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Action-angle approach to the geodesic motions in the homogeneous Sasaki-Einstein space $T^{1,1}$

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Abstract. The complete integrability of geodesics in the homogeneous Sasaki-Einstein space $T^{1,1}$ makes possible the explicit construction of the action-angle variables. This parametrization of the phase space represents a useful tool for developing perturbation theory. We find that two pairs of fundamental frequencies of the geodesic motions are resonant indicating a chaotic behavior when the integrable Hamiltonian is perturbed by a small non-integrable piece.

1. Introduction

Recently Sasaki geometry, as a natural odd-dimensional counterpart of the Kähler geometry, has become of significant interest in some modern studies in mathematics and physics [1, 2]. Sasaki-Einstein (SE) manifolds whose metric cones are Calabi-Yau manifolds find applications in string theory in connection with AdS/CFT correspondence which relates quantum gravity in a certain background to ordinary quantum field theory without gravity. A large class of examples consists of type *IIB* string theory on the background $AdS_5 \times Y_5$ with Y_5 a five-dimensional SE space. A particular interesting class of toric SE structures on $S^2 \times S^3$ have been studied by physicists [3, 4]. An extensively studied case is when Y_5 is $T^{1,1}$ which is homogeneous SE space with $SU(2)^2 \times U(1)$ isometry. The $AdS \times T^{1,1}$ is the first example of a supersymmetric holographic theory based on a compact manifold which is not locally S^5.

The purpose of this paper is to analyze the complete integrability of geodesics of the five-dimensional SE space $T^{1,1}$. We present the action-angle formulation of the geodesic motions in $T^{1,1}$ space. The description of the integrability of geodesics in $T^{1,1}$ in terms of action-angle variables give us a comprehensive description of the dynamics. We find that two pairs of frequencies of the geodesic motions are resonant giving way to chaotic behavior when the integrable Hamiltonian is perturbed by a small non-integrable piece.

The organization of the paper is as follows. In the next Section we present the functionally independent integrals of motions for geodesics in $T^{1,1}$ space. In Section 3 we describe the construction of action-angle variables and evaluate the fundamental frequencies of the geodesic motions. The paper ends with conclusions in Section 4.

2. $T^{1,1}$ space

Until recently, the only five-dimensional SE manifolds that were known explicitly were the round metric on S^5 and the homogeneous metric $T^{1,1}$ on $S^2 \times S^3$.

The metric on $T^{1,1}$ may be written down explicitly by utilizing the fact that it is a $U(1)$ bundle over $S^2 \times S^2$. We choose the coordinates (θ_i, ϕ_i) , $i = 1, 2$ to parametrize the two spheres S^2 in the standard way, while the angle $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber. The metric on $T^{1,1}$ may be written as [5]

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,. \quad (1)$$

In what follows we introduce $\nu = \frac{1}{2}\psi$ so that ν has canonical period 2π.

On a Riemannian manifold (M, g) with local coordinates x^μ and metric $g_{\mu\nu}$ the geodesics are represented by the trajectories of test-particles with proper time Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \,, \quad (2)$$

where p_μ are the canonical momenta conjugate to the coordinate x^μ, $p_\mu = g_{\mu\nu}\dot{x}^\nu$ with overdot denoting proper time derivative.

In the case of $T^{1,1}$ space, the conjugate momenta to the coordinates $(\theta_1, \theta_2, \phi_1, \phi_2, \nu)$ are

$$\begin{aligned}
p_{\theta_1} &= \frac{1}{6}\dot{\theta}_1 \,, \\
p_{\theta_2} &= \frac{1}{6}\dot{\theta}_2 \,, \\
p_{\phi_1} &= \frac{1}{6}\sin^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_1\,\dot{\psi} + \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot{\phi}_2 \,, \\
p_{\phi_2} &= \frac{1}{6}\sin^2\theta_2\,\dot{\phi}_2 + \frac{1}{9}\cos^2\theta_2\,\dot{\phi}_2 + \frac{1}{9}\cos\theta_2\,\dot{\psi} + \frac{1}{9}\cos\theta_1\cos\theta_2\,\dot{\phi}_1 \,, \\
p_\nu &= \frac{2}{9}\,\dot{\nu} + \frac{1}{9}\cos\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_2\,\dot{\phi}_2 \,,
\end{aligned} \quad (3)$$

and the conserved Hamiltonian (2) takes the form:

$$H = 3\left[p_{\theta_1}^2 + p_{\theta_2}^2 + \frac{1}{4\sin^2\theta_1}(2p_{\phi_1} - \cos\theta_1 p_\nu)^2 + \frac{1}{4\sin^2\theta_2}(2p_{\phi_2} - \cos\theta_2 p_\nu)^2\right] + \frac{9}{8}p_\nu^2 \,. \quad (4)$$

Using the complete set of Killing vectors and Killing-Yano tensors of SE space $T^{1,1}$ we find a lot of conserved quantities [6, 7, 8, 9]. However the number of functionally independent integrals of motion is only *five* which means that the geodesic flow on $T^{1,1}$ space is completely integrable, but not superintegrable.

We can choose as functionally independent constants of motion the energy

$$E = H \,, \quad (5)$$

the momenta corresponding to the cyclic coordinates $(\phi_1\,,\, \phi_2\,,\, \nu)$

$$\begin{aligned}
p_{\phi_1} &= c_{\phi_1} \,, \\
p_{\phi_2} &= c_{\phi_2} \,, \\
p_\nu &= c_\nu \,,
\end{aligned} \quad (6)$$

where $(c_{\phi_1}\,,\, c_{\phi_2}\,, c_\nu)$ are some constants, and one of the two total $SU(2)$ angular momentum

$$\vec{J}_i^{\,2} = p_{\theta_i}^2 + \frac{1}{4\sin^2\theta_i}(2p_{\phi_i} - \cos\theta_i p_\nu)^2 + \frac{1}{4}p_\nu^2 \,, \quad i = 1, 2 \,. \quad (7)$$

3. Action-angle variables

The strategy to construct the action-angle variables consists in a few steps [10, 11]. First of all, we fix a level surface $\mathbf{F} = (H, p_{\phi_1}, p_{\phi_2}, p_\nu, \vec{J}_1^{\,2}) = \mathbf{c}$ of the mutually commuting constants of motion (5)–(7). After that, we introduce the generating function for the canonical transformation from the coordinates $(\mathbf{p}, \mathbf{q})$, where $\mathbf{p}$ are the conjugate momenta (3) to the coordinates $\mathbf{q} = (\theta_1, \theta_2, \phi_1, \phi_2, \nu)$, to the action-angle variables $(\mathbf{J}, \mathbf{w})$.

Taking into account that the Hamiltonian (4) has no explicit time dependence, we can write the Hamilton's principal function

$$S(\mathbf{q}, \mathbf{c}) = W(\mathbf{q}, \mathbf{c}) - Et\,, \tag{8}$$

where the Hamilton's characteristic function is

$$W = \sum_i \int p_i dq_i\,. \tag{9}$$

In the case of the geodesic motions in SE space $T^{1,1}$, the variables in the Hamilton-Jacobi equation are separable and we seek a solution with the characteristic function

$$W(\theta_1, \theta_2, \phi_1, \phi_2, \nu) = W_{\theta_1}(\theta_1) + W_{\theta_2}(\theta_2) + W_{\phi_1}(\phi_1) + W_{\phi_2}(\phi_2) + W_\nu(\nu)\,. \tag{10}$$

For completely separable Hamilton-Jacobi equation, considering only closed orbits, the *action variables* are defined as integrals over complete period in the (q_i, p_i) plane:

$$J_i = \oint p_i dq_i = \oint \frac{\partial W_i(q_i; c)}{\partial q_i} dq_i \qquad \text{(no summation)}\,. \tag{11}$$

J_i's form n independent functions of c_i's and can be taken as a set of new constant momenta.

For the cyclic variables $(\phi_1\,,\, \phi_2\,,\, \nu)$ from (6) and (9) we get straight

$$\begin{aligned} W_{\phi_1} &= p_{\phi_1}\phi_1 = c_{\phi_1}\phi_1\,, \\ W_{\phi_2} &= p_{\phi_2}\phi_2 = c_{\phi_2}\phi_2\,, \\ W_\nu &= p_\nu\nu = c_\nu\nu\,, \end{aligned} \tag{12}$$

and the corresponding action variables (11) are easily calculated

$$\begin{aligned} J_{\phi_1} &= 2\pi c_{\phi_1}\,, \\ J_{\phi_2} &= 2\pi c_{\phi_2}\,, \\ J_\nu &= 2\pi c_\nu\,. \end{aligned} \tag{13}$$

Conjugate *angle variables* w_i are given by

$$w_i = \frac{\partial W}{\partial J_i} = \sum_{j=1}^{n} \frac{\partial W_j(q_j; J_1, \cdots, J_n)}{\partial J_i}\,, \tag{14}$$

having a linear evolution in time

$$w_i = \omega_i t + \beta_i\,, \tag{15}$$

with β_i other constants of integration.

Having determined the characteristic functions corresponding to the cyclic variables, the Hamilton-Jacobi equation becomes

$$\begin{aligned} E = \quad & 3\left[\left(\frac{\partial W_{\theta_1}}{\partial \theta_1}\right)^2 + \frac{1}{4\sin^2\theta_1}\left(2c_{\phi_1} - \cos\theta_1 c_\nu\right)^2\right] \\ & + 3\left[\left(\frac{\partial W_{\theta_2}}{\partial \theta_2}\right)^2 + \frac{1}{4\sin^2\theta_2}\left(2c_{\phi_2} - \cos\theta_2 c_\nu\right)^2\right] + \frac{9}{8}c_\nu^2 \,. \end{aligned} \tag{16}$$

In this equation the dependencies on θ_1 and θ_2 are separated and we can set the quantities enter the two square brackets to be constants denoted by $c_{\theta_1}^2$ and respectively $c_{\theta_2}^2$. Consequently we have (16):

$$\frac{\partial W_{\theta_i}}{\partial \theta_i} = \sqrt{c_{\theta_i}^2 - \frac{(2c_{\phi_i} - \cos\theta_i c_\nu)^2}{4\sin^2\theta_i}} \quad , \quad i = 1, 2 \,, \tag{17}$$

and the corresponding action variables are

$$J_{\theta_i} = \oint d\theta_i \sqrt{c_{\theta_i}^2 - \frac{(2c_{\phi_i} - \cos\theta_i c_\nu)^2}{4\sin^2\theta_i}} \quad , \quad i = 1, 2 \,. \tag{18}$$

An efficient procedure to evaluate this integral is to put $\cos\theta_i = t_i$ and extend t_i to a complex variable z. The integral becomes a closed contour integral in the z-plane. The turning points of the t_i-motions are

$$t_{i\pm} = 2\frac{c_{\phi_i} c_\nu \pm c_{\theta_i}\sqrt{4c_{\theta_i}^2 + c_\nu^2 - 4c_{\phi_i}^2}}{4c_{\theta_i}^2 + c_\nu^2} \quad , \quad i = 1, 2 \,, \tag{19}$$

which are real for

$$4c_{\theta_i}^2 + c_\nu^2 - 4c_{\phi_i}^2 \geq 0 \quad , \quad i = 1, 2 \,, \tag{20}$$

and situated in the interval $(-1, +1)$. We cut the complex z-plane from t_- to t_+ and the closed contour integral is a loop enclosing the cut in a clockwise sence. The contour can be deformed to a large circular contour plus two contour integrals about the poles at $z = \pm 1$.

After the standard evaluation of the residues and the contribution of the large contour integral we finally get [12]

$$J_{\theta_i} = 2\pi\left[\frac{1}{2}\sqrt{4c_{\theta_i}^2 + c_\nu^2} - c_{\phi_i}\right] \quad , \quad i = 1, 2 \,. \tag{21}$$

We observe that the constants of motion $J_{\theta_i}, J_{\phi_i}, J_\nu, E$ are connected by the relation:

$$H = E = \frac{3}{4\pi^2}\left[\left(J_{\theta_1} + J_{\phi_1}\right)^2 + \left(J_{\theta_2} + J_{\phi_2}\right)^2 - \frac{1}{8}J_\nu^2\right] . \tag{22}$$

This result is characteristic for the complete integrable systems. The number of conserved quantities equals the number of degrees of freedom and the Hamiltonian depends only on the action variables [10, 11].

Moreover, we observe that the energy depends on J_{θ_i} and J_{ϕ_i} in the combination $J_{\theta_i} + J_{\phi_i}$ meaning that the frequencies of motions in θ_i and ϕ_i are identical:

$$\omega_{\theta_i} = \omega_{\phi_i} = \frac{\partial H}{\partial J_{\theta_i}} = \frac{\partial H}{\partial J_{\phi_i}} = \frac{3}{2\pi^2}\left(J_{\theta_i} + J_{\phi_i}\right) \quad , \quad i = 1, 2 \,. \tag{23}$$

These frequencies are *resonant* or *rational dependent*, the system is degenerate, and this fact has important consequences for developing perturbation theory.

Recall that in a geometric picture of an integrable Hamiltonian system, the whole phase space is foliated into an n-parameter family of invariant tori, on which the the flow is linear with constant frequencies. According to the classical Kolmogorov-Arnold-Moser (KAM) theorem [10], when an integrable Hamiltonian is perturbed by a small non-integrable piece, most KAM tori survive but suffer small alterations. In contrast with the customary case, the resonant tori which have rational ratios of frequencies get destroyed and motion on them becomes chaotic. The chaotic behavior of the perturbed Hamiltonian integrable system $T^{1,1}$ was observed in [13, 14] using numerical simulations in the study of certain classical string configurations in $AdS \times T^{1,1}$.

The angle variables (11) are evaluated from equations (13) and (21) [12]:

$$\begin{aligned}
w_{\theta_i} &= \frac{\partial W}{\partial J_{\theta_i}} = \frac{\partial W_{\theta_i}}{\partial J_{\theta_i}} = -\frac{J_{\theta_i}+J_{\phi_i}}{2\pi} I_1(a_i,b_i,c_i;\cos\theta_i) \quad , \quad i=1,2\,, \\
w_{\phi_i} &= \frac{\partial W}{\partial J_{\phi_i}} = \frac{\partial W_{\theta_i}}{\partial J_{\phi_i}} + \frac{\partial W_{\phi_i}}{\partial J_{\phi_i}} = \frac{\partial W_{\theta_i}}{\partial J_{\phi_i}} + \frac{1}{2\pi}\phi_i \\
&= -\frac{J_{\theta_i}+J_{\phi_i}}{2\pi} I_1(a_i,b_i,c_i;\cos\theta_i) - \frac{2J_{\phi_i}-J_\nu}{8\pi} I_2(a_i+b_i+c_i, b_i+2c_i, c_i;\cos\theta_i-1) \\
&\quad + \frac{2J_{\phi_i}+J_\nu}{8\pi} I_2(a_i-b_i+c_i, b_i-2c_i, c_i;\cos\theta_i+1) + \frac{1}{2\pi}\phi_i \quad , \quad i=1,2\,, \\
w_\nu &= \frac{\partial W}{\partial J_\nu} = \frac{\partial W_{\theta_1}}{\partial J_\nu} + \frac{\partial W_{\theta_2}}{\partial J_\nu} + \frac{\partial W_\nu}{\partial J_\nu} = \frac{\partial W_{\theta_1}}{\partial J_\nu} + \frac{\partial W_{\theta_2}}{\partial J_\nu} + \frac{1}{2\pi}\nu \\
&= \sum_{i=1,2} \frac{2J_{\phi_i}-J_\nu}{16\pi} I_2(a_i+b_i+c_i, b_i+2c_i, c_i;\cos\theta_i-1) \\
&\quad + \sum_{i=1,2} \frac{2J_{\phi_i}+J_\nu}{16\pi} I_2(a_i-b_i+c_i, b_i-2c_i, c_i;\cos\theta_i+1) + \frac{1}{2\pi}\nu\,,
\end{aligned} \tag{24}$$

where:

$$\begin{aligned}
a_i &= J_{\theta_i}^2 + 2J_{\theta_i}J_{\phi_i} - \frac{1}{4}J_\nu^2\,, \\
b_i &= J_{\phi_i}J_\nu\,, \\
c_i &= -\left(J_{\theta_i}+J_{\phi_i}\right)^2 \quad , \quad i=1,2\,.
\end{aligned} \tag{25}$$

In the above equations $I_1(a,b,c;t)$ and $I_2(a,b,c;t)$ are the integrals [15]

$$\begin{aligned}
I_1(a,b,c;t) &= \int \frac{dt}{\sqrt{a+bt+ct^2}} = \frac{-1}{\sqrt{-c}} \arcsin\left(\frac{2ct+b}{\sqrt{-\Delta}}\right), \\
I_2(a,b,c;t) &= \int \frac{dt}{t\sqrt{a+bt+ct^2}} = \frac{1}{\sqrt{-a}} \arctan\left(\frac{2a+bt}{2\sqrt{-a}\sqrt{a+bt+ct^2}}\right),
\end{aligned} \tag{26}$$

evaluated for $c<0$, $\Delta = 4ac-b^2 < 0$ and for $a<0$ respectively. That is the case of the parameters (25) taking for granted (20).

Let us note that the degeneracy of the pairs of frequencies $(\omega_{\theta_i}, \omega_{\phi_i})$ may be removed by a canonical transformation from the variables

$$\{(w_{\theta_1}, J_{\theta_1}), (w_{\theta_2}, J_{\theta_2}), (w_{\phi_1}, J_{\phi_1}), (w_{\phi_2}, J_{\phi_2}), (w_\nu, J_\nu)\}$$

to new action-angle variables

$$\{(\Theta_1, J_{\Theta_1}), (\Theta_2, J_{\Theta_2}), (\Phi_1, J_{\Phi_1}), (\Phi_2, J_{\Phi_2}), (\Xi, J_\Xi)\}\,.$$

For this purpose we introduce the following generating function

$$F(w_{\theta_1}, w_{\phi_1}, w_{\theta_2}, w_{\phi_2}, w_\nu; J_{\Theta_1}, J_{\Phi_1}, J_{\Theta_2}, J_{\Phi_2}, J_\Xi) = \sum_{i=1,2} (w_{\phi_i} - w_{\theta_i}) J_{\Phi_i} + \sum_{i=1,2} w_{\theta_i} J_{\Theta_i} + w_\nu J_\Xi\,, \quad (27)$$

which implies

$$\begin{aligned} \Phi_i &= \frac{\partial F}{\partial J_{\Phi_i}} = w_{\phi_i} - w_{\theta_i} \quad , \quad i = 1, 2\,, \\ \Theta_i &= \frac{\partial F}{\partial J_{\Theta_i}} = w_{\theta_i} \quad , \quad i = 1, 2\,, \\ \Xi &= \frac{\partial F}{\partial J_\Xi} = w_\nu\,, \end{aligned} \quad (28)$$

and

$$\begin{aligned} J_{\phi_i} &= \frac{\partial F}{\partial w_{\phi_i}} = J_{\Phi_i} \quad , \quad i = 1, 2\,, \\ J_{\theta_i} &= \frac{\partial F}{\partial w_{\theta_i}} = J_{\Theta_i} - J_{\Phi_i} \quad , \quad i = 1, 2\,, \\ J_\nu &= \frac{\partial F}{\partial w_\nu} = J_\Xi\,. \end{aligned} \quad (29)$$

The Hamiltonian (22) becomes

$$H = \frac{3}{4\pi^2}(J_{\Theta_1}^2 + J_{\Theta_2}^2 - \frac{1}{8} J_\Xi^2)\,, \quad (30)$$

with the fundamental frequencies

$$\omega_{\Phi_1} = \omega_{\Phi_2} = 0 \quad , \quad \omega_{\Theta_1} = \frac{3}{2\pi^2} J_{\Theta_1} \quad , \quad \omega_{\Theta_2} = \frac{3}{2\pi^2} J_{\Theta_2} \quad , \quad \omega_\Xi = -\frac{3}{16\pi^2} J_\Xi\,. \quad (31)$$

Therefore, in terms of the new transformed variables, the Hamiltonian (30) appears in a form involving only that action variables for which the corresponding frequencies are different from zero.

4. Conclusions

The action-angle approach to geodesic motions in $T^{1,1}$ space gives us a better understanding of the dynamics. The formulation of the integrable system in these variables represents a useful tool for developing perturbation theory.

We observe that for geodesics in $T^{1,1}$ the frequencies of the motions corresponding to θ_i and ϕ_i coordinates are equal. The resonant tori are in general destroyed by an arbitrary small perturbation to the integrable Hamiltonian giving way to chaotic behavior. A similar result has been obtained using numerical simulations [13] which shows that certain classical configurations in $AdS_5 \times T^{1,1}$ are chaotic.

It would be interesting to extend the action-angle formulations to other five-dimensional SE spaces as well to their higher dimensional generalizations relevant for the predictions of the AdS/CFT correspondence.

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