

From regular to irregular: a unified origin for Argyres-Douglas theories

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ABSTRACT: We propose that Argyres-Douglas theories of type $D_p(\mathrm{SU}(N))$ and (A_{p-1}, A_{N-1}) — both realizable as Type A class \mathcal{S} theories with irregular punctures — can be obtained via a sequence of mass deformations from a common ancestor: a class \mathcal{S} theory with only regular punctures. Building on our previous work, this result establishes that these theories ultimately originate from 6d $\mathcal{N} = (1, 0)$ orbi-instanton theories compactified on a torus. The requisite 4d mass deformations are realized as tractable Fayet-Iliopoulos deformations on the 3d mirror quiver. The core of our method is a constructive procedure that utilizes the Euclidean algorithm to define a chain of deformations connecting different $D_p(\mathrm{SU}(N))$ theories. By reversing this chain, we recursively build a “parent” star-shaped quiver for any given (N, p) . This quiver is the 3d mirror theory of the required class \mathcal{S} ancestor. We substantiate our general claims with several detailed examples that explicitly illustrate the deformation procedure.

KEYWORDS: Field Theories in Lower Dimensions, Duality in Gauge Field Theories, Field Theories in Higher Dimensions, Extended Supersymmetry

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1 Introduction and summary

One of the largest known families of 4d $\mathcal{N} = 2$ superconformal field theories (SCFTs) is obtained by compactifying 6d $\mathcal{N} = (2, 0)$ theories on a Riemann surface with punctures. This framework, known as the class \mathcal{S} construction [1, 2], has produced a rich landscape of 4d theories that have been studied extensively over the past fifteen years.¹ In this paper, we consider class \mathcal{S} theories of Type A with both regular and irregular punctures, focusing on the so-called Argyres-Douglas (AD) theories [5–11]. These include $D_p(\text{SU}(N))$ theories [12] (see also [13, 14]) and (A_{p-1}, A_{N-1}) theories [10].

¹See, e.g., [3, 4] for recent reviews.

The $D_p(\text{SU}(N))$ class is realized within the Type A class \mathcal{S} framework by considering a Riemann sphere with one full regular puncture and one irregular (Type I) puncture [11, 15]. These models also admit a geometric engineering realization as a Type IIB string theory compactification on a non-compact Calabi-Yau (CY) threefold, realized as the zero-locus of a single hypersurface singularity in $\mathbb{C}^3 \times \mathbb{C}^*$ [10, 16]. As most of these theories are strongly coupled, an $\mathcal{N} = 2$ Lagrangian description is not yet available. Nevertheless, many of their properties can be computed exactly via their geometric realization.

Although the moduli space of $D_p(\text{SU}(N))$ theories is now well understood, the Renormalization Group (RG) flows between them remain relatively unexplored, particularly those triggered by relevant deformations. RG flows initiated by Higgsing, for instance, are better understood due to the constraints of 't Hooft anomaly matching. A key example is the flow from a $D_{p+N}(\text{SU}(N))$ theory (with $p > 0$) to the (A_{p-1}, A_{N-1}) theory by fully closing the regular puncture. This process involves assigning a vacuum expectation value (VEV) to the $\text{SU}(N)$ moment map and constitutes a Higgs branch RG flow. The same (A_{p-1}, A_{N-1}) theory can also be reached from $D_p(\text{SU}(N))$ via the Maruyoshi-Song (MS) flow [16–19].

In recent years, many properties of the $D_p(\text{SU}(N))$ theories (or more generally, $D_p^b(G)$ theories) have been investigated [16, 20–34]. Notably, in [35] (see also [11, 36]), 3d mirror theories were proposed in the form of $\mathcal{N} = 4$ Lagrangian theories represented by unitary quivers.² Generally, these theories possess a global symmetry algebra $\mathfrak{su}(N) \oplus \mathfrak{u}(1)^{\mu-1}$, where $(\mu - 1)$ is the number of additional mass parameters, excluding the Casimirs of $\text{SU}(N)$.³ The value μ corresponds to the greatest common divisor of p and N .

The purpose of this work is to build on our recent findings in [41], where we demonstrated that all class \mathcal{S} theories of Type A with regular untwisted punctures originate from mass deformations of orbi-instanton theories [42–45]. These parent theories arise from the dimensional reduction on a two-torus of 6d $\mathcal{N} = (1, 0)$ theories realized on M5-branes probing an M9-brane on a $\mathbb{C}^2/\mathbb{Z}_k$ singularity. Our central message is that all Type A class \mathcal{S} theories with untwisted punctures can be recovered from 6d $\mathcal{N} = (1, 0)$ theories on a torus, followed by relevant (mass) deformations in 4d. This approach obviates the need to consider Riemann surfaces of higher genus or with punctures at the 6d level.

Since the resulting 4d theories are often strongly coupled, analyzing these mass deformations directly is a non-trivial task. We circumvent this by mapping the 4d relevant deformation to a Fayet-Iliopoulos (FI) deformation in the corresponding 3d mirror theory. The advantage is that the 3d mirror is a Lagrangian theory, where the effect of the FI term can be reliably studied via the equations of motion [41, 46, 47]. We then posit that the infrared fixed point of the 4d RG flow is the theory whose 3d mirror is the quiver obtained after the FI deformation.

In this work, we show that class \mathcal{S} theories with *irregular* punctures also adhere to this pattern. Specifically, we demonstrate that the $D_p(\text{SU}(N))$ theories can be obtained by activating suitable mass deformations in class \mathcal{S} theories that have only regular punctures. Furthermore, we find that a mass deformation of the $\text{SU}(N)$ symmetry in a $D_p(\text{SU}(N))$

²The 3d mirror theories for most of the $D_p^b(G)$ theories have been found in a series of works [37–40], containing also a detailed study of their conformal manifolds and generalized symmetries.

³This symmetry can get enhanced further to a larger Lie algebra that contains $\mathfrak{su}(N) \oplus \mathfrak{u}(1)^{\mu-1}$ as a subalgebra.

theory induces an RG flow to the (A_{p-1}, A_{N-1}) AD theory. This flow shares its UV and IR fixed points with the MS flow, but manifestly preserves $\mathcal{N} = 2$ supersymmetry.

A central result of this work is the following:

Argyres-Douglas theories of type $D_p(\text{SU}(N))$ and (A_{p-1}, A_{N-1}) share a unified origin. These theories, both characterized as Type A class \mathcal{S} theories with irregular punctures, arise from a common ancestor — a class \mathcal{S} theory with only regular punctures — via a sequence of mass deformations. Combined with our previous work [41], this demonstrates that their ultimate origin lies in the torus compactification of 6d $\mathcal{N} = (1, 0)$ orbi-instanton theories.

Outlook. The $D_p(\text{SU}(N))$ and (A_m, A_n) theories represent some of the simplest instances of geometric engineering in Type IIB string theory. Our results suggest that torus compactifications (possibly with a twist along the cycles of the torus) of 6d theories, when supplemented with all possible relevant deformations in 4d, offer a general method for exploring the landscape of 4d SCFTs with eight supercharges, at least those realizable in string theory. It would therefore be of great interest to generalize our analysis to further test this claim by studying, for instance, other types of class \mathcal{S} theories or other $D_p(G)$ and (G, G') AD theories.

Structure of the paper. The paper is organized as follows. In section 2, we review the application of Fayet-Iliopoulos (FI) deformations to 3d $\mathcal{N} = 4$ theories, based on the analyses in [41, 46, 47]. In section 2.2, we introduce new FI deformation rules that are instrumental to our results.

In section 3, we study various classes of FI deformations applied to the 3d mirrors of $D_p(\text{SU}(N))$ theories, which are crucial to derive our main result. We begin in section 3.1 by showing that the 3d mirror of $D_\mu(\text{SU}(\alpha\mu))$ can be deformed into the mirror of $D_\mu(\text{SU}(\alpha\mu - 1))$, providing a novel example of an FI deformation that transforms one star-shaped quiver into another corresponding to a $D_p(\text{SU}(N))$ theory. Subsequently, in section 3.2, we demonstrate a key result: the mirror of a $D_p(\text{SU}(N))$ theory with $p \leq N$ can be deformed to yield the mirror of $D_N(\text{SU}(p))$. This deformation is a cornerstone of the general ansatz presented in section 4.

In section 4, we propose a general star-shaped quiver that, through specific deformations, flows to the 3d mirror of any $D_p(\text{SU}(N))$ theory. We explain the rationale behind this ansatz and provide several instructive examples. Further examples and a general strategy for deforming these star-shaped quivers are detailed in section 5. Finally, appendix A provides a review of the construction of 3d mirror theories for all $D_p(\text{SU}(N))$ Argyres-Douglas theories, while appendix B reviews the procedure for obtaining any star-shaped quiver by deforming orbi-instanton theories.

Conventions and notation.

1. We primarily consider unitary quiver gauge theories. A circle \circ represents a gauge node, with a label indicating the rank of the group. A square \square represents a flavor node, with its label denoting the number of fundamental hypermultiplets. A line connecting two nodes represents a hypermultiplet in the bifundamental representation. If a line corresponds to multiple hypermultiplets, this will be specified. A loop on a unitary gauge node indicates an adjoint-valued hypermultiplet.

2. We adopt the concept of the “excess number” (\mathbf{e}), following the classification of gauge nodes as balanced, underbalanced, or overbalanced from [48]. For a $U(r)$ gauge group with N_f fundamental hypermultiplets, the excess number is defined as:

$$\mathbf{e}_{U(r)} = N_f - 2r. \tag{1.1}$$

A quiver theory is deemed “good” if all its gauge nodes have $\mathbf{e}_{U(r)} \geq 0$, ensuring that all monopole operators respect the unitarity bound. A node is *balanced* if $\mathbf{e}_{U(r)} = 0$ and *overbalanced* if $\mathbf{e}_{U(r)} > 0$. Conversely, a node is *underbalanced* if $\mathbf{e}_{U(r)} < 0$. A theory is called “ugly” if any gauge group has $\mathbf{e}_{U(r)} = -1$, indicating a monopole operator saturating the unitarity bound. If any gauge group has $\mathbf{e}_{U(r)} < -1$, the theory is “bad”, as it contains monopole operators that violate the unitarity bound. It is important to note that a theory which appears “good” may be revealed to be “bad” when considering monopoles charged under multiple gauge groups, as is the case for certain affine Dynkin-shaped unitary quivers [49].⁴

3. For a $T_\rho[SU(N)]$ theory, if the partition ρ is not specified (e.g., $T[SU(N)]$), it is assumed to be the principal partition $\rho = [1^N]$. We may also drop the $[SU(N)]$ when there is no ambiguity, referring to the theory simply as T_ρ .
4. As detailed in appendix A, for $D_p(SU(N))$ theories, we frequently use the definitions:

$$\mu = \text{GCD}(N, p), \quad N = \mu \mathbf{n}, \quad p = \mu \mathbf{p}. \tag{1.2}$$

Furthermore, when discussing theories with $p \leq N$, we adopt the notation from [35]:

$$x = \left\lfloor \frac{N}{p} \right\rfloor, \quad M = N - p(x + 1), \tag{1.3}$$

and

$$m_A = \mathbf{p}(1 + x) - \mathbf{n}, \quad m_B = \mathbf{n} - \mathbf{p}x, \quad m_G = m_A m_B. \tag{1.4}$$

5. We use the term “parent star-shaped quiver” to refer to the 3d mirror of a 4d class \mathcal{S} theory that serves as the parent theory for a given $D_p(SU(N))$ theory, meaning that the latter can be obtained via a sequence of mass deformations starting from the class \mathcal{S} theory in question. These mirrors are known to be star-shaped quivers according to [52].

2 Rules of Fayet-Iliopoulos and mass deformations

The application of Fayet-Iliopoulos deformations to 3d $\mathcal{N} = 4$ theories was introduced in [46] to study the Higgs branch of 5d SCFTs arising from circle compactifications of orbi-instanton theories. In that context, FI deformations on the magnetic quivers correspond to mass deformations on the 5d Higgs branch, allowing for an analysis of its structure when moving around the extended Coulomb branch. The set of possible deformations on magnetic quivers, first explored in [46, 47], was recently reviewed and extended in [41]. In this section, we briefly review the established rules for FI deformations on unitary magnetic quivers and introduce new deformations that are essential for the present work.

⁴For a similar discussion applied to orthosymplectic magnetic quivers, see [50, 51].

2.1 Review of basic rules

Let us first review the action of an FI parameter ξ on the following quiver:


(2.1)

where Φ is the adjoint chiral in the $\mathcal{N} = 4$ vector multiplet. The superpotential acquires an additional term, namely

$$\mathcal{W} = \tilde{Q}_i \Phi Q^i + \xi \text{Tr } \Phi, \tag{2.2}$$

which modifies the F-term and D-term equations to become

$$\begin{aligned} Q^i \tilde{Q}_i &= \xi I_N, \\ Q^i Q_i^\dagger - \tilde{Q}^{\dagger i} \tilde{Q}_i &= 0. \end{aligned} \tag{2.3}$$

The sum over flavor indices includes all bifundamental hypermultiplets charged under the $U(N)$ gauge node. For a generic quiver composed of multiple $U(N_i)$ gauge groups and only bifundamental matter, this implies that the FI parameters must satisfy the constraint

$$\sum_i N_i \xi_i = 0. \tag{2.4}$$

This condition requires that at least two gauge nodes must have a non-zero FI parameter turned on. Another crucial property is that FI deformations preserve the dimension of the Higgs branch of the quiver.

2.1.1 Fayet-Iliopoulos deformations at nodes of the same rank

The simplest FI deformation involves activating FI parameters for two nodes of the same rank, say $U(k)$. Solving the F-term and D-term equations results in the breaking of all $U(n_i)$ gauge groups to $U(n_i - k)$ along the subquiver connecting the two $U(k)$ nodes, while all other nodes remain unaffected. Among the broken $U(k)$ factors, a diagonal combination survives, leading to the addition of a new $U(k)$ node. This new node is attached to all nodes in the original quiver whose excess number was altered by the deformation, thereby restoring the original balance of the quiver. The multiplicity of the edges connecting this new $U(k)$ node is determined by the number of flavors required to restore the original excess numbers. This operation can be viewed as a modified quiver subtraction, where the subtracted part consists only of $U(k)$ nodes and the balance is restored by a new $U(k)$ node.

Another deformation, analyzed in [41], involves activating FI parameters at nodes with ranks $U(n)$, $U(k)$, and $U(n + k)$ such that $\xi_n = \xi_k$. This can be described as a two-step process. First, a subquiver between the $U(n)$ and $U(n + k)$ nodes is subtracted. After rebalancing, a further subtraction is performed on the resulting quiver between the $U(k)$ node and the original $U(n + k)$ node (which became $U(k)$ after the first step). When performing the deformation in two steps, it is important not to rebalance the rebalancing $U(n)$ node from the first step with the $U(k)$ node from the second. For an explicit example, we refer the reader to [41, section 3.2].

2.2 New Fayet-Iliopoulos deformations

Let us consider the following quiver, in which FI parameters have been activated at the red nodes:

$$\begin{array}{ccccccc}
 \bullet & \circ & \dots & \circ & \bullet & & \\
 1 & 2 & & r & 1 & & \\
 \end{array} \tag{2.5}$$

where we assume that $r \leq h$ so that the whole quiver is “good”. After the deformation, we obtain the quiver

$$\begin{array}{ccccccc}
 \bullet & \circ & \dots & \circ & \bullet & & \\
 1 & 2 & & r-1 & 1 & & \\
 \end{array} \tag{2.6}$$

+ h free hypermultiplets.

The transition from (2.5) to (2.6) can be understood as follows. The FI deformation causes all bifundamental fields along the tail to acquire a VEV. Modulo an $SU(h+1)$ transformation, we can assume that only one of the $U(r) \times U(1)$ bifundamentals gets a VEV. Its components are removed from the spectrum as they pair up with vector multiplets or otherwise become massive. Consequently, all gauge groups along the tail decrease in rank by one unit. In particular, the $U(r)$ node is Higgsed to $U(r-1)$. The remaining h bifundamentals between the $U(r)$ node and the $U(1)$ node become an equal number of $U(r-1) \times U(1)$ bifundamentals, plus h singlets that decouple as free hypermultiplets. This leads to the final state in (2.6).⁵

This process can be rephrased as a quiver subtraction where the subtracted part is an abelian quiver with $r+1$ nodes, removing the $h+1$ bifundamentals in the process. A rebalancing $U(1)$ node is then added, connected only to the $U(r-1)$ node with multiplicity h to preserve the excess number. Since FI deformations preserve the Higgs branch dimension, the initial and final quivers must have Higgs branches of equal dimension, which holds if the h free hypermultiplets in (2.6) are included.

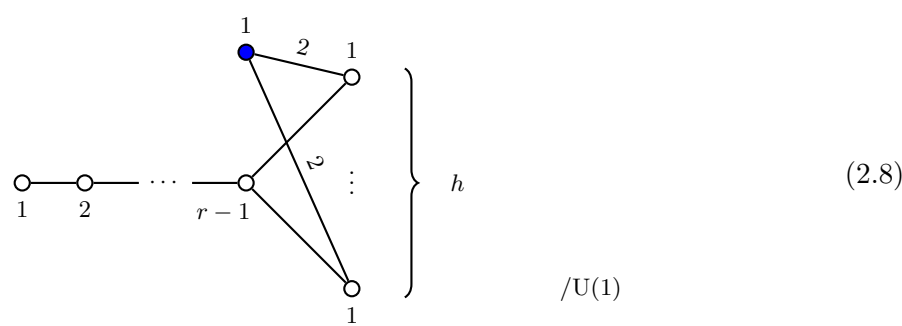
An alternative argument, which is helpful for later sections, proceeds as follows. Let us start with a full tail attached to a bouquet of $h+1$ abelian nodes. For simplicity, we consider the case where one node in the bouquet is connected by a single bifundamental to the other h nodes:

$$\begin{array}{ccccccc}
 & & & \bullet & \circ & & \\
 & & & 1 & 1 & & \\
 & & & \diagdown & \diagup & & \\
 & & & & & & \\
 & & & \vdots & & & \\
 & & & \diagup & \diagdown & & \\
 & & & & & & \\
 \bullet & \circ & \dots & \circ & & & \\
 1 & 2 & & r & & & \\
 \end{array} \left. \vphantom{\begin{array}{c} \bullet \\ 1 \\ \diagdown \\ \vdots \\ \diagup \\ \bullet \\ 1 \end{array}} \right\} h \tag{2.7}$$

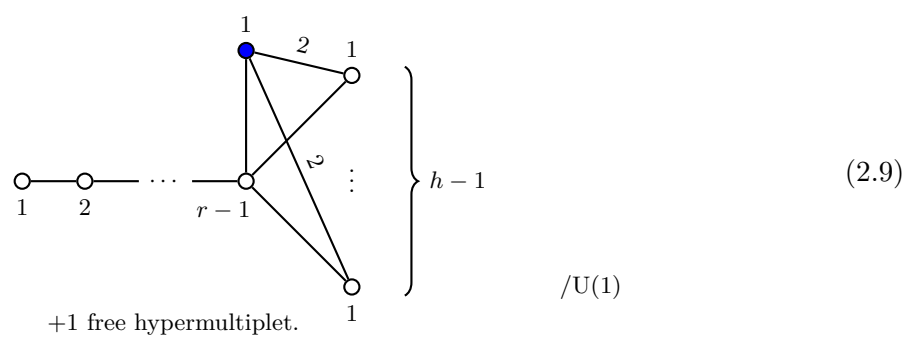
/ $U(1)$

⁵In order to avoid potential confusion, we emphasize again that, in the transition from (2.5) to (2.6), subtracting the abelian quiver reduces the rank of each node by one. This removes the two flanking red $U(1)$ nodes in (2.5) and maps the remaining $U(x)$ nodes to $U(x-1)$. In particular, the $U(2)$ node in (2.5) becomes the $U(1)$ node (2.6), and so on. The new rightmost red node in (2.6), along with edge multiplicity h , is then introduced via rebalancing.

Here, $/U(1)$ denotes the overall $U(1)$ that decouples. We first perform an FI deformation involving the red nodes, which leads to



If we now turn on FI parameters for the blue node and another $U(1)$ node from the remaining h nodes, we obtain



Reiterating this move other $h - 1$ times, we precisely recover the quiver in (2.6). We expect to obtain the same result if we proceed in reverse order: first collapsing h of the $U(1)$ nodes in (2.7) to arrive at (2.5), and then performing the transition from (2.5) to (2.6) as desired.

3 Important classes of deformations

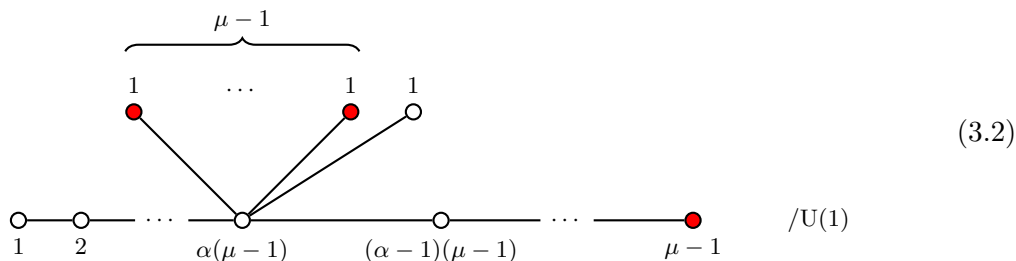
In this section, we analyze several important classes of deformations that are instrumental in deriving our main result in the subsequent section.

3.1 Deforming $D_\mu(SU(\alpha\mu))$ to $D_\mu(SU(\alpha\mu - 1))$

Here, we show that $D_\mu(SU(\alpha\mu - 1))$ can be obtained by a series of mass deformations from $D_\mu(SU(\alpha\mu))$. This statement is equivalent to the claim that the star-shaped parent quiver for $D_\mu(SU(\alpha\mu - 1))$ is identical to that for $D_\mu(SU(\alpha\mu))$. Note that $D_\mu(SU(\alpha\mu))$ is a Lagrangian theory described by

$$D_\mu(SU(\alpha\mu)) : \quad SU(\alpha) - SU(2\alpha) - SU(3\alpha) - \dots - SU(\alpha(\mu - 1)) - [\alpha\mu]. \quad (3.1)$$

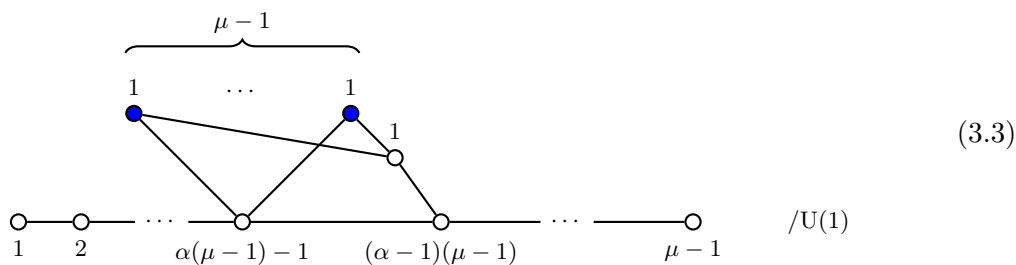
Its 3d mirror theory (see appendix A.2) is the following star-shaped quiver:



We claim that this quiver is, in fact, the star-shaped parent theory for $D_\mu(\text{SU}(\alpha\mu - 1))$.

For convenience, following the convention introduced in [35] and reviewed in appendix A.2, we refer to the right tail consisting of the nodes $U(\alpha(\mu - 1)), U((\alpha - 1)(\mu - 1)), \dots, U(\mu - 1)$ as *Tail A*, while the left tail consisting of the nodes $U(1), U(2), \dots, U(\alpha(\mu - 1))$ will be called *Tail B*. Finally, in the following, the $U(\alpha(\mu - 1))$ gauge node will be called *central node*.

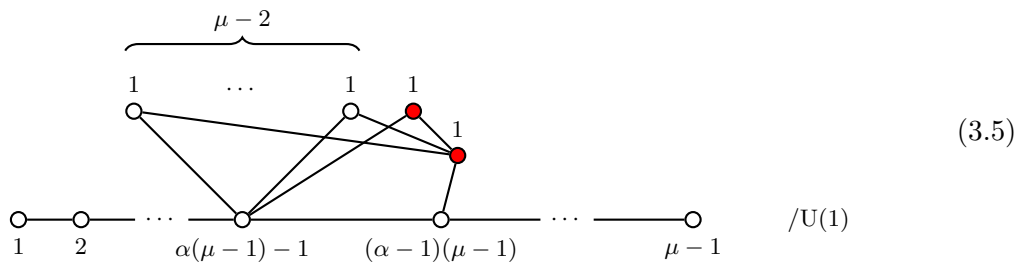
Let us activate FI parameters for the $\mu - 1$ $U(1)$ nodes in the bouquet and the $U(\mu - 1)$ node of Tail A (the red nodes). After the deformation, the quiver becomes



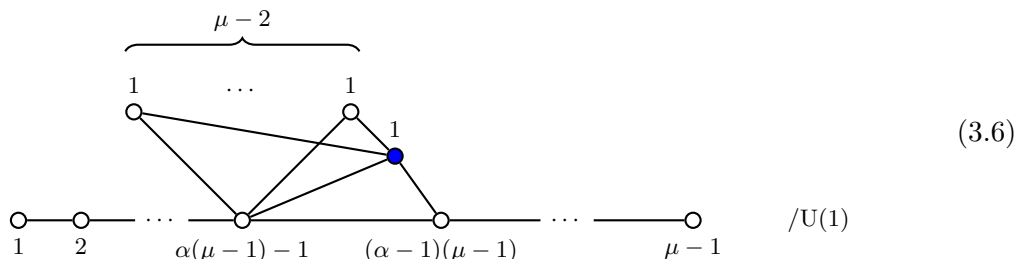
where we introduced $\mu - 1$ blue $U(1)$ nodes to rebalance the quiver after the deformations. In order to obtain the 3d mirror of $D_\mu(\text{SU}(\alpha\mu - 1))$ from (3.3), we need to deform the bouquet of $\mu - 1$ $U(1)$ nodes and reduce them to a single $U(1)$, since Tail A and Tail B of $D_\mu(\text{SU}(\alpha\mu - 1))$ are already present with

$$M = \alpha(\mu - 1) - 1. \tag{3.4}$$

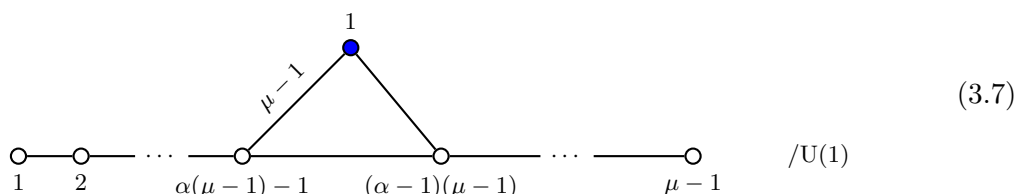
To deform the bouquet of $U(1)$ s, we depict (3.3) as follows:



where we have turned on FI parameters at the **red** nodes. The deformation is implemented by subtracting an A_2 quiver connecting the two nodes, which results in



We can iterate the deformation for all the remaining $U(1)$ s in the bouquet, eventually obtaining



which is the 3d mirror of $D_\mu(\text{SU}(\alpha\mu - 1))$.

Let us summarize this process, which involves a sequence of FI deformations. For instance, consider two $U(1)$ nodes in the bouquet connected to the central node and the rebalancing node. Activating FI parameters for these two nodes and the rebalancing node leads to a reduction in the number of nodes in the bouquet. Iterating this procedure collapses the entire bouquet into a single $U(1)$ node connected with multiplicity $\mu - 1$ to the central node, and also connected to Tail A. The resulting quiver is precisely the 3d mirror of $D_\mu(\text{SU}(\alpha\mu - 1))$.

Note that the special case of $\mu = 2$ can be inferred directly in 4d. This corresponds to deforming the $D_2(\text{SU}(2n))$ theory — which is simply 4d $\mathcal{N} = 2$ $\text{SU}(n)$ SQCD with $2n$ flavors — to the $D_2(\text{SU}(2n - 1))$ theory. As pointed out in [12, section 7.2.1], the latter can be realized at the maximally singular point of $\text{SU}(n)$ SQCD with $2n - 1$ flavors.

3.2 Deforming $D_p(\text{SU}(N))$ with $p \leq N$ to $D_N(\text{SU}(p))$

The mirror theory for $D_p(\text{SU}(N))$ with $p \leq N$, as described in appendix A.2, can be recursively deformed into the mirror of $D_N(\text{SU}(p))$, which is detailed in appendix A.1 under the replacement $N \leftrightarrow p$. In this section, we explain how these deformations are performed.

We begin with the general mirror for $D_p(\text{SU}(N))$, where we activate FI parameters at the red nodes:

(3.8)

where H_{free} is the number of free hypermultiplets, given in (A.8). Subsequently, we use the terminologies of Tail A and Tail B as defined in (A.7). The resulting quiver, after rebalancing, is

(3.9)

The process can be iterated x times by switching on FI deformation at the $U(p-1)$ nodes of Tails A and B, until Tail A disappears. In particular, Tail B reduces to a full tail with final node $U(M - x(p-1)) = U(\mu m_B - 1)$, and the resulting intermediate quiver is

(3.10)

We can apply the new rules described in section 2.2 to the red nodes in (3.10), obtaining the following quiver:

(3.11)

$+H_{\text{free}} + (m_B - 1)$ free hypermultiplets.

It is possible to iterate the procedure m_B times turning on FI deformations at the red $U(1)$ node of the tail on the right and the balancing $U(1)$ node in blue. The multiplicity of the bifundamental connecting the blue node to the tail will decrease until the edge completely disappears, while the multiplicity of the edges between the blue node and the other $U(1)$ s in the complete graph will become $m_G + m_B^2 = \mathfrak{p}m_B$, i.e.

(3.12)

$+H_{\text{free}} + \frac{1}{2}m_B(m_B - 1)$ free hypermultiplets.

The process can also be repeated recursively for all $\mu - 1$ $U(1)$ s left in the complete graph, so that tail B is completely removed. The final quiver is

$+H_{\text{free}} + \frac{1}{2}\mu m_B(m_B - 1)$ free hypermultiplets, (3.13)

where we have already isolated one of the x tails on the left and turned on the FI deformations that we will use next. The iterations are analogous to the previous one, and a tail will increase the multiplicity of the complete graph by \mathfrak{p}^2 , so that we end up with

$+H_{\text{free}} + \frac{1}{2}\mu m_B(m_B - 1) + \frac{1}{2}\mu \mathfrak{p}(\mathfrak{p} - 1)$ free hypermultiplets. (3.14)

Repeating it $(x - 1)$ times, we obtain

$+ \tilde{H}_{\text{free}}$ free hypermultiplets, (3.15)

where

$$\mathfrak{p}m_B + (x - 1)\mathfrak{p}^2 = \mathfrak{p}(n - \mathfrak{p}), \tag{3.16}$$

while

$$\begin{aligned} \tilde{H}_{\text{free}} &= H_{\text{free}} + \frac{1}{2}\mu m_B(m_B - 1) + \frac{1}{2}(x - 1)\mu\mathfrak{p}(\mathfrak{p} - 1) \\ &= \frac{1}{2}\mu(\mathfrak{p} - 1)(\mathfrak{n} - \mathfrak{p} - 1). \end{aligned} \tag{3.17}$$

The quiver in (3.15), together with the \tilde{H}_{free} free hypermultiplets from (3.17), is precisely the mirror of $D_N(\text{SU}(p))$ as described in (A.4) (with the replacement $N \leftrightarrow p$).

Finally, we note that the results of section 3.1 imply that the theories $D_\mu(\text{SU}(\alpha\mu - 1))$, $D_{\alpha\mu-1}(\text{SU}(\mu))$, and $D_{\alpha\mu}(\text{SU}(\mu))$ can all be obtained from the star-shaped quiver in (3.2), which is the 3d mirror of $D_\mu(\text{SU}(\alpha\mu))$.

3.3 Deforming $D_N(\text{SU}(p))$ to (A_{N-1}, A_{p-1})

We also find that, by continuing the procedure described above and deforming the final tail, we obtain a complete graph of $U(1)$ nodes where each edge has multiplicity \mathfrak{np} , along with

$$H_{\text{free}} = \frac{1}{2}\mu(\mathfrak{p} - 1)(\mathfrak{n} - 1) \tag{3.18}$$

free hypermultiplets. This is precisely the mirror theory of (A_{N-1}, A_{p-1}) [35–37]. We therefore conclude that the $D_N(\text{SU}(p))$ theory can be mass-deformed to (A_{N-1}, A_{p-1}) . The relevant deformation corresponds to activating a mass for the $\text{SU}(p)$ global symmetry associated with the full puncture. The IR fixed point of the corresponding RG flow is a class \mathcal{S} theory of the same type, but on a sphere with only an irregular puncture; the regular puncture has been removed by the RG flow. Crucially, this process also modifies the irregular puncture relative to the one describing the UV $D_N(\text{SU}(p))$ theory.

This modification of the puncture is consistent with the description in [16], where it was argued that this flow is equivalent to the one discovered by Maruyoshi and Song [17–19]. However, while that construction involves breaking supersymmetry to $\mathcal{N} = 1$, we find here that an RG flow with the same UV and IR fixed points can be achieved by activating a mass deformation for the global $\text{SU}(p)$ symmetry. This operation manifestly preserves $\mathcal{N} = 2$ supersymmetry throughout the entire RG flow.

We remark that, although the argument in this section implies the existence of this flow only for $N > p$, this restriction is not essential. The RG flow also exists for $N < p$. In that case, however, the FI deformation is more complex than simply deforming away the $T_{[1^p]}[\text{SU}(p)]$ tail in the 3d mirror, as was done here.

4 Star-shaped parent quivers for $D_p(\text{SU}(N))$

We now present our main result: a general construction for the star-shaped parent quivers of $D_p(\text{SU}(N))$ theories.

4.1 General strategy: the Euclidean algorithm

In this section, we propose a general form for the star-shaped quiver that deforms to the 3d mirror of any given $D_p(\text{SU}(N))$ theory.

As established in section 3.2, a $D_p(\text{SU}(N))$ theory with $p \leq N$ can be mass-deformed into the $D_N(\text{SU}(p))$ theory. This implies that both theories share the same star-shaped parent. It is therefore sufficient to construct the parent quiver for the case $p \leq N$.

For convenience, we introduce the following shorthand notation for a full quiver tail of rank r :

$$F_r : \quad \circ_1 \text{---} \circ_2 \text{---} \cdots \text{---} \circ_r \tag{4.1}$$

We propose the following ansatz for the parent star-shaped quiver of $D_p(\text{SU}(N))$ with $p \leq N$:⁶

$$\begin{array}{c}
 \sum_{i=1}^n \mathfrak{h}_i = p \\
 \overbrace{F_{\mathfrak{h}_1} \quad \cdots \quad F_{\mathfrak{h}_n}} \\
 \diagdown \quad \quad \quad \diagup \\
 \circ_1 \text{---} \circ_2 \text{---} \cdots \text{---} \circ_{N-x} \text{---} \circ_{N-x-(p-1)} \text{---} \cdots \text{---} \circ_{N-px} \quad /U(1)
 \end{array} \tag{4.2}$$

where $x = \lfloor N/p \rfloor$. The quiver features a central node $U(N-x)$ connected to several tails. The collection of full tails $F_{\mathfrak{h}_i}$ (where the largest node \mathfrak{h}_i of each tail connects to the central node) is termed the *bouquet*. The rightmost tail, which we denote by Tail A' in (4.21), has nodes whose ranks decrease by $p-1$ at each step, starting from $U(N-x)$ down to $U(N-px)$. The leftmost tail is denoted Tail B' in (4.21). This structure is analogous to the mirror theory of $D_p(\text{SU}(N))$ itself.

The next step in our strategy relies on the result discussed in section 3.2. We leverage this by considering a related theory, $D_N(\text{SU}(p+kN))$ for $k \in \mathbb{Z}_{\geq 1}$. Since $p+kN > N$, we can adapt our ansatz by applying the following substitutions to the parameters in (4.2):

$$p \rightarrow N, \quad N \rightarrow p+kN, \quad \text{and} \quad x \rightarrow \left\lfloor \frac{kN+p}{N} \right\rfloor = k. \tag{4.3}$$

This yields the following ansatz for the parent star-shaped quiver of $D_N(\text{SU}(p+kN))$:

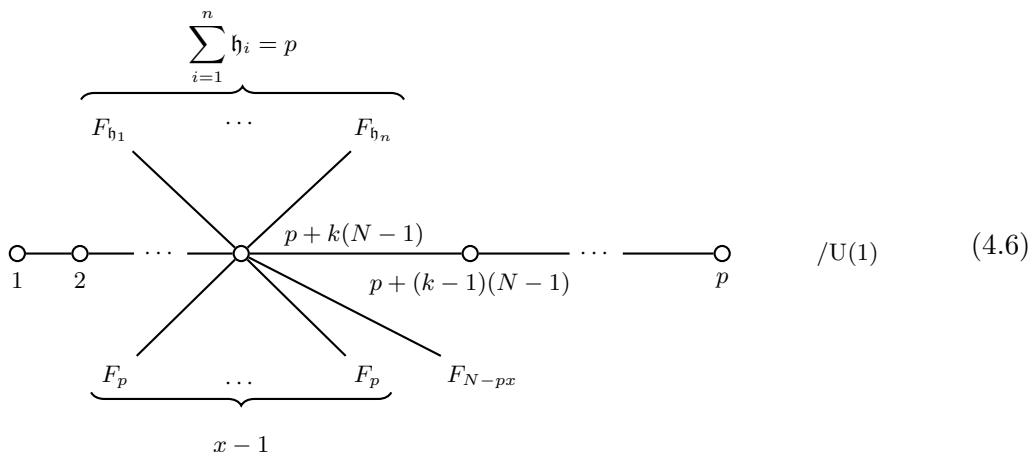
$$\begin{array}{c}
 \sum_{i=1}^{n'} \mathfrak{h}_i = N \\
 \overbrace{F_{\mathfrak{h}_1} \quad \cdots \quad F_{\mathfrak{h}_{n'}}} \\
 \diagdown \quad \quad \quad \diagup \\
 \circ_1 \text{---} \circ_2 \text{---} \cdots \text{---} \circ_{p+k(N-1)} \text{---} \circ_{p+(k-1)(N-1)} \text{---} \cdots \text{---} \circ_p \quad /U(1)
 \end{array} \tag{4.4}$$

By assumption, $p \leq N$. We can therefore partition the bouquet of n' tails, $\{F_{\mathfrak{h}_i}\}$, into two subsets such that

$$\sum_{i=1}^n \mathfrak{h}_i = p, \quad \sum_{i=n+1}^{n'} \mathfrak{h}_i = N-p. \tag{4.5}$$

⁶In subsequent diagrams, an F_1 tail will often be depicted simply as $\circ \text{---} \cdot$.

We require the first subset, with $\sum_{i=1}^n \mathfrak{h}_i = p$, to be identical to the bouquet proposed for $D_p(\text{SU}(N))$ in (4.2). By requiring that the Higgs branch dimensions of the parent quivers (4.2) and (4.4) match the ranks of the corresponding $D_p(\text{SU}(N))$ and $D_N(\text{SU}(p + kN))$ theories, we can uniquely determine the composition of the second subset. It must contain one copy of the tail F_{N-px} and $(x - 1)$ copies of the tail F_p . The complete star-shaped parent quiver for $D_N(\text{SU}(p + kN))$ is therefore



This provides a direct method to construct the parent quiver for $D_N(\text{SU}(p + kN))$ from the parent of $D_p(\text{SU}(N))$. As a point of reference, we note that, for the special case of the Lagrangian theory $D_\mu(\text{SU}(\alpha\mu))$, the mirror is a star-shaped quiver of the form (4.2) where the bouquet consists of μ copies of the F_1 tail, as shown in [35, (5.26)].

To summarize, this procedure establishes a chain of deformations

$$D_p(\text{SU}(N)) \rightarrow D_{p+kN}(\text{SU}(N)) \rightarrow D_N(\text{SU}(p + kN)) \equiv D_{p'}(\text{SU}(N')). \tag{4.7}$$

where the second arrow relies on the result derived in section 3.2. The inverse of this process is

$$D_{p'}(\text{SU}(N')) \rightarrow D_{N' \pmod{p'}}(\text{SU}(p')) = D_p(\text{SU}(N)). \tag{4.8}$$

This observation motivates the use of the following sequence, generated via the *Euclidean algorithm*, which is a standard method to determine the greatest common divisor of two given integers (N and p in this case):

$$\begin{aligned} a_0 = N, \quad a_1 = p, \quad a_2 = N \pmod{p}, \dots, \quad a_j = a_{j-2} \pmod{a_{j-1}}, \\ a_{\bar{n}} = \alpha\mu, \quad a_{\bar{n}+1} = \mu, \quad a_{\bar{n}+2} = 0, \quad \text{with } \mu = \text{GCD}(N, p). \end{aligned} \tag{4.9}$$

This generates a strictly decreasing sequence of non-negative integers $(a_0, a_1, \dots, a_{\bar{n}+1}, 0)$, which terminates at zero. The second-to-last term, $a_{\bar{n}+1}$, is the greatest common divisor, $\mu = \text{GCD}(N, p)$. In terms of the theories, this algorithm defines a chain of deformations

$$D_{a_1}(\text{SU}(a_0)) \rightarrow D_{a_2}(\text{SU}(a_1)) \rightarrow \dots \rightarrow D_\mu(\text{SU}(\alpha\mu)). \tag{4.10}$$

Our main strategy for constructing the parent quiver of $D_p(\text{SU}(N))$ is to reverse this chain:

$$D_\mu(\text{SU}(\alpha\mu)) = D_{a_{\bar{n}+1}}(\text{SU}(a_{\bar{n}})) \rightarrow D_{a_{\bar{n}}}(\text{SU}(a_{\bar{n}-1})) \rightarrow \dots \rightarrow D_{a_1}(\text{SU}(a_0)) = D_p(\text{SU}(N)). \tag{4.11}$$

At each step of this reversed sequence, the parent quiver for a theory $D_{a_{q+1}}(\text{SU}(a_q))$ takes the general form of (4.2), with parameters $N = a_q$, $p = a_{q+1}$, and $x = \lfloor a_q/a_{q+1} \rfloor$. The composition of the bouquet is determined recursively:

$$\text{Bouquet for } D_{a_q}(\text{SU}(a_{q-1})) = \text{Bouquet for } D_{a_{q+1}}(\text{SU}(a_q)) + F_{a_{q+2}} + [(x-1) \times F_{a_{q+1}}], \tag{4.12}$$

where $(x-1) \times F_{a_{q+1}}$ denotes $(x-1)$ copies of the full tail $F_{a_{q+1}}$ attached to the central node. Using the fact that the bouquet for the base case, $D_\mu(\text{SU}(\alpha\mu))$, consists of μ copies of the F_1 tail, we can solve this recursion relation to find the general form of the bouquet:

$$\begin{aligned} \text{Bouquet for } D_{a_q}(\text{SU}(a_{q-1})) &= \mu F_1 + \sum_{j=q}^{\bar{n}} \left[F_{a_{j+2}} + \left(\left\lfloor \frac{a_j}{a_{j+1}} \right\rfloor - 1 \right) F_{a_{j+1}} \right] \\ &= \mu F_1 - F_{a_{q+1}} + \sum_{j=q}^{\bar{n}} \left\lfloor \frac{a_j}{a_{j+1}} \right\rfloor F_{a_{j+1}}. \end{aligned} \tag{4.13}$$

For the specific theory $D_p(\text{SU}(N))$, we have $a_0 = N$ and $a_1 = p$. Setting $q = 1$ in the general solution yields:

$$\begin{aligned} \text{Bouquet for } D_p(\text{SU}(N)) &= \mu F_1 - F_{a_2} + \sum_{j=1}^{\bar{n}} \left\lfloor \frac{a_j}{a_{j+1}} \right\rfloor F_{a_{j+1}} \\ &= \mu F_1 + \left(\left\lfloor \frac{a_1}{a_2} \right\rfloor - 1 \right) F_{a_2} + \sum_{j=2}^{\bar{n}} \left\lfloor \frac{a_j}{a_{j+1}} \right\rfloor F_{a_{j+1}}. \end{aligned} \tag{4.14}$$

Here, $\mu = \text{GCD}(N, p)$, whereas the sequence a_j and the index \bar{n} are determined by the Euclidean algorithm in (4.9). The expression in (4.14) fully specifies the bouquet content, which, when combined with the general structure of (4.2), completely determines the star-shaped parent quiver for the $D_p(\text{SU}(N))$ theory.

This parent quiver is the 3d mirror of a class \mathcal{S} theory of Type A_{N-x-1} associated with a sphere with the following punctures [52]:

$$[1^{N-x}], \quad [(p-1)^x, N - px], \quad c_i \times [N - x - \mathfrak{h}_i, 1^{\mathfrak{h}_i}]. \tag{4.15}$$

The parameters c_i and \mathfrak{h}_i are determined by writing the bouquet composition from (4.14) in the form $\sum_i c_i F_{\mathfrak{h}_i}$. The coefficient c_i gives the multiplicity of the puncture type $[N - x - \mathfrak{h}_i, 1^{\mathfrak{h}_i}]$. This establishes our central claim: any $D_p(\text{SU}(N))$ theory can be obtained from a sequence of mass deformations of the aforementioned class \mathcal{S} theory. It is important to stress that this construction provides one possible star-shaped parent. We do not claim uniqueness, as it is known that some $D_p(\text{SU}(N))$ theories can have multiple distinct parents.

Example: $D_{17\mu}(\text{SU}(26\mu))$ with $\mu \geq 1$. To illustrate this procedure, we determine a star-shaped parent quiver for the $D_{17\mu}(\text{SU}(26\mu))$ theory. The Euclidean algorithm sequence from (4.9) is

$$(a_0, a_1, a_2, a_{\bar{n}=3}, a_4, a_5) = (26\mu, 17\mu, 9\mu, 8\mu, \mu, 0). \tag{4.16}$$

The corresponding sequence of theory deformations from (4.11) reads

$$D_\mu(\text{SU}(8\mu)) \rightarrow D_{8\mu}(\text{SU}(9\mu)) \rightarrow D_{9\mu}(\text{SU}(17\mu)) \rightarrow D_{17\mu}(\text{SU}(26\mu)). \quad (4.17)$$

The bouquet composition of the star-shaped parent quiver at each step can be computed recursively using (4.12), which agrees with the general solution (4.14). The results are summarized below:

Theory	Bouquet
$D_\mu(\text{SU}(8\mu))$	μF_1
$D_{8\mu}(\text{SU}(9\mu))$	$\mu F_1 + 7F_\mu$
$D_{9\mu}(\text{SU}(17\mu))$	$\mu F_1 + 8F_\mu$
$D_{17\mu}(\text{SU}(26\mu))$	$\mu F_1 + 8F_\mu + F_{8\mu}$

(4.18)

Therefore, $D_{17\mu}(\text{SU}(26\mu))$ can be obtained by a series of mass deformations starting from the class \mathcal{S} theory of Type $A_{26\mu-2} = \text{SU}(26\mu - 1)$ associated with a sphere with the following punctures:

$$[1^{26\mu-1}], \quad [17\mu - 1, 9\mu], \quad \mu \times [26\mu - 2, 1], \quad 8 \times [25\mu - 1, 1^\mu], \quad [18\mu - 1, 1^{8\mu}]. \quad (4.19)$$

4.2 Motivation and explanation for ansatz (4.2)

This section provides the motivation and justification for the ansatz (4.2), which was used to derive the star-shaped parent theory. A key validation of this ansatz is that the rank of the $D_p(\text{SU}(N))$ theory (for $N \geq p$), given by

$$\text{rank } D_p(\text{SU}(N)) = \frac{1}{2} [(N - 1)(p - 1) - (\mu - 1)], \quad \mu = \text{GCD}(N, p), \quad (4.20)$$

is precisely equal to the dimension of the Higgs branch of the theory in (4.2) with the bouquet satisfying (4.14). This equality provides a highly non-trivial check on the validity of our proposal. We now present further arguments to support it.

The ansatz (4.2) consists of two main components:

1. The horizontal tails, shown as Tail A' and Tail B' :

Tail B' :

(4.21)

Tail A' :

2. The bouquet of tails, denoted by $F_{\mathfrak{h}_i}$.

We will demonstrate that each component behaves as expected under specific deformations.

First, we show that Tail A' and Tail B' in (4.21) can be deformed into Tail A and Tail B from (A.7), respectively. This is achieved by activating FI parameters for the $U(N - px)$ node in (4.2) and for the $U(\mathfrak{h}_i)$ nodes of the $F_{\mathfrak{h}_i}$ tails, such that

$$\sum_j \mathfrak{h}_j = N - px. \quad (4.22)$$

This is always possible, since the sum of all \mathfrak{h}_i is p , and $N - px$ is always less than p .⁷ This deformation yields Tail A and Tail B connected to some of the $F_{\mathfrak{h}_i}$ tails. An explicit example of this process is the deformation from $D_\mu(\text{SU}(\alpha\mu))$ to $D_\mu(\text{SU}(\alpha\mu - 1))$, discussed in section 3.1. If we view (3.2) as a star-shaped parent quiver for $D_\mu(\text{SU}(\alpha\mu - 1))$ (with $N = \alpha\mu$, $p = \mu$, and $x = \alpha$), the required Tails A and B are explicitly present in (3.3). We also provide further examples in sections 5.3 and 5.5.

To motivate the second component of the ansatz, the bouquet of tails, we focus on the first transition in (4.7): from $D_p(\text{SU}(N))$ to $D_{p+kN}(\text{SU}(N))$. The remainder of this section argues that the star-shaped parent theory for the latter can be obtained by adding k copies of the F_{N-1} tail to the parent theory of the former. Since the second step of the transition was discussed in section 3.2, we only need to focus on this first step. Preliminary evidence for this claim comes from the rank difference:

$$\text{rank } D_{p+kN}(\text{SU}(N)) - \text{rank } D_p(\text{SU}(N)) = \frac{1}{2}kN(N - 1). \tag{4.23}$$

This difference is precisely k times the sum of the ranks of the gauge nodes in an F_{N-1} tail.

From $D_{2n}(\text{SU}(2))$ to a parent of $D_{2n+2}(\text{SU}(2))$. To provide further justification, let us examine the special case of $(A_1, D_{2n}) = D_{2n}(\text{SU}(2))$. As noted in [11, section 4.3], this theory can be embedded into the 4d $\mathcal{N} = 2$ Lagrangian theory

$$[1] - \underbrace{\text{SU}(2) - \dots - \text{SU}(2)}_{n-1} - [2], \tag{4.24}$$

which can be obtained via a mass deformation of the theory

$$[2] - \underbrace{\text{SU}(2) - \dots - \text{SU}(2)}_{n-1} - [2]. \tag{4.25}$$

The latter is an A_1 class \mathcal{S} theory on a sphere with $n + 2$ punctures. Its mirror theory is a star-shaped quiver with a central $\text{U}(2)$ node and $(n + 2)$ tails of type F_1 .

Now, let us consider the $D_{2n+2}(\text{SU}(2))$ theory. To construct it, we couple the T_2 theory (the theory of four free hypermultiplets) to $D_{2n}(\text{SU}(2))$ and gauge the diagonal $\text{SU}(2)$ symmetry:

$$\left([1] - \underbrace{\text{SU}(2) - \dots - \text{SU}(2)}_{n-1} - [2] + [2] - [2] \right) // \text{SU}(2). \tag{4.26}$$

This resulting theory can be obtained by a mass deformation of

$$[2] - \underbrace{\text{SU}(2) - \dots - \text{SU}(2)}_n - [2]. \tag{4.27}$$

⁷By definition $x = \lfloor \frac{N}{p} \rfloor$, so $x \leq \frac{N}{p} < x + 1$. Multiplying by p , we obtain $xp \leq N < xp + p$, from which it follows that $0 \leq N - px < p$. This is also clear by noting that $N - px$ is the element a_2 of the Euclidean sequence, which guarantees that either itself or its partitions are present in the bouquet of tails.

The mirror of this theory is a star-shaped quiver with $(n + 3)$ tails of type F_1 :

$$\begin{array}{c}
 \overbrace{\quad\quad\quad}^{n+3} \\
 \begin{array}{ccc}
 1 & \dots & 1 \\
 \circ & & \circ \\
 & \diagdown & / \\
 & \circ & \\
 & 2 &
 \end{array}
 \end{array}
 \quad /U(1) \tag{4.28}$$

In summary, increasing p by 2 in $D_{p=2n}(SU(2))$ is equivalent to coupling the T_2 theory via $SU(2)$ gauging. In the star-shaped parent theory, this corresponds to adding an F_1 tail.

This concept generalizes to $D_p(SU(N))$. We can increase p by N by coupling the T_N theory to $D_p(SU(N))$ via $SU(N)$ gauging. In the star-shaped parent theory, this operation corresponds to adding an F_{N-1} tail.

To demonstrate this, let us construct the parent theory of $D_{kN}(SU(N))$ starting from the mirror theory of $D_N(SU(N))$. From (3.2), the mirror theory for $D_N(SU(N))$ is

$$\begin{array}{c}
 \overbrace{\quad\quad\quad}^N \\
 \begin{array}{ccc}
 1 & & 1 \\
 \circ & & \circ \\
 & \diagdown & / \\
 & \circ & \\
 & \dots & \\
 1 & 2 & N-1
 \end{array}
 \end{array}
 \quad /U(1) \tag{4.29}$$

Our goal is to find the star-shaped parent theory for $D_{kN}(SU(N))$. To do this, we first consider a related theory:

$$\begin{array}{c}
 \overbrace{\quad\quad\quad}^N \\
 \begin{array}{ccc}
 1 & & 1 \\
 \circ & & \circ \\
 & \diagdown & / \\
 & \circ & \\
 & \dots & \\
 1 & 2 & N
 \end{array}
 \end{array}
 \quad /U(1) \tag{4.30}$$

This theory is equivalent to (4.29) modulo a set of free hypermultiplets, but its central $U(N)$ node is now “ugly” (i.e., it has fewer flavors than twice its rank). Now, we add $(k - 1)$ copies

of the F_{N-1} tail to this quiver, which yields

$$(4.31)$$

We propose that (4.31) is the star-shaped parent theory for $D_{kN}(\text{SU}(N))$. This claim will be substantiated in sections 5.1 and 5.2, where we explicitly demonstrate the deformations from (4.31) to the known mirror theories of $D_8(\text{SU}(4))$ and $D_{3k}(\text{SU}(3))$ for generic k .

General case: from a parent of $D_p(\text{SU}(N))$ to a parent of $D_{p+N}(\text{SU}(N))$. We can apply a similar procedure to the general ansatz (4.2). First, we modify the quiver so that the central node becomes $\text{U}(N)$ instead of $\text{U}(N-x)$. This is achieved by transforming the left tail into an $F_{N-1} = 1-2-\dots-(N-1)$ tail and increasing the rank of each node in the right tail from $N-x-j(p-1)$ to $N-jp$ (for $j = 1, \dots, x$). The resulting quiver is

$$(4.32)$$

This quiver is equivalent to the original ansatz (4.2) modulo free hypermultiplets, a consequence of its “ugly” central node. Adding an F_{N-1} tail to this modified quiver results in the proposed star-shaped parent for $D_{p+N}(\text{SU}(N))$:

$$(4.33)$$

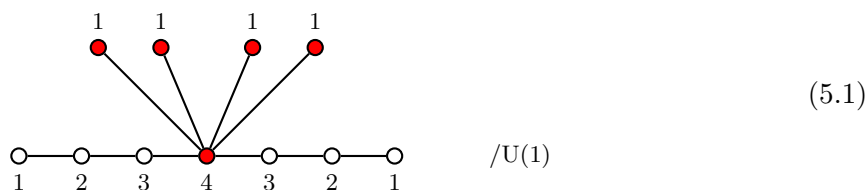
Crucially, this procedure preserves the overall structure of the ansatz (4.2). The fact that adding an F_{N-1} tail to the parent of $D_p(\text{SU}(N))$ yields a theory of the same form for $D_{p+N}(\text{SU}(N))$ provides strong motivation for the general validity of our ansatz.

5 Examples

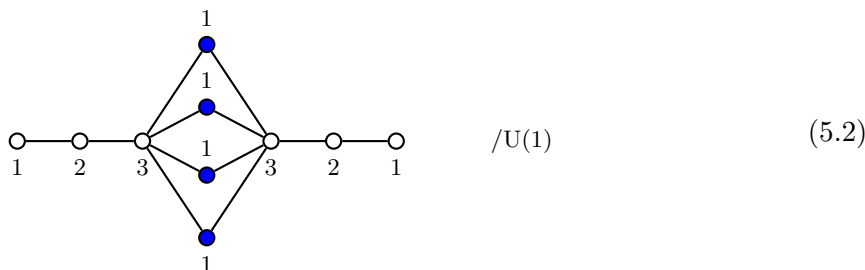
This section provides several explicit examples to demonstrate how the proposed star-shaped parent theories deform into the corresponding mirror theories of $D_p(\text{SU}(N))$.

5.1 Deforming (4.31) $_{N=4,k=2}$ to the mirror theory for $D_8(\text{SU}(4))$

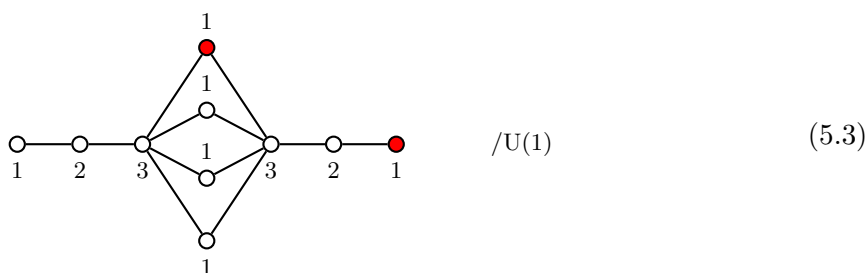
We begin by considering the case of $D_8(\text{SU}(4))$, which corresponds to setting $N = 4$ and $k = 2$ in the parent quiver (4.31):



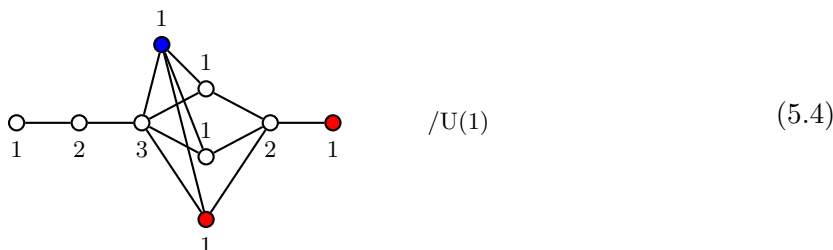
where the nodes at which FI parameters are activated have been colored in red. This deformation leads to the following quiver:



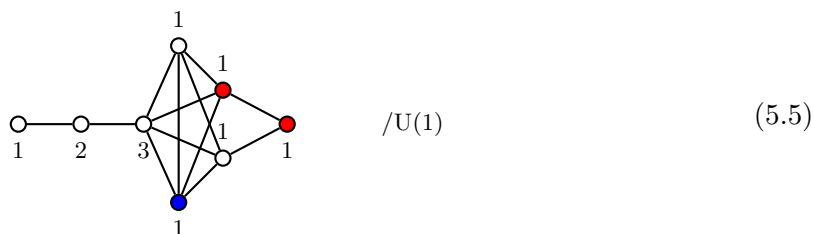
where the blue nodes represent the balancing $U(1)$ factors introduced by the deformation. Next, we activate FI parameters for the red nodes shown in (5.3):



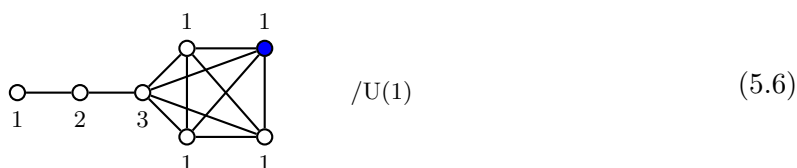
which yields



where the **red** nodes indicate the next set of FI parameters to be activated. The resulting quiver is



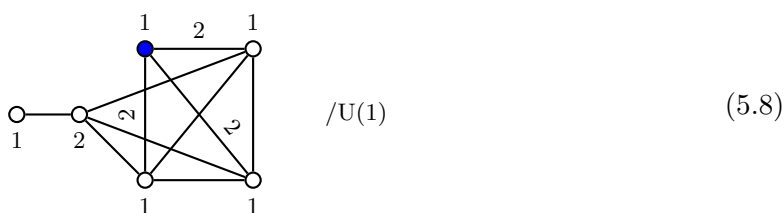
A final deformation at the **red** nodes yields the 3d mirror of $D_8(SU(4))$:



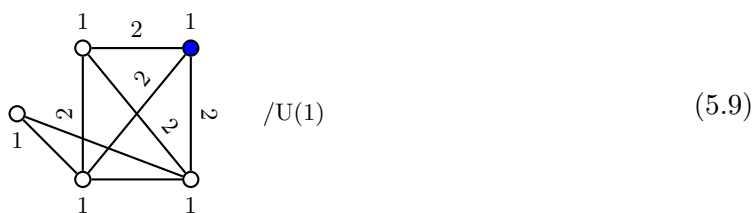
This example can also be used to explicitly demonstrate the result from section 3.3. By activating FI parameters for the $U(1)$ nodes of the full tail and one $U(1)$ node in the complete graph, we can obtain the mirror of (A_7, A_3) . The initial step is



which yields



We note that the balancing node is no longer connected to the full tail. Repeating this deformation with another $U(1)$ node from the complete graph gives

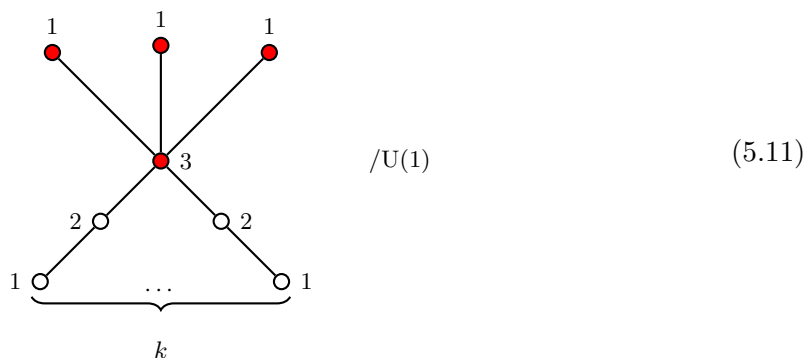


A final deformation results in the complete graph associated with the mirror of (A_7, A_3) , consistent with the findings in [35, (4.11)]:⁸

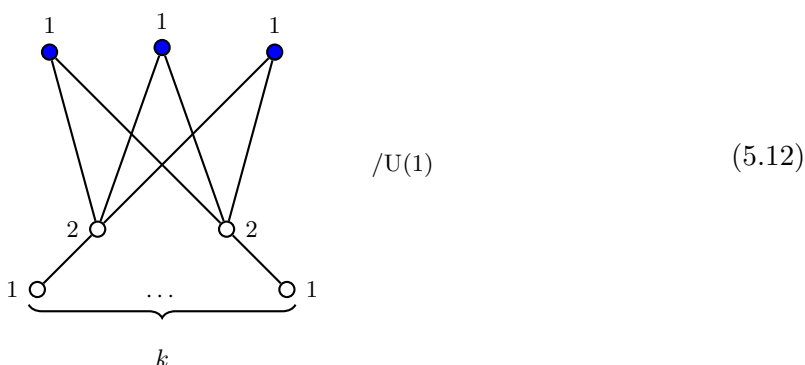


5.2 Deforming (4.31)_{N=3} to the mirror theory for $D_{3k}(\text{SU}(3))$

We now consider the case of $D_{3k}(\text{SU}(3))$, which corresponds to setting $N = 3$ in the parent quiver (4.31) for a generic integer k :

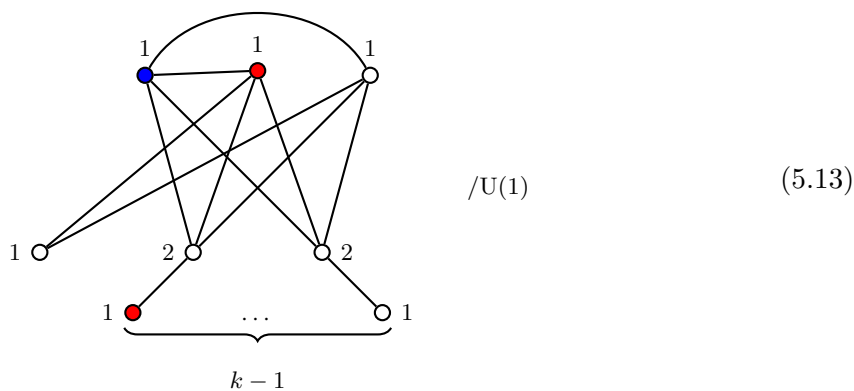


Activating FI parameters at the red nodes and rebalancing the quiver with three blue $U(1)$ nodes yields

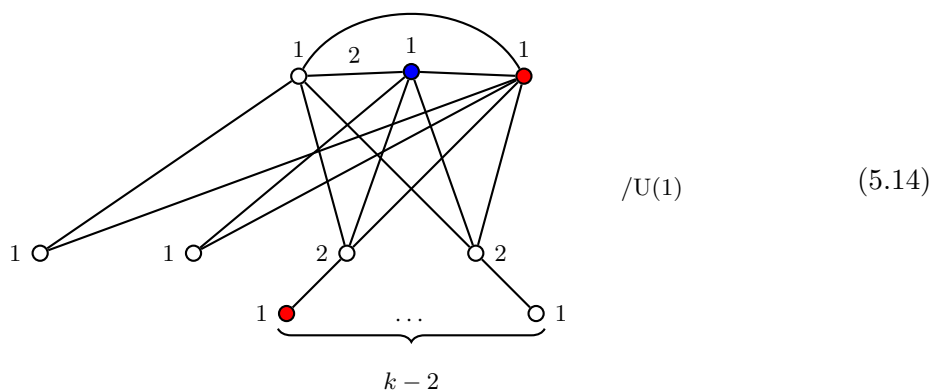


⁸In [35, (4.11)], the authors write the mirror obtained by closing the full puncture of the $D_p(\text{SU}(N))$ theory, while here we are performing a mass deformation of the theory as described in section 3.3. In order to match the mirror, one needs to consider that (A_3, A_7) , which is equivalent to (A_7, A_3) , in [35] is obtained from $D_{12}(\text{SU}(8))$.

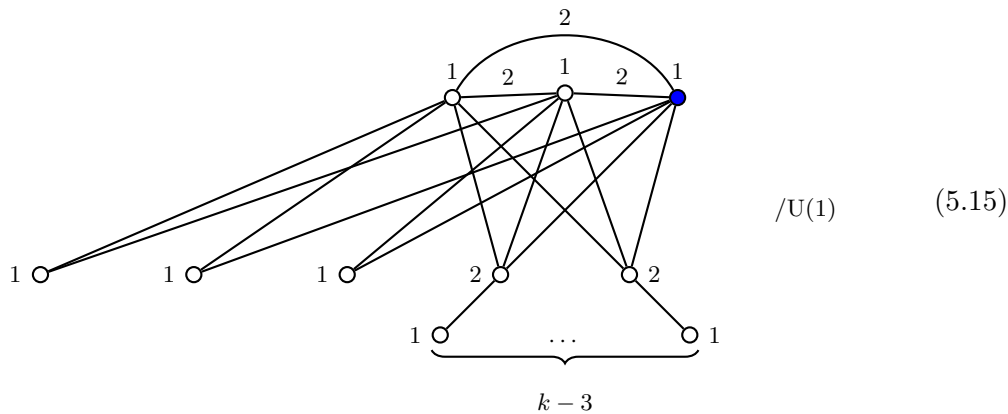
Next, we activate FI parameters for one of the **blue** $U(1)$ nodes and one of the $U(1)$ nodes in an F_2 tail, leading to



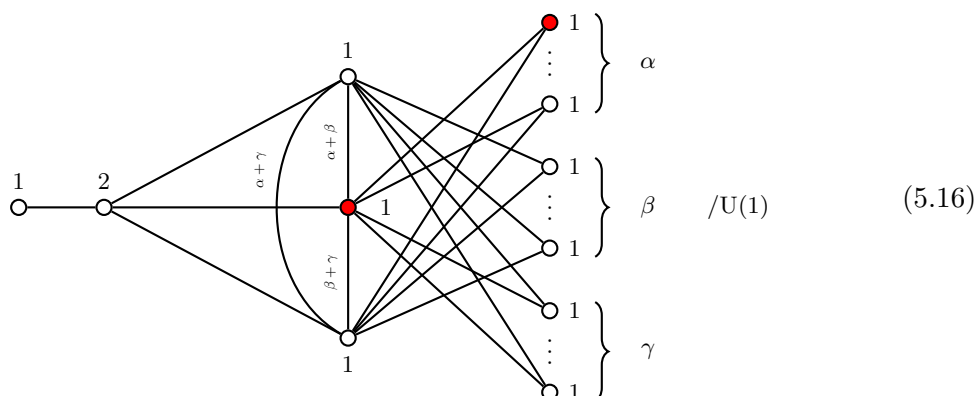
The FI deformation at the **red** nodes increases the multiplicity of the edge connecting the **blue** and **red** nodes by one, resulting in



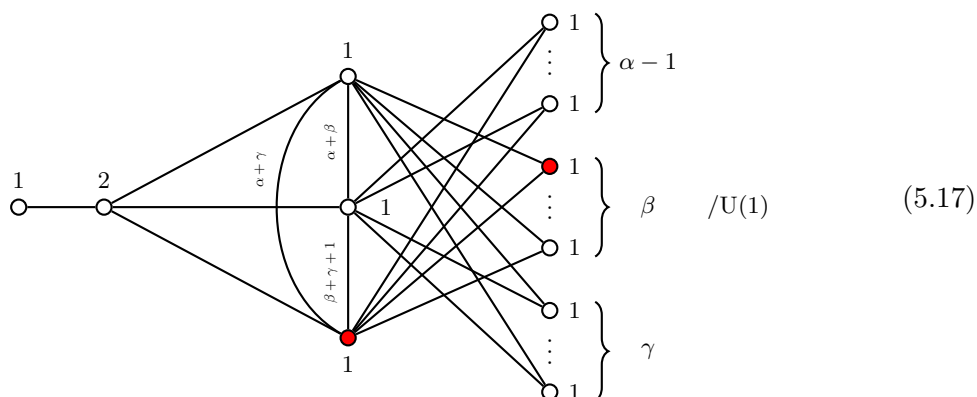
By activating FI parameters for the top-right $U(1)$ node and another $U(1)$ from an F_2 tail, we can reach a quiver where the three top $U(1)$ nodes are connected by edges of multiplicity two:



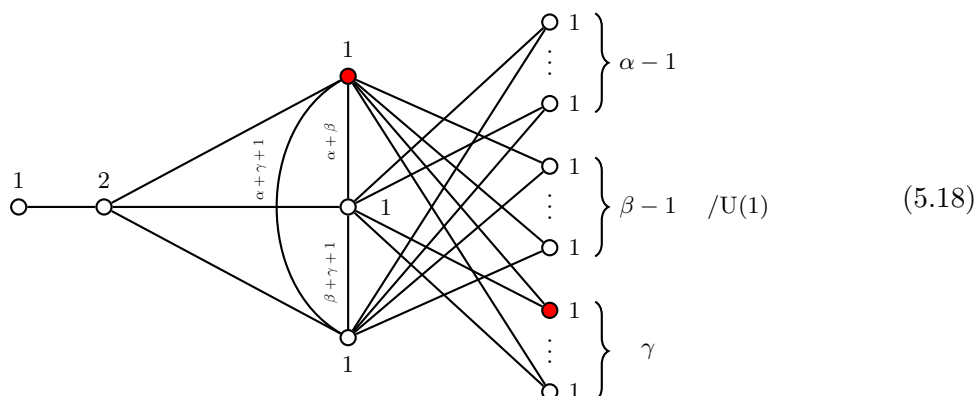
By reiterating this sequence of deformations, we ultimately arrive at



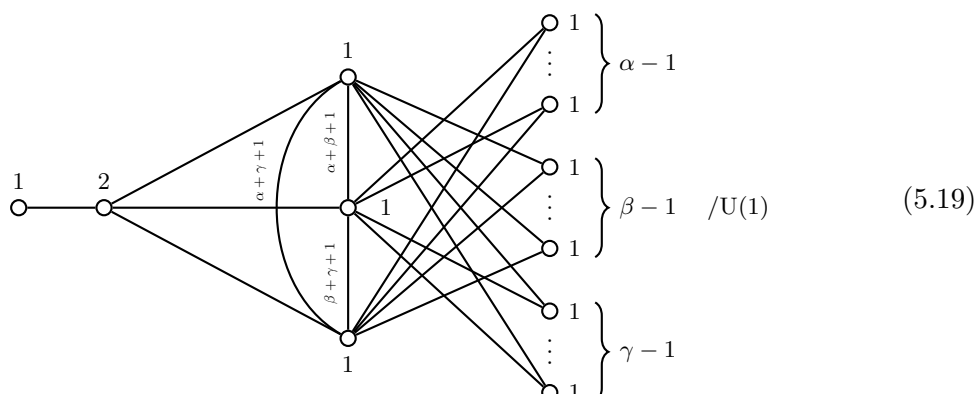
with $\alpha + \beta + \gamma = k - 1$. At this stage, activating FI parameters for the **red** nodes increases the multiplicity of the edge labeled $\beta + \gamma$ by one and reduces the number of nodes in the set labeled α by one. The resulting quiver is



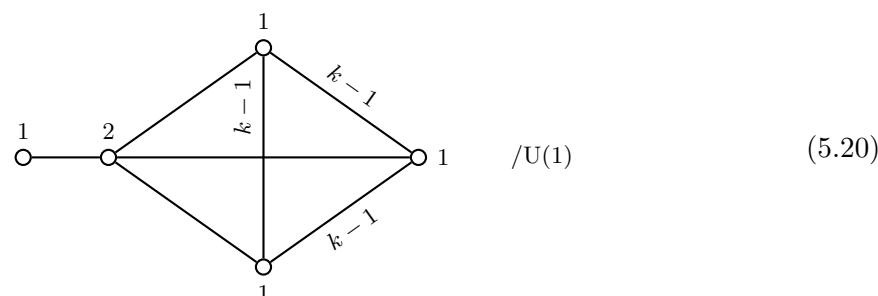
Similarly, an FI deformation at the next set of **red** nodes increases the multiplicity of the edge labeled $\alpha + \gamma$ by one, while reducing the number of nodes in the set labeled β by one:



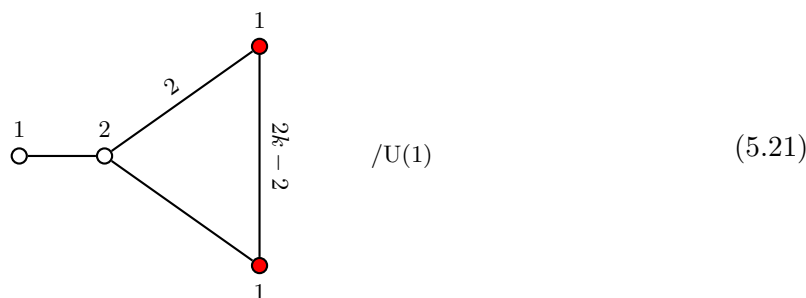
Activating FI parameters at the **red** nodes increases the multiplicity of the edge labeled $\alpha + \beta$ by one and decreases the number of nodes in the set labeled γ by one:



Reiterating this sequence of FI deformations ultimately yields the 3d mirror of $D_{3k}(\text{SU}(3))$:

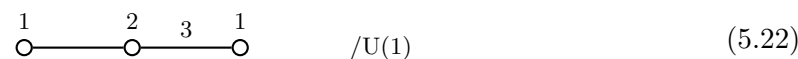


Finally, we note that $D_{3k}(\text{SU}(3))$ can be further deformed to $D_{3k-1}(\text{SU}(3))$. This is achieved by activating FI parameters for two $\text{U}(1)$ nodes in the complete graph of (5.20), which gives



$+(k - 2)$ free hypermultiplets.

Implementing this deformation on the two **red** $\text{U}(1)$ nodes yields the 3d mirror of $D_{3k-1}(\text{SU}(3))$:



$+(3k - 5)$ free hypermultiplets.

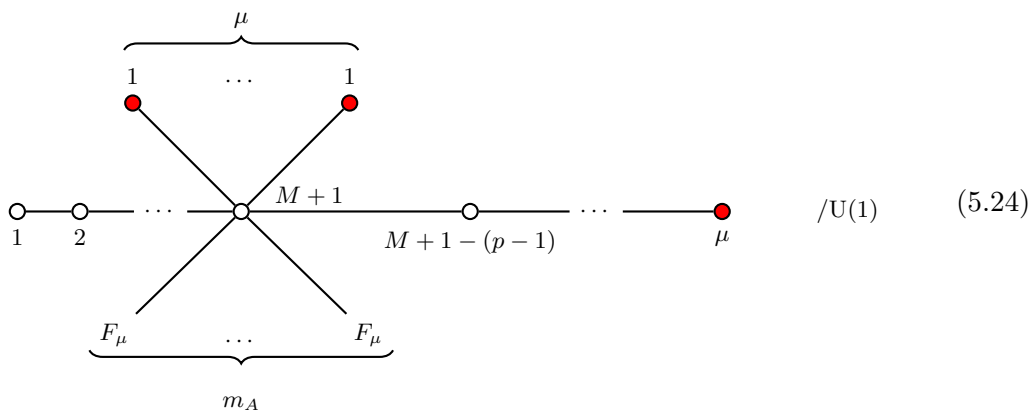
Further deforming the full tail with the single $\text{U}(1)$ on the right leads to the 3d mirror of (A_{3k-2}, A_2) , which consists solely of $3k - 2$ free hypermultiplets (see, e.g., [37, table 2 or (6.35)-(6.36)]).

5.3 Star-shaped parent of $D_p(\text{SU}(N))$ with $p \leq N$ and $m_B = 1$

We now analyze the case where $m_B = (N - px)/\mu = 1$, with $x = \lfloor N/p \rfloor \geq 1$ and $\mu = \text{GCD}(N, p)$. This implies $m_A = p/\mu - 1$. The Euclidean algorithm sequence (4.9) is

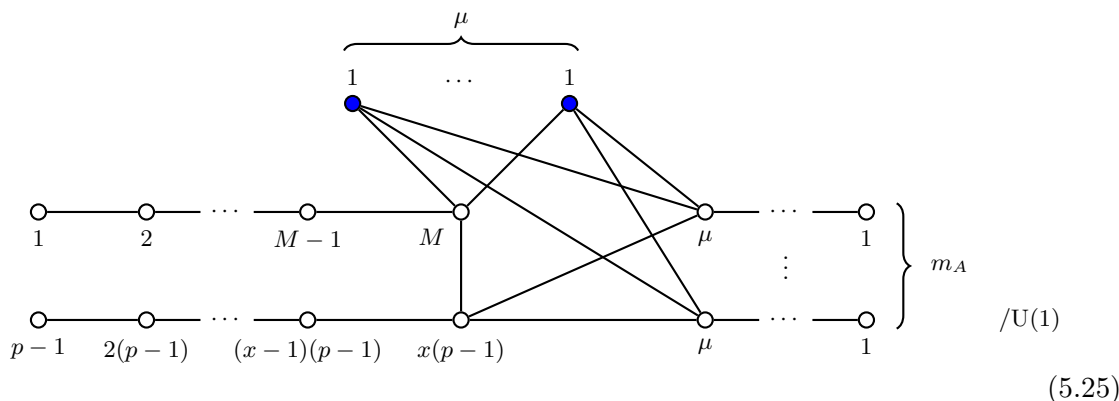
$$a_0 = N = \mu + px, \quad a_1 = p, \quad a_2 = \mu, \quad a_3 = 0. \tag{5.23}$$

From (4.14), the bouquet consists of $\mu F_1 + (p/\mu - 1)F_\mu = \mu F_1 + m_A F_\mu$. Noting that $N - x = M + 1$, the ansatz (4.2) becomes



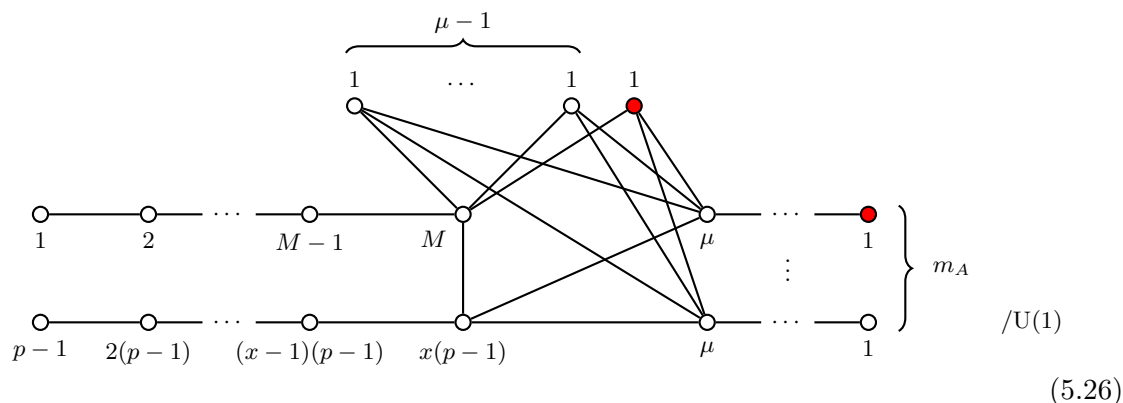
where the red nodes indicate where FI parameters are activated. We will now show that this quiver deforms into the mirror theory for the corresponding $D_p(\text{SU}(N))$, adopting the notation from appendix A.2.

The deformation yields the following quiver:

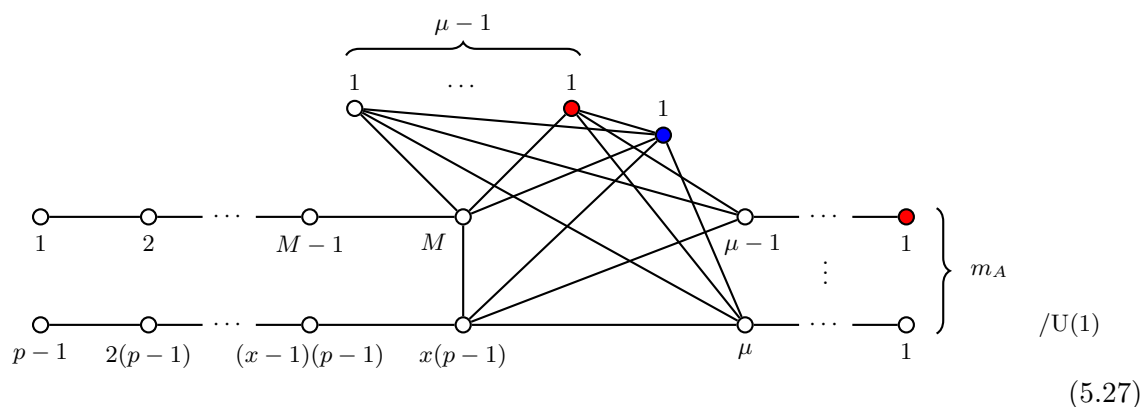


We note that the two tails on the left of (5.25) are already Tail A and Tail B as defined in appendix A.2. Tail A is connected to the m_A tails of type F_μ , while Tail B is only connected to the bouquet of $U(1)$ nodes that arise from rebalancing the quiver. All $U(1)$ s in the bouquet are also connected to the $U(\mu)$ nodes of the F_μ tails. To obtain the mirror of $D_p(\text{SU}(N))$, we must remove the m_A tails of type F_μ . This process will result in a complete graph of $U(1)$ nodes connected to Tail A with multiplicity m_A and to Tail B with multiplicity one. Let us

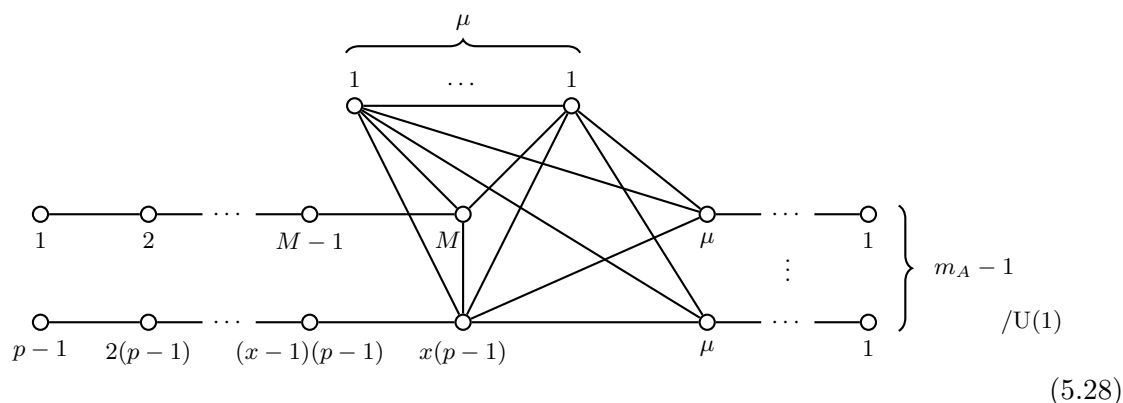
demonstrate this deformation by activating the following FI parameters:



After the deformation, the resulting quiver is



The balancing node in blue is now connected to Tail A, but not to the remaining $F_{\mu-1}$ tails. We can proceed by eliminating the $F_{\mu-1}$ tail by activating FI parameters at another node in the bouquet of $\mu - 1$ U(1)s. At the end of this process, the bouquet of μ U(1) nodes will be connected to Tail A with multiplicity one:



We note that the bouquet of U(1)s is now also connected by an internal edge. This procedure is iterated until all F_{μ} tails are removed. Each iteration increases the multiplicity of the edges connecting the U(1) bouquet to Tail A by one, as well as the multiplicity of the edges

within the complete graph. The final resulting quiver is

$$(5.29)$$

This is precisely the quiver (A.9) for the case $m_B = 1$.

5.4 Star-shaped parent of $D_p(\text{SU}(N))$ with $p \leq N$ and $m_A = 1$

We now consider the case where $m_A = [p(1+x) - N] / \mu = 1$, with $x = \lfloor N/p \rfloor \geq 1$ and $\mu = \text{GCD}(N, p)$. Since $m_A + m_B = p/\mu$, we have $m_B = (N - px) / \mu = p/\mu - 1$. The Euclidean algorithm sequence (4.9) depends on whether m_B is also equal to 1:

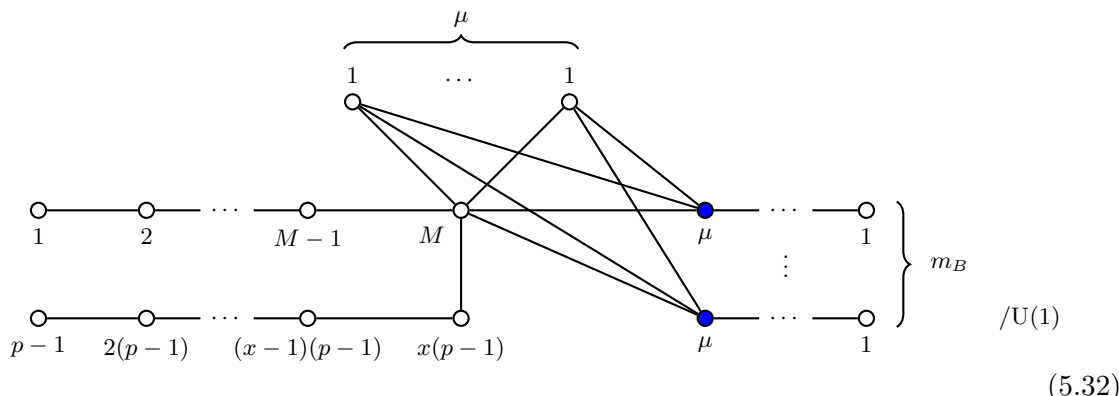
$$\begin{aligned}
 m_A = 1, \quad m_B \neq 1 : \quad & a_0 = N = p(1+x) - \mu, \quad a_1 = p, \quad a_2 = p - \mu = \alpha\mu \quad (\alpha \in \mathbb{Z}_{\geq 2}), \\
 & a_3 = \mu, \quad a_4 = 0, \\
 m_A = 1, \quad m_B = 1 : \quad & a_0 = N = \mu + px, \quad a_1 = p, \quad a_2 = \mu, \quad a_3 = 0.
 \end{aligned}
 \tag{5.30}$$

The case $m_A = m_B = 1$ was discussed in the preceding section. Let us focus on the case $m_A = 1$ and $m_B \neq 1$. From the definitions, this implies $p = (\alpha + 1)\mu$ and $m_B = \alpha$. Using (4.14), we find that the bouquet consists of $\mu F_1 + (\lfloor (\alpha + 1)/\alpha \rfloor - 1)F_{\alpha\mu} + \alpha F_\mu = \mu F_1 + \alpha F_\mu = \mu F_1 + m_B F_\mu$. Writing $N - x = M + 1$, the ansatz (4.2) becomes:

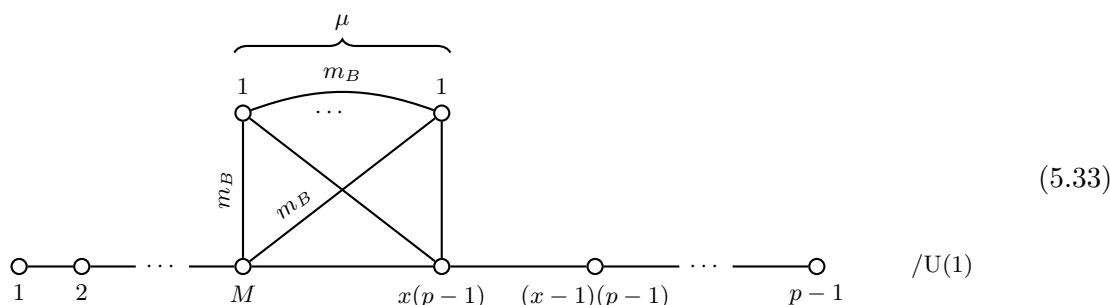
$$(5.31)$$

where we use again the notation from appendix A.2. This quiver is similar to the one in section 5.3, but the previous deformation strategy cannot be used here, because there are not enough $U(1)$ nodes in the bouquet to reduce the right tail. However, there are precisely m_B tails of type F_μ , whose $U(\mu)$ nodes can be used to activate FI parameters along with

the $U(\mu m_B)$ node. After the deformation, the resulting quiver is



This quiver is analogous to (5.25), but with the m_B tails of type F_μ connected to Tail B instead of Tail A. From this point, one can proceed as before by forming a complete graph of the $U(1)$ nodes and attaching it to Tails A and B by eliminating the m_B tails of type F_μ . Each elimination of an F_μ tail increases the multiplicity of the edges connecting Tail B to the complete graph of $U(1)$ s, and also the multiplicity of the edges within the complete graph itself. Finally, the resulting quiver is



This is precisely the quiver (A.9) for $m_A = 1$.

5.5 Star-shaped parent of $D_p(\text{SU}(N))$ with $p = N - 2$

In this section, we explicitly show that the star-shaped parent quiver for $D_{N-2}(\text{SU}(N))$, as prescribed in section 4, deforms into the correct 3d mirror theory for $N \geq 4$. The relevant parameters are:

$$\mu = \begin{cases} 2 & N \text{ even} \\ 1 & N \text{ odd} \end{cases}, \quad x = \left\lfloor \frac{N}{N-2} \right\rfloor = \begin{cases} 2 & N=4 \\ 1 & N>4 \end{cases}, \quad M = \begin{cases} 1 & N=4 \\ N-2 & N>4 \end{cases},$$

$$m_A = \begin{cases} 1 & N=4 \\ \frac{N}{2}-2 & N>4 \text{ even} \\ N-4 & N \text{ odd} \end{cases}, \quad m_B = \begin{cases} 0 & N=4 \\ 1 & N>4 \text{ even} \\ 2 & N \text{ odd} \end{cases}, \quad H_{\text{free}} = \begin{cases} 0 & N \text{ even} \\ \frac{N-5}{2} & N \text{ odd} \end{cases}.$$

(5.34)

Observe that, for $N = 4$ and for even $N > 4$, we have $m_A = 1$ and $m_B = 1$, respectively. Therefore, these cases are included in the analysis described in sections 5.3 and 5.4. We can thus focus just on the cases with odd N .

For odd N , the integer sequence (4.9) reads

$$(a_0, a_1, a_{\bar{n}=2}, a_{\bar{n}+1=3}, a_{\bar{n}+2=4}) = (N, N - 2, 2, 1, 0), \tag{5.35}$$

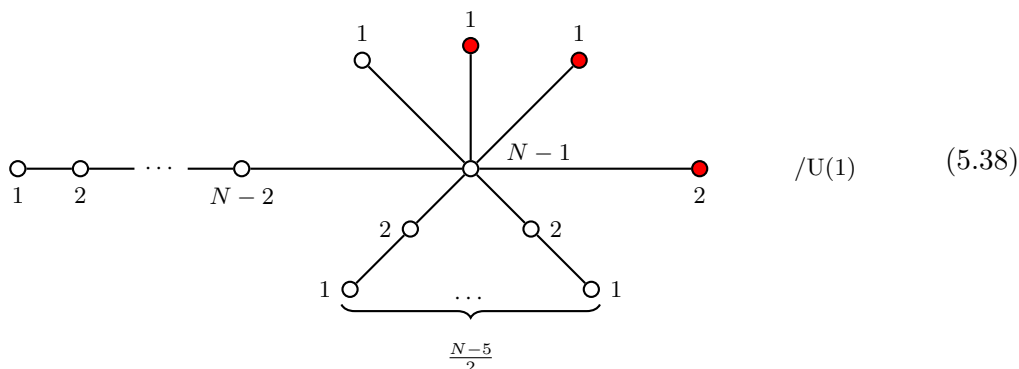
which corresponds to the sequence of deformations

$$D_1(\text{SU}(2)) \rightarrow D_2(\text{SU}(N - 2)) \rightarrow D_{N-2}(\text{SU}(N)). \tag{5.36}$$

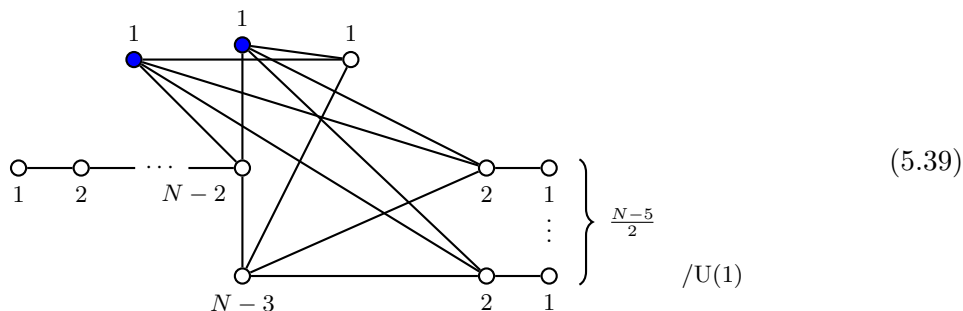
Using (4.14), the bouquet is given by

$$\text{Bouquet for } D_{N-2}(\text{SU}(N)) = 3F_1 + \left(\frac{N-5}{2}\right) F_2. \tag{5.37}$$

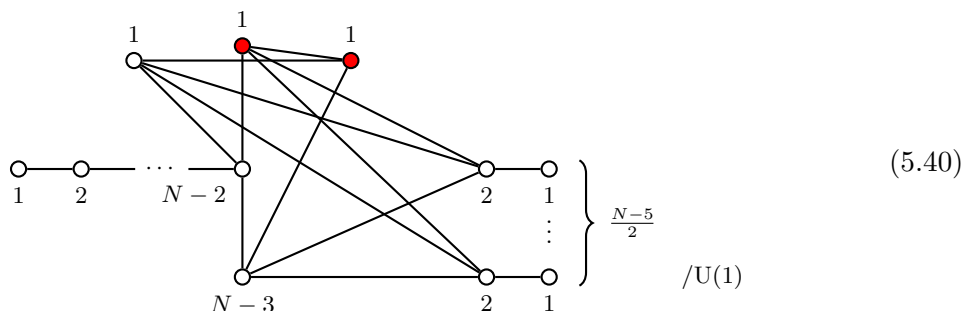
Therefore, from (5.34) and (5.37), the star-shaped parent quiver for $D_{N-2}(\text{SU}(N))$ with odd N is given by



Activating FI parameters on the red nodes yields the deformed quiver



To obtain a quiver analogous to those in sections 5.3 and 5.4, we first reduce the number of $U(1)$ s in the bouquet to $\mu = 1$. We activate FI parameters at the following red nodes:



resulting in

(5.41)

Repeating the deformation at the two remaining $U(1)$ nodes in the bouquet leads to

(5.42)

+1 free hypermultiplet.

We now remove the $(N - 5)/2$ tails of type F_2 by activating FI parameters at the single $U(1)$ in the bouquet and the $U(1)$ in each F_2 tail, i.e.

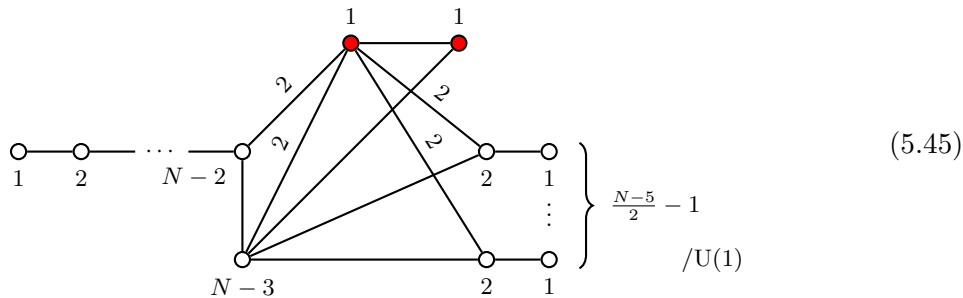
(5.43)

Such deformation leads to the removal of a single F_2 tail as follows:

(5.44)

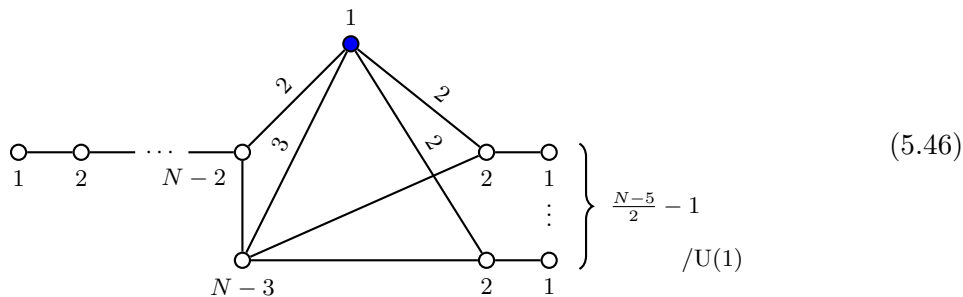
+1 free hypermultiplet.

Activating FI parameters at the two U(1) nodes above, namely



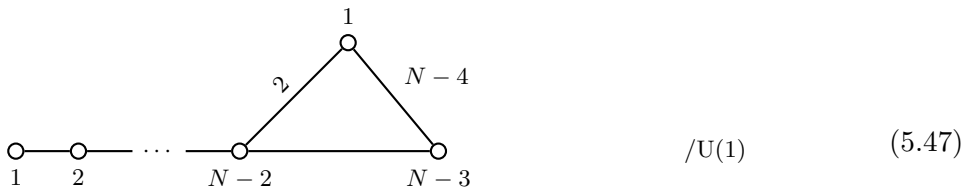
+1 free hypermultiplet,

results in the following quiver:



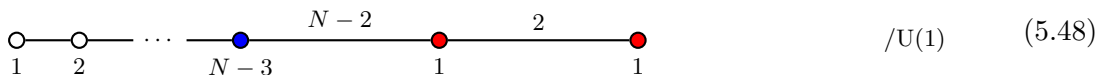
+1 free hypermultiplet.

We conclude that removing an F_2 tail increases the multiplicity of the edge connecting the U(1) bouquet to the U($N - 3$) node by two and generates one free hypermultiplet. This process can be repeated $(N - 5)/2$ times, eventually leading to:



+ $\frac{N-5}{2}$ free hypermultiplets.

This is precisely the mirror of $D_{N-2}(SU(N))$ for odd N . We can also use this example to explicitly test the result from section 3.2. By activating FI parameters at the two U($N - 3$) nodes, we obtain



+ $\frac{N-5}{2}$ free hypermultiplets,

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A Review of $D_p(\text{SU}(N))$ and their 3d mirror theories

In the following sections, we are going to review the constructions of the 3d mirror theories for $D_p(\text{SU}(N))$ theories, which represent the end points of the FI deformations acting on certain star-shaped quivers.

A.1 3d mirror theories for $p \geq N$

Let us consider the 3d mirror theories for $D_p(\text{SU}(N))$ when $p \geq N$. These have been studied in [35] and can be constructed as follows:

1. We construct a complete graph with μ vertices such that each vertex is a $U(1)$ gauge group, and they are connected by an edge with multiplicity

$$m_G = \mathbf{n}(\mathbf{p} - \mathbf{n}) = \frac{1}{\mu^2} N(p - N). \tag{A.1}$$

2. We also add a tail

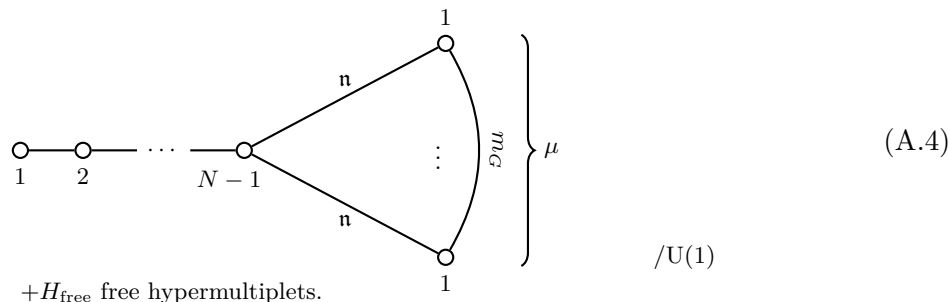
$$\begin{array}{c} \circ \text{---} \circ \text{---} \dots \text{---} \circ \\ 1 \quad 2 \quad \quad \quad N-1 \end{array} \tag{A.2}$$

3. We connect the $U(N - 1)$ node of the tail to all the $U(1)$ s in the complete graph with an edge with multiplicity $\mathbf{n} = \frac{N}{\mu}$.

4. The 3d mirror has also a number of free hypermultiplets given by

$$H_{\text{free}} = \frac{1}{2} \mu (\mathbf{n} - 1)(\mathbf{p} - \mathbf{n} - 1) = \frac{1}{2\mu} (N - \mu)(p - N - \mu). \tag{A.3}$$

We can schematically draw the 3d mirror quiver as



The mirror of (A_{p-N-1}, A_{N-1}) is simply the complete graph of $U(1)$ s together with H_{free} hypermultiplets [35].

A.2 3d mirror theories for $p < N$

In order to construct the mirror theories for $D_p(\text{SU}(N))$ with $p \leq N$, we introduce, together with (1.2), the following quantities:

$$x = \left\lfloor \frac{N}{p} \right\rfloor \quad \text{and} \quad M = N - (x + 1). \tag{A.5}$$

The mirror theory is given by:

1. A complete graph with μ U(1) nodes each with multiplicity

$$m_G = \underbrace{(p(1+x) - n)}_{m_A} \underbrace{(n - px)}_{m_B}. \tag{A.6}$$

2. We construct two tails

$$\begin{array}{l} \text{Tail A:} \quad \bigcirc \text{---} \bigcirc \text{---} \dots \text{---} \bigcirc \text{---} \bigcirc \\ \quad \quad \quad p-1 \quad 2(p-1) \quad \quad \quad (x-1)(p-1) \quad x(p-1) \end{array} \tag{A.7}$$

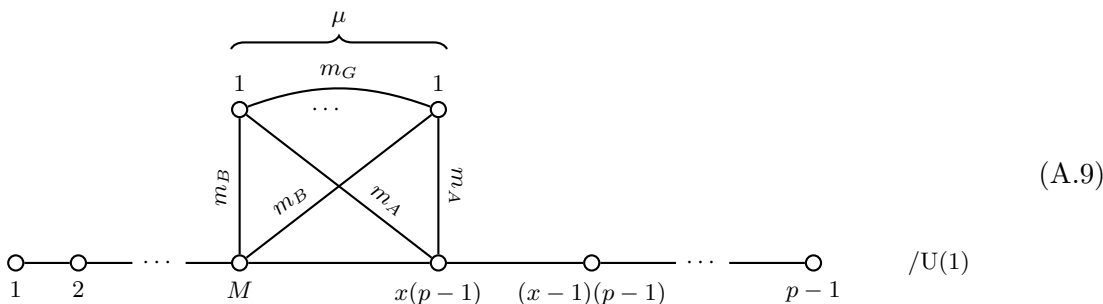
$$\text{Tail B:} \quad \bigcirc \text{---} \bigcirc \text{---} \dots \text{---} \bigcirc \text{---} \bigcirc \\ \quad \quad \quad 1 \quad 2 \quad \quad \quad M-1 \quad M$$

3. Each U(1) node of the complete graph is attached to the U($x(p-1)$) node of Tail A with an edge with multiplicity m_A .
4. Each U(1) node of the complete graph is attached to the U(M) node of Tail B with an edge with multiplicity m_B .
5. The node U($x(p-1)$) node of Tail A is attached to the U(M) node of Tail B with an edge with multiplicity one.
6. There are

$$\begin{aligned} H_{\text{free}} &= \frac{1}{2} \mu (n - xp - 1) (p(1+x) - n - 1) = \frac{1}{2\mu} (N - px - \mu) (p(1+x) - N - \mu) \\ &= \frac{1}{2} \mu (m_B - 1) (m_A - 1) = \frac{1}{2} (\mu(m_G + 1) - p) \end{aligned} \tag{A.8}$$

free hypermultiplets.

Once again, we can draw the quiver schematically as follows:



+ H_{free} free hypermultiplets.

B From orb-instantons to generic star-shaped quivers

In [41], it was shown that combinations of the Fayet-Iliopoulos (FI) deformations reviewed in section 2 allow for the construction of any Type A class \mathcal{S} theory with regular punctures from an orb-instanton theory. While we refer the reader to [41] for a comprehensive study of the deformations that map an orb-instanton theory to a generic star-shaped quiver, this appendix will briefly review the procedure.

B.1 Deformations from orb-instanton theories to star-shaped quivers

Let us consider a generic magnetic quiver for an orb-instanton theory of the form [45] (see also [53]):

$$\begin{array}{ccccccccccc}
 & & & & & & & L & & & \\
 & & & & & & & \circ & & & \\
 \dots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \dots \\
 & & k & & A & & B & & C & & D & & E & & F & & G & & H & &
 \end{array} \tag{B.1}$$

We assume that the $U(F)$ node is balanced, i.e.,

$$2F = E + G + L. \tag{B.2}$$

We can now perform FI deformations involving node F and its surrounding nodes E , G , and L , whose ranks are constrained by (B.2). Implementing the procedure discussed in [41] and reviewed in section 2 leads to the quiver

$$\begin{array}{ccccccccccc}
 & & & & & & & F - G & & & & & & & & & & & & & & \\
 & & & & & & & \circ & & & & & & & & & & & & & & \\
 \dots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \dots \\
 & & k & & A & & B & & C & & D & & G + E - F & & H & & F - E & & & &
 \end{array} \tag{B.3}$$

This is an E_7 -shaped quiver with a single tail, which is inherited from the parent quiver. This resulting quiver can be further deformed into an E_6 -shaped quiver. For example, assuming $B \geq F - G$, we can activate FI parameters for these nodes. If there exists a node in the tail with rank $\tilde{k} = B - F + G$, the FI parameter must also be activated for that node. The resulting quiver after this deformation is

$$\begin{array}{ccccccccccccccc}
 & \\
 & \\
 & \\
 & \\
 & \\
 & \\
 \dots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \dots \\
 & & k - \tilde{k} & & A - \tilde{k} & & F - G & & E - F + G & & H & & D - F + G & & C - F + G & & \tilde{k} & & & & & &
 \end{array} \tag{B.4}$$

This process generates an E_6 -shaped quiver with two tails. One tail originates from the original tail of the E_7 -shaped quiver, with all its ranks reduced by \tilde{k} , while the second tail is new and terminates at a node of rank \tilde{k} . Different types of deformations can produce an E_6 -shaped quiver with three tails, of the general form

$$\begin{array}{ccc}
 & & T_\rho \\
 & & | \\
 & & \circ \\
 & & | \\
 & & N \\
 & & | \\
 T_{\rho_1, \rho_2, \dots, \rho_k} & \text{---} & \circ & \text{---} & T_{\rho'_1, \rho'_2, \dots, \rho'_{k'}} \\
 & & | \\
 & & kN
 \end{array} \tag{B.5}$$

Here, following the notation in [41], we define $T_{\rho_1, \rho_2, \dots, \rho_k}$ as a tail

$$T_{\rho_1} \text{ --- } \underset{N}{\circ} \text{ --- } Y_2 \text{ --- } \underset{2N}{\circ} \text{ --- } Y_3 \text{ --- } \dots \text{ --- } \underset{(k-1)N}{\circ} \text{ --- } Y_k \tag{B.6}$$

where Y_i are T_{ρ_i} tails with their node ranks shifted by $(i-1)N$. For example, if we consider $\rho_i = [1^N]$, the quiver for T_{ρ_i} is

$$\underset{1}{\circ} \text{ --- } \underset{2}{\circ} \text{ --- } \dots \text{ --- } \underset{N-1}{\circ} \text{ --- } \underset{N}{\square} \tag{B.7}$$

The corresponding Y_i is then

$$\underset{1+(i-1)N}{\circ} \text{ --- } \underset{2+(i-1)N}{\circ} \text{ --- } \dots \text{ --- } \underset{iN-1}{\circ} \text{ --- } \underset{iN}{\square} \tag{B.8}$$

and its terminal flavor node is gauged with the first node of Y_{i+1} . If we sequentially activate FI parameters in (B.5) at pairs of nodes with ranks $N, 2N, \dots$, we obtain a star-shaped quiver of the form

$$\begin{array}{c}
 \overbrace{\hspace{10em}}^{k \text{ tails}} \\
 T_{\rho_1} \quad \dots \quad T_{\rho_k} \\
 \diagdown \quad \quad \diagup \\
 \quad \quad N \quad \circ \quad \quad \text{---} \quad T_{\rho} \\
 \diagup \quad \quad \diagdown \\
 T_{\rho'_1} \quad \dots \quad T_{\rho'_{k'}} \\
 \underbrace{\hspace{10em}}_{k' \text{ tails}}
 \end{array} \tag{B.9}$$

This procedure yields a generic star-shaped quiver with an arbitrary number of tails.

B.2 Inverse algorithm from star-shaped quivers to orbi-instanton theories

The pattern of FI deformations that leads from an orbi-instanton theory to a generic star-shaped quiver can be inverted. This allows one to predict the parent E_6 -shaped quiver for a given star-shaped quiver. The inverse transformation generically produces a quiver with underbalanced nodes, which can be made balanced by applying Seiberg-like dualities [54] on the underbalanced node where the FI parameter is turned on [55, 56]. The strategy for a generic star-shaped quiver, such as (B.9), is to propose a candidate E_6 -shaped parent theory of the form (B.5) and then apply dualities to the underbalanced nodes until a “good” quiver is obtained. The result is the parent E_6 -shaped quiver from which the star-shaped quiver in question originates. This procedure can be iterated to trace the lineage back to the parent orbi-instanton mirror quiver, applying dualities at each step of the reconstruction. A full discussion with further details can be found in [41].

Since this inverse algorithm is deterministic (up to dualities), the main focus of this work has been to find a prescription for the parent star-shaped quiver from which a $D_p(SU(N))$ theory can be obtained via FI deformations. This star-shaped quiver can then be traced back to the corresponding orbi-instanton theory by implementing the algorithm described in [41].

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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References

- [1] D. Gaiotto, *$N = 2$ dualities*, *JHEP* **08** (2012) 034 [[arXiv:0904.2715](https://arxiv.org/abs/0904.2715)] [[INSPIRE](#)].
- [2] D. Gaiotto and J. Maldacena, *The gravity duals of $N = 2$ superconformal field theories*, *JHEP* **10** (2012) 189 [[arXiv:0904.4466](https://arxiv.org/abs/0904.4466)] [[INSPIRE](#)].
- [3] M. Aghand et al., *The hitchhiker’s guide to 4d $\mathcal{N} = 2$ superconformal field theories*, *SciPost Phys. Lect. Notes* **64** (2022) 1 [[arXiv:2112.14764](https://arxiv.org/abs/2112.14764)] [[INSPIRE](#)].
- [4] P.C. Argyres, J.J. Heckman, K. Intriligator and M. Martone, *Snowmass White Paper on SCFTs*, [arXiv:2202.07683](https://arxiv.org/abs/2202.07683) [[INSPIRE](#)].
- [5] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **461** (1996) 71 [[hep-th/9511154](https://arxiv.org/abs/hep-th/9511154)] [[INSPIRE](#)].
- [6] P.C. Argyres, M.R. Plesser and A.D. Shapere, *$N = 2$ moduli spaces and $N = 1$ dualities for $SO(n_c)$ and $USp(2n_c)$ superQCD*, *Nucl. Phys. B* **483** (1997) 172 [[hep-th/9608129](https://arxiv.org/abs/hep-th/9608129)] [[INSPIRE](#)].
- [7] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **471** (1996) 430 [[hep-th/9603002](https://arxiv.org/abs/hep-th/9603002)] [[INSPIRE](#)].
- [8] T. Eguchi and K. Hori, *$N = 2$ superconformal field theories in four-dimensions and A-D-E classification*, in the proceedings of the *Conference on the Mathematical Beauty of Physics (In Memory of C. Itzykson)*, Saclay, France, 05–07 June 1996 [[hep-th/9607125](https://arxiv.org/abs/hep-th/9607125)] [[INSPIRE](#)].
- [9] A.D. Shapere and C. Vafa, *BPS structure of Argyres-Douglas superconformal theories*, [hep-th/9910182](https://arxiv.org/abs/hep-th/9910182) [[INSPIRE](#)].
- [10] S. Cecotti, A. Neitzke and C. Vafa, *R-Twisting and 4d/2d Correspondences*, [arXiv:1006.3435](https://arxiv.org/abs/1006.3435) [[INSPIRE](#)].
- [11] D. Xie, *General Argyres-Douglas Theory*, *JHEP* **01** (2013) 100 [[arXiv:1204.2270](https://arxiv.org/abs/1204.2270)] [[INSPIRE](#)].
- [12] S. Cecotti, M. Del Zotto and S. Giacomelli, *More on the $N = 2$ superconformal systems of type $D_p(G)$* , *JHEP* **04** (2013) 153 [[arXiv:1303.3149](https://arxiv.org/abs/1303.3149)] [[INSPIRE](#)].
- [13] S. Cecotti and M. Del Zotto, *Infinitely many $N = 2$ SCFT with ADE flavor symmetry*, *JHEP* **01** (2013) 191 [[arXiv:1210.2886](https://arxiv.org/abs/1210.2886)] [[INSPIRE](#)].
- [14] Y. Wang and D. Xie, *Codimension-two defects and Argyres-Douglas theories from outer-automorphism twist in 6d (2,0) theories*, *Phys. Rev. D* **100** (2019) 025001 [[arXiv:1805.08839](https://arxiv.org/abs/1805.08839)] [[INSPIRE](#)].
- [15] D. Xie and P. Zhao, *Central charges and RG flow of strongly-coupled $N = 2$ theory*, *JHEP* **03** (2013) 006 [[arXiv:1301.0210](https://arxiv.org/abs/1301.0210)] [[INSPIRE](#)].

- [16] S. Giacomelli, *RG flows with supersymmetry enhancement and geometric engineering*, *JHEP* **06** (2018) 156 [[arXiv:1710.06469](#)] [[INSPIRE](#)].
- [17] K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs*, *JHEP* **02** (2017) 075 [[arXiv:1607.04281](#)] [[INSPIRE](#)].
- [18] K. Maruyoshi and J. Song, *Enhancement of Supersymmetry via Renormalization Group Flow and the Superconformal Index*, *Phys. Rev. Lett.* **118** (2017) 151602 [[arXiv:1606.05632](#)] [[INSPIRE](#)].
- [19] P. Agarwal, K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ Deformations and RG flows of $\mathcal{N} = 2$ SCFTs, part II: non-principal deformations*, *JHEP* **12** (2016) 103 [Addendum *ibid.* **04** (2017) 113] [[arXiv:1610.05311](#)] [[INSPIRE](#)].
- [20] M. Del Zotto, I. García Etxebarria and S.S. Hosseini, *Higher form symmetries of Argyres-Douglas theories*, *JHEP* **10** (2020) 056 [[arXiv:2007.15603](#)] [[INSPIRE](#)].
- [21] C. Closset, S. Schäfer-Nameki and Y.-N. Wang, *Coulomb and Higgs Branches from Canonical Singularities: Part 0*, *JHEP* **02** (2021) 003 [[arXiv:2007.15600](#)] [[INSPIRE](#)].
- [22] C. Closset, S. Giacomelli, S. Schäfer-Nameki and Y.-N. Wang, *5d and 4d SCFTs: Canonical Singularities, Trinions and S-Dualities*, *JHEP* **05** (2021) 274 [[arXiv:2012.12827](#)] [[INSPIRE](#)].
- [23] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *Holographic Duals of Argyres-Douglas Theories*, *Phys. Rev. Lett.* **127** (2021) 211601 [[arXiv:2105.11567](#)] [[INSPIRE](#)].
- [24] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *M5-brane sources, holography, and Argyres-Douglas theories*, *JHEP* **11** (2021) 140 [[arXiv:2106.01322](#)] [[INSPIRE](#)].
- [25] S.S. Hosseini and R. Moscrop, *Maruyoshi-Song flows and defect groups of $D_p^b(G)$ theories*, *JHEP* **10** (2021) 119 [[arXiv:2106.03878](#)] [[INSPIRE](#)].
- [26] M.J. Kang, C. Lawrie and J. Song, *Infinitely many 4D $N = 2$ SCFTs with $a=c$ and beyond*, *Phys. Rev. D* **104** (2021) 105005 [[arXiv:2106.12579](#)] [[INSPIRE](#)].
- [27] M.J. Kang, C. Lawrie, K.-H. Lee and J. Song, *Infinitely many 4D $N = 1$ SCFTs with $a=c$* , *Phys. Rev. D* **105** (2022) 126006 [[arXiv:2111.12092](#)] [[INSPIRE](#)].
- [28] L. Bhardwaj, S. Giacomelli, M. Hübner and S. Schäfer-Nameki, *Relative defects in relative theories: Trapped higher-form symmetries and irregular punctures in class S*, *SciPost Phys.* **13** (2022) 101 [[arXiv:2201.00018](#)] [[INSPIRE](#)].
- [29] M.J. Kang et al., *Higgs branch, Coulomb branch, and Hall-Littlewood index*, *Phys. Rev. D* **106** (2022) 106021 [[arXiv:2207.05764](#)] [[INSPIRE](#)].
- [30] I. Bah, F. Bonetti, E. Nardoni and T. Waddleton, *Aspects of irregular punctures via holography*, *JHEP* **11** (2022) 131 [[arXiv:2207.10094](#)] [[INSPIRE](#)].
- [31] M.J. Kang, C. Lawrie, K.-H. Lee and J. Song, *Operator spectroscopy for 4D SCFTs with $a = c$* , *Phys. Rev. D* **107** (2023) 066018 [[arXiv:2210.06497](#)] [[INSPIRE](#)].
- [32] C. Beem et al., *Simplifying the Type A Argyres-Douglas Landscape*, [arXiv:2311.12123](#) [[INSPIRE](#)].
- [33] C. Couzens, M.J. Kang, C. Lawrie and Y. Lee, *Holographic duals of Higgsed $\mathcal{D}_p^b(BCD)$* , [arXiv:2312.12503](#) [[INSPIRE](#)].
- [34] M.J. Kang, C. Lawrie, K.-H. Lee and J. Song, *Landscape of 4d $N = 1$ SCFTs with $a = c$* , *Phys. Rev. D* **111** (2025) 086002 [[arXiv:2412.17895](#)] [[INSPIRE](#)].
- [35] S. Giacomelli, N. Mekareeya and M. Sacchi, *New aspects of Argyres-Douglas theories and their dimensional reduction*, *JHEP* **03** (2021) 242 [[arXiv:2012.12852](#)] [[INSPIRE](#)].

- [36] M. Del Zotto and A. Hanany, *Complete Graphs, Hilbert Series, and the Higgs branch of the 4d $\mathcal{N} = 2$ (A_n, A_m) SCFTs*, *Nucl. Phys. B* **894** (2015) 439 [[arXiv:1403.6523](#)] [[INSPIRE](#)].
- [37] F. Carta, S. Giacomelli, N. Mekareeya and A. Mininno, *Conformal manifolds and 3d mirrors of Argyres-Douglas theories*, *JHEP* **08** (2021) 015 [[arXiv:2105.08064](#)] [[INSPIRE](#)].
- [38] F. Carta, S. Giacomelli, N. Mekareeya and A. Mininno, *Dynamical consequences of 1-form symmetries and the exceptional Argyres-Douglas theories*, *JHEP* **06** (2022) 059 [[arXiv:2203.16550](#)] [[INSPIRE](#)].
- [39] F. Carta, S. Giacomelli, N. Mekareeya and A. Mininno, *A tale of 2-groups: $D_p(USp(2N))$ theories*, *JHEP* **06** (2023) 102 [[arXiv:2208.11130](#)] [[INSPIRE](#)].
- [40] F. Carta, S. Giacomelli, N. Mekareeya and A. Mininno, *Comments on Non-invertible Symmetries in Argyres-Douglas Theories*, *JHEP* **07** (2023) 135 [[arXiv:2303.16216](#)] [[INSPIRE](#)].
- [41] S. Giacomelli, W. Harding, N. Mekareeya and A. Mininno, *All class \mathcal{S} theories of type-A originate from orbi-instantons*, *JHEP* **02** (2025) 185 [[arXiv:2411.03425](#)] [[INSPIRE](#)].
- [42] P.S. Aspinwall and D.R. Morrison, *Point-like instantons on K3 orbifolds*, *Nucl. Phys. B* **503** (1997) 533 [[hep-th/9705104](#)] [[INSPIRE](#)].
- [43] M. Del Zotto, J.J. Heckman, A. Tomasiello and C. Vafa, *6d Conformal Matter*, *JHEP* **02** (2015) 054 [[arXiv:1407.6359](#)] [[INSPIRE](#)].
- [44] J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *Atomic Classification of 6D SCFTs*, *Fortsch. Phys.* **63** (2015) 468 [[arXiv:1502.05405](#)] [[INSPIRE](#)].
- [45] N. Mekareeya, K. Ohmori, Y. Tachikawa and G. Zafrir, *E_8 instantons on type-A ALE spaces and supersymmetric field theories*, *JHEP* **09** (2017) 144 [[arXiv:1707.04370](#)] [[INSPIRE](#)].
- [46] M. van Beest and S. Giacomelli, *Connecting 5d Higgs branches via Fayet-Iliopoulos deformations*, *JHEP* **12** (2021) 202 [[arXiv:2110.02872](#)] [[INSPIRE](#)].
- [47] A. Bourget, S. Giacomelli and J.F. Grimminger, *FI-flows of 3d $N = 4$ Theories*, *JHEP* **04** (2023) 015 [[arXiv:2302.03698](#)] [[INSPIRE](#)].
- [48] D. Gaiotto and E. Witten, *S-Duality of Boundary Conditions In $N = 4$ Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [[arXiv:0807.3720](#)] [[INSPIRE](#)].
- [49] S. Cremonesi, G. Ferlito, A. Hanany and N. Mekareeya, *Coulomb Branch and The Moduli Space of Instantons*, *JHEP* **12** (2014) 103 [[arXiv:1408.6835](#)] [[INSPIRE](#)].
- [50] C. Lawrie, L. Mansi, M. Sperling and Z. Zhong, *Pathway to decay and fission of orthosymplectic quiver theories*, *Phys. Rev. D* **112** (2025) 026025 [[arXiv:2412.15202](#)] [[INSPIRE](#)].
- [51] C. Lawrie, T. Lepper and A. Mininno, *Discrete gauging of 6D SCFTs and wreathed 3D $N = 4$ quivers*, *Phys. Rev. D* **112** (2025) 026035 [[arXiv:2504.03830](#)] [[INSPIRE](#)].
- [52] F. Benini, Y. Tachikawa and D. Xie, *Mirrors of 3d Sicilian theories*, *JHEP* **09** (2010) 063 [[arXiv:1007.0992](#)] [[INSPIRE](#)].
- [53] S. Cabrera, A. Hanany and M. Sperling, *Magnetic quivers, Higgs branches, and 6d $N = (1, 0)$ theories*, *JHEP* **06** (2019) 071 [*Erratum* *ibid.* **07** (2019) 137] [[arXiv:1904.12293](#)] [[INSPIRE](#)].
- [54] I. Yaakov, *Redeeming Bad Theories*, *JHEP* **11** (2013) 189 [[arXiv:1303.2769](#)] [[INSPIRE](#)].
- [55] B. Assel and S. Cremonesi, *The Infrared Physics of Bad Theories*, *SciPost Phys.* **3** (2017) 024 [[arXiv:1707.03403](#)] [[INSPIRE](#)].
- [56] S. Giacomelli et al., *Probing bad theories with the dualization algorithm. Part I*, *JHEP* **04** (2024) 008 [[arXiv:2309.05326](#)] [[INSPIRE](#)].