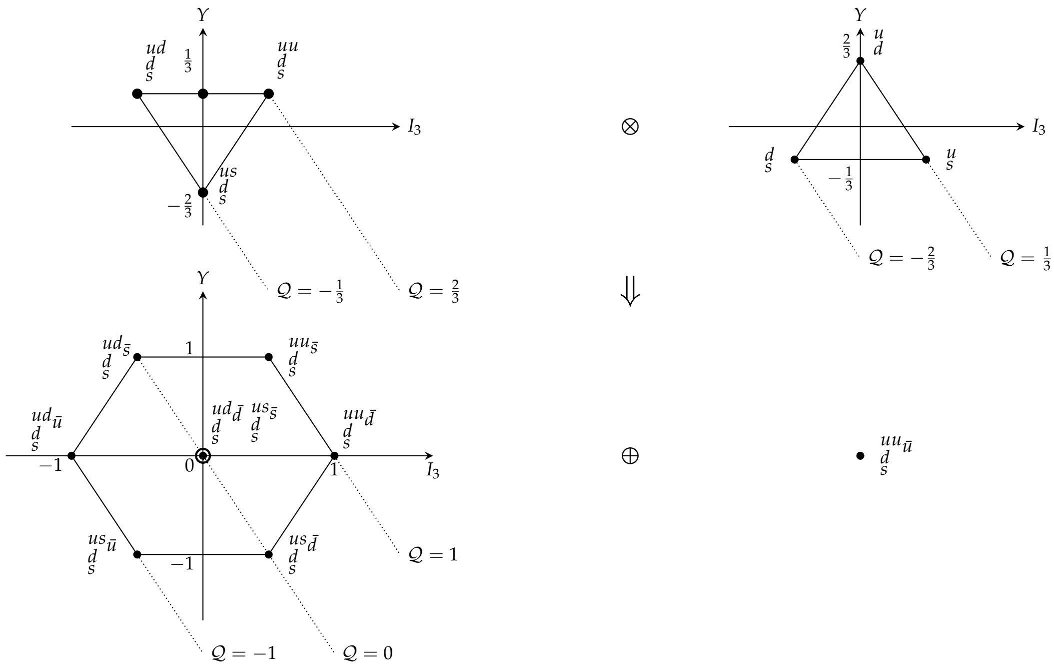


Figure 25. The 8-plet states of pentaquarks for partition [321] and singlet state of pentaquark for partition [222]. They come from $[211]_{\mathbf{3}} \otimes [11]_{\mathbf{3}} = [321]_{\mathbf{8}} \oplus [222]_{\mathbf{1}}$.

6.3.3. $qqqQ\bar{q}$

For pentaquarks with one heavy quark, the flavor symmetry is as follows:

$$\begin{aligned} [1] \otimes [1] \otimes [1] \otimes [11] &= ([111] \oplus 2[21] \oplus [3]) \otimes [11] \\ &= [41] \oplus 2[32] \oplus 3[311] \oplus 3[221], \\ \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}} &= \mathbf{24} \oplus 2 * \mathbf{15} \oplus 3 * \mathbf{6} \oplus 3 * \bar{\mathbf{3}}. \end{aligned} \tag{170}$$

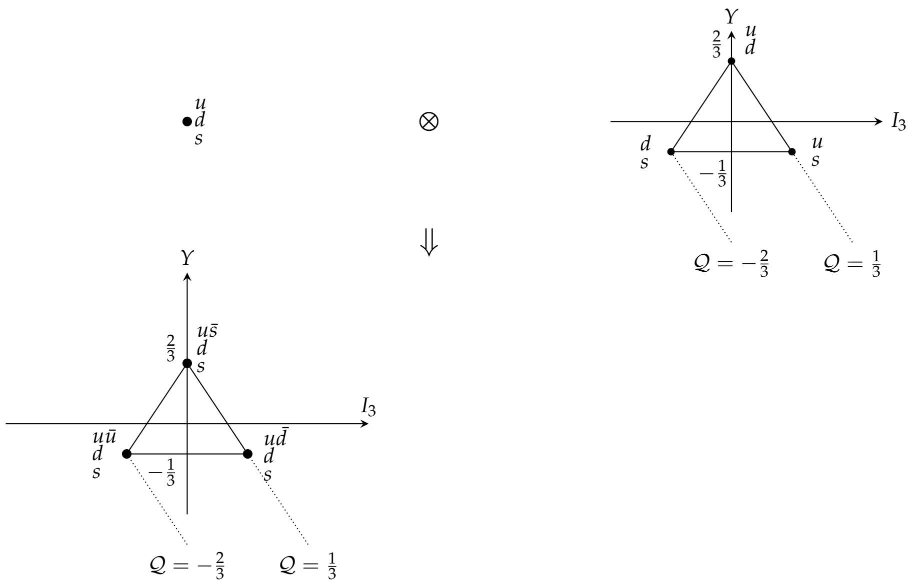


Figure 26. The 3-plet states of pentaquarks for partition [221]. They come from $[111]_{\mathbf{1}} \otimes [11]_{\mathbf{3}} = [221]_{\mathbf{3}}$.

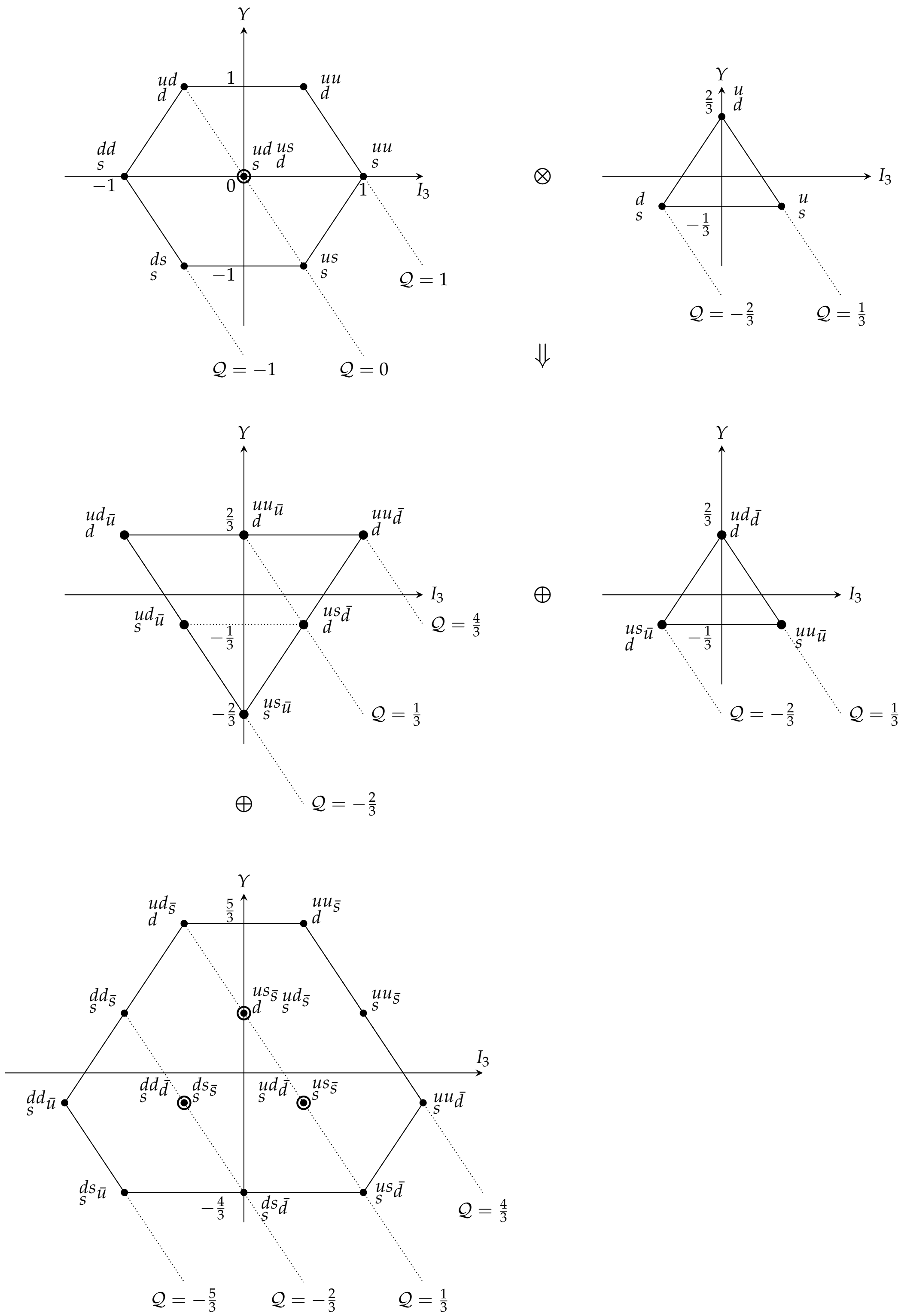


Figure 27. The 6-plet states of pentaquarks for partition [311], 3-plet states of pentaquarks for partition [221], and 15-plet states of pentaquarks for partition [32]. They come from $[21]_{\mathbf{8}} \otimes [11]_{\mathbf{3}} = [311]_{\mathbf{6}} \oplus [221]_{\mathbf{3}} \oplus [32]_{\mathbf{15}}$.

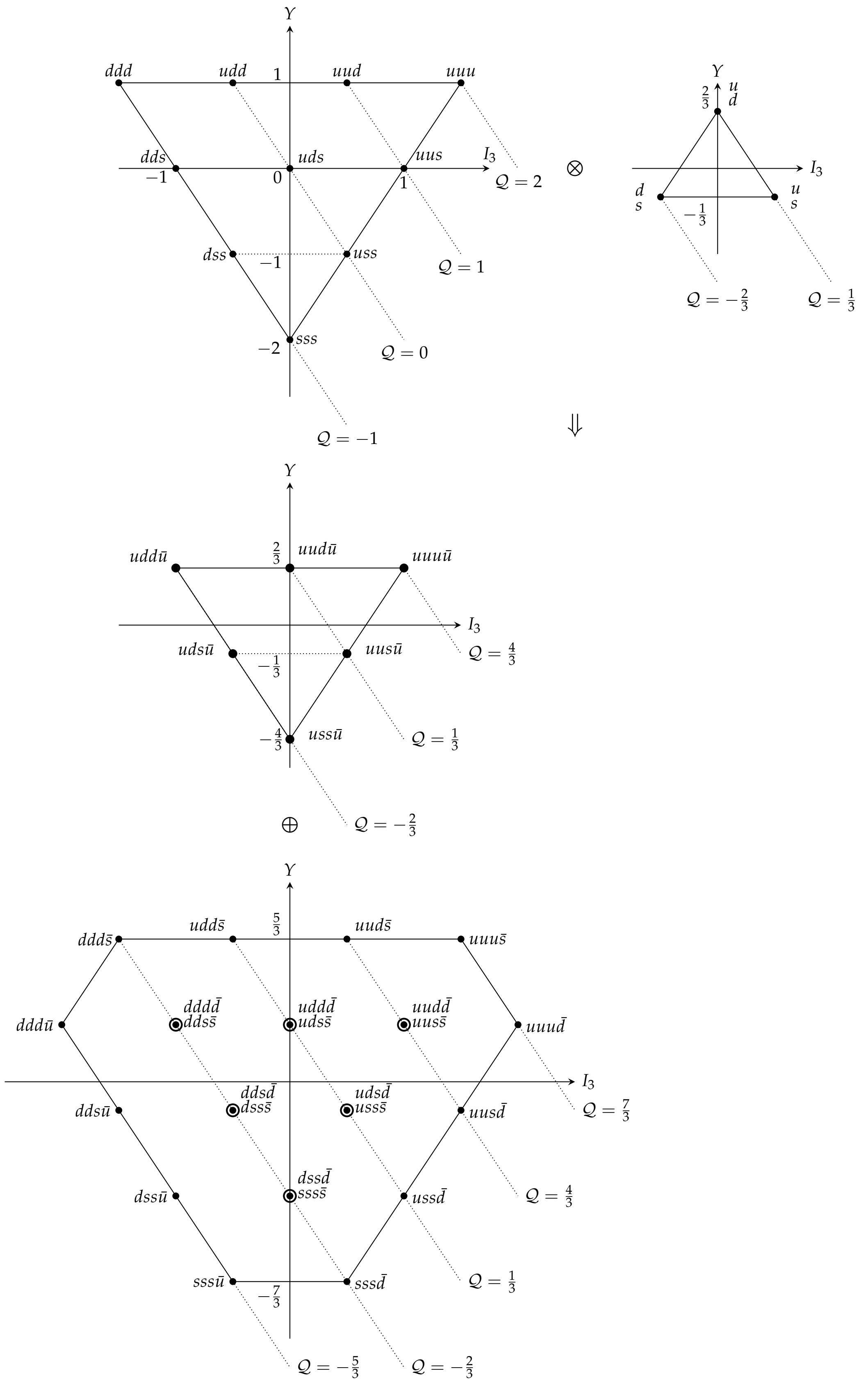


Figure 28. The 6-plet states of pentaquarks for partition [311] and 24-plet states of pentaquarks for partition [41]. They come from $[3]_{\mathbf{10}} \otimes [11]_{\mathbf{3}} = [311]_{\mathbf{6}} \oplus [41]_{\mathbf{24}}$.

6.3.4. $qqqQ\bar{Q}$

In this case, the flavor symmetry is the same as that of the qqq baryon, and the wave functions can be obtained by multiplying $Q\bar{Q}$ by the baryon wave functions. The weight diagram is also the same with that of the qqq baryon, which can be seen in [73,260], so we do not repeat it here.

6.3.5. $qqQQ\bar{q}$

The flavor symmetries in this case are as follows:

$$\begin{gathered}[1]\otimes[1]\otimes[11]=[31]\oplus[22]\oplus 2[211],\\ \mathbf{3}\otimes\mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{15}\oplus\mathbf{6}\oplus 2*\bar{\mathbf{3}}.\end{gathered}\tag{171}$$

The flavor wave functions of the pentaquarks with two heavy quarks are obtained by multiplying the QQ state by the diquark states and coupling, resulting in the four-quark states of the antiquark state using the CG coefficients of SU_3.

$$\chi^{fi}_{IM_I,Y}(qqQQ\bar{q}) = \sum_{M_{I_{\bar{1}}}} C^{IM_I}_{I_2M_I-M_{I_{\bar{1}}},I_{\bar{1}}M_{I_{\bar{1}}}}\chi^{fi}_{I_2M_I-M_{I_{\bar{1}}},Y_2}(qqQQ)\chi_{I_{\bar{1}}M_{I_{\bar{1}}},Y_{\bar{1}}}(\bar{q}).\tag{172}$$

For example, the flavor wave functions for $I=\frac{1}{2}, M_I=\frac{1}{2}, Y=\frac{1}{3}$ are

$$\begin{aligned}
\chi^{f_1}_{\frac{1}{2}\frac{1}{2},\frac{1}{3}}(qqQQ\bar{q}) &= \sqrt{\frac{2}{3}}uuQQ(-\bar{u})-\sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}(udQQ+duQQ)\bar{d},\\
\chi^{f_2}_{\frac{1}{2}\frac{1}{2},\frac{1}{3}}(qqQQ\bar{q}) &= \frac{1}{\sqrt{2}}(udQQ-duQQ)\bar{d},\\
\chi^{f_3}_{\frac{1}{2}\frac{1}{2},\frac{1}{3}}(qqQQ\bar{q}) &= \frac{1}{\sqrt{2}}(usQQ+suQQ)\bar{s},\\
\chi^{f_4}_{\frac{1}{2}\frac{1}{2},\frac{1}{3}}(qqQQ\bar{q}) &= \frac{1}{\sqrt{2}}(usQQ-suQQ)\bar{s},\\
\chi^{f_5}_{\frac{1}{2}\frac{1}{2},\frac{1}{3}}(qqQQ\bar{q}) &= ssQQ\bar{d}.
\end{aligned}\tag{173}$$

6.3.6. Triply Heavy Pentaquark

For $qqQQ\overline{Q}$ ($qQQQ\overline{q}$) systems, the flavor wave function can be written as the product of two parts; the first part is for diquarks, Equation (106) or mesons, Equations (22)–(26)), and the second part is for the heavy quark, the product of the wave functions of three heavy quarks. The weight diagrams of pentaquark $qQQQ\overline{q}$ are the same as that of the $q\bar{q}$ nonet meson [214]. To save space, the expressions are not given here explicitly.

6.3.7. Four Heavy Pentaquarks and Fully Heavy Pentaquark

The wave functions are just the products of those of each quark. It is not necessary to list them here.

The phenomenological model based on the Gursey–Radicati mass formula for hadrons [71] has also been extended to predict the masses of the pentaquark states with or without heavy quarks [72,261–270], the results are presented in Table 8.

Table 8. The parameters and masses of group-state Pentaquarks.

Parameter	M_0	X_1		X_2		X_3
	1510.0 ± 0.33	-56.081 ± 0.05		34.4 ± 0.13		610.12 ± 0.67
	X_s	X_c		X_b		
	248.95 ± 0.27	1355.8 ± 0.22		3200.0 ± 0.14		
states	I (isospin)	J (spin)	$[f]$ (flavor)	Theo. (MeV)	Exp. (MeV)	Ref. (MeV)
$qqqs\bar{c}$	$\frac{1}{2}$	$\frac{1}{2}$	[31]	3000.00 ± 1.97	-	
$qqqs\bar{q}$	0	$\frac{1}{2}$	[22]	1410.90 ± 1.04	$\Lambda(1405)$	1401 [271]
$qqqs\bar{q}$	0	$\frac{3}{2}$	[22]	1514.10 ± 1.43	$\Lambda(1520)$	1518 [271]
$qqsc\bar{c}$	0	$\frac{3}{2}$	[02]	4346.90 ± 1.27	$P_{cs}(4338)$	4303 [272]
$qqsc\bar{c}$	0	$\frac{3}{2}$	[02]	4450.10 ± 1.66	$P_{cs}(4459)$	4419 [272]
$qqqc\bar{c}$	$\frac{1}{2}$	$\frac{1}{2}$	[21]	4368.60 ± 1.67	$P_c(4312)$	4308 [273]
$qqqq\bar{c}$	$\frac{1}{2}$	$\frac{3}{2}$	[21]	4471.80 ± 1.67	$P_c(4457)$	4450 [273]
$qsss\bar{q}$	0	$\frac{3}{2}$	[22]	2012.00 ± 2.06	$\Omega(2012)$	2035 [274]

When flavor–spin symmetry $SU_6^{f\sigma}$ is used, one has the following CG series of the $SU_6^{f\sigma}$ group or the series of the outer product of permutation group S_9.

$$\begin{aligned}
&[1] \otimes [1] \otimes [1] \otimes [1] \otimes [11111] \\
=&[51111] \oplus 4\,[411111] \oplus 3\,[42111] \oplus 8\,[321111] \\
&\oplus 2\,[33111] \oplus 3\,[32211] \oplus 4\,[222111] \oplus [22221],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \bar{\mathbf{6}} \\
=&\mathbf{700} \oplus 4 * \mathbf{56} \oplus 3 * \mathbf{1134} \oplus 8 * \ \mathbf{70} \oplus 2 * \ \mathbf{560} \oplus 3 * \mathbf{540} \oplus 4 * \mathbf{20} \oplus \mathbf{70}.
\end{aligned} \tag{174}$$

The flavor–spin contents in the given symmetry of $SU_6^{f\sigma}$ are listed below.

$$\begin{aligned}
\mathbf{700} =&(\mathbf{8},\mathbf{2}) \oplus (\mathbf{8},\mathbf{4}) \oplus (\mathbf{10},\mathbf{2}) \oplus (\mathbf{10},\mathbf{4}) \oplus (\mathbf{10},\mathbf{6}) \oplus (\overline{\mathbf{10}},\mathbf{2}) \oplus (\mathbf{27},\mathbf{2}) \oplus (\mathbf{27},\mathbf{4}) \oplus \\
&(\mathbf{35},\mathbf{4}) \oplus (\mathbf{35},\mathbf{6}), \\
\mathbf{1134} =&(\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{4}) \oplus 3(\mathbf{8},\mathbf{2}) \oplus 3(\mathbf{8},\mathbf{4}) \oplus (\mathbf{8},\mathbf{6}) \oplus 2(\mathbf{10},\mathbf{2}) \oplus 2(\mathbf{10},\mathbf{4}) \oplus (\mathbf{10},\mathbf{6}) \oplus \\
&(\overline{\mathbf{10}},\mathbf{2}) \oplus (\overline{\mathbf{10}},\mathbf{4}) \oplus 2(\mathbf{27},\mathbf{2}) \oplus 2(\mathbf{27},\mathbf{4}) \oplus (\mathbf{27},\mathbf{6}) \oplus (\mathbf{35},\mathbf{2}) \oplus (\mathbf{35},\mathbf{4}), \\
\mathbf{560} =&(\mathbf{1},\mathbf{4}) \oplus 2(\mathbf{8},\mathbf{2}) \oplus 2(\mathbf{8},\mathbf{4}) \oplus (\mathbf{8},\mathbf{6}) \oplus (\mathbf{10},\mathbf{2}) \oplus (\mathbf{10},\mathbf{4}) \oplus (\overline{\mathbf{10}},\mathbf{2}) \oplus (\overline{\mathbf{10}},\mathbf{4}) \oplus \\
&(\overline{\mathbf{10}},\mathbf{6}) \oplus (\mathbf{27},\mathbf{2}) \oplus (\mathbf{27},\mathbf{4}) \oplus (\mathbf{35},\mathbf{2}), \\
\mathbf{540} =&(\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{4}) \oplus (\mathbf{1},\mathbf{6}) \oplus 3(\mathbf{8},\mathbf{2}) \oplus 3(\mathbf{8},\mathbf{4}) \oplus (\mathbf{8},\mathbf{6}) \oplus (\mathbf{10},\mathbf{2}) \oplus (\mathbf{10},\mathbf{4}) \oplus \\
&(\overline{\mathbf{10}},\mathbf{2}) \oplus (\overline{\mathbf{10}},\mathbf{4}) \oplus 2(\mathbf{27},\mathbf{2}) \oplus (\mathbf{27},\mathbf{4}) \\
\overline{\mathbf{70}} =&(\mathbf{1},\mathbf{2}) \oplus (\mathbf{8},\mathbf{2}) \oplus (\mathbf{8},\mathbf{4}) \oplus (\overline{\mathbf{10}},\mathbf{2}).
\end{aligned} \tag{175}$$

7. Dibaryons

As mentioned in Section 4, based on the 56 ground baryons, several possible dibaryon states were predicted by Dyson and Xuong [50]. According to SU_6 theory, dibaryons can be classified as follows:

$$\begin{aligned}&[3]\otimes[3]=[6]\oplus[51]\oplus[42]\oplus[33],\\&\mathbf{56}\otimes\mathbf{56}=\mathbf{462}\oplus\mathbf{1050}\oplus\mathbf{1134}\oplus\mathbf{490}.\end{aligned}\tag{176}$$

The constituents of dibaryons are fermions; only the two anti-symmetric multiplets exist in nature when the symmetry in the coordinate space is limited to be [6] and totally symmetric.

$$[\mu]=[51],\ [33]\quad(\mathbf{1050},\mathbf{490}).\tag{177}$$

They can be decomposed following $SU_3^f\otimes SU_2^\sigma$ multiplets:

$$\begin{aligned}&\mathbf{1050}=[\mathbf{28},\mathbf{5}]\oplus[\mathbf{35},\mathbf{3}\oplus\mathbf{5}\oplus\mathbf{7}]\oplus[\mathbf{27},\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}]\oplus[\overline{\mathbf{10}},\mathbf{3}]\oplus[\mathbf{10},\mathbf{3}\oplus\mathbf{5}]\oplus[\mathbf{8},\mathbf{1}\oplus\mathbf{3}],\\&\mathbf{490}=[\mathbf{28},\mathbf{1}]\oplus[\mathbf{35},\mathbf{3}]\oplus[\mathbf{27},\mathbf{1}\oplus\mathbf{5}]\oplus[\overline{\mathbf{10}},\mathbf{3}\oplus\mathbf{7}]\oplus[\mathbf{10},\mathbf{3}]\oplus[\mathbf{8},\mathbf{3}\oplus\mathbf{5}]\oplus[\mathbf{1},\mathbf{1}].\end{aligned}\tag{178}$$

For the case of hypercharge $Y=2$, there is no s quark; the $SU_6^{f\sigma}$ symmetry reduces to $SU_4^{\tau\sigma}$:

$$\begin{aligned}&\mathbf{1050}\to\mathbf{140}=[\mathbf{7},\mathbf{5}]\oplus[\mathbf{5},\mathbf{3}\oplus\mathbf{5}\oplus\mathbf{7}]\oplus[\mathbf{3},\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5}]\oplus[\mathbf{1},\mathbf{3}],\\&\mathbf{490}\to\mathbf{50}=[\mathbf{7},\mathbf{1}]\oplus[\mathbf{5},\mathbf{3}]\oplus[\mathbf{3},\mathbf{1}\oplus\mathbf{5}]\oplus[\mathbf{1},\mathbf{3}\oplus\mathbf{7}].\end{aligned}\tag{179}$$

The reported dibaryon $d^*(2380)$ is in the multiplet $[\mathbf{1},\mathbf{7}]$ [50].

7.1. Color

Dibaryons consist of six quarks, and the color symmetry is as follows.

$$\begin{aligned}&[1]\otimes[1]\otimes[1]\otimes[1]\otimes[1]\otimes[1]\\=&([2]\oplus[11])\otimes([2]\oplus[11])\otimes([2]\oplus[11])\\=&[6]\oplus5\ [51]\oplus9\ [42]\oplus10\ [411]\oplus5\ [33]\oplus16\ [321]\oplus5\ [222],\\&\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\\=&\mathbf{28}\oplus5*\mathbf{35}\oplus9*\mathbf{27}\oplus10*\mathbf{10}\oplus5*\mathbf{10}\oplus16*\mathbf{8}\oplus5*\mathbf{1}.\end{aligned}\tag{180}$$

Due to the requirement of color-singlet states, the color symmetry of a dibaryon system can only be taken as **1**; the corresponding partition is [222]. We need to find the irreducible bases of [222] in the *rrggbb* configuration space.

$$\begin{aligned}&\chi_{0,0}^{\sigma_1}=\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array},\quad\chi_{0,0}^{\sigma_2}=\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array},\quad\chi_{0,0}^{\sigma_3}=\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline\end{array},\\&\chi_{0,0}^{\sigma_4}=\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array},\quad\chi_{0,0}^{\sigma_5}=\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array},\end{aligned}\tag{181}$$

$$\begin{aligned}\chi^{c_1}_{[222]}(q^6) =& \frac{1}{\sqrt{36}}[rgbrgb - rgbrbg + rgbgbr - rgbgrb + rgbbrg - rgbbgr \\ & - rbgrgb + rbgrbg - rbggbr + rbggrb - rbgbrg + rbgbgr \\ & + gbrrgb - gbrrbg + gbrgbr - gbrgrb + gbrbrg - gbrbgr \\ & - grbrgb + grbrbg - grbgbr + grbgrb - grbbrg + grbbgr \\ & + brgrgb - brgrbg + brggbr - brggrb + brgbrg - brgbgr \\ & - bgrrgb + bgrrbg - bgrgbr + bgrgrb - bgrbrg + bgrbgr],\end{aligned} \tag{182}$$

$$\begin{aligned}\chi^{c_2}_{[222]}(q^6) =& \frac{1}{\sqrt{96}}[2rgrgbb - rgrbgb - rgrbbg - 2rggrbb + rggbrb + rggbbr \\ & + rgbrgb + rgbrbg - rgbgrb - rgbgbr - rbrggb - rbrgbg \\ & + 2rbrbgg + rbgrgb + rbgrbg - rbgbrg - rbgbgr - 2rbbrgg \\ & + rbbgrg + rbbggr - 2grrgbb + grrbgb + grrbbg + 2grgrbb \\ & - grgbrb - grgbbr - grbrgb - grbrbg + grbgrb + grbgbr \\ & + gbrgrb + gbrgbr - gbrbrg - gbrbgr - gbgrrb - gbgrbr \\ & + 2gbgbrr + gbbrrg + gbbrgr - 2gbbgrr + brrggb + brrgbg \\ & - 2brrbgg - brgrgb - brgrbg + brgbrg + brgbrg + 2brbrgg \\ & - brbgrg - brbggr - bgrgrb - bgrgbr + bgrbrg + bgrbgr \\ & + bggrrb + bggrbr - 2bggbrr - bgbrrg - bgbrgr + 2bgbgrr],\end{aligned} \tag{183}$$

$$\begin{aligned}\chi^{c_3}_{[222]}(q^6) =& \frac{1}{\sqrt{288}}[3rgrbgb - 3rgrbbg - 3rggbrb + 3rggbbr - rgbrgb + rgbrbg \\ & + rgbgrb - rgbgbr + 2rgbbrg - 2rgbbgr - 3rbrggb + 3rbrgbg \\ & + rbgrgb - rbgrbg + 2rbggrb - 2rbggbr + rbgbrg - rbgbgr \\ & - 3rbbgrg + 3rbbggr - 3grrbgb + 3grrbbg + 3grgbrb - 3grgbbr \\ & + grbrgb - grbrbg - grbgrb + grbgbr - 2grbbrg + 2grbbgr \\ & + 2gbrrgb - 2gbrrbg + gbrgrb - gbrgbr - gbrbrg + gbrbgr \\ & - 3gbgrrb + 3gbgrbr + 3gbbrrg - 3gbbrgr + 3brrggb - 3brrgbg \\ & - brgrgb + brgrbg - 2brggrb + 2brggbr - brgbrg + brgbgr \\ & + 3brbgrg - 3brbggr - 2bgrrgb + 2bgrrbg - bgrgrb + bgrgbr \\ & + bgrbrg - bgrbgr + 3bggrrb - 3bggrbr - 3bgbrrg + 3bgbrgr],\end{aligned}$$

$$\begin{aligned}\chi^{c_4}_{[222]}(q^6) =& \frac{1}{\sqrt{96}}[2rrgbgb - 2rrgbbg - 2rrbggb + 2rrbgbg - rgrbgb + rgrbbg \\ & - rggbrb + rggbbr + rgbrgb - rgbrbg + rgbgrb - rgbgbr \\ & + rbrggb - rbrgbg - rbgrgb + rbgrbg + rbgbrg - rbgbgr \\ & - rbbgrg + rbbggr - grrbgb + grrbbg - grgbrb + grgbbr \\ & + grbrgb - grbrbg + grbgrb - grbgbr + 2ggrbrb - 2ggrbbr \\ & - 2ggbrrb + 2ggbrbr - gbrgrb + gbrgbr - gbrbrg + gbrbgr \\ & + gbgrrb - gbgrbr + gbbrrg - gbbrgr + brrggb - brrgbg \\ & - brgrgb + brgrbg + brgbrg - brgbgr - brbgrg + brbggr \\ & - bgrgrb + bgrgbr - bgrbrg + bgrbgr + bggrrb - bggrbr \\ & + bgbrrg - bgbrgr + 2bbrgrg - 2bbrggr - 2bbgrrg + 2bbgrgr],\end{aligned} \tag{184}$$

$$\chi^{c_5}_{[222]}(q^6) = \frac{1}{\sqrt{288}}[4rrggbb - 2rrgbgb - 2rrgbbg - 2rrbggb - 2rrbgbg + 4rrbbgg$$
$$- 2rgrgbb + rgrbgb + rgrbbg - 2rggrbb + rggbrb + rggbbr$$
$$+ rgbrgb + rgbrbg + rgbgrb + rgbgbr - 2rgbbrg - 2rgbbgr$$
$$+ rbrggb + rbrgbg - 2rbrbgg + rbgrgb + rbgrbg - 2rbggrb$$
$$- 2rbggbr + rbgbrg + rbgbgr - 2rbbrgg + rbbgrg + rbbggr$$
$$- 2grrgbb + grrbgb + grrbbg - 2grgrbb + grgbrb + grgbbr$$
$$+ grbrgb + grbrbg + grbgrb + grbgbr - 2grbbrg - 2grbbgr$$
$$+ 4ggrrbb - 2ggrbrb - 2ggrbbr - 2ggbrrb - 2ggbrbr + 4ggbbrr$$
$$- 2gbrrgb - 2gbrrbg + gbrgrb + gbrgbr + gbrbrg + gbrbgr$$
$$+ gbgrrb + gbgrbr - 2gbgbrr + gbbrrg + gbbrgr - 2gbbgrr$$
$$+ brrggb + brrgbg - 2brrbgg + brgrgb + brgrbg - 2brggrb$$
$$- 2brggbr + brgbrg + brgbgr - 2brbrgg + brbgrg + brbggr$$
$$- 2bgrrgb - 2bgrrbg + bgrgrb + bgrgbr + bgrbrg + bgrbgr$$
$$+ bggrrb + bggrbr - 2bggbrr + bgbrrg + bgbrgr - 2bgbgrr$$
$$+ 4bbrrgg - 2bbrgrg - 2bbrggr - 2bbgrrg - 2bbgrgr + 4bbggrr]. \quad (185)$$

7.2. Spin

The spin symmetry is SU^{σ}_2 for a dibaryon composed of six quarks, and its total spins are 3, 2, 1, and 0.

$$\begin{aligned}[1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] &= [6] \oplus 5 * [51] \oplus 9 * [42] \oplus 5 * [33], \\ \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} &= \mathbf{7} \oplus 5 * \mathbf{5} \oplus 9 * \mathbf{3} \oplus 5 * \mathbf{1}.\end{aligned} \quad (186)$$

For the spin wave functions of a six-body system that is constructed by finding the irreps of S_6 in the configuration spaces $\alpha\alpha\alpha\alpha\alpha\alpha$, $\alpha\alpha\alpha\alpha\alpha\beta$, $\alpha\alpha\alpha\alpha\beta\beta$, …, $\beta\beta\beta\beta\beta\beta$, they are called symmetry bases, or one can use the wave functions of baryons in Section 4.2 and the CG coefficients of SU_2. To get the spin wave functions of dibaryons which are physical bases, here we give the specific expression in symmetry bases when $M_S = S$.

$[\sigma_6] = [33], S = 0$:

$$\chi^{\sigma_1}_{0,0} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array},\ \chi^{\sigma_2}_{0,0} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array},\ \chi^{\sigma_3}_{0,0} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array},$$
$$\chi^{\sigma_4}_{0,0} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array},\ \chi^{\sigma_5}_{0,0} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array}. \quad (187)$$

$$
\begin{aligned}
\chi^{\sigma_1}_{0,0} &= \sqrt{\frac{1}{8}}[\beta\alpha\alpha\beta\beta\alpha + \alpha\beta\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha\beta - \beta\alpha\beta\alpha\beta\alpha + \alpha\beta\alpha\beta\alpha\beta - \beta\alpha\alpha\beta\alpha\beta \\
&\quad + \beta\alpha\beta\alpha\alpha\beta - \alpha\beta\alpha\beta\beta\alpha], \\
\chi^{\sigma_2}_{0,0} &= \sqrt{\frac{1}{24}}[2\alpha\alpha\beta\beta\beta\alpha - \beta\alpha\alpha\beta\beta\alpha + 2\beta\beta\alpha\alpha\beta\alpha - 2\beta\beta\alpha\alpha\alpha\beta - \alpha\beta\beta\alpha\beta\alpha + \alpha\beta\beta\alpha\alpha\beta \\
&\quad - 2\alpha\alpha\beta\beta\alpha\beta - \beta\alpha\beta\alpha\beta\alpha + \alpha\beta\alpha\beta\alpha\beta + \beta\alpha\alpha\beta\alpha\beta + \beta\alpha\beta\alpha\alpha\beta - \alpha\beta\alpha\beta\beta\alpha], \\
\chi^{\sigma_3}_{0,0} &= \sqrt{\frac{1}{24}}[2\alpha\beta\beta\beta\alpha\alpha - 2\beta\alpha\beta\beta\alpha\alpha + \beta\alpha\alpha\beta\beta\alpha - 2\beta\alpha\alpha\alpha\beta\beta + 2\alpha\beta\alpha\alpha\beta\beta - \alpha\beta\beta\alpha\beta\alpha \\
&\quad - \alpha\beta\beta\alpha\alpha\beta + \beta\alpha\beta\alpha\beta\alpha - \alpha\beta\alpha\beta\alpha\beta + \beta\alpha\alpha\beta\alpha\beta + \beta\alpha\beta\alpha\alpha\beta - \alpha\beta\alpha\beta\beta\alpha], \\
\chi^{\sigma_4}_{0,0} &= \sqrt{\frac{1}{72}}[2\alpha\beta\beta\beta\alpha\alpha - 2\alpha\alpha\beta\beta\beta\alpha + 2\beta\alpha\beta\beta\alpha\alpha + \beta\alpha\alpha\beta\beta\alpha - 2\beta\alpha\alpha\alpha\beta\beta - 4\beta\beta\alpha\beta\alpha\alpha \\
&\quad + 2\beta\beta\alpha\alpha\beta\alpha + 2\beta\beta\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha\beta\beta - \alpha\beta\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha\beta + 4\alpha\alpha\beta\alpha\beta\beta \\
&\quad - 2\alpha\alpha\beta\beta\alpha\beta - \beta\alpha\beta\alpha\beta\alpha + \alpha\beta\alpha\beta\alpha\beta + \beta\alpha\alpha\beta\alpha\beta - \beta\alpha\beta\alpha\alpha\beta + \alpha\beta\alpha\beta\beta\alpha], \\
\chi^{\sigma_5}_{0,0} &= \sqrt{\frac{1}{36}}[3\beta\beta\beta\alpha\alpha\alpha - \alpha\beta\beta\beta\alpha\alpha + \alpha\alpha\beta\beta\beta\alpha + 3\alpha\alpha\alpha\beta\beta\beta - \beta\alpha\beta\beta\alpha\alpha + \beta\alpha\alpha\beta\beta\alpha \\
&\quad - \beta\alpha\alpha\alpha\beta\beta - \beta\beta\alpha\beta\alpha\alpha - \beta\beta\alpha\alpha\beta\alpha + \beta\beta\alpha\alpha\alpha\beta - \alpha\beta\alpha\alpha\beta\beta - \alpha\beta\beta\alpha\beta\alpha \\
&\quad + \alpha\beta\beta\alpha\alpha\beta - \alpha\alpha\beta\alpha\beta\beta - \alpha\alpha\beta\beta\alpha\beta - \beta\alpha\beta\alpha\beta\alpha - \alpha\beta\alpha\beta\alpha\beta - \beta\alpha\alpha\beta\alpha\beta \\
&\quad + \beta\alpha\beta\alpha\alpha\beta + \alpha\beta\alpha\beta\beta\alpha].
\end{aligned}
\tag{188}
$$

$[\sigma_6] = [42], S = 1$:

$$
\begin{aligned}
&\chi^{\sigma_1}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}, \quad
\chi^{\sigma_2}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2}\end{array}, \quad
\chi^{\sigma_3}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2}\end{array}, \\
&\chi^{\sigma_4}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2}\end{array}, \quad
\chi^{\sigma_5}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \quad
\chi^{\sigma_6}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \\
&\chi^{\sigma_7}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \quad
\chi^{\sigma_8}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \quad
\chi^{\sigma_9}_{1,1} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.
\end{aligned}
\tag{189}
$$

$$
\begin{aligned}
\chi^{\sigma_1}_{1,1} &= \sqrt{\frac{1}{240}}[2\beta\beta\alpha\alpha\alpha\alpha + 2\beta\alpha\beta\alpha\alpha\alpha + 2\beta\alpha\alpha\beta\alpha\alpha - 3\beta\alpha\alpha\alpha\beta\alpha - 3\beta\alpha\alpha\alpha\alpha\beta + 2\alpha\beta\beta\alpha\alpha\alpha \\
&\quad + 2\alpha\beta\alpha\beta\alpha\alpha - 3\alpha\beta\alpha\alpha\beta\alpha - 3\alpha\beta\alpha\alpha\alpha\beta + 2\alpha\alpha\beta\beta\alpha\alpha - 3\alpha\alpha\beta\alpha\beta\alpha - 3\alpha\alpha\beta\alpha\alpha\beta \\
&\quad - 3\alpha\alpha\alpha\beta\alpha\beta - 3\alpha\alpha\alpha\beta\alpha\beta + 12\alpha\alpha\alpha\alpha\beta\beta], \\
\chi^{\sigma_2}_{1,1} &= \sqrt{\frac{1}{144}}[2\beta\beta\alpha\alpha\alpha\alpha + 2\beta\alpha\beta\alpha\alpha\alpha - 2\beta\alpha\alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha\beta\alpha - 3\beta\alpha\alpha\alpha\alpha\beta + 2\alpha\beta\beta\alpha\alpha\alpha \\
&\quad - 2\alpha\beta\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\beta\alpha - 3\alpha\beta\alpha\alpha\alpha\beta - 2\alpha\alpha\beta\beta\alpha\alpha + \alpha\alpha\beta\alpha\beta\alpha - 3\alpha\alpha\beta\alpha\alpha\beta \\
&\quad - 3\alpha\alpha\alpha\beta\beta\alpha + 9\alpha\alpha\alpha\beta\alpha\beta], \\
\chi^{\sigma_3}_{1,1} &= \sqrt{\frac{1}{72}}[2\beta\beta\alpha\alpha\alpha\alpha - \beta\alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha\beta\alpha - 3\beta\alpha\alpha\alpha\alpha\beta - \alpha\beta\beta\alpha\alpha\alpha \\
&\quad + \alpha\beta\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\beta\alpha - 3\alpha\beta\alpha\alpha\alpha\beta - 2\alpha\alpha\beta\beta\alpha\alpha - 2\alpha\alpha\beta\alpha\beta\alpha + 6\alpha\alpha\beta\alpha\alpha\beta], \\
\chi^{\sigma_4}_{1,1} &= \sqrt{\frac{1}{24}}[\beta\alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha\beta\alpha - 3\beta\alpha\alpha\alpha\alpha\beta - \alpha\beta\beta\alpha\alpha\alpha - \alpha\beta\alpha\beta\alpha\alpha \\
&\quad - \alpha\beta\alpha\alpha\beta\alpha + 3\alpha\beta\alpha\alpha\alpha\beta], \\
\chi^{\sigma_5}_{1,1} &= \sqrt{\frac{1}{18}}[\beta\beta\alpha\alpha\alpha\alpha + \beta\alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\beta\alpha\alpha - \beta\alpha\alpha\alpha\beta\alpha + \alpha\beta\beta\alpha\alpha\alpha - \alpha\beta\alpha\beta\alpha\alpha \\
&\quad - \alpha\beta\alpha\alpha\beta\alpha - \alpha\alpha\beta\beta\alpha\alpha - \alpha\alpha\beta\alpha\beta\alpha + 3\alpha\alpha\alpha\beta\beta\alpha], \\
\chi^{\sigma_6}_{1,1} &= \sqrt{\frac{1}{36}}[2\beta\beta\alpha\alpha\alpha\alpha - \beta\alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\beta\alpha\alpha - 2\beta\alpha\alpha\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha\alpha + \alpha\beta\alpha\beta\alpha\alpha \\
&\quad - 2\alpha\beta\alpha\alpha\beta\alpha - 2\alpha\alpha\beta\beta\alpha\alpha + 4\alpha\alpha\beta\alpha\beta\alpha] \\
\chi^{\sigma_7}_{1,1} &= \sqrt{\frac{1}{12}}[\beta\alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\beta\alpha\alpha - 2\beta\alpha\alpha\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha\alpha - \alpha\beta\alpha\beta\alpha\alpha + 2\alpha\beta\alpha\alpha\beta\alpha] \\
\chi^{\sigma_8}_{1,1} &= \sqrt{\frac{1}{12}}[2\beta\beta\alpha\alpha\alpha\alpha - \beta\alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\beta\alpha\alpha - \alpha\beta\beta\alpha\alpha\alpha - \alpha\beta\alpha\beta\alpha\alpha + 2\alpha\alpha\beta\beta\alpha\alpha] \\
\chi^{\sigma_9}_{1,1} &= \sqrt{\frac{1}{4}}[\beta\alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\beta\alpha\alpha - \alpha\beta\beta\alpha\alpha\alpha + \alpha\beta\alpha\beta\alpha\alpha],
\end{aligned} \tag{190}
$$

$[\sigma_6] = [51], S = 2$:

$$
\begin{aligned}
&\chi^{\sigma_1}_{2,2} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}, \;
\chi^{\sigma_2}_{2,2} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}, \;
\chi^{\sigma_3}_{2,2} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&5&6\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}, \\
&\chi^{\sigma_4}_{2,2} = \begin{array}{|c|c|c|c|c|}\hline 1&2&4&5&6\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}, \;
\chi^{\sigma_5}_{2,2} = \begin{array}{|c|c|c|c|c|}\hline 1&3&4&5&6\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}.
\end{aligned} \tag{191}
$$

$$
\begin{aligned}
\chi^{\sigma_1}_{2,2} &= \sqrt{\frac{1}{30}}[5\alpha\alpha\alpha\alpha\alpha\beta - \alpha\alpha\alpha\alpha\beta\alpha - \alpha\alpha\alpha\beta\alpha\alpha - \alpha\alpha\beta\alpha\alpha\alpha - \alpha\beta\alpha\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha\alpha], \\
\chi^{\sigma_2}_{2,2} &= \sqrt{\frac{1}{20}}[4\alpha\alpha\alpha\alpha\beta\alpha - \alpha\alpha\alpha\beta\alpha\alpha - \alpha\alpha\beta\alpha\alpha\alpha - \alpha\beta\alpha\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha\alpha], \\
\chi^{\sigma_3}_{2,2} &= \sqrt{\frac{1}{12}}[3\alpha\alpha\alpha\beta\alpha\alpha - \alpha\alpha\beta\alpha\alpha\alpha - \alpha\beta\alpha\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha\alpha], \\
\chi^{\sigma_4}_{2,2} &= \sqrt{\frac{1}{6}}[2\alpha\alpha\beta\alpha\alpha\alpha - \alpha\beta\alpha\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha\alpha], \\
\chi^{\sigma_5}_{2,2} &= \sqrt{\frac{1}{2}}[\alpha\beta\alpha\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha\alpha],
\end{aligned} \tag{192}
$$

$[\sigma_6] = [6], S = 3$:

$$
\chi^{\sigma_1}_{3,3} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\\hline\end{array} = \alpha\alpha\alpha\alpha\alpha\alpha. \tag{193}
$$

7.3. Flavor

7.3.1. q^6

We adopted SU_3^f flavor symmetry in the flavor space. The coupling of the Young diagrams is the same as that in the SU_3^c space Equation (180), and the flavor symmetry of a dibaryon consisting of six quarks is as follows.

$$\begin{aligned}&[1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] \\ =&[6] \oplus 5 * [51] \oplus 9 * [42] \oplus 10 * [411] \oplus 5 * [33] \oplus 16 * [321] \oplus 5 * [222], \\ &\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \\ =&\mathbf{28} \oplus 5 * \mathbf{35} \oplus 9 * \mathbf{27} \oplus 10 * \mathbf{10} \oplus 5 * \mathbf{10} \oplus 16 * \mathbf{8} \oplus 5 * \mathbf{1}.\end{aligned} \quad (194)$$

The multiplets of dibaryons under SU_3^f flavor symmetry is shown in Figures 29–34.

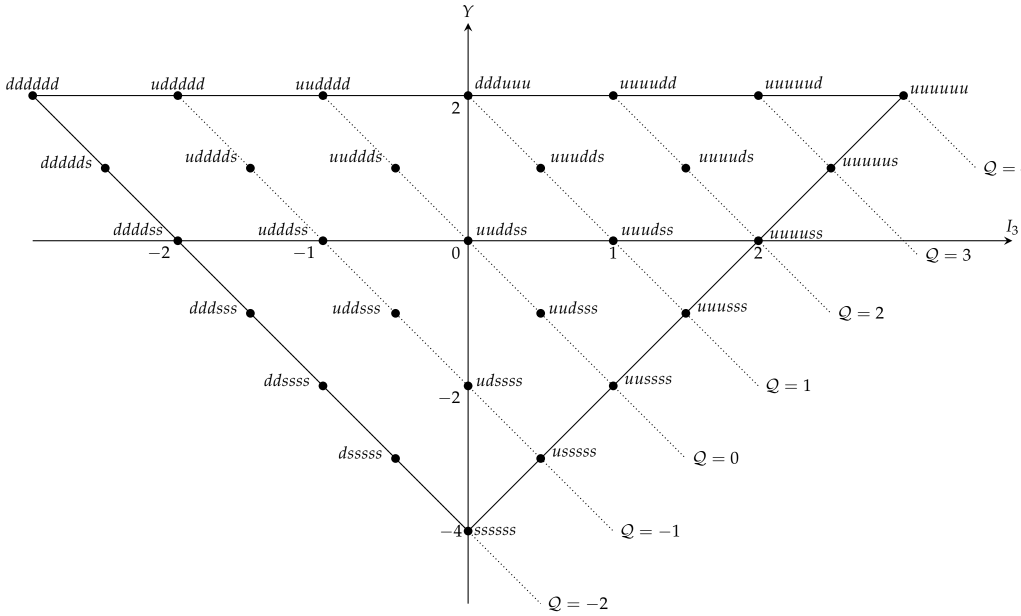


Figure 29. The 28-plet states of dibaryons for the partition [6].

In order to get the flavor wave functions of the six quarks, we can find the irreps of S_6 in the configuration spaces, *uuuuuu*, *uuuuud*, *uuuudd*, *uuuuus*, *uuudds*, …, *ssssss*. The expression in *uuuuuu*, *uuuuud*, *uuuudd*, …, *uuuddd* has been given in Section 7.2 (just replace α with u, β by d), so here we only give the flavor wave function in *uuudds* space with symmetry [321].

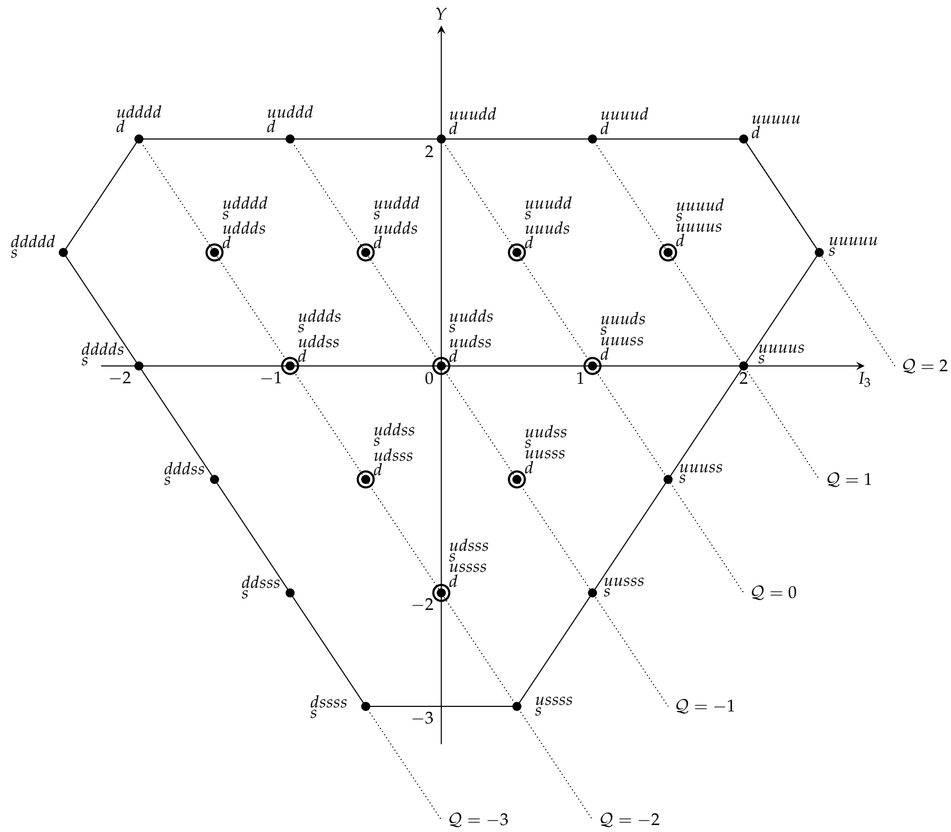


Figure 30. The 35-plet states of dibaryons for the partition [51].

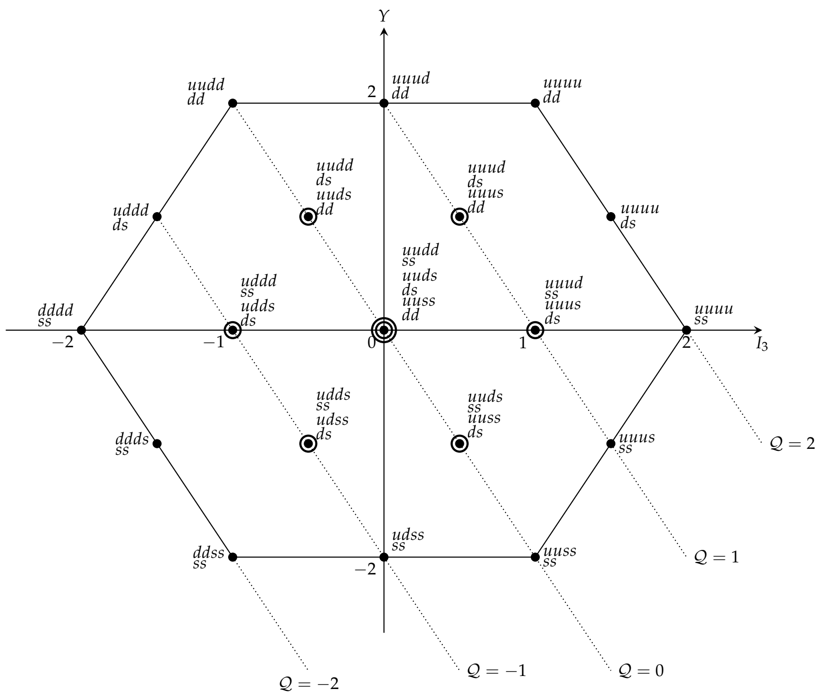


Figure 31. The 27-plet states of dibaryons for the partition [42].

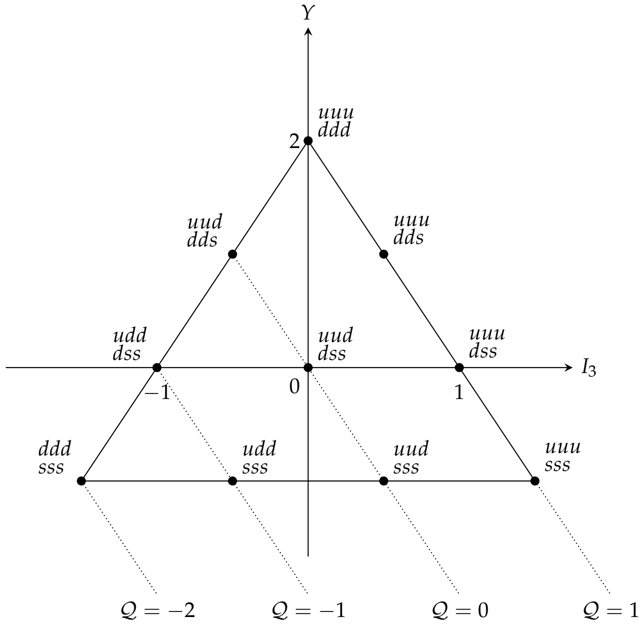


Figure 32. The 10-plet states of dibaryons for the partition [33].

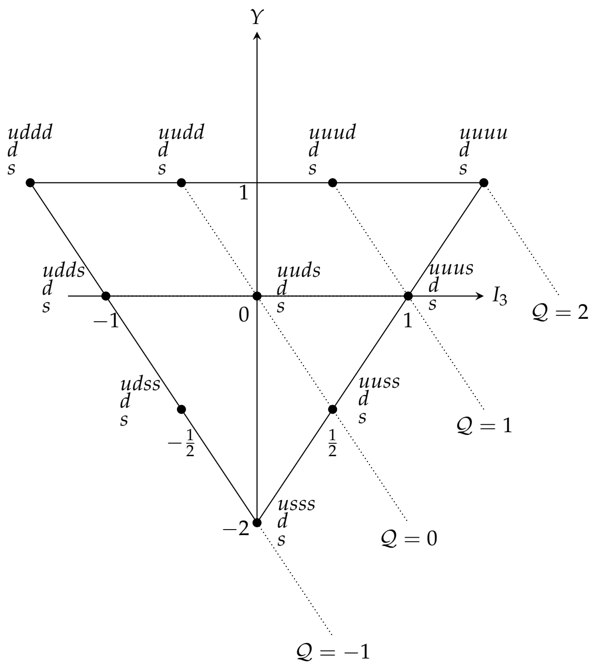


Figure 33. The 10-plet states of dibaryons for the partition [411].

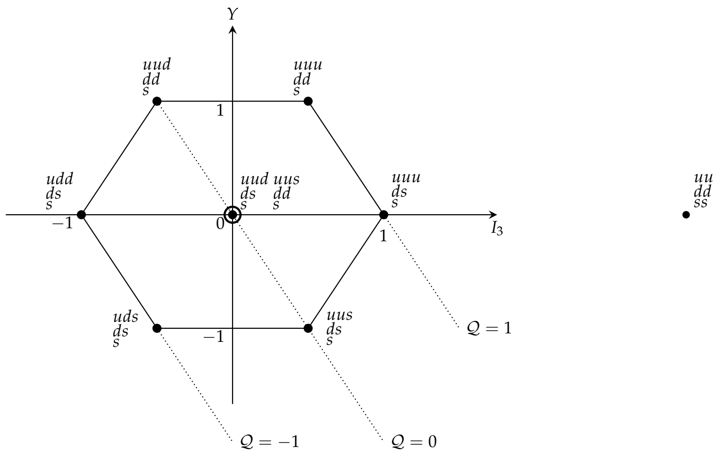


Figure 34. The 8-plet states of dibaryons for the partition [321] and singlet for irrep [222].

$[f] = [321], I = \frac{1}{2}, M_I = \frac{1}{2}, Y = 1$:

$$\begin{aligned}
\chi^{f_1}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = & \sqrt{\frac{1}{540}}[6uuudsd - 12uuudds + 6uuusdd + 2udsuud - udsudu \\
& - udsduu + 2usduud - usdudu - usdduu - sduduu \\
& - sduudu + 2sduuud - sudduu - sududu + 2suduud \\
& - dusduu - dusudu + 2dusuud - dsuduu - dsuudu \\
& + 2dsuuud - 2suuudd - 2suudud + 4suuddu - 2usuudd \\
& - 2usudud + 4usuddu - 2uusudd - 2uusdud + 4uusddu \\
& - 2uudsdu - 2uuddsu + 4uuddus + 4uududs - 2uudusd \\
& - 2uudsud - 2udusud - 2udusdu - 2ududsu + 4ududus \\
& - 2uduusd + 4uduuds - 2duusud - 2duusdu - 2duudsu \\
& + 4duuuds - 2duuusd + 4duudus + 2uddsuu + 2uddusu \\
& - 4udduus + 2ddusuu + 2dduusu - 4dduuus + 2dudsuu \\
& + 2dudusu - 4duduus],
\end{aligned} \tag{195}$$

$$\begin{aligned}
\chi^{f_2}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = & \sqrt{\frac{1}{4320}}[6ddsuuu - 3dsduuu - 3sdduuu - 4udsuud - 7udsudu \\
& + 5udsduu - 4usduud + 11usdudu - 4usdduu - sduduu \\
& - 4sduudu + 8sduuud - 4sudduu + 11sududu - 4suduud \\
& + 5dusduu - 7dusudu - 4dusuud - dsuduu - 4dsuudu \\
& + 8dsuuud - 8suuudd + 4suudud + suuddu - 8usuudd \\
& + 4usudud + usuddu + 16uusudd - 8uusdud - 2uusddu \\
& + 10uudsdu - 8uuddsu + 16uuddus - 32uududs + 16uudusd \\
& - 8uudsud + 4udusud - 5udusdu + 4ududsu - 8ududus \\
& - 8uduusd + 16uduuds + 4duusud - 5duusdu + 4duudsu \\
& + 16duuuds - 8duuusd - 8duudus - uddsuu - 4uddusu \\
& + 8udduus + 2ddusuu + 8dduusu - 16dduuus - dudsuu \\
& - 4dudusu + 8duduus],
\end{aligned} \tag{196}$$

$$\chi^{f_3}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{1440}}[3sdduuu - 3dsduuu + 4udsuud - 5udsudu + udsduu + 4usduud + usdudu - 2usdduu + 3sduduu - 6sduudu + 2sudduu - sududu - 4suduud - dusduu + 5dusudu - 4dusuud - 3dsuduu + 6dsuudu + 8suuudd - 4suudud - suuddu - 8usuudd + 4usudud + usuddu + 4udusud - 5udusdu + 4ududsu - 8ududus - 8uduusd + 16uduuds - 4duusud + 5duusdu - 4duudsu - 16duuuds + 8duuusd + 8duudus + uddsuu + 4uddusu - 8udduus - dudsuu - 4dudusu + 8duduus], \tag{197}$$

$$\chi^{f_4}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{1440}}[6ddsuuu - 3dsduuu - 3sdduuu - 4udsuud + udsudu - 3udsduu - 4usduud + usdudu + 6usdduu - 3sduduu - 2sduudu + 8sduuud + 6sudduu + sududu - 4suduud - 3dusduu + dusudu - 4dusuud - 3dsuduu - 2dsuudu + 8dsuuud - 4suudud + suuddu - 4usudus + usuddu + 8uusdud - 2uusddu - 2uudsdu + 4uuddsu - 16uuddus + 8uudsud - 4udusud + udusdu - 2ududsu + 8ududus - 4duusud + duusdu - 2duudsu + 8duudus - 3uddsuu - 2uddusu + 8udduus + 6ddusuu + 4dduusu - 16dduuus - 3dudsuu - 2dudusu + 8duduus], \tag{198}$$

$$\chi^{f_5}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{480}}[3sdduuu - 3dsduuu + 4udsuud - udsudu - 3udsduu + 4usduud - usdudu - 3sduduu + sududu - 4suduud + 3dusduu + dusudu - 4dusuud + 3dsuduu + 4suudud - suuddu - 4usudud + usuddu - 4udusud + udusdu - 2ududsu + 8ududus + 4duusud - duusdu + 2duudsu - 8duudus + 3uddsuu + 2uddusu - 8udduus - 3dudsuu - 2dudusu + 8duduus], \tag{199}$$

$$\chi^{f_6}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{180}}[6uuudsd - 6uuusdd - udsudu + udsduu - usdudu + usdduu + sduduu - sduudu + sudduu - sududu + dusduu - dusudu + dsuduu - dsuudu + 2suuudd - 2suudud + 2usuudd - 2usudud + 2uusudd - 2uusdud + 2uudsdu - 2uuddsu - 2uudusd + 2uudsud + 2udusud + 2udusdu - 2ududsu - 2uduusd + 2duusud + 2duusdu - 2duudsu - 2duuusd - 2uddsuu + 2uddusu - 2ddusuu + 2dduusu - 2dudsuu + 2dudusu], \tag{200}$$

$$\chi^{f_7}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{1440}}[6ddsuuu - 3dsduuu - 3sdduuu - 6udsuud - 5udsudu + 5udsduu + 6usduud + usdudu - 4usdduu - sduduu + 4sduudu - 4sudduu + sududu + 6suduud + 5dusduu - 5dusudu - 6dusuud - dsuduu + 4dsuudu - 8suuudd + 2suudud + 3suuddu - 8usuudd + 2usudud + 3usuddu + 16uusudd - 4uusdud - 6uusddu - 2uudsdu + 8uuddsu - 16uudusd + 4uudsud - 2udusud + udusdu - 4ududsu + 8uduusd - 2duusud + duusdu - 4duudsu + 8duuusd - uddsuu + 4uddusu + 2ddusuu - 8dduusu - dudsuu + 4dudusu], \tag{201}$$

$$\chi^{f_8}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{480}}[3dsduuu - 3sdduuu + 2udsuud - udsudu - udsduu - 2usduud - 3usdudu + 2usdduu - 3sduduu + 2sduudu + 4sduuud - 2sudduu + 3sududu + 2suduud + dusduu + dusudu - 2dusuud + 3dsuduu - 2dsuudu - 4dsuuud - 8suuudd + 2suudud + 3suuddu + 8usuudd - 2usudud - 3usuddu + 2udusud - udusdu + 4ududsu - 8uduusd - 2duusud + duusdu - 4duudsu + 8duuusd - uddsuu + 4uddusu + dudsuu - 4dudusu], \tag{202}$$

$$\chi^{f_9}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{96}}[2ddsuuu - dsduuu - sdduuu - 2udsuud + udsudu - udsduu + 2usduud - usdudu + sduduu - sududu + 2suduud - dusduu + dusudu - 2dusuud + dsuduu - 2suudud + suuddu - 2usudud + usuddu + 4uusdud - 2uusddu + 2uudsdu - 4uudsud + 2udusud - udusdu + 2duusud - duusdu + uddsuu - 2ddusuu + dudsuu], \tag{203}$$

$$\chi^{f_{10}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{288}}[3sdduuu - 3dsduuu - 2udsuud + udsudu + udsduu + 2usduud - usdudu + 2usdduu - sduduu + 2sduudu - 4sduuud - 2sudduu + sududu - 2suduud - dusduu - dusudu + 2dusuud + dsuduu - 2dsuudu + 4dsuuud + 6suudud - 3suuddu - 6usudud + 3usuddu + 6udusud - 3udusdu - 6duusud + 3duusdu - 3uddsuu + 3dudsuu], \tag{204}$$

$$\chi^{f_{11}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{36}}[udsudu - 2udsuud + udsduu + 2usduud - usdudu - usdduu - sduduu - sduudu + 2sduuud + sudduu + sududu - 2suduud - dusduu - dusudu + 2dusuud + dsuduu + dsuudu - 2dsuuud], \tag{205}$$

$$\chi^{f_{12}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{96}}[dsduuu - 2ddsuuu + sdduuu + udsudu + udsduu + usdudu - 2usdduu + sduduu - 2sduudu - 2sudduu + sududu + dusduu + dusudu + dsuduu - 2dsuudu + suuddu + usuddu - 2uusddu - 2uudsdu + 4uuddsu + udusdu - 2udusdu + duusdu - 2duudsu + uddsuu - 2uddusu - 2ddusuu + 4dduusu + dudsuu - 2dudusu], \tag{206}$$

$$\chi^{f_{13}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{32}}[sdduuu - dsduuu + udsudu - udsduu + usdudu - sduduu - sududu + dusduu - dusudu + dsuduu + suuddu - usuddu - udusdu + 2udusdsu + duusdu - 2duudsu + uddsuu - 2uddusu - dudsuu + 2dudusu], \tag{207}$$

$$\chi^{f_{14}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{32}}[dsduuu - 2ddsuuu + sdduuu + udsudu + udsduu - usdudu - sduduu - sududu + dusduu + dusudu - dsuduu + suuddu + usuddu - 2uusddu + 2uudsdu - udusdu - duusdu - uddsuu + 2ddusuu - dudsuu], \tag{208}$$

$$\chi^{f_{15}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{96}}[3sdduuu - 3dsduuu - udsudu + udsduu + usdudu + 2usdduu - sduduu - 2sduudu - 2sudduu - sududu - dusduu + dusudu + dsuduu + 2dsuudu + 3suuddu - 3usuddu + 3udusdu - 3duusdu - 3uddsuu + 3dudsuu], \tag{209}$$

$$\chi^{f_{16}}_{[321]\frac{1}{2}\frac{1}{2},1}(q^6) = \sqrt{\frac{1}{12}}[udsudu - udsduu - usdudu + usdduu + sduduu - sduudu - sudduu + sududu + dusduu - dusudu - dsuduu + dsuudu]. \tag{210}$$

Here, we suppose the six quarks are located in ground states in coordinate space, which means the orbital symmetry is $[x_6] = 6$, and the color symmetry of the six quark is assumed to be [222]; hence, the allowed states of the dibaryon states should adhere to the following rule:

$$[\sigma_6] \otimes [f_6] = [33]. \tag{211}$$

After referring to the CG series of S_6 [275], there are the following several possible spin–flavor combinations:

$$[\sigma_6] \otimes [f_6] = [51] \otimes [42], [51] \otimes [321], \tag{212}$$

$$[42] \otimes [411], [42] \otimes [33], [42] \otimes [321],$$

$$[33] \otimes [6], [6] \otimes [33]. \tag{213}$$

7.3.2. $qqqqQ$

$$
\begin{aligned}
&[1] \otimes [1] \otimes [1] \otimes [1] \otimes [1] \\
=&([2] \oplus [11]) \otimes ([2] \oplus [11]) \otimes [1] \\
=&[[4] \oplus 3[31] \oplus 2[22] \oplus 3[211]] \otimes [1] \\
=&[5] \oplus [41] \oplus 3[[41] \oplus [32] \oplus [311]] \oplus 2[[32] \oplus [221]] \oplus 3[[221] \oplus [311]] \\
=&[5] \oplus 4[41] \oplus 5[32] \oplus 6[311] \oplus 5[221],
\end{aligned} \tag{214}
$$

$$
\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{21} \oplus 4 * \mathbf{24} \oplus 5 * \mathbf{15} \oplus 6 * \mathbf{6} \oplus 5 * \mathbf{3}. \tag{215}
$$

For singly heavy dibaryons, the color symmetry of a heavy quark is [1], so the symmetry of the remaining five quarks is [221] in the color space. The total symmetry of the five light quarks in orbit–spin–flavor space should be [32], which is the conjugate of [221]. Similar to the previous section, we still rely on tables in the book [275]. The possible spin-flavor combinations of five light quarks are as follows.

$$
\begin{aligned}
[\sigma_5] \otimes [f_5] =&[41] \otimes [41], [41] \otimes [32], [41] \otimes [311], \\
&[32] \otimes [33], [32] \otimes [221], [32] \otimes [311], [32] \otimes [5], [5] \otimes [32].
\end{aligned} \tag{216}
$$

Here, we give the wave functions in the $uudss$ configuration with symmetry [5] and [41] as examples.

$$
\chi^{f_1}_{[5]\frac{3}{2}\frac{3}{2},1}(q^5) = \left| \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline\end{array}\ \begin{array}{|c|c|c|c|c|}\hline u&u&d&s&s\\\hline\end{array} \right\rangle \tag{217}
$$

$$
\begin{aligned}
=&\sqrt{\frac{1}{30}}[uudss + uusds + uussd + uduss + udsus + udssu \\
&+ usuds + ususd + usdus + usdsu + ussud + ussdu \\
&+ duuss + dusus + dussu + dsuus + dsusu + dssuu \\
&+ suuds + suusd + sudus + sudsu + susud + susdu \\
&+ sduus + sdusu + sdsuu + ssuud + ssudu + ssduu],
\end{aligned} \tag{218}
$$

$$
\begin{aligned}
\chi^{f_1}_{[41]\frac{3}{2}\frac{3}{2},1}(q^5) =& \left| \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|c|}\hline u&u&d&s\\\hline s\\\cline{1-1}\end{array} \right\rangle \\
=&\sqrt{\frac{1}{180}}[3uudss + 3uusds - 2uussd + 3uduss + 3udsus - 2udssu \\
&+ 3usuds - 2ususd + 3usdus - 2usdsu - 2ussud - 2ussdu \\
&+ 3duuss + 3dusus - 2dussu + 3dsuus - 2dsusu - 2dssuu \\
&+ 3suuds - 2suusd + 3sudus - 2sudsu - 2susud - 2susdu \\
&+ 3sduus - 2sdusu - 2sdsuu - 2ssuud - 2ssudu - 2ssduu],
\end{aligned} \tag{219}
$$

$$
\begin{aligned}
\chi^{f_2}_{[41]\frac{1}{2}\frac{1}{2},1}(q^5) =& \left| \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|c|}\hline u&u&s&s\\\hline d\\\cline{1-1}\end{array} \right\rangle \\
=&\sqrt{\frac{1}{36}}[2uussd - udssu + 2ususd - usdsu + 2ussud - ussdu \\
&- dussu - dsusu - dssuu + 2suusd - sudsu + 2susud \\
&- susdu - sdusu - sdsuu + 2ssuud - ssudu - ssduu],
\end{aligned} \tag{220}
$$

$$\chi^{f_3}_{[41]\frac{3}{2}\frac{3}{2},1}(q^5) = \left|\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4\\\cline{1-1}\end{array}\;\begin{array}{|c|c|c|c|}\hline u&u&d&s\\\hline s\\\cline{1-1}\end{array}\right\rangle$$
$$=\sqrt{\frac{1}{108}}[3uudss - uusds + 2uussd + 3uduss - udsus + 2udssu$$
$$- usuds + 2ususd - usdus + 2usdsu - 2ussud - 2ussdu$$
$$+ 3duuss - dusus + 2dussu - dsuus + 2dsusu - 2dssuu$$
$$- suuds + 2suusd - sudus + 2sudsu - 2susud - 2susdu$$
$$- sduus + 2sdusu - 2sdsuu - 2ssuud - 2ssudu - 2ssduu], \tag{221}$$

$$\chi^{f_4}_{[41]\frac{1}{2}\frac{1}{2},1}(q^5) = \left|\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4\\\cline{1-1}\end{array}\;\begin{array}{|c|c|c|c|}\hline u&u&s&s\\\hline d\\\cline{1-1}\end{array}\right\rangle$$
$$=\sqrt{\frac{1}{540}}[8uusds + 2uussd - 4udsus - udssu + 8usuds + 2ususd$$
$$- 4usdus - usdsu - 2ussud + 7ussdu - 4dusus - dussu$$
$$- 4dsuus - dsusu - 5dssuu + 8suuds + 2suusd - 4sudus$$
$$- sudsu - 2susud + 7susdu - 4sduus - sdusu - 5sdsuu$$
$$- 2ssuud + 7ssudu - 5ssduu], \tag{222}$$

$$\chi^{f_5}_{[41]\frac{3}{2}\frac{3}{2},1}(q^5) = \left|\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3\\\cline{1-1}\end{array}\;\begin{array}{|c|c|c|c|}\hline u&u&d&s\\\hline s\\\cline{1-1}\end{array}\right\rangle$$
$$=\sqrt{\frac{1}{54}}[2uusds + 2uussd + 2udsus + 2udssu - usuds - ususd$$
$$- usdus - usdsu + ussud + ussdu + 2dusus + 2dussu$$
$$- dsuus - dsusu + dssuu - suuds - suusd - sudus$$
$$- sudsu + susud + susdu - sduus - sdusu + sdsuu$$
$$- 2ssuud - 2ssudu - 2ssduu], \tag{223}$$

$$\chi^{f_6}_{[41]\frac{1}{2}\frac{1}{2},1}(q^5) = \left|\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3\\\cline{1-1}\end{array}\;\begin{array}{|c|c|c|c|}\hline u&u&s&s\\\hline d\\\cline{1-1}\end{array}\right\rangle$$
$$=\sqrt{\frac{1}{270}}[6uudss + 2uusds + 2uussd - 3uduss - udsus - udssu$$
$$- usuds - ususd + 5usdus + 5usdsu + ussud + ussdu$$
$$- 3duuss - dusus - dussu - 4dsuus - 4dsusu - 2dssuu$$
$$- suuds - suusd + 5sudus + 5sudsu + susud + susdu$$
$$- 4sduus - 4sdusu - 2sdsuu - 2ssuud - 2ssudu + 4ssduu], \tag{224}$$

$$\chi^{f_7}_{[41]\frac{3}{2}\frac{3}{2},1}(q^5) = \left|\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2\\\cline{1-1}\end{array}\;\begin{array}{|c|c|c|c|}\hline u&u&d&s\\\hline s\\\cline{1-1}\end{array}\right\rangle$$
$$=\sqrt{\frac{1}{18}}[usuds + ususd + usdus + usdsu + ussud + ussdu$$
$$+ dsuus + dsusu + dssuu - suuds - suusd - sudus$$
$$- sudsu - susud - susdu - sduus - sdusu - sdsuu], \tag{225}$$

$$\chi^{f_8}_{[41]\frac{1}{2}\frac{1}{2},1}(q^5) = \left| \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array} \; \begin{array}{|c|c|c|c|}\hline u & u & s & s \\ \hline d \\ \cline{1-1}\end{array} \right\rangle$$
$$= \sqrt{\frac{1}{180}}[3uduss + 3udsus + 3udssu + usuds + ususd + usdus$$
$$+ usdsu + ussud + ussdu - 3duuss - 3dusus - 3dussu$$
$$- 2dsuus - 2dsusu - 2dssuu - suuds - suusd - sudus$$
$$- sudsu - susud - susdu + 2sduus + 2sdusu + 2sdsuu]. \quad (226)$$

The flavor multiplets of dibaryons with one heavy quark is illustrated in Figures 35–38. In these figures the heavy quark is not shown to simplify the presentation.

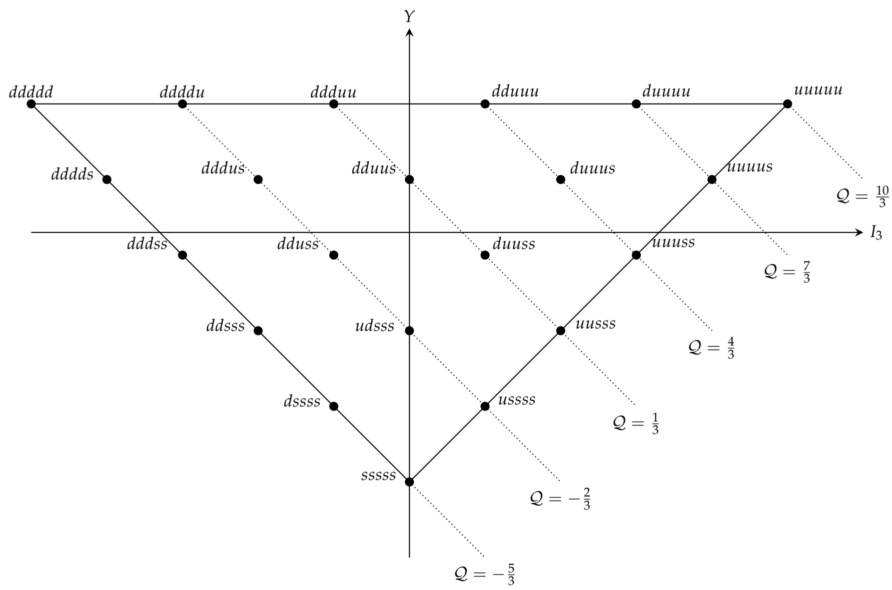


Figure 35. The 21-plet states of dibaryons for the partition [5].

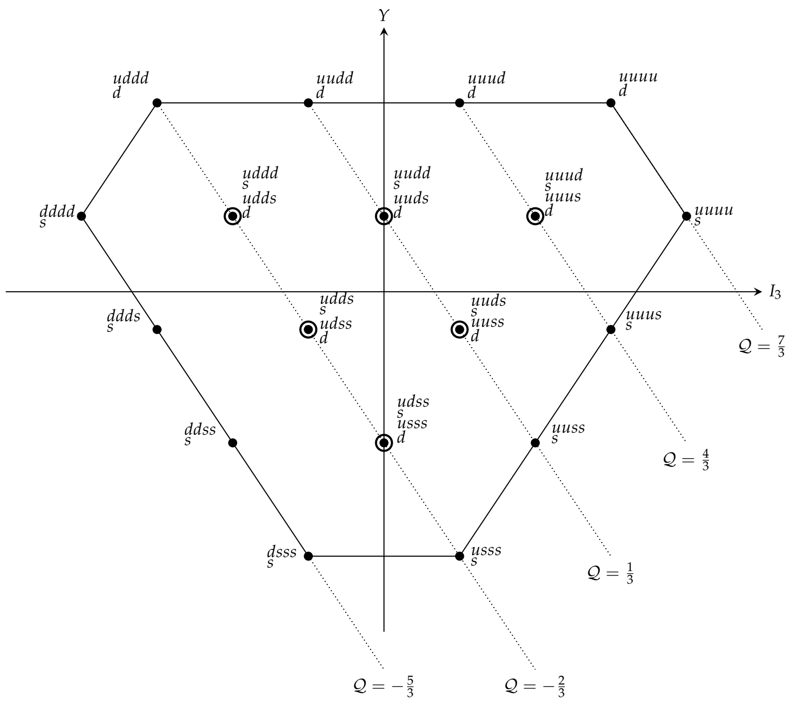


Figure 36. The 24-plet states of the pentaquark for partition [41] of SU^f_3. The heavy quark is not shown.

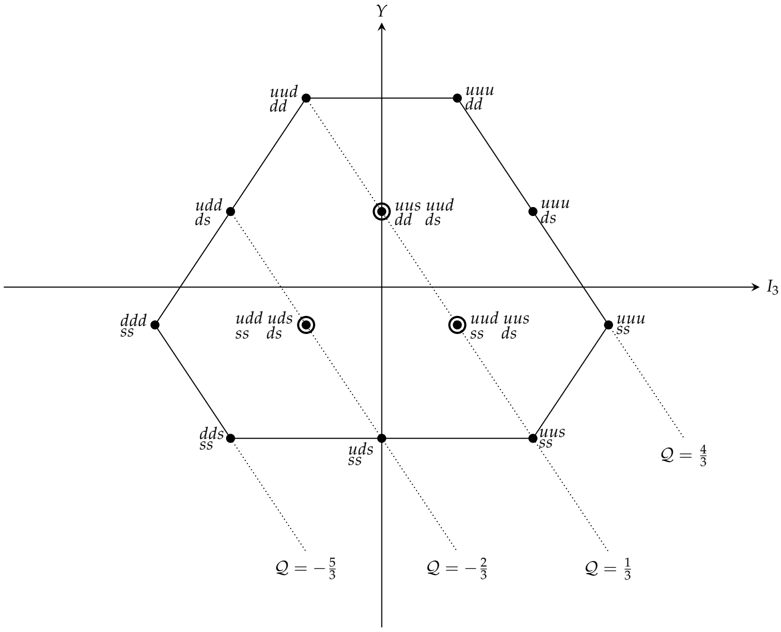


Figure 37. The 15-plet states of the pentaquark for partition [32] of of SU_3^f. The heavy quark is not shown.

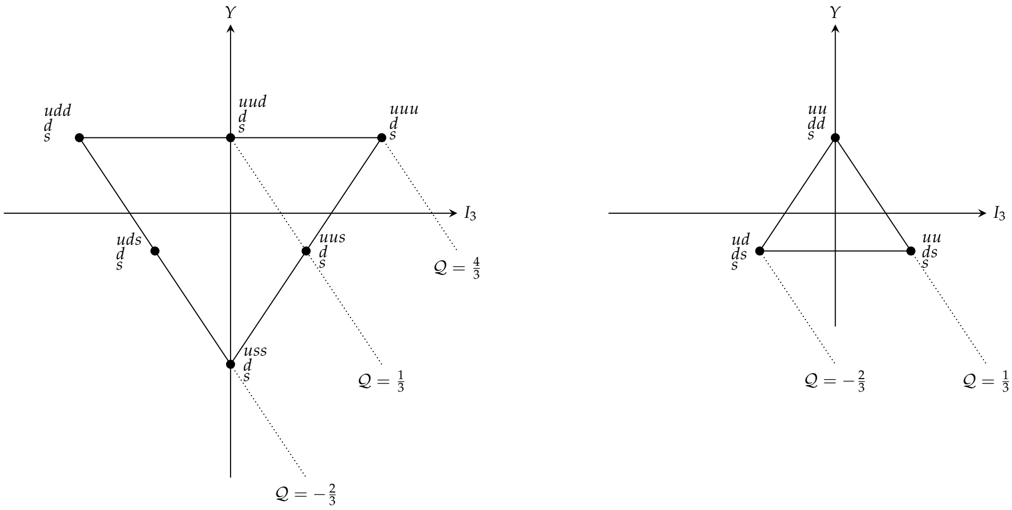


Figure 38. The 6-plet states of the pentaquark for [311] and 3-plet for [221] of SU_3^f. The heavy quark is not shown.

7.3.3. *qqqqQQ*

$$\begin{aligned} [1] \otimes [1] \otimes [1] \otimes [1] =& ([2] \oplus [11]) \otimes ([2] \oplus [11]) = [4] \oplus 3\,[31] \oplus 2\,[22] \oplus 3\,[211], \\ \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} =& \mathbf{15} \oplus 3 * \mathbf{15} \oplus 2 * \mathbf{6} \oplus 3 * \mathbf{3}. \end{aligned} \tag{227}$$

For doubly heavy dibaryons, the wave function of the four light quark is the same as that in Section 6.3.1. We can couple formula Equations (152)–(162) with two heavy quarks so that we can get the flavor wave function of the doubly heavy dibaryons.

Note that if two heavy quarks are the same, we need to consider the symmetry of the light-quark sector and heavy-quark sector at the same time. For example, if the state is *qqqqcc*, due to the constraint of the color singlet, the color symmetry of $q^4 \otimes Q^2$ must be $[22] \otimes [2]$ or $[211] \otimes [11]$. We list the allowed states in Table 9. If the two heavy quarks are *b* and *c*, then we do not need to consider the symmetry of the two heavy quarks.

Table 9. Allowed states of q^4Q^2.

	q^4	Q^2
orbit	$[4]$	$[2]$
color	$[22]$	$[2]$
	$[211]$	$[11]$
spin	$[4], [31], [22]$	$[2], [11]$
flavor	$[4], [31], [22]$	$[2]$
allowed states	$[22]_c[22]_\sigma[4]_f$	$[2]_c[11]_\sigma[2]_f$
	$[22]_c[31]_\sigma[31]_f$	
	$[22]_c[22]_\sigma[22]_f$	
	$[22]_c[4]_\sigma[22]_f$	
	$[211]_c[31]_\sigma[4]_f$	$[11]_c[2]_\sigma[2]_f$
	$[211]_c[31]_\sigma[31]_f$	
	$[211]_c[22]_\sigma[31]_f$	
	$[211]_c[31]_\sigma[22]_f$	
	$[211]_c[4]_\sigma[31]_f$	

There are no signals of doubly heavy dibaryons reported so far. However, several possible states are predicted. $\Xi^*_{cc}\Xi$ with resonance mass 5081 MeV and decay width 0.3 MeV and the $\Xi_c\Xi^*_c$ state with the mass 5213 MeV and decay width 19.8 MeV were predicted in the $\Lambda\Omega_{cc}$ scattering channel, which may be two promising dibaryon candidates to be searched for in future experiments [276].

7.3.4. $qqqQQQ$ and $qqQQQQ$

For dibaryon $qqqQQQ$, consisting of three light quarks and three heavy quarks, the flavor wave functions can be obtained from the product of the light part and the heavy part. The light part can be found in Section 4.3; the heavy part is just the product of the wave functions of three quarks.

For dibaryon $qqQQQQ$, with two light quarks and four heavy quarks, the flavor wave functions are the product of the wave functions of the light diquarks and the wave functions of the heavy quarks, we still need to consider the symmetry of both the light-quark and heavy-quark sectors together. For example, the wave function for the $qqqQQQ$ state with quantum numbers $Y = 0, I = 1, M_I = 1, C = 2, B = -1$ is

$$\chi^f_{011,2,-1} = \sqrt{\frac{1}{3}}[uus + usu + suu]ccb. \tag{228}$$

7.3.5. $qQQQQQ$ and $QQQQQQ$

In these cases, the flavor wave functions are just the product of the wave functions of each quark. There is no report about fully heavy dibaryons so far, but many theoretical predictions have been made, and it is believed that $\Omega_{ccc}\Omega_{ccc}$ and $\Omega_{bbb}\Omega_{bbb}$ can form bound states [277,278].

The Gürsey–Radicati mass formula has also been applied to predict the mass of dibaryon states, the results are listed in Table 10. Although there is no sufficient experimental information on these dibaryons, the results still have a certain guiding significance for our future research on six-quark systems [279–281].

Table 10. The parameters and masses of group-state dibaryons.

parameter	M_0	X_1			X_2		X_3
	2534.2 ± 0.70	-6.8206 ± 0.03			61.780 ± 0.07		-303.62 ± 0.21
	X_s	X_c			X_b		
	65.882 ± 0.11	1180.3 ± 0.18			-4392.0 ± 0.18		
dibaryon	Y (hypercharge)	I (isospin)	J (spin)	$[f]$ (flavor)	Theo. (MeV)	Exp. (MeV)	Ref. (MeV)
D_{01}	2.0	0	1	[33]	1902.83 ± 1.79	1875.6 [43]	1876 [282]
D_{10}	2.0	1	0	[42]	1883.75 ± 1.77	-	1883 [282]
D_{03}	2.0	0	3	[33]	2520.63 ± 2.54	2380 [28–33]	2400 ± 200 [283]
D_{30}	2.0	3	0	[6]	2406.16 ± 3.57	-	2430 [284]
D_{12}	2.0	1	2	[42]	2254.43 ± 2.22	-	2168 [282]
D_{21}	2.0	2	1	[51]	2216.28 ± 2.64	-	2179.8 [282]
$N\Omega$	-1.0	$\frac{1}{2}$	2	[321]	3200.55 ± 1.62	-	2590 ± 170 [285]
$\Omega\Omega$	-4.0	0	0	[6]	3239.64 ± 3.21		3282 [286]
H	0	0	0	[222]	2534.24 ± 0.7	-	$2225 \backsim 2234$ [287]
$\Lambda_c\Lambda_c$	$\frac{4}{3}$	0	0	[0]	4460.80 ± 1.39	-	-
$\Lambda_b\Lambda_b$	$\frac{4}{3}$	0	0	[0]	$10{,}884.22 \pm 1.39$	-	-
$N\Omega_{ccc}$	1.0	$\frac{1}{2}$	2	[21]	6134.31 ± 2.16	-	5992.5 [288]
$N\Omega_{bbb}$	1.0	$\frac{1}{2}$	2	[21]	$15{,}769.45 \pm 2.16$	-	16,010.9 [288] -
$\Omega_{ccc}\Omega_{ccc}$	0.0	0	0	[0]	9616.23 ± 1.79	-	9684.8 [289]
$\Omega_{bbb}\Omega_{bbb}$	0.0	0	0	[0]	$28{,}886.51 \pm 1.79$	-	28,680.7 [289]
$\Omega_{ccc}\Omega_{bbb}$	0.0	0	0	[0]	$19{,}251.37 \pm 1.79$	-	19,090.4 [289]

For the system of double-photon D_{01}, which is a system composed of a neutron and a proton, its isospin and spin quantum numbers are 0 and 1, respectively, and its mass is 1907 MeV. This is the deuteron, and the experimental value of the deuteron is 1876 MeV. D_{10} is a system composed of two neutrons, and it is generally believed that there may be a zero-energy resonant state formed by two neutrons. The mass formula provides a mass that is slightly higher than the threshold. The calculated mass of D_{03} is 2310 MeV, corresponding to the d^* currently found in the experiments. With a mass of 2380 MeV, our previous work indicated that the $d^*(D_{03})$ was a tightly bound six-quark system instead of a loosely bound deuteron-like system of two Δs [290–296]. The mass of D_{30} is expected to be 2418 MeV, but this state is difficult to observe in experiments because of its large decay width. A recent study revealed that vector meson exchange interactions played a dominant role between the u/d quarks in the light dibaryons like NN, D_{30}, D_{03} [297]. The preliminary experiment gave negative results [298], possibly due to the large width ($\Gamma > 200$ MeV) [299].

The mass of D_{12} is near the threshold, which is 2159 MeV. This state has been reported in experiments, with a mass of 2144–2148 MeV [300,301]. The energy of D_{21} is larger than that of D_{12} ($N\Delta$ with $IJ^P = 12^+$); the same conclusion is also obtained with the analysis of the matrix elements within the color magnetic interaction, so it is harder for D_{21} to form a bound stated than D_{12} [282].

Research on $N\Omega$ can be traced back to 1987. The relativistic quark model calculations by Goldman et al. indicated that the $N\Omega$ dibaryon with quantum numbers $S = -3, I = \frac{1}{2}, J = 2$ could appear as a narrow resonance [302]. Oka predicted several

$J^P = 2^+$ dibaryon resonances with a quark cluster model, and the result showed that there is no quark exchange contribution within the $N\Omega$ system, and coupling of other channels like $\Lambda\Xi^*$, $\Sigma\Xi^*$, $\Sigma^*\Xi$ is crucial [303]. A similar result was obtained with a chiral quark model and quark delocalization color screening model. The two models also show that there is no dibaryon resonance in the channels composed of octet–octet baryons, but dibaryon resonance appears in the channels composed of octet–decuplet and decuplet–decuplet baryons [304]. Possible $N\Omega$-like dibaryons with heavy quarks were also predicted in the two models [288]. Lattice QCD simulations employing $(2+1)$ quark flavors conducted by the HAL QCD collaboration demonstrated $N\Omega$ to be a bound state at different pion mass [305,306]. The experimental search of $N\Omega$ has been conducted by the STAR Cooperation at the Relativistic Heavy Ion Collider (RHIC) in Au +Au collisions. The measurement of the $p\Omega$ correlation function demonstrated positive scattering length for the $p\Omega$ interaction and supported the possible existence of a bound $p\Omega$ state [35]. Theoretically, femtoscopic correlation functions can be used to probe the strong interactions within exotic hadrons [13,307]. Very recently, the result of the energy spectrum, scattering phase shifts, and correlation functions indicated that $p\Omega$ may be a bound state [308].

The masses obtained by the formula are generally lower for the states with strangeness, especially for H particles, which are below the corresponding threshold. However, this state has not been found in experiments to this day [309–315]. This may be due to the fact that the flavor symmetry of SU_3 is not a good symmetry, and the introduction of isospin and supercharge terms in the formula considering breaking is still insufficient. Additionally, we assume that the system is in the orbital ground state, and all orbital space effects are replaced by constants, which is a strong approximation.Nowadays, H baryon is usually considered as a bound state [313,316–318]. More information about dibaryons can be seen in [319–324].

8. Summary

In this article, dynamical symmetry in hadron physics is investigated, and the mass formulas for hadrons are derived. These formulas exhibit excellent descriptive power for the ground states of hadrons and are notably model-independent, enabling robust predictive capabilities that have achieved significant success in describing baryons and dibaryons.

The non-perturbative dynamics of QCD in the low-energy regime remain analytically challenging, yet multi-quark systems provide a unique platform to probe quark confinement, chiral symmetry breaking, and other non-perturbative phenomena. By studying these systems, we aim to elucidate the fundamental hadron formation mechanisms—including distinguishing between compact multi-quark states and molecular states—thereby transcending the traditional $q\bar{q}$ meson and qqq baryon frameworks predicted by constituent quark models.

To enable detailed hadron characterization, we present wave functions for two- to six-quark systems across various degrees of freedom. These wave functions are constructed based on underlying symmetry principles, with strict adherence to phase consistency. Specifically, a consistent phase convention is adopted throughout the construction, following the methodology outlined in Ref. [70]. For brevity, not all configuration-specific wave functions are explicitly listed here; they can be derived using the methods described in this work or obtained directly from the authors upon request.

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Appendix A. Finding the Irreducible Representation Bases of S_4 in the *abbc* Configuration

As an appendix, we gave a simple example to show how to find irreducible representations of the S_4 group based on the eigen function method (EFM) in Ref. [70], which is a general one to find the irreducible representation of various groups.

For the representation space with the *abbc* configuration of S_4, the dimension of the space is 12, and the 12 bases are listed in Table A1.

Table A1. The index of bases and the effect of permutation operators on the bases.

	φ_1 *abbc*	φ_2 *abcb*	φ_3 *acbb*	φ_4 *babc*	φ_5 *bacb*	φ_6 *bbac*	φ_7 *bbca*	φ_8 *bcab*	φ_9 *bcba*	φ_{10} *cabb*	φ_{11} *cbab*	φ_{12} *cbba*
(12)	φ_4	φ_5	φ_{10}	φ_1	φ_2	φ_6	φ_7	φ_{11}	φ_{12}	φ_3	φ_8	φ_9
(13)	φ_6	φ_{11}	φ_8	φ_4	φ_{10}	φ_1	φ_{12}	φ_3	φ_9	φ_5	φ_2	φ_7
(14)	φ_{12}	φ_7	φ_9	φ_{10}	φ_5	φ_{11}	φ_2	φ_8	φ_3	φ_4	φ_6	φ_1
(23)	φ_1	φ_3	φ_2	φ_6	φ_8	φ_4	φ_9	φ_5	φ_7	φ_{11}	φ_{10}	φ_{12}
(24)	φ_3	φ_2	φ_2	φ_6	φ_8	φ_4	φ_5	φ_6	φ_4	φ_{12}	φ_{11}	φ_{10}
(34)	φ_2	φ_3	φ_2	φ_6	φ_8	φ_4	φ_6	φ_9	φ_8	φ_{10}	φ_{12}	φ_{11}

In the EFM, the irreducible representation bases can be obtained by finding the eigen functions of some operators. So the first step is to find the CSCO-I, the complete set of commuting operators in the class space of the group. The class space is spanned by class operators, which are sums of group elements belong to the same class. Collecting all the CSCO-Is of the group and its subgroups forms the CSCO-II of the group if the group chain is a canonical one. $S_f \supset S_{f-1} \supset \ldots \supset S_2$ is the canonical group chain. If some irreducible representations appear more than once, new operators are needed. In the configuration space, the operators come from the state permutation group and its subgroups. In this example, the operators needed are the following:

$$
\begin{aligned}
C &= (12)+(13)+(23)+(14)+(24)+(34), \\
C_{s1} &= (12)+(13)+(23), \\
C_{s2} &= (12), \\
\mathcal{C}_s &= (i_1 i_2)+(i_1 i_3)+(i_2 i_3),
\end{aligned}
\tag{A1}
$$

where $i_1 = a, i_2 = b, i_3 = b, i_4 = c$. The CSCO-I of $\mathcal{S}_4$ is the same as CSCO-I of S_4, and $\mathcal{S}_2$ does not exist. The set of operators in Equation (A1) is the CSCO in the configuration space. The matrices of the operators can be read from Tables A1 and A2. Solving the eigen equations of these operators,

$$\begin{pmatrix} C \\ C_{s1} \\ C_{s2} \\ \mathcal{C}_s \end{pmatrix} \psi^{(\nu)k}_{m_1,m_2} = \begin{pmatrix} \nu \\ m_1 \\ m_2 \\ k \end{pmatrix} \psi^{(\nu)k}_{m_1,m_2}, \tag{A2}$$

$$\psi^{(\nu)k}_{m_1,m_2} = \sum_{i=1}^{12} a_i \varphi_i, \tag{A3}$$

the simultaneous eigenfunctions $\psi^{(\nu)k}_{m_1,m_2}$ of these operators are the irreducible representation bases of S_4 in the configuration space. ν, m_1, m_2, k denote the eigenvalues of operators $C, C_{s1}, C_{s2}, \mathcal{C}_s$.

Table A2. The effect of state permutation operators on the bases.

	$\boldsymbol{\varphi_1}$ ***abbc***	$\boldsymbol{\varphi_2}$ ***abcb***	$\boldsymbol{\varphi_3}$ ***acbb***	$\boldsymbol{\varphi_4}$ ***babc***
$\mathcal{C}_s$	$\varphi_4 + \varphi_6 + \varphi_1$	$\varphi_5 + \varphi_7 + \varphi_2$	$\varphi_8 + \varphi_9 + \varphi_3$	$\varphi_1 + \varphi_6 + \varphi_4$
	φ_5 *bacb*	φ_6 *bbac*	φ_7 *bbca*	φ_8 *bcab*
$\mathcal{C}_s$	$\varphi_2 + \varphi_7 + \varphi_5$	$\varphi_1 + \varphi_4 + \varphi_6$	$\varphi_2 + \varphi_5 + \varphi_7$	$\varphi_3 + \varphi_9 + \varphi_8$
	φ_9 *bcba*	φ_{10} *cabb*	φ_{11} *cbab*	φ_{12} *cbba*
$\mathcal{C}_s$	$\varphi_3 + \varphi_8 + \varphi_9$	$\varphi_{11} + \varphi_{12} + \varphi_{10}$	$\varphi_{10} + \varphi_{12} + \varphi_{11}$	$\varphi_{10} + \varphi_{11} + \varphi_{12}$

$$\begin{aligned}
\psi^{(6)3}_{3,1} &= \left| \young(1234)\ \young(abbc) \right\rangle \\
&= \sqrt{\frac{1}{12}}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 + \varphi_7 + \varphi_8 + \varphi_9 + \varphi_{10} + \varphi_{11} + \varphi_{12}), \\
\psi^{(2)3}_{3,1} &= \left| \young(123,4)\ \young(abb,c) \right\rangle \\
&= \frac{1}{6}(3\varphi_1 - \varphi_2 - \varphi_3 + 3\varphi_4 - \varphi_5 + 3\varphi_6 - \varphi_7 - \varphi_8 - \varphi_9 - \varphi_{10} - \varphi_{11} - \varphi_{12}), \\
\psi^{(2)3}_{0,1} &= \left| \young(124,3)\ \young(abb,c) \right\rangle \\
&= \sqrt{\frac{1}{18}}(2\varphi_2 - \varphi_3 + 2\varphi_5 + 2\varphi_7 - \varphi_8 - \varphi_9 - \varphi_{10} - \varphi_{11} - \varphi_{12}), \\
\psi^{(2)3}_{0,-1} &= \left| \young(134,2)\ \young(abb,c) \right\rangle \\
&= \sqrt{\frac{1}{6}}(\varphi_3 + \varphi_8 + \varphi_9 - \varphi_{10} - \varphi_{11} - \varphi_{12}),
\end{aligned}$$

$$
\begin{aligned}
\psi^{(2)0}_{3,1} &= \left| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline a&b&c\\\hline b\\\cline{1-1}\end{array} \right\rangle \\
&= \sqrt{\frac{1}{18}}(\varphi_2+\varphi_3+\varphi_5-2\varphi_7+\varphi_8-2\varphi_9+\varphi_{10}+\varphi_{11}-2\varphi_{12}), \\
\psi^{(2)0}_{0,1} &= \left| \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline a&b&c\\\hline b\\\cline{1-1}\end{array} \right\rangle \\
&= \frac{1}{12}(3\varphi_1+\varphi_2+4\varphi_3+3\varphi_4+\varphi_5-6\varphi_6-2\varphi_7-5\varphi_8+\varphi_9+4\varphi_{10}-5\varphi_{11}+\varphi_{12}), \\
\psi^{(2)0}_{0,-1} &= \left| \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline a&b&c\\\hline b\\\cline{1-1}\end{array} \right\rangle \\
&= \sqrt{\frac{1}{48}}(3\varphi_1+3\varphi_2+2\varphi_3-3\varphi_4-3\varphi_5-\varphi_8-\varphi_9-2\varphi_{10}+\varphi_{11}+\varphi_{12}), \qquad \text{(A4)} \\
\psi^{(0)0}_{0,1} &= \left| \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\ \begin{array}{|c|c|}\hline a&b\\\hline b&c\\\hline\end{array} \right\rangle \\
&= \sqrt{\frac{1}{24}}(\varphi_1+\varphi_2-2\varphi_3+\varphi_4+\varphi_5-2\varphi_6-2\varphi_7+\varphi_8+\varphi_9-2\varphi_{10}+\varphi_{11}+\varphi_{12}), \\
\psi^{(0)0}_{0,-1} &= \left| \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\ \begin{array}{|c|c|}\hline a&b\\\hline b&c\\\hline\end{array} \right\rangle \\
&= \sqrt{\frac{1}{8}}(\varphi_1-\varphi_2-\varphi_4+\varphi_5+\varphi_8-\varphi_9-\varphi_{11}+\varphi_{12}), \\
\psi^{(-2)0}_{0,1} &= \left| \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline a&b\\\hline b\\\cline{1-1} c\\\cline{1-1}\end{array} \right\rangle \\
&= \frac{1}{4}(\varphi_1-\varphi_2+\varphi_4-\varphi_5-2\varphi_6+2\varphi_7+\varphi_8-\varphi_9+\varphi_{11}-\varphi_{12}), \\
\psi^{(-2)0}_{0,-1} &= \left| \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline a&b\\\hline b\\\cline{1-1} c\\\cline{1-1}\end{array} \right\rangle \\
&= \sqrt{\frac{1}{48}}(3\varphi_1-\varphi_2-2\varphi_3-3\varphi_4+\varphi_5-\varphi_8+3\varphi_9+2\varphi_{10}+\varphi_{11}-3\varphi_{12}), \\
\psi^{(-2)0}_{-3,-1} &= \left| \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline a&b\\\hline b\\\cline{1-1} c\\\cline{1-1}\end{array} \right\rangle \\
&= \sqrt{\frac{1}{6}}(\varphi_2-\varphi_3-\varphi_5+\varphi_8+\varphi_{10}-\varphi_{11}).
\end{aligned}
$$

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