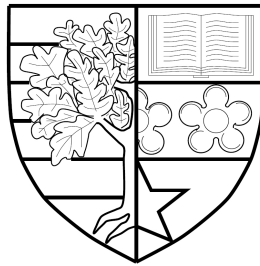


HIGHER GAUGE THEORY WITH STRING 2-GROUPS
AND HIGHER POINCARÉ LEMMA

by

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Abstract

This thesis is concerned with the mathematical formulations of higher gauge theory. Firstly, we develop a complete description of principal 2-bundles with string 2-group model of Schommer-Pries, which is obtained by defining principal smooth 2-group bundles as internal functors in the weak 2-category \mathbf{Bibun} of Lie groupoids, right principal smooth bibundles and bibundle maps. Furthermore, this formalism allows us to construct the known string Lie 2-algebra by differentiating this model of the string 2-group. Generalizing the differentiation process, we provide Maurer-Cartan forms leading us to higher non-abelian Deligne cohomology, encoding the kinematical data of higher gauge theory together with their (finite) gauge symmetries. Secondly, we prove the non-abelian Poincaré lemma in higher gauge theory in two different ways. That is, we show that every flat local connective structure in strict principal 2-bundles is gauge trivial. The first proof is based on the result by Jacobowitz, which explains solvability conditions for equations of differential forms. The second is an extension of a proof by T. Voronov and yields the explicit gauge parameters connecting a flat local connective structure to the trivial one. Finally, we develop a method that shows how higher flatness appears as a necessary integrability condition of a linear system by translating the usual matrix product into categorified settings. Moreover, we comment how this notion can be also generalized to the case of higher principal bundles with connective structures.

Dedicated to my wife Jemi, and sons Abel and Henok.

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Chapter 1

Introduction

Most of the materials in this chapter are taken from the introductions of the papers [29] and [30] with a little expansion and re-arrangements.

Higher gauge theory [3, 6, 70] is an interesting generalization of ordinary gauge theory that describes consistent parallel transport of higher dimensional objects. This requires the introduction of higher form potentials, and the usual no-go theorems [26] concerning non-abelian higher form theories are circumvented by categorifying the underlying mathematical structures in ordinary gauge theory. In particular, parallel transport of extended objects [1, 6, 55], which arises e.g. in string and M-theory, where point particles are replaced by one-, two- and five-dimensional objects demands the higher structures: (weak) Lie n -groups [4, 95] and their corresponding Lie n -algebras [2, 67, 72]. Accordingly, these objects have great significance to illustrate the kinematical data on higher principal bundles. In recent years, different approaches have been used to discuss bundles in different perspectives [1, 6, 8, 70, 92]. For instance, in [34], the authors have described strict principal 2-bundles over Lie groupoids using Lie 2-groupoid generalized morphisms, whereas in [65], the authors have discussed the notion by replacing the base manifold by categorified space.

The most pressing issues in the study of higher principal bundles are identifying the “appropriate” (weak) Lie n -group that can be differentiated to the corresponding Lie n -algebra. Consequently, knowing the integrating Lie n -algebra is important in order to provide a complete description of higher gauge theory on higher principal

bundles. But, this was not an easy task even for the simplest cases of semistrict Lie 2-algebras, [42]. As a result, in order to work with string Lie 2-group bundles with connective structures, either we have to incorporate infinite dimensional spaces [7, 70] or consider a more general 2-group object called smooth 2-groups [71]. There is in fact an equivalence between a smooth 2-group and a weak Lie 2-group [93, 95, 94]. Since smooth 2-groups form the most general notion of 2-group objects, which includes ordinary Lie groups as well as strict Lie 2-groups, the notion of principal 2-bundles with structure smooth 2-groups is very comprehensive, and we think that studying principal smooth 2-group bundles might answer many open problems in theoretical physics and higher geometry, e.g. the lack of solutions to higher gauge equations which are truly non-abelian and the study of their integrating Lie 2-algebra of Morita equivalent smooth 2-groups. Specifically, no higher principal bundle with connection is known that is not gauge equivalent to the trivially embedded abelian gerbe with connection. This is particularly unfortunate because knowing such a solution would lead to immediate progress in higher gauge theory, both on the mathematical and the physical aspects.

Therefore, it is important to consider another sort of generalization of the current formulations of higher gauge theory by studying principal 2-bundles with smooth 2-groups. We focus our attention in particular on the smooth 2-group model of the string groups given by Schommer-Pries [71].

This 2-group model of the string group is interesting for a number of reasons. First, recall that the most relevant examples of non-abelian monopoles on \mathbb{R}^3 and instantons on \mathbb{R}^4 form connections on principal bundles with structure group $\mathrm{SU}(2)$, where this gauge group is intrinsically linked to the spin groups $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ of the isotropy groups $\mathrm{SO}(3)$ and $\mathrm{SO}(4)$ of the underlying spacetimes. Correspondingly, one might expect the higher version of the spin group, the string group, to be relevant in the description of higher monopoles and instantons.

Second, to work on principal 2-bundles with string Lie 2-algebra valued connective structures, we need to know the “appropriate” smooth 2-group (finite dimen-

sional) which differentiates to the string Lie 2-algebra. But, since $H_{\text{smooth}}^3(\mathbf{G}, \mathbf{A}) = 0$, for a compact Lie group \mathbf{G} and an abelian Lie group \mathbf{A} , cf. [57, 39], it can not be a semistrict Lie 2-group in the sense of [4]. This motivates us to consider a more general 2-group objects, and the aforementioned smooth string 2-group model.

The main goals of this thesis are:

- 1) To identify the “appropriate” categorified group, which can be differentiated to the string Lie 2-algebra and pursue the differentiation process to obtain the string Lie 2-algebra, and to describe the kinematical data of higher gauge theory on principal 2-bundles. That is, we want to illustrate the formalisms of connective structures, and corresponding gauge transformations so that the results can be applied to construct self-dual string solutions.
- 2) In ordinary principal bundles, to show a one-to-one correspondence between the trivial connection and a class of flat connections, we need the Poincaré lemma. This idea can be extended to principal 2-and 3-bundles, in order to develop gauge equivalences between a class of flat local connective structures and the trivial ones. So, in this thesis we present two proofs of the higher Poincaré lemma for strict principal 2-bundles¹.
- 3) The linear system $(d + A)g = 0$, where g is a matrix group valued smooth function and A is a matrix Lie algebra valued differential 1-form implies that $A = dg g^{-1}$, that is A is pure gauge. Moreover, it can only have a solution if the curvature $F := dA + \frac{1}{2}[A, A]$ vanishes. The Frobenius theorem or, equivalently, the Poincaré lemma then states that the condition of flatness is sufficient for the existence of a solution. It is then interesting to see how these statements can be generalized to the categorified case. Do there exist analogous statements in principal 2-and 3-bundles and in what sense?

¹By a strict principal 2-bundle, we mean a principal 2-bundle whose structure 2-group is a strict Lie 2-group.

1.1 Main results

Broadly speaking, this thesis contains three major results. We present them as follows.

1.1.1 Principal \mathcal{S}_λ^w -bundles and differentiation

Currently, there is no systematic treatment of constructing the string Lie 2-algebra by differentiating a smooth 2-group. The result in Theorem 5.2.5 gives the full description of the string Lie 2-algebra obtained by differentiating the weak Lie 2-group \mathcal{S}_λ^w . Furthermore, this construction enables us to present equivalent gauge transformations

$$\beta \otimes \omega' = \omega \otimes \beta + d_K \beta , \quad (1.1a)$$

$$\psi' = \psi - d_K \zeta - \lambda^{0,3}(\beta, \omega', \omega') + \lambda^{0,3}(\omega, \beta, \omega') - \lambda^{0,3}(\omega, \omega, \beta) . \quad (1.1b)$$

This in turn is important when developing the complete features of higher gauge theory on principal \mathcal{S}_λ^w -bundles as shown in (5.49) and (5.51).

Returning to the topic of differentiating the weak Lie 2-group \mathcal{S}_λ^w , the main technicality is the argument to obtain the linearized non-trivial 3-cocycle $\lambda^{0,3} \in H^3(\mathfrak{g}, \mathbb{R}) \cong H_{dR}^3(\mathbf{G})$, which is crucial to obtain the non-trivial μ_3 in Theorem 5.2.5. To be precise, the linearized 3-cocycle $\lambda^{0,3}$ must not be cohomologous to zero in order to obtain a non-trivial μ_3 . We justify this by taking into account of the integration procedure developed in [32]. Hence, Theorem 5.2.5 provides another method of constructing the string Lie 2-algebra.

1.1.2 Higher Poincaré lemma

The difficulty in proving the higher Poincaré lemma, “flat local connective structures are gauge equivalent to the trivial connective structure” is that the Frobenius theorem can not be readily extended. That is, it does not seem possible to give a definition of involutive higher distributions using the notion of smooth manifolds. See Appendix E for a proposed hypothesis on this aspect, but further research is needed to give a full picture of involutive higher distributions. This might be clear

by using the notion of NQ -manifolds. Fortunately, for the purpose of the completion of Subsection 6.1.3, we apply a generalization of the reformulation of Frobenius theorem as an equation of differential forms by Jacobowitz [40]. This result is sufficient to establish the first proof of higher Poincaré lemma in Theorem 6.1.3.

The second proof generalizes a proof by T.Voronov [90]. Here, the explicit gauge parameters relating a flat connection to the trivial one are constructed from the Cauchy problem. We find a nice generalizations of this using the Cauchy problems

$$\begin{aligned} \dot{g} &= -A_t g + g\mathfrak{t}(\Lambda_t) , \\ \dot{\Lambda}_x &= g^{-1} \triangleright B_t + d_x \Lambda_t + (g^{-1} A_x g + g^{-1} d_x g) \triangleright \Lambda_t \end{aligned} \tag{1.2a}$$

with initial conditions

$$g(x, 0) = \mathbb{1}_G \quad \text{and} \quad \Lambda(x, 0) = 0 \quad \text{for} \quad x \in U , \tag{1.2b}$$

which are obtained from the gauge transformations of local connective structures in (6.3).

Both proofs are for strict principal 2-bundles over smooth manifolds, which can be possibly extended to principal 3-bundles is mentioned, cf. Subsection 6.1.6.

1.1.3 Higher integrability

The idea in formulating higher integrability conditions comes from critical observations of the ordinary cases. Section 6.2 presents the categorified analogue of the notion of having a matrix group for a crossed module $\mathbf{H} \xrightarrow{\mathfrak{t}} \mathbf{G}$ of matrix Lie groups is to have an underlying A_∞ -algebra structure. Thus, if the products of this structure extend to the corresponding Lie groups, then one can write down a linear system involving local connective structure (A, B) on principal 2-bundles as

$$\left(\tilde{m}_1 + \tilde{m}_2(A, -) - \tilde{m}_2(B, -) \right) (g + X + Y) = 0 , \tag{1.3}$$

where \tilde{m}_1 and \tilde{m}_2 are products in the 2-term A_∞ -algebra, g is a (matrix group) G -valued smooth function and X and Y are $\text{Lie}(G)$ -and $\text{Lie}(H)$ -valued differential

forms, respectively, cf. Lemma 6.2.5. Therefore, Theorem 6.2.6 shows that the connective structure (A, B) is gauge equivalent to the trivial one and the corresponding curvatures vanish.

1.2 Thesis outline

The thesis is structured as follows. Chapter 2 contains introductory material, the next three chapters (3 to 5) are based on [30] and Chapter 6 is derived from our paper [29]. Below, we give a brief outline of each chapter.

In Chapter 2, we review background material with emphasis on the weak 2-category \mathbf{Bibun} . Moreover, in order to provide a different view on the concepts and terms in the later chapters, we discuss the weak 2-category of Lie groupoids, generalized morphisms and their 2-morphisms between the generalized morphisms \mathbf{Gen} , which is biequivalent to \mathbf{Bibun} .

In Chapter 3, we deal with internal categories in \mathbf{Bibun} , which are important for introducing the basic properties of principal smooth 2-group bundles. Furthermore, the concept of internalization is vital in the generalization of the notions on ordinary principal bundles. For instance, Lie groupoid functors are considered as internal functors in \mathbf{Bibun} , while their natural transformations induce internal transformations. Equivalently, the broad notion of internal categories in \mathbf{Bibun} subsumes Lie groupoids, weak Lie 2-groups, smooth 2-groups, and Lie 2-groupoids, cf. Subsection 3.2.2. Consequently, the axioms of internal functors and natural transformations in Subsections 3.2.3 and 3.2.4 are generalizations of smooth 2-group 1- and 2-homomorphism axioms introduced in [71].

In Chapter 4, we begin by considering smooth 2-group bundles as objects of a slice weak 2-category. Then we define principal smooth 2-group bundles as internal functors in \mathbf{Bibun} in which we illustrate the case for ordinary Lie group principal bundles and strict principal 2-bundles. Finally, as a main result of the chapter, we give cocycle and coboundary relations of principal \mathcal{S}_λ^w -bundles, cf. Theorems 4.3.4 and 4.3.5.

In Chapter 5, we start by differentiating the weak Lie 2-group \mathcal{S}_λ^w to the known

string Lie 2-algebra, this procedure is known to be the adjoint of the integration functor in the sense of [32], see also [77]. Knowing the corresponding Lie 2-algebra helps us to construct \mathcal{S}_λ^w -valued Deligne cohomology, which is important to give a complete description of higher gauge theory on principal 2-bundles with structure 2-group \mathcal{S}_λ^w , cf. Section 5.3.

Chapter 6 treats the proofs of higher Poincaré lemma by making use of the Cauchy problems obtained from the gauge transformations for local connective structures on strict principal 2-bundles. Moreover, we present a way of extending the classical integrability condition on principal bundles to the case of strict principal 2-bundles. We also conjecture the construction of corresponding 2-term A_∞ -algebra from a 2-term L_∞ -algebra obtained from the differential crossed module of the structure Lie 2-group.

Finally, in Chapter 7, we summarize the key results and sketch an example showing how we may possibly apply our constructions in order to feature solutions of self-dual strings, though a more general solution needs future study.

1.3 Preliminaries on weak 2-categories

Here, with the assumption that the reader has a prerequisite knowledge of category theory, we present the following review of weak 2-categories as first given in [10]. We will use these terminologies in this thesis.

A weak 2-category [10] \mathcal{C} consists of:

- a class of objects a, b, \dots ,
- for any two objects $a, b \in \mathcal{C}$, there is a hom-category $\mathcal{C}(a, b)$ whose objects are all 1-morphisms of the form $f, g : a \rightarrow b$, while morphisms $\xi : f \Rightarrow g$ are 2-morphisms in \mathcal{C} , together with the multiplication \circ (also called vertical multiplication of 2-morphisms in \mathcal{C}), and
- for any objects $a, b, c \in \mathcal{C}$, and terminal category² $\mathcal{C}(*, *)$, there are functors³

$$\mathbf{u}_a : * \rightarrow \mathcal{C}(a, a) \text{ and } \otimes_{a,b,c} : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \rightarrow \mathcal{C}(a, c)$$

²Here, $*$ is the terminal object in \mathcal{C} , for simplicity, we also denote $\mathcal{C}(*, *)$ by $*$.

³These are usually called as the unit and horizontal multiplication functors.

such that the two multiplication laws of 2-morphisms called the vertical multiplication \circ , (which is always associative)

$$\left(\begin{array}{ccc} & f & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & b \\ & \curvearrowleft & \\ & g & \end{array} \begin{array}{c} \Downarrow \xi \\ \Downarrow \chi \end{array} \right) = \left(\begin{array}{ccc} & f & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & b \\ & \curvearrowleft & \\ & g & \end{array} \begin{array}{c} \Downarrow \chi \circ \xi \end{array} \right)$$

Figure 1.1: Vertical multiplication $\chi \circ \xi$ of 2-morphisms

and the horizontal multiplication \otimes ,

$$\left(\begin{array}{ccc} & f_1 & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & b \\ & \curvearrowleft & \\ & g_1 & \end{array} \begin{array}{c} \Downarrow \xi \end{array} \right) \left(\begin{array}{ccc} & f_2 & \\ & \curvearrowright & \\ b & \xrightarrow{\quad} & c \\ & \curvearrowleft & \\ & g_2 & \end{array} \begin{array}{c} \Downarrow \chi \end{array} \right) = \left(\begin{array}{ccc} & f_2 \otimes f_1 & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & c \\ & \curvearrowleft & \\ & g_2 \otimes g_1 & \end{array} \begin{array}{c} \Downarrow \chi \otimes \xi \end{array} \right)$$

Figure 1.2: Horizontal multiplication $\chi \otimes \xi$ of 2-morphisms

need to satisfy the interchange law $(\chi_2 \otimes \chi_1) \circ (\xi_2 \otimes \xi_1) = (\chi_2 \circ \xi_2) \otimes (\chi_1 \circ \xi_1)$ as depicted below

$$\begin{aligned} \left(\begin{array}{ccc} & f_1 & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & b \\ & \curvearrowleft & \\ & g_1 & \end{array} \begin{array}{c} \Downarrow \xi_1 \\ \Downarrow \chi_1 \end{array} \right) \left(\begin{array}{ccc} & f_2 & \\ & \curvearrowright & \\ b & \xrightarrow{\quad} & c \\ & \curvearrowleft & \\ & g_2 & \end{array} \begin{array}{c} \Downarrow \xi_2 \\ \Downarrow \chi_2 \end{array} \right) &= \left(\begin{array}{ccc} & f_2 \otimes f_1 & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & c \\ & \curvearrowleft & \\ & g_2 \otimes g_1 & \end{array} \begin{array}{c} \Downarrow \xi_2 \otimes \xi_1 \\ \Downarrow \chi_2 \otimes \chi_1 \end{array} \right) \\ &= \left(\begin{array}{ccc} & f_1 & \\ & \curvearrowright & \\ a & \xrightarrow{\quad} & b \\ & \curvearrowleft & \\ & g_1 & \end{array} \begin{array}{c} \Downarrow \chi_1 \circ \xi_1 \end{array} \right) \left(\begin{array}{ccc} & f_2 & \\ & \curvearrowright & \\ b & \xrightarrow{\quad} & c \\ & \curvearrowleft & \\ & g_2 & \end{array} \begin{array}{c} \Downarrow \chi_2 \circ \xi_2 \end{array} \right) \end{aligned}$$

Figure 1.3: Interchange law for 2-morphisms

Furthermore, since the horizontal multiplication is associative up to natural 2-isomorphisms called the associator a , and the unit laws for multiplication of 1-morphisms hold up to the natural 2-isomorphisms called the right r and left l unitors, we need these natural 2-isomorphisms to satisfy the coherence axioms, cf. [10].

Remark 1.3.1. *If all the natural 2-isomorphisms are trivial, the weak 2-category \mathcal{C} is called a strict 2-category.*

1.4 Conventions and notations

Throughout this thesis all structures are assumed to be smooth and finite dimensional. For example, \mathbf{Man}^∞ denotes the category of finite dimensional smooth manifolds and smooth maps, and whenever necessary the field is either \mathbb{R} or \mathbb{C} , and the open covers are good covers. Moreover, we use the following notations:

- $\mathbf{Set} :=$ the category of sets, and functions between them.
- $\mathbf{Man}^\infty\mathbf{Cat} :=$ the strict 2-category of categories in \mathbf{Man}^∞ , functors and natural transformations,
- $\mathbf{LieGrpd} :=$ the strict 2-category of Lie groupoids, Lie groupoid functors and natural transformations,
- $\mathbf{Bibun} :=$ the weak 2-category of Lie groupoids, right principal bibundles and bibundle maps, and
- $\mathbf{Gen} :=$ the weak 2-category of Lie groupoids, Lie groupoid generalized morphisms and 2-morphisms between generalized morphisms.

Chapter 2

Lie groupoids and bibundles

The main objective of this chapter is to give a review on the weak 2-categories \mathbf{Bibun} and \mathbf{Gen} . We follow closely the discussions in [71], [51] and [12]. See also [58] for bibundles of topological groupoids.

We begin our discussion with the strict 2-category $\mathbf{LieGrpd}$, and then the weak 2-categories \mathbf{Bibun} and \mathbf{Gen} . We will denote the objects of these 2-categories by \mathcal{G} , \mathcal{H} , \mathcal{K} , \mathcal{J} , \dots , but we use three different notations for the 1-morphisms in \mathbf{Bibun} :

- right principal $(\mathcal{G}, \mathcal{H})$ -bibundle B ,
- right principal bibundles Φ_0 , Φ_1 , Ψ_0 , and Ψ_1 if we want to emphasise the morphism nature, or
- simply a right principal bibundle $B : \mathcal{G} \rightarrow \mathcal{H}$.

Moreover, the collection of all right principal $(\mathcal{G}, \mathcal{H})$ -bibundles is denoted by $\mathbf{Bibun}(\mathcal{G}, \mathcal{H})$, and as usual it constitutes the hom-categories with objects right principal $(\mathcal{G}, \mathcal{H})$ -bibundles and bibundle maps as morphisms. We will not prove here that it forms a category. We refer the reader to [12] for related discussions. See also [11] for general axioms of weak 2-categories.

2.1 Lie groupoids and their actions

This section is devoted to the 2-category $\mathbf{LieGrpd}$ of Lie groupoids, functors and natural transformations between the functors. Moreover, we will give some examples

to motivate the definitions and results. The original material on Lie groupoids is available in [53]. To discuss gauge theories, we have to describe various actions on fields. Such actions are most naturally described using the language of groupoids. We begin by stating the definition of groupoids and Lie groupoids.

2.1.1 Preliminaries on Lie groupoids

A groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is a small category that consists the sets of invertible arrows \mathcal{G}_1 and objects \mathcal{G}_0 together with the five structure maps called the source \mathbf{s} , target \mathbf{t} , identity \mathbf{id} , inverse \mathbf{inv} and multiplication \circ :

$$\begin{array}{c}
 \mathcal{G}_1 \times_{\mathbf{s}, \mathbf{t}} \mathcal{G}_1 \xrightarrow{\circ} \mathcal{G}_1 \begin{array}{c} \xrightarrow{\mathbf{s}} \\ \xrightarrow{\mathbf{t}} \end{array} \mathcal{G}_0 \\
 \uparrow \mathbf{inv} \\
 \mathcal{G}_1 \xrightarrow{\mathbf{id}} \mathcal{G}_0
 \end{array}$$

satisfying all the axioms of category, see [89, 50] for the discussions of basic category theory.

Definition 2.1.1. *A Lie groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is a groupoid internal to \mathbf{Man}^∞ . That is, its set of objects \mathcal{G}_0 and morphisms \mathcal{G}_1 are smooth manifolds and the structure maps \mathbf{s} , \mathbf{t} , \mathbf{id} , \mathbf{inv} and \circ are all smooth. In particular, we need \mathbf{s} and \mathbf{t} to be surjective submersions in order to have the pullback $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ as a smooth manifold, since submersion maps are transversal, which is a sufficient for the existence of pullbacks.*

We give some examples of Lie groupoids, which are important for later discussions. A reader curious about Lie groupoids and their constructions may consult [53]. Our discussion here depends on this material.

Example 2.1.2. *As a first example, for any smooth manifold X , the discrete Lie groupoid $X \rightrightarrows X$ is a Lie groupoid with the trivial structure maps.*

Example 2.1.3. *For any Lie group G , we have a Lie groupoid $G \rightrightarrows *$ with the structure maps $\mathbf{s}(g) = \mathbf{t}(g) = *$, Lie group multiplication \circ , and the trivial map \mathbf{id} .*

Example 2.1.4. *Another important example, which generalizes the previous two examples, is the action groupoid that arises from a smooth right-action of a Lie group \mathbf{G} on a smooth manifold X . The action groupoid $X//\mathbf{G}$ has as its set of objects X and set of morphisms $X \times \mathbf{G}$ with structure maps*

$$\begin{aligned} \mathfrak{s}(x, g) &= x , & \mathfrak{t}(x, g) &= xg , & \mathfrak{id}(x) &= (x, \mathbb{1}_{\mathbf{G}}) , \\ (x, g) \circ (\tilde{x}, \tilde{g}) &= (\tilde{x}, \tilde{g}g) , \end{aligned} \tag{2.1}$$

whenever $\mathfrak{s}(x, g) = x = \tilde{x}\tilde{g}$.

Example 2.1.5. *For any smooth manifold X , we have the pair groupoid $X \times X \rightrightarrows X$; whose set of objects are X and set of morphisms $X \times X$ together with the structure maps*

$$\begin{aligned} \mathfrak{s}(x, y) &= x , & \mathfrak{t}(x, y) &= y , \\ \mathfrak{id}(x) &= (x, x) , & (x, y) \circ (y, z) &= (x, z) , \end{aligned} \tag{2.2}$$

for any $x, y, z \in X$.

Example 2.1.6. *Lastly, for an open covering $U = \bigsqcup U_i \longrightarrow X$ of a smooth manifold X , we have the Čech groupoid $U \times_X U \rightrightarrows U$ with the structure maps*

$$\begin{aligned} \mathfrak{s}(y_1, y_2) &= y_1 , & \mathfrak{t}(y_1, y_2) &= y_2 , & \mathfrak{id}(y_1) &= (y_1, y_1) , \\ (y_1, y_2) \circ (y_2, y_3) &= (y_1, y_3) . \end{aligned} \tag{2.3}$$

We denote it by $\check{C}(U)$, as it is constructed using the open cover U .

Since Lie groupoids are internal categories in \mathbf{Man}^∞ , we recall their internal functors, natural transformations and actions on the objects of \mathbf{Man}^∞ .

2.1.2 The strict 2-category LieGrpd

Analogously to functors between categories, we present internal functors between Lie groupoids. They are important to discuss weak-equivalences and generalized Lie groupoid morphisms in the later sections.

A functor of Lie groupoids $p : \mathcal{G} \longrightarrow \mathcal{H}$ consists of two smooth maps $p_0 : \mathcal{G}_0 \longrightarrow \mathcal{H}_0$ and $p_1 : \mathcal{G}_1 \longrightarrow \mathcal{H}_1$ such that they commute with all the structure

maps of \mathcal{G} and \mathcal{H} . Now to motivate the definition, we give some examples, which are available in [53].

Example 2.1.7. *A functor p between discrete Lie groupoids $X \rightrightarrows X$ and $Z \rightrightarrows Z$ is induced by a smooth map $f : X \rightarrow Z$ of smooth manifolds as $p_0 = p_1 = f$.*

Example 2.1.8. *Let X and Z be smooth manifolds and let \mathbf{G} be a Lie group with right-actions on both X and Z . Then any \mathbf{G} -equivariant map $f : X \rightarrow Z$ induces a functor p with $p_0 = f$ and $p_1 = f \times \text{id}_{\mathbf{G}}$ between the action groupoids $X//\mathbf{G}$ and $Z//\mathbf{G}$. In general, due to the commutativity of the structure maps of the respective groupoids, any Lie groupoid functors between two action groupoids are of this type.*

Now in order to finalize our discussion of the 2-category LieGrpd , we provide here the definition of natural transformations. Similar to Lie groupoid functors, which are internal functors in Man^∞ , the natural transformations are also smooth maps.

Let $p, q : \mathcal{G} \rightarrow \mathcal{H}$ be Lie groupoid functors. A natural transformation from p to q is a smooth map $T : \mathcal{G}_0 \rightarrow \mathcal{H}_1$ such that for all $g \in \mathcal{G}_1$ the following digram is commutative.

$$\begin{array}{ccc}
 p_0(\mathbf{s}(g)) & \xrightarrow{p_1(g)} & p_0(\mathbf{t}(g)) \\
 T(\mathbf{s}(g)) \downarrow & & \downarrow T(\mathbf{t}(g)) \\
 q_0(\mathbf{s}(g)) & \xrightarrow{q_1(g)} & q_0(\mathbf{t}(g))
 \end{array}$$

Figure 2.1: Lie groupoid natural transformation T

Remark 2.1.9. *Here, for any $x \in \mathcal{G}_0$, $T(x)$ is an isomorphism, since it is in \mathcal{G} .*

All together, we have the well known strict 2-category LieGrpd with objects Lie groupoids, 1-morphisms Lie groupoid functors and 2-morphisms the natural transformations. We recall this as follows.

Proposition 2.1.10. *Lie groupoids together with Lie groupoid functors and natural transformations form a 2-category LieGrpd which is a full subcategory of $\text{Man}^\infty \text{Cat}$.*

2.1.3 2-pullbacks in LieGrpd

This section recalls basic definitions of 2-pullbacks of Lie groupoids. Additional materials of the topic is also found in [56]. Here, we begin our discussion by defining 2-pullbacks in LieGrpd.

The 2-pullback of two Lie groupoid functors, $p : \mathcal{G} \rightarrow \mathcal{H}$ and $q : \mathcal{K} \rightarrow \mathcal{H}$, denoted by $\mathcal{G} \times_{\mathcal{H}} \mathcal{K}$ has

- objects of the form (x, h, y) , where $x \in \mathcal{G}_0$, $y \in \mathcal{K}_0$ and $h : p_0(x) \rightarrow q_0(y)$ in \mathcal{H}_1 , and
- morphisms from (x_1, h_1, y_1) to (x_2, h_2, y_2) are pairs (g, k) with $g : x_1 \rightarrow x_2$ and $k : y_1 \rightarrow y_2$ in \mathcal{G}_1 and \mathcal{K}_1 such that the following diagram commutes.

$$\begin{array}{ccc}
 p_0(x_1) & \xrightarrow{p_1(g)} & p_0(x_2) \\
 h_1 \downarrow & & \downarrow h_2 \\
 q_0(y_1) & \xrightarrow{q_1(k)} & q_0(y_2)
 \end{array}$$

Figure 2.2: Commutative diagram on morphisms of a 2-pullback in LieGrpd

Moreover, the groupoid multiplication is defined component-wise:

$$(x_1, h_1, y_1) \xrightarrow{(g_1, k_1)} (x_2, h_2, y_2) \xrightarrow{(g_2, k_2)} (x_3, h_3, y_3) , \quad (2.4)$$

gives a morphism $(g_2 \circ g_1, k_2 \circ k_1)$:

$$g_2 \circ g_1 : x_1 \rightarrow x_3 , \quad \text{and} \quad k_2 \circ k_1 : y_1 \rightarrow y_3 , \quad (2.5)$$

from (x_1, h_1, y_1) to (x_3, h_3, y_3) , and the commutativity of Figure 2.2 trivially follows.

Note that also the inverse of a morphism (g, k) is (g^{-1}, k^{-1}) .

Here, one can easily see that the groupoid obtained from the above construction may not be necessarily a Lie groupoid. In particular, the set of objects may not always be a smooth manifold. Hence, to remedy this, we need either \mathbf{spr}_2 or \mathbf{tpr}_2 in

Figure 2.3 to be a surjective submersion¹. In fact, the set of morphisms $(\mathcal{G} \times_{\mathcal{H}} \mathcal{K})_1 := \mathcal{G}_1 \times_{\mathcal{H}_0, \mathfrak{s}} \mathcal{H}_1 \times_{\mathcal{H}_0, \mathfrak{s}q_1} \mathcal{K}_1$ is always a smooth manifold, since \mathfrak{s} and \mathfrak{t} are surjective submersions.

$$\begin{array}{ccc}
 \mathcal{G}_0 \times_{\mathcal{H}_0, \mathfrak{p}_0} \mathcal{H}_1 \times_{\mathcal{H}_0, \mathfrak{s}p_2} \mathcal{K}_0 & \longrightarrow & \mathcal{K}_0 \\
 \downarrow & & \downarrow q_0 \\
 \mathcal{G}_0 \times_{\mathcal{H}_0, \mathfrak{p}_0} \mathcal{H}_1 & \xrightarrow{\quad \mathfrak{s}p_2 \quad} & \mathcal{H}_0
 \end{array}$$

Figure 2.3: The set of objects of the 2-pullback $\mathcal{G} \times_{\mathcal{H}} \mathcal{K}$

As depicted in Figure 2.3, the set of objects will also be a smooth manifold if either p_0 or q_0 is a surjective submersion. Therefore, one can impose this condition on one of the Lie groupoid functors p or q , in order to make $\mathcal{G} \times_{\mathcal{H}} \mathcal{K}$ a Lie groupoid. See [85] for more general cases.

Remark 2.1.11. *In the above construction, the name 2-pullback implies that the 2-commutativity of the following pullback square. That is, it is commutative up to a natural transformation $T : (\mathcal{G} \times_{\mathcal{H}} \mathcal{K})_0 \longrightarrow \mathcal{H}_1$, defined by $T(g, h, k) = k$.*

$$\begin{array}{ccc}
 \mathcal{G} \times_{\mathcal{H}} \mathcal{K} & \longrightarrow & \mathcal{K} \\
 \downarrow & \nearrow T & \downarrow q \\
 \mathcal{G} & \xrightarrow{\quad p \quad} & \mathcal{H}
 \end{array}$$

Figure 2.4: 2-commutative pullback square of Lie groupoid functors p and q

Typically, T is specified as part of the structure of the 2-pullback, and it is universal among diagrams of the form Figure 2.4. If T is the trivial natural transformation, then the 2-pullback becomes the ordinary pullback.

In general, in a any (weak) 2-category, a 2-pullback is a 2-commuting square, and its universality property determines the uniqueness of the 2-morphism, see [85] for discussions of 2-pullbacks in any 2-category.

¹Here, $\mathfrak{s}p_2$ and $\mathfrak{t}p_2$ are the compositions of the source and the target maps with the obvious projection map onto the second factor.

2.1.4 Lie groupoid actions

Recall that Lie groupoids are groupoid objects in $\mathbf{Man}^\infty\mathbf{Cat}$, which are also internal categories in \mathbf{Man}^∞ . Therefore, it is natural to discuss their actions on the objects of the original category \mathbf{Man}^∞ . See [22] and [66] for general discussions on the actions of internal categories. In particular, the latter deals with actions of internal 2-categories.

Here, we review Lie groupoid actions on smooth manifolds, similar to the usual action of groups on objects of \mathbf{Set}^2 . The actions of Lie groupoids are useful to define bundles and bibundles.

The action Lie groupoid shown in Example 2.1.4 is constructed by considering the right-action of any Lie group on a smooth manifold X . Similarly, here we can generalize this construction by taking the action of any Lie groupoid. Now let us begin our discussion by recalling the definition of the actions of Lie groupoids.

Definition 2.1.12. *A smooth right-action of a Lie groupoid \mathcal{G} on a smooth manifold X consists of smooth maps*

$$\sigma : X \longrightarrow \mathcal{G}_0, \quad \text{and} \quad \cdot : X_{\sigma} \times_{\mathcal{G}_0, t} \mathcal{G}_1 \longrightarrow X, \quad (2.6)$$

satisfying the following three conditions

- (i) $\sigma(x \cdot g) = \mathfrak{s}(g)$, for all $(x, g) \in X_{\sigma} \times_{\mathcal{G}_0, t} \mathcal{G}_1$,
- (ii) $x \cdot \mathbf{id}_{\sigma(x)} = x$,
- (iii) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$.

Similarly, we can define the left-action by using the source map. For the sake of simplification, we write xg instead of $x \cdot g$. The smooth map σ usually called the base/moment map.

Example 2.1.13. *For any Lie groupoid \mathcal{G} , there is always a natural right-action of*

²Set is the usual category of sets and functions.

\mathcal{G} both on the set of objects \mathcal{G}_0 and morphisms \mathcal{G}_1 as described below.

$$\begin{aligned} \sigma &:= \text{id} : \mathcal{G}_0 \longrightarrow \mathcal{G}_0 & \text{and} & & \mathfrak{s} : \mathcal{G}_1 \cong \mathcal{G}_0 \times_{\text{id}, \mathfrak{t}} \mathcal{G}_1 \longrightarrow \mathcal{G}_0 , \\ \sigma &:= \mathfrak{s} : \mathcal{G}_1 \longrightarrow \mathcal{G}_0 & \text{and} & & \circ : \mathcal{G}_1 \times_{\mathfrak{s}, \mathfrak{t}} \mathcal{G}_1 \longrightarrow \mathcal{G}_1 . \end{aligned} \quad (2.7)$$

Based on the action of Lie groupoids, we can construct the semidirect product Lie groupoid, which is a generalization of the action Lie groupoid in Example 2.1.4.

Definition 2.1.14. *For a right-action of a Lie groupoid \mathcal{G} on X , the semidirect product groupoid $X \rtimes \mathcal{G}$ has objects X and morphisms $X \times_{\sigma, \mathfrak{t}, \mathcal{G}_0} \mathcal{G}_1$, together with the source, target and multiplication maps*

$$\mathfrak{s}(x, g) = xg , \quad \mathfrak{t}(x, g) = x , \quad (x, g_1) \circ (y, g_2) = (x, g_1 g_2) , \quad (2.8)$$

provided that $y = xg_1$ and $\mathfrak{s}(g_1) = \mathfrak{t}(g_2)$.

Remark 2.1.15. *In Definition 2.1.14, the pullback $X \times_{\sigma, \mathfrak{t}, \mathcal{G}_0} \mathcal{G}_1$ as depicted below is a smooth manifold, since \mathfrak{t} is a surjective submersion.*

$$\begin{array}{ccc} X \times_{\sigma, \mathfrak{t}, \mathcal{G}_0} \mathcal{G}_1 & \xrightarrow{\quad\quad\quad} & \mathcal{G}_1 \\ \downarrow & & \downarrow \mathfrak{t} \\ X & \xrightarrow{\quad\quad\quad \sigma \quad\quad\quad} & \mathcal{G}_0 \end{array}$$

Figure 2.5: Arrow part of semidirect product groupoid

In the following section, we will discuss a special type of Lie groupoid actions and bibundles within the framework of finite dimensional Lie groupoids, which are important for the construction of the string 2-group model by Schommer-Pries [71] in Chapter 3. We will follow closely [71] and [51]. See also [58] for bibundles of topological groupoids.

2.2 The weak 2-category Bibun

Here, we review the bicategory **Bibun**. Moreover, we shall discuss equivalences and 2-pullbacks in this 2-category. Our main references are [71] and [51].

We begin our discussion by stating the definitions of Lie groupoid bibundles. Next, we will give some examples of right principal bibundles to illustrate the definition. Finally, we state a theorem that characterizes **Bibun**. This theorem might also give us partial insight in proving the biequivalence between **Bibun** and **Gen**, which was conjectured in [27]. We will not give the proof of this conjecture here. But, we leave it for future research.

2.2.1 Right principal-bibundles

Now, let us define bibundles and right principal bibundles as a generalizations of principal bundles. Recall that, for a Lie group G , a principal G -bundle B over a smooth manifold X consists of an action of G and a smooth map. Analogously, in order to define bibundles, we generalize this by considering Lie groupoid actions and by taking X to be any Lie groupoid acting on the left of B . See the references [71], [51] and [12] for discussions on Lie groupoid bibundles.

Definition 2.2.1. *A $(\mathcal{G}, \mathcal{H})$ -bibundle is a smooth manifold B equipped with a left-action of \mathcal{G} and a right-action of \mathcal{H} (as shown below)*

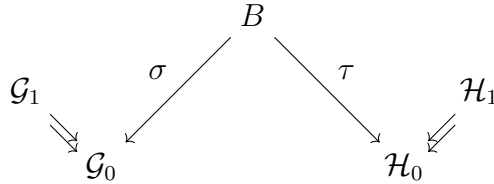


Figure 2.6: $(\mathcal{G}, \mathcal{H})$ -bibundle B

satisfying the following three conditions:

- (i) $\sigma(xh) = \sigma(x)$, for all $(x, h) \in B_{\tau} \times_{\mathcal{H}_0, \mathfrak{t}} \mathcal{H}_1$,
- (ii) $\tau(gx) = \tau(x)$, for all $(g, x) \in \mathcal{G}_1 \times_{\mathcal{G}_0, \mathfrak{s}} B$,
- (iii) $(gx)h = g(xh)$, for all $g \in \mathcal{G}_1$, $h \in \mathcal{H}_1$ and $x \in B$.

In this definition, conditions (i) and (ii) show that σ and τ respectively are \mathcal{H} - and \mathcal{G} -invariant, and (iii) gives the commutativity of the two actions. Furthermore, we follow [12, 51, 71] to provide the definition of right principal bibundles.

Definition 2.2.2. A $(\mathcal{G}, \mathcal{H})$ -bibundle B is right principal if

(i) the base map $\sigma : B \rightarrow \mathcal{G}_0$ is a surjective submersion, and

(ii) the smooth map $B \times_{\tau \times_{\mathcal{H}_0, \mathfrak{t}} \mathcal{H}_1} \rightarrow B \times_{\mathcal{G}_0} B$ defined by $(b, h) \mapsto (b, bh)$ is a diffeomorphism.

Pictorially, a right principal $(\mathcal{G}, \mathcal{H})$ -bibundle is described by

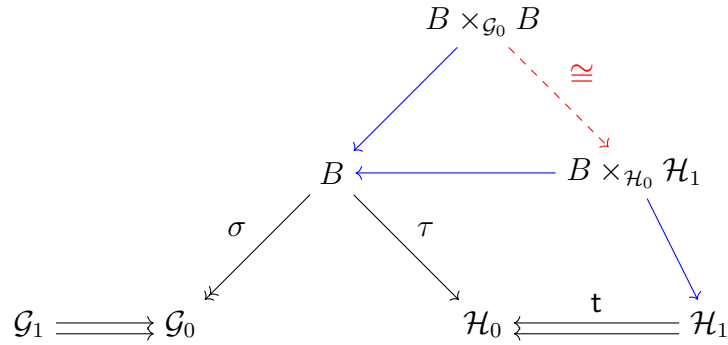


Figure 2.7: Right principal $(\mathcal{G}, \mathcal{H})$ -bibundle B

Here, the double arrow on σ is to denote it is a surjective submersion.

In this definition, condition (ii) implies the action of \mathcal{H} on the fibers of $\sigma : B \rightarrow \mathcal{G}_0$ is free and transitive. The term right principal bibundle generalizes the known terms bispaces, smooth maps, Lie groupoid functors and principal bundles.

2.2.2 Examples of right principal bibundles

Below, we give some examples by considering a particular cases on the Lie groupoids \mathcal{G} and \mathcal{H} , see [51] and [12] for additional examples of Lie groupoid bibundles.

The first source of examples come from Lie groupoid functors. That is, if $p : \mathcal{G} \rightarrow \mathcal{H}$ is a Lie groupoid functor, then one can construct the right principal bibundle \hat{p} corresponding to p , which is called the bundlization of p . We depict \hat{p} as follows

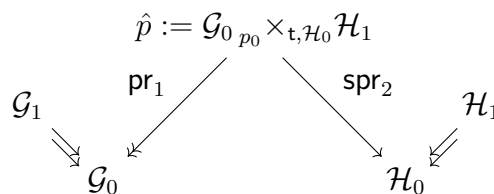


Figure 2.8: A right principal $(\mathcal{G}, \mathcal{H})$ -bibundle as a bundlization of p

where \mathfrak{s} is the source map in \mathcal{H} and \mathfrak{pr}_1 and \mathfrak{pr}_2 are the obvious projections. The left and right-actions of \mathcal{G} and \mathcal{H} on \hat{p} , respectively are given by

$$g(x, h) := (\mathfrak{s}(g), p_1(g)h) \quad \text{and} \quad (x, h_1)h_2 := (x, h_1h_2), \quad (2.9)$$

for $g \in \mathcal{G}_1$, $h, h_{1,2} \in \mathcal{H}_1$ and $(x, h) \in \hat{p}$, with $\mathfrak{s}(g) = x$, and $\mathfrak{t}(h_2) = \mathfrak{s}(h_1)$. Therefore, any Lie groupoids functor gives rise to a corresponding right principal bibundle.

It is clear that \hat{p} is a smooth manifold since \mathfrak{t} is a surjective submersion. Moreover, one can easily check that all the conditions in Definition 2.2.2 are satisfied. For instance, to check right principality of the \mathcal{H} -action, it suffices to verify the two conditions. It is clear that the action is transitive and free. That is for any two elements $(x, h_1), (x, h_2) \in \hat{p}$, we have a unique element $h_1h_2^{-1} \in \mathcal{H}_1$ such that $(x, h_1)(h_1^{-1}h_2) = (x, h_2)$. And as we can easily see it from the definition of \hat{p} , the base map \mathfrak{pr}_1 is also a surjective submersion. Thus, \hat{p} is a right principal $(\mathcal{G}, \mathcal{H})$ -bibundle.

In particular, if $p = \text{id}_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{G}$, then we evidently have the identity right principal bibundle³ from a Lie groupoid \mathcal{G} to itself as depicted below:

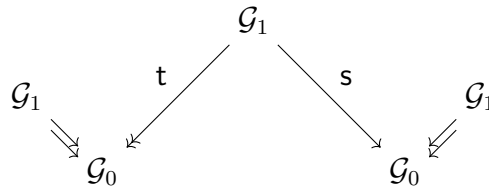


Figure 2.9: The identity principal bibundle 1 as the bundlization of $\text{id}_{\mathcal{G}}$

Consequently, smooth maps between smooth manifolds and Lie group homomorphisms are also included as right principal bibundles obtained from bundlization. Conversely, a right principal bibundle between discrete⁴ Lie groupoids $X \rightrightarrows X$ and $Y \rightrightarrows Y$ reduces to a smooth map $X \rightarrow Y$ by condition (ii) in Definition 2.2.2, which implies that the total space is X . That is, right principal bibundles between discrete Lie groupoids arise from a bundlization of smooth maps between manifolds. Similarly, right principal bibundles between Lie groupoids $\mathcal{G} = (\mathbf{G} \rightrightarrows *)$ and $\mathcal{H} = (\mathbf{H} \rightrightarrows *)$ for Lie groups \mathbf{G} and \mathbf{H} arise from bundlization of Lie groupoid

³We denote the identity right principal bibundle by 1.

⁴By discrete, we shall always mean categorically discrete and not topologically.

functors corresponding to Lie group homomorphisms.

2.2.3 Smooth bibundle maps

In order to define morphisms between right principal bibundles, let us first define equivariant smooth maps between Lie groupoid functors. See also [58] for discussions on equivariant continuous maps between topological groupoid functors.

Definition 2.2.3 ([58]). *Let $p : \mathcal{G} \rightarrow \mathcal{K}$ and $q : \mathcal{H} \rightarrow \mathcal{J}$ be Lie groupoid functors. A smooth map χ between a right principal $(\mathcal{G}, \mathcal{H})$ -bibundle B_1 and a right principal $(\mathcal{K}, \mathcal{J})$ -bibundle B_2*

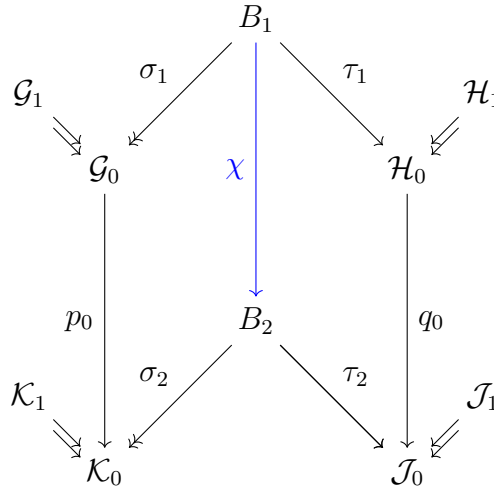


Figure 2.10: (p, q) -equivariant map

is called (p, q) -equivariant if it satisfies

$$p_0 \circ \sigma_1 = \sigma_2 \circ \chi, \quad q_0 \circ \tau_1 = \tau_2 \circ \chi, \quad \text{and} \quad \chi(gh) = p_1(g)\chi(x)q_1(h), \quad (2.10)$$

for all $g \in \mathcal{G}_1$, $x \in B_1$, and $h \in \mathcal{H}_1$ with $\mathfrak{t}(g) = \sigma_1(x)$ and $\tau_1(x) = \mathfrak{s}(h)$.

Remark 2.2.4. *A morphism between two right principal $(\mathcal{G}, \mathcal{H})$ -bibundles is an $(\text{id}_{\mathcal{G}}, \text{id}_{\mathcal{H}})$ -equivariant smooth map χ , hence a diffeomorphism χ satisfying*

$$\sigma_1 = \sigma_2 \circ \chi, \quad \tau_1 = \tau_2 \circ \chi, \quad \text{and} \quad \chi(gh) = g\chi(x)h. \quad (2.11)$$

We now have all the ingredients to discuss the weak 2-category **Bibun** and its

hom-categories. Therefore, we state the following results without proof, cf. [12]. The following results are based on the basic axioms of bicategories in [11].

Lemma 2.2.5. *The collection of right principal $(\mathcal{G}, \mathcal{H})$ -bibundles as objects and bibundle isomorphisms as morphisms together with the usual smooth map composition and the identity map as identity morphism form a category $\mathbf{Bibun}(\mathcal{G}, \mathcal{H})$.*

Thus, $\mathbf{Bibun}(\mathcal{G}, \mathcal{H})$ is a category under the appropriate notions of morphisms, composition and identity morphism. This constitutes the first step in constructing the weak 2-category \mathbf{Bibun} . Moreover, one can give a functor $u_{\mathcal{G}} : * \rightarrow \mathbf{Bibun}(\mathcal{G}, \mathcal{G})$, where $*$ is the terminal category and \mathcal{G} is any object in \mathbf{Bibun} .

The other points we need to address to show that \mathbf{Bibun} is a weak 2-category are the bibundle multiplication, the associator and the unitors, which are defined below. We follow closely the general discussions in [11].

The multiplication on objects can be defined using pullbacks over \mathcal{H}_0 . We give the construction of right principal bibundle multiplication as discussed in [71].

Let B_1 and B_2 be objects in $\mathbf{Bibun}(\mathcal{G}, \mathcal{H})$ and $\mathbf{Bibun}(\mathcal{H}, \mathcal{K})$, respectively. Then the product $B_2 \otimes B_1$ is given by

$$B_2 \otimes B_1 := (B_1 \times_{\mathcal{H}_0} B_2) / \mathcal{H}, \tag{2.12a}$$

which is a smooth manifold, since σ_2 is surjective submersion and the \mathcal{H} -action is free on B_1 . The quotient is defined by the induced right \mathcal{H} -action $(x, y)h = (xh, h^{-1}y)$. Pictorially we have

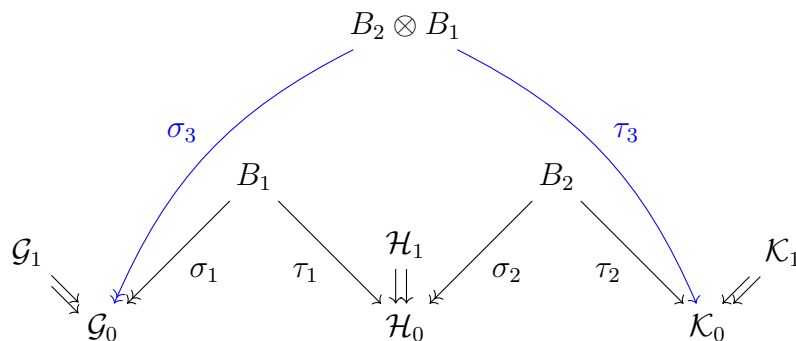


Figure 2.11: Bibundle multiplication

together with the smooth maps

$$\sigma_3[(x, y)] := \sigma_1(x) , \quad \text{and} \quad \tau_3[(x, y)] := \tau_2(y) . \quad (2.12b)$$

Moreover, the left-action of \mathcal{G} and right-action of \mathcal{K} on $B_2 \otimes B_1$ along these base maps are defined by

$$\begin{aligned} \mathcal{G}_1 \times_{\mathcal{G}_0, \sigma_3} (B_2 \otimes B_1) &\longrightarrow B_2 \otimes B_1 , & (g, [(x, y)]) &\longmapsto [(gx, y)] , \\ (B_2 \otimes B_1) \times_{\tau_3, \mathcal{K}_0, \iota} \mathcal{K}_1 &\longrightarrow B_2 \otimes B_1 , & ([x, y], k) &\longmapsto [(x, yk)] . \end{aligned} \quad (2.12c)$$

It is clear that this bibundle is right principal, since σ_3 is a surjective submersion and the action of \mathcal{K} is free and transitive. Thus, using the language of hom-categories, we shall state the following lemmata. See also [11] for general discussions on weak 2-categories.

Lemma 2.2.6 (Bibundle multiplication). *For any objects \mathcal{G}, \mathcal{H} , and \mathcal{K} in \mathbf{Bibun} we can define a bifunctor $\otimes_{\mathcal{G}, \mathcal{H}, \mathcal{K}}: \mathbf{Bibun}(\mathcal{G}, \mathcal{H}) \times \mathbf{Bibun}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathbf{Bibun}(\mathcal{G}, \mathcal{K})$.*

Moreover, the multiplication \otimes is associative up to a right principal bibundle natural isomorphism. That is if B_1, B_2, B_3 are right principal $(\mathcal{G}, \mathcal{H})$ -, $(\mathcal{H}, \mathcal{K})$ - and $(\mathcal{K}, \mathcal{J})$ -bibundles, respectively, then there is a natural bibundle 2-isomorphism:

$$\mathfrak{a}_{B_1, B_2, B_3} : (B_1 \otimes B_2) \otimes B_3 \Longrightarrow B_1 \otimes (B_2 \otimes B_3) . \quad (2.13)$$

This can be restated as follows.

Lemma 2.2.7. *For any objects $\mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{J}$, there is a natural isomorphism (called the associator)*

$$\mathfrak{a}_{\mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{J}} : \otimes_{\mathcal{G}, \mathcal{K}, \mathcal{J}} \circ (\otimes_{\mathcal{G}, \mathcal{H}, \mathcal{K}} \times 1) \Longrightarrow \otimes_{\mathcal{G}, \mathcal{H}, \mathcal{J}} \circ (1 \times \otimes_{\mathcal{H}, \mathcal{K}, \mathcal{J}}) , \quad (2.14)$$

satisfying the Pentagon identity, cf. [11]. The naturality of the associator arises from the universal property of pullbacks.

Furthermore, by taking the identity right principal bibundle 1 as $(\mathcal{G}, \mathcal{G})$ -bibundle for any Lie groupoid \mathcal{G} , we can show that it is both left and right identity for the

bibundle multiplication \otimes up to natural bibundle 2-isomorphisms called right and left unitors l and r , respectively. Therefore, if B is a right principal $(\mathcal{G}, \mathcal{H})$ -bibundle and 1 is the identity right principal $(\mathcal{H}, \mathcal{H})$ -bibundle, then the right principal $(\mathcal{G}, \mathcal{H})$ -bibundle $1 \otimes B$ is isomorphic to B . That is there exists a bibundle isomorphism,

$$r : B \longrightarrow (B \times_{\mathcal{H}_0} \mathcal{H}_1) / \mathcal{H} , \quad x \longmapsto [x, \text{id}_{\tau(x)}] , \quad (2.15)$$

which is an equivariant diffeomorphism⁵. We can also state this as follows by using the hom-categories and the respective functors between them.

Lemma 2.2.8. *For any objects \mathcal{G} and \mathcal{H} in Bibun , one can define a pair of natural isomorphisms*

$$\begin{aligned} l_{\mathcal{G}, \mathcal{H}} : \otimes_{\mathcal{G}, \mathcal{G}, \mathcal{H}} \circ (u_{\mathcal{G}} \times 1) &\Longrightarrow 1 , \\ r_{\mathcal{G}, \mathcal{H}} : \otimes_{\mathcal{G}, \mathcal{H}, \mathcal{H}} \circ (1 \times u_{\mathcal{H}}) &\Longrightarrow 1 \end{aligned} \quad (2.16)$$

called left and right unitors satisfying the Triangle identity, cf. [11].

Finally, using the Lemmata 2.2.5, 2.2.6, 2.2.7, and 2.2.8, we can conclude that Bibun is a weak 2-category. We summarize this by the following proposition.

Proposition 2.2.9 ([58]). *The collection of Lie groupoids as objects, right principal smooth bibundles as 1-morphisms and bibundle isomorphisms as 2-morphisms form the weak 2-category Bibun .*

In the following section, we present the relation between the strict 2-category LieGrpd and the weak 2-category Bibun . Let us first discuss bibundle maps induced by natural transformations.

2.2.4 Bibundle maps induced from natural transformations

We have explained that bundlization takes Lie groupoid functors, which are 1-morphisms in the strict 2-category LieGrpd , into 1-morphisms in the weak 2-category Bibun . In order to extend this to a functor, we have to consider also the natural

⁵Note that, the natural 2-isomorphisms r and l are interchanged here, due to our definition of the multiplication bifunctor.

transformations, which are the 2-morphisms in LieGrpd . The following result gives the correspondence between natural transformations and bibundle isomorphisms.

Proposition 2.2.10 ([52]). *Bibundle maps between bundlizations \hat{p} and \hat{q} are in one-to-one correspondence with natural transformations $T : p \Rightarrow q$.*

The proposition asserts that bibundle isomorphisms between \hat{p} and \hat{q} are induced from the natural transformation between the corresponding Lie groupoid functors p and q . Therefore, combining with bundlization, there is a strict 2-functor $F : \text{LieGrpd} \rightarrow \text{Bibun}$, which is identity on the level of objects, bundlization on the level of 1-morphisms and the induced bibundle maps on the level of 2-morphisms. That is, if $T : p \Rightarrow q$ is a natural transformation between the Lie groupoid functors p and q , then the induced bibundle map χ has the form

$$\begin{array}{ccccc}
 & & \hat{p} = \mathcal{G}_0 \times_{p_0} \times_{\mathcal{H}_0, \mathfrak{t}} \mathcal{H}_1 & & \\
 & \swarrow \text{pr}_1 & \downarrow \chi & \searrow \text{spr}_2 & \\
 \mathcal{G}_1 & & & & \mathcal{H}_1 \\
 \Downarrow & & & & \Downarrow \\
 & \swarrow \text{pr}_1 & & \searrow \text{spr}_2 & \\
 & & \hat{q} = \mathcal{G}_0 \times_{q_0} \times_{\mathcal{H}_0, \mathfrak{t}} \mathcal{H}_1 & & \\
 & \swarrow \text{pr}_1 & & \searrow \text{spr}_2 & \\
 \mathcal{G}_0 & & & & \mathcal{H}_0
 \end{array}$$

Figure 2.12: The bibundle map χ induced from the natural transformation T

$$\chi(x, h) := (x, T(x) \circ h), \text{ where } (x, h) \in \mathcal{G}_0 \times_{p_0} \times_{\mathcal{H}_0, \mathfrak{t}} \mathcal{H}_1.$$

2.2.5 Facts about Bibun

We now state a very nice characterization of right principal bibundles arising from bundlization, which also assures that there are right principal bibundles that can not be obtained from Lie groupoid functors. This shows that the set of right principal bibundles is larger than that of Lie groupoid functors. Of course, this assertion is evident from the existence of weak-equivalences in LieGrpd , cf. Appendix A. The following proposition is based on [58] and [51].

Proposition 2.2.11 ([58]). *For a right principal $(\mathcal{G}, \mathcal{H})$ -bibundle B , the smooth map $\sigma : B \rightarrow \mathcal{G}_0$ admits a smooth section if and only if B is diffeomorphic to a bundlization of some Lie groupoid functor $p : \mathcal{G} \rightarrow \mathcal{H}$.*

Proof. Suppose that a right principal $(\mathcal{G}, \mathcal{H})$ -bibundle B has a smooth section $\tilde{\sigma} : \mathcal{G}_0 \rightarrow B$. Then we have to construct a Lie groupoid functor $p : \mathcal{G} \rightarrow \mathcal{H}$ and a diffeomorphism $\chi : B \rightarrow \hat{p}$.

Thus, the functor p can be defined by $p_0(x) = \tau(\tilde{\sigma}(x))$ on the objects. Moreover, since B is right principal (free and transitive action of \mathcal{H}), for any pair of elements $b_1, b_2 \in B$, there is a unique element $h \in \mathcal{H}_1$ such that $b_2 = b_1 h$. Thus, for any $g \in \mathcal{G}_1$, by considering $\tilde{\sigma}(s(g))$ and $g\tilde{\sigma}(t(g))$ in B , we have a unique element $p_1(g) := h \in \mathcal{H}_1$ such that

$$g\tilde{\sigma}(t(g)) = \tilde{\sigma}(s(g))h . \quad (2.17)$$

This defines p on morphisms and one can easily check that p is a Lie groupoid functor.

Thus, by taking the bundlization of p as

$$\hat{p} = \mathcal{G}_0 \times_{p_0} \times_{\mathcal{H}_0, t} \mathcal{H}_1 , \quad (2.18)$$

we can define the diffeomorphism $\chi : B \rightarrow \hat{p}$ by $\chi(b) := (\sigma(b), h)$, where $b = \tilde{\sigma}(\sigma(b))h$, and its smooth inverse $\chi^{-1}(x, h) := \tilde{\sigma}(x)h$. Here, it is clear that $t(h) = \tau(\tilde{\sigma}(\sigma(b))) = p_0(\sigma(b))$, and χ is an equivariant smooth map, cf. Definition ??.

Conversely, let $\chi : \hat{p} \rightarrow B$ be a diffeomorphism between the two right principal $(\mathcal{G}, \mathcal{H})$ -bibundles. Then we can define a smooth section $\tilde{\sigma} : \mathcal{G}_0 \rightarrow B$ by $\tilde{\sigma}(x) := \chi(x, \text{id}_x)$, which is possible by using the commutativity of the following diagram and the smooth section of \hat{p} .

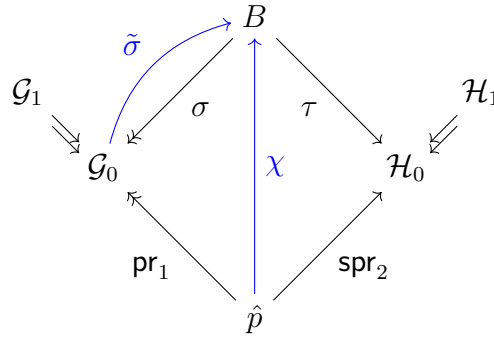


Figure 2.13: Smooth section of a right principal bibundle diffeomorphic to \hat{p}

□

Let us give additional characterizations of the weak 2-category **Bibun**. In particular, we discuss here the 2-pullbacks and equivalences in **Bibun**. We will use these properties to define internal categories in this weak 2-category. See [52] and [51] for discussions of 2-pullbacks and equivalences in **Bibun**.

Definition 2.2.12. *A right principal $(\mathcal{G}, \mathcal{H})$ -bibundle B is an equivalence if there exists a right principal $(\mathcal{H}, \mathcal{G})$ -bibundle B^{-1} (defined by reversing the roles of σ and τ) such that $B \otimes B^{-1} \cong 1$ and $B^{-1} \otimes B \cong 1$. In this case we call B and B^{-1} biprincipal bibundles, since they are both right and left principal. Two Lie groupoids \mathcal{G} and \mathcal{H} are equivalent, if there is such a right principal bibundle equivalence between them.*

Note that the weak 2-category **Bibun** can be regarded as the 2-category of “stacky manifolds”, cf. [12]. In particular, bibundle equivalence amounts to Morita equivalence of Lie groupoids. Moreover, we have the following result, which relates weak-equivalences in **LieGrpd** and equivalences in **Bibun**. Accordingly, this proposition can be counted as the main reason of constructing the weak 2-category **Bibun**.

Proposition 2.2.13 ([58]). *Let $p : \mathcal{G} \rightarrow \mathcal{H}$ be a weak-equivalence in **LieGrpd**. Then the bundlization \hat{p} is a right principal bibundle equivalence. Moreover, every right principal bibundle equivalence $B \in \mathbf{Bibun}(\mathcal{G}, \mathcal{H})$ is of this type.*

This proposition gives that all right principal bibundle equivalences are obtained from weak-equivalences in **LieGrpd**. Therefore, the weak 2-category **Bibun** gives us

more freedom in order to invert Morita equivalences, which is the main advantage of constructing higher categories at the expense of non-trivial natural isomorphisms. We present some examples of equivalences in **Bibun**, we begin by recalling the definition of Lie group crossed modules.

Definition 2.2.14. *A crossed module of Lie groups $(\mathbf{H} \xrightarrow{\mathbf{t}} \mathbf{G}, \triangleright)$ is a pair of Lie groups \mathbf{G} and \mathbf{H} together with a group homomorphism $\mathbf{t} : \mathbf{H} \rightarrow \mathbf{G}$ and an action by automorphism \triangleright of \mathbf{G} on \mathbf{H} . The group homomorphism and the action satisfy the following compatibility conditions for all $g \in \mathbf{G}$ and $h, h_{1,2} \in \mathbf{H}$:*

$$\mathbf{t}(g \triangleright h) = g\mathbf{t}(h)g^{-1} \quad \text{and} \quad \mathbf{t}(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}. \quad (2.19)$$

The first condition guarantees equivariance of \mathbf{t} with respect to the actions of \mathbf{G} on itself by conjugation and on \mathbf{H} by \triangleright , while the second condition tells us the homomorphism \mathbf{t} induces the conjugation action on \mathbf{H} , and it is called the Peiffer identity.

Now as a first example of bibundle equivalence, let us consider the Lie 2-group $\mathcal{G} = (\mathbf{G} \times \mathbf{G} \rightrightarrows \mathbf{G})$, which corresponds to the crossed module $\mathbf{G} \xrightarrow{\mathbf{t}=\text{id}} \mathbf{G}$, cf. Subsection 3.3.3. This Lie 2-group is Morita equivalent to the trivial Lie 2-group $(* \rightrightarrows *)$ and the bibundle equivalence reads as⁶

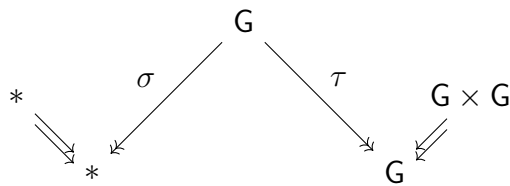
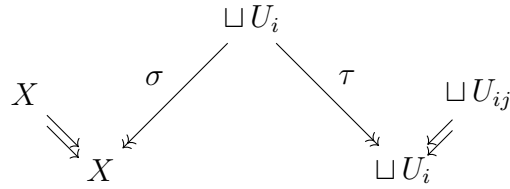


Figure 2.14: The right principal $(*, \mathbf{G} \times \mathbf{G} \rightrightarrows \mathbf{G})$ -bibundle

The second example comes from the Čech groupoid $\check{C}(U) = (\sqcup U_{ij} \rightrightarrows \sqcup U_i)$ of a covering (U_i) of a smooth manifold X , where $U_{ij} := U_i \cap U_j$. This groupoid is equivalent to the discrete groupoid $X \rightrightarrows X$ as depicted below.

⁶The smooth map τ is the identity map on \mathbf{G} , while σ is the trivial map.


 Figure 2.15: Biprincipal $(X, \check{C}(U))$ -bibundle

Remark 2.2.15. *The right principal bibundle equivalence in Figure 2.15 can also be generalized for any Lie groupoid \mathcal{G} by considering the Čech groupoid with respect to a cover $U \rightarrow \mathcal{G}_0$, cf. Theorem 2.2.17.*

Proposition 2.2.16 ([71], [52]). *Let B_1 and B_2 be right principal $(\mathcal{G}, \mathcal{H})$ - and $(\mathcal{K}, \mathcal{H})$ - bibundles together with surjective submersions $B_1 \rightarrow \mathcal{H}_0$ and $B_2 \rightarrow \mathcal{H}_0$. Then the 2-pullback $B_1 \times_{\mathcal{H}} B_2$ in **Bibun** exists.*

The proposition gives us the existence of 2-pullbacks in **Bibun**. The proof of this result is available in [71] and [52]. The 2-pullback of B_1 and B_2 in **Bibun** exists if both right principal bibundles B_1 and B_2 are equivalences.

Moreover, in Proposition 2.2.16, if the Lie groupoid \mathcal{H} is a discrete groupoid, then the 2-pullback in **Bibun** is the same as the 2-pullback in **LieGrpd**. We will use this result in Chapter 3 to define internal categories in **Bibun**.

The following theorem gives an additional characterization of **Bibun**, which is also important to prove biequivalence between the two bicategories **Bibun** and **Gen**. See also [27] and [23] for previous considerations on the correspondence between these 2-categories.

Theorem 2.2.17. *The weak 2-category **Bibun** satisfies the following conditions.*

- (i) *Any right principal bibundle B is isomorphic to $\hat{p} \otimes \hat{q}^{-1}$, where p and q are Lie groupoid functors, in particular q is a weak-equivalence in **LieGrpd**.*
- (ii) *There is a unique bibundle map between any two biprincipal bibundles.*

In the following section, we review the weak 2-category **Gen** of Lie groupoids, generalized morphisms and maps between generalized morphisms.

2.3 Lie groupoid generalized morphisms

We recall some basic definition and properties of the weak 2-category \mathbf{Gen} of Lie groupoids, generalized morphisms and morphisms between them, cf. [27, 23]. Our aim is to use the correspondence between right principal bibundles in \mathbf{Bibun} and Lie groupoid generalized morphisms in \mathbf{Gen} . Constructing an indirect proof (using Theorem 2.2.17) of the biequivalence between these weak 2-categories will not be covered in this thesis, but we shall consider it in future research. In fact, here, we will use the correspondence constructed in [27] and the direct proof given in [23]. Let us first give the definition of Lie groupoid generalized morphisms.

A generalized morphism $\mathfrak{r} : \mathcal{G} \longrightarrow \mathcal{H}$ consists of a Lie groupoid \mathcal{K} , a functor p and a weak-equivalence r . It is denoted by $\mathcal{G} \xleftarrow{r} \mathcal{K} \xrightarrow{p} \mathcal{H}$.

Remark 2.3.1. *Here, r is also a Lie groupoid functor as it is a weak-equivalence between the Lie groupoids \mathcal{G} and \mathcal{K} . And the middle Lie groupoid \mathcal{K} is weakly-equivalent to \mathcal{G} .*

Similar to right principal bibundle maps, we can define morphisms between generalized morphisms, which are a special type of generalized morphisms between the middle Lie groupoids. These will constitute the 2-morphisms in the weak 2-category \mathbf{Gen} , see also [27, 23].

A 2-morphism $\mathfrak{T} : \mathfrak{r}_1 \Longrightarrow \mathfrak{r}_2$ between two generalized morphisms $\mathcal{G} \xleftarrow{r} \mathcal{K} \xrightarrow{p} \mathcal{H}$ and $\mathcal{G} \xleftarrow{r'} \mathcal{K}' \xrightarrow{q} \mathcal{H}$ consists of a Lie groupoid \mathcal{E} and two weak-equivalences s and s' such that the diagram

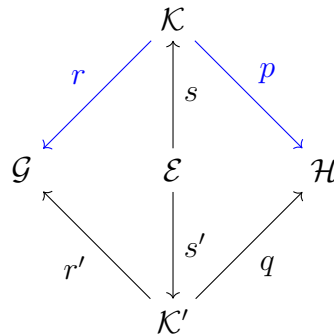


Figure 2.16: Morphism between Lie groupoid generalized morphisms

2-commutes. That is, there are two natural transformations S and T such that

$$S : rs \Longrightarrow r's' , \quad \text{and} \quad T : ps \Longrightarrow qs' . \quad (2.20)$$

Moreover, the multiplication $\mathfrak{s} \otimes \mathfrak{r}$ of two generalized morphisms $\mathfrak{r} : \mathcal{G} \rightarrow \mathcal{H}$ and $\mathfrak{s} : \mathcal{H} \rightarrow \mathcal{J}$ can be defined by taking the 2-pullback⁷ as depicted below:

$$\begin{array}{ccccc}
 \mathcal{G} & \xleftarrow{r} & \mathcal{K} & \xrightarrow{p} & \mathcal{H} \\
 & & \uparrow \text{pr}_1 & & \uparrow r' \\
 & & \mathcal{K} \times_{\mathcal{H}} \mathcal{K}' & \xrightarrow{\text{pr}_2} & \mathcal{K}' \\
 & & \downarrow u & & \downarrow q \\
 & & & & \mathcal{J}
 \end{array}$$

Figure 2.17: Multiplication $\mathfrak{s} \otimes \mathfrak{r}$ of Lie groupoid generalized morphisms

By considering the functors $s = rpr_1$ and $u = qpr_2$ between the respective Lie groupoids, and as a composition of weak-equivalences is also a weak-equivalence, the multiplication $\mathfrak{s} \otimes \mathfrak{r}$ is a generalized morphism from \mathcal{G} to \mathcal{J} .

The above multiplication is also associative up to a natural isomorphism obtained from the universality of 2-pullbacks called the associator. Moreover, for any Lie groupoid \mathcal{G} , the identity (up to a natural 2-morphism) generalized morphism id is given by using the identity Lie groupoid functor on \mathcal{G} . That is $\mathcal{G} \xleftarrow{\text{id}} \mathcal{G} \xrightarrow{\text{id}} \mathcal{G}$. Thus, we state the following proposition.

Proposition 2.3.2 ([27]). *The collection of Lie groupoids as objects, Lie groupoid generalized morphisms as 1-morphisms and morphisms between generalized morphisms as 2-morphisms form a weak 2-category Gen .*

In the following section, we will review biequivalence between the two weak 2-categories Bibun and Gen , which is useful to define the multiplication in \mathcal{S}_λ , cf.

⁷Here, since r' is a weak-equivalence, then the 2-pullback $\mathcal{K} \times_{\mathcal{H}} \mathcal{K}'$ is a Lie groupoid, cf. Subsection 2.1.3.

Chapter 3 using these formalisms. See [27] for the original treatment of biequivalences in these weak 2-categories and [23] for the more general case.

2.4 Comments on biequivalence between Bibun and Gen

In [27], the author has conjectured a biequivalence between the two weak 2-categories **Bibun** and **Gen**. The author has also shown the correspondence between right principal bibundles and Lie groupoid generalized morphisms. For the purpose of this thesis this result is enough.

A biequivalence between **Bibun** and **Gen** is a functor

$$F : \mathbf{Bibun} \longrightarrow \mathbf{Gen} \tag{2.21}$$

such that F is surjective up to equivalence on objects in **Bibun** and locally an equivalence. That is for any Lie groupoids \mathcal{G} and \mathcal{H} the respective hom-categories $\mathbf{Bibun}(\mathcal{G}, \mathcal{H})$ and $\mathbf{Gen}(\mathcal{G}, \mathcal{H})$ are equivalent.

Proposition 2.4.1. *([27]) There is a 1-1 correspondence between Lie groupoid generalized morphisms from \mathcal{G} to \mathcal{H} and right principal $(\mathcal{G}, \mathcal{H})$ -bibundles.*

Therefore, we will use the notion of Lie groupoid generalized morphisms in order to define principal smooth 2-group bundles as generalized internal functors in Chapter 4. Moreover, in Chapter 3, one can use this formalism to discuss the structure maps of the Schommer-Pries smooth 2-group model \mathcal{S}_λ .

Chapter 3

Categories in **Bibun** and the Schommer-Pries model of the string group

The material in this chapter is also available in our paper [30].

In this chapter we will apply the notion of internalization in order to explore internal categories in **Bibun**. Furthermore, it is essential to study smooth 2-groups in general and the Schommer-Pries model of the string group in particular. Categorical groups constitute the basic ingredients of higher principal bundles. Here, we shall discuss smooth 2-groups as internal categories in **Bibun**. We will use the notion of smooth 2-groups in order to define smooth 2-group bundles as internal functors in Chapter 4. Moreover, we will give a self-contained review of the string group model by Schommer-Pries. Now, we begin the chapter by recalling differentiable hypercohomology of Lie groups as first treated in [75].

3.1 Simplicial manifolds and differentiable hypercohomology

This section briefly reviews Segal-Mitchison group cohomology [75]. Recall that given a simplicial set $S_\bullet = \bigcup_{m=0}^{\infty} S_m$, we have face and degeneracy maps

$f_i^m : S_m \rightarrow S_{m-1}$ and $d_i^m : S_m \rightarrow S_{m+1}$, $0 \leq i \leq m$, which satisfy the suitable simplicial identities. The former can be combined into a boundary operator $\partial : S_m \rightarrow S_{m-1}$ via $\partial = \sum_{j=0}^m (-1)^j f_j^m$. This boundary operator induces a coboundary operator on functions f on S_\bullet via $(\delta f)(s) := f(\partial s)$ for $s \in S_\bullet$.

3.1.1 Simplicial manifolds and simplicial covers

Now we are interested in simplicial objects in \mathbf{Man}^∞ and hence simplicial maps between them are the collection of morphisms in \mathbf{Man}^∞ . See [86] for discussions of simplicial objects in any category \mathcal{C} . Here, we follow the sequential definition of simplicial objects, which can be generalized by employing the nerve construction for small categories. Thus, roughly speaking, a simplicial manifold X_\bullet is a sequence of objects in \mathbf{Man}^∞ together with the usual face and degeneracy maps between them.

In many constructions of category theory, and in particular in higher category theory, it is more convenient to use the nerve of a category than the actual category itself. We use this to recall differentiable hypercohomology of compact Lie groups. First, we review two standard examples of the nerve construction and its applications in computing the geometric data of the corresponding simplicial object.

The first example is for a smooth manifold X together with a good cover $V_1 = \sqcup_i (V_i)$, one can define a simplicial manifold by taking the *nerve of the Čech groupoid*, which is the fibered product¹

$$V_\bullet = \bigcup_{m=0}^{\infty} V_1^{[m+1]} = \bigcup_{m=1}^{\infty} \underbrace{V_1 \times_X V_1 \times_X \dots \times_X V_1}_{(m)\text{-times}}. \quad (3.1)$$

Maps from $V_1^{[m]}$ into a sheaf \mathcal{S} over X are called Čech $(m-1)$ -cochains. These cochains together with the corresponding simplicial coboundary operator $\delta_{\check{C}}$ form a complex. Thus, the Čech cohomology with values in the sheaf \mathcal{S} is simply the cohomology of that complex.

The second example is obtained by considering the simplicial manifold $\mathbf{G}_\bullet = \bigcup_{m=0}^{\infty} \mathbf{G}^{\times m}$, for any compact Lie group \mathbf{G} considered as a Lie groupoid ($\mathbf{G} \rightrightarrows *$),

¹If $\pi_i : V_i \rightarrow X$ define the cover, then the fibered product is defined as $V_i \times_X V_j := \{(i, j, x) | \pi_i(x) = \pi_j(x)\}$.

cf. Example 2.1.3 together with the face and degeneracy maps²

$$\begin{aligned} \mathbf{f}_i^m(g_0, \dots, g_m) &= \begin{cases} \mathbf{f}_i^m(g_1, \dots, g_m) & \text{if } i = m, \\ \mathbf{f}_i^m(g_0, \dots, g_{m-1}) & \text{if } i = 0, \\ (g_0, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_m) & \text{else,} \end{cases} \\ \mathbf{d}_i^m(g_0, \dots, g_m) &= (g_0, \dots, g_{i-1}, g_i, g_i, \dots, g_m), \end{aligned} \quad (3.2)$$

we give the double complex, see [35], which discusses the Čech cohomology of the Lie groupoid $(\mathbf{G} \rightrightarrows *)$.

We now present the definition of a simplicial cover, which is also important to review differentiable hypercohomology of a compact Lie group \mathbf{G} . See [20] and [91] for relevant explanations of simplicial good covers of a compact Lie group \mathbf{G} and their constructions.

Definition 3.1.1 ([20]). *A simplicial cover (V_\bullet, I_\bullet) of a simplicial manifold X_\bullet is a simplicial set I_\bullet together with a simplicial manifold V_\bullet , which is a good covering of X_\bullet . That is, for each $j \in I_m$, we have*

$$\mathbf{f}_i^m(V_{m,j}) \subset V_{m-1, \mathbf{f}_i^m(j)} \quad \text{and} \quad \mathbf{d}_i^m(V_{m,j}) \subset V_{m+1, \mathbf{d}_i^m(j)}, \quad (3.3)$$

where $0 \leq i \leq m$ and $0 \leq i \leq m+1$, respectively.

Now, we briefly show how one can construct a simplicial cover of $\mathbf{G} = \mathbf{Spin}(n)$, for $n \geq 3$. Here, we consider the simplicial manifold $\mathbf{G}_\bullet = \bigcup_{m=0}^{\infty} \mathbf{G}^{\times m}$, which is the nerve of the Lie groupoid $(\mathbf{G} \rightrightarrows *)$. Thus, following [20], we present the covering for the case $n = 3$, and this can be readily generalized to arbitrary n .

Example 3.1.2. *An element $g \in \mathbf{Spin}(3) \cong \mathbf{SU}(2)$ is parameterized by a real vector (x^1, x^2, x^3, x^4) of length 1 as follows:*

$$g = \begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ -x^3 + ix^4 & x^1 - ix^2 \end{pmatrix}. \quad (3.4)$$

²Note that our symbols for these maps differ from another widespread choice, we have used here \mathbf{d}_i for degeneracy and \mathbf{f}_i for face maps.

A convenient cover of $\mathrm{SU}(2)$ is given by $V_1 = V_1^{[1]} = \sqcup_{i \in I_1} V_{1,i}$ with $I_1 = \{1, \dots, 8\}$ and

$$\begin{aligned} V_{1,1} &= \{g \in \mathrm{SU}(2) \mid x^1 \geq 0\}, & V_{1,2} &= \{g \in \mathrm{SU}(2) \mid x^1 < 0\}, \\ V_{1,3} &= \{g \in \mathrm{SU}(2) \mid x^2 \geq 0\}, & V_{1,4} &= \{g \in \mathrm{SU}(2) \mid x^2 < 0\}, \\ V_{1,5} &= \{g \in \mathrm{SU}(2) \mid x^3 \geq 0\}, & V_{1,6} &= \{g \in \mathrm{SU}(2) \mid x^3 < 0\}, \\ V_{1,7} &= \{g \in \mathrm{SU}(2) \mid x^4 \geq 0\}, & V_{1,8} &= \{g \in \mathrm{SU}(2) \mid x^4 < 0\}. \end{aligned} \quad (3.5)$$

The index set I_1 is now trivially extended to a simplicial set I_\bullet by using multiindices:

$$I_2 = \{(i_1, i_2, i_3) \mid i_{1,2,3} \in I_1\}, \quad I_3 = \{(j_1, j_2, j_3, j_4) \mid j_{1,2,3,4} \in I_2\}, \quad \text{etc.} \quad (3.6)$$

The actions of the face \mathbf{f}_i^m and degeneracy maps \mathbf{d}_i^m obviously satisfy the conditions in Definition 3.1.1. Note that the I_m 's are finite and carry a total order induced by the lexicographic ordering of indices.

The simplicial cover V_\bullet is then obtained from the preimages of the face maps of the nerve of $\mathrm{BSU}(2)$:

$$\begin{aligned} V_{2,(i_1,i_2,i_3)} &:= (\mathbf{f}_0^2)^{-1}(V_{1,i_1}) \cap (\mathbf{f}_1^2)^{-1}(V_{1,i_2}) \cap (\mathbf{f}_2^2)^{-1}(V_{1,i_3}), \\ V_{3,(j_1,j_2,j_3,j_4)} &:= (\mathbf{f}_0^3)^{-1}(V_{2,j_1}) \cap (\mathbf{f}_1^3)^{-1}(V_{2,j_2}) \cap (\mathbf{f}_2^3)^{-1}(V_{2,j_3}) \cap (\mathbf{f}_3^3)^{-1}(V_{2,j_4}), \end{aligned} \quad (3.7)$$

etc, where $j_{1,2,3} \in I_1$ and $i_{1,2,3,4} \in I_2$.

The lexicographic ordering of indices allows us to introduce a section ϕ_1 of $\pi : V_1 \rightarrow \mathrm{Spin}(3)$. In particular, $\phi_1(g)$ is the element $v_i \in V_{1,i}$ with $\pi(v_i) = g$ and i is the smallest index. This, in turn, leads us to construct the weak Lie 2-group \mathcal{S}_λ^w . cf. Subsection 3.4.3.

Remark 3.1.3. An index $j_1 \in I_2$ is identified with the triple indices (i_1, i_2, i_3) in $I_1 \times I_1 \times I_1$.

We now recall, the differentiable hypercohomology as a Čech cohomology of the Lie groupoid $(\mathbf{G} \rightrightarrows *)$. For cohomology of strict Lie 2-groups, please refer [35].

3.1.2 Differentiable hypercohomology

To combine this simplicial complex with that arising from the Čech groupoid, we need to consider a simplicial cover of \mathbf{G}_\bullet as stated in Definition 3.1.1. Our definition will come with somewhat more structure than that of [71], cf. [20]. We also denote the differential arising as a coboundary map of this simplicial complex by δ_N , which is defined by in terms of the face maps.

Now given an abelian Lie group \mathbf{A} , we can consider the hypercohomology of smooth \mathbf{A} -valued Čech cochains on \mathbf{G}_\bullet , where the differentials are induced by the two simplicial structures. We have the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & & & \\
 & & \delta_N \uparrow & & & & \\
 & & \mathcal{C}^\infty(V_3^{[1]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \cdots & & \\
 & & \delta_N \uparrow & & \delta_N \uparrow & & \\
 & & \mathcal{C}^\infty(V_2^{[1]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \mathcal{C}^\infty(V_2^{[2]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \cdots \\
 & & \delta_N \uparrow & & \delta_N \uparrow & & \delta_N \uparrow \\
 & & \mathcal{C}^\infty(V_1^{[1]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \mathcal{C}^\infty(V_1^{[2]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \mathcal{C}^\infty(V_1^{[3]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \cdots \\
 & & \delta_N \uparrow & & \delta_N \uparrow & & \delta_N \uparrow & & \delta_N \uparrow \\
 & & \mathcal{C}^\infty(V_0^{[1]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \mathcal{C}^\infty(V_0^{[2]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \mathcal{C}^\infty(V_0^{[3]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} & \mathcal{C}^\infty(V_0^{[4]}, \mathbf{A}) & \xrightarrow{\delta_{\check{C}}} \cdots
 \end{array}$$

Figure 3.1: The double complex in computing $H_{\text{SM}}^3(\mathbf{G}, \mathbf{A})$

where each V_\bullet is a simplicial good covering of \mathbf{G}_\bullet , and

$$V_n^{[m]} = \underbrace{V_n \times_{G \times n} V_n \times_{G \times n} \cdots \times_{G \times n} V_n}_{m\text{-times}}, \quad n \geq 0, \quad m \geq 1. \quad (3.8)$$

Moreover, the differential $\delta_{\check{C}}$ is the Čech differential while

$$\delta_N : \mathcal{C}^\infty(V_n^{[m]}, \mathbf{A}) \longrightarrow \mathcal{C}^\infty(V_{n+1}^{[m]}, \mathbf{A}) \quad (3.9a)$$

is defined by the sum of pullbacks of the face maps as:

$$\delta_N := \sum_{i=0}^n (-1)^i (f_i^n)^* , \quad n \geq 0 . \quad (3.9b)$$

Note that V_0 covers the point space $*$ and therefore the bottom line of the diagram above is trivial. For simplicity, we shall label the (m, n) -cochains by $\mathcal{C}^{m,n}(\mathbf{A}) := \mathcal{C}^\infty(V_n^{[m+1]}, \mathbf{A})$ in the following. The Segal-Mitchison cohomology groups [75] are now the total cohomology of this double complex, which is independent of the the choice of the simplicial cover, cf. [20, 75]. The underlying differential is

$$\delta_{\text{SM}} = \delta_{\check{C}} + (-1)^m \delta_N : \bigsqcup_{m=0}^k \mathcal{C}^\infty(V_{k-m}^{[m+1]}, \mathbf{A}) = \bigsqcup_{m=0}^k \mathcal{C}^{m, k-m}(\mathbf{A}) \rightarrow \bigsqcup_{m=0}^{k+1} \mathcal{C}^{m, k+1-m}(\mathbf{A}) , \quad (3.10)$$

where m is the Čech degree of the cochain that δ_{SM} acts on.

As an example, consider a representative λ of an element in $H_{\text{SM}}^3(\mathbf{G}, \mathbf{A})$. Such an element encodes a model for the string group as shown in Section 3.4.2. It is given by four smooth maps³

$$\lambda = (\lambda^{3,0} = 0, \lambda^{2,1}, \lambda^{1,2}, \lambda^{0,3}) , \quad (3.11a)$$

where the cocycle condition $\delta_{\text{SM}}\lambda = 0$ reads as

$$0 = \delta_{\check{C}}\lambda^{2,1} , \quad \delta_N\lambda^{2,1} = \delta_{\check{C}}\lambda^{1,2} , \quad \delta_N\lambda^{1,2} = \delta_{\check{C}}\lambda^{0,3} , \quad \delta_N\lambda^{0,3} = 0 . \quad (3.11b)$$

Evidently, the map $\lambda^{2,1}$ defines an element in Čech cohomology $\check{H}^2(\mathbf{G}, \mathbf{A})$ and therefore, encodes an abelian gerbe [38, 59, 92] over \mathbf{G} . Moreover, we shall always work with a normalized 3-cocycle $\lambda = (\lambda^{2,1}, \lambda^{1,2}, \lambda^{0,3})$. Hence, we give the following remark.

Remark 3.1.4. *A 3-cocycle λ is said to be normalized if all the components*

$$\lambda^{2,1} : V_1^{[3]} \rightarrow \mathbf{A} , \quad \lambda^{1,2} : V_2^{[2]} \rightarrow \mathbf{A} \quad \text{and} \quad \lambda^{0,3} : V_3^{[1]} \rightarrow \mathbf{A} . \quad (3.12)$$

³For comparison, λ_1 , λ_2 and λ_3 , δ_h and δ_v in [71] correspond to $\lambda^{2,1}$, $\lambda^{1,2}$, $\lambda^{0,3}$, $\delta_{\check{C}}$ and δ_N , respectively.

are so. That is, if they satisfy the conditions

$$\lambda^{2,1}(v_i, v_i, v_j) = \lambda^{2,1}(v_i, v_j, v_j) = 0, \quad \lambda^{1,2}(y_i, y_i) = 0, \quad \text{and} \quad \lambda^{0,3}(z_a) = 0, \quad (3.13)$$

whenever $\pi(\mathbf{f}_0^2(\mathbf{f}_0^3(z_a)))$, or $\pi(\mathbf{f}_2^2(\mathbf{f}_2^3(z_a)))$, or $\pi(\mathbf{f}_0^2(\mathbf{f}_3^3(z_a)))$ ($= \mathbb{1}_G$).

Note that the construction of normalized cocycles is always possible by refining the simplicial covering, cf. [93].

3.2 Internalization in Bibun

Recall that internalization can also be considered as the method of categorification within a 2-category. In this section, we deploy this method in order to explain internal categories in Bibun. Usually, a category $\mathcal{D} = (\mathcal{D}_0, \mathcal{D}_1)$ internal to another category \mathcal{C} is a pair $(\mathcal{D}_0, \mathcal{D}_1)$ of objects in \mathcal{C} together with source, target, identity and multiplication morphisms in \mathcal{C} such that the usual compatibility conditions between these structure maps for categories hold. Fully analogously, one can define internal functors and internal natural transformations, which are important for describing principal 2-bundles.

3.2.1 Categories internal to Bibun

In order to define principal 2-bundles with smooth structure 2-groups, we need the notion of a category internal to Bibun.

The concept of an internal category has been weakened in the past to allow for categories internal to strict 2-categories [31]. Here, we need a slight extension of this to weak 2-categories in order to define categories internal to Bibun.

In dealing with internal categories, a more technical issue is that of the existence of 2-pullbacks, which do not all exist in Bibun, cf. Proposition 2.2.16, similarly to the case of LieGrpd. We can circumvent this problem by applying the aforementioned proposition. Therefore, we are now ready to define the notion of categories internal to Bibun.

A category \mathcal{C} internal to **Bibun** is a pair of Lie groupoids \mathcal{C}_0 and \mathcal{C}_1 together with right principal bibundles

$$\mathbf{s}, \mathbf{t} : \mathcal{C}_1 \rightrightarrows \mathcal{C}_0, \quad \text{id} : \mathcal{C}_0 \rightarrow \mathcal{C}_1, \quad B_c : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1, \quad (3.14)$$

called the source, target, identity and composition morphisms, respectively. We demand that the \mathcal{C}_0 -action base maps on both \mathbf{s} and \mathbf{t} are surjective submersions, which guarantees the existence of the pullback $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ in **Bibun**, cf. Proposition 2.2.16. And the following diagrams are required to be commutative:

$$\begin{array}{ccc} \mathcal{C}_1 & \xleftarrow{\text{pr}_1} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{\text{pr}_2} & \mathcal{C}_1 \\ \mathbf{t} \downarrow & & \downarrow B_c & & \downarrow \mathbf{s} \\ \mathcal{C}_0 & \xleftarrow{\mathbf{t}} & \mathcal{C}_1 & \xrightarrow{\mathbf{s}} & \mathcal{C}_0 \end{array} \quad \begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{\text{id}} & \mathcal{C}_1 \\ & \searrow 1 & \downarrow \mathbf{s}, \mathbf{t} \\ & & \mathcal{C}_0 \end{array} \quad (3.15)$$

Figure 3.2: Commutative diagram of categories in **Bibun**

We also have the bibundle natural isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} , which accounts for the 2-commutativity of the following diagrams

$$\begin{array}{ccc} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \\ & \swarrow B_c \times 1 & \searrow 1 \times B_c \\ \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{\mathbf{a}} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \\ & \searrow B_c & \swarrow B_c \\ & \mathcal{C}_1 & \end{array} \quad \begin{array}{ccccc} \mathcal{C}_0 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{\text{id} \times 1} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xleftarrow{1 \times \text{id}} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_0 \\ & \searrow & \downarrow B_c & \swarrow & \swarrow \\ & & \mathcal{C}_1 & & \mathcal{C}_1 \\ & \swarrow \text{pr}_2 & & \swarrow \text{pr}_1 & \\ & & & & \end{array}$$

Figure 3.3: Bibundle natural isomorphisms

called the associator, the left- and right-unitors, respectively. Coherence of the associator and the unitors amounts to the (internal) Pentagon identity,

$$\begin{array}{ccc}
 & [B_c \otimes (1 \times B_c)] \otimes (B_c \times 1 \times 1) & \\
 & \swarrow^{(a \otimes 1) \circ \cong} & \searrow_{(a \otimes 1) \circ \cong} \\
 B_c \otimes [(B_c \times 1) \otimes (B_c \times 1 \times 1)] & & B_c \otimes [(1 \times B_c) \otimes (1 \times 1 \times B_c)] \\
 \downarrow 1 \otimes (a \times 1) & & \uparrow (1 \otimes (1 \times a)) \circ \cong \\
 B_c \otimes [(B_c \times 1) \otimes (1 \times B_c \times 1)] & \xrightarrow{(a \otimes 1) \circ \cong} & [B_c \otimes (1 \times B_c)] \otimes (1 \times B_c \times 1)
 \end{array}$$

Figure 3.4: The internal Pentagon identity

as well as the (internal) Triangle identity,

$$\begin{array}{ccc}
 [B_c \otimes (B_c \times 1)] \otimes (1 \times \text{id} \times 1) & \xrightarrow{(a \times 1) \otimes 1} & [B_c \otimes (1 \times B_c)] \otimes (1 \times \text{id} \times 1) \\
 \searrow_{(1 \otimes (r \times 1)) \circ \cong} & & \swarrow_{(1 \otimes (1 \times l)) \circ \cong} \\
 & B_c &
 \end{array}$$

Figure 3.5: The internal Triangle identity

In the above diagrams, we suppressed arrows for the isomorphisms \cong between bibundles arising from the non-associativity of horizontal or bibundle multiplication \otimes in **Bibun**.

3.2.2 Examples

Now, in order to illustrate our definition of internal categories, we give some examples.

The first example is a Lie group G , which can be considered as an internal category with all the structure morphisms s , t and id are trivial bibundles, while B_c is the bundlization of the Lie group multiplication. Hence, the bibundle isomorphisms a , r and l are all equalities. We denote this by

$$(G \rightrightarrows G) \rightrightarrows (* \rightrightarrows *) , \tag{3.16}$$

with the trivial structure maps⁴.

The second example is obtained by denoting a Lie groupoid \mathcal{G} as

$$(\mathcal{G}_1 \rightrightarrows \mathcal{G}_1) \rightrightarrows (\mathcal{G}_0 \rightrightarrows \mathcal{G}_0) , \quad (3.17)$$

whose structure morphisms are obtained by bundlizations. In particular, the multiplication B_c becomes the bundlization of the Lie groupoid composition.

Finally, recall that, roughly a Lie 2-groupoid is a (weak) 2-category in which all the set of objects, 1-morphisms and 2-morphisms are smooth manifolds, cf [4, 95].

Thus, it can be expressed as an internal category in **Bibun** as

$$(\mathcal{G}_1 \rightrightarrows \mathcal{G}_0) \rightrightarrows (X \rightrightarrows X) . \quad (3.18)$$

See [4, 95] for general discussions of Lie 2-groupoids. Therefore, Lie 2-groupoids are also another examples of internal categories in **Bibun**, in which case the structure morphisms are obtained by bundlizations.

In all the above examples, the structure morphisms are bundlizations of Lie groupoid functors, hence all the natural isomorphisms are induced from natural transformations. In Section 3.3, we will provide a more substantial example of internal categories whose structure maps might not be necessarily obtained from bundlizations.

3.2.3 Internal functors

Analogous to functors between any two categories, we now define internal functors between internal categories. See e.g. [4] for strict version of internal functors. But, our internal functors can be taken as homomorphisms of general bicategories. Given two categories \mathcal{C} and \mathcal{D} internal to **Bibun**, an internal functor $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ consists of right principal bibundles $\Phi_i : \mathcal{C}_i \rightarrow \mathcal{D}_i$, for $i = 0, 1$ and bibundle isomorphisms $\Phi_{2,c}$ and $\Phi_{2,id}$ such that the following diagrams (2-)commute:

⁴In general, one can use the functors $\text{Man}^\infty \xrightarrow{\text{discrete}} \text{LieGrpd} \xrightarrow{\text{bundlization}} \text{Bibun}$.

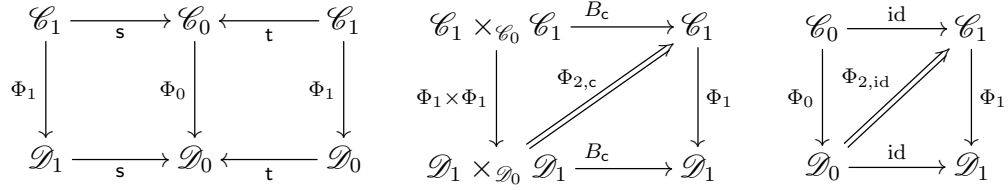


Figure 3.6: (2-)commutative diagrams of internal functors

These bibundle isomorphisms have to satisfy coherence axioms, which amount to the following diagrams commuting⁵:

$$\begin{array}{ccc}
 & B_c \otimes [(\Phi_1 \times \Phi_1) \otimes (B_c \times 1)] & \\
 & \uparrow 1 \otimes (\Phi_{2,c} \times 1) & \searrow \Phi_{2,c} \circ \cong \\
 B_c \otimes [(B_c \times 1) \otimes (\Phi_1 \times \Phi_1 \times \Phi_1)] & & \Phi_1 \otimes (B_c \times (B_c \times 1)) \\
 \downarrow a \circ \cong & & \downarrow 1 \otimes a \\
 [B_c \otimes (1 \times B_c)] \otimes (\Phi_1 \times \Phi_1 \times \Phi_1) & & \Phi_1 \otimes (B_c \times (1 \times B_c)) \\
 & \searrow (1 \otimes (1 \times \Phi_{2,c})) \circ \cong & \nearrow \Phi_{2,c} \circ \cong \\
 & B_c \otimes [(\Phi_1 \times \Phi_1) \otimes (1 \times B_c)] &
 \end{array}$$

Figure 3.7: Coherence axiom-I of internal functors

$$\begin{array}{ccc}
 & B_c \otimes [(\Phi_1 \times \Phi_1) \otimes [(1 \times \text{id}) \otimes (1, s)]] & \\
 & \uparrow 1 \otimes (1 \times \Phi_{2,\text{id}}) & \searrow \Phi_{2,c} \circ \cong \\
 B_c \otimes [(1 \times \text{id}) \otimes [(\Phi_1 \times \Phi_0) \otimes (1, s)]] & & [\Phi_1 \otimes B_c] \otimes [(1 \times \text{id}) \otimes (1, s)] \\
 & \searrow r & \nearrow (1 \otimes r) \circ \cong \\
 & \Phi_1 & \\
 & \nearrow l & \searrow (1 \otimes l) \circ \cong \\
 B_c \otimes [(\text{id} \times 1) \otimes [(\Phi_0 \times \Phi_1) \otimes (t, 1)]] & & [\Phi_1 \otimes B_c] \otimes [(\text{id} \times 1) \otimes (t, 1)] \\
 & \searrow 1 \otimes (\Phi_{2,\text{id}} \times 1) & \nearrow \Phi_{2,c} \circ \cong \\
 & B_c \otimes [(\Phi_1 \times \Phi_1) \otimes [(\text{id} \times 1) \otimes (t, 1)]] &
 \end{array}$$

Figure 3.8: Coherence axiom-II of internal functors

⁵Note that in these diagrams, the structure 1- and 2-morphisms in \mathcal{C} and \mathcal{D} are labelled by the same symbols.

We again suppressed additional arrows for isomorphisms arising from the non-associativity of horizontal composition in **Bibun**. Moreover, we write (B_1, B_2) for the morphism $(B_1 \times B_2) \otimes \Delta$, where Δ is the diagonal morphism $\Delta : \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$. The first diagram contains bibundles from $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ to \mathcal{D}_1 , while the second diagram contains bibundles from \mathcal{C}_1 to \mathcal{D}_1 .

Similarly, we give here the definition of internal natural transformations. Discussions of internal natural transformations between internal 2-categories is also available in [4].

3.2.4 Internal natural transformations

Given two internal functors Φ and Ψ between categories \mathcal{C} and \mathcal{D} internal to **Bibun**, a natural transformation $\chi : \Phi \Rightarrow \Psi$ consists of a right principal $(\mathcal{C}_0, \mathcal{D}_1)$ -bibundle χ_1 together with a bibundle isomorphism χ_2 rendering the diagrams

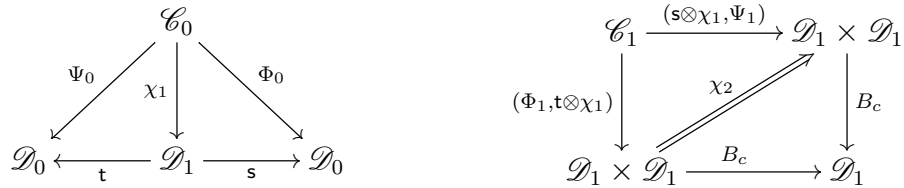


Figure 3.9: (2-)commutative diagram of internal natural transformations

(2-)commutative.

In addition, we have coherence rules amounting to the commutative diagrams

$$\begin{array}{ccc}
 & B_c \otimes [(1 \times B_c) \otimes [(\Phi_1 \times \chi_1 \times \Psi_1) \otimes (1 \times (\mathbf{t}, 1))]] & \\
 & \swarrow a^{-1} & \nwarrow 1 \otimes (\chi_2 \times 1) \\
 [B_c \otimes (B_c \times 1)] \otimes [(\Phi_1 \times \chi_1 \times \Psi_1) \otimes (1 \times (\mathbf{t}, 1))] & & \\
 \downarrow \cong \circ (1 \otimes (1 \times \chi_2)) \circ \cong & & B_c \otimes [(1 \times B_c) \otimes [(\Phi_1 \times \Phi_1 \times \chi_1) \otimes ((\mathbf{t}, 1) \times 1)]] \\
 [B_c \otimes (B_c \times 1)] \otimes [(\chi_1 \times \Psi_1 \times \Psi_1) \otimes (1 \times (1, \mathbf{s}))] & & \downarrow \cong \circ a \\
 \downarrow a & & [B_c \otimes (B_c \times 1)] \otimes [(\Phi_1 \times \Phi_1 \times \chi_1) \otimes ((\mathbf{t}, 1) \times 1)] \\
 [B_c \otimes (1 \times B_c)] \otimes [(\chi_1 \times \Psi_1 \times \Psi_1) \otimes (1 \times (1, \mathbf{s}))] & & \downarrow (1 \otimes (1 \times \Phi_{2,c})) \circ \cong \\
 & & B_c \otimes [(\Phi_1 \times \chi_1) \otimes [(B_c \times 1) \otimes ((\mathbf{t}, 1) \times 1)]] \\
 & \swarrow (1 \otimes (\Psi_{2,c} \times 1)) \circ \cong & \swarrow \chi_2 \\
 & B_c \otimes [(\chi_1 \times \Psi_1) \otimes [(B_c \times 1) \otimes (1, (1, \mathbf{s}))]] &
 \end{array}$$

Figure 3.10: Internal natural transformation: axiom I

$$\begin{array}{ccc}
 & \chi_1 & \\
 \swarrow \text{id} \cong & & \searrow (r \circ \cong)^{-1} \\
 B_c \otimes [(\text{id} \times 1) \otimes (\Phi_0, \mathbf{t} \otimes \chi_1)] & & B_c \otimes [(1 \times \text{id}) \otimes (\mathbf{s} \otimes \chi_1, \Psi_0)] \\
 \downarrow 1 \otimes (\Psi_{2,\text{id}} \times 1) & & \downarrow 1 \otimes (1 \times \Phi_{2,\text{id}}) \\
 B_c \otimes [(\Phi_1, \mathbf{t} \otimes \chi_1) \otimes \text{id}] & \xrightarrow{\chi_2 \otimes \text{id}_{\text{id}}} & B_c \otimes [(\mathbf{s} \otimes \chi_1, \Psi_1) \otimes \text{id}]
 \end{array}$$

Figure 3.11: Internal natural transformation: axiom II

The first diagram gives isomorphisms between right principal bibundles from $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ to \mathcal{D}_1 and on this Lie groupoid we have $(1 \times (\mathbf{t}, 1)) = ((1, \mathbf{s}) \times 1)$. The second diagram describes isomorphisms between right principal bibundles from \mathcal{C}_0 to \mathcal{D}_1 and involves the bibundle isomorphism

$$\chi_2 \otimes \text{id}_{\text{id}} : B_c \otimes ((\Phi_1, \mathbf{t} \otimes \chi_1) \otimes \text{id}) \Rightarrow B_c \otimes ((\mathbf{s} \otimes \chi_1, \Psi_1) \otimes \text{id}) . \quad (3.19)$$

3.3 2-group objects in Bibun

Having the concept of internalization, one can define categorified groups in any weak 2-category with finite products and terminal object. The 2-groups we are interested in are obtained as an internal categories in the weak 2-category **Bibun**. We therefore begin by presenting a brief review of 2-group objects, see e.g. [4, 71].

3.3.1 Preliminaries on 2-groups

Definition 3.3.1 ([4, 71]). *A 2-group object in a weak 2-category \mathcal{C} with finite products and terminal object $*$ is an object \mathcal{G} in \mathcal{C} together with the 1-morphisms $m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and $e : \mathcal{G} \rightarrow *$ and the natural 2-isomorphisms*

$$\begin{aligned} \mathbf{a} : m \otimes (m \times 1) &\Rightarrow m \otimes (1 \times m) , \\ \mathbf{l} : m \otimes (e \times 1) &\Rightarrow 1 \quad \text{and} \quad \mathbf{r} : m \otimes (1 \times e) \Rightarrow 1 , \end{aligned} \tag{3.20}$$

such that the 1-morphism $(\text{pr}_1, m) : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ has to be an equivalence and the natural 2-isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} must satisfy the (internal) Pentagon and Triangle identities in \mathcal{C} , cf. [71].

For instance, considering the ordinary category **Set** as a strict 2-category with identity 2-morphisms gives the definition of a group, which is just an object \mathbf{G} of **Set** together with the multiplication, identity and inversion morphisms $m : \mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$, $e : * \rightarrow \mathbf{G}$, and $\text{inv} : \mathbf{G} \rightarrow \mathbf{G}$ such that the following diagrams are commutative⁶:

⁶Here Δ is the diagonal map.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 G \times G \times G & \xrightarrow{\text{id}_G \times m} & G \times G \\
 \downarrow m \times \text{id}_G & & \downarrow m \\
 G \times G & \xrightarrow{m} & G
 \end{array} & &
 \begin{array}{ccc}
 * \times G = G \times * & \xrightarrow{\text{id}_G \times e} & G \times G \\
 \downarrow e \times \text{id}_G & \searrow = & \downarrow m \\
 G \times G & \xrightarrow{m} & G
 \end{array} \\
 \\
 \begin{array}{ccc}
 G & \xrightarrow{\Delta} & G \times G \\
 \searrow & & \downarrow \text{id}_G \times \text{inv} \\
 & & G \times G \\
 & & \downarrow m \\
 & & G \\
 & \searrow e & \\
 & * &
 \end{array}
 \end{array}$$

Figure 3.12: Group axioms

Similarly, one can consider a Lie group as a group object in Man^∞ .

Let us now restrict the above definition to a weak 2-category Bibun .

3.3.2 Smooth 2-groups

We now recall the definition of smooth 2-groups as first studied in [71], which are internal categories in Bibun $s, t : \mathcal{G} \rightrightarrows (* \rightrightarrows *)$ in Bibun , cf. Subsection 3.2.1. Explicitly, we have the following definition.

Definition 3.3.2 ([71]). *A smooth 2-group is given by a Lie groupoid \mathcal{G} together with a right principal bibundles $B_m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and $e : (* \rightrightarrows *) \rightarrow \mathcal{G}$ as well as bibundle natural isomorphisms*

$$\begin{aligned}
 \mathbf{a} : B_m \otimes (B_m \times 1) &\rightrightarrows B_m \otimes (1 \times B_m), \\
 \mathbf{l} : B_m \otimes (e \times 1) &\rightrightarrows 1 \quad \text{and} \quad \mathbf{r} : B_m \otimes (1 \times e) \rightrightarrows 1
 \end{aligned} \tag{3.21}$$

such that the right principal bibundle $(\text{pr}_1, B_m) : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ is an equivalence and the bibundle natural isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} have to satisfy certain coherence axioms; the (internal) Pentagon and Triangle identities, cf. Figure 3.4 and Figure 3.5, see also [71].

Note that, these identities are obtained by considering the two obvious 2-isomorphisms

$$(B_m \otimes (B_m \times 1)) \otimes (B_m \times 1 \times 1) \quad B_m \otimes ((1 \times B_m) \otimes (1 \times 1 \times B_m)) \quad (3.22)$$

and

$$B_m \otimes ((B_m \otimes (1 \times \mathbf{e})) \times 1) \quad B_m \otimes (1 \times 1) \quad (3.23)$$

in Figure 3.4 and Figure 3.5 with B_c replaced by B_m . Moreover, here we are using our notion of the identity right principal bibundle 1 in contrary to the notation $\text{id}_\mathcal{X}$ in Definition 3.3.1.

The definition of smooth 2-groups includes all the previously known notions of Lie 2-groups [4]. See also [71] for more examples of smooth 2-groups. Here, we present the some of them.

3.3.3 Examples of smooth 2-groups

First, a Lie group \mathbf{G} , regarded as a Lie groupoid ($\mathbf{G} \rightrightarrows *$) is a smooth 2-group. Here, B_m is the bundlization of the Lie group multiplication in \mathbf{G} .

Second, crossed modules of Lie groups ($\mathbf{H} \xrightarrow{\mathbf{t}} \mathbf{G}$) as shown in Definition 2.2.14 give rise to strict Lie 2-groups, which are also smooth 2-groups, cf. [4]. Consider the Lie groupoid ($\mathbf{G} \times \mathbf{H} \rightrightarrows \mathbf{G}$), with structure maps

$$\mathbf{s}(g, h) := \mathbf{t}(h)g, \quad \mathbf{t}(g, h) := g \quad \text{and} \quad \text{id}_g := (g, \mathbb{1}_\mathbf{H}), \quad (3.24a)$$

vertical multiplication

$$(g, h) \circ (\mathbf{t}(h)g, h') = (g, h'h), \quad (3.24b)$$

and the horizontal multiplication

$$g \otimes g' := gg' \quad \text{and} \quad (g, h) \otimes (g', h') := (gg', h(g \triangleright h')), \quad (3.24c)$$

where $g, g' \in \mathbf{G}$, $h, h' \in \mathbf{H}$ and $\triangleright: \mathbf{G} \times \mathbf{H} \rightarrow \mathbf{H}$ is the action in the crossed module of Lie groups. One can even show categorical equivalence between crossed modules of

Lie groups and strict Lie 2-groups, cf. [4].

Finally, weak Lie 2-groups, that is, weak 2-groups internal to $\mathbf{Man}^\infty\mathbf{Cat}$ are also examples of smooth 2-groups in which case all the structure morphisms are again obtained by bundlizations. Moreover, the natural isomorphisms are induced from the natural transformations in $\mathbf{Man}^\infty\mathbf{Cat}$. Later, we will discuss a more general example of smooth 2-groups whose structure morphisms are not all obtained from Lie groupoid functors, cf. Subsection 3.4.2.

3.3.4 Homomorphisms and central extension of smooth 2-groups

One of the methods of constructing the string group $\mathbf{String}(n)$ involves the classification of the exact sequence as extension of $\mathbf{Spin}(n)$. Therefore, classifying smooth 2-group extensions of $\mathbf{Spin}(n)$ is important to develop a smooth 2-group model, cf. [71]. That is, he considered a more general extension, which is in the categorified settings, as done in [71, Def. 75]. We will describe this notion based on this reference. An earlier approach in the context of (symmetric) categorified group extensions is also available in [16, 24, 25, 33, 44]. We now give the definition of smooth 2-group homomorphisms in order to review the classification of central extension of $\mathbf{Spin}(n)$ given in [71].

A homomorphism Φ between smooth 2-groups \mathcal{G} and \mathcal{H} is an internal functor in \mathbf{Bibun} . And hence it consists of a right principal bibundle $\Phi_1 : \mathcal{G} \rightarrow \mathcal{H}$ and 2-isomorphisms $\Phi_m : B_m \otimes (\Phi_1 \times \Phi_1) \rightarrow \Phi_1 \otimes B_m$ and $\Phi_0 : \mathbf{e} \rightarrow \Phi_1 \otimes \mathbf{e}$ satisfying the coherence axioms, cf. Figure 3.6. see also [71].

An extension of a smooth 2-group \mathcal{G} by a smooth 2-group \mathcal{A} consists of a smooth 2-group \mathcal{E} together with homomorphisms $\Phi : \mathcal{A} \rightarrow \mathcal{E}$, $\Psi : \mathcal{E} \rightarrow \mathcal{G}$ and 2-homomorphism, (which is an internal natural transformation) $\chi : \Psi \otimes \Phi \Rightarrow 0$ such that \mathcal{E} is a principal \mathcal{A} -bundle over \mathcal{G} , cf. Chapter 4.

We are interested in central extensions of a smooth 2-group $\mathcal{G} = (\mathbf{G} \rightrightarrows \mathbf{G})$ (compact semisimple Lie group \mathbf{G}) by a smooth 2-group $\mathcal{A} = (\mathbf{A} \rightrightarrows *)$ (abelian Lie group \mathbf{A}), which forms the weak 2-category, denoted by $\mathbf{Ext}(\mathcal{G}, \mathcal{A})$, [71]. This weak

2-category can be encoded in a Segal-Mitchison cocycle $H_{\text{SM}}^3(\mathbf{G}, \mathbf{A})$, and gives the following theorem:

Theorem 3.3.3. [71, Theorem 1] *Let \mathbf{G} be a compact Lie group and \mathbf{A} be an abelian Lie group, viewed as a trivial \mathbf{G} -module. Then the isomorphism class of central extension of smooth 2-groups*

$$\mathcal{A} \rightarrow \mathcal{E} \rightarrow \mathcal{G} , \tag{3.25}$$

where $\mathcal{G} = (\mathbf{G} \rightrightarrows \mathbf{G})$ and $\mathcal{A} = (\mathbf{A} \rightrightarrows *)$ are in natural bijection with $H_{\text{SM}}^3(\mathbf{G}, \mathbf{A})$.

Theorem 3.3.3 gives a bijection between smooth 2-group central extension \mathcal{E} of \mathcal{G} by \mathcal{A} and the characteristic class $[\lambda] \in H_{\text{SM}}^3(\mathbf{G}, \mathbf{A})$. But, as conjectured in [79], for a compact Lie group \mathbf{G} and an abelian Lie group \mathbf{A} , $H_{\text{SM}}^3(\mathbf{G}, \mathbf{A}) \cong H^4(\mathbf{BG}, K(\pi_1(\mathbf{A}), 0))$, where $K(\pi_1(\mathbf{A}), 0)$ is the Eilenberg-MacLane space, while $\pi_1(\mathbf{A})$ is the fundamental group of \mathbf{A} .

In particular, if $\mathbf{G} = \text{Spin}(n)$ and $\mathbf{A} = \mathbf{U}(1)$, then $K(\pi_1(\mathbf{A}), 0) = \mathbb{Z}$ and $H^4(\mathbf{BSpin}(n), \mathbb{Z}) \cong \mathbb{Z}$, for $n \geq 3$; and the generator corresponds to the string group $\text{String}(n)$, cf. [71, 63, 79]. See also [87, 18] for discussions on the integral cohomology of spin groups and their classifying spaces. The next section discusses the construction of this 2-group model $\text{String}(n)$ by Schommer-Pries [71].

3.4 The string 2-group model of Schommer-Pries

In this section, we will review the string 2-group model of Schommer-Pries, and later we will explain why this smooth 2-group is in fact equivalent to a Lie quasi-groupoid⁷ with the trivial space of objects, due to the nature of the multiplication B_m . In fact it is well known that every smooth 2-group is equivalent to a weak Lie 2-group, see [94, 95]. For discussions of general quasi-categories consult [13]. Now let us start the section by recalling general remarks on the string group.

⁷This is due to the fact that the multiplication B_m does not give a unique object, but there is always an isomorphism between them.

3.4.1 General remarks on string groups

The string group $\mathbf{String}(n)$ is a 3-connected cover of the spin group $\mathbf{Spin}(n)$, for $n \geq 3$. It fits within the Whitehead tower of the orthogonal group $\mathbf{O}(n)$. Recall that each space $X^{(i+1)}$ of the Whitehead tower of a space X is constructed by killing the i -th homotopy group $\pi_i(X^{(i)})$, hence it lifts X into $(i-1)$ -connected space $X^{(i)}$. Thus, it consists of a sequence of spaces

$$* \rightarrow \dots \xrightarrow{\mu_{i+1}} X^{(i)} \xrightarrow{\mu_i} \dots \xrightarrow{\mu_3} X^{(2)} \xrightarrow{\mu_2} X^{(1)} \xrightarrow{\mu_1} X, \quad (3.26)$$

where the maps μ_i induce isomorphisms on all homotopy groups in degree $k \geq i$ and $\pi_k(X^{(i)}) = 0$ for $k < i$. In the case of $X = \mathbf{O}(n)$, we have

$$\dots \rightarrow \mathbf{String}(n) \rightarrow \mathbf{Spin}(n) \rightarrow \mathbf{Spin}(n) \rightarrow \mathbf{SO}(n) \hookrightarrow \mathbf{O}(n). \quad (3.27)$$

Moreover, the string group can be constructed for any compact Lie group \mathbf{G} with $\pi_3(\mathbf{G}) \cong \mathbb{Z}$ as a based loop space of a homotopy fiber of half of the first Pontryagin class $\frac{1}{2}\mathbf{p}_1 \in H^4(\mathbf{BSpin}(n), \mathbb{Z}) \cong \mathbb{Z}$, cf. [32].

As a result, the string group is only defined up to homotopy, and therefore the group structure can only be determined up to homotopy equivalences. Moreover the smooth structure on the string group is not determined at all, this is the reason for having various models. The first geometric model as a topological group was constructed by Stolz [83] and Stolz and Teichner in [84]. Because π_1 and π_3 of $\mathbf{String}(n)$ both vanish, the string group cannot be a finite-dimensional Lie group [15]. Rather, it is an infinite-dimensional Lie group. To remedy this, considering categorified group models is vital, [71] and [61].

Moreover, in order to restrict the arbitrariness in the definition of the string group, one is naturally led to Lie 2-group models of the string group [4]. These are Lie 2-groups endowed with a Lie 2-group homomorphism to $\mathbf{Spin}(n)$, regarded as a Lie 2-group. A first such model was constructed in [7], which is a strict but infinite-dimensional Lie 2-group. Closely related is the construction of [37], which is obtained by integrating the string Lie 2-algebra as a simplicial manifold. There is

also an infinite-dimensional model as a strict Lie 2-group [61], obtained by smoothing the original Stolz-Teichner construction. The model we shall be mostly interested in this section is constructed by Schommer-Pries [71]. He modeled the string group by a 2-group object \mathcal{S}_λ in Bibun, which is equivalent to a weak Lie 2-group within the framework of finite dimensional Lie groupoids. In the following we shall give the detail construction of \mathcal{S}_λ and its equivalent weak Lie 2-group counterpart \mathcal{S}_λ^w .

3.4.2 The string group model

As it is stated in Theorem 3.3.3, cf. [71], the construction of the model of the string group corresponds to the 3-cocycle λ , we denote it by \mathcal{S}_λ . In particular, we are interested in the non-trivial components of $\lambda = (\lambda^{2,1}, \lambda^{1,2}, \lambda^{0,3})$ in $H_{\text{SM}}^3(\mathbf{G}, \mathbf{A})$ with $\mathbf{G} = \text{Spin}(n)$ and $\mathbf{A} = \text{U}(1)$, for $n \geq 3$, cf. (3.11). That is the smooth 2-group corresponding to λ , in the central extension of $(\mathbf{G} \rightrightarrows \mathbf{G})$ by $(\mathbf{A} \rightrightarrows *)$.

Therefore, in this section, we study the explicit construction of the smooth 2-group \mathcal{S}_λ in Bibun, for a compact semisimple Lie group \mathbf{G} and an abelian Lie group \mathbf{A} (\mathbf{G} -module with trivial action).

Thus, starting from a simplicial cover V_\bullet of \mathbf{G} as shown in Subsection 3.1.2, the 3-cocycle λ contains the non-trivial smooth maps

$$\lambda^{2,1} : V_1^{[3]} \rightarrow \mathbf{A} , \quad \lambda^{1,2} : V_2^{[2]} \rightarrow \mathbf{A} \quad \text{and} \quad \lambda^{0,3} : V_3^{[1]} \rightarrow \mathbf{A} . \quad (3.28)$$

But, as remarked in Subsection 3.1.2, the map $\lambda^{2,1}$ is in fact a Čech 2-cocycle and defines an \mathbf{A} -bundle gerbe over \mathbf{G} . Identifying bundle gerbes with central groupoid extensions, [38, 59, 92]

$$\mathcal{S}_\lambda := V_1^{[2]} \times \mathbf{A} \rightrightarrows V_1 , \quad (3.29a)$$

underlying the smooth 2-group corresponding to λ , cf. Theorem 3.3.3. Here the source, target, identity and composition smooth maps are given by

$$\mathbf{s}(v_0, v_1, a) := v_1 , \quad \mathbf{t}(v_0, v_1, a) := v_0 , \quad \mathbf{id}(v_0) := (v_0, v_0, 0) , \quad (3.29b)$$

and

$$(v_0, v_1, a) \circ (v_1, v_2, b) := (v_0, v_2, a + b + \lambda^{2,1}(v_0, v_1, v_2)) \quad (3.29c)$$

for $v_{0,1,2} \in V_1$ and $a, b \in \mathbf{A}$.

To specify the smooth 2-group structure, note that there is a Lie groupoid functor (f_2^2, f_0^2) from the Lie groupoid $\mathcal{C}_2 := (V_2^{[2]} \times \mathbf{A}^{\times 2} \rightrightarrows V_2)$ to $\mathcal{S}_\lambda \times \mathcal{S}_\lambda$, since it is a weak-equivalence in LieGrpd , upon bundlization, it becomes an equivalence in Bibun , cf. Proposition 2.2.13. The same is true for the Lie groupoid functor $(f_2^2 f_2^3, f_2^2 f_0^3, f_0^2 f_0^3)$ from $\mathcal{C}_3 := (V_3^{[2]} \times \mathbf{A}^{\times 3} \rightrightarrows V_3)$ to $\mathcal{S}_\lambda^{\times 3}$.

These yield the right principal bibundles

$$B_2 : \mathcal{S}_\lambda \times \mathcal{S}_\lambda \rightarrow \mathcal{C}_2 \quad \text{and} \quad B_3 : \mathcal{S}_\lambda \times \mathcal{S}_\lambda \times \mathcal{S}_\lambda \rightarrow \mathcal{C}_3 . \quad (3.30)$$

Furthermore, we have Lie groupoid functors

$$\mathcal{C}_2 \xrightarrow{m} \mathcal{S}_\lambda , \quad \mathcal{C}_3 \xrightarrow{p} \mathcal{S}_\lambda \quad \text{and} \quad \mathcal{C}_3 \xrightarrow{q} \mathcal{S}_\lambda , \quad (3.31)$$

defined on the set of morphisms by

$$m(y_0, y_1, a_0, a_1) := (f_1^2(y_0), f_1^2(y_1), a_0 + a_1 + \lambda^{1,2}(y_0, y_1)) , \quad (3.32a)$$

$$p_1(z_0, z_1, a_0, a_1, a_2)$$

$$:= (f_1^2 f_1^3(z_0), f_1^2 f_1^3(z_1), a_0 + a_1 + a_2 + \lambda^{1,2}(f_1^3(z_0), f_1^3(z_1)) + \lambda^{1,2}(f_3^3(z_0), f_3^3(z_1))) , \quad (3.32b)$$

$$q_1(z_0, z_1, a_0, a_1, a_2)$$

$$:= (f_1^2 f_2^3(z_0), f_1^2 f_2^3(z_1), a_0 + a_1 + a_2 + \lambda^{1,2}(f_0^3(z_0), f_0^3(z_1)) + \lambda^{1,2}(f_2^3(z_0), f_2^3(z_1))) , \quad (3.32c)$$

for $y_{0,1} \in V_2^{[2]}$, $z_{0,1} \in V_3^{[2]}$ and $a_{0,1,2} \in \mathbf{A}$.

Thus, after bundlization and multiplication with the previous right principal

bibundles, we obtain the right principal bibundles⁸

$$\begin{aligned} B_m &: \mathcal{S}_\lambda \times \mathcal{S}_\lambda \rightarrow \mathcal{S}_\lambda , \\ B_p &: \mathcal{S}_\lambda \times \mathcal{S}_\lambda \times \mathcal{S}_\lambda \rightarrow \mathcal{S}_\lambda , \\ B_q &: \mathcal{S}_\lambda \times \mathcal{S}_\lambda \times \mathcal{S}_\lambda \rightarrow \mathcal{S}_\lambda . \end{aligned} \tag{3.33}$$

Moreover, one can also consider the natural transformation

$$T : V_3 \longrightarrow V_1^{[2]} \times \mathbf{A} , \tag{3.34a}$$

defined by

$$T(z_0) = (\mathbf{f}_1 \mathbf{f}_1(z_0), \mathbf{f}_1 \mathbf{f}_2(z_0), \lambda^{0,3}(z_0)) , \quad z_0 \in V_3 , \tag{3.34b}$$

and which satisfy

$$q_1(z_0, z_1, a_0, a_1, a_2) \circ T(z_1) = T(z_0) \circ p_1(z_0, z_1, a_0, a_1, a_2) \tag{3.34c}$$

due to $\delta_{\tilde{\mathcal{C}}} \lambda^{0,3} = \delta_N \lambda^{1,2}$.

Thus, the natural transformation T induces the right principal bibundle natural isomorphism $\mathbf{a} : B_p \Rightarrow B_q$, called the associator, because the right principal bibundles B_p and B_q can be identified with $B_m \otimes (B_m \times 1)$ and $B_m \otimes (1 \times B_m)$, respectively. Moreover, here \mathbf{a} is completely determined by T since \mathbf{a} is the horizontal multiplication of T with the identity isomorphism on the right principal $(\mathcal{S}_\lambda^{\times 3}, \mathcal{C}_3)$ -bibundle B_3 .

It remains to define the unit \mathbf{e} as well as the left- and right-unitors \mathbf{l} and \mathbf{r} . Both unitors are trivial (i.e. the identity bibundle isomorphism) and up to isomorphism, the unit \mathbf{e} is uniquely defined as the bundlization \mathbf{e} of the trivial Lie groupoid functor

$$(* \rightrightarrows *) \longrightarrow \mathcal{S}_\lambda , \tag{3.35}$$

⁸These can be also obtained by using the notion of Lie groupoid generalized morphisms, cf. Proposition 2.4.1.

which takes $*$ to $v_i \in V_{1,i}$ with $\pi(v_i) = \mathbb{1}_G$, where $\pi : V_1 \rightarrow G$.

Let us now briefly verify that we indeed constructed a smooth 2-group. For this, we need to check that the bibundle (pr_1, B_m) is an equivalence and that the internal Pentagon Identity is satisfied. The former is relatively clear, because B_2 and thus also (pr_1, B_2) are right principal bibundle equivalences. One then readily checks that⁹

$$(1 \times \hat{m}) : \mathcal{S}_\lambda \times \mathcal{C}_2 \rightarrow \mathcal{S}_\lambda \times \mathcal{S}_\lambda \quad (3.36)$$

is a bibundle equivalence. It is obvious that the associator only affects the A-part of the Lie groupoids \mathcal{S}_λ , \mathcal{C}_2 and \mathcal{C}_3 , and therefore the internal Pentagon Identity reduces to the equation

$$\begin{aligned} \lambda^{0,3}(v_1, v_2, v_3) + \lambda^{0,3}(v_0, v_1 v_2, v_3) + \lambda^{0,3}(v_0, v_1, v_2) \\ = \lambda^{0,3}(v_0 v_1, v_2, v_3) + \lambda^{0,3}(v_0, v_1, v_2 v_3) , \end{aligned} \quad (3.37)$$

where $v_{0,1,2,3} \in V_1^{[1]}$. This is precisely the equation $\delta_N \lambda^{0,3} = 0$, which holds since λ is a Segal-Mitchison 3-cocycle.

Finally, note that the interchange law, which is the compatibility condition for the vertical and horizontal multiplications, follows from $\delta_N \lambda^{2,1} = \delta_{\hat{C}} \lambda^{1,2}$.

Remark 3.4.1. *Although we are interested in the smooth 2-group model of the string group \mathcal{S}_λ given by the central extension of the smooth 2-group $(\text{Spin}(n) \rightrightarrows \text{Spin}(n))$ by $(A \rightrightarrows *)$, the above construction of this extension as well as most of our following discussion readily generalizes to arbitrary compact Lie group G .*

In the following section we will explain the weak Lie 2-group \mathcal{S}_λ^w , which is equivalent to the smooth 2-group \mathcal{S}_λ , [93]. See also [95, 94] for discussions of the correspondence between groups in higher categories (which are also called categorified groups, [4, 49]) and weak Lie 2-groups.

⁹Here, \hat{m} is the bundlization of the functor m .

3.4.3 From the smooth 2-group model to a weak 2-group model

Recall that it has been shown in [93] that smooth 2-groups are equivalent to Lie 2-quasigroupoids with a single object, which are given by certain Kan simplicial manifolds. It differs to a weak Lie 2-group, which is a weak 2-group object internal to $\mathbf{Man}^\infty\mathbf{Cat}$, is that in the latter case, horizontal multiplication of objects and morphisms yields unique objects, which is not true in the case of Lie 2-quasigroupoids.

In particular, consider the horizontal multiplication of two objects (v_0, v_1) by the composition bibundle B_m in the smooth string 2-group model, the result is a set of isomorphic objects given by $\{\tau(b) | b \in B_m : \sigma(b) = (v_0, v_1)\}$.

To give the underlying multiplication functor explicitly, we proceed as follows. Without restriction in the cases we are interested in, we assume a simplicial cover V_\bullet as constructed in Example 3.1.2. In particular, the simplicial index set I_\bullet has now a total order with each subset of the simplicial set having a lowest element. We can now use these lowest elements to fix ambiguities, like defining preferred horn fillers and fixing a unique identity object in \mathcal{S}_λ .

First, consider the surjective submersion $(f_2^2, f_0^2) : V_2 \rightarrow V_1 \times V_1$. For each element $(v_0, v_1) \in V_1 \times V_1$, we can now choose the element $v_a \in V_2$ over (v_0, v_1) with the lowest position according to the obvious lexicographic ordering of patch multiindices. This defines a smooth map $\phi_2 : (V_1 \times V_1) \rightarrow V_2$ satisfying

$$f_0^2 \phi_2(v_0, v_1) = f_0^2(v_a) = v_1 \quad \text{and} \quad f_2^2 \phi_2(v_0, v_1) = f_2^2(v_a) = v_0 . \quad (3.38)$$

In the language of quasigroupoids and Kan complexes, the map ϕ_2 picks a horn filler in V_2 for the horn $(v_0, v_1) \in V_1 \times V_1$. Applying the face map f_1^2 to this horn filler then yields a preferred horizontal multiplication:

$$v_0 \otimes v_1 := f_1^2 \phi_2(v_0, v_1) = f_1^2(v_a) . \quad (3.39)$$

Since the lexicographic ordering on V_2 arises from that on V_1 , we evidently have a relation between $\phi_2 : V_1 \times V_1 \rightarrow V_2$ and $\phi_1 : \mathbf{G} \rightarrow V_1$:

Proposition 3.4.2. *The horizontal multiplication is completely induced from the product on \mathbf{G} :*

$$v_0 \otimes v_1 := f_1^2 \phi_2(v_0, v_1) = \phi_1(\pi(v_0)\pi(v_1)) \quad (3.40)$$

for all $v_{0,1} \in V_1$.

Corollary 3.4.3. *We have the following identities:*

$$\begin{aligned} \pi(v_0 \otimes v_1) &= \pi(v_0)\pi(v_1) , \\ v_0 \otimes v_1 \cong v_2 \otimes v_3 &\Rightarrow v_0 \otimes v_1 = v_2 \otimes v_3 , \\ (v_0 \otimes v_1) \otimes v_2 &= v_0 \otimes (v_1 \otimes v_2) , \end{aligned} \quad (3.41)$$

for all $v_{0,1,2} \in V_1$.

Proof. The first relation follows from the fact that ϕ_1 is a section of π and therefore $\pi \circ \phi_1 = \text{id}_{\mathbf{G}}$. The second and third relations are then direct consequences of the first one. \square

We can now readily extend $\phi_2 : V_1 \times V_1 \rightarrow V_2$ to higher fibered products as done in the following lemma:

Lemma 3.4.4. *The smooth map $\phi_2 : V_1^{[2]} \times V_1^{[2]} \rightarrow V_2^{[2]}$ with*

$$\phi_2((v_0, v_1), (v_2, v_3)) := (\phi_2(v_0, v_2), \phi_2(v_1, v_3)) , \quad (3.42)$$

where $v_{0,1,2,3} \in V_1$ with $\pi(v_0) = \pi(v_1)$ and $\pi(v_2) = \pi(v_3)$, renders the following diagram commutative:

$$\begin{array}{ccc} V_2^{[2]} \times \mathbf{A} \times \mathbf{A} & \xleftarrow{\phi_2 \times \text{id}_{\mathbf{A} \times \mathbf{A}}} & V_1^{[2]} \times V_1^{[2]} \times \mathbf{A} \times \mathbf{A} \\ \Downarrow & & \Downarrow \\ V_2 & \xleftarrow{\phi_2} & V_1 \times V_1 \end{array}$$

Figure 3.13: Lie groupoid functor ϕ_2

and we can replace the bibundle B_m with the bundlization of this functor, making the horizontal multiplication uniquely defined.

We also have a surjective submersion $(f_0^3, f_2^3, f_3^3) : V_3 \rightarrow V_2 \times V_2 \times V_2$, and we define a smooth map $\phi_3 : V_1 \times V_1 \times V_1$ as the horn filler of $\phi_2(v_1, v_2)$, $\phi_2(v_0, v_1 \otimes v_2)$ and $\phi_2(v_0, v_1)$ with the lowest lexicographic position satisfying

$$\begin{aligned} f_0^3 \phi_3(v_0, v_1, v_2) &= \phi_2(v_1, v_2) , & f_0^2 f_0^3 \phi_3(v_0, v_1, v_2) &= v_2 \\ f_2^3 \phi_3(v_0, v_1, v_2) &= \phi_2(v_0, v_1 \otimes v_2) , & f_2^2 f_0^3 \phi_3(v_0, v_1, v_2) &= v_1 , \\ f_3^3 \phi_3(v_0, v_1, v_2) &= \phi_2(v_0, v_1) , & f_2^2 f_2^3 \phi_3(v_0, v_1, v_2) &= v_0 . \end{aligned} \quad (3.43)$$

Altogether, we arrive at the following theorem.

Theorem 3.4.5. *The Lie groupoid $\mathcal{S}_\lambda := V_1^{[2]} \times \mathbf{A} \rightrightarrows V_1$, together with the identity-assignment*

$$I : (* \rightrightarrows *) \rightarrow \mathcal{S}_\lambda , \quad I_0(*) := \mathbb{1}_{\mathcal{S}_\lambda} := \phi_1(\mathbb{1}_G) , \quad I_1(*) := \text{id}_{\mathbb{1}_{\mathcal{S}_\lambda}} := (\mathbb{1}_{\mathcal{S}_\lambda}, \mathbb{1}_{\mathcal{S}_\lambda}, 0) , \quad (3.44a)$$

the multiplication

$$\begin{aligned} v_0 \otimes v_1 &:= f_1^2 \phi_2(v_0, v_1) = \phi_1(\pi(v_0)\pi(v_1)) , \\ (v_0, v_1, a_0) \otimes (v_2, v_3, a_1) &:= (v_0 \otimes v_2, v_1 \otimes v_3, a_0 + a_1 + \lambda^{1,2}(\phi_2(v_0, v_2), \phi_2(v_1, v_3))) , \end{aligned} \quad (3.44b)$$

the vertical composition

$$(v_0, v_1, a_0) \circ (v_1, v_2, a_1) := (v_0, v_2, a_0 + a_1 + \lambda^{2,1}(v_0, v_1, v_2)) , \quad (3.44c)$$

the unitors

$$l_v = (v, \mathbb{1}_{\mathcal{S}_\lambda} \otimes v, 0) = (v, \phi_1(\pi(v)), 0) , \quad r_v = (v, v \otimes \mathbb{1}_{\mathcal{S}_\lambda}, 0) = (v, \phi_1(\pi(v)), 0) , \quad (3.44d)$$

and associator

$$\begin{aligned} \mathbf{a}_{v_0, v_1, v_2} &= (f_1^2 f_2^3(\phi_3(v_0, v_1, v_2)) , f_1^2 f_1^3(\phi_3(v_0, v_1, v_2)) , \lambda^{0,3}(\phi_3(v_0, v_1, v_2))) \\ &= (v_0 \otimes v_1 \otimes v_2 , v_0 \otimes v_1 \otimes v_2 , \lambda^{0,3}(\phi_3(v_0, v_1, v_2))) , \end{aligned} \quad (3.44e)$$

where $v_{0,1,2,3} \in V_1$ and $a_{0,1} \in \mathbf{A}$, forms a weak Lie 2-group, which we denote by \mathcal{S}_λ^w .

Note that since the unitors are non-trivial, \mathcal{S}_λ^w is *not* a semistrict Lie 2-group in the sense of [42].

This description of the smooth string 2-group model \mathcal{S}_λ as its equivalent weak Lie 2-group counterpart \mathcal{S}_λ^w will simplify the explicit computations leading to the cocycle descriptions of principal \mathcal{S}_λ -bundles with connective structure in the next chapter. Therefore, in the next chapter we will work with the weak Lie 2-group \mathcal{S}_λ^w for computing the explicit cocycle and coboundary relations.

Chapter 4

Principal smooth 2-group bundles

This chapter is an expansion of the paper [30, Section 3-5].

Higher principal bundles have been extensively studied using the tools of homotopification and internalization, see [6, 92], which employ categorifying the ingredients of ordinary bundles. In particular, the group actions can be replaced by actions of categorical groups, whereas smooth maps are substituted by internal functors. This chapter aims to generalize the previously known terms and concepts of higher principal bundles by defining smooth 2-group bundles as internal functors in **Bibun**. The main objectives of this chapter are:

- to define smooth 2-group bundles as internal functors in **Bibun**, which generalize the known formalism of principal Lie 2-group bundles as internal functors in $\mathbf{Man}^\infty\mathbf{Cat}$, cf. [92, 42].
- to give the explicit cocycle and coboundary relations of principal \mathcal{S}_λ^w -bundles.

Here, we start by recalling some preliminary concepts and terms.

4.1 Preliminaries

In this section, we review principal bundles as objects of a slice category, cf. Appendix B. Let us begin our discussion by giving an overview on the category of G -objects $\mathbf{Man}^\infty//G$ from [45].

4.1.1 Category of \mathbf{G} -objects

For the category \mathbf{Man}^∞ and a Lie group \mathbf{G} , one can construct a new category $\mathbf{Man}^\infty//\mathbf{G}$, [45].

The category $\mathbf{Man}^\infty//\mathbf{G}$ has objects (X, \triangleright) , where X is an object in \mathbf{Man}^∞ and $\triangleright: \mathbf{G} \times X \rightarrow X$ is a smooth \mathbf{G} -action on X . A morphism $f: (Z, \triangleright) \rightarrow (X, \triangleright)$ between any two objects is an equivariant smooth map $f: Z \rightarrow X$ in \mathbf{Man}^∞ . That is, smooth maps subject to the commutativity of

$$\begin{array}{ccc}
 \mathbf{G} \times Z & \xrightarrow{\triangleright} & Z \\
 \text{id}_{\mathbf{G}} \times f \downarrow & & \downarrow f \\
 \mathbf{G} \times X & \xrightarrow{\triangleright} & X
 \end{array}$$

Figure 4.1: Morphisms between \mathbf{G} -spaces X and Z

Furthermore, the category $\mathbf{Man}^\infty//\mathbf{G}$ contains all objects of \mathbf{Man}^∞ , since for any smooth manifold X , one can define the trivial \mathbf{G} -action. We denote this by $\text{pr}_2: \mathbf{G} \times X \rightarrow X$. Moreover, under mild assumptions, the slice category¹ \mathbf{Man}^∞/X has finite products. Thus, we can also consider the group object $\mathbf{G}_X = (\mathbf{G} \times X \rightarrow X)$ in this category. This enables us to consider a principal \mathbf{G} -bundle P over X as a \mathbf{G} -space (P, \triangleright) together with a smooth map $\pi: (P, \triangleright) \rightarrow (X, \text{pr}_2)$ in $\mathbf{Man}^\infty//\mathbf{G}$. We illustrate this in the following subsection.

4.1.2 Ordinary principal bundles

Given a smooth manifold $X \in \mathbf{Man}^\infty$, the (generalized) bundles over X are objects in the slice category \mathbf{Man}^∞/X . That is, a generalized bundle is a smooth manifold P with a smooth morphism $P \rightarrow X$. See also [45] for the general discussion of the topic.

To obtain a principal bundle P over X with structure Lie group \mathbf{G} , we have to demand that there is a (principal) group action of \mathbf{G} on P and that the bundle is locally trivial with typical fiber \mathbf{G} . The first condition is implemented as follows. We

¹cf. Appendix B.

switch to \mathbf{G} -objects in $\mathbf{Man}^\infty//\mathbf{G}$ and it is clear that X together with the trivial \mathbf{G} -action is an object of this category. Then a principal \mathbf{G} -bundle consists of \mathbf{G} -objects (P, \triangleright) and (X, \mathbf{pr}_2) together with a morphism of \mathbf{G} -objects $\pi : (P, \triangleright) \rightarrow (X, \mathbf{pr}_2)$ and a diffeomorphism $(\triangleright, \mathbf{pr}_2) : \mathbf{G} \times P \rightarrow P \times_X P$ in \mathbf{Man}^∞/X . This is equivalently expressed as principal \mathbf{G}_X -bundles in \mathbf{Man}^∞/X by considering the group object $\mathbf{G}_X = (\mathbf{G} \times X \rightarrow X)$ and expressing P as \mathbf{G}_X -object $\pi : P \rightarrow X$ in \mathbf{Man}^∞/X .

To implement the second condition, we define a cover of X as a surjective submersion $\kappa \in \mathbf{Man}^\infty(Y, X)$. While not all pullbacks exist in \mathbf{Man}^∞ , those along surjective submersions do. For simplicity and for reasons of familiarity, let us restrict ourselves to ordinary covers $\kappa : U \twoheadrightarrow X$ given by a disjoint union of patches, $U := \sqcup_i U_i$. We then demand that the pullback bundle $\kappa^*P \rightarrow U$ is \mathbf{G} -equivariantly diffeomorphic to the bundle $\mathbf{G} \times U \rightarrow U$ as an object of \mathbf{Man}^∞/U .

We also need a description of the principal bundle P in terms of descent data or transition functions. For this, we use the \mathbf{G} -equivariant diffeomorphism $\rho_i : P|_{U_i} \rightarrow \mathbf{G} \times U_i$ to define transition functions. Note that the diffeomorphism is of the form $\rho_i(x) = (g_i(x), \pi(x))$ for $x \in \pi^{-1}(U_i)$. Then the expression

$$g_{ij}(x) := g_i^{-1}(x)g_j(x) , \quad (4.1a)$$

for $x \in \pi^{-1}(U_i \cap U_j)$ depends only on $\pi(x)$ since

$$g_i^{-1}(hx)g_j(hx) = g_i^{-1}(x)h^{-1}hg_j(x) = g_i^{-1}(x)g_j(x) . \quad (4.1b)$$

Thus we obtain a function $g_{ij} : U_i \cap U_j \rightarrow \mathbf{G}$, which satisfies the condition

$$g_{ij}g_{jk} = g_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset . \quad (4.1c)$$

Therefore, the (g_{ij}) form a Čech 1-cocycle with respect to the cover U . Similarly, one readily shows that diffeomorphic principal bundles P and P' subordinate to the same cover U are described by transition functions (g_{ij}) and (g'_{ij}) , which are related by $g_{ij}\gamma_j = \gamma_i g'_{ij}$, for some local smooth functions $\gamma_i : U_i \rightarrow \mathbf{G}$. The (γ_i) form the Čech coboundaries linking the Čech cocycles (g_{ij}) and (g'_{ij}) .

Alternatively, one can regard the principal bundle P as a functor from the Čech groupoid $U^{[2]} \rightarrow U$ with $U^{[2]} := \sqcup_{i,j} U_i \cap U_j$ to the Lie groupoid $\mathbf{BG} = (\mathbf{G} \rightrightarrows *)$. One readily sees that this functor is encoded in a Čech 1-cocycle² (g_{ij}) :

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{(g_{ij})} & \mathbf{G} \\ \Downarrow & & \Downarrow \\ U & \longrightarrow & * \end{array}$$

Figure 4.2: \mathbf{G} -valued Čech 1-cocycles encoding principal \mathbf{G} -bundle

Moreover, two functors corresponding to diffeomorphic principal bundles are related by a natural isomorphism, which in turn gives rise to a Čech coboundary.

In the following section we give a description of smooth 2-group bundles as objects of a slice weak 2-category, $\mathbf{Bibun}/\mathcal{X}$, for an object \mathcal{X} in \mathbf{Bibun} . See [71] for related discussions.

4.2 Smooth 2-group bundles as \mathcal{G} -objects

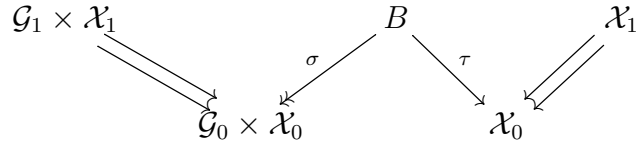
Let us now generalize the above discussion to the categorified setting. This yields higher principal bundles as special kinds of stacks, which were already defined in [71], and we recall the relevant definitions in the following. For a related approach, see also [8] and [9].

Note that a 2-space is a category internal to \mathbf{Man}^∞ and here, we restrict our attention to Lie groupoids, that is, groupoids internal to \mathbf{Man}^∞ . The 2-bundles over a 2-space \mathcal{X} are then simply elements of (a subcategory of) the slice weak 2-category $\mathbf{Bibun}/\mathcal{X}$, cf. [8]. This is in fact a more general approach of the previous subsection. Let us state first the definition of group objects and \mathcal{G} -stacks in the weak 2-category \mathbf{Bibun} , for a smooth 2-group \mathcal{G} .

For a smooth 2-group \mathcal{G} , we can trivially regard it as a 2-group object $\mathcal{G}_{\mathcal{X}}$ in $\mathbf{Bibun}/\mathcal{X}$ as follows³

²This description of principal bundles is understood due to [76].

³See Appendix B for definitions of slice bicategories.


 Figure 4.3: A smooth 2-group object \mathcal{G}_X

Similarly, we can define \mathcal{G} -stacks and \mathcal{G}_X -objects. See also C for the action of 2-group objects on the objects of the weak 2-category.

Definition 4.2.1 ([71]). *Given a smooth 2-group \mathcal{G} , a smooth \mathcal{G} -stack is a \mathcal{G} -object in Bibun , which is an object in the weak 2-category Bibun/\mathcal{G} , cf. Appendix C.*

We also define \mathcal{G} -stacks over other smooth stacks \mathcal{X} , which are the objects of Bibun .

Definition 4.2.2 ([71]). *A smooth 2-group over a smooth stack \mathcal{X} is a 2-group object in Bibun/\mathcal{X} . Given a 2-group object \mathcal{G}_X over a smooth stack \mathcal{X} , a smooth \mathcal{G}_X -stack over \mathcal{X} is a \mathcal{G}_X -object in Bibun/\mathcal{X} .*

Finally, let us impose the condition of local triviality to arrive at higher principal bundles. To this end, we need to introduce covers and discuss pullbacks to the patches in the covers. It will be sufficient for us to work with covering right principal bibundles arising from bundlization of 2-covers as defined in [6]. For a more general perspective, see [71]. Since the 2-spaces we want to cover are Lie groupoids, we demand that our cover is also a Lie groupoid $\mathcal{U} = (\mathcal{U}_1 \rightrightarrows \mathcal{U}_0)$ internal to Man^∞ , together with a functor $\kappa : \mathcal{U} \rightarrow \mathcal{X}$ such that the contained smooth maps $\kappa_1 : \mathcal{U}_1 \rightarrow \mathcal{X}_1$ and $\kappa_0 : \mathcal{U}_0 \rightarrow \mathcal{X}_0$ are surjective submersions, cf. Subsection 2.1.3. The bundlization of such a 2-cover gives rise to the right principal bibundle

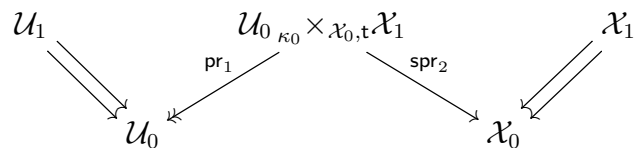


Figure 4.4: 2-covers of smooth stacks as bundlization

Pullbacks exist for surjective submersions, and thus they exist along the corresponding bundlizations, see Proposition 2.2.16, for the existence of pullbacks of right principal bibundles. For details, see also e.g. [8].

Definition 4.2.3. *Given a smooth 2-group \mathcal{G} over a smooth stack \mathcal{X} , a principal \mathcal{G} -bundle over \mathcal{X} is a smooth $\mathcal{G}_{\mathcal{X}}$ -stack \mathcal{P} over \mathcal{X} such that there exists a covering right principal bibundle $\kappa : \mathcal{U} \rightarrow \mathcal{X}$ with $\kappa^*(\mathcal{P})$ being \mathcal{G} -equivariantly equivalent to $\mathcal{U} \times \mathcal{G}$ as a smooth \mathcal{G} -stack over \mathcal{U} .*

Altogether we have the following picture:

$$\begin{array}{ccccccc}
 \mathcal{U}_1 \times \mathcal{G}_1 & & \kappa^* \mathcal{P}_1 & & \mathcal{P}_1 & & \mathcal{G}_1 \times \mathcal{X}_1 \\
 \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
 \mathcal{U}_0 \times \mathcal{G}_0 & \xleftarrow{B_{\text{eq}}} & \kappa^* \mathcal{P}_0 & \xleftarrow{B_{\kappa^*}} & \mathcal{P}_0 & \xleftarrow{B_{\kappa}} & \mathcal{G}_0 \times \mathcal{X}_0 \\
 \nwarrow & & \uparrow & \nearrow & \uparrow & & \nwarrow \\
 & & B_{\kappa^* \mathcal{P}} & \xrightarrow{\eta} & B_{\mathcal{P}} & & \\
 & & \downarrow & & \downarrow & & \\
 \mathcal{U}_1 & \xrightarrow{B_{\mathcal{U} \times \mathcal{G}}} & \mathcal{U}_0 & \xrightarrow{B_{\kappa}} & \mathcal{X}_0 & \xrightarrow{B_{\mathcal{G}_{\mathcal{X}}}} & \mathcal{X}_1
 \end{array}$$

Figure 4.5: The covering right principal bibundle B_{κ}

where $B_{\mathcal{P}}$ is a $\mathcal{G}_{\mathcal{X}}$ -object in $\mathbf{Bibun}/\mathcal{X}$, η is a bibundle isomorphism, B_{eq} is a bibundle equivalence and $B_{\mathcal{U} \times \mathcal{G}} \otimes B_{\text{eq}} \cong B_{\kappa^* \mathcal{P}}$.

Now let us work through two examples in somewhat more detail: ordinary principal \mathbf{G} -bundles and strict principal 2-bundles over a manifold X (considered as an object in \mathbf{Bibun}), where the structure 2-group is a crossed module of Lie groups. Let us begin with the ordinary principal bundles.

Here, we consider a principal bundle $\pi : P \rightarrow X$ with structure Lie group \mathbf{G} over a manifold X with cover $\kappa : U \rightarrow X$. We have an isomorphism⁴ $\rho_i : \mathbf{G} \times \sqcup_i U_i \rightarrow \kappa^* P$ such that $\pi \circ \rho_i$ is the obvious projection. To regard these as principal bundles in the sense of Definition 4.2.3, we first trivially extend the group object $\mathbf{G}_X = (\mathbf{G} \times X \rightarrow X)$ to a 2-group object over a Lie groupoid, by promoting $\mathbf{G} \times X$ and X to discrete groupoids $\mathcal{G} = (\mathbf{G} \times X \rightrightarrows \mathbf{G} \times X)$ and $\mathcal{X} = (X \rightrightarrows X)$. The projection on \mathbf{G}_X induces

⁴ $\kappa^* P$ is the pullback bundle over the cover κ .

an obvious functor between \mathcal{G} and \mathcal{X} , which we can bundlize to the following smooth 2-group over \mathcal{X} :

$$\begin{array}{ccccc}
 \mathbf{G} \times X & & \mathbf{G} \times X & & X \\
 \Downarrow & & \xleftarrow{=} & \xrightarrow{\text{pr}_2} & \Downarrow \\
 \mathbf{G} \times X & & & & X
 \end{array}$$

Figure 4.6: 2-group object in Bibun/\mathcal{X}

To obtain a covering bibundle B_κ of \mathcal{X} , we proceed similarly. We trivially extend a cover $\kappa : U \twoheadrightarrow X$ to the discrete 2-cover $(\kappa, \kappa) : (U \rightrightarrows U) \twoheadrightarrow (X \rightrightarrows X)$, and bundlize the result:

$$\begin{array}{ccccc}
 U & & U & & X \\
 \Downarrow & & \xleftarrow{=} & \xrightarrow{\kappa} & \Downarrow \\
 U & & & & X
 \end{array}$$

Figure 4.7: The bundlization of (κ, κ)

Similarly, all the other smooth maps can be bundlized to obtain the right principal bibundles of the corresponding Lie groupoid functors between discrete Lie groupoids. Hence, in Figure 4.5, η is trivial.

Now let us consider strict principal 2-bundles over X with strict structure Lie 2-group $\mathcal{G} = (\mathbf{G} \times \mathbf{H} \rightrightarrows \mathbf{G})$. Again, we promote X and its cover U to discrete Lie groupoids. Then we have the obvious Lie groupoid functor from $\mathcal{G} \times \mathcal{X}$ to \mathcal{X} , which we bundlize to obtain the right principal bibundle

$$\begin{array}{ccccc}
 (\mathbf{G} \times \mathbf{H}) \times X & & \mathbf{G} \times X & & X \\
 \Downarrow & & \xleftarrow{=} & \xrightarrow{\text{pr}_2} & \Downarrow \\
 \mathbf{G} \times X & & & & X
 \end{array}$$

Figure 4.8: Right principal $((\mathbf{G} \times \mathbf{H}) \times X \rightrightarrows X, X)$ -bibundle as a 2-group object over X

Moreover, as covering bibundle, we choose again Figure 4.7. Recall that a strict principal \mathcal{G} -bundle over X can be regarded as a 2-space \mathcal{P} fibered over X , whose

pullback along the cover is equivalent to the bundle $\mathcal{G} \times (U \rightrightarrows U)$, cf. e.g. [92]. Thus, bundlization allows us to obtain biprincipal bibundles, which also enables us to fill in all the remaining right principal bibundles in Figure 4.5 as bundlizations of the respective Lie groupoid functors.

4.3 Smooth 2-group bundles as internal functors

Here, we discuss principal smooth 2-group bundles as internal functors in \mathbf{Bibun} defined from the Čech groupoid (considered as internal category) to the smooth 2-group \mathcal{G} .

4.3.1 General remarks

For a smooth 2-group \mathcal{G} , we give a description of principal \mathcal{G} -bundles over an object \mathcal{X} of \mathbf{Bibun} in terms of generalized Čech cocycles.

Let $\kappa : \mathcal{U} \rightarrow \mathcal{X}$ be a covering right principal bundle of an object $\mathcal{X} \in \mathbf{Bibun}$. We can construct the Čech (2)-groupoid⁵ $\check{\mathcal{C}}(\mathcal{U})$ of $\mathcal{U}_1 \rightrightarrows \mathcal{U}_0$ as the category internal to \mathbf{Bibun} . Correspondingly, we construct $\mathbf{B}\mathcal{G}$ of the smooth 2-group \mathcal{G} as a category in \mathbf{Bibun} , cf. Subsection 3.3.2. We then have the following definition, which generalizes the usual Čech description of principal (2)-bundles.

Definition 4.3.1. *Let \mathcal{G} be a smooth 2-group and let \mathcal{X} be an object in \mathbf{Bibun} . A principal \mathcal{G} -bundle over \mathcal{X} subordinate to a cover \mathcal{U} of \mathcal{X} is a functor internal to \mathbf{Bibun} and defined from the Čech groupoid $\check{\mathcal{C}}(\mathcal{U})$ to the delooping $\mathbf{B}\mathcal{G}$ of the smooth structure 2-group \mathcal{G} . Moreover, two principal \mathcal{G} -bundles over \mathcal{X} subordinate to a cover \mathcal{U} are called equivalent, if there is a natural isomorphism between their corresponding functors.*

Altogether, we depict it by the following diagram:

⁵See [34] and references therein.

$$\begin{array}{ccccc}
 & & \Phi_1 & & \\
 & & \downarrow & & \\
 \mathcal{U}_{1,0} & \rightrightarrows & \mathcal{U}_{1,1} & \xrightarrow{\chi} & \mathcal{G}_0 \rightrightarrows \mathcal{G}_1 \\
 \downarrow s & & \downarrow t & & \downarrow \\
 \mathcal{U}_0 & \rightrightarrows & \mathcal{U}_1 & \xrightarrow[\Psi_0]{\Phi_0} & * \rightrightarrows * \\
 & & & & \downarrow \\
 & & & & \Psi_1
 \end{array}$$

 Figure 4.9: Principal \mathcal{G} -bundles as internal functors in **Bibun**

Here, Φ and Ψ are internal functors, while χ is an internal natural isomorphism and s and t are right principal bibundles⁶. We will not prove here how this definition is equivalent to Definition 4.2.3, it needs a future study.

As a particular example, one can consider the trivial case by taking the principal \mathcal{G} -bundle $\Phi^{\mathbb{1}}$ whose $\Phi_1^{\mathbb{1}}$ -component is given by the bundlization of the Lie groupoid functor which maps all of $\mathcal{U}_{1,0}$ to $\mathbb{1}_{\mathcal{G}} \in \mathcal{G}_0$ and all of $\mathcal{U}_{1,1}$ to $\text{id}_{\mathbb{1}_{\mathcal{G}}} \in \mathcal{G}_1$, hence, we have the following definition.

Definition 4.3.2. *A trivial principal \mathcal{G} -bundle is a principal \mathcal{G} -bundle which is equivalent to the principal \mathcal{G} -bundle $\Phi^{\mathbb{1}}$.*

Now let us explain how ordinary principal bundles over a smooth manifold X with structure Lie group \mathbf{G} fits into this definition.

4.3.2 Example: Ordinary principal \mathbf{G} -bundles

If \mathcal{X} is the discrete Lie groupoid $X \rightrightarrows X$, then we can also choose the cover \mathcal{U} to be discrete. In this case, the smooth right principal bibundles s and t collapse to smooth maps between $\mathcal{U}_{1,0}$ and \mathcal{U}_0 . Moreover, the Čech groupoid $\check{\mathcal{C}}(\mathcal{U})$ can be reduced to the Čech groupoid of an ordinary cover $\mathcal{U}_0 = U = \sqcup_i U_i$ of X and the composition of compatible elements in $\mathcal{U}_{1,0} = U^{[2]} = \sqcup_{i,j} U_i \cap U_j$ is the (bundlization of) the usual composition of double overlaps. The smooth 2-group \mathcal{G} is now the discrete Lie groupoid $\mathbf{G} \rightrightarrows \mathbf{G}$ with the multiplication right principal bibundle B_m , which is the bundlization of the Lie group multiplication, trivially lifted to a Lie groupoid

⁶In principle, the maps $(\mathcal{G}_1 \rightrightarrows \mathcal{G}_0) \rightrightarrows (* \rightrightarrows *)$ are also given by bibundles, but since the target is trivial, they collapse to trivial maps.

functor. Given this initial data, the right principal bibundles contained in Φ and Ψ reduce to smooth maps $(g_{ij}) : U^{[2]} \rightarrow \mathbf{G}$. Their composition with multiplication appearing in the Figure 3.6 is encoded by the right principal bibundles

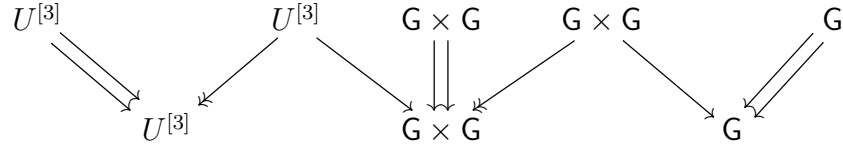


Figure 4.10: Bibundle multiplication between the bundlizations of $(g_{ij} \times g_{ik})$ and B_m

with

$$U^{[3]} := \sqcup_{i,j,k} U_{ij} \times_M U_{jk} = \sqcup_{i,j,k} U_i \cap U_j \cap U_k . \quad (4.2)$$

Notice that, here the second right principal bibundle is the bundlization of the Lie group multiplication. This gives the right principal bibundle

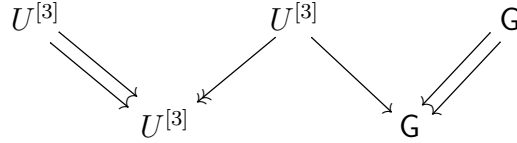


Figure 4.11: The right principal bibundle $B_m \otimes (g_{ij} \times g_{jk})$

Altogether, we recover the usual Čech cocycles encoding transition functions of a principal \mathbf{G} -bundle over X subordinate to the cover U :⁷

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x) , \quad x \in U_i \cap U_j \cap U_k . \quad (4.3)$$

Analogously, we can derive the coboundary relation by considering the bundlization of the smooth map $(g_i) : (U_i) \rightarrow \mathbf{G}$. Thus, if (g_{ij}) and (g'_{ij}) are the cocycles corresponding to the functors Φ and Ψ , then we have

$$g_i(x)g'_{ij}(x) = g_{ij}(x)g_j(x) . \quad (4.4)$$

Hence, the (g_i) form a Čech coboundary.

⁷Note that (4.3) is due to the fact that the bibundle isomorphism $\Phi_{2,m}$ is trivial.

4.3.3 Example: Strict principal 2-bundles

As a preparation for discussing principal 2-bundles with the smooth 2-group model \mathcal{S}_λ of the string group as their structure 2-group, let us also go through the explicit derivation of the cocycle and coboundary relations for the case of principal 2-bundles with strict structure 2-group using Definition 4.3.1.

The relevant 2-cover is again derived from the Čech groupoid of the underlying smooth manifold X as a category internal to **Bibun** as done in the previous subsection. The structure 2-group is given by a strict Lie 2-group $\mathcal{G} = ((\mathbf{G} \times \mathbf{H}) \rightrightarrows \mathbf{G})$, which is regarded as a category $(\mathcal{G} \rightrightarrows *)$ internal to **Bibun** with the right principal bibundle B_c being the monoidal product in the strict Lie 2-group \mathcal{G} , cf. Subsection 3.3.3.

In this case, the right principal bibundle Φ_1 (and Ψ_1) no longer collapses straightforwardly. To simplify the discussion, we assume that the cover U is sufficiently fine so that $U^{[2]} = \sqcup_{i,j} U_{ij}$ is contractible. Then the right principal bibundle Φ_1 reads as

$$\begin{array}{ccccc}
 U^{[2]} & & \Phi_1 = U^{[2]} \times \mathbf{H} & & \mathbf{G} \times \mathbf{H} \\
 \swarrow & & \searrow \sigma & & \swarrow \tau \\
 & & U^{[2]} & & \mathbf{G} \\
 \searrow & & & & \swarrow \\
 & & & & \mathbf{G}
 \end{array}$$

Figure 4.12: The right principal bibundle Φ_1

where σ is the projection and the smooth right action is defined by

$$(i, j, x, h_1)(g, h_2) := (i, j, x, h_1 h_2) . \quad (4.5)$$

Here, the right principal bibundle Φ_1 is now necessarily trivial over $U^{[2]}$ and therefore diffeomorphic to a bundlization, cf. Subsection 2.2.2. Instead of using this fact, let us come to this conclusion by explicitly working through the details.

Note that τ is fully fixed by its image of elements $(i, j, x, \mathbb{1}_{\mathbf{H}}) \in U^{[2]} \times \mathbf{H}$, because the right-action fixes the remaining part of τ . In particular,

$$(i, j, x, h) = (i, j, x, \mathbb{1}_{\mathbf{H}}) (\tau(i, j, x, \mathbb{1}_{\mathbf{H}}), h) , \quad (4.6a)$$

and thus by (i) of the definition of Lie groupoid right-action in Subsection 2.1.4, we have

$$\tau(i, j, x, h) = \mathfrak{s}(\tau(i, j, x, \mathbb{1}_H), h) = \mathfrak{t}(h)\tau(i, j, x, \mathbb{1}_H) . \quad (4.6b)$$

We therefore define

$$g_{ij}(x) := \tau(i, j, x, \mathbb{1}_H) , \quad (4.6c)$$

implying⁸ $\tau(i, j, x, h) = \mathfrak{t}(h)g_{ij}(x)$.

Altogether, we see that the right principal bibundle Φ_1 is simply the bundlization of the Lie groupoid functor

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{(g_{ij}, \mathbb{1}_H)} & \mathbf{G} \ltimes \mathbf{H} \\ \Downarrow & & \Downarrow \\ U^{[2]} & \xrightarrow{g_{ij}} & \mathbf{G} \end{array}$$

Figure 4.13: The Lie groupoid functor inducing the right principal bibundle Φ_1

This is to be expected as the right principal bibundle is trivial over $U^{[2]}$ and the smooth map σ therefore allows for global section.

Let us now consider the appropriate version of the second diagram in Figure 3.6, which encodes (weak) compatibility of the internal functor with the bibundle multiplication. We depict it as

$$\begin{array}{ccccc} & & U^{[3]} \times \mathbf{H} & & \\ & \swarrow \sigma & \downarrow \chi & \searrow \tau \otimes \tau & \\ U^{[3]} & \xrightarrow{\quad} & U^{[3]} & \xrightarrow{\quad} & \mathbf{G} \xleftarrow{\quad} \mathbf{G} \ltimes \mathbf{H} \\ & \swarrow \sigma & \downarrow & \searrow \tau \circ \text{pr}_{13} & \\ & & U^{[3]} \times \mathbf{H} & & \end{array}$$

Figure 4.14: The bibundle isomorphism χ

where $U^{[3]} = \sqcup_{i,j,k} U_{ijk}$. Now the smooth map χ is compatible with the principal smooth right-action and the projections σ , thus it is fully determined by the

⁸Here, we again abuse of notation by using \mathfrak{t} for both the target map and the crossed module homomorphism $\mathbf{H} \xrightarrow{\mathfrak{t}} \mathbf{G}$.

component $h : U^{[3]} \rightarrow \mathbf{H}$ defined implicitly according⁹ to

$$\chi(i, j, k, x, \mathbb{1}_{\mathbf{H}}) = (i, j, k, x, h_{ijk}^{-1}(x)) , \quad (i, j, k, x, \mathbb{1}_{\mathbf{H}}) \in U^{[3]} \times \mathbf{H} , \quad (4.7)$$

where we chose to invert h_{ijk} for consistency with conventions e.g. in [92]. The condition that $\tau \otimes \tau = (\tau \circ \mathbf{pr}_{13}) \circ \chi$ then directly translates into the expression

$$\mathbf{t}(h_{ijk}(x))g_{ij}(x)g_{jk}(x) = g_{ik}(x) , \quad x \in U_{ijk} . \quad (4.8)$$

Also, the coherence axiom in Figure 3.7 amounts to

$$\chi_{ikl} \circ (\chi_{ijk} \otimes \mathbf{id}_{\Phi_{kl}}) = \chi_{ijl} \circ (\mathbf{id}_{\Phi_{ij}} \otimes \chi_{jkl}) , \quad (4.9)$$

where the restrictions of $\chi : U^{[3]} \times \mathbf{H} \rightarrow U^{[3]} \times \mathbf{H}$ to

$$\chi_{ijk} : U_{ijk} \times \mathbf{H} \rightarrow U_{ijk} \times \mathbf{H} , \quad (4.10a)$$

and of $\Phi : (U^{[2]} \rightrightarrows U^{[2]}) \rightarrow (\mathbf{G} \times \mathbf{H} \rightrightarrows \mathbf{G})$ to

$$\Phi_{ij} : (U_{ij} \rightrightarrows U_{ij}) \rightarrow (\mathbf{G} \times \mathbf{H} \rightrightarrows \mathbf{G}) \quad (4.10b)$$

are used¹⁰. Now evaluating (4.9) on $(i, j, k, l, x, \mathbb{1}_{\mathbf{H}})$ using the formulas (3.24) and (4.7), we obtain the relation

$$h_{ikl}h_{ijk} = h_{ijl}(g_{ij} \triangleright h_{jkl}) . \quad (4.11)$$

Equations (4.8) and (4.11) are the usual cocycle relations for a principal 2-bundle with strict structure 2-group, cf. [68, 92].

Similarly, we can derive the coboundary relations by using the same argument as the construction of the cocycle relations discussed above. Here we give an overview of the procedure.

⁹Hence, $\chi(i, j, k, x, h) := \chi(i, j, k, x, \mathbb{1}_{\mathbf{H}})(\mathbb{1}_{\mathbf{G}}, h)$.

¹⁰Thus, $\mathbf{id}_{\Phi_{ij}}$ is the bibundle isomorphism induced by the trivial natural transformation $\mathbf{id}_{g_{ij}} = (g_{ij}, \mathbb{1}_{\mathbf{H}})$.

Given two such cocycles (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) , we can consider an internal natural transformation between them. Such a natural transformation χ is encoded in a right principal bibundle B from $U \rightrightarrows U$ to $\mathbf{G} \ltimes \mathbf{H} \rightrightarrows \mathbf{G}$ and a bibundle isomorphism χ as shown below.

$$\begin{array}{ccccc}
 & & B_m \otimes (\Phi_1, (\mathfrak{t} \otimes B)) & & \\
 & \swarrow & \parallel \chi & \searrow^{\tau \otimes \tau} & \\
 U^{[2]} \rightrightarrows U^{[2]} & & & & \mathbf{G} \leftarrow \mathbf{G} \ltimes \mathbf{H} \\
 & \nwarrow & \downarrow & \nearrow_{\tau \otimes \tau} & \\
 & & B_m \otimes ((\mathfrak{s} \otimes B), \Psi_1) & &
 \end{array}$$

Figure 4.15: Internal natural transformation χ

Here as usual, B_m is the bundlization of the monoidal multiplication in the strict 2-group $\mathbf{G} \ltimes \mathbf{H} \rightrightarrows \mathbf{G}$ and we use again the standard notation

$(B_1, B_2) := (B_1 \times B_2) \otimes \Delta$, where $\Delta : \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ is the appropriate diagonal right principal bibundle. Following arguments analogous to those given above, B is diffeomorphic to $U \times \mathbf{H}$ and the smooth map $\tau : U \times \mathbf{H} \rightarrow \mathbf{G}$ is fully determined by the smooth maps $\gamma_i(x) := \tau(i, x, \mathbb{1}_{\mathbf{H}})$. Moreover, the right principal bibundles related by χ are diffeomorphic to $U^{[2]} \times \mathbf{H}$, and the diffeomorphism χ is fixed by the smooth maps $\chi_{ij}(i, j, x, \mathbb{1}_{\mathbf{H}}) := (i, j, x, h_{ij}^{-1})$, where $h : U^{[2]} \rightarrow \mathbf{H}$. Therefore, by evaluating $\tau \otimes \tau = (\tau \otimes \tau) \circ \chi$ at $(i, j, x, \mathbb{1}_{\mathbf{H}})$, we obtain

$$\gamma_i g'_{ij} = \mathfrak{t}(h_{ij}) g_{ij} \gamma_j. \quad (4.12a)$$

Moreover, the commutative diagram in Figure 3.10 simplifies a bit, because all the associators are trivial. Therefore, by evaluating the bibundle isomorphisms in the expression

$$\chi_{ik} \circ (\chi_{ijk} \otimes \text{id}_{\gamma_k}) = (\text{id}_{\gamma_i} \otimes \chi'_{ijk}) \circ (\chi_{ij} \otimes \text{id}_{\Psi_{jk}}) \circ (\text{id}_{\Phi_{ij}} \otimes \chi_{jk}) \quad (4.12b)$$

at $(i, j, k, x, \mathbb{1}_H) \in U^{[3]} \times H$, we obtain

$$h_{ik}h_{ijk} = (\gamma_i \triangleright h'_{ijk})h_{ij}(g_{ij} \triangleright h_{jk}) . \quad (4.12c)$$

Equations (4.12a) and (4.12c) are the usual coboundary relations for a principal 2-bundle with strict structure 2-group, as found e.g. in [6, 68].

4.3.4 Cocycle description of principal string 2-group bundles

For the differentiation of the string 2-group model \mathcal{S}_λ , we need descent data for principal \mathcal{S}_λ -bundles in terms of Čech cocycles and Čech coboundaries. Let us develop these in the following. We restrict ourselves to principal \mathcal{S}_λ -bundles over ordinary manifolds subordinate to a good cover U , and consequently, the covering groupoid $\mathcal{U} = (U \rightrightarrows U)$ is discrete. Following the discussion in Subsection 4.3.1, we start from the diagram

$$\begin{array}{ccccc}
 & & \Phi_1 & & \\
 & & \curvearrowright & & \\
 U^{[2]} & \rightrightarrows & U^{[2]} & \xrightarrow{\chi} & \mathcal{S}_\lambda \\
 & & \downarrow \Psi_1 & & \downarrow \\
 U & \rightrightarrows & U & \xrightarrow{\Phi_0} & * \leftarrow * \\
 & & \downarrow \Psi_0 & & \\
 & & & &
 \end{array}$$

s t

Figure 4.16: Principal \mathcal{S}_λ -bundles as internal functors in Bibun

where $\mathcal{S}_\lambda \rightrightarrows (* \leftarrow *)$ is a category internal to Bibun with multiplication given by the right principal bibundle B_m . The information contained in the functors Φ and Ψ as well as in the natural isomorphism χ , together with the coherence conditions will yield the appropriate generalization of Čech cochains, cocycles and coboundaries describing principal \mathcal{S}_λ -bundles and their isomorphisms.

Now recall that since U is a good cover, $U^{[2]}$ is contractible. This implies that the right principal bibundles Φ_1 and Ψ_1 are both trivial bundles over $U^{[2]}$ admitting a global smooth section. By Proposition 2.2.11, this implies that Φ and Ψ are

isomorphic to bundlizations of some Lie groupoid functors. Moreover, because of Proposition 2.2.10, the bibundle map χ can be given by a smooth natural transformation between the corresponding Lie groupoid functors. The only right principal bibundle which is not a bundlization here is the multiplication B_m . It appears in the coherence diagrams for internal functors in Figure 3.7 and for internal natural transformations in Figure 3.10 with $B_c = B_m$.

An explicit evaluation of the composition of bibundle isomorphisms in Figure 3.7 is rather cumbersome. To simplify our discussion, we therefore choose to switch to the weak Lie 2-group model \mathcal{S}_λ^w of the string 2-group model given in Section 3.4.3. Our principal 2-bundles will therefore be weak principal 2-bundles in the sense of [42], which are given by weak 2-functors internal to $\mathbf{Man}^\infty\mathbf{Cat}$ from the Čech 2-groupoid to the delooping of \mathcal{S}_λ^w . From there, we also recall the following proposition:

Proposition 4.3.3 ([42], Prop. 3.15). *Every weak principal 2-bundle Φ is equivalent to its normalization, which is given by a normalized weak functor:*

$$\Phi_{ii} := \mathbb{1}_{\mathcal{S}_\lambda} , \quad (4.13)$$

$$\chi_{iij} := \mathbb{l}_{\Phi_{ij}} , \quad \chi_{ijj} := \mathbb{r}_{\Phi_{ij}} , \quad (4.14)$$

which maps to the unit \mathcal{S}_λ^w on $U_i \cap U_i$ and to the left and right unitors \mathbb{l} and \mathbb{r} on $U_i \cap U_i \cap U_j$ and $U_i \cap U_j \cap U_j$.

We will give the consequences of this proposition below; for more details, see [42]. Using these, we arrive at the following theorem.

Theorem 4.3.4. *The functor Φ defining a (normalized) principal \mathcal{S}_λ^w -bundle is described by a 1-cochain $(v_{ij}) \in \mathcal{C}^\infty(U^{[2]}, V_1)$ together with a 2-cochain (v_{ijk}) in $\mathcal{C}^\infty(U^{[3]}, V_1^{[2]} \times \mathbf{A})$ such that*

$$v_{ijk} = (v_{ik}, v_{ij} \otimes v_{jk}, a_{ijk}) , \quad v_{ii} = \mathbb{1}_{\mathcal{S}_\lambda} , \quad v_{iij} = \mathbb{l}_{v_{ij}} , \quad v_{ijj} = \mathbb{r}_{v_{ij}} , \quad (4.15a)$$

$$\begin{aligned}
 & a_{ikl} + a_{ijk} + \lambda^{1,2}(\phi_2(v_{ik}, v_{kl}), \phi_2(v_{ij} \otimes v_{jk}, v_{kl})) \\
 & = a_{ijl} + a_{jkl} + \lambda^{1,2}(\phi_2(v_{ij}, v_{jl}), \phi_2(v_{ij}, v_{jk} \otimes v_{kl})) + \lambda^{0,3}(\phi_3(v_{ij}, v_{jk}, v_{kl})) ,
 \end{aligned} \tag{4.15b}$$

where $a_{ijk} \in \mathcal{C}^\infty(U^{[3]}, \mathbf{A})$. We call the data (v_{ij}, a_{ijk}) a degree-2 Čech cocycle over the cover U with values in \mathcal{S}_λ^w .

Proof. The expression in (4.15a) is readily derived from (v_{ijk}) encoding the natural isomorphism¹¹

$$\Phi_{2,ijk} : \Phi_{1,ij} \otimes \Phi_{1,jk} \Rightarrow \Phi_{1,ik} , \tag{4.16}$$

cf. Figure 3.6 together with Proposition 4.3.3. The coherence axioms of this natural isomorphism read as

$$\begin{aligned}
 & v_{ikl} \circ (v_{ijk} \otimes \text{id}_{v_{kl}}) = v_{ijl} \circ (\text{id}_{v_{ij}} \otimes v_{jkl}) \circ \mathbf{a}_{v_{ij}, v_{jk}, v_{kl}} , \\
 & v_{ijj} \circ (\text{id}_{v_{ij}} \otimes \text{id}_{\mathbb{1}_{\mathcal{S}_\lambda}}) = \mathbf{r}_{v_{ij}} \quad \text{and} \quad v_{iij} \circ (\text{id}_{\mathbb{1}_{\mathcal{S}_\lambda}} \otimes \text{id}_{v_{ij}}) = \mathbf{l}_{v_{ij}} ,
 \end{aligned} \tag{4.17}$$

cf. Figure 3.7. Now in (4.17), the last two equations are identities, cf. (3.44d). And the first expression reduces to

$$\begin{aligned}
 & (v_{il}, (v_{ij} \otimes v_{jk}) \otimes v_{kl}, a_{ikl} + a_{ijk} + \lambda^{1,2}(\phi_2(v_{ik}, v_{kl}), \phi_2(v_{ij} \otimes v_{jk}, v_{kl})) + \\
 & + \lambda^{2,1}(v_{il}, v_{ik} \otimes v_{kl}, (v_{ij} \otimes v_{jk}) \otimes v_{kl})) \\
 & = (v_{il}, (v_{ij} \otimes v_{jk}) \otimes v_{kl}, a_{ijl} + a_{jkl} + \lambda^{1,2}(\phi_2(v_{ij}, v_{jl}), \phi_2(v_{ij}, v_{jk} \otimes v_{kl})) + \\
 & + \lambda^{2,1}(v_{il}, v_{ij} \otimes v_{jl}, v_{ij} \otimes (v_{jk} \otimes v_{kl})) + \lambda^{0,3}(\phi_3(v_{ij}, v_{jk}, v_{kl})) + \\
 & + \lambda^{2,1}(v_{il}, v_{ij} \otimes (v_{jk} \otimes v_{kl}), (v_{ij} \otimes v_{jk}) \otimes v_{kl})) .
 \end{aligned} \tag{4.18}$$

The identities of Corollary 3.4.3 together with the fact that we are working with normalized cocycles cause $\lambda^{2,1}$ to drop out of (4.18). The remaining non-trivial part of this equation then yields the equation on the 2-cochain (a_{ijk}) . \square

It is now similarly straightforward to describe the natural 2-isomorphism χ given the coboundary relations between two Čech 2-cocycles.

Theorem 4.3.5. *The natural isomorphism $\chi : \Phi \Rightarrow \Psi$ giving an equivalence relation between (normalized) principal \mathcal{S}_λ^w -bundles Φ and Ψ described by 2-cocycles (v_{ij}, a_{ijk})*

¹¹These are notations from the definition of internal natural functors, see Figure 3.6.

and (v'_{ij}, a'_{ijk}) is captured by 0-cochain $s(\beta_i) \in \mathcal{C}^\infty(U, V_1)$ and 1-cochains $(\beta_{ij}) \in \mathcal{C}^\infty(U^{[2]}, V_1^{[2]} \times \mathbf{A})$ such that

$$\beta_{ij} = (\beta_i \otimes v'_{ij}, v_{ij} \otimes \beta_j, \alpha_{ij}) , \quad \beta_{ii} = r_{\beta_i}^{-1} \circ l_{\beta_i} = \text{id}_{\phi_1(\pi(\beta_i))} , \quad (4.19a)$$

$$\begin{aligned} & \alpha_{ik} + a_{ijk} + \lambda^{1,2}(\phi_2(v_{ik}, \beta_k), \phi_2(v_{ij} \otimes v_{jk}, \beta_k)) \\ &= \alpha_{ij} + a'_{ijk} + \alpha_{jk} + \lambda^{1,2}(\phi_2(\beta_i, v'_{ik}), \phi_2(\beta_i, v'_{ij} \otimes v'_{jk})) \\ & \quad + \lambda^{0,3}(\beta_i, v'_{ij}, v'_{jk}) - \lambda^{0,3}(v_{ij}, \beta_j, v'_{jk}) + \lambda^{0,3}(v_{ij}, v_{jk}, \beta_k) , \end{aligned} \quad (4.19b)$$

where $\alpha_{ij} \in \mathcal{C}^\infty(U^{[2]}, \mathbf{A})$. We call the data (β_i, α_{ij}) a degree-2 Čech coboundary over the cover U with values in \mathcal{S}_λ^w .

Proof. The first expression in (4.19a) can be directly obtained from the defining diagram for β_{ij} in Figure 3.9. The coherence axioms then read as

$$\begin{aligned} \beta_{ik} \circ (v_{ijk} \otimes \text{id}_{\beta_k}) &= (\text{id}_{\beta_i} \otimes v'_{ijk}) \circ a_{\beta_i, v'_{ij}, v'_{jk}} \circ (\beta_{ij} \otimes \text{id}_{v'_{jk}}) \circ \\ & \quad \circ a_{v_{ij}, \beta_j, v'_{jk}}^{-1} \circ (\text{id}_{v_{ij}} \otimes \beta_{jk}) \circ a_{v_{ij}, v_{jk}, \beta_k} , \end{aligned} \quad (4.20a)$$

$$\beta_{ii} \circ (\text{id}_{1_{\mathcal{S}_\lambda}} \otimes \text{id}_{\beta_i}) = (\text{id}_{\beta_i} \otimes \text{id}_{1_{\mathcal{S}_\lambda}}) \circ r_{\beta_i}^{-1} \circ l_{\beta_i} ,$$

cf. Figure 3.10 and 3.11. Here, the second expression in (4.20a) yields the identity

$$\begin{aligned} \beta_{ii} = r_{\beta_i}^{-1} \circ l_{\beta_i} &= (\phi_1(\pi(\beta_i)), \phi_1(\pi(\beta_i)), \lambda^{2,1}(\phi_1(\pi(\beta_i)), \beta_i, \phi_1(\pi(\beta_i)))) \\ &= \text{id}_{\phi_1(\pi(\beta_i))} . \end{aligned} \quad (4.20b)$$

Moreover, the part in $V_1^{[2]}$ of the first expression in (4.20a) also yields an identity.

But, the A-component gives

$$\begin{aligned}
 & \alpha_{ik} + a_{ijk} + \lambda^{1,2}(\phi_2(v_{ik}, \beta_k), \phi_2(v_{ij} \otimes v_{jk}, \beta_k)) \\
 & \quad + \lambda^{2,1}(\beta_i \otimes v'_{ik}, v_{ik} \otimes \beta_k, (v_{ij} \otimes v_{jk}) \otimes \beta_k) \\
 & = \alpha_{ij} + a'_{ijk} + \alpha_{jk} + \lambda^{1,2}(\phi_2(\beta_i, v'_{ik}), \phi_2(\beta_i, v'_{ij} \otimes v'_{jk})) + \lambda^{0,3}(\beta_i, v'_{ij}, v'_{jk}) \\
 & \quad + \lambda^{1,2}(\phi_2(\beta_i \otimes v'_{ij}, v'_{jk}), \phi_2(v_{ij} \otimes \beta_j, v'_{jk})) - \lambda^{0,3}(v_{ij}, \beta_j, v'_{jk}) \\
 & \quad + \lambda^{1,2}(\phi_2(v_{ij}, \beta_j \otimes v'_{jk}), \phi_2(v_{ij}, v_{jk} \otimes \beta_k)) + \lambda^{0,3}(v_{ij}, v_{jk}, \beta_k) + \\
 & \quad + \lambda^{2,1}(\beta_i \otimes v'_{ik}, \beta_i \otimes (v'_{ij} \otimes v'_{jk}), (\beta_i \otimes v'_{ij}) \otimes v'_{jk}) \\
 & \quad + \lambda^{2,1}(\beta_i \otimes (v'_{ij} \otimes v'_{jk}), (\beta_i \otimes v'_{ij}) \otimes v'_{jk}, (v_{ij} \otimes \beta_j) \otimes v'_{jk}) \\
 & \quad + \lambda^{2,1}((\beta_i \otimes v'_{ij}) \otimes v'_{jk}, (v_{ij} \otimes \beta_j) \otimes v'_{jk}, v_{ij} \otimes (\beta_j \otimes v'_{jk})) \\
 & \quad + \lambda^{2,1}((v_{ij} \otimes \beta_j) \otimes v'_{jk}, v_{ij} \otimes (\beta_j \otimes v'_{jk}), v_{ij} \otimes (v_{jk} \otimes \beta_k)) \\
 & \quad + \lambda^{2,1}(v_{ij} \otimes (\beta_j \otimes v'_{jk}), v_{ij} \otimes (v_{jk} \otimes \beta_k), (v_{ij} \otimes v_{jk}) \otimes \beta_k) .
 \end{aligned} \tag{4.21}$$

Just as in the proof of Theorem 4.3.4, terms containing $\lambda^{2,1}$ drop out due to identities from Corollary 3.4.3. The same is true for the third and fourth term containing $\lambda^{1,2}$. The remaining part is then the coboundary condition on the α_{ij} . \square

The above two theorems give us the cocycle and coboundary relations of principal \mathcal{S}_λ^w -bundles defined as internal functor in **Bibun**.

Moreover, in Chapter 2, we have mentioned a result on the biequivalence between the two bicategories **Bibun** and **Gen**. Consequently, in the next section, we will give a new approach of describing principal smooth 2-group bundles as generalized internal functors.

4.4 Comments on principal smooth 2-group bundles as generalized functors

Recall that a principal **G**-bundle over a smooth manifold X can be described by Lie groupoid generalized morphisms (in the sense of [94]) from the discrete Lie groupoid

$X \rightrightarrows X$ to $\mathbf{G} \rightrightarrows *$. Thus, if $U \hookrightarrow X$ is a good cover, then we have

$$(X \rightrightarrows X) \xleftarrow{\kappa} \check{C}(U) \xrightarrow{(g_{ij}, \mathbb{1}_{\mathbf{G}})} (\mathbf{G} \rightrightarrows *) \quad (4.22)$$

in $\mathbf{LieGrpd}$. Note that this idea can be generalized for principal smooth 2-group bundles by considering the so called generalized internal functors in \mathbf{Bibun} . See Appendix D for the definition of generalized internal functors. That is a smooth 2-group principal \mathcal{G} -bundle over a Lie groupoid might be described by using internal functors in \mathbf{Bibun} . In [34], they have defined principal strict Lie 2-group bundles over Lie groupoids as generalized Lie 2-groupoid morphisms, which are a special type of generalized internal functors in \mathbf{Bibun} , cf. Subsection 3.2.2.

Therefore, using generalized internal functors gives us a more general description. See Appendix D and [34] for the relation between Lie 2-groupoid generalized morphisms and generalized internal functors.

Here, we only give a comment without any further explanation. The detail discussion of the topic is beyond the scope of this thesis, and has to be postponed to future research. Furthermore, it is also logical to ask how we can establish equivalences between the definitions of principal smooth 2-group bundles in this chapter.

Chapter 5

Differentiation and gauge theory

Basically, this chapter is taken from [30, Section 5].

Here, we differentiate the weak Lie 2-group \mathcal{S}_λ^w to the known string Lie 2-algebra based on [78]. See also [42] for the differentiation of semistrict Lie 2-groups. Then using the aforementioned literature, we shall show that the functor from the category of smooth manifolds \mathbf{Man}^∞ to the category of \mathcal{S}_λ^w -valued descent data on surjective submersions of $\mathbb{R}^{0|1} \times X \longrightarrow X$ is parametrized by elements of an NQ-manifold, which corresponds to the string Lie 2-algebra. The adjoint of this functor can be viewed as the integration functor, which was first discussed in [77]. Furthermore, we present the full description of principal \mathcal{S}_λ^w -bundles with connective structures by establishing the Maurer-Cartan forms, and the non-abelian Deligne cohomology. Using this, we present the kinematical data of higher gauge theory on this principal 2-bundle. We begin the chapter by recalling some preliminary terms from [2] and [48].

5.1 Preliminaries

Let us review some background materials from [2] and [48]. In particular, we present definitions and examples of 2-term L_∞ -algebras and their correspondence with Lie 2-algebras and NQ-manifolds.

5.1.1 L_∞ -algebras and the string Lie 2-algebra

In order to discuss the differentiation of \mathcal{S}_λ^w to the string Lie 2-algebra, first let us define 2-term L_∞ -algebras and their equivalent counterpart Lie 2-algebras.

Recall that a 2-term L_∞ -algebra L is a 2-term complex of graded vector spaces

$$\mathfrak{a} \xrightarrow{\mu_1} \mathfrak{b} , \quad (5.1a)$$

together with the graded linear maps

$$\mu_1 : \mathfrak{a} \rightarrow \mathfrak{b} , \quad \mu_2 : \mathfrak{b} \wedge \mathfrak{b} \rightarrow \mathfrak{b} , \quad \mu_2 : \mathfrak{a} \wedge \mathfrak{b} \rightarrow \mathfrak{a} , \quad \mu_3 : \mathfrak{b} \wedge \mathfrak{b} \wedge \mathfrak{b} \rightarrow \mathfrak{a} , \quad (5.1b)$$

(of degree $-1, 0, +1$, respectively) satisfying the following higher homotopy Jacobi identities:

$$\mu_1(\mu_2(b, a)) = \mu_2(b, \mu_1(a)) , \quad (5.2a)$$

$$\mu_2(\mu_1(a_1), a_2) = \mu_2(a_1, \mu_1(a_2)) , \quad \mu_2(a, b) = -\mu_2(b, a) , \quad (5.2b)$$

$$\begin{aligned} \mu_1(\mu_3(b_1, b_2, b_3)) &= -\mu_2(\mu_2(b_1, b_2), b_3) - \mu_2(\mu_2(b_3, b_1), b_2) \\ &\quad - \mu_2(\mu_2(b_2, b_3), b_1) , \end{aligned} \quad (5.2c)$$

$$\begin{aligned} \mu_3(\mu_1(a), b_1, b_2) &= -\mu_2(\mu_2(b_1, b_2), a) - \mu_2(\mu_2(a, b_1), b_2) \\ &\quad - \mu_2(\mu_2(b_2, a), b_1) , \end{aligned} \quad (5.2d)$$

$$\begin{aligned}
 & \mu_2(\mu_3(b_1, b_2, b_3), b_4) - \mu_2(\mu_3(b_4, b_1, b_2), b_3) - \mu_2(\mu_3(b_3, b_4, b_1), b_2) \\
 & \quad - \mu_2(\mu_3(b_2, b_3, b_4), b_1) = \mu_3(\mu_2(b_1, b_2), b_3, b_4) - \mu_3(\mu_2(b_2, b_3), b_4, b_1) \\
 & \quad + \mu_3(\mu_2(b_3, b_4), b_1, b_2) - \mu_3(\mu_2(b_4, b_1), b_2, b_3) - \mu_3(\mu_2(b_1, b_3), b_2, b_4) \\
 & \quad - \mu_3(\mu_2(b_2, b_4), b_1, b_3) , \tag{5.2e}
 \end{aligned}$$

for all $a, a_{1,2} \in \mathfrak{a}$ and $b, b_{1,2,3,4} \in \mathfrak{b}$.

A 2-term L_∞ -algebra $\mathfrak{a} \xrightarrow{\mu_1} \mathfrak{b}$ is called strict if $\mu_3 = 0$, otherwise it is semistrict. Moreover, a 2-term L_∞ -algebra $\mathfrak{a} \xrightarrow{\mu_1} \mathfrak{b}$ is called skeletal if $\mu_1 = 0$.

To motivate the above definitions, let us present two examples of 2-term L_∞ -algebras. The first is the known string Lie 2-algebra, which is an example of a skeletal semistrict L_∞ -algebra, and the second is also a semistrict L_∞ -algebra, which is equivalent to the string Lie 2-algebra of $\mathfrak{so}(4)$.

Example 5.1.1. *For a compact semisimple Lie group G , the string Lie 2-algebra is the best known example of semistrict 2-term L_∞ -algebras. It is given by $\mathfrak{a} = \mathbb{R}$ and $\mathfrak{b} = \mathfrak{g}$, hence denoted by $\mathbb{R} \xrightarrow{\mu_1} \mathfrak{g}$, together with the products*

$$\begin{aligned}
 \mu_1 &= 0 , \quad \mu_2(a, b) = 0 , \quad \mu_2(b_1, b_2) = [b_1, b_2]_{\mathfrak{g}} , \\
 \mu_3(b_1, b_2, b_3) &= \langle b_1, [b_2, b_3]_{\mathfrak{g}} \rangle , \quad a \in \mathbb{R}, \quad b, b_{1,2,3} \in \mathfrak{g} , \tag{5.3}
 \end{aligned}$$

where $\langle -, [-, -] \rangle$ is the Killing form of the Lie algebra \mathfrak{g} of G .

We now give the second example, which is also equivalent¹ to the string Lie 2-algebra $\mathbb{R} \rightarrow \mathfrak{so}(4)$. Moreover, the following construction can be generalized to an n -sphere.

Example 5.1.2. *Consider the sphere S^3 embedded in \mathbb{R}^4 as $|x| = 1$. We would like to define the corresponding a 2-term L_∞ -algebra in terms of the Cartesian coordinates x^μ , $\mu = 1, 2, 3, 4$. All our equations below hold modulo $|x|^2 = 1$. The generators of the corresponding vector spaces are given by*

$$\begin{aligned}
 C^\infty(S^3) \supset \mathfrak{a} &= \langle 1, x^\mu, x^\mu x^\nu \rangle , \\
 \Omega^1(S^3) \supset \mathfrak{b} &= \langle dx^\mu, x^\mu dx^\nu, x^{[\mu} dx^{\nu]} \rangle . \tag{5.4}
 \end{aligned}$$

¹In the sense of [2], cf. Proposition 5.1.5

To calculate the higher brackets, we consider the volume

$$\text{vol}_{S^3} = \frac{1}{3!} \varepsilon_{\mu_1 \mu_2 \mu_3 \mu_4} x^{\mu_1} dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} =: \varpi \quad (5.5)$$

form of S^3 . Then the Hamiltonian vector fields $X_{\vartheta_{\mu\nu}} = X_{\vartheta_{\mu\nu}}^\kappa \frac{\partial}{\partial x^\kappa}$ should be in TS^3 and therefore $X_{\vartheta_{\mu\nu}}^\kappa x^\kappa = 0$. To determine $X_{\vartheta_{\mu\nu}}$, we use the equation $\iota_{X_{\vartheta_{\mu\nu}}} \varpi = d\vartheta_{\mu\nu}$. Using the usual identities for the totally antisymmetric tensors of $\mathfrak{so}(4)$, we find that

$$X_{\vartheta_{\mu\nu}} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} , \quad (5.6)$$

and the Lie algebra of Hamiltonian vector fields is $\mathfrak{so}(4)$, as expected. Moreover, the non-trivial linear maps are

$$\begin{aligned} \mu_1 &:= d , \\ \mu_2(\vartheta_{\mu\nu}, \vartheta_{\kappa\lambda}) &:= \delta_{\nu\kappa} \vartheta_{\mu\lambda} - \delta_{\mu\kappa} \vartheta_{\nu\lambda} - \delta_{\nu\lambda} \vartheta_{\mu\kappa} + \delta_{\mu\lambda} \vartheta_{\nu\kappa} , \\ \mu_3 &:= R(-, \mu_2(-, -)) , \end{aligned} \quad (5.7)$$

where²

$$R(\vartheta_{\mu\nu}, \vartheta_{\kappa\lambda}) := \frac{1}{2} (\varepsilon_{\nu\kappa\lambda\rho} x^\rho x^\mu - \varepsilon_{\mu\kappa\lambda\rho} x^\rho x^\nu + \varepsilon_{\lambda\mu\nu\rho} x^\rho x^\kappa - \varepsilon_{\mu\nu\kappa\rho} x^\rho x^\lambda) . \quad (5.8)$$

Note that all generators are central except for $\vartheta_{\mu\nu} = x^{[\mu} dx^{\nu]}$.

To state the correspondence between 2-term L_∞ -algebras and Lie 2-algebras, we recall the definition of the latter here.

Definition 5.1.3. A Lie 2-algebra \mathcal{L} is an internal category in the category of Lie algebras. That is, it consists of a 2-vector space³ $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1)$ together with the following structure maps

i) a skew symmetric bilinear functor, called the bracket:

$$[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L} , \quad (5.9a)$$

²Note that the paper [64] contains a typographical error, and the last two signs of $R(\vartheta_{\mu\nu}, \vartheta_{\kappa\lambda})$ are interchanged in the discussion of S^3 .

³See [2] for discussions on 2-vector spaces.

ii) a completely antisymmetric natural isomorphism called the Jacobiator:

$$\mathcal{J}_{x_1, x_2, x_3} : [[x_1, x_2], x_3] \Rightarrow [x_1, [x_2, x_3]] + [[x_1, x_3], x_2] , \quad (5.9b)$$

for all $x_1, x_2, x_3 \in \mathcal{L}_0$ such that these maps need to satisfy the coherence identities, cf. [2].

The Jacobiator is an element of \mathcal{L}_1 , this is also evident from the notion of internalization. It is also well known that Lie 2-algebras are equivalent to 2-term L_∞ -algebras, cf. [2]. Now in order to switch from one formalism to the other, we state the following proposition.

Proposition 5.1.4 ([2]). *The categories of 2-term L_∞ -algebras and Lie 2-algebras are equivalent.*

Moreover, 2-term L_∞ -algebras can be classified by the skeletal ones. Correspondingly, skeletal 2-term L_∞ -algebras are equivalent to Lie 2-algebras. The following proposition provides this correspondence.

Proposition 5.1.5 ([2]). *Every 2-term L_∞ -algebra is equivalent to a skeletal 2-term L_∞ -algebra.*

Example 5.1.6. *The semistrict skeletal 2-term L_∞ -algebra $\mathbb{R} \rightarrow \mathfrak{so}(4)$ is equivalent to the one in Example 5.1.2. An explicit construction of Lie 2-algebra equivalence using [2] is not difficult. But, here we follow an indirect approach. Using the $\vartheta_{\mu\nu}$, we can define orthonormal basis of the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ corresponding to the forms*

$$\vartheta_1 := \vartheta_{12} - \vartheta_{34} , \quad \vartheta_2 := \vartheta_{13} + \vartheta_{24} , \quad \vartheta_3 := \vartheta_{14} - \vartheta_{23} , \quad (5.10)$$

$$\tilde{\vartheta}_1 := \vartheta_{12} + \vartheta_{34} , \quad \tilde{\vartheta}_2 := \vartheta_{13} - \vartheta_{24} , \quad \tilde{\vartheta}_3 := \vartheta_{14} + \vartheta_{23} , \quad (5.11)$$

which are left-invariant over $S^3 \cong \mathrm{SU}(2)$. Moreover, all the non-trivial linear maps are given by

$$\mu_2(\vartheta_i, \vartheta_j) := -2\varepsilon_{ijk}\vartheta_k , \quad \mu_2(\tilde{\vartheta}_i, \tilde{\vartheta}_j) := 2\varepsilon_{ijk}\tilde{\vartheta}_k , \quad (5.12)$$

$$\mu_3(\vartheta_i, \vartheta_j, \vartheta_k) := \langle \vartheta_i, \mu_2(\vartheta_j, \vartheta_k) \rangle = -2\varepsilon_{ijk} , \quad (5.13a)$$

$$\mu_3(\tilde{\vartheta}_i, \tilde{\vartheta}_j, \tilde{\vartheta}_k) := \langle \tilde{\vartheta}_i, \mu_2(\tilde{\vartheta}_j, \tilde{\vartheta}_k) \rangle = 2\varepsilon_{ijk} , \quad (5.13b)$$

where $\langle -, - \rangle$ is the inner product on \mathbb{R}^4 . This concludes construction of the semistrict skeletal 2-term L_∞ -algebra $\mathbb{R} \longrightarrow \mathfrak{so}(4)$ using the 2-term L_∞ -algebra in Example 5.1.2. Furthermore, as mentioned earlier, this can be extended to any n -sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$, for $n > 3$.

5.1.2 2-term L_∞ -algebras as NQ-manifold

Recall that an NQ-manifold is a non-negatively graded manifold endowed with a degree 1 vector field Q satisfying $Q^2 = 0$. It plays an important role in differential geometry. For instance, in the differentiation of Lie n -groupoids [78]. In this section, we review the description of 2-term L_∞ -algebras as NQ-manifolds. Indeed, the former are particular examples of NQ-manifolds concentrated in degree 2, see [46, 88], and [14] for more general cases. For the sake of our discussion in this section we present the following definition.

Definition 5.1.7. *Let V and W be finite dimensional vector spaces over the field of complex numbers \mathbb{C} with coordinates x^α and y^a . Then the shifted vector space $V[1] \oplus W[2]$ is a degree 2 NQ-manifold, where the notation implies that elements in $V[1]$ and $W[2]$ come with homogeneous gradings of 1 and 2, respectively. The vector field Q is necessarily of the form*

$$Q = -f_a^\alpha y^a \frac{\partial}{\partial x^\alpha} - \frac{1}{2} f_{\alpha\beta}^\gamma x^\alpha x^\beta \frac{\partial}{\partial x^\gamma} - f_{ab}^a x^\alpha y^b \frac{\partial}{\partial y^a} - \frac{1}{3!} f_{\alpha\beta\gamma}^a x^\alpha x^\beta x^\gamma \frac{\partial}{\partial y^a} \quad (5.14)$$

with some structure constants $f_{\dots} \in \mathbb{C}$. The expression (5.14) defines graded anti-symmetric multilinear brackets on the shifted vector space $V[0] \oplus W[1]$. Introducing

the grade-carrying bases (τ_α) and (t_a) on $V[0]$ and $W[1]$, respectively, we have

$$\begin{aligned}\mu_1(t_a) &= f_a^\alpha \tau_\alpha, & \mu_2(\tau_\alpha, \tau_\beta) &= f_{\alpha\beta}^\gamma \tau_\gamma, \\ \mu_2(\tau_\alpha, t_a) &= f_{a\alpha}^b t_b, & \mu_3(\tau_\alpha, \tau_\beta, \tau_\gamma) &= f_{\alpha\beta\gamma}^a t_a.\end{aligned}\tag{5.15}$$

Note that the linear maps μ_i are of degree $i - 2$ and the condition $Q^2 = 0$ yields the usual higher or homotopy Jacobi relations, cf. (5.2) between the μ_i defining a 2-term L_∞ -algebra, cf. [47, 48, 88].

In particular, if we consider a n -dimensional ordinary Lie algebra \mathfrak{g} with generators t_a as 2-term L_∞ -algebra, we will obtain the usual Chevalley-Eilenberg description of Lie algebras. And in this case the homological vector field Q takes the form

$$Q = -\frac{1}{2} f_{ab}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c},\tag{5.16}$$

where ξ^a is the basis element of $\mathfrak{g}[1]$ dual to t_a .

5.2 Functor from Man^∞ to the category of descent data

In order to differentiate the weak Lie 2-group \mathcal{S}_λ^w , we use the method of differentiation suggested by Ševera [78], see also [42]. The goal of this section is to Lie differentiate the weak Lie 2-group \mathcal{S}_λ^w into the known string Lie 2-algebra. Consequently, we have two main results in this section: Theorem 5.2.5, which gives us a complete summary of the differentiation, and Theorem 5.2.6 presents equivalence relations of local connective structures.

We start our discussion by reviewing the differentiation of a Lie group \mathbf{G} to its Lie algebra $\text{Lie}(\mathbf{G})$.

5.2.1 Differentiation of a Lie group to its Lie algebra

This subsection is borrowed from [42], see also [41].

Consider a Lie group \mathbf{G} as a Lie 2-group ($\mathbf{G} \rightrightarrows *$). Since the object space is trivial, it suffices to differentiate only the set of morphisms \mathbf{G} . We also assume that \mathbf{G} is a matrix Lie group. Thus, we can use the usual diffeomorphism from the neighbourhood U_0 of $0 \in \text{Lie}(\mathbf{G})$ to the neighbourhood $U_{\mathbb{1}_{\mathbf{G}}}$ of $\mathbb{1}_{\mathbf{G}} \in \mathbf{G}$. Moreover, since we are interested in infinitesimal neighbourhoods, we can consider elements in $\text{Lie}(\mathbf{G})[1]$ multiplied by coordinates θ of $\mathbb{R}^{0|1}$. Thus, we have

$$(\mathbb{1}_{\mathbf{G}} + \omega\theta)^{-1} = \mathbb{1}_{\mathbf{G}} - \omega\theta , \quad (5.17a)$$

$$(\mathbb{1}_{\mathbf{G}} + \omega_1\theta_0)(\mathbb{1}_{\mathbf{G}} + \omega_2\theta_1) = \mathbb{1}_{\mathbf{G}} + \omega_1\theta_0 + \omega_2\theta_1 - (\omega_1 \cdot \omega_2)\theta_0\theta_1 , \quad (5.17b)$$

where

$$\omega_1 \cdot \omega_2 + \omega_1 \cdot \omega_2 = [\omega_1, \omega_2] , \quad (5.17c)$$

is the graded Lie bracket and $\omega, \omega_{1,2} \in \text{Lie}(\mathbf{G})[1]$.

In addition to this, we shall write the product of an element $g \in \mathbf{G}$ and $\omega \in \text{Lie}(\mathbf{G})[1]$ as $g\omega$. Thus, to provide a complete description of the Lie algebra $\text{Lie}(\mathbf{G})$ of \mathbf{G} as discussed in [42], we use a \mathbf{G} -valued descent datum on the surjective submersion $\mathbb{R}^{0|1} \times X \rightarrow X$, which is just a smooth map

$$g(\theta_0, \theta_1) : \mathbb{R}^{0|2} \times X \rightarrow \mathbf{G} , \quad (5.18a)$$

satisfying

$$g(\theta_0, \theta_1)g(\theta_1, \theta_2) = g(\theta_0, \theta_2) . \quad (5.18b)$$

Consequently, we aim to verify that the functor from the category of smooth manifolds \mathbf{Man}^∞ to the category of such \mathbf{G} -valued descent data is parametrized by the NQ -manifold $\text{Lie}(\mathbf{G})[1]$ concentrated in degree 1. We now review the properties of the descent data in (5.18).

Proposition 5.2.1. *If we let $g(\theta_0, 0) := g(\theta_0)$ and $g(\theta_1, 0) := g(\theta_1)$, then we have*

$$g(\theta_0, \theta_1) = g(\theta_0)g(\theta_1)^{-1} , \quad (5.19a)$$

which gives also

$$g(0, 0) = \mathbb{1}_G, \quad g(\theta_0) = \mathbb{1}_G + \omega\theta_0. \quad (5.19b)$$

From these one can also obtain

$$g(\theta_0, \theta_1) = \mathbb{1}_G + \omega(\theta_0 - \theta_1) + \frac{1}{2}[\omega, \omega]\theta_0\theta_1, \quad (5.19c)$$

where $\mathbb{1}_G \in G$, $\omega \in \text{Lie}(G)[1]$ and $g(\theta_0, \theta_1)^{-1} = g(\theta_1, \theta_0)$.

Now we have the Lie algebra $\text{Lie}(G)$ as a vector space. Thus, we are left to obtain its Lie bracket. To do that, we consider the generator of the action of the semigroup $\text{Hom}(\mathbb{R}^{01}, \mathbb{R}^{01}) = \mathbb{R}^{11}$ onto $\text{Lie}(G)[1]$, which gives us the Q -vector field. And as mentioned earlier, the action of this vector field can be identified with the Chevalley-Eilenberg differential of $\text{Lie}(G)$, encoding the Lie bracket. We denote this differential by d_K and its action on smooth functions f on \mathbb{R}^{0k} reads as

$$d_K f(\theta_0, \theta_1, \dots, \theta_{k-1}) := \frac{d}{d\varepsilon} f(\theta_0 + \varepsilon, \theta_1 + \varepsilon, \dots, \theta_{k-1} + \varepsilon), \quad (5.20)$$

and its application on the descent data

$$g(\theta_0, \theta_1) = \mathbb{1}_G + \omega(\theta_0 - \theta_1) + \frac{1}{2}[\omega, \omega]\theta_0\theta_1 \quad (5.21)$$

yields $d_K\omega = -\frac{1}{2}[\omega, \omega]$, which in turn yields the Chevalley-Eilenberg differential

$$Q\xi^a = -\frac{1}{2}f_{bc}^a \xi^b \wedge \xi^c, \quad (5.22)$$

where the ξ^a 's are the coordinate functions on $\text{Lie}(G)[1]$, cf. (5.16).

Thus, we have reviewed that the functor from Man^∞ to the category of G -valued decent datum on the surjective submersions $\mathbb{R}^{01} \times X \rightarrow X$ as defined in (5.18) is parameterized by $\text{Lie}(G)[1]$. In the next subsection, we shall extend this result to the weak Lie 2-group \mathcal{S}_λ^w -valued descent datum, where the parameterizing elements will form an NQ -manifold concentrated in degree 2, which is the known string Lie 2-algebra $\mathbb{R} \rightarrow \mathfrak{g}$, cf. Theorem 5.2.5. Equivalently, the theorem can be also consid-

ered as another method of constructing the string Lie 2-algebra, which is just by differentiating the weak Lie 2-group \mathcal{S}_λ^w . See [5] and [67] for previous works on the construction of the string Lie 2-algebra from geometric and categorical perspectives, respectively.

5.2.2 Differentiation of the weak Lie 2-group \mathcal{S}_λ^w

Now similar to the differentiation above, we apply Ševera's [78] differentiation technique on the weak Lie 2-group \mathcal{S}_λ^w ,

$$\mathcal{S}_\lambda^w := V_1^{[2]} \times \mathbf{A} \rightrightarrows V_1 \quad (5.23)$$

to establish the string Lie 2-algebra, cf. Theorem (5.2.5). We also follow closely the differentiation in [42], but we have to generalize due to the non-trivial unitors in \mathcal{S}_λ^w . Thus, we consider a functor from the category of smooth manifolds \mathbf{Man}^∞ to the category of \mathcal{S}_λ^w -valued descent data on the surjective submersions $\mathbb{R}^{01} \times X \rightarrow X$, and this functor is parametrized by an NQ-manifold concentrated in degree 2, which corresponds to the string Lie 2-algebra. Consequently, the higher products on the resulting 2-term complex of vector spaces is encoded by the Chevelley-Eilenberg differential induced by the generator $\frac{\partial}{\partial \theta}$ of the action of $\mathbf{Hom}(\mathbb{R}^{01}, \mathbb{R}^{01})$ on \mathbb{R}^{01} , which gives the Q -vector field on the shifted vector space $\mathbf{Lie}(\mathbf{A})[2] \oplus \mathbf{Lie}(\mathbf{G})[1]$.

Hence, in order to obtain the 2-term complex of vector spaces, let us begin our discussion as usual by using the local diffeomorphisms between the respective neighbourhoods. Likewise since we are interested in infinitesimal neighbourhoods, $U_{\mathbb{1}_{\mathcal{S}_\lambda^w}}$ is assumed to be contained in $V_{1,i}$ containing $\mathbb{1}_{\mathcal{S}_\lambda^w}$, for some $i \in I_1$. And we write the elements as $\mathbb{1}_{\mathcal{S}_\lambda^w} + \omega\theta$ for $\exp(\omega\theta)$ and $a\theta$ for $\exp(a\theta)$ with $\omega \in \mathbf{Lie}(\mathbf{G})[1] \cong \mathbf{Lie}(V_1)$ and $a \in \mathbf{Lie}(\mathbf{A})[1]$. Similarly, one can linearize all the structure maps in \mathcal{S}_λ^w . In particular, we use the linearized horizontal multiplication in order to give meaning for products of the form $g \otimes \omega$, for $g \in V_1$ and $\omega \in \mathbf{Lie}(\mathbf{G})[1]$. Here, we will not discuss in detail the linearization of the structure maps of \mathcal{S}_λ^w , because our aim is to construct the string Lie 2-algebra as a 2-term L_∞ -algebra, cf. Example 5.1.1. See [42] for the discussion on linearization of the structure maps for the semistrict case.

Thus, after having a 2-term complex of vector spaces $\text{Lie}(\mathbf{A}) \rightarrow \text{Lie}(\mathbf{G})$, the next step is to give a complete description of the L_∞ -algebra structure. For this we shall consider the functor from \mathbf{Man}^∞ to the category of \mathcal{S}_λ^w -valued descent data on the surjective submersions $\mathbb{R}^{0|1} \times X \twoheadrightarrow X$. Such a descent data can be given by a weak normalized⁴ functor, which is encoded in a degree 2-Čech 2-cocycles (v, a) . These cocycles consist of V_1 -valued 1-cochains together with $(V_1^{[2]} \times \mathbf{A})$ -valued 2-cochains

$$v(\theta_0, \theta_1) \quad \text{and} \quad v(\theta_0, \theta_1, \theta_2) = (v(\theta_0, \theta_2), v(\theta_0, \theta_1) \otimes v(\theta_1, \theta_2), a(\theta_0, \theta_1, \theta_2)) . \quad (5.24)$$

But, as we are interested in infinitesimal neighbourhoods, open sets in V_1 are fully contained within one of the open sets, cf. Example 3.1.2. Thus, because $v(\theta_0, \theta_1)$ depends smoothly on the Graßmann variables, it lies on the same cover $V_{1,i}$ as $\mathbb{1}_{\mathcal{S}_\lambda}$. This open set contains an infinitesimal neighbourhood of $\mathbb{1}_{\mathcal{S}_\lambda}$, and we have $\phi_1(\pi(v(\theta_0, \theta_1))) = v(\theta_0, \theta_1)$. Hence, the cocycle relation is reduced to

$$v(\theta_0, \theta_1) \otimes v(\theta_1, \theta_2) = v(\theta_0, \theta_2) . \quad (5.25a)$$

This, in turn, renders the $\lambda^{1,2}$ -terms to vanish in the cocycle condition (4.15) for the $a(\theta_0, \theta_1, \theta_2)$, cf. Remark 3.1.4. Therefore, we have

$$\begin{aligned} a(\theta_0, \theta_2, \theta_3) + a(\theta_0, \theta_1, \theta_2) = \\ a(\theta_0, \theta_1, \theta_3) + a(\theta_1, \theta_2, \theta_3) + \lambda^{0,3}(\phi_3(v(\theta_0, \theta_1), v(\theta_1, \theta_2), v(\theta_2, \theta_3))) , \end{aligned} \quad (5.25b)$$

where $\phi_3 : V_1 \times V_1 \times V_1 \rightarrow V_3$ is a smooth map, cf. Subsection 3.4.3. As one might expect due to the form of the surjective submersion, the principal \mathcal{S}_λ^w -bundle we are dealing with here is trivial. Consequently, we have the following result.

Lemma 5.2.2. *The \mathcal{S}_λ^w -valued cochain (β, α) with the expansion*

$$\beta(\theta_0) := v(\theta_0, 0) \quad \text{and} \quad \alpha(\theta_0, \theta_1) := a(\theta_0, \theta_1, 0) \quad (5.26)$$

forms a coboundary as defined in Theorem 4.3.5, trivializing the principal \mathcal{S}_λ^w -bundle

⁴ $v(\theta_0, \theta_0) = v(0, 0) = \mathbb{1}_{\mathcal{S}_\lambda^w}$, $a(\theta_0, \theta_0, \theta_1) = a(\theta_0, \theta_1, \theta_1) = 0$.

described by (5.25). Moreover, we have

$$\beta(0) = \mathbb{1}_{\mathcal{S}_\lambda} , \quad \alpha(\theta_0, 0) = 0 , \quad \alpha(0, \theta_0) = 0 . \quad (5.27)$$

Proof. The proof follows by direct computation, and using the fact that the $\lambda^{m,n}$ vanishes if an argument is given by a sequence of degeneracy maps acting on $\mathbb{1}_{\mathcal{S}_\lambda}$. Note in particular that $a(\theta_0, \theta_0, \theta_1) = a(\theta_0, \theta_1, \theta_1) = 0$ due to the normalization of (v, a) . \square

Now by considering the parameterizing elements $\omega \in \text{Lie}(\mathbf{G})[1]$ and $\psi \in \text{Lie}(\mathbf{A})[2]$, we can give the expansions of the cochain (β, α) in (5.26) in terms of the Graßmann variables by making use of the respective local diffeomorphisms⁵. This leads us to the following proposition.

Proposition 5.2.3. *For parameterizing elements $\omega \in \text{Lie}(\mathbf{G})[1]$ and $\psi \in \text{Lie}(\mathbf{A})[2]$, we have*

$$\beta(\theta_0) = \mathbb{1}_{\mathcal{S}_\lambda} + \omega\theta_0 \quad \text{and} \quad \alpha(\theta_0, \theta_1) = \psi\theta_0\theta_1 , \quad (5.28)$$

Proof. The first expansion is a direct consequence of $\beta(0) = \mathbb{1}_{\mathcal{S}_\lambda}$, the second also follows from $\alpha(0, 0) = \alpha(\theta_0, 0) = \alpha(0, \theta_1) = 0$. \square

Moreover, using the expressions for the coboundary (β, α) in (5.28), we can compute the Graßmann expansions of the cocycle components in (5.25). And since we are working on infinitesimal neighbourhoods of $\mathbb{1}_{\mathcal{S}_\lambda}$, the horizontal multiplication collapses to the ordinary group multiplication in \mathbf{G} . Thus, we have the following proposition.

Proposition 5.2.4. *The expansions of the cocycle components are given by*

$$v(\theta_0, \theta_1) = \mathbb{1}_{\mathcal{S}_\lambda} + \omega(\theta_0 - \theta_1) + \frac{1}{2}[\omega, \omega]\theta_0\theta_1 , \quad (5.29a)$$

$$a(\theta_0, \theta_1, \theta_2) = \psi(\theta_0\theta_1 + \theta_1\theta_2 - \theta_0\theta_2) + \lambda^{0,3}(\omega, \omega, \omega)\theta_0\theta_1\theta_2 , \quad (5.29b)$$

where $\lambda^{0,3}(\omega, \omega, \omega)$ is the obvious linearization of $\lambda^{0,3}$ around $(\mathbb{1}_{\mathcal{S}_\lambda}, \mathbb{1}_{\mathcal{S}_\lambda}, \mathbb{1}_{\mathcal{S}_\lambda})$.

⁵The local diffeomorphisms between the neighbourhoods in V_1 and $T_{\mathbb{1}_{\mathcal{S}_\lambda}}V_1$, and \mathbf{A} and $\text{Lie}(\mathbf{A})[2]$.

Proof. Here, (5.29a) follows from the normalization $v(\theta_0, \theta_0) = v(0, 0) = \mathbb{1}_{\mathcal{S}_\lambda}$ together with the expressions in (5.28), and (5.25a). Similarly, to obtain (5.29b), we use the coboundary relations for the morphisms

$$\alpha(\theta_0, \theta_2) + a(\theta_0, \theta_1, \theta_2) = \alpha(\theta_0, \theta_1) + \alpha(\theta_1, \theta_2) + \lambda^{0,3}(v(\theta_0, \theta_1), v(\theta_1, \theta_2), \beta(\theta_2)) , \quad (5.30)$$

and (5.28). But, by Remark 3.1.4

$$\lambda^{0,3}(v(\theta_0, \theta_1), v(\theta_1, \theta_2), \beta(\theta_2)) =: \lambda^{0,3}(\omega, \omega, \omega)\theta_0\theta_1\theta_2 . \quad (5.31)$$

Hence, (5.29b) follows. \square

Thus, applying the induced differential d_K by the relevant generator of the action of $\text{Hom}(\mathbb{R}^{01}, \mathbb{R}^{01})$ induces an action on (5.29), which is given by

$$d_K\omega = -\frac{1}{2}[\omega, \omega] , \quad (5.32a)$$

$$d_K\psi = -\lambda^{0,3}(\omega, \omega, \omega) . \quad (5.32b)$$

This action in turn yields the Chevalley-Eilenberg differential on $\text{Lie}(\mathbf{G})[1] \oplus \text{Lie}(\mathbf{A})[2]$ due to the vector field Q , which in turn give us all the linear maps μ_i of degree $(i-2)$ on a 2-term L_∞ -algebra $\text{Lie}(\mathbf{A}) \rightarrow \text{Lie}(\mathbf{G})$, cf. Definition 5.1.7, and the condition $Q^2 = 0$ gives the higher homotopy Jacobi identities between the μ_i 's. In particular, for $\mathbf{G} = \text{Spin}(n)$, $n \geq 3$ and $\mathbf{A} = \text{U}(1)$, the linear map μ_3 is non-trivial, since it corresponds to the non-trivial 3-cocycle $\lambda^{0,3} \in H^3(\mathfrak{spin}(n), \mathfrak{u}(1))$. The latter is induced from the generator half of the first Pontryagin class $\frac{1}{2}\mathbf{p}_1 \in H^4(B\text{Spin}(n), \mathbb{Z}) \cong \mathbb{Z}$, cf. [32]. We summarize our findings by the following theorem.

Theorem 5.2.5. *The Lie 2-algebra of the string 2-group $\mathcal{S}_\lambda^{\mathbf{w}}$ with $\mathbf{G} = \text{Spin}(n)$, $n \geq 3$ and $\mathbf{A} = \text{U}(1)$ is the string Lie 2-algebra, which is the 2-term L_∞ -algebra $\mathbb{R} \rightarrow \mathfrak{g}$, (where $\text{Lie}(\text{U}(1)) = \mathbb{R}$ and $\text{Lie}(\mathbf{G}) = \mathfrak{g}$) together with the non-trivial higher products*

$$\mu_2(x_1, x_2) = [x_1, x_2] , \quad \mu_3(x_1, x_2, x_3) = k\langle x_1, [x_2, x_3] \rangle \quad (5.33)$$

for some $k \in \mathbb{R}$. Here $\langle -, - \rangle$ and $[-, -]$ denote the inner product and the Lie bracket on \mathfrak{g} , respectively.

Proof. Since, \mathbf{G} is compact and semisimple, it has the canonical 3-form $\langle -, [-, -] \rangle$. Furthermore, the non-trivial 3-cocycle $\lambda^{0,3} \in H^3(\mathfrak{g}, \mathbb{R})$ is necessarily of the form $\lambda^{0,3}(-, -, -) = k\langle -, [-, -] \rangle$ for some non-zero $k \in \mathbb{R}$. \square

Theorem 5.2.5 is due to the fact that, the class of equivalent skeletal semistrict 2-term L_∞ -algebras $\mathbb{R} \rightarrow \mathfrak{g}$ is constructed with $[\mu_3] \in H^3(\mathfrak{g}, \mathbb{R}) \cong H_{dR}^3(\mathbf{G})$, cf. [2]. In other words, this class corresponds to the canonical 3-form on \mathbf{G} called the Killing form. Hence, for all non-zero $k \in \mathbb{R}$, $\lambda^{0,3}(x_1, x_2, x_3) \sim k\langle x_1, [x_2, x_3] \rangle$.

The following subsection gives the equivalence relations on the local connective structures (ω, ψ) , which are important to derive the gauge transformations for connective structures on principal \mathcal{S}_λ^w -bundles. From now on until the end of this chapter, we take $\mathbf{A} = \mathbf{U}(1)$, and $\mathbf{G} = \mathbf{Spin}(n)$ and we denote their corresponding Lie algebras by $\mathfrak{u}(1)$ and \mathfrak{g} , respectively.

5.2.3 Equivalence relations

We now extend the above differentiation process to derive equivalence relations on local connective structures. Later we will replace the Chevalley-Eilenberg differential by the de Rham differential and this will give us gauge transformations for connective structures on principal \mathcal{S}_λ^w -bundles, and further the full underlying Deligne cohomology of these bundles. The idea of using Deligne cohomology with values in categorified Lie groups was first used in [42]. The first treatment of ordinary principal bundles is also found in [21]. Here, we follow closely the results in [42] to give these descriptions.

Thus, given a principal \mathcal{S}_λ^w -bundle on $\mathbb{R}^{0|1} \times X \rightarrow X$ in terms of Čech 2-cocycles (v, a) , we now use a coboundary (β, α) , to derive a relation with another Čech 2-cocycle (v', a') . That is

$$(\beta(\theta_0), \alpha(\theta_0, \theta_1)) : (v(\theta_0, \theta_1), a(\theta_0, \theta_1, \theta_2)) \rightarrow (v'(\theta_0, \theta_1), a'(\theta_0, \theta_1, \theta_2)) . \quad (5.34)$$

And this can be translated on the level of parameterizing elements to a relation of the form

$$(\beta(\theta_0), \alpha(\theta_0, \theta_1)) : (\omega, \psi) \rightarrow (\omega', \psi'). \quad (5.35)$$

Here, we are interested in the explicit expressions that correspond to (5.34) and (5.35). Consequently, the new Čech 2-coboundary (β, α) is necessarily of the form

$$\beta(\theta_0) = \beta - d_K \beta \theta_0 \quad \text{and} \quad \alpha(\theta_0, \theta_1) = \zeta(\theta_1 - \theta_0) + d_K \zeta \theta_0 \theta_1, \quad (5.36)$$

where $\beta \in V_1$ and $\zeta \in \mathfrak{u}(1)[1]$. These expressions are evident from the fact that $\beta(0) = \beta$, $\alpha(0, 0) = 0$, and $\alpha(\theta_0, \theta_0) = 0$.

Thus, the new coboundary as shown in (5.36) is constructed in the neighbourhood of an arbitrary point $\beta \in V_1$, which is different from the one given in (5.28). Accordingly, by applying this coboundary, we can relate the normalized 2-cocycles (v, a) and (v', a') as

$$v(\theta_0, \theta_1) \otimes \beta(\theta_1) = \beta(\theta_0) \otimes v'(\theta_0, \theta_1), \quad (5.37a)$$

$$\begin{aligned} \alpha(\theta_0, \theta_2) + a(\theta_0, \theta_1, \theta_2) &= \alpha(\theta_0, \theta_1) + a'(\theta_0, \theta_1, \theta_2) + \alpha(\theta_1, \theta_2) \\ &+ \lambda^{0,3}(\beta(\theta_0), v'(\theta_0, \theta_1), v'(\theta_1, \theta_2)) - \lambda^{0,3}(v(\theta_0, \theta_1), \beta(\theta_1), v'(\theta_1, \theta_2)) \\ &+ \lambda^{0,3}(v(\theta_0, \theta_1), v(\theta_1, \theta_2), \beta(\theta_2)). \end{aligned} \quad (5.37b)$$

Therefore, using (5.29) and (5.36), the expression (5.37b) reduces to

$$\begin{aligned} &\psi(\theta_0 \theta_1 + \theta_1 \theta_2 - \theta_0 \theta_2) \\ &= (\psi' + d_K \zeta + \lambda^{0,3}(\beta, \omega', \omega') - \lambda^{0,3}(\omega, \beta, \omega') + \lambda^{0,3}(\omega, \omega, \beta))(\theta_0 \theta_1 + \theta_1 \theta_2 - \theta_0 \theta_2), \end{aligned} \quad (5.38a)$$

$$\begin{aligned} &\lambda^{0,3}(\omega, \omega, \omega) \\ &= \lambda^{0,3}(\omega', \omega', \omega') + \lambda^{0,3}(\beta, [\omega, \omega], \omega) + \dots - \lambda^{0,3}(d_K \beta, \omega', \omega') - \dots \end{aligned} \quad (5.38b)$$

Moreover, from (5.37a) and (5.38a), we obtain

$$\beta \otimes \omega' = \omega \otimes \beta + d_K \beta, \quad (5.39a)$$

$$\psi' = \psi - d_K \zeta - \lambda^{0,3}(\beta, \omega', \omega') + \lambda^{0,3}(\omega, \beta, \omega') - \lambda^{0,3}(\omega, \omega, \beta). \quad (5.39b)$$

The second equation of (5.38b) is then automatically satisfied. Next we summarize our findings in the following theorem.

Theorem 5.2.6. *The 2-coboundaries in (5.36) relating the 2-cocycles (v, a) and (v', a') , which corresponds to the descent data for principal \mathcal{S}_λ^w -bundles on surjective submersions $\mathbb{R}^{0|1} \times X \rightarrow X$ are parametrized by elements $\beta \in V_1$ and $\zeta \in \mathfrak{u}(1)[1]$. Moreover, the corresponding moduli (ω, ψ) and (ω', ψ') are related by the expressions in (5.39).*

Now we have a complete description of the string Lie 2-algebra, thus we can discuss connective structures on principal 2-bundles with the string 2-groups.

5.3 Higher gauge theory with the string 2-groups

This section is devoted to a complete description of higher gauge theory with the string 2-groups by using the tensor product of an L_∞ -algebras with the graded differential algebra of local differential forms $\Omega^\bullet(U)$ of a smooth manifold X , which is important to give the local description of higher gauge theory using the Maurer-Cartan equation on the new L_∞ -algebra obtained from this tensor product. The first treatment of the topic is available in [70]. See also [42] for related discussions on semistrict higher gauge theory with local connective structures.

5.3.1 Local description with infinitesimal gauge symmetries

The local description of higher gauge theory with the string 2-groups is readily given without the above considerations and below we briefly recall how, cf. e.g. [42]. The string Lie 2-algebra of a compact semisimple Lie group \mathbf{G} is known to be $\mathfrak{u}(1) \rightarrow \mathfrak{g}$, cf. Theorem 5.2.5 with non-trivial higher products $\mu_2(x_1, x_2) = [x_1, x_2]$ and $\mu_3(x_1, x_2, x_3) = k\langle x_1, [x_2, x_3] \rangle$, $x_i \in \mathfrak{g}$ and $k \in \mathbb{R}$. This 2-term L_∞ -algebra can be tensored with the graded differential algebra of differential forms $\Omega^\bullet(U)$ on a good cover U of a smooth manifold X . The result is another L_∞ -algebra, $\tilde{\mathbf{L}}$ whose

degree k -element

$$\ell_k \in ((\Omega^k(U) \otimes \mathfrak{g}) \oplus (\Omega^{k+1}(U) \otimes \mathfrak{u}(1))) , \quad (5.40)$$

for $k \geq -1$ together with the higher products $\tilde{\mu}_i$ obtained by tensoring the higher products μ_i on the string Lie 2-algebra with the differential of $\Omega^\bullet(U)$.

Recall that in any L_∞ -algebra, we can define homotopy Maurer-Cartan elements. Consequently, here we consider the Maurer-Cartan elements $\phi \in ((\Omega^1(U) \otimes \mathfrak{g}) \oplus (\Omega^2(U) \otimes \mathfrak{u}(1)))$, which are elements that satisfy the *homotopy Maurer-Cartan equation*

$$\sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{i!} \tilde{\mu}_i(\phi, \dots, \phi) = 0 . \quad (5.41)$$

This expression exhibits a gauge symmetry, parameterized at infinitesimal level by a degree 0 element in $\gamma \in \tilde{\mathbb{L}}$, which transforms Maurer-Cartan elements to Maurer-Cartan elements:

$$\phi \rightarrow \phi + \delta\phi \quad \text{with} \quad \delta\phi = \sum_i \frac{(-1)^{i(i-1)/2}}{(i-1)!} \tilde{\mu}_i(\gamma, \phi, \dots, \phi) . \quad (5.42)$$

Hence, from (5.41), by considering our 2-term L_∞ -algebra $\mathfrak{u}(1) \rightarrow \mathfrak{g}$, we obtain

$$-\tilde{\mu}_1(\phi) - \frac{1}{2}\tilde{\mu}_2(\phi, \phi) + \frac{1}{3!}\tilde{\mu}_3(\phi, \phi, \phi) = 0 , \quad (5.43)$$

Moreover, (5.42) gives

$$\delta\phi = \tilde{\mu}_1(\gamma) + \tilde{\mu}_2(\gamma, \phi) - \frac{1}{2}\tilde{\mu}_3(\gamma, \phi, \phi) . \quad (5.44)$$

In more explicit form, we have the following proposition, which gives us the expressions of Maurer-Cartan elements and the infinitesimal gauge transformations using the local connective structures.

Proposition 5.3.1. *The homotopy Maurer-Cartan elements of $\tilde{\mathbb{L}}$ are given by pairs*

$A \in \Omega^1(U) \otimes \mathfrak{g}$ and $B \in \Omega^2(U) \otimes \mathfrak{u}(1)$ satisfying the equations

$$\begin{aligned}\mathcal{F} &:= dA + \frac{1}{2}\mu_2(A, A) = 0, \\ H &:= dB - \frac{1}{3!}\mu_3(A, A, A) = 0.\end{aligned}\tag{5.45}$$

Infinitesimal gauge transformations are parameterized by pairs $x \in \Omega^0(U) \otimes \mathfrak{g}$ and $\zeta \in \Omega^1(U) \otimes \mathfrak{u}(1)$ and act according to

$$\delta A = dx + \mu_2(A, x) \quad \text{and} \quad \delta B = -d\zeta + \frac{1}{2}\mu_3(x, A, A).\tag{5.46}$$

Proof. These expressions directly follow from (5.43) and (5.44) by substituting $\phi = A - B$ and $\gamma = x + \zeta$. □

Having discussed the infinitesimal gauge symmetries, we now present the finite gauge transformations. This can be done by using non-abelian Deligne cohomology. Moreover, note that under these gauge transformations, flat local connective structures⁶ remain flat.

5.3.2 Non-abelian Deligne cohomology with values in the string 2-group

The full global description of non-abelian gauge theory of principal n -bundles with connective structures is governed by non-abelian Deligne cohomology. We begin here by reviewing the case of ordinary principal bundles with connections before presenting the details for principal \mathcal{S}_λ^w -bundles. The following discussion is based on [69], see also [42].

Given a Lie group \mathbf{G} with Lie algebra \mathfrak{g} , a principal \mathbf{G} -bundle with connection over a smooth manifold X with respect to a good cover $U = \sqcup_i U_i \rightarrow X$ is described by a non-abelian Deligne 1-cocycle with values in \mathbf{G} . Such a 1-cocycle consists of \mathbf{G} -valued transition smooth functions (g_{ij}) on the fibered product $U \times_X U = \sqcup_{i,j} U_i \cap U_j$

⁶Here, by flat connective structures we mean, local connective structures that satisfy (5.45).

and \mathfrak{g} -valued one-forms (A_i) on U satisfying

$$g_{ij}g_{jk} = g_{ik} \quad \text{and} \quad A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij} . \quad (5.47)$$

Note that the cocycle conditions glue together the local data contained in (A_i) to a global connection. One can also obtain an expression that relates any two equivalent 1-cocycles. Thus, two such Deligne 1-cocycles (g_{ij}, A_i) and (g'_{ij}, A'_i) are considered equivalent, if there exist \mathbf{G} -valued smooth functions (g_i) on U called Deligne 1-coboundary that satisfy

$$g'_{ij} = g_i^{-1}g_{ij}g_j \quad \text{and} \quad A'_i = g_i^{-1}A_i g_i + g_i^{-1}dg_i . \quad (5.48)$$

The above relations are called the coboundary conditions; they describe finite gauge transformations of the 1-cocycles.

Analogously, we can now present the explicit form of Deligne 2-cocycles and 2-coboundaries for principal $\mathcal{S}_\lambda^{\mathbf{w}}$ -bundles by using the results in Section 5.2. As usual, in the following, X denotes the base smooth manifold of the principal $\mathcal{S}_\lambda^{\mathbf{w}}$ -bundles and $U = \sqcup_i U_i \rightarrow X$ is a good cover of X . Here, we will follow closely the discussions in [42].

Definition 5.3.2. *Let \mathbf{G} be a compact semisimple Lie group with Lie algebra \mathfrak{g} . Furthermore, let $\text{Lie}(\mathcal{S}_\lambda^{\mathbf{w}}) = (\mathfrak{g} \times \mathfrak{u}(1) \rightrightarrows \mathfrak{g})$ be the Lie 2-algebra of the weak 2-group model $\mathcal{S}_\lambda^{\mathbf{w}}$ over \mathbf{G} . A Deligne 2-cocycle with values in $\mathcal{S}_\lambda^{\mathbf{w}}$, which describes a principal $\mathcal{S}_\lambda^{\mathbf{w}}$ -bundle with connective structure, is then given by $\mathcal{S}_\lambda^{\mathbf{w}}$ -valued Čech 2-cocycle (v_{ij}, a_{ijk}) together with local differential forms $A = (A_i) \in \Omega^1(U, \mathfrak{g})$, $B = (B_i) \in \Omega^2(U, \mathfrak{u}(1))$ and $\zeta = (\zeta_{ij}) \in \Omega^1(U \times_X U, \mathfrak{u}(1))$, satisfying the following cocycle relations:*

$$\pi(v_{ik}) = \pi(v_{ij} \otimes v_{jk}) , \quad v_{ii} = \mathbb{1}_{\mathcal{S}_\lambda} , \quad (5.49a)$$

$$\begin{aligned} a_{ikl} + a_{ijk} + \lambda^{1,2}(\phi_2(v_{ik}, v_{kl}), \phi_2(v_{ij} \otimes v_{jk}, v_{kl})) &= a_{ijl} + a_{jkl} \\ &+ \lambda^{1,2}(\phi_2(v_{ij}, v_{jl}), \phi_2(v_{ij}, v_{jk} \otimes v_{kl})) + \lambda^{0,3}(\phi_3(v_{ij}, v_{jk}, v_{kl})) , \end{aligned} \quad (5.49b)$$

$$\pi(v_{ij})A_i = A_j\pi(v_{ij}) + d\pi(v_{ij}) , \quad (5.49c)$$

$$B_i = B_j - d\zeta_{ij} - \lambda^{0,3}(v_{ij}, A_i, A_i) + \lambda^{0,3}(A_j, v_{ij}, A_i) - \lambda^{0,3}(A_j, A_j, v_{ij}) , \quad (5.49d)$$

$$\zeta_{kj} + \lambda^{0,3}(A_j, v_{ji}, v_{ij}) = \zeta_{ij} + \zeta_{ki} + da_{jik} + \lambda^{0,3}(v_{ji}, A_k, v_{ik}) - \lambda^{0,3}(v_{ji}, v_{ik}, A_k) . \quad (5.49e)$$

The expressions in (5.49) are called the Deligne 2-cocycle conditions, thus the quintuples $(v_{ij}, a_{ijk}, A_i, B_i, \zeta_{ij})$ together with these relations are called the Deligne 2-cycles for principal \mathcal{S}_λ^w -bundles with connective structures. In particular, (5.49a) and (5.49b) are the previously obtained Čech cocycle conditions, (5.49c) and (5.49d) are the gauge transformations of A and B taken on double overlaps, whereas (5.49e) gives us compatibility conditions for the transformations of the local 1-forms ζ_{ij} on triple overlaps. Moreover, on each open set U_i , we can give the *fake curvature*⁷ 2- and 3-forms as follows

$$\mathcal{F}_i := dA_i + \frac{1}{2}[A_i, A_i] = 0 , \quad \text{and} \quad H_i := dB_i - \frac{1}{3!}\mu_3(A_i, A_i, A_i) . \quad (5.50)$$

Let us now give the coboundary relations, which are Deligne 1-cochains relating two equivalent 2-cocycles.

Definition 5.3.3. *A Deligne 2-coboundary between two Deligne 2-cocycles $(v_{ij}, a_{ijk}, A_i, B_i, \zeta_{ij})$ and $(v'_{ij}, a'_{ijk}, A'_i, B'_i, \zeta'_{ij})$ consists of $(\beta_i, \zeta_i, \alpha_{ij})$ for $\beta = (\beta_i) \in \mathcal{C}^\infty(U, V_1)$, $\zeta = (\zeta_i) \in \Omega^1(U, \mathfrak{u}(1))$ and $\alpha = (\alpha_{ij}) \in \mathcal{C}^\infty(U \times_X U, \mathfrak{U}(1))$ satisfying*

$$\beta_i \otimes v'_{ij} = v_{ij} \otimes \beta_j , \quad \beta_{ii} = \text{id}_{\phi_1(\pi(\beta_i))} , \quad (5.51a)$$

$$\begin{aligned} & \alpha_{ik} + a_{ijk} + \lambda^{1,2}(\phi_2(v_{ik}, \beta_k), \phi_2(v_{ij} \otimes v_{jk}, \beta_k)) \\ &= \alpha_{ij} + a'_{ijk} + \alpha_{jk} + \lambda^{1,2}(\phi_2(\beta_i, v'_{ik}), \phi_2(\beta_i, v'_{ij} \otimes v'_{jk})) \\ & \quad + \lambda^{0,3}(\beta_i, v'_{ij}, v'_{jk}) - \lambda^{0,3}(v_{ij}, \beta_j, v'_{jk}) + \lambda^{0,3}(v_{ij}, v_{jk}, \beta_k) , \end{aligned} \quad (5.51b)$$

$$\pi(\beta_i)A'_i = A_i\pi(\beta_i) + d\pi(\beta_i) , \quad (5.51c)$$

$$B'_i = B_i - d\zeta_i - \lambda^{0,3}(\beta_i, A'_i, A'_i) + \lambda^{0,3}(A_i, \beta_i, A'_i) - \lambda^{0,3}(A_i, A_i, \beta_i) , \quad (5.51d)$$

⁷Recall that, the fake curvature \mathcal{F} has to vanish to make sense of the expressions in (5.49d) and (5.49e).

$$\begin{aligned}
 & \zeta_{ji} - \lambda^{0,3}(A_i, v_{ij}, \beta_j) + \lambda^{0,3}(A_i, \beta_i, v'_{ij}) + \lambda^{0,3}(\beta_i, v'_{ij}, A'_j) + \zeta_j \\
 & = \zeta'_{ji} - \lambda^{0,3}(v_{ij}, A_j, \beta_j) + \lambda^{0,3}(v'_{ij}, \beta_j, A'_j) + d\alpha_{ij} + \lambda^{0,3}(\beta_i, A'_i, v'_{ij}) + \zeta_i .
 \end{aligned} \tag{5.51e}$$

If such a coboundary exists, then the two Deligne 2-cocycles $(v_{ij}, a_{ijk}, A_i, B_i, \zeta_{ij})$ and $(v'_{ij}, a'_{ijk}, A'_i, B'_i, \zeta'_{ij})$ are called equivalent. Moreover, similar to the Deligne 2-cocycle relations, the first two expressions (5.51a) and (5.51b) are Čech coboundary relations, while (5.51c), (5.51d) and (5.51e) are the coboundary relations of the respective gauge transformations given above.

As a consistency check, one can consider the Deligne 2-cocycle relations and remove one index, say k . Then relabel all affected cochains as their corresponding parts of a 2-coboundary, e.g. $v_{ik} \rightarrow v_i \rightarrow \beta_i$. The resulting relations have to agree with the relations for a 2-coboundary between a Deligne 2-cocycle and the trivial Deligne 2-cocycle, which they do. Furthermore, one can also give the gauge transformations of the curvatures given in (5.50) on each open set U_i as

$$\mathcal{F}'_i = \beta_i^{-1} \mathcal{F}_i \beta_i , \quad \text{and} \quad H'_i = H_i . \tag{5.52}$$

The above two definitions provide all necessary details for a global description of the kinematical part of higher gauge theory with the string 2-group \mathcal{S}_λ^w as structure 2-group. In Chapter 7, we will present a possible application of this long construction.

Chapter 6

Higher Poincaré lemma and higher integrability

This chapter is basically taken from [29].

The aim of this chapter is to prove the higher Poincaré lemma, (see Theorem 6.1.3) and generalize the description of the classical integrability condition on connections of principal bundles to local connective structures on principal 2-bundles, (see Theorem 6.2.6). Moreover, we sketch how these statements and proofs can be established to principal 3-bundles. We believe that the results in this chapter provide important insights in how to generalize classical concepts and terms into the categorified settings.

6.1 The Poincaré lemma on higher gauge theory

As the Poincaré lemma is a local statement, we shall be merely interested in the local description of higher gauge theories. That is, we consider local connective structures on principal 2- and 3-bundles, which are encoded in certain differential forms on an open contractible patch of a smooth manifold. We ignore all issues related to glueing these local objects to global ones.

The local description of higher gauge theory is readily derived, cf. e.g. [42]. Consider the tensor product of the differential graded algebra of differential forms $\Omega^\bullet(U)$ on a good cover U of a smooth manifold with a semistrict gauge Lie 2-algebra,

cf. Section 5.3.

Here, we restrict our attention to the case of principal 2- and 3-bundles with strict structure 2- and 3-groups. It is not easy to imagine that an analogous statement fails to hold in the semistrict case or for higher principal n -bundles, and a proof for these cases along similar lines to the ones below should exist.

Recall that the classical Poincaré lemma on 1-forms states that every closed 1-form on a contractible domain is exact. In the language of principal $U(1)$ -bundles, this statement means a flat connection is gauge equivalent to the trivial connection on the trivial bundle. A more general form of this statement arises by taking arbitrary Lie group G , which results in a non-abelian Poincaré lemma. In this section, we aim to discuss the proofs of analogous statements on strict principal 2-bundles in two different ways.

6.1.1 A generalized Poincaré lemma

The usual Poincaré lemma states that the equation $d\alpha = \beta$ involving some p - and $p + 1$ -forms α and β can be solved in an open, contractible region if and only if $d\beta = 0$. In [40], Jacobowitz presented a generalization of this statement which we briefly review below. The precise definition of having local solutions is as follows.

Definition 6.1.1. *We say that the equation $d\omega = \Psi_{p+1}(x, \omega)$ for a p -form ω is solvable in a region D , if for each $x \in D$ and for each $\omega_0 \in \wedge^p T^*M|_x$, there is an open neighbourhood $U_x \subset D$ and an $\omega \in \Omega^p(U_x)$ such that $d\omega = \Psi_{p+1}(x, \omega)$ and $\omega|_x = \omega_0$.*

The generalized Poincaré lemma reads then as follows.

Proposition 6.1.2. *The equation $d\omega = \Psi_{p+1}(x, \omega)$ is solvable in a region D , if for all $x \in D$ there is a neighbourhood U_x such that for all $\omega_0 \in \Omega^p(U_x)$ with $d\omega_0 = \Psi_{p+1}(\omega_0)$ at x , we have $d\Psi_{p+1}(\omega_0) = 0$ at x . This statement generalizes to systems of such equations with forms ω of varying degree.*

The proof found in [40] is a generalization of the usual proof of the Frobenius theorem.

Recall that the ordinary Frobenius theorem states that an involutive distribution \mathcal{D} on a manifold X (i.e. a smoothly varying family of subspaces of the tangent bundle, on whose sections the Lie bracket of vector fields closes) corresponds to a regular foliation of X by submanifolds N . In modern language, the distribution is the annihilator of a differential ideal generated by 1-forms. Such a differential ideal comes with integral submanifolds. That is, for each point $p \in X$, we have an embedding $i : N_p \hookrightarrow X$ such that $p \in N_p$ and $i^*\alpha = 0$ for any form α in the differential ideal. These integral submanifolds correspond to the leaves of the foliation of X .

It does not seem to be completely clear how to generalize this picture to higher forms. The equation $d\omega = \Psi_{p+1}(x, \omega)$ is certainly again encoded in a differential ideal which, however, is no longer generated exclusively by 1-forms. Such an ideal forms an exterior differential system, which admits integral submanifolds if and only if Cartan's test is passed, cf. [19]. One issue with Cartan's test is that it does not work in the smooth, but only in the real analytic category. In Appendix E, we present some partial generalization of the notion of distribution, which correspond to a differential ideal. The conditions of Cartan's test, however, do not seem to have a clear interpretation in the context of generalized distributions.

Now let us review connective structures on local principal 2-bundles with strict structure 2-group. Our main reference for this subsection is [6].

6.1.2 Local flat connective structures on strict principal 2-bundles

A principal 2-bundle is essentially the non-abelian generalization of a gerbe, see [17, 1, 8]. Connective structures on principal 2-bundles were discussed in detail in [6]. Here, we will only need the local description over an open, contractible patch U of a smooth manifold X and the only non-trivial data will be the local connective structure over the patch U .

Principal 2-bundles come with a structure Lie 2-group. The most general Lie 2-groups are notoriously difficult to handle, and we therefore restrict our attention in this paper to strict such 2-groups. These are well-known to be equivalent to crossed

modules of Lie groups, cf. [4], See also Definition 2.2.14.

Now applying the tangent functor to a crossed module of Lie groups, one can obtain its corresponding crossed module of Lie algebras. We recall its definition.

A crossed module of Lie algebras $(\mathfrak{h} \xrightarrow{\mathfrak{t}} \mathfrak{g}, \triangleright)$ is a pair of Lie algebras \mathfrak{g} and \mathfrak{h} together with a Lie algebra homomorphism $\mathfrak{t} : \mathfrak{h} \rightarrow \mathfrak{g}$ and an action by derivation \triangleright of \mathfrak{g} on \mathfrak{h} . The compatibility conditions here read as:

$$\mathfrak{t}(x \triangleright y) = [x, \mathfrak{t}(y)] \quad \text{and} \quad \mathfrak{t}(y_1) \triangleright y_2 = [y_1, y_2] \quad (6.1)$$

for all $x \in \mathfrak{g}$ and $y, y_{1,2} \in \mathfrak{h}$.

Here, we only need the local description of connective structures on strict principal 2-bundles as explained in [6]. Given an open, contractible patch U of a smooth manifold X , a local connective structure over U of a principal 2-bundle with structure crossed module $(\mathbf{H} \xrightarrow{\mathfrak{t}} \mathbf{G}, \triangleright)$ is given by a $\text{Lie}(\mathbf{G})$ -valued 1-form A together with a $\text{Lie}(\mathbf{H})$ -valued 2-form B over U . The corresponding curvatures read

$$\mathcal{F} := dA + \frac{1}{2}[A, A] - \mathfrak{t}(B) \quad \text{and} \quad H := dB + A \triangleright B . \quad (6.2)$$

An equivalence relation on local connective structures is given by gauge transformations, which are parameterized by a \mathbf{G} -valued function g together with a $\text{Lie}(\mathbf{H})$ -valued 1-form Λ as follows:

$$\begin{aligned} A &\mapsto \tilde{A} := g^{-1}Ag + g^{-1}dg - \mathfrak{t}(\Lambda) , \\ B &\mapsto \tilde{B} := g^{-1} \triangleright B - d\Lambda - \tilde{A} \triangleright \Lambda - \frac{1}{2}[\Lambda, \Lambda] , \\ \mathcal{F} &\mapsto \tilde{\mathcal{F}} := g^{-1}\mathcal{F}g , \\ H &\mapsto \tilde{H} := g^{-1} \triangleright H - \tilde{\mathcal{F}} \triangleright \Lambda . \end{aligned} \quad (6.3)$$

If a connective structure is to describe a consistent parallel transport of a 1-dimensional object along a surface, the curvature \mathcal{F} , also called “fake curvature” has to vanish. Note that the equation $\mathcal{F} = 0$ is invariant under gauge transformations (6.3), and recall that a local connective structure (A, B) is flat, if $\mathcal{F} = 0$ and $H = 0$. Note also that a flat connective structure remains flat under gauge transformations (6.3). We

now have all the ingredients in order to explain the proofs of the higher Poincaré lemma on principal 2-bundles with strict structure 2-groups. The following subsection contains the first method.

6.1.3 Poincaré lemma for strict principal 2-bundles: Proof-I

The higher Poincaré lemma for principal 2-bundles reads as follows:

Theorem 6.1.3. *For any flat local connective structure (A, B) on a contractible patch U and any point $p \in U$, there is a neighbourhood U_p of p such that (A, B) is pure gauge. That is, it can be written as*

$$\begin{aligned} A &= g^{-1}dg - \mathfrak{t}(\Lambda) , \\ B &= -d\Lambda - A \triangleright \Lambda - \frac{1}{2}[\Lambda, \Lambda] , \end{aligned} \tag{6.4}$$

for some \mathbf{G} -valued function g and \mathfrak{h} -valued 1-form Λ on $U_p \subset U$.

Proof. For simplicity, we assume that \mathbf{G} and \mathbf{H} are matrix groups. The proof is, however, readily extended to the general case. We can rewrite equations (6.4) as

$$\begin{aligned} dg^{-1} &= -Ag^{-1} - \mathfrak{t}(\Lambda)g^{-1} =: \Psi_1(g, \Lambda) , \\ d\Lambda &= -B - A \triangleright \Lambda - \frac{1}{2}[\Lambda, \Lambda] =: \Psi_2(g, \Lambda) . \end{aligned} \tag{6.5}$$

We regard (6.5) as a system of equations of the form $d\omega = \Psi_{p+1}(\omega, x)$ with $\dim(\mathbf{G})$ 0-forms and $\dim(\text{Lie}(\mathbf{H}))$ 1-forms. To apply Proposition 6.1.2, we merely have to show that $d\Psi_1(g_0, \Lambda_0) = 0$ and $d\Psi_2(g_0, \Lambda_0) = 0$ at any $x \in U$ if $\mathcal{F} = H = 0$ as well as $dg_0^{-1} = \Psi_1(g_0, \Lambda_0)$ and $d\Lambda_0 = \Psi_2(g_0, \Lambda_0)$ at x . We compute

$$\begin{aligned} d\Psi_1(g_0, \Lambda_0)|_x &= (-dAg_0^{-1} + A \wedge dg_0^{-1} - \mathfrak{t}(d\Lambda_0)g_0^{-1} + \mathfrak{t}(\Lambda_0) \wedge dg_0^{-1})|_x \\ &= (A \wedge Ag_0^{-1} - \mathfrak{t}(B)g_0^{-1} + (A + \mathfrak{t}(\Lambda_0)) \wedge \Psi_1(g_0, \Lambda_0) - \mathfrak{t}(\Psi_2(g_0, \Lambda_0))g_0^{-1})|_x \\ &= 0 , \end{aligned} \tag{6.6a}$$

and

$$\begin{aligned}
 d\Psi_2(g_0, \Lambda_0)|_x &= (-dB - dA \triangleright \Lambda_0 + A \triangleright d\Lambda_0 - [d\Lambda_0, \Lambda_0])|_x \\
 &= (A \triangleright B + (A \wedge A - \mathfrak{t}(B)) \triangleright \Lambda_0 + (A + \mathfrak{t}(\Lambda_0)) \triangleright \Psi_2(g_0, \Lambda_0))|_x \quad (6.6b) \\
 &= 0 .
 \end{aligned}$$

This proves that $d\Psi_1(g_0, \Lambda_0) = 0$ and $d\Psi_2(g_0, \Lambda_0) = 0$ at any $x \in U$ for all (g_0, Λ_0) satisfying (6.5) at x whenever $\mathcal{F} = 0$ and $H = 0$. Therefore, applying Proposition 6.1.2 on (6.5) shows that being flatness guarantees the solvability of (6.5), and hence, also (6.4) on U . According to Definition 6.1.1, this means that there is a solution in a neighbourhood $U_p \subset U$ of each point $p \in U$. \square

Remark 6.1.4. *Note that, in this proof we have assumed that (g_0, Λ_0) satisfying (6.5) at $x \in U$ exists, for a contractible patch U , see also [40].*

6.1.4 Poincaré lemma for strict principal 2-bundles: Proof-II

We now come to the second proof of the Poincaré lemma, which yields the explicit gauge transformation trivializing a flat local connective structure. Our proof will be a direct generalization of that of [90], where the author constructs a solution to a Cauchy problem, relating pullbacks of flat connections along homotopic maps by a gauge transformation. On a contractible patch of a smooth manifold, flat connective structures are therefore gauge equivalent to pullbacks along constant maps. This implies that they are locally pure gauge. Thus, we state it as follows.

Theorem 6.1.5. *(Higher Poincaré lemma) Flat local connective structures are gauge equivalent to the trivial connective structure.*

Now, let us set up our notations first. Let U be an open contractible patch of a smooth manifold X . Over $U \times [0, 1]$, let (A, B) be a local connective structure with underlying crossed module of Lie groups $(\mathbf{H} \xrightarrow{\mathfrak{t}} \mathbf{G}, \triangleright)$ with the corresponding differential crossed module $(\mathfrak{h} \xrightarrow{\mathfrak{t}} \mathfrak{g}, \triangleright)$. To simplify our notation, we assume that \mathbf{G} and \mathbf{H} are matrix groups. We decompose the differential forms A and B according

to

$$A = A_x + dt A_t \quad \text{and} \quad B = B_x + dt B_t , \quad (6.7)$$

where $\frac{\partial}{\partial t} \lrcorner A_x = 0$ and $\frac{\partial}{\partial t} \lrcorner B_x = 0$. Similarly, we decompose the exterior derivative

$$d\omega = d_x \omega + dt \frac{\partial}{\partial t} \omega = d_x \omega + dt \dot{\omega} . \quad (6.8)$$

We are interested in solutions $g \in \mathcal{C}^\infty(U \times [0, 1], \mathbf{G})$ and $\Lambda \in \Omega^1(U \times [0, 1], \mathfrak{h})$ to the following Cauchy problem, which arises by considering gauge transformations of the components \tilde{A}_t and \tilde{B}_t to 0, cf. (6.3):

$$\begin{aligned} \dot{g} &= -A_t g + g \mathfrak{t}(\Lambda_t) , \\ \dot{\Lambda}_x &= g^{-1} \triangleright B_t + d_x \Lambda_t + (g^{-1} A_x g + g^{-1} d_x g) \triangleright \Lambda_t \end{aligned} \quad (6.9a)$$

with initial conditions

$$g(x, 0) = \mathbb{1}_{\mathbf{G}} \quad \text{and} \quad \Lambda(x, 0) = 0 \quad \text{for} \quad x \in U . \quad (6.9b)$$

Remark 6.1.6. *These Cauchy problems are the general ones. For instance, one can take $\Lambda_t = 0$ to get another, but too strict one.*

Proposition 6.1.7. *Let (g, Λ) be a solution to the Cauchy problem (6.9). Then*

$$-g_1^{-1} dg_1 + \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} \mathcal{F}g) = g_1^{-1} A_x|_{t=1} g_1 - A_x|_{t=0} - \mathfrak{t}(\Lambda_x)|_{t=1} , \quad (6.10)$$

where $g_1 := g(x, 1)$ and \mathcal{F} is the fake curvature of the local connective structure (A, B) .

Proof. First, using (6.9a), we readily compute

$$\frac{\partial}{\partial t} (g^{-1} d_x g) = -g^{-1} (d_x A_t) g + \mathfrak{t} (g^{-1} d_x g \triangleright \Lambda_t) + d_x \mathfrak{t}(\Lambda_t) , \quad (6.11)$$

and

$$\frac{\partial}{\partial t} (g^{-1} A_x g) = g^{-1} \left(\dot{A}_x + [A_t, A_x] \right) g + \mathfrak{t}(g^{-1} A_x g \triangleright \Lambda_t) . \quad (6.12)$$

Moreover,

$$g_1^{-1}dg_1 = (g^{-1}d_x g)|_{t=1} \quad \text{and} \quad d_x g|_{t=0} = 0 . \quad (6.13)$$

Now we would like to rewrite (6.11) and (6.12) in terms of the fake curvature of (A, B) . Note that

$$\begin{aligned} \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} \mathcal{F} g) &= \int_0^1 dt g^{-1} \left(-d_x A_t + \dot{A}_x + [A_t, A_x] - \mathfrak{t}(B_t) \right) g \\ &= \int_0^1 dt \frac{\partial}{\partial t} (g^{-1} d_x g) + \int_0^1 dt \frac{\partial}{\partial t} (g^{-1} A_x g) \\ &\quad - \int_0^1 dt \mathfrak{t} \left(g^{-1} \triangleright B_t + d_x \Lambda_t + (g^{-1} A_x g + g^{-1} d_x g) \triangleright \Lambda_t \right) . \end{aligned} \quad (6.14)$$

Using (6.9a) and (6.13), we can further simplify this to

$$\int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} \mathcal{F} g) = g_1^{-1} dg_1 + \int_0^1 dt \frac{\partial}{\partial t} (g^{-1} A_x g) - \int_0^1 dt \frac{\partial}{\partial t} \mathfrak{t}(\Lambda_x) , \quad (6.15)$$

which is obviously equivalent to (6.10). \square

Next, we prove an analogous statement involving the 3-form curvature H of (A, B) :

Proposition 6.1.8. *Let (g, Λ) be a solution to the Cauchy problem (6.9). Then*

$$\begin{aligned} d_x \Lambda_1 + g_1^{-1} dg_1 \triangleright \Lambda_x|_{t=1} - (\Lambda_x \wedge \Lambda_x)|_{t=1} + (g_1^{-1} A_x|_{t=1} g_1) \triangleright \Lambda_x|_{t=1} \\ = - \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} \triangleright H - (g^{-1} \mathcal{F} g) \triangleright \Lambda) + g_1^{-1} \triangleright B_x|_{t=1} - B_x|_{t=0} , \end{aligned} \quad (6.16)$$

where $g_1 := g(x, 1)$, $\Lambda_1 = \Lambda(x, 1)$ and \mathcal{F} and H are the fake and 3-form curvatures of a local connective structure (A, B) .

Proof. In this case we have

$$d_x \Lambda_x|_{t=0} = 0 \quad \text{and} \quad d_x \Lambda_1 = d_x \Lambda_x|_{t=1} . \quad (6.17)$$

Moreover, by direct differentiation and using (6.9a), we obtain

$$\frac{\partial}{\partial t} (d_x \Lambda_x) = d_x (g^{-1} \triangleright B_t + d_x \Lambda_t + (g^{-1} A_x g + g^{-1} d_x g) \triangleright \Lambda_t) , \quad (6.18a)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} (g^{-1} d_x g \triangleright \Lambda_x) &= (-g^{-1} (d_x A_t) g + \mathfrak{t}(g^{-1} d_x g \triangleright \Lambda_t) + d_x \mathfrak{t}(\Lambda_t)) \triangleright \Lambda_x \\ &+ g^{-1} d_x g \triangleright (g^{-1} \triangleright B_t + d_x \Lambda_t + (g^{-1} A_x g + g^{-1} d_x g) \triangleright \Lambda_t) . \end{aligned} \quad (6.18b)$$

Thus, considering the expressions of the fake and the 3-curvatures of a local connective structure (A, B) yields

$$\begin{aligned} &\int_0^1 dt \frac{\partial}{\partial t} \lrcorner \left(-g^{-1} \mathcal{F} g \triangleright \Lambda + g^{-1} \triangleright H \right) \\ &= \int_0^1 dt \left(g^{-1} d_x A_t g \triangleright \Lambda_x - g^{-1} \left(\dot{A}_x + [A_t, A_x] \right) g \triangleright \Lambda_x \right) \\ &+ \int_0^1 dt \left(-g^{-1} (d_x A_x + A_x \wedge A_x) g \triangleright \Lambda_t + g^{-1} \mathfrak{t}(B_x) g \triangleright \Lambda_t \right) \\ &+ \int_0^1 dt \left(g^{-1} \mathfrak{t}(B_t) g \triangleright \Lambda_x + g^{-1} \triangleright \left(\dot{B}_x + A_t \triangleright B_x - d_x B_t - A_x \triangleright B_t \right) \right) . \end{aligned} \quad (6.19)$$

But by direct differentiation and using (6.9a) we have

$$\begin{aligned} \frac{\partial}{\partial t} \left((g^{-1} A_x g) \triangleright \Lambda_x \right) &= \left(g^{-1} \left(\dot{A}_x + [A_t, A_x] \right) g \right) \triangleright \Lambda_x + \mathfrak{t}(g^{-1} A_x g \triangleright \Lambda_t) \triangleright \Lambda_x \\ &+ (g^{-1} A_x g) \triangleright (g^{-1} \triangleright B_t + d_x \Lambda_t + g^{-1} A_x g \triangleright \Lambda_t + g^{-1} d_x g \triangleright \Lambda_t) , \end{aligned} \quad (6.20a)$$

and

$$\frac{\partial}{\partial t} (g^{-1} \triangleright B_x) = (g^{-1} A_t - \mathfrak{t}(\Lambda_t) g^{-1}) \triangleright B_x + g^{-1} \triangleright \dot{B}_x . \quad (6.20b)$$

Now applying (6.9a), after combining (6.18a), (6.18b), (6.20a) and (6.20b), gives

$$\begin{aligned}
 & \int_0^1 dt \frac{\partial}{\partial t} \lrcorner \left(-g^{-1} \mathcal{F}g \triangleright \Lambda + g^{-1} \triangleright H \right) \\
 &= \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (-d_x \Lambda_x) + \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (-g^{-1} d_x g \triangleright \Lambda_x) \\
 &+ \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (-g^{-1} A_x g) \triangleright \Lambda_x + \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} \triangleright B_x) \\
 &+ \int_0^1 dt \left(\mathfrak{t}(\dot{\Lambda}_x) \triangleright \Lambda_x \right) . \tag{6.21}
 \end{aligned}$$

After simplification of (6.21) using (6.9b) and (6.17), we finally arrive at

$$\begin{aligned}
 & d_x \Lambda_1 + g_1^{-1} d g_1 \triangleright \Lambda_x|_{t=1} - (\Lambda_x \wedge \Lambda_x)|_{t=1} + (g_1^{-1} A_x|_{t=1} g_1) \triangleright \Lambda_x|_{t=1} \\
 &= - \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (-g^{-1} \mathcal{F}g \triangleright \Lambda + g^{-1} \triangleright H) + g_1^{-1} \triangleright B_x|_{t=1} - B_x|_{t=0} . \tag{6.22}
 \end{aligned}$$

□

We can now follow [90] further and consider homotopic maps $h_{0,1}(x) : U \rightrightarrows V$ between local patches U and V of some smooth manifolds. Let $h(x, t) : U \times [0, 1] \rightarrow V$ with $h(x, 0) = h_0(x)$ and $h(x, 1) = h_1(x)$ be a homotopy satisfying $\frac{\partial}{\partial t} h(x, t)|_{t=0,1} = 0$. Because the pullback is compatible with the wedge product and the exterior derivative, Propositions 6.1.7 and 6.1.8 yield the following corollary.

Corollary 6.1.9. *The pullbacks of a local connective structure (A, B) on the patch V of some manifold along homotopic maps $h_{0,1} : U \rightrightarrows V$ are related as follows:*

$$-g_1^{-1} d g_1 + \mathfrak{t}(\Lambda_{1,x}) + \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} h^*(\mathcal{F})g) = g_1^{-1} h_1^*(A_x) g_1 - h_0^* A_x , \tag{6.23a}$$

$$\begin{aligned}
 & d_x \Lambda_1 + (g_1^{-1} h_1^*(A_x) g_1) \triangleright \Lambda_{1,x} + (g_1^{-1} d g_1) \triangleright \Lambda_{1,x} - (\Lambda_{1,x} \wedge \Lambda_{1,x}) \\
 &= - \int_0^1 dt \frac{\partial}{\partial t} \lrcorner (g^{-1} \triangleright h^* H - g^{-1} h^*(\mathcal{F})g \triangleright \Lambda) + g_1^{-1} \triangleright h_1^* B_x - h_0^* B_x , \tag{6.23b}
 \end{aligned}$$

where h denotes a homotopy between h_0 and h_1 with $\frac{\partial}{\partial t} h(x, t)|_{t=0,1} = 0$, (g, Λ) is a solution of the Cauchy problem (6.9) and $g_1 = g(x, 1)$, $\Lambda_1 = \Lambda(x, 1)$. In particular, the pullbacks for flat connective structures are gauge equivalent.

This corollary can now be used to prove the Poincaré lemma. Consider an open

contractible patch U of a smooth manifold and regard it as a subset of some vector space \mathbb{R}^d containing the origin 0_U . We are interested in the homotopy $h(x, t) : U \times [0, 1] \rightarrow U$ with $h(x, t) = xtk(t)$ between U and the point $0_U \in U$, where $k(t)$ is a smooth function such that $k'(t)|_{t=0,1} = 0$, $k(0) = 0$ and $k(1) = 1$. Note that the pullback of the connective structure on U along h_0 vanishes, which implies our main result Theorem 6.1.5.

6.1.5 Local flat connective structures on principal 3-bundles

Principal 3-bundles are one step further in the categorification of principal bundles, and form non-abelian generalizations of 2-gerbes. The full description of principal 3-bundles with connective structures is found in [69], see also [55, 43] for partial earlier accounts.

Principal 3-bundles use Lie 3-groups as structure 3-groups, and we shall restrict ourselves to semistrict 3-groups for simplicity. Just as strict Lie 2-groups are categorically equivalent to crossed modules of Lie 2-groups, semistrict Lie 3-groups are equivalent to 2-crossed modules of Lie groups. We therefore start by recalling the latter notion [28].

Definition 6.1.10. *A 2-crossed module of Lie groups is a normal complex of Lie groups (i.e. a complex of Lie groups in which each image of \mathfrak{t} is a normal subgroup of the next group)*

$$\mathbf{L} \xrightarrow{\mathfrak{t}} \mathbf{H} \xrightarrow{\mathfrak{t}} \mathbf{G} , \quad (6.24)$$

together with an action, \triangleright , of \mathbf{G} on \mathbf{H} and \mathbf{L} by automorphism as well as a \mathbf{G} -equivariant binary map $\{\cdot, \cdot\} : \mathbf{H} \times \mathbf{H} \rightarrow \mathbf{L}$ satisfying the following conditions. For all $h, h_{1,2,3} \in \mathbf{H}$, $g \in \mathbf{G}$ and $\ell, \ell_{1,2} \in \mathbf{L}$, we have

$$(i) \quad \mathfrak{t}(g \triangleright \ell) = g \triangleright \mathfrak{t}(\ell) \quad \text{and} \quad \mathfrak{t}(g \triangleright h) = g\mathfrak{t}(h)g^{-1},$$

$$(ii) \quad \mathfrak{t}(\{h_1, h_2\}) = (h_1 h_2 h_1^{-1}) (\mathfrak{t}(h_1) \triangleright h_2^{-1}),$$

$$(iii) \quad \{\mathfrak{t}(\ell_1), \mathfrak{t}(\ell_2)\} = \ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1} := [\ell_1, \ell_2],$$

$$(iv) \quad \{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\} (\mathfrak{t}(h_1) \triangleright \{h_2, h_3\}),$$

$$(v) \{h_1, h_2 h_3\} = \{h_1, h_2\} \{h_1, h_3\} \{t(\{h_1, h_3\})^{-1}, t(h_1) \triangleright h_2\},$$

$$(vi) \{h, t(\ell)\} = (\{t(\ell), h\})^{-1} \ell (t(h) \triangleright \ell^{-1}).$$

The map $\{\cdot, \cdot\}$ is called the Peiffer lifting and measures the failure of $(\mathbf{H} \xrightarrow{t} \mathbf{G}, \triangleright)$ to be a crossed module. Sometimes we use $\mathbf{L} \rightarrow \mathbf{H} \rightarrow \mathbf{G}$ to denote 2-crossed modules. Lie 2-crossed modules are generalizations of Lie crossed modules. In particular, if the Peiffer lifting is trivial, then (6.24) becomes a crossed module. Moreover, we can obtain a Lie crossed module from the given Lie 2-crossed module in (6.24) by considering $(\mathbf{L} \xrightarrow{t} \mathbf{H}, \triangleright)$ together with the induced action

$$h \triangleright \ell := \ell \{t(\ell)^{-1}, h\} \quad (6.25)$$

for all $h \in \mathbf{H}$ and $\ell \in \mathbf{L}$. Note that here all the conditions in Definition 2.2.14 follow using (i), (ii) and (vi) from Definition 6.1.10.

Applying the tangent functor to the normal sequence (6.24), we obtain the axioms for 2-crossed modules of Lie algebras. Here, let us recall the definition of differential Lie 2-crossed modules.

Let $(\mathfrak{l}, \mathfrak{h}, \mathfrak{g})$ be a triple of Lie algebras. A 2-crossed module of Lie algebras (or a differential Lie 2-crossed module) is a normal complex¹ of Lie algebras

$$\mathfrak{l} \xrightarrow{t} \mathfrak{h} \xrightarrow{t} \mathfrak{g}, \quad (6.26)$$

together with actions \triangleright of \mathfrak{g} on \mathfrak{l} and \mathfrak{h} by derivation as well as a \mathfrak{g} -equivariant bilinear map, $\{\cdot, \cdot\} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{l}$ satisfying the conditions

$$(i) \quad t(x \triangleright z) = x \triangleright t(z) \text{ and } t(x \triangleright y) = [x, t(y)],$$

$$(ii) \quad t(\{y_1, y_2\}) = [y_1, y_2] - t(y_1) \triangleright y_2,$$

$$(iii) \quad \{t(z_1), t(z_2)\} = [z_1, z_2],$$

$$(iv) \quad \{[y_1, y_2], y_3\} = t(y_1) \triangleright \{y_2, y_3\} + \{y_1, [y_2, y_3]\} \\ - t(y_2) \triangleright \{y_1, y_3\} - \{y_2, [y_1, y_3]\},$$

¹i.e. a complex in which the image of each term is an ideal of the next

- (v) $\{y_1, [y_2, y_2]\} = \{\mathfrak{t}(\{y_1, y_2\}), y_3\} - \{\mathfrak{t}(\{y_1, y_3\}), y_2\},$
 (vi) $-\{\mathfrak{t}(z), y\} = \{y, \mathfrak{t}(z)\} + \mathfrak{t}(y) \triangleright z,$

for every $x \in \mathfrak{g}$, $y, y_{1,2,3} \in \mathfrak{h}$, and $z, z_{1,2} \in \mathfrak{l}$.

Note also that a Lie 2-crossed module of Lie groups can be partially linearized to obtain more general actions, as e.g. the action of \mathbf{G} onto \mathfrak{h} . More details on 2-crossed modules can be found in [55, 69].

The local description of a connective structure on a principal 3-bundle is now readily given, cf. [69].

Let U be a contractible patch of a smooth manifold X . A local connective structure over U of a principal 3-bundle with structure 2-crossed module $(\mathbf{L} \rightarrow \mathbf{H} \rightarrow \mathbf{G}, \triangleright, \{\cdot, \cdot\})$ can be expressed as a triple of Lie algebra valued forms (A, B, C) , where $A \in \Omega^1(U, \text{Lie}(\mathbf{G}))$, $B \in \Omega^2(U, \text{Lie}(\mathbf{H}))$ and $C \in \Omega^3(U, \text{Lie}(\mathbf{L}))$. Corresponding curvatures are defined according to

$$\begin{aligned} \mathcal{F} &:= dA + \frac{1}{2}[A, A] - \mathfrak{t}(B), \\ \mathcal{H} &:= dB + A \triangleright B - \mathfrak{t}(C), \quad G := dC + A \triangleright C + \{B, B\}. \end{aligned} \tag{6.27}$$

Gauge transformations act on the Lie algebra valued forms according to

$$\begin{aligned} A &\mapsto \tilde{A} := g^{-1}Ag + g^{-1}dg - \mathfrak{t}(\Lambda), \\ B &\mapsto \tilde{B} := g^{-1} \triangleright B - (d + \tilde{A} \triangleright)\Lambda - \frac{1}{2}\mathfrak{t}(\Lambda) \triangleright \Lambda - \mathfrak{t}(\Sigma), \\ C &\mapsto \tilde{C} := g^{-1} \triangleright C - \left((d + \tilde{A} \triangleright) + \mathfrak{t}(\Lambda) \triangleright \right) \Sigma + \{\tilde{B} + \frac{1}{2}(d + \tilde{A} \triangleright)\Lambda \\ &\quad + \frac{1}{2}[\Lambda, \Lambda], \Lambda\} + \{\Lambda, \tilde{B} - \frac{1}{2}(d + \tilde{A} \triangleright)\Lambda - \frac{1}{2}[\Lambda, \Lambda]\}, \\ \mathcal{F} &\mapsto \tilde{\mathcal{F}} := g^{-1}\mathcal{F}g, \\ \mathcal{H} &\mapsto \tilde{\mathcal{H}} := g^{-1} \triangleright \mathcal{H} - \tilde{\mathcal{F}} \triangleright \Lambda, \\ G &\mapsto \tilde{G} := g^{-1} \triangleright G - \left(\tilde{\mathcal{F}} \triangleright (\Sigma - \frac{1}{2}\{\Lambda, \Lambda\}) \right) + \{\Lambda, \tilde{\mathcal{H}}\} \\ &\quad - \{\tilde{\mathcal{H}}, \Lambda\} - \{\Lambda, \tilde{\mathcal{F}} \triangleright \Lambda\}, \end{aligned} \tag{6.28}$$

where g is a \mathbf{G} -valued function and Λ and Σ are $\text{Lie}(\mathbf{H})$ and $\text{Lie}(\mathbf{L})$ -valued 1- and 2-forms, respectively.

For consistency of the parallel transport described by this local connective structure, it is necessary that both the 2- and 3-form fake curvatures \mathcal{F} and \mathcal{H} vanish. Moreover, a local connective structure (A, B, C) is said to be flat, if all curvatures vanish: $\mathcal{F} = 0$, $\mathcal{H} = 0$ and $G = 0$. Again, note that as in the case of principal 2-bundles, flat connective structures on principal 3-bundles remain flat under the gauge transformations (6.28).

6.1.6 Comments on the proofs of Poincaré lemma for principal 3-bundles

An interesting aspect of our proof in Subsection 6.1.4 is that it is not necessary to extend the interval $[0, 1]$ used in the case of ordinary principal 2-bundles to $[0, 1]^2$. The latter arises if one wants to define the general transport 2-functor from the path 2-groupoid to the delooping of the strict Lie 2-group corresponding to the crossed module $\mathbf{H} \rightarrow \mathbf{G}$, cf. [73].

Therefore, and since all the terms in the formulas contained in our proof have clear meanings, one can in principle readily generalize our proof to the case of local connective structures on principal 3-bundles. Let us here concisely summarize and give the important steps in proving the Poincaré lemma on principal 3-bundles.

We start from a local connective structure $(\hat{A}, \hat{B}, \hat{C})$ on $U \times [0, 1]$, where U is a contractible patch of a smooth manifold. Let $\mathbf{L} \rightarrow \mathbf{H} \rightarrow \mathbf{G}$ be the relevant 2-crossed module and $\text{Lie}(\mathbf{L}) \rightarrow \text{Lie}(\mathbf{H}) \rightarrow \text{Lie}(\mathbf{G})$ the corresponding linearization. Then the Poincaré lemma reads as follows.

Theorem 6.1.11. *For any flat local connective structure (A, B, C) on a patch U and any point $p \in U$, there is a neighbourhood $U_p \subset U$ of p such that (A, B, C) is pure gauge. That is, it can be written as*

$$\begin{aligned}
 A &= g^{-1}dg - \mathbf{t}(\Lambda) , \\
 B &= -(d + A \triangleright)\Lambda - \frac{1}{2}\mathbf{t}(\Lambda) \triangleright \Lambda - \mathbf{t}(\Sigma) , \\
 C &= -((d + A \triangleright) + \mathbf{t}(\Lambda) \triangleright)\Sigma + \{B + \frac{1}{2}(d + A \triangleright)\Lambda + \frac{1}{2}[\Lambda, \Lambda], \Lambda\} \\
 &\quad + \{\Lambda, B - \frac{1}{2}(d + A \triangleright)\Lambda - \frac{1}{2}[\Lambda, \Lambda]\} ,
 \end{aligned} \tag{6.29}$$

for some \mathbf{G} -valued function g , $\text{Lie}(\mathbf{H})$ -valued 1-form Λ and $\text{Lie}(\mathbf{L})$ -valued 2-form Σ on U_p .

Proof. The proof is fully analogous to that of theorem 6.1.3, but considerably more involved. We therefore only outline the computations. First, we rewrite (6.29) as follows.

$$\begin{aligned}
 dg^{-1} &= -Ag^{-1} - \mathbf{t}(\Lambda)g^{-1} =: \Psi_1(g, \Lambda, \Sigma) , \\
 d\Lambda &= -B - A \triangleright \Lambda - \frac{1}{2}\mathbf{t}(\Lambda) \triangleright \Lambda - \mathbf{t}(\Sigma) =: \Psi_2(g, \Lambda, \Sigma) , \\
 d\Sigma &= -C - A \triangleright \Sigma - \mathbf{t}(\Lambda) \triangleright \Sigma + \{B + \frac{1}{2}(d + A \triangleright)\Lambda + \frac{1}{2}[\Lambda, \Lambda], \Lambda\} \\
 &\quad + \{\Lambda, B - \frac{1}{2}(d + A \triangleright)\Lambda - \frac{1}{2}[\Lambda, \Lambda]\} =: \Psi_3(g, \Lambda, \Sigma) .
 \end{aligned} \tag{6.30}$$

Proposition 6.1.2 guarantees that (6.30) are solvable on U , if $d\Psi_{1,2,3}(g_0, \Lambda_0, \Sigma_0)$ vanish at any $x \in U$ if $\mathcal{F} = \mathcal{H} = G = 0$ as well as

$$dg_0^{-1}|_x = \Psi_1(g_0, \Lambda_0, \Sigma_0)|_x , \quad d\Lambda_0|_x = \Psi_2(g_0, \Lambda_0, \Sigma_0)|_x , \quad d\Sigma_0|_x = \Psi_3(g_0, \Lambda_0, \Sigma_0)|_x . \tag{6.31}$$

We now have to rewrite $d\Psi_{1,2,3}(g_0, \Lambda_0, \Sigma_0)$ in terms of quantities which we know at x . The exterior derivative will hit either a potential n -form or a gauge parameter. The exterior derivatives of the gauge parameters are given in (6.30) and the exterior derivatives of the potential n -forms can be rewritten using the flatness equations $\mathcal{F} = \mathcal{H} = G = 0$.

Putting everything together, we find after a lengthy calculation that given (6.31), $d\Psi_{1,2,3}(g_0, \Lambda_0, \Sigma_0)$ indeed vanish for flat local connective structures. Again, solvability of (6.30) over U implies that for all $p \in U$ there exists a neighbourhood U_p over which (6.30) have a solution. \square

Moreover, the second method of proof follows by constructing the Cauchy problem, which is again given by equations stating that the components of the connective

structures along dt can be gauged away. Here, we have

$$\begin{aligned} \dot{g} &= -A_t g + g \mathfrak{t}(\Lambda_t) , \\ \dot{\Lambda}_x &= g^{-1} \triangleright B_t + d_x \Lambda_t + (g^{-1} A_x g + g^{-1} d_x g) \triangleright \Lambda_t - \mathfrak{t}(\Sigma_t) , \\ \dot{\Sigma}_x &= g^{-1} \triangleright C_t + \dots , \end{aligned} \tag{6.32}$$

where \dots stands for terms easily read off from equations (6.28). As their explicit forms are not illuminating, we suppress them here. This will then lead to statements analogous to Propositions 6.1.7 and 6.1.8, which are of the form

$$\int_0^1 dt \frac{\partial}{\partial t} \lrcorner (\tilde{K}) = \tilde{P}|_{t=1} - \hat{P}|_{t=0} . \tag{6.33}$$

Here, \hat{P} is the potential n -form for $n = 1, 2, 3$ and \hat{K} is the corresponding curvature $n + 1$ -form. Furthermore, \tilde{P} and \tilde{K} denote gauge transformed objects.

These equations describe the relation between pullbacks of a local connective structure along homotopic maps. In particular, they imply that the pullbacks of flat local connective structures along homotopic maps are gauge equivalent. Considering again the homotopy $h(x, t) : U \times [0, 1] \rightarrow U$ with $h(x, t) = xtk(t)$ implies that flat local connective structures are pure gauge.

6.2 Higher integrability

Recall that for a matrix Lie group \mathbf{G} , and its Lie algebra \mathfrak{g} , the linear system

$$\nabla g := (d + A)g = 0 , \tag{6.34}$$

where A is a \mathfrak{g} -valued 1-form, g is a \mathbf{G} -valued smooth function and d is a differential arises in the expressions of gauge fields of classical gauge theory. Moreover, the linear system in (6.34) has solution if and only if the curvature $F = dA + \frac{1}{2}[A, A]$ vanishes.

Proposition 6.2.1. *The linear system in (6.34) has a solution if and only if $\nabla^2 = 0$.*

Proof. Now acting with ∇ on (6.34), we obtain $\nabla(\nabla g) = 0$, which is equivalent to

$Fg := (\nabla)^2g = 0$. Note that the product in $Fg = 0$ is just an ordinary matrix product. Multiplying by g^{-1} from the right, we see that the existence of a solution g requires that the curvature F of ∇ to vanish. Moreover, re-arranging equation (6.34) directly yields the relation $A = gdg^{-1}$, implying $F = 0$. \square

In this section, we demonstrate how these statements and conditions can be translated to linear systems involving local connective structures on principal 2-bundles with strict structure 2-group. To do these we use the idea of constructing L_∞ -algebras from A_∞ -algebras by antisymmetrization of the products, cf. [81, 82, 80]. This process resolves the issue of establishing a higher analogue of having a matrix Lie group structure.

The approach is motivated by the observations of the linear integrable systems (6.34) that uses the matrix Lie algebra \mathfrak{g} , which is a particular example of a 2-term A_∞ -algebra.

Recall that an associative 2-term A_∞ -algebra is a graded vector space $\mathcal{A} := \mathcal{A}_{-1} \oplus \mathcal{A}_0 := \mathfrak{h} \oplus \mathfrak{g}$ together with “products” $m_1 : \mathcal{A} \rightarrow \mathcal{A}$ and $m_2 : \mathcal{A}^{\otimes 2} \rightarrow \mathcal{A}$ of degrees 1 and 0, respectively, such that

$$m_1 \circ m_1 = 0, \quad m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathbb{1} + \mathbb{1} \otimes m_1), \quad m_2 \circ (\mathbb{1} \otimes m_2 - m_2 \otimes \mathbb{1}) = 0. \quad (6.35)$$

The first equation says that m_1 is a differential, the second equation states the compatibility between this differential and the product m_2 and the third equation implies that the product m_2 is associative. We use the usual sign convention for the maps m_i :

$$(m_i \otimes m_j)(a_1 \otimes a_2) = (-1)^{\tilde{m}_j \tilde{a}_1} m_i(a_1) \otimes m_j(a_2), \quad (6.36)$$

where \tilde{a}_1 denotes the total parity of $a_1 \in \mathcal{A}^{\otimes i}$ and $\tilde{m}_j := 2 - j$.

In the next subsection, we will explain the construction of a 2-term A_∞ -algebra from the corresponding 2-term L_∞ -algebra at least for the very particular case. The general statement needs further study.

6.2.1 Higher integrability on strict principal 2-bundles

In order to write down equation (6.34), it is crucial to have a matrix Lie group such that the expressions dg and Ag make sense. Analogously, we consider a crossed module of matrix Lie groups $\mathbf{H} \xrightarrow{\mathbf{t}} \mathbf{G}$, which yields matrix products between elements of $\mathfrak{g} := \text{Lie}(\mathbf{G})$ and \mathbf{G} as well as $\mathfrak{h} := \text{Lie}(\mathbf{H})$ and \mathbf{H} . The Lie brackets are recovered by antisymmetrization of the matrix product. For our construction, we need a further product which turns into the action $\triangleright: \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$ upon antisymmetrization. As we will explain now, the right context to look for such a product is an associative 2-term A_∞ -algebra.

Thus, if we antisymmetrize the products m_i to antisymmetric products μ_i , we obtain a 2-term L_∞ -algebra, cf. [81, 82, 80]. Associative 2-term L_∞ -algebras, in turn, are equivalent to crossed modules of Lie algebras. In more explicit form, the group homomorphism \mathbf{t} is identified with $m_1 = \mu_1$, the commutator on $\mathfrak{g} := \mathcal{A}_0$ is given by $\mu_2: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, the action of \mathfrak{g} onto $\mathfrak{h} := \mathcal{A}_{-1}$ is given by $\mu_2: \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$ and the commutator on \mathfrak{h} is given by $\mu_2 \circ (\mu_1 \otimes \mathbb{1})$. Altogether, we conclude that the higher analogue of demanding a matrix Lie algebra structure instead of merely a Lie algebra structure implies to ask for an A_∞ -algebra underlying the L_∞ -algebra corresponding to the crossed module of Lie algebras. Finally, we also need that the A_∞ -product can be continued to products between the A_∞ -algebra and the crossed module of Lie groups $\mathbf{H} \xrightarrow{\mathbf{t}} \mathbf{G}$, such that we have products of the form

$$m_2: \mathcal{A}_0 \times \mathbf{G} \rightarrow \mathcal{A}_0, \quad m_2: \mathcal{A}_0 \times \mathbf{H} \rightarrow \mathcal{A}_{-1} \quad \text{and} \quad m_2: \mathcal{A}_{-1} \times \mathbf{G} \rightarrow \mathcal{A}_{-1}. \quad (6.37)$$

We now arrived at a complete higher analogue of having a matrix Lie group structure. Here we give a non-trivial example to illustrate the above construction.

Example 6.2.2. *Consider the crossed module of Lie groups $(\mathbf{H} \xrightarrow{\mathbf{t}} \mathbf{G})$ with $\mathbf{G} = \mathbf{H} = \text{GL}(n, \mathbb{C})$, $\mathbf{t} = \text{id}$, and the action \triangleright is just the adjoint action. Then the product m_2*

on the respective elements can be defined by

$$m_2(a, b) := \begin{cases} ab & a, b \in \mathfrak{G} \cup \text{Lie}(\mathfrak{G}) \\ ab & a \in \mathfrak{G} \cup \text{Lie}(\mathfrak{G}), \quad b \in \mathfrak{H} \cup \text{Lie}(\mathfrak{H}), \\ ab^{-1} & a \in \text{Lie}(\mathfrak{H}), \quad b \in \mathfrak{G}, \\ -ab & a \in \text{Lie}(\mathfrak{H}), \quad b \in \text{Lie}(\mathfrak{G}). \end{cases} \quad (6.38)$$

This gives us at least one example for the construction given above.

In general, it is not clear to us how to establish such an A_∞ -algebra for an arbitrary crossed module of Lie algebras $\mathfrak{h} \rightarrow \mathfrak{g}$, but we strongly suspect that there is such a construction, cf. Example 6.2.2. So, we provide the following conjecture.

Conjecture 6.2.3. *For a differential crossed module $\mathfrak{h} \rightarrow \mathfrak{g}$, there is a 2-term A_∞ -algebra $\mathcal{A} := \mathcal{A}_{-1} \oplus \mathcal{A}_0 \supseteq \mathfrak{h} \oplus \mathfrak{g}$ together with “products” $m_1 : \mathcal{A} \rightarrow \mathcal{A}$ and $m_2 : \mathcal{A}^{\otimes 2} \rightarrow \mathcal{A}$ of degrees 1 and 0, respectively, such that antisymmetrization of the products m_i gives the antisymmetric products μ_i of the corresponding 2-term L_∞ -algebra.*

In this section, instead of using the conjecture, we could impose a restriction to crossed modules admitting such a construction. This set is not empty, as the above example shows.

Remark 6.2.4. *The A_∞ -algebra $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0$, resulting from this construction may contain the crossed module as a 2-term L_∞ -subalgebra and therefore might be larger than $\mathfrak{g} \oplus \mathfrak{h}$.*

Now to deal with local connective structures and their curvatures, we have to allow for differential forms on some contractible region U taking values in the subalgebra $\mathfrak{h} \oplus \mathfrak{g}$ of the A_∞ -algebra $\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0$. Recall that $\Omega^\bullet(U)$ is a differential graded algebra, can be viewed as an A_∞ -algebra and there is a natural tensor product between differential graded algebras and A_∞ -algebras. This product yields an A_∞ -algebra $\tilde{\mathcal{A}} := \Omega^\bullet(U) \otimes \mathcal{A}$ where the total degree of an element is the sum of the degree in \mathcal{A} and its form degree. The products are given by

$$\tilde{m}_1(a) := da + (-1)^p m_1(a) \quad \text{and} \quad \tilde{m}_2 = m_2 \quad (6.39)$$

for $a \in \Omega^p(U) \otimes \mathcal{A}$. For the purpose of simplification, we shall write $a * b := \tilde{m}_2(a, b)$. To rewrite these products in terms of the maps \mathbf{t} and \triangleright of the crossed module which are independent of the form degree, we choose the convention of moving all form degrees to the left. Whenever two odd elements are moved past each other, a sign has to be inserted. For example, we have

$$\begin{aligned}
 A * B + B * A &:= \tilde{m}_2(A, B) + \tilde{m}_2(B, A) \\
 &= dx^\mu \wedge dx^\nu \wedge dx^\kappa (m_2(A_\mu, \frac{1}{2}B_{\nu\kappa}) - m_2(\frac{1}{2}B_{\nu\kappa}, A_\mu)) \\
 &= dx^\mu \wedge dx^\nu \wedge dx^\kappa (A_\mu \triangleright \frac{1}{2}B_{\nu\kappa}) \\
 &=: A \triangleright B ,
 \end{aligned} \tag{6.40}$$

where we used some coordinates (x^μ) on U to illustrate the case. In the following subsection, we will provide a brief explanation how we employ the above construction on proving the statements of higher integrability conditions on principal 2-bundles with strict structure Lie 2-groups.

6.2.2 Higher flatness as integrability condition

The above constructions now suggest a higher generalization of the covariant derivative $d + A$ to the operator

$$\nabla := \tilde{m}_1 + A * -B * , \tag{6.41}$$

where we inserted a sign for convenience. This operator will act on formal sums consisting of differential forms with values in \mathbf{G} , \mathfrak{g} and \mathfrak{h} . The detailed action is given in the following lemma.

Lemma 6.2.5. *For $g \in \Omega^0(U) \otimes \mathbf{G}$, $X \in \Omega^p(U) \otimes \mathfrak{g}$ and $Y \in \Omega^q(U) \otimes \mathfrak{h}$, we have the following equations:*

$$\begin{aligned}
 \nabla(g + X + Y) &= dg + dX + dY + (-1)^q \mathbf{t}(Y) + Ag + AX + A * Y \\
 &\quad - B * g - B * X ,
 \end{aligned} \tag{6.42a}$$

$$\nabla^2(g + X + Y) = \mathcal{F} * (g + X + Y) - H * (g + X + Y) . \quad (6.42b)$$

Proof. The expression in (6.42a) is a direct consequence of the definitions of ∇ in (6.41) and \tilde{m}_1 and \tilde{m}_2 .

To compute (6.42b), recall that \tilde{m}_1 satisfies by definition a Leibniz rule

$$\tilde{m}_1(a * b) := \tilde{m}_1(a) * b + (-1)^{\tilde{a}} a * \tilde{m}_1(b) . \quad (6.43)$$

Then we obtain

$$\begin{aligned} \nabla^2(g + X + Y) &= \nabla(\tilde{m}_1(g + X + Y) + A * (g + X + Y) - B * (g + X + Y)) \\ &= \tilde{m}_1(A * (g + X + Y) - B * (g + X + Y)) + \\ &\quad + A * \tilde{m}_1(g + X + Y) - B * \tilde{m}_1(g + X + Y) + \\ &\quad + A * A * (g + X + Y) - A * B * (g + X + Y) + \\ &\quad - B * A * (g + X) - B * A * Y - B * B * (g + X) \\ &= \tilde{m}_1(A) * (g + X + Y) - \tilde{m}_1(B) * (g + X + Y) + \\ &\quad + A * A * (g + X + Y) - A * B * (g + X) - B * A * (g + X) \\ &= \mathcal{F}(g + X) + \mathcal{F} * Y - H * (g + X) , \end{aligned} \quad (6.44)$$

which is the required expression, since $H * Y = 0$. \square

Now we have everything at our disposal to consider the higher analogue of the linear system (6.34) in the context of local connective structures on principal 2-bundles with structure strict Lie 2-groups. And hence we have the following theorem, which is also the main result of this section.

Theorem 6.2.6. *For $g \in \Omega^0(U) \otimes \mathfrak{G}$, and $\Lambda \in \Omega^1(U) \otimes \mathfrak{h}$, the expression*

$$\nabla(g - \Lambda * g) = 0 \quad (6.45)$$

implies that the local connective structure (A, B) is pure gauge and that the curvature $\nabla^2 = (\mathcal{F} - H)$ vanishes.

Proof. Using Lemma 6.2.5 with $X = 0$ and $Y = -\Lambda * g$, we obtain

$$\nabla(g - \Lambda * g) = dg - (d\Lambda) * g - \Lambda * (dg) + \mathfrak{t}(\Lambda)g + Ag - A * \Lambda * g - B * g = 0 . \quad (6.46)$$

We can split this expression by form degree into

$$0 = dg + Ag + \mathfrak{t}(\Lambda)g , \quad (6.47a)$$

and

$$B * g = (-d\Lambda - A \triangleright \Lambda) * g - \Lambda * (dg + Ag) . \quad (6.47b)$$

The expression in (6.47a) states that A is pure gauge, whereas (6.47b) needs further simplification. Thus, to reformulate it, we need $-\frac{1}{2}[\Lambda, \Lambda]$, which can be obtained from

$$\begin{aligned} \Lambda * \mathfrak{t}(\Lambda) &= \frac{1}{2}(\Lambda * \mathfrak{t}(\Lambda) + \Lambda * \mathfrak{t}(\Lambda) - \tilde{m}_1(\Lambda * \Lambda)) \\ &= \frac{1}{2}(\Lambda * \mathfrak{t}(\Lambda) - \mathfrak{t}(\Lambda) * \Lambda) = -\frac{1}{2}\mathfrak{t}(\Lambda) \triangleright \Lambda \\ &= -\frac{1}{2}[\Lambda, \Lambda] , \end{aligned} \quad (6.48)$$

where we use the Leibniz rule as \tilde{m}_1 is a derivation with respect to \tilde{m}_2 and the Peiffer identity². Thus, using (6.48) together with $Y * g * g^{-1} = Y$, we can simplify the expression in (6.47b) as

$$B = -d\Lambda - A \triangleright \Lambda - \frac{1}{2}[\Lambda, \Lambda] . \quad (6.49)$$

Therefore, from (6.47a) and (6.49), we conclude that the local connective structure (A, B) is pure gauge. A local connective structure which is pure gauge is clearly flat. Equivalently, Lemma 6.2.5 implies

$$0 = \nabla^2(g - \Lambda * g) = (\mathcal{F} - H) * (g - \Lambda * g), \quad (6.50)$$

which also leads to the same conclusion, since the expression $(g - \Lambda * g)$ is invertible

²cf. Definition 2.2.14

with inverse $(g^{-1} + \Lambda * g^{-1})$. See also [36] for the general discussion on the inverse of a sum of matrices. \square

Altogether, we have shown how the usual solution and integrability condition for the linear system (6.34) can be translated to the categorified case (6.45) by means of an associative 2-term A_∞ -algebra.

6.2.3 Comments on principal 3-bundles

Finally, let us comment on the case of a local connective structure on principal 3-bundles. Again, the extension of our results in the previous subsection to the case of local connective structures on principal 3-bundles is more or less a mere technicality. One starts from an associative 3-term A_∞ -algebra whose products extend to a 2-crossed module of matrix Lie groups. The covariant derivative is extended by adding a 3-form potential and the generalizations of the linear system (6.45) is rather straightforward. The same holds for the derivation of the analogous statements in Lemma 6.2.5 and Theorem 6.2.6.

Chapter 7

Summary and application

In this chapter we present a summary of the main results and a possible application of our constructions.

On the one hand, for spin groups $\mathrm{Spin}(n)$, the nature of the string group $\mathrm{String}(n)$, $n \geq 3$ is topological, on the other hand it has applications in mathematical physics and higher geometry. Thus, in order to use it as structure 2-group of principal 2-bundles, we need a smooth structure. Therefore, we considered the smooth 2-group model by Schommer-Pries as reviewed in Chapter 3, which is finite dimensional and allows us to do differential geometry, hence the construction of the known string Lie 2-algebra, Theorem 5.2.5. Moreover, the concept of internalization was used to define principal smooth 2-group bundles as internal functors and their cocycle descriptions, cf. Chapter 4, which are generalizations of principal Lie 2-group bundles in [92] as internal functors in $\mathrm{Man}^\infty\mathrm{Cat}$. Furthermore, in Chapter 4, we have commented on a more general description of principal smooth 2-group bundles over Lie groupoids as generalized internal functors, cf. Appendix D. This might lead to a more general notion in this context.

The purpose of the explicit description of the equations of higher gauge theory with string 2-group in Chapter 5 might open a door to construct a particular example of solutions. Here, we want to apply this for the simplest case by considering a principal \mathcal{S}_λ^w -bundle over the base manifold $X = \mathbb{R}^4 \setminus \{0\}$, with a covering $U \rightarrow X$ where $\mathbf{G} = \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. This example is taken from our paper [30, Section 6].

One of the obvious candidate for dynamical constraint on such connective structures is the self-dual string equation in \mathbb{R}^4 . In higher gauge theory, the self-dual string is known to satisfy the equations¹,

$$\mathcal{F} = dA + \frac{1}{2}[A, A] = 0 \quad \text{and} \quad H = dB - \frac{1}{3!}\mu_3(A, A, A) = *d\Phi, \quad (7.1)$$

where the local connective structures on a principal \mathcal{S}_λ^w -bundle is described by a $\mathfrak{spin}(4)$ -valued 1-form A together with a $\mathfrak{u}(1)$ -valued 2-form B . Additionally, we have a $\mathfrak{u}(1)$ -valued function Φ .

Note that the first expression in (7.1) is the *fake-curvature condition*, which implies that A is pure gauge, cf. Section 6.1 and we write

$A = \pi(v)^{-1}d\pi(v)$ for some $v \in \mathcal{C}^\infty(U, V_1)$, cf. Chapter 5. Moreover, since² $d\mu_3(-, -, -) = 0$, we have $d*d\Phi = 0$. Hence, Φ is a harmonic function on $\mathbb{R}^4 \setminus \{0\}$. Thus, we can set $\Phi = \frac{1}{R^2}$, where R is the distance from the origin in \mathbb{R}^4 . Consequently, we can obtain two extreme solutions (A, B) that satisfy the second equation in (7.1).

One is $v = \mathbb{1}_{\mathcal{S}_\lambda^w}$, hence $A = 0$ and

$$B = \frac{3}{8}dx^\mu \wedge dx^\nu \varepsilon_{\mu\nu\kappa\lambda} \frac{x^\kappa \left(R^2 \arctan\left(\frac{r^\lambda}{x^\lambda}\right) - r^\lambda x^\lambda \right)}{R^2 (r^\lambda)^3}, \quad r^\lambda = \sqrt{|x|^2 - (x^\lambda)^2}. \quad (7.2)$$

The other being $B = 0$ and

$$\pi(v) = \left(\frac{1}{R} \begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ -x^3 + ix^4 & x^1 - ix^2 \end{pmatrix}, \mathbb{1} \right). \quad (7.3)$$

The first one is a reformulation of the solution given in [60], the second one is an adaptation of the standard gerbe over $S^3 \sim \mathbf{SU}(2)$. Note that the content of $\pi(v)$ in the second solution can be partially gauged into the other $\mathbf{SU}(2)$ contained in $\mathbf{Spin}(4)$.

It is also easy to deduce that the above two solutions are gauge equivalent as

¹See cf. e.g. [68] or [62].

²In the string Lie 2-algebra, $\mu_3 \in H^3(\mathfrak{g}, \mathbb{R}) \cong H_{dR}^3(\mathbf{G})$.

one would expect. This is because, we have the same Φ , hence it is gauge invariant and thus, the same holds trivially for H . Since both solutions have the same Φ and thus the same H , this implies that they are gauge equivalent.

Altogether, we found that a solution to the non-abelian self-dual string equation (7.1) is gauge equivalent to the usual abelian one. The reason for this was the imposed fake curvature relation $\mathcal{F} = 0$, which guarantees that parallel transport of strings along surfaces is reparameterization invariant, see e.g. [74]. In this case, this equation has no physical relevance for self-dual strings, as the string is perpendicular to the space $\mathbb{R}^4 \setminus \{0\}$, there is no parallel transport within $\mathbb{R}^4 \setminus \{0\}$. For more details on this point, see [62]. A study of non-abelian self-dual string solutions which do not satisfy the fake curvature condition is beyond the scope of this thesis, and we therefore postpone it to future work.

Finally, the proofs of the Poincaré lemma and the integrability conditions on higher principal bundles, cf. Chapter 6 introduce useful approaches to establish generalizations of the known terms and notions to higher structures that might be of interest to both physicists and mathematicians working on higher geometry. For instance, the conjecture in Section 6.2 might give ample motivation for further research in the area.

To finalize this thesis, I would like to mention one important point. The most difficult step in constructing a more general solution of an equation in higher gauge theory in this context is finding the “appropriate” model semistrict 2-term L_∞ -algebra, which is equivalent to the string Lie 2-algebra.

Appendix A

Weak-equivalences in LieGrpd

A Lie groupoid functor $p : \mathcal{G} \rightarrow \mathcal{H}$ is a weak-equivalence if the smooth map

$$\mathrm{tpr}_1 : \mathcal{G}_0 \times_{p_0} \times_{\mathcal{H}_0, \mathrm{t}} \mathcal{H}_1 \rightarrow \mathcal{H}_0 \quad (\text{A.1})$$

is a surjective submersion. And the following diagram is a pullback of smooth manifolds.

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{p_1} & \mathcal{H}_1 \\ (\mathrm{s}, \mathrm{t}) \downarrow & & \downarrow (\mathrm{s}, \mathrm{t}) \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow{p_0 \times p_0} & \mathcal{H}_0 \times \mathcal{H}_0 \end{array}$$

In other words, p is a weak-equivalence if it is a fully faithful, essentially surjective Lie groupoid functor.

Appendix B

Slice (2)-categories

We now give the definition of slice (bi)categories.

Let \mathcal{C} be a small category together with an object $X \in \mathcal{C}$. The slice category of \mathcal{C} over the object X is a category denoted by \mathcal{C}/X and has objects $f : Y \rightarrow X$, for any object $Y \in \mathcal{C}$, which are morphisms in \mathcal{C} , and morphisms between two objects $f : Y \rightarrow X$ and $g : Z \rightarrow X$ are commuting triangles of the form

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & & Z \end{array}$$

i.e. elements $h \in \mathcal{C}(Y, Z)$ with $g \circ h = f$. This can be generalized to bicategories as follows.

For a weak 2-category \mathcal{C} and an object $X \in \mathcal{C}$, the slice weak 2-category \mathcal{C}/X consists of the following data. The objects of \mathcal{C}/X are the 1-morphisms of \mathcal{C} with codomain X . The 1-morphisms of \mathcal{C}/X between objects $f : Y \rightarrow X$ and $g : Z \rightarrow X$ are pairs (h, χ) , where $h : Y \rightarrow Z$ is a 1-morphism of \mathcal{C} and $\chi : f \Rightarrow gh$ is a 2-isomorphism in \mathcal{C} :

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & & X \end{array} \quad \begin{array}{c} \chi \\ \nearrow \\ \searrow \end{array}$$

Finally, consider two 1-morphisms (h_1, χ_1) and (h_2, χ_2) between $f : Y \rightarrow X$ and $g : Z \rightarrow X$ in \mathcal{C}/X . The 2-morphisms from (h_1, χ_1) to (h_2, χ_2) are 2-morphisms $\xi : h_1 \Rightarrow h_2$ of \mathcal{C} such that $\chi_2(f) = (g\xi)\chi_1(f)$.

Appendix C

The weak 2-category $\mathbf{Bibun//\mathcal{G}}$

The objective of this appendix is to give the definition of the weak 2-category $\mathbf{Bibun//\mathcal{G}}$ and to remark on the definition of principal \mathcal{G} -bundles using objects of the weak 2-category $\mathbf{Bibun//\mathcal{G}}$. We start by defining action of 2-group objects in any 2-category \mathcal{C} as discussed in [71].

Definition C.1. *Given a 2-group object \mathcal{G} in a weak 2-category \mathcal{C} with finite products, a \mathcal{G} -object in \mathcal{C} is an object \mathcal{X} in \mathcal{C} together with a 1-morphism $f_{\triangleright} : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$ as well as the 2-isomorphisms $\mathbf{a}_{\alpha} : f_{\triangleright} \otimes (\mathbf{m} \times \text{id}_{\mathcal{X}}) \Longrightarrow f_{\triangleright} \otimes (\text{id}_{\mathcal{G}} \times f_{\triangleright})$ and $\mathbf{l}_{\triangleright} : \alpha \otimes (\mathbf{e} \times \text{id}_{\mathcal{X}}) \Longrightarrow \text{id}_{\mathcal{X}}$, such that the following diagrams 2-commute:*

$$\begin{array}{ccc}
 \mathcal{G} \times \mathcal{G} \times \mathcal{X} & \xrightarrow{\mathbf{m} \times \text{id}_{\mathcal{X}}} & \mathcal{G} \times \mathcal{X} \\
 \text{id}_{\mathcal{G}} \times f_{\triangleright} \downarrow & \swarrow \mathbf{a}_{\triangleright} & \downarrow f_{\triangleright} \\
 \mathcal{G} \times \mathcal{X} & \xrightarrow{f_{\triangleright}} & \mathcal{X}
 \end{array}
 \qquad
 \begin{array}{ccc}
 * \times \mathcal{X} & \xrightarrow{\mathbf{e} \times \text{id}_{\mathcal{X}}} & \mathcal{G} \times \mathcal{X} \\
 \text{pr}_2 \downarrow & \swarrow \mathbf{l}_{\triangleright} & \downarrow f_{\triangleright} \\
 \mathcal{X} & \xrightarrow{\text{id}_{\mathcal{X}}} & \mathcal{X}
 \end{array}$$

Moreover, certain coherence axioms for $\mathbf{a}_{\triangleright}$ and $\mathbf{l}_{\triangleright}$ are satisfied, cf. [71].

We now restrict the above notion to \mathbf{Bibun} in order to discuss the weak 2-category $\mathbf{Bibun//\mathcal{G}}$.

Let \mathcal{G} be a smooth 2-group. Then we can define the weak 2-category $\mathbf{Bibun//\mathcal{G}}$ as follows:

- objects are \mathcal{G} -objects in \mathbf{Bibun} , that is quadruples $(\mathcal{X}, B_{\triangleright}, \mathbf{a}_{\triangleright}, \mathbf{l}_{\triangleright})$, where

$$B_{\triangleright} : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X} \tag{C.1}$$

is a right principal bibundle, can be called as the left action of \mathcal{G} on \mathcal{X} , and $\mathbf{a}_\triangleright$ and $\mathbf{l}_\triangleright$ are bibundle isomorphisms

$$\mathbf{a}_\triangleright : B_\triangleright \otimes (B_m \times 1) \Longrightarrow B_\triangleright \otimes (1 \times B_\triangleright) , \quad (\text{C.2})$$

$$\mathbf{l}_\triangleright : B_\triangleright \otimes (\mathbf{e} \times 1) \Longrightarrow 1 \quad (\text{C.3})$$

satisfying some coherence conditions. See [71] for the definition of action of a group object and the coherence conditions.

- a 1-morphism between $(\mathcal{X}, B_\triangleright, \mathbf{a}_\triangleright, \mathbf{l}_\triangleright)$ and $(\mathcal{Z}, B_\triangleright, \mathbf{a}_\triangleright, \mathbf{l}_\triangleright)$ is a pair (B, χ) , which consists of a right principal bibundle $B : \mathcal{X} \rightarrow \mathcal{Z}$ and a bibundle isomorphism $\chi : B_\triangleright \otimes (1 \times B) \Longrightarrow B \otimes B_\triangleright$ subject to a certain coherence condition, cf. [71].
- a 2-morphism between the 1-morphisms (B_1, χ) and (B_2, η) is a bibundle isomorphism $\xi : B_1 \Longrightarrow B_2$ subject to a commutative diagram, cf. [71].

Thus, a principal smooth 2-group bundle over a smooth manifold X consists of objects $(\mathcal{P}, B_\triangleright)$ and (X, \mathbf{pr}_2) in $\mathbf{Bibun}/\mathcal{G}$ together with a 1-morphism (B_π, χ) of these objects $B_\pi : (\mathcal{P}, B_\triangleright) \rightarrow (X, \mathbf{pr}_2)$ and an equivalence $(B_\triangleright, \mathbf{pr}_2) : \mathcal{G} \times \mathcal{P} \rightarrow \mathcal{P} \times_X \mathcal{P}$ in \mathbf{Bibun}/X , see also Proposition 2.2.13 and Section 4.2.

Appendix D

Generalized internal functors in Bibun

Here we introduce the notion of generalized internal functors in **Bibun**, which is important to generalize the previously discussed smooth 2-group principal 2-bundles, see also Subsection 4.4. So, similar to generalized morphisms in **LieGrpd**, we can define generalized internal functors in **Bibun** as follows.

Definition D.1. *Let \mathcal{C} and \mathcal{D} be internal categories in **Bibun**, then a generalized internal functor $\mathfrak{R} : \mathcal{C} \rightarrow \mathcal{D}$ consists of internal functors in **Bibun***

$$\mathcal{C} \xleftarrow{\Delta} \mathcal{E} \xrightarrow{\Phi} \mathcal{D} , \tag{D.1}$$

where \mathcal{E} is an internal category equivalent to \mathcal{C} , and Δ is an equivalence, which is an equivalence of internal categories keeping the smooth structures on them. Thus, as right principal bibundles, we demand both Δ_0 and Δ_1 are bibundle equivalences.

This gives a more general description of the previously known constructions. A particular case of this definition is also found in [34]. The authors have discussed Lie 2-groupoid generalized morphisms in order to give a general definition of strict principal 2-bundles¹ over Lie groupoids. Recall that Lie 2-groupoids are special examples of internal categories in **Bibun**.

¹By strict principal 2-bundles we mean principal 2-bundles with strict structure 2-groups.

Now let us discuss some examples by considering the particular cases.

Example D.2 ([34], Ordinary principal \mathbf{G} -bundles). *Consider a principal \mathbf{G} -bundle P over a smooth manifold X . In the language of generalized internal functors, we express it as*

$$\begin{array}{ccccc}
 X \rightrightarrows X & & P \times_X P \rightrightarrows P \times_X P & & \mathbf{G} \rightrightarrows \mathbf{G} \\
 \begin{array}{c} \text{s} \parallel \text{t} \\ \Downarrow \\ X \rightrightarrows X \end{array} & \xleftarrow{\Delta} & \begin{array}{c} \text{s} \parallel \text{t} \\ \Downarrow \\ P \rightrightarrows P \end{array} & \xrightarrow{\Psi} & \begin{array}{c} \text{s} \parallel \text{t} \\ \Downarrow \\ * \rightrightarrows * \end{array}
 \end{array}$$

where all the bibundles are bundlizations of smooth maps. To be precise, the right principal bibundles Ψ_i , for $i = 0, 1$ are bundlizations of the smooth maps obtained from the principal left \mathbf{G} -action on P and Δ_i , for $i = 0, 1$ are also bundlizations of the surjective submersion $\pi : P \rightarrow X$, which is an equivalence. Here, if we take a good cover $U \hookrightarrow X$ and the corresponding cocycles (g_{ij}) , we will have an expression of principal \mathbf{G} -bundles as generalized internal functors in **Bibun** as we commented in Section 4.4.

Example D.3 ([34], strict principal 2-bundles). *Let $\mathbf{H} \xrightarrow{\text{t}} \mathbf{G}$ be a crossed module of Lie groups. Then for a strict principal 2-bundle $\pi : P \rightarrow X$ with covering $U \hookrightarrow X$, and corresponding cocycles (g_{ij}, h_{ijk}) , we can define a generalized internal functor*

$$\begin{array}{ccccc}
 X \rightrightarrows X & & U^{[2]} \times \mathbf{H} \times \mathbf{H} \rightrightarrows U^{[2]} \times \mathbf{H} & & \mathbf{G} \times \mathbf{H} \rightrightarrows \mathbf{G} \\
 \begin{array}{c} \text{s} \parallel \text{t} \\ \Downarrow \\ X \rightrightarrows X \end{array} & \xleftarrow{\Delta} & \begin{array}{c} \text{s} \parallel \text{t} \\ \Downarrow \\ U \rightrightarrows U \end{array} & \xrightarrow{\Phi} & \begin{array}{c} \text{s} \parallel \text{t} \\ \Downarrow \\ * \rightrightarrows * \end{array}
 \end{array}$$

where the source and target maps in $U^{[2]} \times \mathbf{H} \times \mathbf{H} \rightrightarrows U^{[2]} \times \mathbf{H}$ are

$$\text{s}(x_{ij}, h_1, h_2) = (x_{ij}, h_1) , \quad \text{t}(x_{ij}, h_1, h_2) = (x_{ij}, h_2) , \quad (\text{D.2})$$

and all the bibundles are bundlizations of the Lie groupoid functors

$$\begin{aligned}
 \Delta_0(x_{ij}, h) &= x, & \Delta_1(x_{ij}, h_1, h_2) &= x, \\
 \Phi_0(x_{ij}, h) &= g_{ij}(x), & \Phi_1(x_{ij}, h_1, h_2) &= (\mathfrak{t}(h_{ijk})g_{ij}g_{ik}, h_1h_2),
 \end{aligned}
 \tag{D.3}$$

for all $h, h_{1,2} \in \mathbf{H}$, $x_{ij} \in U^{[2]}$.

Appendix E

Higher distributions leading to differential ideals

In this appendix, we briefly present a relation between certain higher distributions and differential ideals, generalizing the correspondence between ordinary involutive distributions and differential ideals generated by 1-forms. This is a first step towards a generalized Frobenius theorem.

Recall that a distribution \mathcal{D} is a smoothly varying family of subspaces \mathcal{D}_x of the fibers $T_x M$ of the tangent bundle of some manifold M . It is involutive if the Lie algebra of vector fields closes on sections of \mathcal{D} . That is, for any point $x \in M$, there is a neighbourhood U_x and vector fields $X_1, \dots, X_r \in \mathfrak{X}(U_x)$ such that the X_i are linearly independent and at each point $y \in U_x$, \mathcal{D}_y is spanned by the X_i . Extending these vector fields to a local basis X_1, \dots, X_d of TM , we have

$$[X_i, X_j] = f_{ij}^k X_k \quad \text{with} \quad \overline{f_{ij}^k} = 0 \quad (\text{E.1})$$

where the f_{ij}^k are functions on U_p and overlined and underlined indices \overline{i} and \underline{i} denote indices $i \leq r$ and $i > r$, respectively.

Recall that by the Frobenius theorem, such an involutive distribution induces a regular foliation of the manifold M .

The Lie algebra of vector fields in (E.1) has a dual Chevalley–Eilenberg algebra,

which is encoded in the relations

$$d\theta^k = -\frac{1}{2}f_{ij}^k\theta^i \wedge \theta^j, \quad (\text{E.2})$$

where the 1-forms θ^i locally span T^*M and satisfy $\theta^i(X_j) = \delta_j^i$. Note that because of $f_{ij}^k = 0$, the 1-forms θ^i form a differential ideal.

This yields the modern formulation of the Frobenius theorem, which states that for a differential ideal on a manifold M which is generated by 1-forms, there are submanifolds $e : N_x \hookrightarrow M$ for each point $x \in M$ such that $x \in N_x$ and $e^*\alpha = 0$ for any α in the differential ideal.

Let us now generalize the correspondence between certain distribution and differential ideals. We start by recalling some basic facts on multivector fields.

Consider a patch U of a d -dimensional manifold M together with the set of multivector fields $\mathfrak{X}^\bullet(U) := \Gamma(TU) \oplus \Gamma(\wedge^2 TU) \oplus \dots \oplus \Gamma(\wedge^d TU)$. On $\mathfrak{X}^\bullet(U)$, there is a natural generalization of the Lie bracket, which fulfills the Leibniz rule with respect to the \wedge -product:

Definition E.1. *The Schouten–Nijenhuis bracket is the bilinear extension to $\mathfrak{X}^\bullet(U)$ of*

$$\begin{aligned} [V_1 \wedge \dots \wedge V_m, W_1 \wedge \dots \wedge W_n]_S := \\ \sum_{i,j=1}^{m,n} (-1)^{i+j} [V_i, W_j] \wedge V_1 \wedge \dots \wedge \hat{V}_i \wedge \dots \wedge V_m \wedge W_1 \wedge \dots \wedge \hat{W}_j \wedge \dots \wedge W_n, \end{aligned} \quad (\text{E.3})$$

where $V_i, W_j \in \mathfrak{X}^1(U)$ and $\hat{}$ indicates an omission.

Note that the Schouten–Nijenhuis bracket turns the complex $\mathfrak{X}^\bullet(U)$ into a graded Lie algebra L_0 . This graded Lie algebra has a dual Chevalley–Eilenberg algebra description in terms of forms in $\Omega^\bullet(U)$. Given a local basis θ^i, ξ^a, \dots of linearly independent 1-forms, 2-forms, ..., spanning $T_x^*U, \wedge^2 T_x^*U, \dots$ at every $x \in U$ we have

$$d\theta^i = -\frac{1}{2}f_{jk}^i\theta^j \wedge \theta^k, \quad d\xi^a = -d_{ib}^a\theta^i \wedge \xi^b, \quad \dots, \quad (\text{E.4})$$

where the f_{jk}^i are the structure constants of the Lie algebra of vector fields and the

additional structure constants d_{ib}^a are functions on U determined by the f_{jk}^i . As the θ^i, ξ^a, \dots form a complete basis, we can also write these relations as

$$\begin{aligned} d\theta^i &= -\frac{1}{2}\tilde{f}_{jk}^i\theta^j \wedge \theta^k + \tilde{t}_a^i\xi^a, \\ d\xi^a &= -\tilde{d}_{ib}^a\theta^i \wedge \xi^b - \frac{1}{3!}\tilde{c}_{ijk}^a\theta^i \wedge \theta^j \wedge \theta^k, \\ &\dots \end{aligned} \tag{E.5}$$

where $\xi^a = m_{ij}^a\theta^i \wedge \theta^j$ and

$$f_{jk}^i = \tilde{f}_{jk}^i + \tilde{t}_a^i m_{jk}^a, \quad d_{ib}^a m_{jk}^b = \tilde{d}_{ib}^a m_{jk}^b + \tilde{c}_{ijk}^a, \quad \dots \tag{E.6}$$

Equation (E.5) describes the Chevalley–Eilenberg algebra of a strong homotopy Lie algebra, see [48, 54] for a definition and more details.

Proposition E.2. *The tilded structure constants in (E.5) define a strong homotopy Lie algebra on the graded vector space of multivector fields $\mathfrak{X}^\bullet(U)$.*

In particular, in terms of a basis $X_i \in \mathfrak{X}^1(U), Y_a \in \mathfrak{X}^2(U), \dots$ dual to that of $\Omega^\bullet(U)$ used above, we have the following higher brackets:

$$\begin{aligned} \mu_1(Y_a) &= \tilde{t}_a^i X_i, & \mu_2(X_i, X_j) &= \tilde{f}_{ij}^k X_k, \\ \mu_2(X_i, Y_a) &= \tilde{d}_{ia}^b Y_b, & \mu_3(X_i, X_j, X_k) &= \tilde{c}_{ijk}^a Y_a, \\ &\dots \end{aligned} \tag{E.7}$$

The two underlying Chevalley–Eilenberg complexes of the Lie algebra L_0 given by the Schouten–Nijenhuis bracket and any L_∞ -algebra on $\mathfrak{X}^\bullet(U)$ given by a rewriting as in (E.5) are essentially identical. Therefore, there is an L_∞ -algebra isomorphisms between these, which motivates the following definition.

Definition E.3. *An L_∞ -algebra associated to the Lie algebra L_0 is an L_∞ -algebra-structure on $\mathfrak{X}^\bullet(U)$ with higher brackets as in (E.7) obtained by a rewriting of the underlying Chevalley–Eilenberg algebra of L_0 as in (E.5).*

Finally, note that we can truncate the structures introduced above from $\mathfrak{X}^\bullet(U)$ to multivector fields of a maximal degree n . In particular, we can evidently truncate

the Schouten–Nijenhuis bracket to the complex

$$\mathfrak{X}_{(n)}(U) = TU \longleftarrow \wedge^2 TU \longleftarrow \wedge^3 TU \longleftarrow \cdots \longleftarrow \wedge^n TU \quad (\text{E.8})$$

by setting

$$[X_1 \wedge \cdots \wedge X_{k_1}, Y_1 \wedge \cdots \wedge Y_{k_2}] := 0 \quad (\text{E.9})$$

for $X_i, Y_i \in \mathfrak{X}^1(U)$ and $k_1 + k_2 > n + 1$. The associated L_∞ -algebras come then with higher brackets satisfying

$$\mu_k(X_1, \dots, X_k) := 0 \quad (\text{E.10})$$

for homogeneously graded $X_i \in \mathfrak{X}^{|X_i|} \subset \mathfrak{X}_{(n)}(U)$ and $k > n + 1$ or $|X_1| + \cdots + |X_k| > n + 1$.

We now come to a generalization of the notion of distribution based on multi-vector fields.

Definition E.4. *An n -distribution on a d -dimensional manifold M with $n \leq d$ is a sequence of distributions $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ such that \mathcal{D}_i is a distribution in $\wedge^i TM$.*

The notion of a pre-involutive distribution is now defined as follows:

Definition E.5. *An n -distribution \mathcal{D} on a manifold M is called pre-involutive, if there is an L_∞ -algebra associated to L_0 , which closes on \mathcal{D} .*

In the case $n = 1$, the above two definitions trivially reduce to those of an ordinary distribution and an ordinary involutive distribution.

In the following, let again $X_i \in \mathfrak{X}^1(U)$, $Y_a \in \mathfrak{X}^2(U)$, ... form a local basis spanning TU , $\wedge^2 TU$, ... and let X_i , $i \leq r_1$, Y_a , $a \leq r_2$, ... span a pre-involutive n -distribution $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n)$. We shall again underline indices larger than r_i and overline indices that are less or equal to r_i . Using this notation, we can characterize the structure constants of L_∞ -algebras on pre-involutive n -distributions in more detail.

Lemma E.6. *The closure of an L_∞ -algebra associated to L_0 on a pre-involutive n -distribution is equivalent to its structure constants $s_{\beta_1 \dots \beta_k}^\alpha = (\tilde{t}_a^i, \tilde{f}_{ij}^k, \tilde{d}_{ia}^b, \tilde{c}_{ijk}^a, \dots)$*

satisfying

$$s_{\beta_1 \dots \beta_k}^\alpha = 0 . \quad (\text{E.11})$$

Let us now switch to the dual picture and consider the Chevalley–Eilenberg description of the above n -term L_∞ -algebra. That is, we have a local basis of forms $\theta^i \in \Omega^1(U)$, $\xi^a \in \Omega^2(U)$, ... with $i_{X_i} \theta^j = \delta_i^j$, $i_{Y_a} \xi^b = \delta_a^b$, etc. Closure of an associated L_∞ -algebra on a pre-involutive n -distribution amounts here to the following:

Theorem E.7. *The forms θ^i, ξ^a, \dots spanning the annihilators of the distributions contained in a pre-involutive n -distribution generate a differential ideal.*

Proof. The Chevalley–Eilenberg description of the L_∞ -algebra associated to L_0 is of the form

$$d\omega^\alpha = \sum_k s_{\beta_1 \dots \beta_k}^\alpha \omega^{\beta_1} \wedge \dots \wedge \omega^{\beta_k} \quad (\text{E.12})$$

for general forms $\omega^\alpha \in \Omega^1(U) \oplus \dots \oplus \Omega^n(U)$. With Lemma (E.6), we conclude that

$$-d\omega^\alpha = \sum_k s_{\beta_1 \beta_2 \dots \beta_k}^\alpha \omega^{\beta_1} \wedge \omega^{\beta_2} \wedge \dots \wedge \omega^{\beta_k} , \quad (\text{E.13})$$

which states that the ω^α generate a differential ideal. □

Note that in the case $n = 1$, this is just the familiar statement that the annihilator of an integrable distribution spans a differential ideal.

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