



Rekindling of de Broglie–Bohm Pilot Wave Theory in the Late Twentieth Century: A Personal Account

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Abstract

David Bohm published his “Suggested Interpretation of Quantum Theory in Terms of Hidden Variables” some twenty five years after Louis de Broglie first presented his similar Pilot Wave theory of quantum mechanics. In the following 30 years what became known as the de Broglie–Bohm approach to quantum theory was to a large extent ignored within the physics community. Even David Bohm himself became somewhat disillusioned with the lack of impact of his interpretation of quantum theory and he directed his interest elsewhere. But some 27 years after Bohm had published his interpretation of quantum theory, interest was rekindled in part by new, detailed calculations that demonstrated clearly and graphically, exactly how his interpretation explained quantum phenomena in terms of well defined individual particle trajectories. These computations encompassed two-slit interference, quantum tunnelling, neutron interferometry, Wheeler’s delayed choice experiment, orbital and intrinsic angular momentum, quantum measurement and Einstein–Podolsky–Rosen nonlocal correlations for orbital angular momentum, intrinsic angular momentum and correlated particle interferometry. Since then, the acceptance of the validity of de Broglie–Bohm theory has steadily grown, as has the interest in the consequences of the approach. For my contribution to the current celebratory volume I was asked to provide a personal review specifically of this novel work within its historical context of the last quarter of the twentieth century.

Keywords Quantum trajectories · Quantum potential · Tunnelling · Quantum measurement

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1 Introduction

During my undergraduate joint degree in Physics and Philosophy, undertaken at the University of Warwick (1970–1973), neither in quantum theory courses nor the philosophy courses, had there been discussion of the work of Louis de Broglie and David Bohm in the interpretation of quantum theory. The situation was similar in my postgraduate course in History and Philosophy of Science, undertaken at Cambridge in the mid seventies. At Cambridge, I wrote my dissertation on Niels Bohr's interpretation of quantum theory within the context of his wider philosophical background. It was clear to me that Bohr did not develop his philosophical approach to the interpretation of quantum theory on the basis of the physics, but rather, he saw within the quantum theory an opportunity to impose his particular philosophical predilections. Bohr has not been alone in doing this. The remarkable success of quantum theory rests on its ability to predict precisely the possible outcomes of experiments and their statistics. Nobody doubts this, but the fact that the standard formalism itself gives no clue as to how the individual observed results, and their statistical distributions, actually come about allows all-comers to exploit the latitude afforded to find support for their particular philosophical positions. Bohr argued that any ambiguity in the division of the world into subject and object, in any context, imposes the demand for complementary description. Different such divisions result in incompatible, but equally valid, complementary descriptions. In the context of quantum theory, Bohr concluded that "there is no quantum world, there is only an abstract quantum physical description". For Bohr, the task of physics is not to say how the world actually is, physics must only be concerned with what we can legitimately say, within the limitations of the classical conceptual scheme that we must use to ensure unambiguous communication. As I found later, the argument that Bohr's views are not an inescapable consequence of the quantum formalism is strongly supported by the existence of the de Broglie–Bohm interpretation of quantum mechanics.

I joined David Bohm's small theoretical physics group at Birkbeck College in London in 1977 as a PhD student. The intellectual environment in Bohm's group was open, both sceptical and tolerant. Bohm attracted visitors from around the world keen to discuss his wider ranging philosophical ideas. At that time in Bohm's group there was a rather small, diverse group of part time, self-funded research students who turned up sporadically, mainly in the evenings. Research students were not strongly directed and worked independently on various mathematical approaches to quantum theory ranging from category theory and algebraic topological methods, to catastrophe theory. The permanent faculty members of the group were just David Bohm and Basil Hiley. There was no discussion of Bohm's 1952 hidden variables theory [1, 2]. Instead, the main motivation in Bohm's group at that time was to develop a mathematical formulation of the process of explication, or becoming, within the implicate order as a new paradigm for physics. Obtaining funding to work in Bohm's group during that period was almost impossible, there was no traditional Research Council funding, no full time PhD students and no postdocs. But having taught for 2 years I was eligible

for a personal Science Research Council “Instant Award” and this funded my doctoral studies, full time. Birkbeck only admitted part-time undergraduate students and so David Bohm gave undergraduate lectures on quantum theory, that I was happy to attend, in the evening to a small group of students, often in his office, frequently referring to the well-thumbed manuscript of his Quantum Theory text book whilst scribbling notes on his small blackboard. These were no ordinary undergraduate lectures, they were informal and largely discursive, going well beyond traditional quantum theory courses with the conceptual understanding of the wide ranging ideas he discussed of foremost importance. Even more informally, there were also occasional seminars, that took place in the physics post-graduates’ room, again on wide ranging and speculative ideas, often delivered by people visiting Bohm.

My interest in Bohm’s 1952 “Causal Interpretation of Quantum Theory” was stimulated, not within Bohm’s group, but whilst browsing in Dillon’s bookshop opposite Birkbeck College, by a chance encounter with J. F. Belinfante’s 1973 book entitled “A survey of hidden variables theories” [3]. Belinfante devotes a thirty-two page chapter to what he refers to as “Bohm’s 1951 theory” and has a further eight appendices in which he works out some of the details for the non relativistic theory and the extension of the approach to quantum electrodynamics. I stood for a long while in the bookshop reading, quite shocked that nobody at Birkbeck at that time really had any interest in this work, before deciding to buy the book. It became clear to me that Bohm had provided a clear counter example for many of the claims concerning that which quantum theory had supposedly, inescapably taught us. Contra Bohr, and every quantum theory text book at the time, here was an approach to quantum mechanics in which there was a definite world of particles following trajectories and well-defined fields evolving in space and time. In de Broglie–Bohm theory there is no need for a new epistemology, the focus instead is on a modified ontology. Here was a version of quantum theory that did tell how the world is, or might be. By simply mathematically decomposing the Schrödinger equation into a continuity equation for the position probability density and an equation that bore a resemblance to the classical Hamilton–Jacobi equation but with an additional quantum potential term, Bohm had recast quantum mechanics in a form as close as was possible to classical mechanics, but not in a vain attempt to reinstate Newton. Bohm’s motivation was not reactionary, he did not wish to reinstate a classical world, albeit with hitherto unknown forces, instead, presenting quantum theory in this new form was designed to bring into sharp focus the revolutionary novelty of quantum mechanics.

Bohm did not believe his interpretation was the final word, rather he sought to exploit its radical clarity to guide the search for a deeper theory of implicate processes from which space, time and quantum theory would emerge. Bohm’s quantum world was well defined, but it exhibited a wholeness completely absent in classical physics. The wholeness finds concrete expression in the contextuality and non locality so strikingly apparent in Bohm’s theory. In Newtonian particle mechanics, once the positions and momenta of a set of particles are specified, along with any forces (which are specified as preassigned functions of the coordinates) acting between them, the motion of the system is completely determined. In the same scenario, in Bohm’s theory, one needs to specify the positions of

the particles and the classical forces acting but, crucially, it is also necessary to specify the wave function of the system within its configuration space. The momenta of the particles, along with their other properties, are not freely assignable, as they are in Newtonian mechanics, instead they are fixed by the configuration space wave function. In Bohm's theory, the actual configuration of the system evolves from a well-defined (but uncontrollable) initial configuration along a unique trajectory in configuration space under the influence of non local forces of a quantum origin whose form is governed by the wave function (the quantum forces are not preassigned functions of the coordinates).

Classical mechanics may also be formulated within a configuration space, but it need not be: the theory can be completely specified in ordinary space. Quantum mechanics, on the other hand, *cannot* be formulated in ordinary three dimensional space and this is possibly its most revolutionary characteristic. It is only the fact that for a single particle configuration space is identical with ordinary three space that leads to the rather misleading, informal picture of the Schrödinger wave moving through our everyday space and time as do classical waves. A set of similar particles has a configuration $(\mathbf{x};t)$, where for n particles $(\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2 \dots \mathbf{x}_n)$. The complex probability amplitude for a configuration, \mathbf{x} , evolves continuously governed by the Schrödinger wave function $\psi(\mathbf{x}, t)$. In the non relativistic case the wave function evolves according to the configuration space Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = \hat{H}\psi(\mathbf{x}, t) \quad (1)$$

where \hat{H} is the system's Hamiltonian operator. The actual configuration of a given system is unknown and uncontrollable but nonetheless is assumed, in de Broglie–Bohm theory, to be well defined. The Schrödinger equation allows the calculation of the time evolution of the wave function and consequently of the probabilities of different configurations of the system according to Born's rule. Bohm's reformulation allows the calculation of how a definite initial configuration changes with time as it carves out a specific trajectory in the configuration space. All of the predictions of standard quantum theory are reproduced in Bohm's theory with individual results furnished with a unique, individual back story. The form of the configuration space trajectory, and by projection into real space, that of the individual particle trajectories, is determined according to the guidance formula originally introduced by de Broglie, specifying the velocity, \mathbf{v} , which may be written

$$\mathbf{v} = \frac{\hbar}{m} \Im \left(\frac{\nabla \psi}{\psi} \right) \quad (2)$$

Given the configuration space wave function ψ as the solution of Schrödinger's equation, Eq. 2 is all that is needed to determine the evolution of all the coordinates of the particles (and by extension fields) that make up the system under study. The version of de Broglie–Bohm theory based solely on the first order in time Eq. 2 is often referred to as the first-order theory, or pilot wave theory, and all of the trajectory calculations that I carried out, as a practical matter, simply calculated ψ and then integrated Eq. 2 for an ensemble of initial configurations.

In 1952, Bohm gave a second order in time account of quantum motion, couched in terms of non local potentials, forces and accelerations that operate in everyday space. Bohm proceeded by substituting

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t)e^{\frac{i}{\hbar}S(\mathbf{x}, t)}, \quad (3)$$

where R and S are real functions, into the Schrödinger equation (1) and separating the real and imaginary parts, whence the motion can be described in a way, similar in some respects, to the Hamilton–Jacobi form of classical mechanics. One finds:

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{1}{2m}(\nabla S(\mathbf{x}, t))^2 + U + Q = 0, \quad (4)$$

where U represents the potentials associated with any classical forces acting and Q plays the role of a potential (the quantum potential) and is given by

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R(\mathbf{x}, t)}{R(\mathbf{x}, t)} \quad (5)$$

and

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla \cdot \left(\rho(\mathbf{x}, t) \frac{\nabla S(\mathbf{x}, t)}{m} \right) = 0, \quad (6)$$

that plays the role of a continuity equation for the configuration space probability density

$$\rho(\mathbf{x}, t) = \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t). \quad (7)$$

In this second order account of de Broglie–Bohm theory, a force arises which has both classical and quantum components, dependent on both classical and quantum potentials

$$\mathbf{F} = -\nabla(U + Q). \quad (8)$$

The force acting on an individual particle depends on the classical potentials U (distinguished by the fact they are fixed, pre assigned functions of the particle coordinates) and on the quantum potential whose functional form is not pre assigned but rather depends on the configuration space wave function.

The much debated “paradoxes” and problems originally associated with attempts to understand quantum mechanics were expunged in de Broglie–Bohm theory, but hardly anybody at the time was prepared to use, or further develop, such a prosaic theory. The series of novel and detailed computations, reviewed here, illustrated in graphic detail exactly how de Broglie–Bohm theory was able to explain archetypical puzzling quantum phenomena in terms of a well-defined, deterministic and

continuous, but inescapably nonlocal and contextual world. The first computation at Birkbeck in 1978 was of particle trajectories for the two slit experiment. Then, inspired by the film loops of wave packet scattering produced by Goldberg, Schey and Schwarz using a numerical integration of the Schrödinger equation [4], I generated similar motion pictures that illustrated how de Broglie–Bohm theory could account for tunnelling and other square potential scattering phenomena. Furthermore, these calculations were easily extended, simply by changing the initial conditions in the numerical integration of the Schrödinger equation, to describe the behaviour of neutrons in interferometers. The explanations of quantum phenomena that we gave then were of the second-order in time, or pseudo Newtonian type: the deviations of the quantum trajectories from classical expectations were accounted for by the action of the quantum potential in accordance with Eq. 8. This presentation of the calculations in second order form simply followed the pattern of Bohm’s 1952 papers. I was amazed by the ease with which such a clear and distinct picture of the otherwise mysterious quantum world could be produced using de Broglie–Bohm theory. Extending the calculations to demystify ever more quantum puzzles became my obsession.

Describing these calculations at various conferences around the world, I was surprised by the fact that even if people had heard of de Broglie–Bohm theory, they were convinced it was somehow wrong or otherwise inadequate. By the 1980’s, Bohm himself had long since given up on presenting his 1952 theory at conferences, he once told me that he was tired of explaining that his approach was perfectly consistent, of answering, once again, ill thought out objections made by people who had not seriously studied his work, but still being ignored nonetheless. Many believed, with Max Born, that Bohm had been “slain not only philosophically but physically as well” [5]. In spite of John Bell’s demonstration of the contrary case [6], many assumed von Neumann had shown that all hidden variables theories were impossible, others that the theory was to be rejected because it was non local (!), arbitrary or inelegant. But the most common objection was that the experimental content was the same as that in standard quantum theory so there was nothing to be gained by adding unobservable trajectories. In regard to the last point Bohm himself had argued, ever since he first proposed his approach, that a similar criticism could equally have been applied to the standard interpretation of quantum theory. Imagine, he would say, that de Broglie’s approach had been adopted by the physics community in the 1920’s, quantum trajectories would never have been considered controversial, the notion of continuity and causality operating in a well defined non local reality would not be thought outlandish. Now, imagine that within this context it was proposed that quantum trajectories were not possible, that the only reality was at the classical level of the results of measurements. That particular outcomes were irreducibly just random events that nonetheless were statistically distributed according to Born’s rule. Only then, without trajectories, would all of the difficulties of interpretation of quantum theory arise.¹ However, recent experiments, for example [8] and [9], that exploit “weak” measurements of momentum, have produced

¹ Bohm’s argument was eloquently elaborated by Jim Cushing in his book entitled: “Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony” [7].

statistical reconstructions of “trajectories or flows” for the two-slit experiment similar to the individual particle trajectories deduced in the de Broglie–Bohm theory. But the interpretation of the aforementioned experiments is controversial [10], not least because they reconstruct photon “trajectories” (from ensemble measurements) where Bohm’s 1952 interpretation does not entail trajectories for quantum field “particles” such as photons.

As already mentioned above, the prevailing attitude in the physics community was such that funding was hard to come by for research on Bohm’s theory, his group at Birkbeck had no Research Council grants. I had even been advised by eminent physicists not to waste my career by working on Bohm’s ideas as I would never get a permanent academic position on the basis of such research work. It appeared that the end of my Science Research Council Instant Award would be the end of my research on de Broglie–Bohm theory. But fortunately, my work had been noticed by Jean-Pierre Vigi er. Vigi er had worked with Louis de Broglie and also published research with David Bohm in 1954 showing how the ψ^2 distribution would naturally arise as an equilibrium distribution as a result of the action of supposed sub-quantum fluctuations [11]. Bohm initially referred to his interpretation as “The Causal Interpretation” and Vigi er would refer to the modified de Broglie–Bohm theory as the “Causal Stochastic Interpretation”. He was a champion of the de Broglie–Bohm interpretation and tirelessly sought experiments that would verify it, or at least demonstrate its undoubted superiority. Vigi er visited Birkbeck in 1983 and, during discussion after his seminar, he invited me to work in his group at the Institute Henri Poincar e (IHP) in Paris. I was fortunate enough to obtain a British Royal Society European Fellowship to support my period in Paris. The Institute Henri Poincar e was where Louis de Broglie had worked, although there was no visible public recognition of this fact. His office and desk were still there and I was delighted to make use of them. Vigi er had de Broglie’s leather armchair in his office and he would always invite visitors to “sit in de Broglie’s chair”. Vigi er’s group was even smaller than Bohm’s, before I joined there were two post doctoral researchers there: Anastasios Kyprianidis and Dimitri Sardelis. Whilst in Vigi er’s group in Paris I extended the computations in de Broglie–Bohm theory to simulate the neutron interferometry experiments performed, for example, by Helmut Rauch’s group in Vienna [12]. In the latter part of my time in Paris, in collaboration with Peter Holland and Anastasios Kyprianidis, and afterwards continued following my appointment to a lectureship at what was then Portsmouth Polytechnic in England, using the approach of Bohm, Schiller and Tiomno (BST), the computations were extended to explain the behaviour of spin-one-half particles in spin superposition, spin measurement and Einstein, Podolsky and Rosen (EPR) scenarios. Computations were also carried out whilst at Portsmouth to describe the behaviour, according to Bohm’s approach, of quantum fields—including energy exchange between particle and field during a quantum transition, orbital angular momentum measurement, EPR entanglement of orbital angular momentum in hydrogen-like atoms and non locality in correlated particle interferometry. The key features of the de Broglie–Bohm theory that are illustrated by these calculations are discussed in some detail in the following.

2 Solutions for a Free Gaussian Wave Packet

In his appendix G, Belinfante studied the de Broglie–Bohm solutions for a free Gaussian packet (see [3], p. 195). This simple case is particularly interesting as the trajectories can be calculated analytically. It is also of particular interest since all of the early computations of de Broglie–Bohm trajectories were founded upon initial states that were coherent superpositions of time-dependent Gaussian wave packets. The solution of the Schrödinger equation for a free Gaussian wave packet with the form given in Eq. 9 at $t = 0$, is given by Eq. 10.

$$\psi_0(\mathbf{x}) = (2\pi\sigma_0^2)^{-\frac{3}{4}} \exp\left(i\mathbf{k}_0 \cdot \mathbf{x} - \frac{\mathbf{x}^2}{4\sigma_0^2}\right), \quad (9)$$

$$\psi_t(\mathbf{x}) = (2\pi\sigma_t^2)^{-\frac{3}{4}} \exp\left[i\mathbf{k}_0 \cdot \left(\mathbf{x} - \frac{1}{2}\mathbf{u}t\right) - \frac{(\mathbf{x} - \mathbf{u}t)^2}{4\sigma_0\sigma_t}\right], \quad (10)$$

where \mathbf{k}_0 is the central wave number, σ_0 is the dispersion at $t = 0$ and

$$s_t = \sigma_0 \left(1 + \frac{i\hbar t}{2m\sigma_0^2}\right). \quad (11)$$

In position space, the probability distribution is given by

$$\rho_t(\mathbf{x}) = (2\pi\sigma_t^2)^{-\frac{3}{2}} \exp\left[-\frac{(\mathbf{x} - \mathbf{u}t)^2}{2\sigma_t^2}\right], \quad (12)$$

with dispersion σ_t given by

$$\sigma_t = \sigma_0 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2\right]^{\frac{1}{2}}, \quad (13)$$

whereas in momentum space, the momentum probability distribution is given by

$$\rho_t(p) = (2\pi\sigma_p^2)^{-\frac{3}{2}} \exp\left[-\frac{(\mathbf{p} - \hbar\mathbf{k}_0)^2}{2\sigma_p^2}\right], \quad (14)$$

with time independent dispersion

$$\sigma_p = \frac{\hbar}{2\sigma_0}. \quad (15)$$

$\rho_t(x)$ and $\rho_t(p)$ are the distributions that would be found in a series of separate position and momentum measurements each carried out on an identically prepared

system with the wave function given in Eq. 10, at time t . The minimum uncertainty is satisfied only at $t = 0$, when $\sigma_t \sigma_p = \frac{\hbar}{2}$. The Bohm momentum is given by

$$\pi_t = \hbar \mathbf{k}_0 + \frac{(\mathbf{x} - \mathbf{u}t)\hbar^2 t}{4m\sigma_0^2 \sigma_t^2}, \quad (16)$$

which at $t = 0$ has the value \mathbf{k}_0 for any initial particle position. Thereafter, the Bohm momentum has a linear dependence on position retaining the value \mathbf{k}_0 at the packet centre. Writing $X = x - \mathbf{u}t$, Belinfante found the possible particle trajectories by integrating Eq. 16 resulting in

$$X = X_0 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2 \right]^{\frac{1}{2}}. \quad (17)$$

The measured momentum distribution is given by Eq. 14. This difference in the distributions underlines the fact that the values assigned in the Bohm theory to particular quantities derived from the wave function (such as momentum and spin) are, in general, not the eigenvalues of the corresponding operator, such as are found on measurement of the particular quantities. Of course, if the wave function is already an eigenfunction of an operator then the Bohm assigned values are the associated eigenvalues. This is possible as the Bohm theory has a dynamical theory of measurement that shows exactly how the assigned (non eigenvalue) values evolve to become the eigenvalues (or measured values) as the measurement process takes place. Clearly, in Bohm's theory, a measurement does not simply reveal a pre-existing value for the measured quantity. Instead, a quantum measurement introduces an interaction that entangles the measured system with the measuring apparatus, both of which are treated quantum mechanically, in such a way that by observing the apparatus coordinate it is possible to infer the corresponding value associated with the measured system.

For the case of the Gaussian wave packet, the distribution of the Bohm momentum is given by

$$\rho(\pi_t) = (2\pi\sigma_{\pi t}^2)^{-\frac{3}{2}} \exp \left[-\frac{(\pi_t - \hbar \mathbf{k})^2}{2\sigma_{\pi t}^2} \right]^{-\frac{1}{2}}, \quad (18)$$

with dispersion

$$\sigma_{\pi_t} = \sigma_p \left[1 + \left(\frac{2m\sigma_0^2}{\hbar t} \right)^2 \right]^{-\frac{1}{2}}, \quad (19)$$

which as $t \rightarrow \infty$ approaches the measured momentum distribution, whereas at the same time the position distribution spreads over all space. The position-momentum uncertainty relation clearly does not apply to the Bohm distributions of positions

and momenta, but it does apply, in Bohm's theory, as it does in standard quantum theory, to the scatter in the results of position measurements and momentum measurements carried out on different members of an ensemble of identically prepared systems.

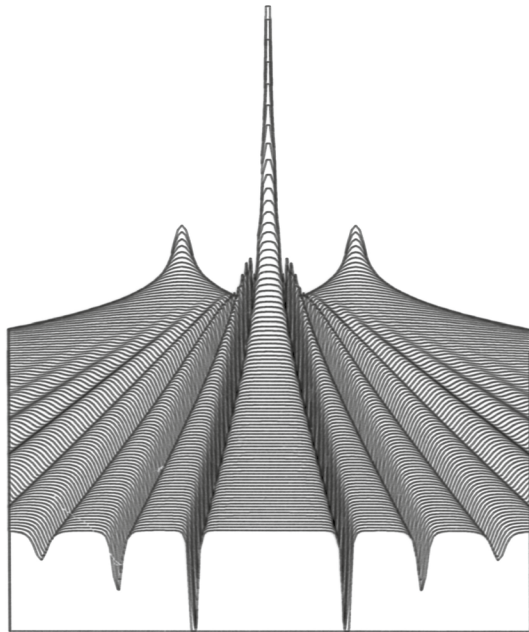
Consider the case in which the initial spatial wave packet is held, at $t = 0$, in the ground state of a narrow harmonic oscillator potential around at $x = 0$, with zero central momentum. Then, the initial state is a minimum uncertainty Gaussian wave packet with central momentum, $p_0 = 0$. In this case, at $t = 0$, the Bohm momentum for any initial particle position is just the central momentum. On removal of the potential this wave packet spreads very rapidly. Time-of-flight measurements could then be employed to estimate the momentum. Initially, there is no correlation between the position of the particle and its momentum but, during the free evolution of the wave packet, the position of the particle becomes correlated with its momentum according to Eq. 16. In this "measurement" example, the measured value is the momentum and the apparatus coordinate is the position of the particle, since by observing the position at time t one can deduce (an approximate) value of the momentum. Bohm showed that all quantum measurements follow this pattern, ultimately position is measured and the measurement interaction is designed to introduce a correlation between the measured value and the measuring apparatus pointer state (position); the system and the apparatus become entangled.

3 The Two Slit Experiment

On further reading I found that Belinfante has a section (2.7) with the heading "How to tell through which slit a particle came that is observed in the interference pattern behind several slits." He discusses the fact that in principle it is possible to integrate the Schrödinger equation through the slit system although he suggests "it would not be easy to really do it (say, on a computer)". In fact it was pretty easy once a simple model had been devised.

Over coffee one afternoon in the Birkbeck coffee shop I discussed Belinfante's work with Basil Hiley and a long standing part-time PhD student of David Bohm's named Chris Philippidis. Chris was aware of Bohm's hidden variables theory but had not worked on it, whilst, at first, Basil was not interested, as he thought (in common with most of the physics community) that the theory was deficient in some unspecified way, or just not of interest. We talked about the two-slit scenario and decided to develop a model that would allow the quantum potential and the trajectories between the slits and the screen to be plotted. Philippidis produced a quantum potential plot, whilst I calculated the trajectories using a simple model that had two separated and spreading, time dependent, one-dimensional Gaussian wave packets, as defined in Eq. 10, to model the wave function along the relevant axis, (parallel to the plane of the slits) in which the interference pattern developed. Motion perpendicular to the slits was parameterised by the time. There was no need to explicitly integrate the Schrödinger equation to get the wave function. Although anybody who had read Bohm's 1952 papers knew such trajectories were possible, actually

Fig. 1 The quantum potential for the two slit experiment



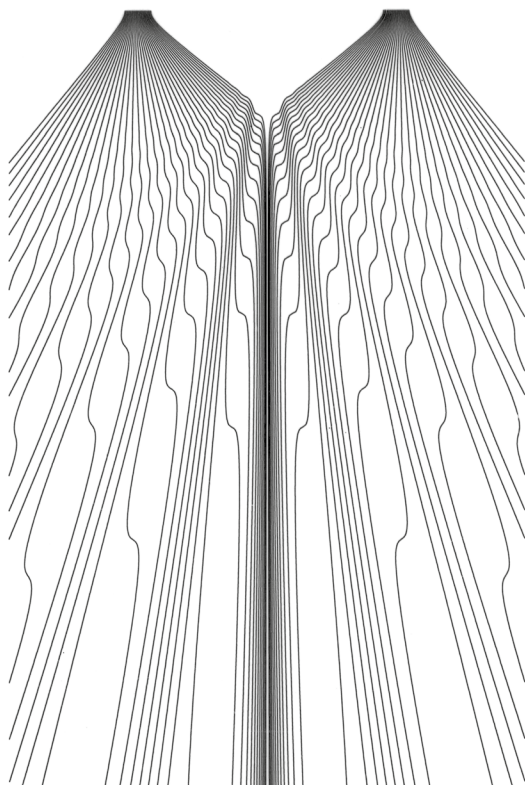
seeing them was quite different. Clearly, particle trajectories were not incompatible with the formation of wave-like interference patterns after all. Both Bohm and Hiley became quite excited when they first saw the quantum potential, Fig. 1 and trajectory, Fig. 2 plots. It was decided that we should publish in *Il Nuovo Cimento* [13], as we suspected (correctly at the time) that the establishment American and British journals would not be sympathetic to this type of research.

Feynman had suggested that the two-slit experiment encapsulates the only mystery of quantum mechanics and argued that it was impossible to “cook up” an explanation of the interference pattern in terms of hidden variables, he stated that “nature herself doesn’t know which way the electron was going to go”. In particular, Feynman argued that no such explanation could ever account for the loss of interference that occurs on detecting through which slit a particle passed. But, de Broglie–Bohm theory accomplished the “impossible” as had been clearly demonstrated in the plots of the trajectories for the two-slit experiment [13].² The same model as used in the two-slit calculations was also used later to discuss the de Broglie–Bohm point of view on Wheeler’s delayed choice experiment in 1985. The paper [14] used the two slit trajectories to illustrate the argument, although the crossed beam trajectories, reproduced in Fig. 3, were also available [15].

In Wheeler’s thought experiment, the two slits are imagined to be bonded onto the surface of a convex lens so that the beams emanating from each slit cross on their way to detectors (the two separate detectors are arranged so that they each

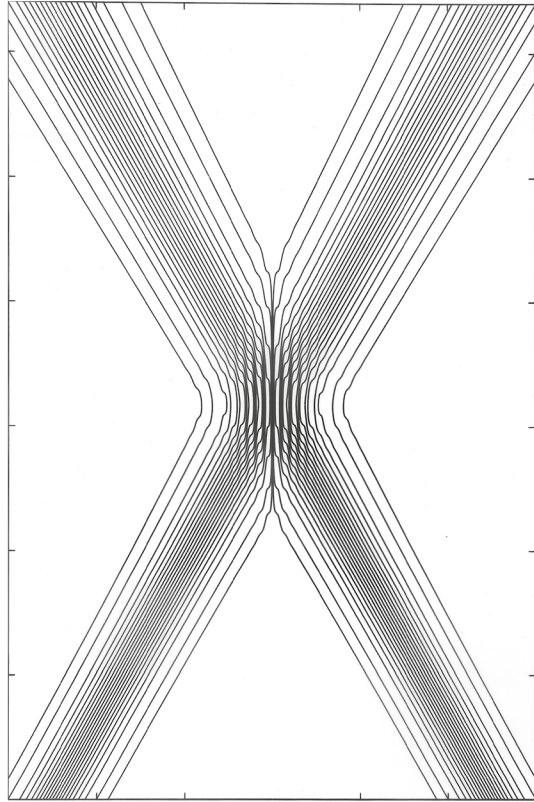
² Animations of the two slit experiment, with and without a detector, prepared for the *Infinite Potential* documentary film about the life and ideas of David Bohm can be viewed at <https://www.infinitepotential.com/animation>. See also https://youtu.be/X_ccsoGkaxM and <https://youtu.be/Ygq4ASmyUBg>.

Fig. 2 Trajectories for the two slit experiment



“look” at just one of the slits). Wheeler proposed that a measurement choice, made after the particle was known to have already passed the plane of the slits could nonetheless affect what happened at the slits at the earlier time. The experimenter could either choose to observe interference in the region of overlap in which case the particle must have passed, in some sense, through both slits at the earlier time, or make no observation other than which detector receives the particle. In the latter case, the common sense assumption being that the particle passed, at the earlier time, through only that slit towards which the receiving detector is pointing. But common sense is a poor guide in the quantum realm. Since there are no trajectories at all in standard quantum theory, strictly speaking, even when a detector fires, one can make no statement about how it got to the detector concerned. Of course, in Bohm’s deterministic theory, just as in classical theory, there is no question of choices made at a given point in time affecting what has already happened in the past. The definite configuration of the entire universe evolves deterministically along a trajectory according to the evolution of the universal configuration space wave function. As Bohm pointed out, in this case, the essential difference was that Wheeler chose to connect the past with the future through the evolution of the two slits’ wave packet states, so if a detector looking at a particular slit were to register the particle, then the particle must have come from that specific slit, along with the wave packet itself. In de Broglie–Bohm theory, the past is connected with the future through particle trajectories

Fig. 3 Trajectories for Wheeler’s delayed choice experiment. Two wave packets approach and then separate. The trajectories swap packets through the region of interference. Time increases down the page. The gap in the emerging trajectories is an artefact of the computation



and in the delayed choice experiment the particle swaps wave packets through the region of interference where the packets from each slit overlap; the particle received by a particular detector has actually passed through the slit it is *not* looking at.³

In Wheeler’s crossed beam experiment, the trajectory packet-swapping behaviour persists even in the case that the two wave packet states may be distinguished by some parameter, such as momentum or angular momentum. For example, if approaching the overlap region, one beam is associated with “spin up” and the other with “spin down”, then there is no spatial interference, but as the particle swaps packets through the overlap region it must adopt the spin state associated with its new packet. Along an individual the trajectory, the spin is rotated through π radians, even though there is no magnetic field present. (See below for a discussion of angular momentum and spin in de Broglie–Bohm theory.) Should the beams be distinguishable instead by their momentum, for example when the particle in just one beam has been accelerated before entering the overlap region, then a time dependent

³ An animated representation of the de Broglie–Bohm trajectories in Wheeler’s delayed choice experiment can be seen at <https://youtu.be/D4q-XxUymQI>, along with the quantum potential and trajectories at <https://youtu.be/muilsKIZ428>.

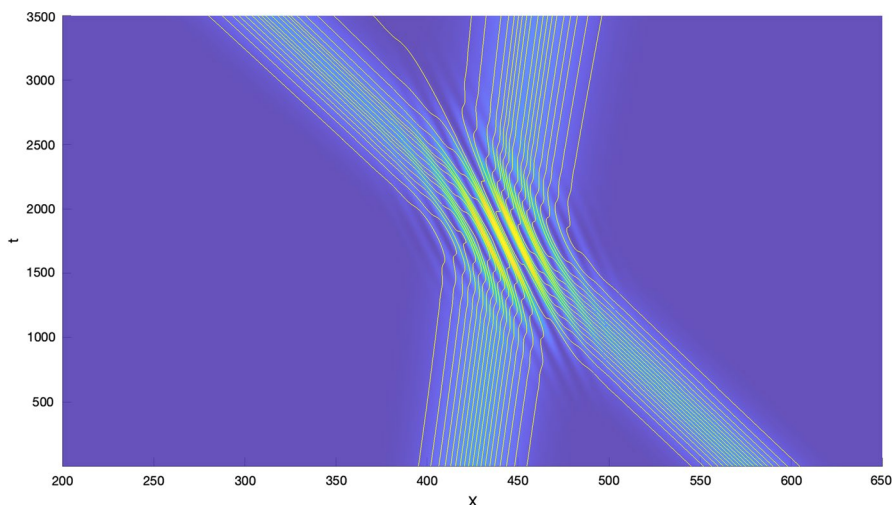


Fig. 4 Two wave packets, distinguishable by their momenta, produce a time dependent interference pattern in the region of overlap. The particle switches packets, changing its momentum as it does so. Time increases up the page

interference pattern occurs in said overlap region. Nonetheless, as shown in Fig. 4, the trajectories still do not cross, just as before, they switch packets as they “bounce back” in the direction from which they came. Along a trajectory the Bohm momentum changes from one value to the other: a particle starting off in the slow (fast) beam emerges with the opposite fast (slow) velocity. Integrated over time the spatial interference pattern is washed out, but the interference itself can still be observed using a streak camera that is blind to the momenta [16], demonstrating that distinguishable paths interfere nonetheless. In the case of light, should the two superposed beams be, say, red or blue respectively, then, in any photon trajectory theory the no crossing effect must persist and a photon which sets out red (blue), say, would emerge from the region of overlap in the other beam and with the other colour, blue (red). Of course, if a blue (or a red) colour filter were to be inserted in front of the detector, then no interference figure would be recorded as only one of the beams would impinge on the detector. The crucial factor is not whether the paths are distinguishable, but whether they are distinguishable by the detector.

In the second order version of de Broglie–Bohm theory, deviations from classical motions are accounted for by accelerations and torques of purely quantum origin that arise, without artifice, when quantum theory is recast in the manner presented by Bohm in 1952. The first-order version of the theory, relying solely on Eq. 2, eschews all talk of accelerations, instead, Eq. 2 is regarded as a form of law that determines the changes in the configuration of the quantum system. The packet swapping behaviour in the crossed beam experiments just described is simply a consequence of this law, no further explanation is required.

Quantum theory is the fundamental theory from which classical behaviour must be derived in the appropriate limit. de Broglie–Bohm theory has no problem

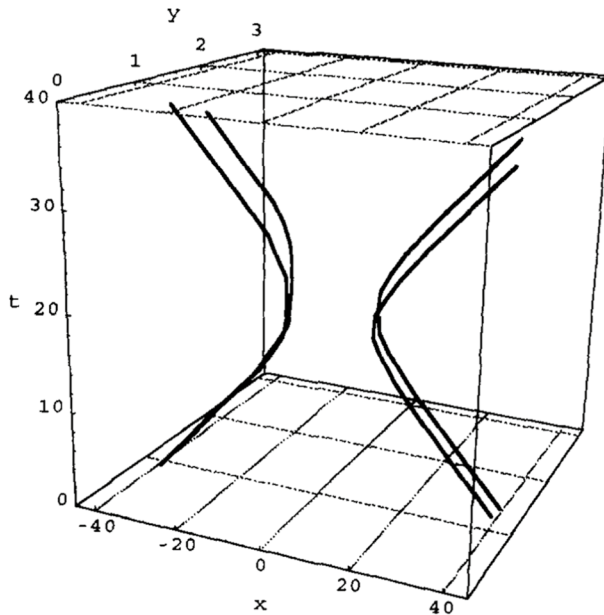


Fig. 5 Configuration space trajectories in the Wheeler experiment with a detector present. the detector coordinate is such that $\phi_L(y) = \phi_R(y)$

accounting for the existence of a definite classical world, since the world is already definite in the quantum description. However, the two versions of de Broglie–Bohm theory tell a different story of the classical limit. In the second order version, the classical limit has particles with properties just as described in Newtonian theory, classical behaviour arises as the quantum accelerations and torques become negligible and only classical potentials have significant impact on particle behaviour. In the first order version of the theory, in the classical limit, particles still possess only their positions: it is the behaviour of the associated wave function that explains why the traditional Newtonian description is possible in which particles appear to possess intrinsic properties other than position.

Feynman's objection to the possibility of a hidden variable explanation of the two slit experiment, referred to above, is addressed quite naturally in de Broglie–Bohm theory by giving a full quantum mechanical treatment of both the particle and the device used to detect through which slit the particle passes in a particular run of the experiment. The inclusion of the measuring device requires a quantum description that is now necessarily formulated in the configuration space of the whole system that is relevant in this case, particle plus detector. The behaviour of the whole system now depends on the entangled configuration space wave function

$$\Psi(x, z, t) = \psi_L(x, t)\phi_L(y, t) + \psi_R(x, t)\phi_R(y, t) \quad (20)$$

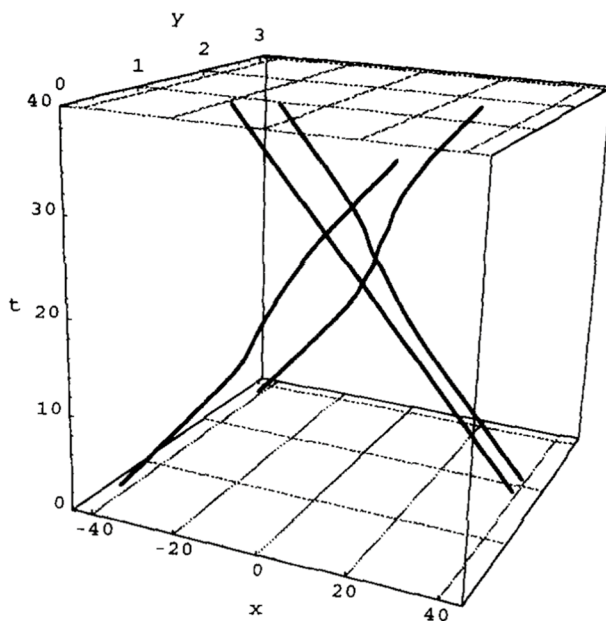


Fig. 6 Configuration space trajectories in the Wheeler experiment with a detector present and the detector coordinate is such that $\phi_L(y) \neq \phi_R(y)$

where $\psi_{L,R}$ represents the particle, with coordinate x , in the left (L) or right (R) beam and $\phi_{L,R}$ represents the detector with coordinate y , seeing the particle in the left or right slit. The behaviour of the particle in the two slit experiment, as in Wheeler's variation, now depends on the nature of the detector states. If the detector works unambiguously then the wave functions $\phi_L(y)$ and $\phi_R(y)$ will become completely separated along the detector coordinate axis, y , as the particle passes the slits. Consequently, in the configuration space of the whole system, two non overlapping and hence non interfering channels form. As a result, the particle interference pattern does not form and the particle trajectories are simply those associated with each slit separately which cross in the real space of the experiment.⁴

Consider now the consequence of adding such a detector in the Wheeler experiment. If $\phi_{L,R}$ only become separated along the z axis sometime after the particle has already passed the overlap region (the detector operates slowly) then there will be some trajectories for which $\phi_L(y) = \phi_R(y)$ at the time of overlap and in these cases the x particle trajectories will not cross, instead they 'bounce back' swapping wave packets on the way to the detectors. The configuration space trajectories for this case are shown in Fig. 5. In surprising behaviour that Englert et al. first identified [17] and described as surreal, the implication is that, even though the detector state is

⁴ An animated representation of the configuration space behaviour of the two slit plus detector system can be seen at www.infinite.potential.com along with other de Broglie-Bohm simulations on the "Animations" tab.

subsequently read as ϕ_L , the particle trajectory may in fact, according to de Broglie–Bohm theory, have passed along the path associated with ϕ_R . For other values of the detector coordinate the trajectories are as shown in Fig. 6, and the particle trajectories cross in real space implying that they match the classical expectation of corresponding with the final detector states. Although counterintuitive, the de Broglie–Bohm behaviour described here is no more surreal than any other non local behaviour that arises when a many particle configuration space wave function is entangled. The computations of the specific configuration space trajectories that arise in this scenario were published in 1993 [18].

In the single particle case, where configuration space is isomorphic to three space, it is tempting to imagine a physical wave actually passing through the two slits and there are terms in the configuration space wave function representing just that. But, in terms of understanding the nature of de Broglie–Bohm theory it is deeply misleading; the particle itself does not have “its own” independent wave function once it is entangled with the which-slit detector. It is possible to create a single particle wave function by projecting out of the configuration space onto the single particle axis, but how the particle waves (and hence the particle trajectories) behave in this case is determined only in the configuration space. Any attempt to define truly single particle wave equations that, taken together, could reproduce the configuration space wave function is doomed to failure.

4 Quantum Tunnelling

Extending the numerical integration technique used by Goldberg, Schey and Schwarz [4], and given access to the University of London’s mainframe computers, it was quite straightforward, even in 1982, to create computer animated motion pictures showing how de Broglie–Bohm theory (in second order form) accounted for square potential phenomena such as tunnelling, by calculating the quantum potential and trajectories associated with wave packet scattering from square (or indeed any shaped) potentials [19]. This work was extended to include computer generated motion pictures representing a simple model of Mach–Zehnder-type neutron interferometry in 1985 [20].⁵ The simple one dimensional model described above could be extended to describe Mach–Zehnder interference simply by adding an additional wave packet at $t = 0$ approaching the potential region from the opposite side. The two packets approaching the potential region from either side created a model of the convergence of the beams on the last set of crystal planes. The phase of the second packet could be adjusted (just as in a Mach–Zehnder interferometer) such that all of the wave function (and hence the trajectories) emerges on one or other side of the potential region. The animated movies made quite an impact at many conferences in

⁵ The original 16 mm films created in the early 1980’s covering scattering from square potentials, neutron interferometers and spin superposition in neutron interferometers can be viewed at <https://youtu.be/OQmYsik-Q74> and <https://youtu.be/3hq8RylizgQ>.

Europe and in North America. At that time, computer animated movies were quite novel in themselves, and certainly unique in the context of de Broglie–Bohm theory.

In the movies, at $t = 0$, the wave function takes the form of a Gaussian packet, as given by Eq. 10, at this time, as noted above, the de Broglie–Bohm particle momentum is k_0 , and its energy is $E_0 = \frac{k_0^2}{2m}$, regardless of the position in the packet. The choice of $t = 0$ is an arbitrary one, other choices, for example $t < 0$, will see a range of different momenta and energies for the Bohm particle—some initial positions of the particle will be associated with energies that are greater than the magnitude of the potential, but these will be in the rear of the wave packet and will nonetheless not tunnel through but be reflected in front of the barrier. Integrating the wave function forwards in time, at first the de Broglie–Bohm momentum depends on position according to Eq. 16 and the de Broglie–Bohm momentum distribution, given by Eq. 18 starts to approach the quantum mechanical momentum distribution given by Eq. 14. So, as the packet approaches the potential those particles in the forward part of the packet become accelerated, whereas those in the rear of the packet are decelerated. Furthermore, the potential seen by the particle is severely modified due to the presence of the quantum potential. As the incoming and reflected wave packets overlap, a series of ridges and troughs form in the quantum potential in front of the potential that serve to reflect some of the particles, without ever reaching the region of the classical potential. Those particles arriving at the region where the classical potential is applied, at first meet an effective potential (quantum plus classical) that is reduced. A combination of these effects allows some of the particles in the forward part of the wave packet to pass through the potential whereas classically, if the magnitude of the potential, V , is greater than the initial energy of the particles, E_0 , no particle could enter the potential region: instead all such particles would be reflected.⁶ It was obvious from the form of the quantum trajectories that the transmitted wave packet was constructed from the forward part of the incident wave packet, which consequently is ahead of where the incident packet would be in the absence of the potential. The appearance that the tunnelled packet had moved more quickly than would the same packet in the absence of the potential was unremarkable and easily explained. Furthermore, where standard quantum theory struggled to define a tunnelling time, it was also obvious how to define, if not measure, the tunnelling time for each de Broglie–Bohm trajectory.

5 Quantum Measurement of Angular Momentum in de Broglie–Bohm Theory

Following the application of de Broglie–Bohm theory to the quantum phenomena of interference and then tunnelling, the next problem to tackle was the measurement problem. In his discussion of the quantum measurement process, in his 1951

⁶ A more recent animated representation of the de Broglie–Bohm account of quantum tunnelling can be seen at <https://youtu.be/NC6yVkbAIZk>. A comparison of classical and quantum trajectories for a smooth Gaussian potential can be viewed at <https://youtu.be/sT4RDIXTGSQ>.

text [21], David Bohm had taken the example of the measurement of the spin of a spin one half particle along a particular direction by means of a Stern–Gerlach (SG) device. Bohm’s discussion was very clear, using the Pauli theory, he showed how an initial Gaussian wave packet with spin orientation definite along the x -direction, would be split by the impulsive action of an inhomogeneous magnetic field along the z -direction, into two packets with opposite momenta in the z -direction, each associated with one of the possible outcomes. The account given by Bohm shows exactly how a measurement interaction brings about a correlation between the measured variable (the spin) and the apparatus coordinate (the particle’s position). He follows this with an extensive discussion of what is known today as decoherence, although he does not use that term to describe it, showing how interaction with the measuring device necessarily destroys the coherence of the two separating packets that result from the measurement interaction. Given the dynamics of measurement, and decoherence, the essence of the measurement problem remains to explain how a single definite result of the measurement in question comes about when the initial quantum state is not an eigenstate of the observable being measured. Since in de Broglie–Bohm theory the configuration of a quantum system is always definite, it is clear that the particular outcome observed in a measurement interaction must be no exception. Bohm also reformulated the original Einstein, Podolsky and Rosen (EPR) thought experiment, in terms of spin correlations. His reformulation allowed the thought experiment to become actual when formulated in terms of polarisation measurements with entangled polarisation states of photons.

In order to give a trajectory account of this measurement process a de Broglie–Bohm account of angular momentum was necessary. Such an account, for the case of intrinsic angular momentum, or spin, had been provided by Bohm, whilst in Brazil in 1955, in collaboration with Schiller and Tiomno, [22] and whilst working in Paris, detailed calculations using BST theory were completed. However, before discussing the case of spin and its measurement, we first consider the possibly less contentious case of the measurement of angular momentum in hydrogen-like atoms. Bohm’s analysis of the measurement of the magnetic moment of a spin one-half particle can clearly also be applied to the measurement of the magnetic moment arising from orbital angular momentum. Similarly, an EPR-Bohm experiment can be described with correlated orbital angular momentum states. Detailed calculations, applying the de Broglie–Bohm theory to such scenarios were first presented (in second order form) in a paper [23] reporting a collaboration with Zahid Malik whilst at the University of Portsmouth in southern England in 1993. The essential results are reviewed here.

Firstly, consider the motion of the electron in a hydrogen-like atom. The simultaneous eigenstates of L^2 and L_z , stationary states of a hydrogen-like atom, are normally written in the following way

$$\psi_{nlm} = R_{nl}(r)P_l^m(\cos \theta)e^{im\phi}. \quad (21)$$

The R ’s are the Laguerre polynomials, the P_l^m are the spherical harmonics, and n, l, m are the quantum numbers associated with the total energy, the square angular momentum and its z component. The electron coordinates are (r, θ, ϕ) relative to the

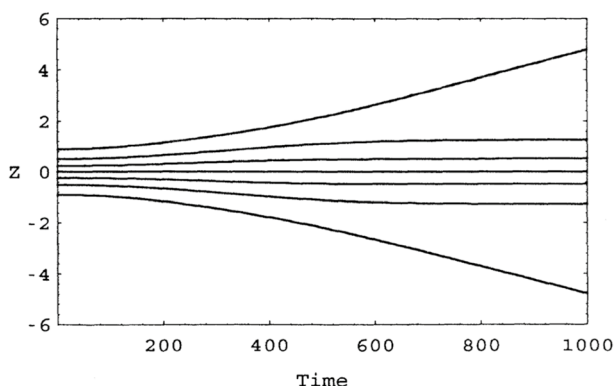


Fig. 7 A set of trajectories for the angular momentum measurement carried out on the atom in the state given by Eq. 24

nucleus of the atom. Applying de Broglie–Bohm theory, the momentum is the gradient of the phase S

$$\mathbf{p} = \nabla S = \frac{m\hbar}{r \sin \theta} \mathbf{e}_\phi \quad (22)$$

and the well defined angular momentum vector is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -\frac{m\hbar}{\sin \theta} \mathbf{e}_\theta \quad (23)$$

with rectangular coordinate components $(-m\hbar \cot \theta \cos \phi, -m\hbar \cot \theta \sin \phi, m\hbar)$. Clearly the values attributed to the angular momentum components in de Broglie–Bohm theory are not restricted to the eigenvalues of the associated operators. There are well defined electron trajectories for the eigenstates in which the electron is either standing still ($m = 0$), or circling the z axis in a clockwise sense (m positive) or anticlockwise sense (m negative) in a plane parallel to the xy plane. This, at first sight strange, motion can be completely accounted for by the action of the angle dependent quantum potential and the associated quantum torques that arise in the extension of de Broglie–Bohm theory to angular momentum. The effective force, that is the combination of the classical forces and those arising from the action of the quantum potential are exactly those needed to maintain the motion of the electron within the atom hitherto described. Once again, this underlies the fact that in de Broglie–Bohm theory the values assigned to observables (the “beables” in Bell’s sense) [24] are not restricted to the eigenvalues of the operators representing the observables. But, during a measurement interaction, analysis reveals that the definite value assigned in de Broglie–Bohm theory, evolves along the particle trajectory to become identical with one of the expected eigenvalues, which eigenvalue is realised depends on the values of the initial coordinates.

In the case of the measurement of the z -component of angular momentum in hydrogen-like atoms, the z coordinate of the nucleus of the atom is the significant apparatus coordinate. If the initial z -dependence of the atom is assumed to be

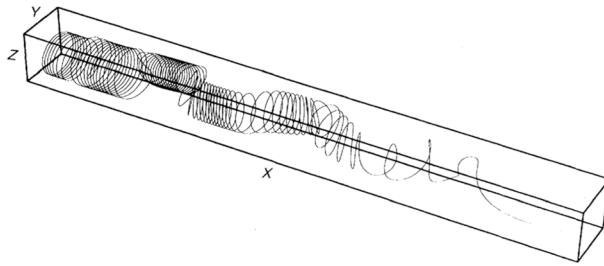


Fig. 8 The motion of the electron (time added parametrically to the x axis) as the atom enters the packet associated with $m = 0$.

described by a Gaussian wave packet centred on $z = 0$ then, assuming an impulsive interaction with the magnetic field, a superposition of a set of three Gaussian packets separated along the z direction is produced, one for each possible outcome. Then, if the packets separate faster than they spread in time, by observing the position of the atom it is possible to infer the value of the angular momentum of the electron. The quantum theory of the measurement process produces a wave function that is a superposition of the possible outcomes, the de Broglie–Bohm theory ensures that in any given case just one result actually occurs. For example, if the initial wave function of the internal coordinates of the atom is a superposition of the three simultaneous degenerate eigenstates of the angular momentum operators \hat{L}^2 and \hat{L}_z corresponding to $l = 1$,

$$\sum_{j=-1}^{j=1} c_j \psi_j(r, \theta, \phi), \quad (24)$$

then three separated packets emerge from the impulsive interaction. If the initial amplitudes are taken to be $c_{+1} = c_{-1} = \frac{1}{2}$ whilst $c_0 = \frac{1}{\sqrt{2}}$ this corresponds with an initial eigenstate of the \hat{L}_x operator, with eigenvalue one. As the packets separate, in the field-free region after the impulsive interaction has occurred, the trajectory of the nucleus and of the electron within the atom evolve together in a correlated manner. The atomic trajectories are shown in Fig. 7. If, for example, the initial nuclear coordinate is in the twenty five percent of the wave packet furthest from the centre in the positive direction of the z axis, then the atom moves in the positive z -direction to enter the $m = 1$ wave packet. As it does so, the motion of the electron evolves continuously from that associated with the $+1$ eigenvalue of \hat{L}_x , in which it is circling about the atomic x axis, towards the end result in which it is circling about the atomic z axis in a positive sense. Similarly, if the nuclear coordinate is in the twenty five percent of the packet at the other extreme, the atom moves in the negative z direction whilst the motion of the electron evolves to circle the z axis in the anti-clockwise sense ($m = -1$). If the atom coordinate is within the remaining centre part of the wave packet, the atom enters the central wave packet whilst the electron's motion comes to halt ($m = 0$), see Fig. 8. If, in an ensemble of initial electron positions the distribution matches that of the wave function associated with the $+1$

eigenvalue of \hat{L}_x , then after the packets have separated the distribution of electron positions in each of the three packets becomes that associated with the relevant eigenstate of the \hat{L}_z operator.

Interestingly, the outcome of the measurement that we have just described is solely determined by the initial position of the atom (the apparatus coordinate) with respect to the z axis within the initial wave packet. The measurement interaction, through the action of the magnetic field gradient, brings about an entanglement of the internal (electron) and the external (atomic) wave functions such that at the end of the process, by observing the position of the atom, it is possible to infer the angular momentum state of the electron. As the atom progresses along its particular trajectory the electron distribution in an ensemble of systems is appropriately transformed.

These calculations from the late twentieth century, revealed that to describe such a process as a “measurement” seems to be a misnomer. In a classical measurement process the value revealed is just the actual pre existing value, but here this is not the case: instead system and apparatus evolve together in a correlated manner to create anew the value that is observed. The wave function of the combined system is transformed in the measurement process and de Broglie–Bohm theory gives a clear meaning to the notion of the disturbance of the system that the measurement brings about. In the case discussed here, if a subsequent measurement of \hat{L}_x were to be carried out on any of the wave packets emerging from the \hat{L}_z measurement then any of the three possible outcomes for \hat{L}_x could occur, now depending only the x component of the atom. The “disturbance” is such that there is no memory in the system and no possibility to create dispersion-free ensembles using a sequence of three measurements of \hat{L}_x, \hat{L}_y and \hat{L}_z . In the same way de Broglie–Bohm theory also gives a clear account of the nature of contextuality [25]. A particle behaving classically, with an initial positive orbital angular momentum component along the z direction, would definitely yield a positive result on measurement—even following an interim measurement of its x component. Furthermore, for a classical particle a sequence of orbital angular momentum component measurements would yield the same appropriate outcomes irrespective of the order in which they are carried out.

6 Angular Momentum Correlations in the Einstein–Podolsky–Rosen–Bohm Experiment

Once again, following Bohm’s masterful exegesis in his 1951 textbook, in which he formulated the EPR argument for the case of correlations of spin one-half particles, it was easy enough, to modify the discussion for the case of angular momentum correlations. In our paper we considered two parallel SG magnets each aligned along a z axis and space-like separated along a perpendicular axis with respect to the source. Let atom labelled, B , be in the left wing and atom, A , in the right wing. Both atoms have angular momentum $j = 1$. The entangled wave function, a simultaneous eigenstate of the total z -component of angular momentum and the total angular

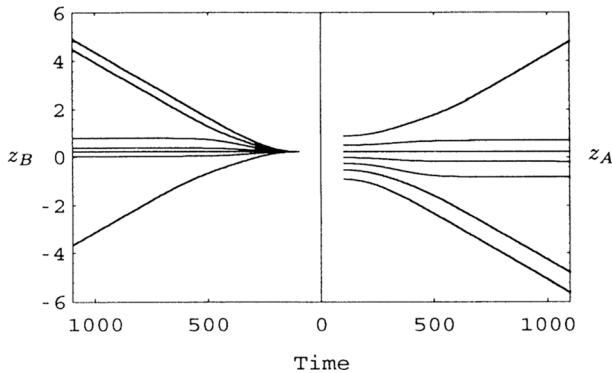


Fig. 9 A set of trajectories for scenario one in which the z component of angular momentum for both atoms is subject to simultaneous measurement. The initial value of z_B is fixed at 0.24 (in arbitrary units) whilst each trajectory has a different value of z_A . The outcome for the measurement on atom B also depends on the initial value of z_A . The trajectories are paired such that the lower trajectory for atom B is paired with the upper trajectory for atom A and so on

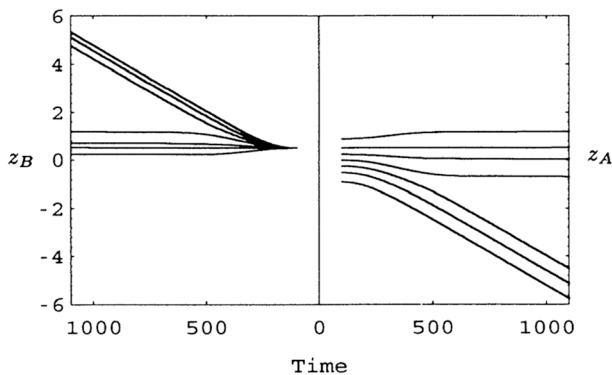


Fig. 10 As for Fig. 9 but for this figure the fixed initial value of $z_B = 0.512$ (in arbitrary units). Changing the initial position of atom B changes the outcome for atom A

momentum with eigenvalues each zero, of the pair of particles before the impulsive action of the magnetic fields takes place is given by

$$\Psi_0 = \psi_p^A(z_A)\psi_p^B(z_B)\frac{1}{\sqrt{3}}\sum_{j=-1}^1(-1)^{j+1}\psi_{e+j}^A\psi_{e-j}^B, \quad (25)$$

where $\psi_p^A(z_A)\psi_p^B(z_B)$ are the (localised Gaussian) wave functions describing the positions of atoms A (on the right) and B (on the left) with respect to the measurement axis, z , and $\psi_e^A(r, \theta, \phi), \psi_e^B(r, \theta, \phi)$ are the electron wave functions. After the impulsive interaction the wave function becomes

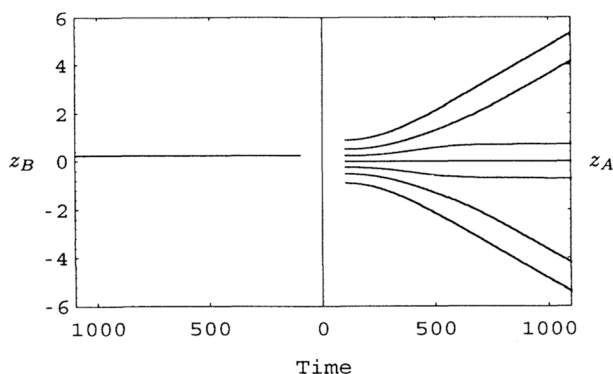


Fig. 11 A set of trajectories when the z component of angular momentum measurement is only carried out on atom A . The initial value of z_B is fixed at 0.24 (in arbitrary units) whilst each trajectory has a different value of z_A . The atomic trajectories are independent but the electron trajectories are correlated

$$\Psi(t) = \frac{1}{\sqrt{3}} \sum_{j=-1}^1 (-1)^{j+1} \psi_{p+j}^A(z_A) \psi_{p-j}^B(z_B) \psi_{e+j}^A \psi_{e-j}^B, \quad (26)$$

where ψ_{pj}^A , with $j = -1, 0, +1$, indicates the atomic wave function moving in the negative sense, remaining stationary or moving in the positive sense with respect to the measurement axis z . The interaction brings about a correlation between the atomic and the electronic wave functions. With wave function 26 it is clear that the individual de Broglie–Bohm velocities for any of the four particles depend on the coordinates of all of the particles. Detailed calculations confirmed that the outcome of the measurement, for example for particle A , depends not only on its own atomic coordinate (z_A) but also on the coordinate of atom B , (z_B). This can be seen clearly in Figs. 9 and 10. Just as in the single atom Stern Gerlach measurement described above, the motion of the electrons within the atoms is correlated with the position of the atom and in an ensemble the electron distribution is similarly correlated.

In order to bring out exactly how the nonlocality and contextuality in de Broglie–Bohm theory operates in this case, two specific EPR scenarios were studied. Firstly, the scenario in which both particles are measured along the same direction (z), and secondly the scenario in which atom B in the left wing of the experiment is not interacted with, whilst that on the right wing is subjected to a Stern-Gerlach angular momentum measurement in the z direction. For the first scenario the wave function of the two particles is entangled, a set of possible atomic trajectories is shown in Fig. 9. The diagram shows that with the initial position of atom B fixed, the motion of atom B depends also on the initial position of atom A .

In the first scenario, in which both atoms are subjected to measurement interactions, the atomic trajectories are correlated such that the outcome for either atom depends not only on its own position but also on that of the other atom, whilst the electronic motions are correlated in such a way as to ensure the appropriate

trajectories and ensemble distributions are acquired. Comparison of Figs. 9 and 10 makes this abundantly clear.

In the second scenario the trajectories of the atoms are independent. As can be seen in Fig. 11, atom B, in the left wing, not subjected to an SG field has rectilinear motion, whilst the atom in the right wing (A) displays the same trajectories as for an SG measurement on a single particle. There is no correlation of the atomic trajectories when only one of the atoms is subjected to measurement. Comparison with Fig. 9 shows that the trajectory of atom A depends on whether the SG field at the location of B is on or off. Simply turning the field on or off at the location of B affects the motion of the distant atom, A. As we emphasised in 1993, according to special relativity, in some frames of reference the field at B may be judged as off at the time that A is measured, whereas in other frames of reference it may be judged as on at that time. When the field at B is off, the nonlocality appears only in the motion of the electrons. Although the trajectories of the atoms are independent, the motion of the electron in the atom not subjected to measurement is non locally correlated with the electron motion in the measured atom. The electron motion is such that in an ensemble of systems the distributions of the electrons within each atom become those associated with appropriate correlated angular momentum wave functions. For instance, were it to be the case that atom A should enter the packet associated with $L_z = +1$, whence its electron trajectory ends up circling the z axis in a positive sense, then atom B (not subjected to measurement) is not deflected but nonetheless its electron trajectory evolves, under the action of non local forces, so that it ends up circling the z axis in a negative sense with $L_z = -1$. Even though there are no forces of classical origin operating on the electron not measured, its motion is subjected to non local quantum forces.

7 Intrinsic Angular Momentum: The Bohm, Schiller and Tiomno Theory for Spin One Half Particles

The Bohm theory for intrinsic angular momentum, or spin, as developed by Bohm, Schiller and Tiomno and based upon Pauli's two-component spinor theory, takes a rather different form to the theory for orbital angular momentum, discussed above. In BST theory a spin vector is attributed to the particle but, in contrast to the theory for orbital angular momentum discussed above (the computations for which in fact came later), there are neither internal wave functions, nor internal coordinates. Instead, the theory is based upon the orientation angles of the intrinsic magnetic moment of the particle that are derived in a similar way to the orientation angles of the orbital angular momentum vector given in Eq. 5, that is from the local values of the expectation of the quantum mechanical spin operators. The Pauli hamiltonian for a neutral, spin one-half particle of mass m with (anomalous) magnetic moment μ , subject to magnetic field \vec{B} and scalar potential V , can be written

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V + \mu\vec{\sigma} \cdot \vec{B}, \quad (27)$$

where $\vec{\sigma}$ represents the Pauli operator. The Shrödinger–Pauli equation may then be written in terms of the two-component spinor Ψ as

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi. \quad (28)$$

The spin operator \hat{s} may be written in terms of the Pauli $\vec{\sigma}$ operator according to

$$\hat{s} = \frac{\hbar}{2}\vec{\sigma}, \quad (29)$$

and a spin vector may be defined as

$$\vec{s} = \frac{\hbar}{2} \frac{\Psi^\dagger \vec{\sigma} \Psi}{\Psi^\dagger \Psi}. \quad (30)$$

This vector defines the direction in space in which a measurement of the spin, at that location, would yield the eigenvalue result $\frac{\hbar}{2}$ with probability one. Following usual procedures, the Pauli current is found to be

$$\vec{j} = \frac{\hbar}{m} \Im(\Psi^\dagger \nabla \Psi), \quad (31)$$

from which we may define a velocity

$$\vec{v} = \frac{\vec{j}}{\rho}, \quad (32)$$

where

$$\rho = \Psi^\dagger \Psi, \quad (33)$$

defines the probability density. For a given evolution of a Pauli spinor, and accepting that in Bohm's theory a particle always has a definite (but unknown) position, Eqs. 30, 31, 32 and 33, evaluated at the position of the particle, are all that is required to define a particle trajectory and spin vector orientations along the trajectory, once the initial position is specified. Taken alone, these equations constitute the first order form of the theory.

Following the second order approach of Bohm's 1952 publications, Bohm, Schiller and Tiomno (BST) further elaborated the theory in second-order “Newtonian” form, couched in terms of potentials, forces and torques, by writing an arbitrary magnitude Pauli spinor in terms of Euler orientation angles θ , ϕ and ψ , according to

$$\Psi = Re^{\frac{i\psi}{2}} \begin{pmatrix} \cos \frac{\theta}{2} e^{i\frac{\phi}{2}} \\ i \sin \frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix}. \quad (34)$$

The velocity, expressed in terms of the Euler angles, for the neutral particle considered here, is then given by

$$\vec{v} = \frac{\hbar}{2m}(\nabla\psi + \cos\theta\nabla\phi). \quad (35)$$

The full set of equations of motion, written in these terms, can be obtained by substituting 34 into 28. This yields the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0, \quad (36)$$

and an energy equation

$$\frac{\hbar}{2}\left(\frac{\partial\psi}{\partial t} + \cos\theta\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}mv^2 + Q + Q_s + \frac{2\mu}{\hbar}\vec{\mu} \cdot \vec{s} + V = 0, \quad (37)$$

with Q , the quantum potential familiar from Bohm's treatment of the Schrödinger theory given by Eq. 5, and Q_s a spin dependent additional potential

$$Q_s = -\frac{\hbar^2}{8m}[(\nabla\theta) + \sin^2\theta(\nabla\phi)^2]. \quad (38)$$

The equation of motion of the spin vector can be written

$$\frac{d\vec{s}}{dt} = \vec{T} - \frac{2\mu}{\hbar}(\vec{s} \times \vec{H}), \quad (39)$$

where there is an additional quantum torque term, \vec{T} , given by

$$\vec{T} = \frac{1}{m\rho}\vec{s} \times \sum_i \frac{\partial}{\partial x_i} \left(\rho \frac{\partial \vec{s}}{\partial x_i} \right). \quad (40)$$

For non stationary spin states, the particle trajectories become spin dependent and the quantum torque produces a purely quantum mechanical precession of the spin vector, even in the absence of magnetic fields. The trajectories are, of course, the same in the first order in time and the second order in time versions of the theory, as are the directions defined by Eq. 30.

Both Bohm's original scalar theory and the BST spinor theory are simply re-castings of the scalar Schrödinger theory and the spinor Pauli theory respectively. Both yield exactly the same measurement statistics as the standard quantum theories. Bohm's scalar Schrödinger theory retains the classical notion that particles have definite positions under all circumstances and the BST theory simply also retains the classical notion that particles with magnetic moments also have well defined orientations. But neither the position nor the orientation evolve according to classical expectations. For non stationary states, trajectories are not necessarily rectilinear in classical force free regions neither are orientations static even in magnetic field free regions.

Just like de Broglie's (1927) and Bohm's (1952) interpretation of the scalar Schrödinger theory, for many years the BST theory was either rejected for no good

reason, or ignored. But, in a series of papers published some 30 years after the original BST theory, detailed and specific calculations were carried out showing exactly how the theory could account for all quantum phenomena associated spin-one-half particles in terms of deterministic and well defined motions. The calculations were carried out in Vigier's group in Paris in 1987. Spin dependent trajectories and spin vector orientations along the trajectories showed for the first time exactly how the BST theory explains: spin measurement [26]; spin superposition in neutron interferometry [27], and Einstein–Podolsky–Rosen spin correlations [28, 29]. In each of these early spin one-half particle calculations, spin dependent trajectories and associated spin orientations were calculated for non stationary superpositions of spin states in the absence of any magnetic fields.

The quantum states used in the single, spin-one-half particle calculations, have the form of a superposition of “spin up” and “spin down” states:

$$\Psi = \psi_1(z, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_2(z, t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (41)$$

or

$$\Psi = \begin{pmatrix} \psi_1(z, t) \\ \psi_2(z, t) \end{pmatrix} = \begin{pmatrix} R_1 e^{\frac{iS_1}{\hbar}} \\ R_2 e^{\frac{iS_2}{\hbar}} \end{pmatrix}. \quad (42)$$

where $\psi_1(z, t)$ and $\psi_2(z, t)$ take the form of one-dimensional Gaussian wave packets (as given in Eqs. 10, and 11), and R_1, R_2, S_1 and S_2 are real functions of z and t .

During the periods covered by the calculations no magnetic fields were present, nevertheless the spin vectors, as defined in BST theory according to Eq. 30, were shown to precess along sets of spin dependent trajectories calculated using Eq. 32. In the quantum state given by Eq. 41 the velocity of the particle, according to Eq. 35, is simply the probability-density-weighted average of the particle velocities associated with each component of the Pauli spinor separately:

$$v = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho}, \quad (43)$$

where $\rho = \Psi^\dagger \Psi$, $\rho_1 = \psi_1^* \psi_1$, $\rho_2 = \psi_2^* \psi_2$, $v_1 = \frac{\hbar}{m} \Im \frac{\nabla \psi_1}{\psi_1}$, and $v_2 = \frac{\hbar}{m} \Im \frac{\nabla \psi_2}{\psi_2}$. Where the spatial support of ψ_1 is disjoint from that of ψ_2 , the particle velocity is either v_1 or v_2 and the spin is stationary—either up or down, depending on the location of the particle—which must be in one or the other packet. Should ψ_1 and ψ_2 overlap in space there can be no spatial interference under these circumstances as the spin states are orthogonal and this fact removes the interference terms in the probability density. The probability density is given by

$$\Psi^\dagger \Psi = |\psi_1|^2 + |\psi_2|^2. \quad (44)$$

Coherence is nonetheless evident in the spatial variation of the spin direction in the region of overlap, given by

$$\phi = S_1 - S_2 - \frac{\pi}{2}, \quad (45)$$

and

$$\theta = 2 \arctan \frac{\sqrt{\rho_2}}{\sqrt{\rho_1}} = 2 \arctan \frac{R_2}{R_1}, \quad (46)$$

with the symbols as defined in Eq. 42.

Given that, in the single particle case, trajectories may not cross in space, the form of the BST trajectories can easily be deduced (perhaps with hindsight) without detailed computation in most cases. For instance, consider the Stern-Gerlach measurement of spin. For a particle prepared in a spin x eigenstate and whose spin is subsequently measured along the z direction, the z spin measurement may be modelled with ψ_1 and ψ_2 , initially coincident but subsequently separating along the direction in which the impulsive measuring Stern-Gerlach field gradient previously acted, z . (In an impulsive measurement, the field gradient imparts an opposite momenta kick along the z direction to each packet.) According to Eq. 43, the $z = 0$ plane splits the trajectories into two groups: those with initial position $z > 0$ move towards the positive z direction, whilst those with $z < 0$ move in the opposite direction. This also follows from the fact that the trajectories may not cross. As the wave packets separate, the spin vector rotates, in accordance with Eq. 46, such that in one beam $\theta \rightarrow 0$ whilst $\theta \rightarrow \pi$ in the other beam.⁷ Spin is the measured quantity, whereas the particle position plays the role of the measuring apparatus coordinate. Just as in the case of orbital angular momentum, the outcome of the measurement in any individual case is determined by the initial position of the particle in the packet. So, once again, the outcome of the measurement is determined not by the value of variable being measured (the spin), but by the specific initial value of the apparatus coordinated (the particle's position).

Another scenario, illustrating spin superposition has ψ_1 and ψ_2 , in Eq. 41, initially distanced and non overlapping along the z axis, but approaching each other (such as to create a crossed beam configuration when time is used to parameterise motion in the y direction). Effectively, this is a time reversal of the spin measurement scenario. Whilst the packets do not overlap, the spin associated with each packet is stationary and the possible trajectories are simply those associated with each packet in isolation and independent of spin. As the packets start to overlap the spin precesses, again even in the absence of a magnetic field, so that at the instant the packets are coincident, $\theta \rightarrow \frac{\pi}{2}$, on each trajectory, whilst the particle's motion in the z direction is momentarily halted. Equation 43 guarantees that the trajectories cannot cross the plane of symmetry at $z = 0$. The motion as the packets separate again is the reverse of the motion as they converged. The trajectories bounce back from the overlap region, whilst their spin rotates so that those with initial direction $\theta = 0$ rotate to $\theta = \pi$, and those with initial direction $\theta = \pi$ rotate to $\theta = 0$. The particle trajectories

⁷ An animation of the Stern-Gerlach interaction can be viewed at <https://youtu.be/vtJvdn5IMyo>.

swap packets and hence spins. Of course, no meaningful statement can be made concerning particle trajectories in the usual interpretation of quantum theory; observing a particle in, say, packet ψ_1 at the end of the process *does not* allow the tempting and classically common sense deduction that it was in ψ_1 all along. Similarly, the absence of spatial interference in the region of overlap cannot be “explained” as due to the particle paths being distinguishable by their spins; a point emphasised in BST theory by the fact that the paths are not those expected using classical intuition. Although the signature of coherent superposition is absent in the spatial distribution of the trajectories, it is clearly present in the spatial distribution of the spin direction in the xy plane, the angle ϕ (see [27]). The significant factor in determining interference is not whether the paths themselves are distinguishable in some parameter but whether the paths are distinguishable *at the detector*.

It was easy enough, in 1987, to extend the spin measurement calculations to describe BST behaviour in the Einstein-Podolsky-Rosen-Bohm spin correlation experiment [28]. After a long exchange with a referee, a review of the work on spin was also eventually published a year later, (with J-P Vigi er), in Nature [29]. It was only after the modified publication was published that the exacting referee revealed himself as David Bohm! The analysis and description of the EPR-Bohm experiment for the case of entangled spin one-half particles goes very much along the lines already discussed for the case of EPR-Bohm entangled angular momentum correlations already discussed above (but without the internal motions).⁸

EPR’s original argument concerned the lack of completeness of quantum mechanics and rested upon the notion of “elements of reality”: if without in any way disturbing a system we can predict with probability unity the value of a dynamical quantity, then that quantity is an element of reality. In the de Broglie–Bohm description of the EPR scenario the measurement on one of the particles, that allows the prediction of the outcome of the measurement on the other particle, does indeed disturb it through the action of the nonlocal quantum potential and quantum torque. In responding to the EPR argument Bohr suggested that there is no question of a disturbance of the distant particle, as for him there is simply one unanalysable system described through the quantum mechanical configuration space wave function. That which Bohr characterised as the unanalysable wholeness of quantum phenomena finds its expression in de Broglie–Bohm theory in its nonlocal features characterised instead as undivided wholeness. But both Bohr and Bohm emphasise that the wholeness of quantum phenomena is an unavoidable consequence of the necessity of a configuration space description. As we pointed out in 1987, the angular momentum correlation statistics are identical whether obtained in simultaneous or staggered measurements, but they result from quite different evolutions at the level of the individual particles which make up an ensemble. The non-local mechanism at work in the two different cases is not the same, but they cannot be experimentally distinguished. As noted earlier, according to relativity, some observers may judge the measurements as simultaneous, whilst those in relative motion may well judge them

⁸ An illustration of de Broglie–Bohm theory applied to the case of EPR-like nonlocality arising in correlated particle interferometry, hence avoiding the complication of extra spin degrees of freedom, was also published in 1990 [30].

as staggered. The implication is that the detailed trajectories and angular momentum motions attributed to the particles in de Broglie–Bohm theory by observers in different frames of reference are quite different. Of course, Bohm was well aware of the tension between the uncontrollable and unobservable quantum non locality exhibited in individual particle motions as described here and relativistic constraints. But, Bohm argued, that if quantum non locality implies a preferred frame (in which the nonlocal connection could be described as instantaneous) this does not conflict with relativistic requirements provided that the preferred frame is not experimentally distinguishable. A theory may be relativistically covariant at the level of the predicted experimental statistics but nonetheless be underpinned by a non relativistically covariant theory of individual processes.⁹

In the last years of the twentieth century whilst at Portsmouth, working with Ph.D. student M.M. Lam, detailed calculations were carried out, following the work of Bohm, Hiley and Kaloyerou [31], to elucidate the behaviour of quantum scalar fields according to Bohm's original approach first outlined in an appendix to his 1952 publications. In Bohm's approach, there are no trajectories associated with quantum fields, instead the fields are well defined and continuous, evolving according to deterministic field equations that differ by additional quantum terms from their classical counterparts.

Our calculations showed, using simple models, how well-defined, free quantum fields evolved in different quantum states: even free fields behave non locally in Bohm's approach [32]. Further work showed how in Bohm theory simplified models of field–matter interactions could account for the nature of the absorption and emission of energy by fields and particles during quantum transitions, and the anti-correlation of detectors in a single photon field, without the need for discrete field-photons following trajectories, and no need for any discontinuity [33]. In Bohm's quantum field theory photons are merely excitations of well-defined continuous fields, not particles following trajectories. Once again, local behaviour in the configuration space of the system, now spanned by all the particle coordinates and all of the field coordinates, projected into real space trajectories and field motions that are both contextual and nonlocal.

8 Conclusion

The pioneering, detailed calculations, produced late last century and discussed in this review, made it abundantly clear just how de Broglie–Bohm theory could account for otherwise puzzling quantum phenomena in terms of a single definite world in which well-defined particles and fields evolve deterministically, governed by the evolution of the entire system's configuration space wave function. The computations reviewed here were described in conformity with Bohm's 1952 second

⁹ More recently than the calculations described here, a number of ways of developing covariant de Broglie–Bohm theories have been proposed in order to address these problems.

order presentation of his theory. At the time such calculations as those described here were almost unique, certainly within the then fairly small foundations of quantum theory community.¹⁰ They contributed to the rekindling of de Broglie–Bohm theory and kept the flame alight in perhaps the darkest time. In the current century, a small industry in such calculations has blossomed, as has interest in, and acceptance of, de Broglie–Bohm quantum theory, stimulated by the work of many authors, and including the publication of several highly influential books on the subject. Since the work described here was first carried out, following John Bell’s earlier, significant and highly influential presentation and promotion of de Broglie–Bohm theory, several authors, notably Antony Valentini and also Detleff Dürr, Sheldon Goldstein and Nino Zanghi, amongst others, have argued that the de Broglie–Bohm theory is better presented as a first order theory, shorn of its arcane second order accelerations. This may well be, but it does not invalidate the work we have described, as, although the ontologies differ in some respects, the trajectories are the same in either case and it is simply the explanation of their form that differs. The detailed work reviewed here (taken either way) clearly demonstrated the clarity and validity of the de Broglie–Bohm theory through its detailed computations of the definite individual quantum motions that allow an explanation in terms of a single and precise conceptual scheme of otherwise mysterious quantum phenomena. It underlined that it was the arbitrary, unnecessary and sensational rejection of the commonplace notion of a definite ontology of trajectories and field evolutions that lead to the creation of the problems of the interpretation of quantum theory—and the plethora of attempts to resolve them. The simple existence of de Broglie–Bohm theory should make us cautious in assessing claims concerning that which quantum theory has taught us about the world. It can, at the very least be used in this way as a “touchstone”, to distinguish what an author may wish for, from that which necessarily follows. The early computations clearly illustrated the manner in which de Broglie–Bohm theory actually works, that it is essentially and irreducibly a configuration space theory, in principle of the entire universe, that exhibits a wholeness manifested through the contextuality and non locality of the definite and well defined individual motions derived from the system’s configuration space trajectory. Contextuality and non locality, so clearly apparent in de Broglie–Bohm theory, are fundamental in quantum theory and today it is clear, both theoretically and experimentally, that in quantum theory connection is not limited by proximity. Of course, de Broglie–Bohm theory is only one of a number of contenders in the interpretation of quantum theory, but it is one that describes a single, well-defined world in which fields and particles evolve in a nonlocal, continuous and deterministic way to explain the individual processes that give rise to the statistical predictions of quantum theory, and it is one that provides a fascinating world to explore.

¹⁰ There had been some early work on quantum mechanical streamlines in the physical chemistry field, for example by Hirschfelder et al. [34]. Apparently, Hirschfelder soon became discouraged and did not further pursue this line of research.

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