Dynamical system analysis of Bianchi-I spacetimes in $f(R)$ gravity

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Based on our original work published in Ref.[1], we investigate an autonomous system analysis in terms of new expansion-normalized variables for homogeneous and anisotropic Bianchi-I spacetimes in $f(R)$ gravity in the presence of anisotropic matter. It is demonstrated that with a suitable choice of the evolution parameter, the Einstein's equations are reduced to an autonomous 5-dimensional system of ordinary differential equations for the new variables. Furthermore, for a large class of functions $f(R)$, which includes several cases commonly considered in the literature, all the fixed points are polynomial roots, and thus they can be determined with good accuracy and classified for stability. In addition, typically for these cases, any fixed point corresponding to isotropic solutions in the presence of anisotropic matter will be unstable. The assumption of a perfect fluid as source and or the vacuum cases imply some dimensional reductions and even more simplifications. In particular, it is found that the vacuum solutions of $f(R) = R^{\delta+1}$ with δ a constant are governed by an effective bi-dimensional phase space which can be constructed analytically, leading to an exactly soluble dynamics. It is also shown that several results already reported in the literature can be re-obtained in a more direct and easy way by exploring our dynamical formulation.

Keywords: Modified gravity theories; autonomous system analysis

1. Introduction

According to recent various cosmological observations including Type Ia Supernovae, cosmic microwave background (CMB) radiation and large scale structure, in addition to inflation[2–4] in the early universe, the current expansion of the universe is also accelerating. This is the so-called dark energy problem. There are two representative approaches to study the issue of dark energy. One is to introduce some unknown matter called dark energy with the negative pressure in general relativity.

The other is to modify gravity at large scales (for reviews of dark energy and modified gravity theories, see, e.g., Refs.[5–17]). As one of the popular modified gravity theories, $f(R)$ gravity has been proposed.[18–20]

In this article, based on our original reference[1], we extend the so-called expansion-normalized variables[21] to write down the dynamical equations of $f(R)$ gravity for a homogeneous and anisotropic Bianchi-I metric in the presence of an anisotropic fluid, as a 5-dimensional system of ordinary differential equations. We show that some further assumptions may lead to considerable simplifications in the equations, and for several examples we end up with analytically soluble systems. For the sake of illustration, we consider explicitly the case of $f(R) = R^{1+\delta}$. We demonstrate that the formulation of[22,23] is recovered in the isotropic matter limit. Moreover, in a simpler and more direct way, we re-derive some uniqueness and stability properties of the Starobinsky's isotropic inflationary scenario in R^2 gravity,[24–26] which is consistent with the Planck 2018 results.[27,28]

The article is organized as follows. In section 2, we describe the dynamical equations for a Bianchi-I cosmology in $f(R)$ gravity in the presence of an anisotropic fluid. We discuss the isotropic fluid limit and introduce the new expansion-normalized variables for the system. In Section 3, we present cosmological applications. Finally, we summarize our results in Section 4.

2. Bianchi-I cosmology in $f(R)$ gravity with anisotropic fluid

The action describing $f(R)$ gravity is expressed as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M, \tag{1}$$

where $\kappa = 8\pi G$, $c = \hbar = 1$, and S_M stands for the usual matter contributions to the totalaction.

We consider the homogeneous and anisotropic Bianchi-I metric, which can be conveniently cast for our purposes in the following form[29–31]

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} e^{2\beta_i(t)}(dx^i)^2, \tag{2}$$

where $a(t)$ is the average scale factor and the three functions β_i, which characterize the anisotropies, are such that $\beta_1 + \beta_2 + \beta_3 = 0$. It is more convenient to employ the variables

$$\beta_\pm = \beta_1 \pm \beta_2. \tag{3}$$

The total amount of anisotropy in the metric (2) is given by the quantity

$$\sigma^2 = \dot\beta_1^2 + \dot\beta_2^2 + \dot\beta_3^2 = \frac{3}{2}\dot\beta_+^2 + \frac{1}{2}\dot\beta_-^2. \tag{4}$$

For $\sigma = 0$, one can show that the spatial coordinates x^i can be suitably rescaled to recast the Bianchi-I metric in the standard Friedmann-Lemaitre-Robertson-Walker

(FLRW) form. The Ricci scalar for the metric (2) reads

$$R = 6\dot{H} + 12H^2 + \sigma^2, \tag{5}$$

where the average Hubble parameter H is given by the standard expression

$$H = \frac{\dot{a}}{a}. \tag{6}$$

We assume the presence of an anisotropic barotropic fluid with energy momentum tensor parametrized as[32]

$$T_\mu^\nu = \mathrm{diag}\left(-\rho, p_1, p_2, p_3\right) = \mathrm{diag}\left(-\rho, \omega_1\rho, \omega_2\rho, \omega_3\rho\right), \tag{7}$$

and we define the anisotropic equation of state as

$$p_i = (\omega + \mu_i)\rho, \tag{8}$$

with $i = 1, 2, 3$, where ω is the average barotropic parameter and $\omega_i = \omega + \mu_i$, with $\mu_1 + \mu_2 + \mu_3 = 0$ by construction. We parameterize our fluid by the constants ω and $\mu_\pm = \mu_1 \pm \mu_2$.[31]

The dynamics of the Bianchi-I metric (2) under $f(R)$ gravity action (1), in the presence of and anisotropic barotropic fluid with energy-momentum tensor (7), can be described by the following set of equations,[31]

$$3H^2 = \frac{\kappa}{f'}\left(\rho + \frac{Rf' - f}{2\kappa} - \frac{3Hf''\dot{R}}{\kappa}\right) + \frac{\sigma^2}{2} \quad , \tag{9}$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left(\omega\rho + \frac{\dot{R}^2 f''' + \left(2H\dot{R} + \ddot{R}\right) f''}{\kappa} - \frac{Rf' - f}{2\kappa}\right) - \frac{\sigma^2}{2}, \tag{10}$$

$$\ddot{\beta}_\pm + \left(3H + \frac{\dot{R}f''}{f'}\right)\dot{\beta}_\pm = \frac{\kappa\rho}{F}\mu_\pm, \tag{11}$$

$$\dot{\rho} + \left(3H\left(1 + \omega\right) + \boldsymbol{\delta}\cdot\dot{\boldsymbol{\beta}}\right)\rho = 0, \tag{12}$$

where $i = 1, 2, 3$, and

$$\boldsymbol{\delta}\cdot\dot{\boldsymbol{\beta}} = \mu_1\dot{\beta}_1 + \mu_2\dot{\beta}_2 + \mu_3\dot{\beta}_3 = \frac{3}{2}\mu_+\dot{\beta}_+ + \frac{1}{2}\mu_-\dot{\beta}_-. \tag{13}$$

Notice that in the presence of a perfect fluid, we have $\mu_+ = \mu_- = 0$ and the two equations (11) for β_+ and β_- can be substituted with

$$\dot{\sigma} + \left(3H + \frac{\dot{R}f''}{f'}\right)\sigma = 0. \tag{14}$$

In this case, there is no anisotropy in the matter sector and the single variable σ is sufficient to describe the total amount of metric anisotropy in the system.

In general, we have four functions of time $H(t)$, $\rho(t)$, $\beta_{\pm}(t)$ governing the dynamics. The existence of the constraint equation (9) implies that only three of them are indeed independent. Without loss of generality, we can choose them to be, for instance, $H(t)$ and $\beta_{\pm}(t)$. Given some specific form of the function $f(R)$, they can be determined by solving equations (10) and (11). The fluid energy density $\rho(t)$ can then be found using the energy constraint (9).

The traditional expansion-normalized variables were initially introduced for a better dynamical analysis of the standard FLRW model, see, e.g., Ref.[21]. Here, we expand the variables already introduced in[22,23] to include the case of the anisotropic barotropic fluid (7). In this regard, let us introduce the monotonically increasing variable

$$N = \epsilon \ln a, \tag{15}$$

known as the logarithmic time, where ϵ is defined to be $+1$ for expanding universe and -1 for a contracting one. Without loss of generality, we choose the scale factor at $t = 0$ to be $a_0 = 1$. Therefore, as time progresses in the forward (positive) direction, the logarithmic time N becomes positive and goes towards $+\infty$ in case of both expanding and contracting universes. One can notice that

$$\dot{N} = \epsilon H, \tag{16}$$

so that $\dot{N}$ is effectively always positive, justifying the use of N as the dimensionless evolution variable for both expanding and contracting universes. On the other hand, around a bounce or a turnaround point, this argument is not valid though and the expanding and contracting branches must be considered separately.

The expansion-normalized dynamical variables suitable for the equations (9) - (12) are the following dimensionless combinations

$$u_1 = \frac{\dot{R} f''}{f' H},\ u_2 = \frac{R}{6H^2},\ u_3 = \frac{f}{6f'H^2}, \tag{17}$$
$$u_4^+ = \frac{\dot{\beta}_+^2}{4H^2},\ u_4^- = \frac{\dot{\beta}_-^2}{12H^2},\ u_5 = \frac{\kappa\rho}{3f'H^2}.$$

in terms of which the energy constraint (9) reads simply

$$g = 1 + u_1 - u_2 + u_3 - u_4^+ - u_4^- - u_5 = 0, \tag{18}$$

from where we have that one of the expansion-normalized variables can always be eliminated. Unless otherwise stated, we always choose the matter content variable u_5 to be expressed in terms of the others dynamical variables. The variable

$$u_4 = u_4^+ + u_4^- = \frac{\sigma^2}{6H^2} \tag{19}$$

is also relevant for our purposes. It is important to stress that the variables u_4^+ and u_4^- are both non-negative by construction. Now, let us introduce the quantity

$$\gamma(R) = \frac{f'}{Rf''}, \tag{20}$$

which contains the information about the form of $f(R)$. Knowing the form of $f(R)$, γ can be determined in terms of the dynamical variables u_2, u_3 by inverting the relation

$$\frac{u_2}{u_3} = \frac{Rf'}{f}. \tag{21}$$

We return to the question of the invertibility of (21) in the last section. The 5-dimensional system of autonomous first order differential equations fully equivalent to (10) - (12) is given by

$$\begin{aligned}
\epsilon\frac{du_1}{dN} &= 1 + u_2 - 3u_3 - u_4 - 3\omega u_5 \\
&\quad -u_1\left(u_1 + u_2 - u_4\right), && (22)\\
\epsilon\frac{du_2}{dN} &= u_1 u_2 \gamma\left(\frac{u_2}{u_3}\right) - 2u_2\left(u_2 - u_4 - 2\right), && (23)\\
\epsilon\frac{du_3}{dN} &= u_1 u_2 \gamma\left(\frac{u_2}{u_3}\right) - u_3\left(u_1 + 2u_2 - 2u_4 - 4\right), && (24)\\
\epsilon\frac{du_4^+}{dN} &= -2u_4^+\left(1 + u_1 + u_2 - u_4\right) + 3\mu_+\sqrt{u_4^+}u_5, && (25)\\
\epsilon\frac{du_4^-}{dN} &= -2u_4^-\left(1 + u_1 + u_2 - u_4\right) + \mu_-\sqrt{3u_4^-}u_5, && (26)\\
\epsilon\frac{du_5}{dN} &= -u_5\Bigg(3\omega - 1 + u_1 + 2u_2 - 2u_4 && (27)\\
&\quad +3\mu_+\sqrt{u_4^+} + \mu_-\sqrt{3u_4^-}\Bigg).
\end{aligned}$$

Notice that differentiating (18) with respect to N and using the equations (22)-(27), we have

$$\epsilon\frac{dg}{dN} = -(u_1 + 2u_2 - 2u_4^+ - 2u_4^- - 1)g, \tag{28}$$

showing that the constraint $g = 0$ is indeed conserved along the solutions of our equations and the system (22) - (27) is effectively 5-dimensional.

The case of $f(R) = R^{1+\delta}$, with $\delta \neq 0$, is particularly important in our next examples. For this choice of $f(R)$, one has simply

$$\gamma = \delta^{-1}, \tag{29}$$

and the equations (23) and (24) can be considerably simplified. In this case, the right-handed side of the equations (22) - (27) involves only second degree polynomials in u_1, u_2, and u_3, and forth degree in $\sqrt{u_4^-}$ and $\sqrt{u_4^+}$. Hence, the task of finding the fixed points of our system reduce to finding polynomial roots, which may be performed in general with good accuracy. Notice that there are other relevant choices for $f(R)$ leading to polynomial fixed points. Besides of the trivial extension $f(R) = \alpha R^{1+\delta} + \Lambda$, with α and Λ constants, for which (29) also holds. For $f(R) = R^a + \alpha R^b$, with $a \neq b$ constants, we have

$$\gamma = \frac{u_2}{(b+a-1)u_2 - abu_3}. \tag{30}$$

Notice that, as in the exponential case, the function γ does not depend on the parameter α. This, of course, does not mean that the dynamics in insensitive to the value of α, since the expansion-normalized variables (17) depend explicitly on α. The case $a = 1$ and $b = 2$ is the original Starobinsky inflationary scenario,[4] and for the vacuum case our approach reduces to that one considered recently in.[33]

3. Applications to cosmology

In the following, we consider the expanding universes ($\epsilon = 1$). We investigate the case $f(R) = R^{1+\delta}$, whose main motivations from a cosmological perspective can be found in,[22,23,34,35] for instance. The case with $\delta = 0$ is obviously pure GR, for which the corresponding system is lower-dimensional, and our approach simply does not apply. The case logarithmic case $f(R) = \ln R$ must be treated separately. Hence, we start considering $\delta \neq 0$ and $\delta \neq -1$. Since we will deal with vacuum solutions, we set $u_5 = 0$ in the equations (18) and (22) - (27). In this case, notice that (25) and (26) can be combined in only one equation for u_4. We can use (18) to write u_3 as

$$u_3 = u_2 - u_1 + u_4 - 1, \tag{31}$$

and we are left with only three dynamical variables u_1, u_2, and u_4. Now, there is an interesting point to notice[36] about the specific choice $f(R) = R^{1+\delta}$, with $\delta \neq -1$, namely that

$$\frac{u_2}{u_3} = \frac{Rf'}{f} = 1 + \delta, \tag{32}$$

which combined with the constraint (31) implies

$$\delta u_2 = (1+\delta)(u_1 - u_4 + 1), \tag{33}$$

and we are left in fact with a two-dimensional phase space spanned by the variables u_1 and u_4. The corresponding dynamical equations in this case are

$$\frac{du_1}{dN} = \phi_1(u_1, u_4) \tag{34}$$
$$= -\delta^{-1}(1+2\delta)(u_1 - u_1^*)(u_1 - u_4 + 1),$$

$$\frac{du_4}{dN} = \phi_4(u_1, u_4) \tag{35}$$
$$= -2\delta^{-1}(1+2\delta)u_4\,(u_1 - u_4 + 1),$$

where

$$u_1^* = \frac{2(\delta - 1)}{1 + 2\delta}. \tag{36}$$

The phase space (u_1, u_4) associated with the system (34) - (35) has some interesting features. For instance, it has an one-dimensional invariant subspace (a continuous line of fixed points) corresponding to the straight line $u_1 - u_4 = -1$. However, from (31) we have that $u_3 = u_2$ on this line, which implies from (17) and (32) that $R = 0$ on $u_1 - u_4 = -1$. Besides of this invariant straight line, we find the isolated fixed $(u_1^*, 0)$, for $\delta \neq -\frac{1}{2}$. The case $\delta = -\frac{1}{2}$ is also discussed separately.

The stability of the isolated fixed point can be inferred from the linearization of (34) - (35). The Jacobian matrix of (34) - (35) at the point $(u_1^*, 0)$ reads

$$\left(\frac{\partial(\phi_1, \phi_4)}{\partial(u_1, u_4)} \right) = -\delta^{-1}(4\delta - 1) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \tag{37}$$

from where we see that such fixed point is stable for $\delta > \frac{1}{4}$ or for $\delta < 0$. For the stability of the invariant straight line, we can consider the divergence of the vector field (ϕ_1, ϕ_2). One has

$$\nabla \cdot \boldsymbol{\phi} = \frac{\partial \phi_1}{\partial u_1} + \frac{\partial \phi_4}{\partial u_4} = \delta^{-1}((1 + 2\delta)u_4 + 4\delta - 1) \tag{38}$$

on the invariant line. Recalling that $u_4 \geq 0$, we find that the invariant line is entirely repulsive (positive divergence) for $\delta > \frac{1}{4}$ or for $\delta \leq -\frac{1}{2}$. For $-\frac{1}{2} < \delta \leq \frac{1}{4}$, we can obtain some attractive segments, depending on the value of u_4. We return to the physical interpretation of this $R = 0$ invariant line in a following sub-section. The case $\delta = -\frac{1}{2}$ is particularly curious, since the isolated fixed point is absent and we get a second one-dimensional invariant line, namely $u_1 = 0$, which is also entirely repulsive. On the other hand, the case $f(R) = \ln R$ cannot be incorporated in the present analysis since (32) is not valid for $\delta \to -1$, and in fact we have a three-dimensional phase space for such case.

The solutions of (34) and (35) are curves on the plane (u_1, u_4), and it turns out that such curves can be determined analytically. Notice that the solutions are such that

$$\frac{u_4'}{u_1'} = \frac{2u_4}{u_1 - u_1^*}, \tag{39}$$

which can be integrated as

$$u_4 = c\,(u_1 - u_1^*)^2\,, \tag{40}$$

with arbitrary c. Thus, the phase space trajectories of all solutions of (34) and (35) are simply parabolas centered in the isolated fixed point, irrespective of the value of δ, provided the fixed point exists. Since we know the trajectories graphs, one can infer the dynamics direction and, consequently, the dynamical properties of the fixed point and the invariant line, directly form the equations (34) and (35) as follows. Consider the phase space function $L = u_1 - u_4 + 1$. It is clear that $L = 0$ is the invariant line. On the other hand, $L = c$ constant is a parallel line located below the invariant line if $c > 0$, or above if $c < 0$. The invariant line is the boundary between two semiplanes with reverse dynamics direction, and the dynamical properties of the

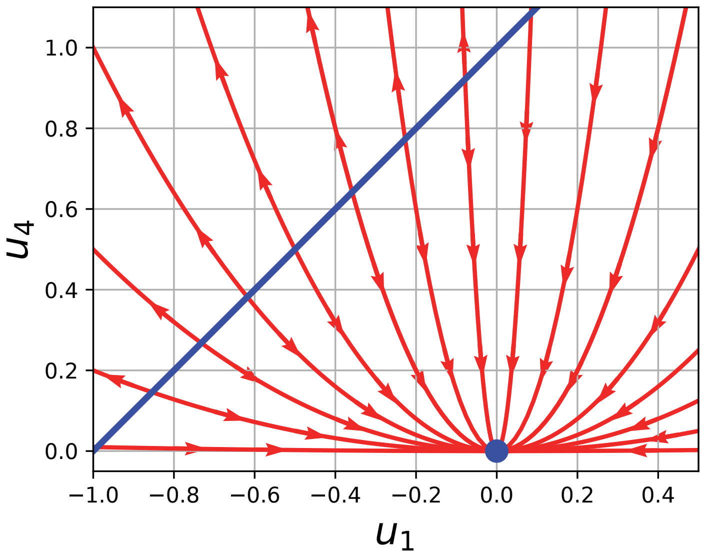

Fig. 1. Phase space for the system (34) - (35), for $\delta = 1$. The fixed point $(0, 0)$ is located in the semiplane below the critical line. The solutions are restricted to parabolas centered in the attractive fixed point. The region below the invariant line corresponds to the attraction basin of the fixed point. Any solution starting there tends to the fixed point asymptotically. All solutions starting in the region above the critical line will diverge to infinity. Notice that the critical line is entirely repulsive. Such phase space is rather generic, it is essentially the same for all theories of the type $f(R) = R^{1+\delta}$ such that the fixed point is attractive and is located below the invariant line.

fixed point and of the invariant line depend on the relative position between then, see Figs 1 and 2, which correspond, respectively, to the cases $\delta = 1$ and $\delta = \frac{1}{10}$. The former is the important case of the Starobinsky's inflationary scenario with $f(R) = R^2$.

We note that the solutions are constrained to the parabolas (40), the exact solutions of (34) and (35) boils down to a simple quadrature of a rational function

$$\frac{d\bar{u}_1}{c\bar{u}_1^3 - \bar{u}_1^2 - (u_1^* + 1)\bar{u}_1} = -\delta^{-1}(1 + 2\delta)dN, \tag{41}$$

with $u_1 = \bar{u}_1 + u_1^*$. For the case $\delta = -\frac{1}{2}$, u_4 is a constant and (34) also reduces to a simple rational quadrature. It has been shown that the vacuum solutions for the $f(R) = R^{1+\delta}$ case, for $\delta \neq -1$, are exactly soluble.

Since the stable fixed points of a cosmological model correspond to the cosmological histories which dominate the asymptotic evolution of the system, it worth to look more closely on them. By using (31) and (32), we see that the isolated fixed points are given by

$$u_3 = \frac{4\delta - 1}{\delta(1 + 2\delta)}, \tag{42}$$

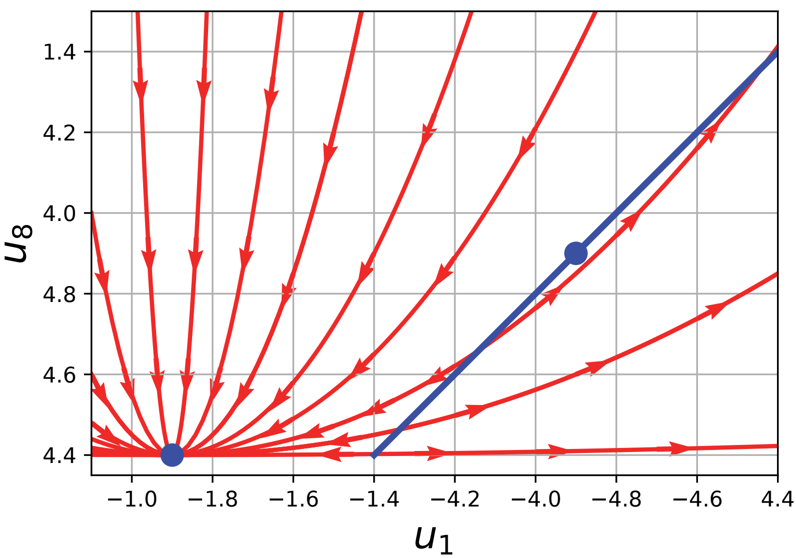

Fig. 2. Phase space for the system (34) - (35), for $\delta = \frac{1}{10}$. The fixed point $\left(-\frac{3}{2}, 0\right)$ is now located in the semiplane above the critical line. The solutions are also restricted to parabolas centered in the attractive fixed point. However, the attraction basin of the fixed point is now in the region above the critical line. Notice that the invariant line in this case has an attractive and a repulsive segment located, respectively, above and below the depicted point $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The divergence (38) always vanishes in limit points between attractive and repulsive segments like this one.

with $\delta > \frac{1}{4}$ or $\delta < 0$. From the definition of u_3 and (5), we have that (42) implies that

$$\dot{H} = \Delta H^2, \tag{43}$$

where

$$\Delta = \frac{\delta - 1}{\delta(1 + 2\delta)}. \tag{44}$$

It is clear that for $\delta = 1$, the stable fixed point corresponds to de Sitter solution with $a(t) = e^{Ht}$, with constant H. (The case $H = 0$ corresponds to the flat Minkowski spacetime). This is namely the well known Starobinsky's inflationary solution. For $\delta \neq 1$, the solutions are

$$H(t) = \frac{H_0}{1 - \Delta H_0(t - t_0)}, \tag{45}$$

where $H(t_0) = H_0$, which interpretation is straightforward. For $\Delta > 0$, which corresponds to $-\frac{1}{2} < \delta < 0$ or $\delta > 1$, we have a future finite time big rip singularity, while for $\Delta < 0$ ($\delta < -\frac{1}{2}$ or $0 < \delta < 1$), the Hubble parameter H decreases as t^{-1} for large t, *i.e.*, the solution asymptotically tends to a power law expansion.

The Starobinsky's R^2 inflationary scenario is unique among the $F(R) = R^{1+\delta}$ theories of gravity, since only for $\delta = 1$ the stable de Sitter fixed point $(0, 0)$ is available, a result indeed known for a long time, see,[24–26] for example. We can, however, easily prove a stronger result for generic $f(R)$ theories. The de Sitter solution $a(t) = e^{Ht}$, with constant and arbitrary H, implies $u_1 = u_4 = 0$, and also

$$R = 12H^2, \tag{46}$$

which, on the other hand, determine that $u_2 = 2$ and $u_3 = \frac{2f}{Rf'}$ and, hence, the constraint (18) reads

$$Rf'(R) = 2f(R). \tag{47}$$

Since we assume that de Sitter solution exists for arbitrary H, we have from (46) that it should exist for any $R > 0$, and hence equation (47) can be seen as a ordinary differential equation for $f(R)$, which unique solution is $f(R) = \alpha R^2$, establishing in this way a stronger result: the case R^2 is unique among all vacuum $f(R)$ theories with respect to the existence of a de Sitter solution with arbitrary H. The condition (47) was first obtained by Barrow and Ottewill in[24] by using a more intricate approach, but here we see that it appears from a very simple analysis of fixed points.

It is noted that in the case with anisotropic fluids, all isotropic fixed points are unstable (there is no asymptotically stable isotropic solutions in the presence of anisotropic matter). It is also mentioned that several known results as the existence of vacuum Kasner-like solutions for $-\frac{1}{2} \le \delta \le \frac{1}{4}$[34, 35] are found. Moreover, for the case of exponential gravity,[37–40] $f(R) = e^{\alpha R}$, all isotropic fixed points in the presence of an anisotropic barotropic fluid are unstable.

4. Summary

In this article, we have introduced a new set of expansion-normalized variables for homogeneous and anisotropic Bianchi-I spacetimes in $f(R)$ gravity in the presence of anisotropic matter. In terms of these new dynamical variables, the full set of Einstein's equations boils down to a 5-dimensional phase space. As applications of the proposed dynamical approach, we have explicitly explored the $f(R) = R^{1+\delta}$ modified theory of gravity, and shown that its vacuum dynamics is exactly solvable. Furthermore, in a easier and more direct way, we have re-obtained several well known results for this particular choice of $f(R)$ such as Bleyer and Schmidt isotropic solutions.[41–43] We have also extended a uniqueness result for Starobisnki inflationary scenario, namely that the case R^2 is unique among all vacuum $f(R)$ theories with respect to the existence of a de Sitter solution with arbitrary H, a result obtained previously by Barrow and Ottewill by using a more intricate approach.[24]

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