Characters of D=4 Conformal Supersymmetry

V.K. Dobrev
Institute of Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

Abstract

We give character formulae for the positive energy unitary irreducible representations of the N-extended D=4 conformal superalgebras su(2,2/N). Using these we also derive decompositions of long superfields as they descend to the unitarity threshold.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, cf. extensive bibliography in [1] . This makes the classification of the UIRs of the conformal superalgebras very important. The classification was given first for the $D = 4$ superconformal algebras $su(2,2/1)$ [2] and $su(2,2/N)$ (for arbitrary N) [3]. Recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1,2$) was given in [4] (some results being conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [5]. Finally, the cases $D = 9,10,11$ were treated by finding the UIRs of $osp(1/2n)$, [6]. Once we know the UIRs of a (super-)algebra the next question is to find their characters, since these give the spectrum which is important for the applications. This is the question we address in this paper for the UIRs of $D = 4$ conformal superalgebras $su(2,2/N$ using results from [7, 3, 8, 9]. The present paper is a compact version of [1] to which we refer for more extended introduction.

2. Representations of D=4 conformal supersymmetry

The conformal superalgebras in $D = 4$ are $\mathcal{G} = su(2,2/N)$. The even subalgebra of $\mathcal{G}$ is $\mathcal{G}_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, j_1\, ,\, j_2\, ;\, z\, ;\, r_1\, , \ldots , r_{N-1}\,] \tag{1}$$

where d is the conformal weight, j_1, j_2 are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, z represents the $u(1)$ subalgebra which is central for $\mathcal{G}_0$ (and for $N = 4$ is central for $\mathcal{G}$ itself), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or R) symmetry algebra $su(N)$.

We need to recall the root system of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$ - for definiteness - as used in [8]. The positive root system Δ^+ is comprised from α_{ij}, $1 \leq i < j \leq 4+N$. The even positive root system $\Delta^+_{\bar{0}}$ is comprised from α_{ij}, with $i,j \leq 4$ and $i,j \geq 5$; the odd positive root system $\Delta^+_{\bar{1}}$ is comprised from α_{ij}, with $i \leq 4, j \geq 5$. The simple roots are chosen as in (2.4) of [8]:

$$\gamma_1 = \alpha_{12}\ , \ \gamma_2 = \alpha_{34}\ , \ \gamma_3 = \alpha_{25}\ , \ \gamma_4 = \alpha_{4,4+N}\ , \ \gamma_k = \alpha_{k,k+1}\ , \ 5 \leq k \leq 3+N, \tag{2}$$

where γ_3, γ_4 are odd, the rest are even. Thus, the Dynkin diagram is:

$$\underset{1}{\bigcirc} \text{---} \underset{3}{\bigotimes} \text{---} \underset{5}{\bigcirc} \text{---} \cdots \text{---} \underset{3+N}{\bigcirc} \text{---} \underset{4}{\bigotimes} \text{---} \underset{2}{\bigcirc} \tag{3}$$

This is a non-distinguished simple root system with two odd simple roots [11].

We consider lowest weight Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where the lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathcal{H}$ and is in 1-to-1 correspondence with the signature χ. If a Verma module V^Λ is irreducible then it gives the lowest weight irrep L_Λ with the same weight. If a Verma module V^Λ is reducible then it contains a maximal invariant submodule I^Λ and the lowest weight irrep L_Λ with the same weight is given by factorization: $L_\Lambda = V^\Lambda / I^\Lambda$ [10]. The reducibility conditions were given by Kac [10]. There are submodules which are generated by the singular vectors related to all even simple roots [8]. These generate an even invariant submodule I^Λ_c present in all Verma modules that we consider and which must be factored out. Thus, we shall consider also the factor-modules:

$$\tilde{V}^\Lambda = V^\Lambda / I^\Lambda_c \tag{4}$$

The Verma module reducibility conditions for the $4N$ odd positive roots of $\mathcal{G}^{\mathbb{C}}$ were derived in [7, 8] adapting the results of Kac [10], (for $k = 1, \ldots, N$,)

$$d = d^1_{Nk} - z\delta_{N4}\ , \quad d^1_{Nk} \equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N, \tag{5a}$$

$$d = d^2_{Nk} - z\delta_{N4}\ , \quad d^2_{Nk} \equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N, \tag{5b}$$

$$d = d^3_{Nk} + z\delta_{N4}\ , \quad d^3_{Nk} \equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N, \tag{5c}$$

$$d = d^4_{Nk} + z\delta_{N4}\ , \quad d^4_{Nk} \equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N, \tag{5d}$$

where $m_k \equiv \sum_{i=k}^{N-1} r_i\ , \ k < N, \ m_N \equiv 0, \ m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} k r_k\ .$

We next recall the result of [3] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$\begin{aligned} d &\geq d_{\max} = \max(d^1_{N1}, d^3_{NN}) \ , & (6a)\\ d &= d^4_{NN} \geq d^1_{N1} \ , \quad j_1 = 0 \ , & (6b)\\ d &= d^2_{N1} \geq d^3_{NN} \ , \quad j_2 = 0 \ , & (6c)\\ d &= d^2_{N1} = d^4_{NN} \ , \quad j_1 = j_2 = 0 \ , & (6d) \end{aligned}$$

where $d_{\max}$ is the threshold of the continuous unitary spectrum. Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$.

Next we note that if $d > d_{\max}$ the factorized Verma modules are irreducible and coincide with the UIRs L_Λ . These UIRs are called **long** in the modern literature, cf., e.g., [12, 13, 14, 15, 16, 17, 18]. Analogously, the cases when $d = d_{\max}$ in (6a) are called also **semi-short** UIRs, cf., e.g., [12, 14], while the cases (6b,c,d) are also called **short** UIRs, cf., e.g., [13, 14, 15, 16, 17, 18].

Next consider in more detail the UIRs at the four distinguished reducibility points determining the UIRs list above: d^1_{N1} , d^2_{N1} , d^3_{NN} , d^4_{NN} which occur for the following odd roots, resp.:

$$\alpha_{3,4+N} = \gamma_2 + \gamma_4 \ , \quad \alpha_{4,4+N} = \gamma_4 \ , \quad \alpha_{15} = \gamma_1 + \gamma_3 \ , \quad \alpha_{25} = \gamma_3 \ . \quad (7)$$

We note a partial ordering of these four points:

$$d^1_{N1} > d^2_{N1} \ , \qquad d^3_{NN} > d^4_{NN} \ . \quad (8)$$

Due to this ordering *at most two* of these four points may coincide.

First we consider the situations in which *no two* of the distinguished four points coincide. There are four such situations:

$$\begin{aligned} &\mathbf{a} : d = d_{\max} = d^1_{N1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN} \ , & (9a)\\ &\mathbf{b} : d = d^2_{N1} > d^3_{NN} \ , \quad j_2 = 0 \ , & (9b)\\ &\mathbf{c} : d = d_{\max} = d^3_{NN} = d^c \equiv 2 + 2j_1 - z + 2m/N > d^1_{N1} \ , & (9c)\\ &\mathbf{d} : d = d^4_{NN} > d^1_{N1} \ , \quad j_1 = 0 \ . & (9d) \end{aligned}$$

We shall call these cases **single-reducibility-condition (SRC)** Verma modules or UIRs, depending on the context. The cases (9a,c) are semi-short, the cases (9b,d) - short. The corresponding factorized Verma modules $\tilde{V}^\Lambda$ have only one invariant odd submodule which has to be factorized in order to obtain the UIRs. These odd embeddings and factorizations are:

$$\tilde{V}^\Lambda \to \tilde{V}^{\Lambda+\beta} \ , \qquad L_\Lambda = \tilde{V}^\Lambda / I^\beta \ , \quad (10)$$

where we use the convention [7] that arrows point to the oddly embedded module, and we give only the cases for β that we shall use later:

$$\begin{aligned} \beta &= \alpha_{3,4+N} \ , && \text{for } (9a), \quad j_2 > 0, && (11a) \\ &= \alpha_{3,4+N} + \alpha_{4,4+N} \ , && \text{for } (9a), \quad j_2 = 0, && (11b) \\ &= \alpha_{15} \ , && \text{for } (9c), \quad j_1 > 0, && (11c) \\ &= \alpha_{15} + \alpha_{25} \ , && \text{for } (9c), \quad j_1 = 0 && (11d) \end{aligned}$$

The weight shifts $\Lambda' = \Lambda + \beta$, when β is an odd root are called **odd reflections** in [8], and for future reference will be denoted as:

$$\hat{s}_\beta \cdot \Lambda \ \equiv \ \Lambda + \beta \ , \quad (\beta,\beta) = 0, \ (\Lambda,\beta) \neq 0 \ . \qquad (12)$$

We consider now the four situations in which *two* distinguished points coincide:

$$\begin{aligned} \mathbf{ac}: \quad d &= d_{\max} = d^{ac} \ \equiv \ 2 + j_1 + j_2 + m_1 = d^1_{N1} = d^3_{NN} \ , && (13a) \\ \mathbf{ad}: \quad d &= d^1_{N1} = d^4_{NN} \ = \ 1 + j_2 + m_1 \ , \quad j_1 = 0 \ , && (13b) \\ \mathbf{bc}: \quad d &= d^2_{N1} = d^3_{NN} \ = \ 1 + j_1 + m_1 \ , \quad j_2 = 0 \ , && (13c) \\ \mathbf{bd}: \quad d &= d^2_{N1} = d^4_{NN} \ = \ m_1 \ , \quad j_1 = j_2 = 0 \ . && (13d) \end{aligned}$$

We shall call these **double-reducibility-condition (DRC)** Verma modules or UIRs. The case (13a) is a semi-short UIR, while the other cases are short.
The odd embedding diagrams and factorizations for the DRC modules are:

$$\begin{array}{ccc} \tilde{V}^{\Lambda+\beta'} & & \\ \uparrow & & \\ \tilde{V}^{\Lambda} & \longrightarrow & \tilde{V}^{\Lambda+\beta} \end{array} \qquad L_\Lambda \ = \ \tilde{V}^\Lambda / I^{\beta,\beta'} \ , \quad I^{\beta,\beta'} \ = \ I^\beta \cup I^{\beta'} \qquad (14)$$

and we give only the cases for β, β' to be used later:

$$\begin{aligned} (\beta,\beta') &= (\alpha_{15}, \alpha_{3,4+N}), && \text{for } (13a), \quad j_1 j_2 > 0 && (15a) \\ &= (\alpha_{15}, \alpha_{3,4+N} + \alpha_{3,4+N}), && \text{for } (13b), \quad j_1 > 0, \ j_2 = 0 && (15b) \\ &= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N}), && \text{for } (13c), \quad j_1 = 0, \ j_2 > 0 && (15c) \\ &= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N} + \alpha_{3,4+N}), && \text{for } (13d), \quad j_1 = j_2 = 0 && (15d) \end{aligned}$$

3. Character formulae of positive energy UIRs

3.1. Character formulae: generalities

In the beginning of this subsection we follow [19]. Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\Delta}$, simple root system $\hat{\pi}$. Let Γ, (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i, $(\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i, \alpha_i))$. Let V be a lowest weight module with lowest weight Λ and lowest weight vector v_0. It has the following decomposition:

$$V = \underset{\mu\in\Gamma_+}{\oplus} V_\mu \quad , \qquad V_\mu = \{u \in V \mid Hu = (\lambda + \mu)(H)u, \ \forall\ H \in \mathcal{H}\} \tag{16}$$

(Note that $V_0 = \mathbb{C} v_0$.) Let $E(\mathcal{H}^*)$ be the associative Abelian algebra consisting of the series $\sum_{\mu\in\mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geq \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$.
Then the (formal) character of V is defined by:

$$ch\ V \quad = \quad \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\Lambda + \mu) \quad = \quad e(\Lambda) \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\mu) \tag{17}$$

The character formula for Verma modules is [19]:

$$ch\ V^\Lambda \quad = \quad e(\Lambda) \prod_{\alpha\in\Delta^+} (1 - e(\alpha))^{-1} \tag{18}$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) \ = \ \lambda - (\lambda, \alpha^\vee)\alpha\,, \quad \lambda \in \hat{\mathcal{H}}^*\,, \quad \alpha \in \hat{\Delta} \tag{19}$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$ch\ L_\Lambda \quad = \quad \sum_{w\in W} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda}\,, \quad \Lambda \in -\Gamma_+ \tag{20}$$

where the dot $\cdot$ action is defined by $w \cdot \lambda = w(\lambda - \rho) + \rho$.
In the case of basic classical Lie superalgebras (except $osp(1/2N)$) the character formula for Verma modules is [10]:

$$ch\ V^\Lambda \ = \ e(\Lambda) \left(\prod_{\alpha\in\Delta_{\bar{0}}^+} (1 - e(\alpha))^{-1} \right) \left(\prod_{\alpha\in\Delta_{\bar{1}}^+} (1 + e(\alpha)) \right) \tag{21}$$

Note that this may be written as:

$$ch\ V^\Lambda \ = \ ch\ V_0^\Lambda\ ch\ \hat{V}^\Lambda\ , \qquad ch\ \hat{V}^\Lambda \ \equiv \ \prod_{\alpha\in\Delta_{\bar{1}}^+} (1 + e(\alpha))\ , \tag{22}$$

where $\hat{V}^\Lambda \equiv \left(U(\mathcal{G}_+^{I\!\!C})/(\mathcal{G}_+^{I\!\!C})_{(0)}\right) \widetilde{|\Lambda\rangle}$, $ch\ V_0^\Lambda$ is the character of the restriction of V^Λ to the even subalgebra. Obviously, $\hat{V}^\Lambda$ may be viewed as the result of all possible application of the $4N$ odd generators $X^+_{a,4+k}$ on $|\Lambda\rangle$. Thus, $\hat{V}^\Lambda$ has 2^{4N} states. Explicitly, the basis of $\hat{V}^\Lambda$ is [9]:

$$\Psi_{\bar\varepsilon} = \left(\prod_{k=N}^{1} (X^+_{1,4+k})^{\varepsilon_{1,4+k}} \right) \left(\prod_{k=N}^{1} (X^+_{2,4+k})^{\varepsilon_{2,4+k}} \right) \times \\ \times \left(\prod_{k=1}^{N} (X^+_{3,4+k})^{\varepsilon_{3,4+k}} \right) \left(\prod_{k=1}^{N} (X^+_{4,4+k})^{\varepsilon_{4,4+k}} \right) \widetilde{|\Lambda\rangle}\,, \quad \varepsilon_{aj} = 0,1, \tag{23}$$

where $\bar\varepsilon$ denotes the set of all ε_{ij}.

The odd null conditions entwine with the even null conditions as we shall see. The even null conditions carry over from the even null conditions of $\tilde{V}^\Lambda$:

$$(X_1^+)^{1+2j_1}\,|\Lambda\rangle \;=\; 0\,, \tag{24a}$$
$$(X_2^+)^{1+2j_2}\,|\Lambda\rangle \;=\; 0\,, \tag{24b}$$
$$(X_j^+)^{1+r_{N+4-j}}\,|\Lambda\rangle \;=\; 0\,, \quad j=5,...,N+3 \tag{24c}$$

where by $|\Lambda\rangle$ we denote the lowest weight vector of the UIR L_Λ.

For future use we introduce additional notation:

$$\varepsilon_i \;=\; \sum_{k=1}^{N} \varepsilon_{i,4+k}\,, \quad i=1,2,3,4\,, \qquad \varepsilon \;=\; \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4\,. \tag{25}$$

3.2. Character formulae for the long UIRs

As we mentioned if $d > d_{\max}$ there are no further reducibilities, and the UIRs $L_\Lambda = \tilde{V}^\Lambda$ are called *long* since $\hat{L}_\Lambda$ may have the maximally possible number of states 2^{4N} (including the vacuum state). However, the actual number of states may be less than 2^{4N} states due to the fact that - depending on the values of j_a and r_k - not all actions of the odd generators on the vacuum would be allowed. The latter is obvious from explicit signature of the state $\Psi_{\bar\varepsilon}$ [1]:

$$\chi\,(\Psi_{\bar\varepsilon}) = [\,d+\tfrac12\varepsilon\,;\, j_1+\tfrac12(\varepsilon_2-\varepsilon_1),\, j_2+\tfrac12(\varepsilon_4-\varepsilon_3)\,;\, z+\epsilon_N(\varepsilon_3+\varepsilon_4- \\ -\varepsilon_1-\varepsilon_2)\,;\ldots, r_i+\varepsilon_{1,N+4-i}-\varepsilon_{1,N+5-i}+\varepsilon_{2,N+4-i}- \\ -\varepsilon_{2,N+5-i}-\varepsilon_{3,N+4-i}+\varepsilon_{3,N+5-i}-\varepsilon_{4,N+4-i}+\varepsilon_{4,N+5-i}\,,\ldots]\,. \tag{26}$$

Thus, only if $j_1, j_2 \geq N/2$ and $r_i \geq 4$ (for all i) the number of states is 2^{4N} [3], and the character formula may be symbolically[1] written as:

$$ch\ L_\Lambda \ = \ ch\ L^0_\Lambda\ ch\ \hat{V}^\Lambda\ , \qquad j_1, j_2 \geq N/2,\ \ r_i \geq 4,\ \forall\, i\ , \tag{27}$$

where $ch\ L^0_\Lambda$ denotes the character of the restriction of L_Λ to the even subalgebra.

The general formula for $ch\ L_\Lambda$ shall be written in a similar symbolic fashion:

$$ch\ L_\Lambda \ = \ ch\ L^0_\Lambda\ ch\ \hat{L}_\Lambda\ . \tag{28}$$

Moreover, from now on we shall write only the formulae for $ch\ \hat{L}_\Lambda$. Thus, formula (27) may be written equivalently as:

$$ch\ \hat{L}_\Lambda \ = \ ch\ \hat{V}^\Lambda\ , \quad j_1, j_2 \geq N/2, \quad r_i \geq 4, \forall\, i\ . \tag{29}$$

If the auxiliary conditions in (27) are not fulfilled then a careful analysis is necessary. To simplify the exposition we classify the states by the following quantities:

$$\begin{aligned}
\varepsilon^c_j \ &\equiv \ \varepsilon_1 - \varepsilon_2\ , \qquad \varepsilon^a_j \ \equiv \ \varepsilon_3 - \varepsilon_4\ , \qquad\qquad (30)\\
\varepsilon^i_r \ &\equiv \ \varepsilon_{1,5+i} + \varepsilon_{2,5+i} + \varepsilon_{3,4+i} + \varepsilon_{4,4+i} - \varepsilon_{1,4+i} - \varepsilon_{2,4+i} - \varepsilon_{3,5+i} - \varepsilon_{4,5+i}\ ,
\end{aligned}$$

$i = 1, \ldots, N-1$. This gives the following necessary conditions on ε_{ij} for a state to be allowed:

$$\varepsilon^c_j \ \leq \ 2j_1\ , \tag{31a}$$
$$\varepsilon^a_j \ \leq \ 2j_2\ , \tag{31b}$$
$$\varepsilon^i_r \ \leq \ r_{N-i}\ , \quad i = 1, \ldots, N-1\ . \tag{31c}$$

These conditions are also sufficient only for $N = 1$. The exact conditions are:

Criterion: The necessary and sufficient conditions for the state $\Psi_{\bar{\varepsilon}}$ of level ε to be allowed are that conditions (31) are fulfilled and that the state is a descendant of an allowed state of level $\varepsilon - 1$. $\diamondsuit$

The second part of the Criterion will eliminate first of all impossible chiral (or anti-chiral) states which happen when some ε_{aj} contribute to opposing sides of the inequalities in (31a) and (31c), (or (31b) and (31c)) and $j_1 = r_i = 0$, (or $j_2 = r_i = 0$). For the lack of space we omit examples of such impossible states and their combinations given in [1].

[1] We say symbolically, since if we expand the odd part of the character we get the expansion of the corresponding superfield in components, and each component has its own even character. However, we do not lose information using this symbolically factorized form which has the advantage of brevity.

Summarizing the discussion so far, the general character formula may be written as follows:

$$ch\ \hat{L}_\Lambda \ = \ ch\ \hat{V}^\Lambda \ - \ \mathcal{R}\,,\ d \ > \ d_{\max}\,,\ \mathcal{R} = e(\hat{V}^\Lambda_{\rm excl}) = \sum_{\substack{\rm excluded \\ \rm states}} e(\Psi_{\bar\varepsilon})\,, \qquad (32)$$

$$e(\Psi_{\bar\varepsilon}) \ = \ \left(\prod_{k=N}^{1} e(\alpha_{1,4+k})^{\varepsilon_{1,4+k}} \right) \left(\prod_{k=N}^{1} e(\alpha_{2,4+k})^{\varepsilon_{2,4+k}} \right) \times$$
$$\times \left(\prod_{k=1}^{N} e(\alpha_{3,4+k})^{\varepsilon_{3,4+k}} \right) \left(\prod_{k=1}^{N} e(\alpha_{4,4+k})^{\varepsilon_{4,4+k}} \right)\,,$$

where the counter-terms denoted by $\mathcal{R}$ are determined by $\hat{V}^\Lambda_{\rm excl}$ which is the collection of all states (i.e., collection of ε_{jk}) which violate the Criterion.

Finally, we consider two important conjugate special cases. Namely, there are only N anti-chiral states that can be built from the generators $X^+_{4,4+k}$ alone:

$$X^+_{4,5+N-k}\, X^+_{4,6+N-k} \cdots X^+_{4,4+N}\, |\Lambda\rangle\ ,\ \ k = 1,\ldots,N\ ,\ j_2 = r_i = 0,\ \forall\, i\ . \qquad (33)$$

This follows from (31c) which for such states becomes $\varepsilon_{4,4+N-i} \ \leq \ \varepsilon_{4,5+N-i}$ for $i = 1,\ldots,N-1$. The chiral sector of R-symmetry scalars with $j_1 = 0$ is obtained from the above by conjugation.

3.3. Character formulae of SRC UIRs

• **a** $\quad d \ = \ d^1_{N1} \ = \ d^a \ \equiv \ 2 + 2j_2 + z + 2m_1 - 2m/N \ > \ d^3_{NN}$.

• Let first $j_2 > 0$. In these semi-short SRC cases holds the odd null condition:

$$P_{3,4+N}\,|\Lambda\rangle \ = \ \left(2j_2 X^+_{3,4+N} - X^+_4 X^+_2\right)\,|\Lambda\rangle \ = \ 0\ . \qquad (34)$$

Clearly, condition (34) means that the generator $X^+_{3,4+N}$ is eliminated from the basis that is built on the lowest weight vector $|\Lambda\rangle$. Thus, for $N = 1$ and if $r_1 > 0$ for $N > 1$ the character formula is:

$$ch\ \hat{L}_\Lambda \ = \ \prod_{\substack{\alpha\in\Delta^+_{\bar 1} \\ \alpha\neq\alpha_{3,4+N}}} (1 + e(\alpha)) \ - \ \mathcal{R}\ , \qquad j_2 r_1 > 0\ . \qquad (35)$$

There are no counter-terms when $j_1 \ \geq \ N/2, j_2 \ \geq \ (N-1)/2$ and $r_i \ \geq 4$ (for all i), and then the number of states is 2^{4N-1}.

Formula (35) may be described by using the odd reflection (12) with $\beta = \alpha_{3,4+N}$:

$$ch\ \hat{L}_\Lambda \ = \ ch\ \hat{V}^\Lambda \ - \ \frac{1}{1+e(\alpha_{3,4+N})}\ ch\ \hat{V}^{\hat{s}_{\alpha_{3,4+N}}\cdot\Lambda} \ - \ \mathcal{R} = \tag{36a}$$

$$= \sum_{\hat{s}\in\hat{W}_{\alpha_{3,4+N}}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot ch\ \hat{V}^\Lambda \ - \ \mathcal{R}\ , \tag{36b}$$

where $\hat{W}_\beta \ \equiv \ \{1, \hat{s}_\beta\}$, and we have formalized further by introducing notation for the action of an odd reflection on characters:

$$\hat{s}_\beta\ \cdot ch\ V^\Lambda = \frac{1}{1+e(\beta)}\ ch\ V^{\hat{s}_\beta\cdot\Lambda} = \frac{1}{1+e(\beta)}\ ch\ V^{\Lambda+\beta} = \frac{e(\beta)}{1+e(\beta)}\ ch\ V^\Lambda\ . \tag{37}$$

It is natural to use $\hat{W}_\beta$ since only the identity element and the generator $\hat{s}_\beta$ act nontrivially because the action $\hat{s}_\beta$ on characters is nilpotent:

$$(\hat{s}_\beta)^2\ \cdot ch\ V^\Lambda = 0\,. \tag{38}$$

In fact, we shall need more general formula for the action of odd reflections on polynomials $\mathcal{P}$ from $E(\mathcal{H}^*)$. Thus, instead of (37) we shall define the action of $\hat{s}_\beta$ on $\mathcal{P}$ as a homogeneity operator treating $e(\beta)$ as a variable:

$$\hat{s}_\beta\cdot\mathcal{P} \ \equiv \ e(\beta)\,\frac{\partial}{\partial e(\beta)}\ \mathcal{P}\,, \tag{39}$$

where β may be a root or the sum of roots. Obviously, if $\mathcal{P}$ is a monomial which contains a multiplicative factor $1+e(\beta)$ the action (39) is equivalent to (37).

We shall show that in many cases character formulae (35),(36) may be written as follows:

$$ch\ \hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_\beta} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\Big(ch\ \hat{V}^\Lambda \ - \ \mathcal{R}_{\rm long}\Big)\ , \tag{40}$$

where $\mathcal{R}_{\rm long}$ represents the counter-terms for the long superfields for the same values of j_1 and r_i as Λ, while the value of j_2 is zero when j_2 from Λ is zero, otherwise it is any $j_2 \geq N/2$.

Writing (35) as (36) (or (40)) may look as a complicated way to describe the cancellation of a factor from the character formula for $\hat{V}^\Lambda$, however, first of all it is related to the structure of $\tilde{V}^\Lambda$ given by (10), and furthermore may be interpreted - when there are no counter-terms - as the following decomposition:

$$\hat{V}^\Lambda = \hat{L}_\Lambda \ \oplus \hat{L}_{\Lambda+\beta}\,, \tag{41}$$

for $\beta = \alpha_{3,4+N}$. Indeed, for generic signatures $\hat{L}_{\Lambda+\beta}$ is isomorphic to $\hat{L}_\Lambda$ as a vector space (this is due to the fact that $V^{\Lambda+\beta}$ has the same reducibilities as V^Λ, cf. [1], they differ only by the vacuum state. Thus, when there are no counter-terms, both $\hat{L}_\Lambda$ and $\hat{L}_{\Lambda+\beta}$ have the same 2^{4N-1} states.

It is more important that there is a similar decomposition valid for many cases beyond the generic, i.e., we have:

$$\left(\hat{L}_{\rm long}\right)_{|_{d=d^a}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} , \quad N = 1 \text{ or } r_1 > 0 \text{ for } N > 1 , \qquad (42)$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of j_1 and r_i as Λ, while the value of j_2 has to be specified, and equality is as vector spaces.

For $N > 1$ there are possible additional truncations of the basis. Let i_0 be an integer such that $0 \leq i_0 \leq N-1$, and $r_i = 0$ for $i \leq i_0$, and if $i_0 < N-1$ then $r_{i_0+1} > 0$. Now for $i_0 > 0$ the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the basis. This follows from:

$$P_{3,4+N-i}\,|\Lambda\rangle = \left(2j_2 X^+_{3,4+N-i} - X^+_{4,4+N-i}\,X^+_2\right)\,|\Lambda\rangle = 0\,, \quad i \leq i_0\,. \qquad (43)$$

From the above follows that for $i_0 > 0$ the decomposition (42) can not hold. Indeed, the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the irrep $\hat{L}_\Lambda$ due to the fact that we are at a reducibility point, but there is no reason for them to be eliminated from the long superfield. Certainly, some of these generators are present in the second term $\hat{L}_{\Lambda+\alpha_{3,4+N}}$ in (42), but that would be only those which in the long superfield were in states of the kind: $\Phi\, X^+_{3,4+N}\,|\Lambda\rangle$, and, certainly, such states do not exhaust the occurrence of the discussed generators in the long superfield. Symbolically, instead of the decomposition (42) we shall write:

$$\left(\hat{L}_{\rm long}\right)_{|_{d=d^a}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus \hat{L}'_\Lambda\,, \qquad N > 1\,,\ i_0 > 0\,, \qquad (44)$$

where we have represented the excess states by the last term with prime stressing that this is not a genuine irrep, but just a book-keeping device. Formulae as (44) in which not all terms are genuine irreps shall be called *quasi-decompositions.*

The corresponding character formula is:

$$ch\ \hat{L}_\Lambda \ = \prod_{\substack{\alpha\in\Delta^+_{\bar 1}\\ \alpha\neq\alpha_{3,5+N-k}\\ k=1,\ldots,1+i_0}} (1+e(\alpha)) \ - \ \mathcal{R} \ = \qquad (45a)$$

$$= \sum_{\hat{s}\in\hat{W}^a_{i_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot ch\ \hat{V}^\Lambda\ -\ \mathcal{R}\ = \tag{45b}$$

$$= \sum_{\hat{s}\in\hat{W}^a_{i_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \Big(ch\ \hat{V}^\Lambda\ -\ \mathcal{R}_{\rm long}\Big)\ , \tag{45c}$$

$$\hat{W}^a_{i_0}\ \equiv\ \hat{W}_{\alpha_{3,N+4}}\times\hat{W}_{\alpha_{3,N+3}}\times\cdots\times\hat{W}_{\alpha_{3,N+4-i_0}}\ .$$

The restrictions (31) used to determine the counter-terms are, of course, with $\varepsilon_{3,5+N-k}=0,\ \ k=1,\ldots,1+i_0$. Formulae (35),(36),(40) are special cases of (45a,b,c), resp., for $i_0=0$. The maximal number of states in $\hat{L}_\Lambda$ is 2^{4N-1-i_0}. This is the number of states that is obtained from the action of the Weyl group $\hat{W}^a_{i_0}$ on $ch\ \hat{V}^\Lambda$, while the actual counter-term is obtained from the action of the Weyl group on $\mathcal{R}_{\rm long}$.

• Let now $j_2=0$. Then all null conditions above follow from (24b), so these conditions do not mean elimination of the mentioned vectors. In this situation we have the following null condition [1]:

$$X^+_{3,4+N}\,X^+_{4,4+N}\,|\Lambda\rangle\ =\ X^+_4\,X^+_2\,X^+_4\,|\Lambda\rangle\ =\ 0\ . \tag{46}$$

The state in (46) and all of its 2^{4N-2} descendants are zero for any N. Thus, the character formula is similar to (36), but with $\alpha_{3,4+N}$ replaced by $\beta_{12}=\alpha_{3,4+N}+\alpha_{4,4+N}$:

$$ch\ \hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_{\beta_{12}}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\Big(ch\ \hat{V}^\Lambda\ -\ \mathcal{R}_{\rm long}\Big)\ ,\quad N=1 \text{ or } r_1>0\ , \tag{47}$$

where $\hat{W}_{\beta_{12}}\ \equiv\ \{1,\beta_{12}\}$. Note that for $N=1$ formula (47) is equivalent to (35).

Here holds a decomposition similar to (42):

$$\Big(\hat{L}_{\rm long}\Big)_{|d=d^a} = \hat{L}_\Lambda\ \oplus\hat{L}_{\Lambda+\beta_{12}}\ ,\quad N=1 \text{ or } r_1>0 \text{ for } N>1\ , \tag{48}$$

where $\hat{L}_{\rm long}$ is with the same values of $j_1,j_2(=0),r_i$ as Λ. Note, however, that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ belongs to type **b** below.

There are more eliminations for $N>1$ when $i_0>0$ and the decomposition (48) does not hold. Instead, there is a quasi-decomposition similar to (44). We can be more explicit in the case when all $r_i=0$. In that case there are only N anti-chiral states given in (33). Thus the character formula is:

$$ch\ \hat{L}_\Lambda = \sum_{k=1}^N\prod_{i=1}^k e(\alpha_{4,5+N-i})\ +\ \prod_{\substack{\alpha\in\Delta^+_{\bar 1}\\ \varepsilon_1+\varepsilon_2>0}} (1+e(\alpha))\ -\ \mathcal{R}\ ,\quad j_2=r_i=0,\forall\, i \tag{49}$$

• **b** $d = d^2_{N1} = z + 2m_1 - 2m/N > d^3_{NN}$, $j_2 = 0$. Here holds the odd null condition:

$$X^+_4 \, |\Lambda\rangle = X^+_{4,4+N} \, |\Lambda\rangle \ = \ 0 \, . \tag{50}$$

Since $j_2 = 0$ from (24b) and (50) follows the additional null condition:

$$X^+_{3,4+N} \, |\Lambda\rangle = [X^+_2, X^+_4] \, |\Lambda\rangle = 0 \, . \tag{51}$$

For $N > 1$ and $r_1 > 2$ each of these UIRs enters as the second term in decomposition (48).
Further, for $N > 1$ there are additional recursive null conditions if $r_i = 0$, $i \leq i_0 < N$ which follow from (24c) and (51):

$$X^+_{3,4+N-i} \, |\Lambda\rangle = [X^+_{3,5+N-i}, X^+_{4+N-i}] \, |\Lambda\rangle = 0 \, , \quad r_j = 0, \ 1 \leq j \leq i \leq i_0 \tag{52a}$$

$$X^+_{4,4+N-i} \, |\Lambda\rangle = [X^+_{4,5+N-i}, X^+_{4+N-i}] \, |\Lambda\rangle = 0 \, , \quad r_j = 0, \ 1 \leq j \leq i \leq i_0 \tag{52b}$$

Thus, $2(1+i_0)$ generators $X^+_{3,5+N-k}$, $X^+_{4,5+N-k}$, $k = 1, \ldots, 1+i_0$, are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-2-2i_0}$. The corresponding character formula is:

$$ch \ \hat{L}_\Lambda \ = \sum_{\hat{s} \in \hat{W}^b_{i_0}} (-1)^{\ell(\hat{s})} \ \hat{s} \cdot ch \ \hat{V}^\Lambda \ - \ \mathcal{R} \, , \quad j_2 = r_i = 0, \ i \leq i_0 \, , \tag{53a}$$

$$\hat{W}^b_{i_0} \ \equiv \ \hat{W}^a_{i_0} \ \times \ \hat{W}_{\alpha_{4,N+4}} \times \hat{W}_{\alpha_{4,N+3}} \times \cdots \times \hat{W}_{\alpha_{4,N+4-i_0}} \tag{53b}$$

where determining the counter-terms we use $\varepsilon_{a,4+k} = 0$, $a = 3, 4$, $k = 1, \ldots, 1+i_0$.
The case of R-symmetry scalars ($i_0 = N-1$) should be called chiral since all anti-chiral generators are eliminated.

• **c** $d = d^3_{NN} = d^c \equiv 2 + 2j_1 - z + 2m/N > d^1_{N1}$.
• **d** $d = d^4_{NN} = -z + 2m/N > d^1_{N1}$, $j_1 = 0$.

These cases are conjugate to the cases **a**,**b**, resp. All results may be obtained by the substitutions (for $a = 1, 2$, $k = 1, ..., N$):

$$j_1 \longleftrightarrow j_2 \, , \ r_i \longleftrightarrow r_{N-i} \, , z \longleftrightarrow -z, \alpha_{a,4+k} \longleftrightarrow \alpha_{4-a,N+5-k}$$

and so we shall omit them here, cf. [1].

3.4. Character formulae of DRC UIRs

Let first $N > 1$ and $r_1 r_N - 1 > 0$, (i.e., $i_0 = i'_0 = 0$). Then holds the following character formula:

$$ch \ \hat{L}_\Lambda \ = \sum_{\hat{s} \in \hat{W}_{\beta,\beta'}} (-1)^{\ell(\hat{s})} \ \hat{s} \cdot ch \ \hat{V}^\Lambda \ - \ \mathcal{R} \, , \tag{54a}$$

$$\hat{W}_{\beta,\beta'} \quad \equiv \quad \hat{W}_{\beta} \ \times \hat{W}_{\beta'} \tag{54b}$$

The above formula is proved in [1] similarly to what we had in the SRC cases, however, it takes into account the richer structure given explicitly already in the paper [7]. The proof is not valid for $N = 1$, nevertheless, formula (54) holds also then for the case (15a), cf. Appendix A.1. of [1].

• **ac** $d = d_{\max} = d^1_{N1} = d^3_{NN} = d^{ac} \equiv 2 + j_1 + j_2 + m_1$. In these semi-short DRC cases hold the null condition (34) and its conjugate. In addition, for $N > 1$ if $r_i = 0, i = 1, \dots, i_0$, holds (43) and if $r_{N-i} = 0$, $i = 1, \dots, i'_0$, holds the conjugate to (43).

There are two basic situations. The first is when $i_0 + i'_0 \leq N-2$. This means that not all r_i are zero and all eliminations are as described separately for cases •**a** and •**c**. These semi-short UIRs may be called Grassmann-analytic following [14], since odd generators from different chiralities are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-2-i_0-i'_0}$.

The second is when $i_0 + i'_0 \leq N - 2$ does not hold which means that all r_i are zero, and in fact we have $i_0 = i'_0 = N - 1$ and all generators $X^+_{1,4+k}$ and $X^+_{3,4+k}$ are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is 2^{2N}.

• For $j_1 j_2 > 0$ the character formulae are combinations of (45) and its conjugate [1]:

$$ch\ \hat{L}_\Lambda \quad = \sum_{\hat{s} \in \hat{W}^{ac}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s} \cdot \Big(ch\ \hat{V}^\Lambda \ - \ \mathcal{R}_{\text{long}} \Big)\ , \quad N \geq 1, \tag{55a}$$

$$\begin{aligned} \hat{W}^{ac}_{i_0,i'_0} \quad \equiv \quad & \hat{W}^a_{i_0} \ \times \ \hat{W}^c_{i'_0}\ , \qquad j_1 j_2 > 0\ , \\ & \text{either} \ \ i_0 + i'_0 \leq N - 2, \\ & \quad r_i = 0,\ i = 1, 2, \dots, i_0, N - i'_0, N - i'_0 + 1, \dots, N - 1, \\ & \quad r_i > 0,\ i = i_0 + 1, N - i'_0 - 1\ , \\ & \text{or} \ \ i_0 = i'_0 = N - 1\ , \quad r_i = 0,\ \forall i\ . \end{aligned} \tag{55b}$$

The last subcase is of R-symmetry scalars.

For $N > 1$ and $i_0 = i'_0 = 0$ formula (55) is equivalent to (54) with $\beta = \alpha_{15}$, $\beta' = \alpha_{3,4+N}$. Also holds the following decomposition:

$$\Big(\hat{L}_{\text{long}}\Big)_{|d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\alpha_{3,4+N}}\ ,\ \ r_1\, r_{N-1} > 0\,, \tag{56}$$

$\hat{L}_{\text{long}}$ being a long superfield with the same values of r_i as Λ and with $j_1, j_2 \geq N/2$.

• For $j_1 > 0, j_2 = 0$ the character formulae are combinations of (47) and the conjugate to (45) [1]:

$$ch\ \hat{L}_\Lambda \ = \ \sum_{\hat{s}\in\hat{W}^{a'c}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot ch\ \hat{V}^\Lambda \ - \ \mathcal{R} \ = \tag{57a}$$

$$= \ \sum_{\hat{s}\in\hat{W}^{a'c}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \Big(ch\ \hat{V}^\Lambda \ - \ \mathcal{R}_{\rm long}\Big)\ , \quad r_1>0\,, \tag{57b}$$

$$\hat{W}^{a'c}_{i'_0} \ \equiv \ \hat{W}_{\beta_{12}} \ \times \ \hat{W}^{c}_{i'_0}\,, \qquad \beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}\,. \tag{57c}$$

For $i_0 = i'_0 = 0$ holds the decomposition:

$$\Big(\hat{L}_{\rm long}\Big)_{|d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\beta_{12}}\ ,\ \ r_1\,r_{N-1}>0\,, \tag{58}$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of $j_2(=0), r_i$ as Λ and with $j_1 \geq N/2$. Note that the UIR $\hat{L}_{\Lambda+\alpha_{15}}$ is also of the type **ac** under consideration, while the last two UIRs are short from type **bc** considered below.

For R-symmetry scalars we combine (49) and the conjugate to (45a):

$$ch\ \hat{L}_\Lambda = \sum_{k=1}^{N}\prod_{i=1}^{k} e(\alpha_{4,5+N-i})\ +\ \prod_{\substack{\alpha\in\Delta^+_{\bar 1}\\ \alpha\neq\alpha_{1,4+k}\,,\\ k=1,\ldots,N\\ \varepsilon_2>0}} (1+e(\alpha))\ -\ \mathcal{R}\,, \tag{59}$$

$$r_i = 0,\ \forall\, i\ , \qquad N>1.$$

- The case $j_1 = 0, j_2 > 0$ is obtained from the previous one by conjugation. Here for $i_0 = i'_0 = 0$ holds the decomposition:

$$\Big(\hat{L}_{\rm long}\Big)_{|d=d^{ac}} = \hat{L}_\Lambda \ \oplus\ \hat{L}_{\Lambda+\alpha_{3,4+N}} \ \oplus\ \hat{L}_{\Lambda+\beta_{34}} \ \oplus\ \hat{L}_{\Lambda+\alpha_{3,4+N}+\beta_{34}}\ ,\ \ r_1 r_{N-1}>0 \tag{60}$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of $j_1(=0), r_i$ as Λ and with $j_2 \geq N/2$. Note that the UIR $\hat{L}_{\Lambda+\alpha_{3,4+N}}$ is again of the type **ac** under consideration, while the last two UIRs are actually from type **ad** considered below.

- For $j_1 = j_2 = 0$ the character formulae are combinations of (47) and its conjugate:

$$ch\ \hat{L}_\Lambda \ = \ \sum_{\hat{s}\in\hat{W}^{a'c'}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \Big(ch\ \hat{V}^\Lambda \ - \ \mathcal{R}_{\rm long}\Big)\ , \quad r_1 r_{N-1}>0\,, \tag{61a}$$

$$\hat{W}^{a'c'}_{i'_0} \equiv \hat{W}_{\beta_{12}} \times \hat{W}_{\beta_{34}} . \tag{61b}$$

For $i_0 = i'_0 = 0$ holds the decomposition:

$$\left(\hat{L}_{\rm long}\right)_{|d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}_{\Lambda+\beta_{12}+\beta_{34}} , \quad r_1\, r_{N-1} > 0 , \tag{62}$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of $j_1(=0), j_2(=0), r_i$ as Λ. Note that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ is of the type **bc**, $\hat{L}_{\Lambda+\beta_{34}}$ is of the type **ad**, $\hat{L}_{\Lambda+\beta_{12}+\beta_{34}}$ is of the type **bd**, these three being considered below.

For R-symmetry scalars we combine (49) and its conjugate:

$$\begin{aligned} ch\ \hat{L}_\Lambda \ =\ & \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) \ +\ \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) \ + \\ & + \prod_{\substack{\alpha\in\Delta_1^+ \\ \varepsilon_1+\varepsilon_2>0 \\ \varepsilon_3+\varepsilon_4>0}} (1+e(\alpha)) \ -\ \mathcal{R} , \qquad r_i = 0,\ \forall\, i , \quad N>1 . \end{aligned} \tag{63}$$

$\bullet$ **ad** $\quad d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1$, $j_1 = 0$. In these short DRC cases hold the three null conditions (34), and the conjugates to (50) and (51). In addition, for $N>1$ if $r_i = 0, i = 1,\dots,i_0$, hold (43) and if $r_{N-i} = 0, i = 1,\dots,i'_0$, hold the conjugate of (52).

If $i_0 + i'_0 \le N-2$ all eliminations are as described separately for cases $\bullet$ **a** and $\bullet$ **d**. All these are Grassmann-analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-3-i_0-2i'_0}$. Interesting subcases are the so-called BPS states, cf., [20, 21, 14, 17, 22, 23, 24, 25]. They are characterized by the number κ of odd generators which annihilate them - then the corresponding case is called $\frac{\kappa}{4N}$-BPS state. For example consider $N = 4$ and $\frac{1}{4}$-BPS cases with $z = 0 \Rightarrow d = 2m/N$. One such case is obtained for $i_0 = 1, i'_0 = 0, j_2 > 0$, then $d = \frac{1}{2}(2r_2 + 3r_3)$, $r_1 = 0$, $r_2 > 0$, $r_3 = 2(1 + j_2)$.

For $j_2\, m_1 > 0$ the character formula is a combination of (45) and the conjugate of (53):

$$ch\ \hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{ad}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot ch\ \hat{V}^\Lambda \ -\ \mathcal{R} , \qquad N>1, \tag{64a}$$

$$\begin{aligned} \hat{W}^{ad}_{i_0,i'_0} \ \equiv\ & \hat{W}^a_{i_0} \times \hat{W}^d_{i'_0} , \qquad j_2\, m_1 > 0 , \\ & r_i = 0,\ i = 1,2,\dots,i_0, N-i'_0, N-i'_0+1,\dots,N-1, \\ & r_i > 0,\ i = i_0+1, N-i'_0-1 . \end{aligned} \tag{64b}$$

For $i_0 = i'_0 = 0$ some of these UIRs appear (up to two times) in the decomposition (60) [1].

For $j_2 = 0\,, m_1 > 0$ the character formula is a combination of (47) and the conjugate of (53a):

$$ch\ \hat{L}_\Lambda \ = \ \sum_{\hat{s}\in\hat{W}^{a'd}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot ch\ \hat{V}^\Lambda \ - \ \mathcal{R}\ , \qquad N > 1, \tag{65a}$$

$$\hat{W}^{a'd}_{i'_0} \ \equiv \ \hat{W}_{\beta_{12}} \ \times \ \hat{W}^d_{i'_0}\ , \tag{65b}$$

where $\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}$. For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (62) or (60) [1].

In the case of R-symmetry scalars we have $i_0 = i'_0 = N-1$, $\kappa = 3N$ and all generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $X^+_{3,4+k}$ are eliminated. Here holds $d = -z = 1 + j_2$. These anti-chiral irreps form one of the three series of **massless** UIRs; they are denoted χ^+_s , $s = j_2 = 0, \frac{1}{2}, 1, \ldots$, in Section 3 of [3]. Besides the vacuum they contain only N states in $\hat{L}_\Lambda$ given by (33) for $k = 1, \ldots, N$. These should be called ultrashort UIRs. The character formula can be written most explicitly:

$$ch\ \hat{L}_\Lambda = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i})\,, \qquad j_1 = r_i = 0,\ \forall\, i\,, \quad N \geq 1\,, \tag{66}$$

and it is valid for any j_2 .

• **bc** $d = d^2_{N1} = d^3_{NN} = 1+j_1+m_1$, $j_2 = 0$, $z = 2m/N - m_1 + 1 + j_1$. This case is conjugate to the previous one and all results may be obtained as in the SRC conjugate cases.

• **bd** $d = d^2_{N1} = d^4_{NN} = m_1$, $j_1 = j_2 = 0$, $z = 2m/N - m_1$. In these short DRC cases hold the four null conditions (50), (51), and their conjugates.

For $N = 1$ this is the trivial irrep with $d = j_1 = j_2 = z = 0$, since, we have the null conditions: $X^+_k\,|\Lambda\rangle = 0$ for all simple root generators (and consequently for all generators) and the irrep consists only of the vacuum $|\Lambda\rangle$.

For $N > 1$ the situation is non-trivial. In addition to the mentioned conditions, and if $r_i = 0$, $i = 1, \ldots, i_0$, hold (52) and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, hold the conjugates of (52).

If $i_0 + i'_0 \leq N - 2$ all eliminations are as described separately for cases • **b** and • **d**. These are also Grassmann-analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-4-2i_0-2i'_0}$. For $N = 4$ for the BPS cases we take $z = \frac{1}{2}(r_3 - r_1) = 0 \Rightarrow d = 2r_1 + r_2$. In the $\frac{1}{4}$-BPS case we have $i_0 = i'_0 = 0$, $r_1 = r_3 > 0$.

For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (62) [1].
Most interesting is the case $i_0+i'_0 = N-2$, then there is only one non-zero r_i, namely, $r_{1+i_0} = r_{N-1-i'_0} > 0$, while the rest r_i are zero. Thus, the Young tableau parameters are: $m_1 = r_{1+i_0}$, $m = (1+i_0)r_{1+i_0}$.
An important subcase is when $d = m_1 = 1$, then $m = i_0+1 = N-1-i'_0$, $r_i = \delta_{mi}$, and these irreps form the third series of **massless** UIRs. In Section 3 of [3] they are denoted χ'_n, $n = m \geq \frac{1}{2}N$, $(z = 2n/N-1)$, χ'^{+}_n, $n = N-m \geq \frac{1}{2}N$, $(z = 1-2n/N)$. Note that for even N there is the coincidence: $\chi'_n = \chi'^{+}_n$, where $n = m = N-m = N/2$. Here we shall parametrize these UIRs by the parameter $i_0 = 0, 1, \ldots, N-1$.
Another subcase here are $\frac{1}{2}$-BPS states for even N with $z = 0 \Rightarrow d = m_1 = 2m/N \Rightarrow i_0 = i'_0 = N/2-1 \Rightarrow m_1 = r_{N/2}$, $m = \frac{N}{2}r_{N/2}$. These are also massless only if $r_{N/2} = 1$, which is the self-conjugate case: χ'_n, $n = N/2$. For $N = 4$ we have: $i_0 = i'_0 = 1$, $r_1 = r_3 = 0$, $r_2 > 0$, which is also massless if $r_2 = 1$.
Finally, in the case of R-symmetry scalars we have $i_0 = i'_0 = N-1$ and all $4N$ odd generators are eliminated, all quantum numbers are zero, (cf. (13d)), and this is the trivial irrep (as for $N = 1$).
For $m_1 > 0$ the character formula is a combination of (53) and its conjugate:

$$ch\ \hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{bd}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot ch\ \hat{V}^\Lambda - \mathcal{R}\,, \qquad N > 1\,, \tag{67a}$$

$$\hat{W}^{bd}_{i_0,i'_0} \equiv \hat{W}^{b}_{i_0} \times \hat{W}^{d}_{i'_0}\,, \tag{67b}$$
$$r_i = 0,\ i = 1,2,\ldots,i_0, N-i'_0, N-i'_0+1,\ldots,N-1,$$
$$r_i > 0,\ i = i_0+1, N-i'_0-1\,,$$

where $\mathcal{R}$ designates the counter-terms due to our Criterion, in particular, due to (31) taken with $\varepsilon_{a,N+1-k} = 0$, $a = 1,2$, $k = 1,\ldots,1+i'_0$, $\varepsilon_{bj} = 0$, $b = 3,4$, $k = j,\ldots,1+i_0$.
Also for the third series of massless UIRs have an explicit character formula without counter-terms. Fix the parameter $i_0 = 0,1,\ldots,N-2$. Then there are only the following states in $\hat{L}_\Lambda$:

$$X^+_{2,N+4-j} \cdots X^+_{2,N+4-i_0}\,|\Lambda\rangle\,, \quad j = 0,1,\ldots,i_0\,, \tag{68a}$$
$$X^+_{4,4+k} \cdots X^+_{4,N+3-i_0}\,|\Lambda\rangle\,, \quad k = 1,\ldots,N-1-i_0\,, \tag{68b}$$

altogether N states besides the vacuum [1]. The character formula is:

$$ch\ \hat{L}_\Lambda = 1 + \sum_{j=0}^{i_0}\prod_{i=j}^{i_0} e(\alpha_{2,N+4-i}) + \sum_{k=1}^{N-1-i_0}\prod_{i=k}^{N-1-i_0} e(\alpha_{4,4+i})\,,$$
$$i_0 = 0,1,\ldots,N-2\,, \qquad r_i = \delta_{i,i_0+1}\,. \tag{69}$$

References

[1] V.K. Dobrev, hep-th/0406154, 78 pages.

[2] M. Flato and C. Fronsdal, Lett. Math. Phys. **8**, 159 (1984).

[3] V.K. Dobrev and V.B. Petkova, Phys. Lett. **162B**, 127 (1985).

[4] S. Minwalla, Adv. Theor. Math. Phys. **2**, 781 (1998).

[5] V.K. Dobrev, J. Phys. **A35** 7079-7100 (2002) hep-th/0201076.

[6] V.K. Dobrev and R.B. Zhang, Phys. Atom. Nuclei, to appear; hep-th/0402039.

[7] V.K. Dobrev and V.B. Petkova, Lett. Math. Phys. **9**, 287 (1985).

[8] V.K. Dobrev and V.B. Petkova, Fortschr. d. Phys. **35**, 537-572 (1987).

[9] V.K. Dobrev and V.B. Petkova, Lect. Notes in Phys., vol. 261 (1986) p. 291 & 300.

[10] V.G. Kac, Lect. Notes in Math. **676** (Springer-Verlag, Berlin, 1978) pp. 597-626.

[11] V.G. Kac, Adv. Math. **26**, 8-96 (1977); Comm. Math. Phys. **53**, 31-64 (1977).

[12] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, hep-th/9904017.

[13] S. Ferrara and E. Sokatchev, Lett. Math. Phys. **52**, 247 (2000); hep-th/0010117.

[14] S. Ferrara and E. Sokatchev, Int. J. Theor. Phys. **40**, 935 (2001).

[15] G. Arutyunov, B. Eden and E. Sokatchev, Nucl. Phys. **B619**, 359 (2001).

[16] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, J. High En. Phys. 05 (2001) 042.

[17] B. Eden and E. Sokatchev, Nucl. Phys. **B618**, 259 (2001).

[18] P.J. Heslop and P.S. Howe, Phys. Lett. **516B**, 367 (2001); hep-th/0307210.

[19] J. Dixmier, *Enveloping Algebras*, (North Holland, New York, 1977).

[20] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, hep-th/9912007.

[21] S. Ferrara and E. Sokatchev, hep-th/0003051; hep-th/0007058; hep-th/0110174.

[22] A.V. Ryzhov, J. High En. Phys. 11 (2001) 046; hep-th/0307169.

[23] E. D'Hoker and A.V. Ryzhov, J. High En. Phys. 02 (2002) 047.

[24] G. Arutyunov and E. Sokatchev, Nucl. Phys. **B635**, 3-32 (2002); hep-th/0209103.

[25] E. D'Hoker, P. Heslop, P. Howe and A.V. Ryzhov, J. High En. Phys. 04 (2003) 038.